Computing Techniques and Mathematical Tools (CTMT)

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Who will Teach

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Optimization and search techniques

What is Optimization Techniques

- **Definition:** Optimization is the process of making something as effective, perfect, or functional as possible. In mathematical terms, it involves finding the maximum or minimum value of a function subject to constraints.

Applications

- Supply chain and logistics (e.g., minimizing transportation costs)
- Finance (e.g., portfolio optimization)
- Engineering (e.g., design optimization)

Example

- Consider Company has two product **A** and **B**. The profit for each product is ₹40 and ₹30, respectively.
- The company has 120 hours of labor and 150 kg of raw material available.
- Each product **A** requires 2 hours of labor and 3 kg of material, and each product **B** requires 1 hour of labor and 2 kg of material.
- **Objective**: How many units of each product should be produced to maximize profit

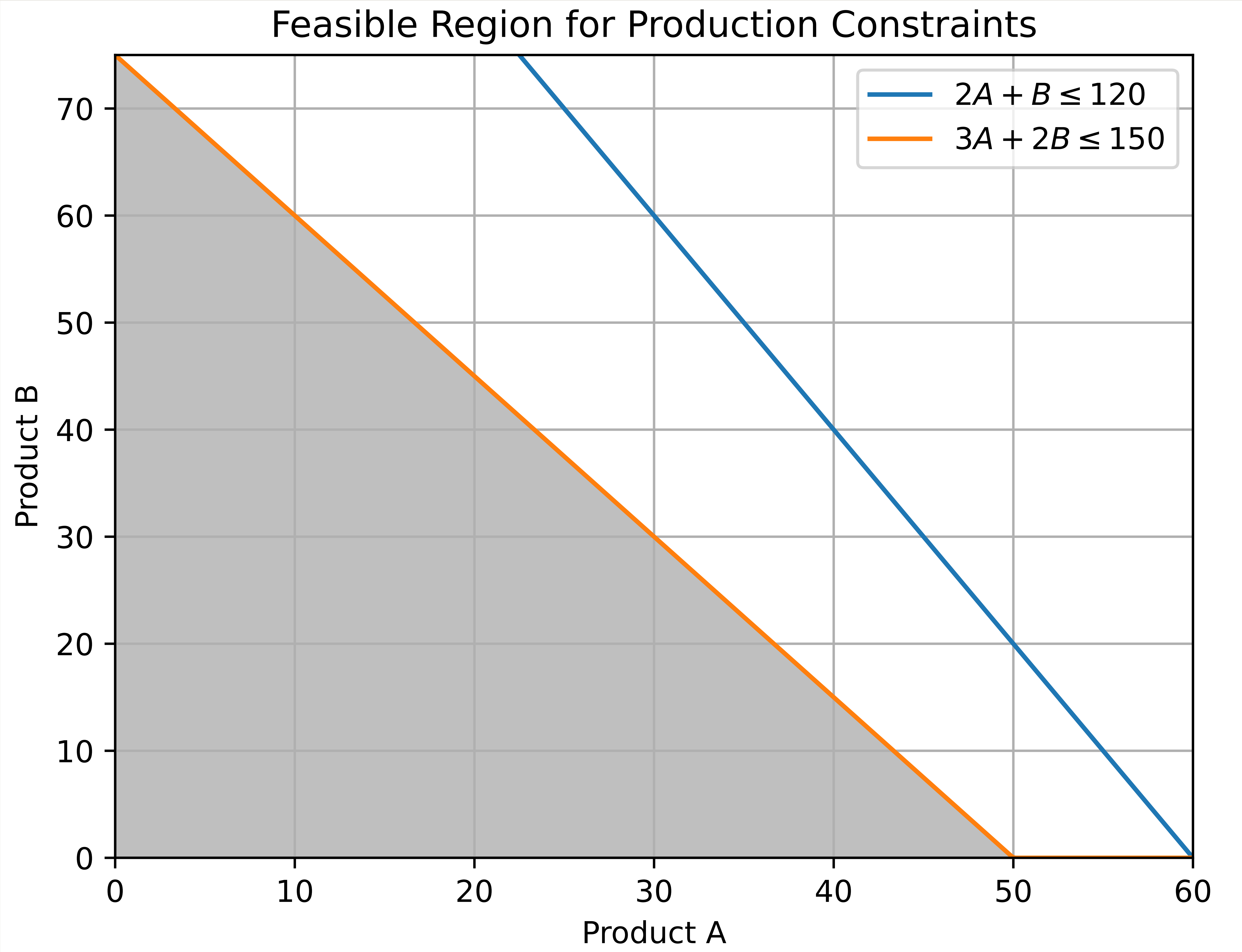
Given Data

- Profit for product A: ₹40 per unit
- Profit for product B: ₹30 per unit
- Total labor hours available: 120 hours
- Total raw material available: 150 kg
- Labor requirement for product A: 2 hours per unit
- Labor requirement for product B: 1 hour per unit
- Material requirement for product A: 3 kg per unit
- Material requirement for product B: 2 kg per unit

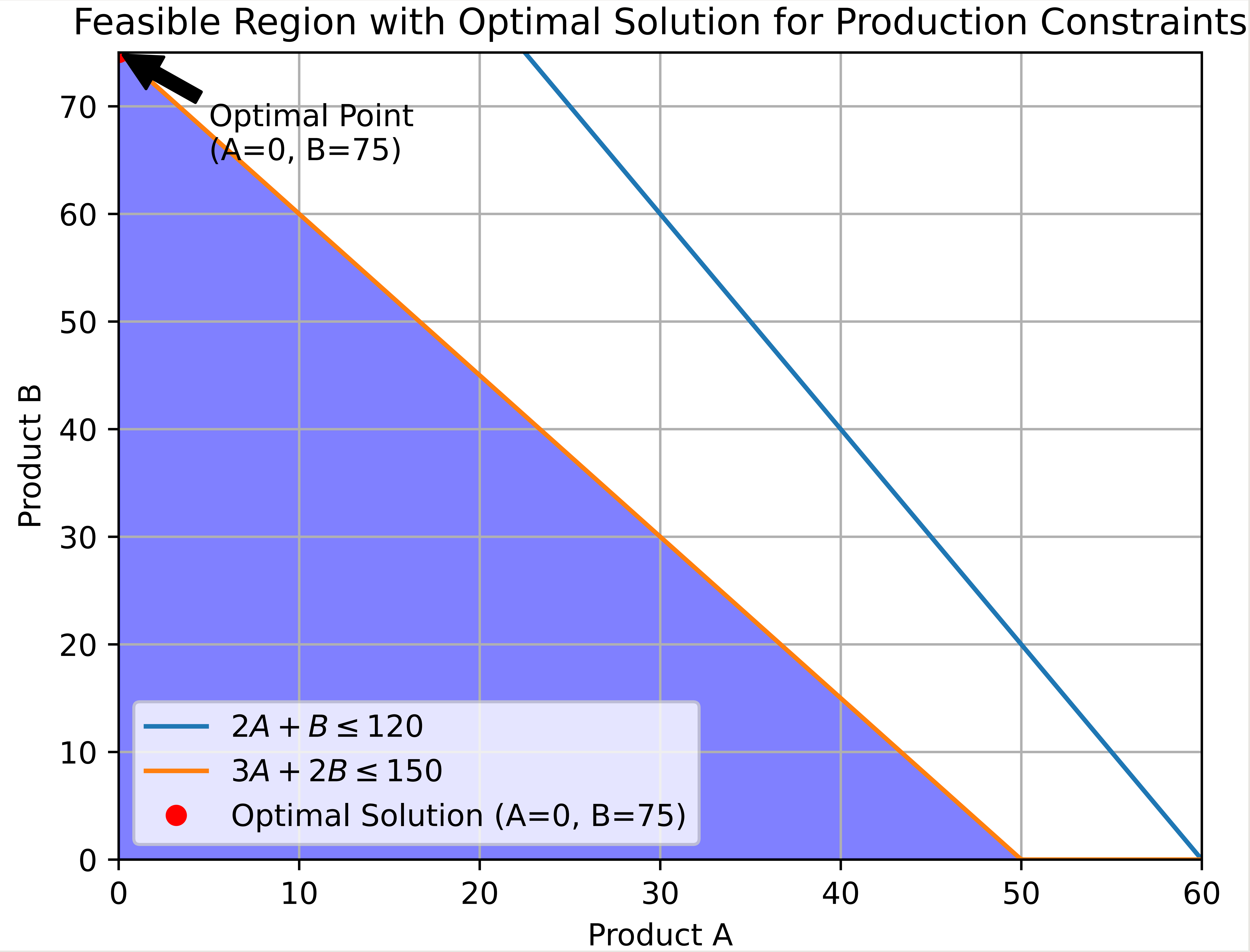
Mathematical Representation

$$
\begin{aligned}
\text{Maximize} \quad & P = 40A + 30B \\
\text{subject to} \quad & \mathbf{C1}: 2A + B \le 120 \\
& \mathbf{C2}: 3A + 2B \le 150 \\
& A \ge 0 \\
& B \ge 0
\end{aligned}
$$

Constraints Plot



Constraints Plot



IO

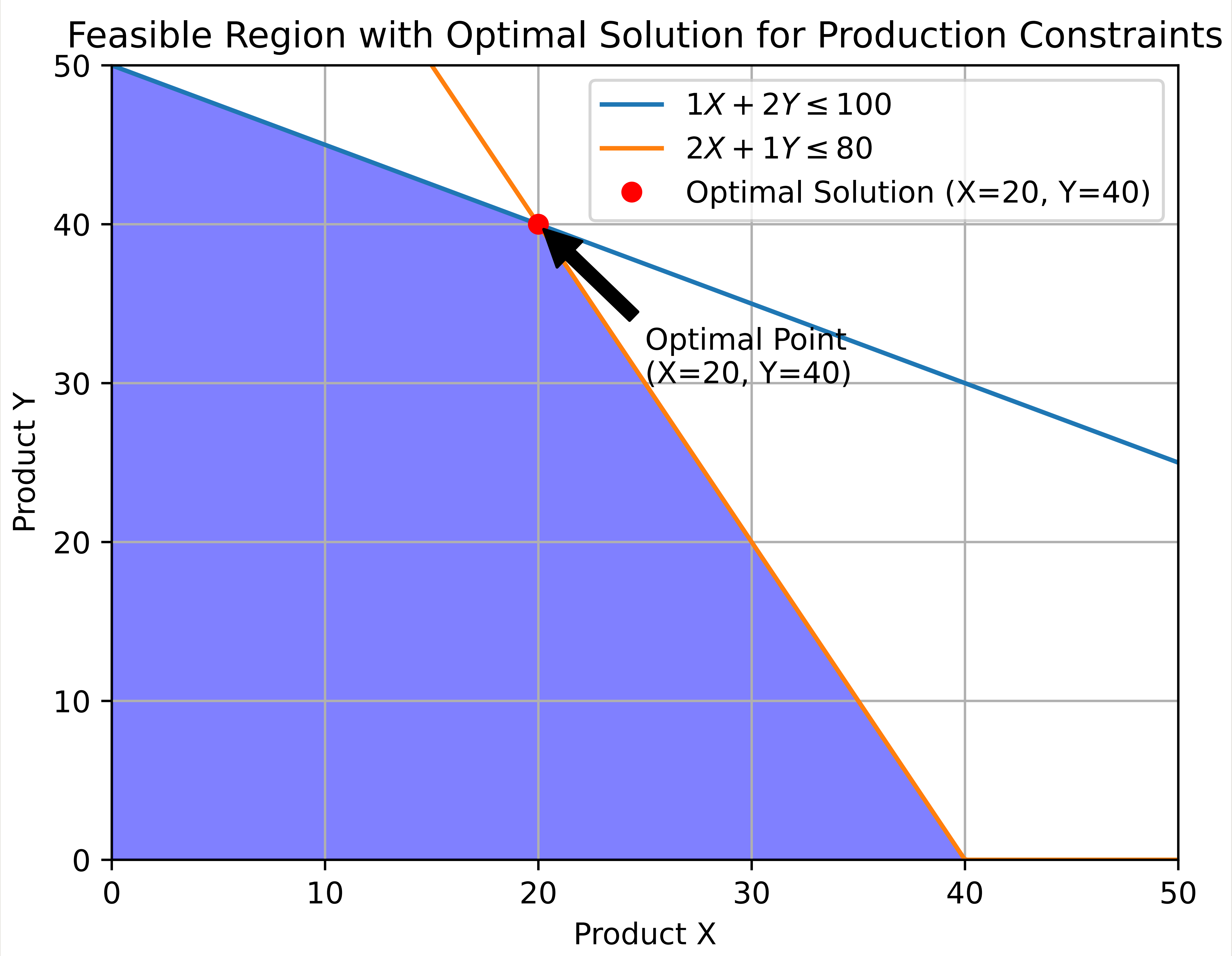
Example 2

- Consider Company has two product **X** and **Y**. The profit for each product is ₹50 and ₹40, respectively.
- The company has 100 hours of labor and 80 kg of raw material available.
- Each product **X** requires 1 hours of labor and 2 kg of material, and each product **Y** requires 2 hour of labor and 1 kg of material.
- **Objective**: How many units of each product should be produced to maximize profit

Mathematical Representation

$$
\begin{aligned}
\text{Maximize} \quad & P = 50X + 40Y \\
\text{subject to} \quad & \mathbf{C1}: 1X + 2Y \le 100 \\
& \mathbf{C2}: 2X + 1Y \le 80 \\
& X \ge 0 \\
& Y \ge 0
\end{aligned}
$$

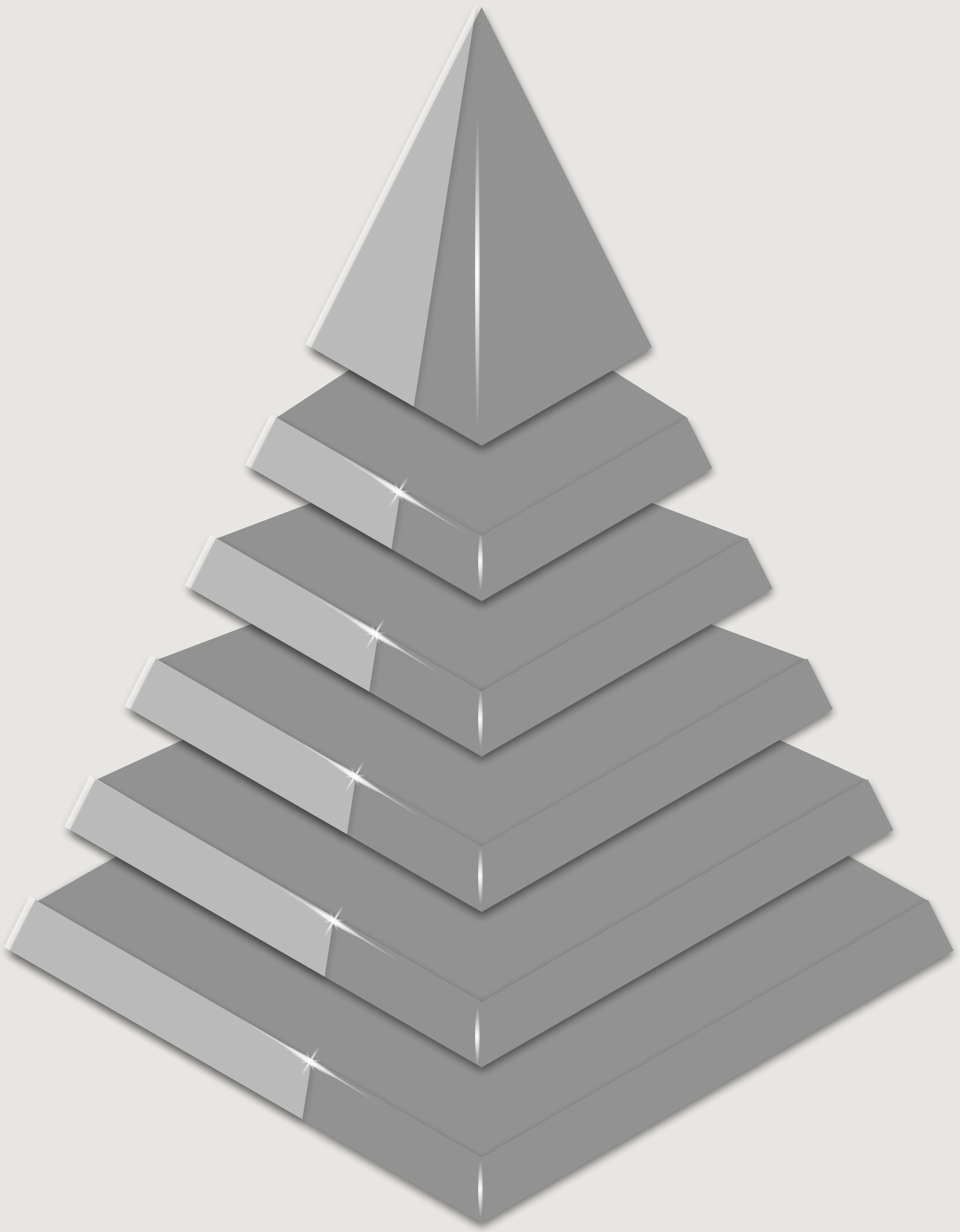
Constraints Plot



Optimal Problem Formulation

- **Step 1:** Design Variables (Identifying variables, which are primarily varied during the optimization process.)
- **Step 2:** Constraints: identify the constraints associated with the optimization problem.
  - Functional relationships among the design variables and other design parameters satisfying certain physical phenomenon and certain resource limitations.
- **Step 3:** Objective Function: objective function in terms of the design variables and other problem parameters.
- **Step 4:** Variable Bounds: procedure is to set the minimum and the maximum bounds on each design variable.

Flow Chart

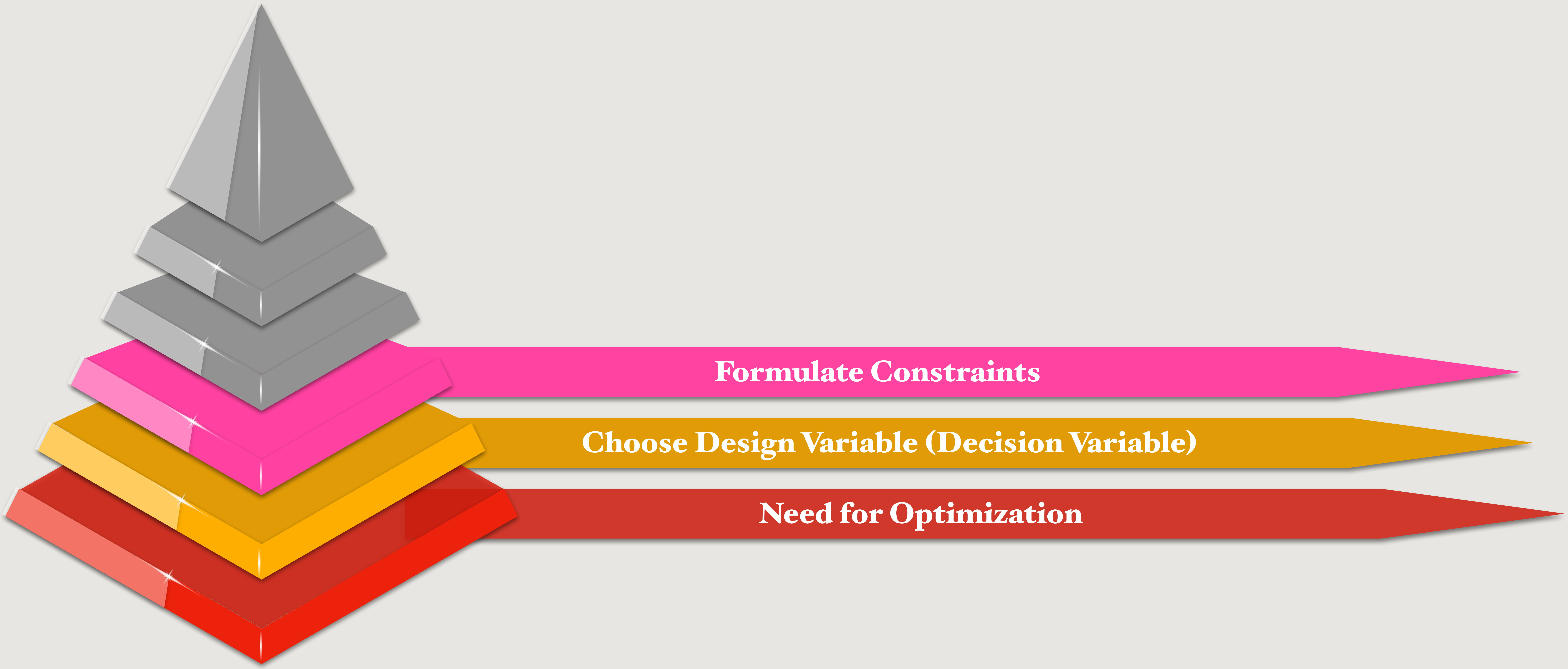


Flow Chart

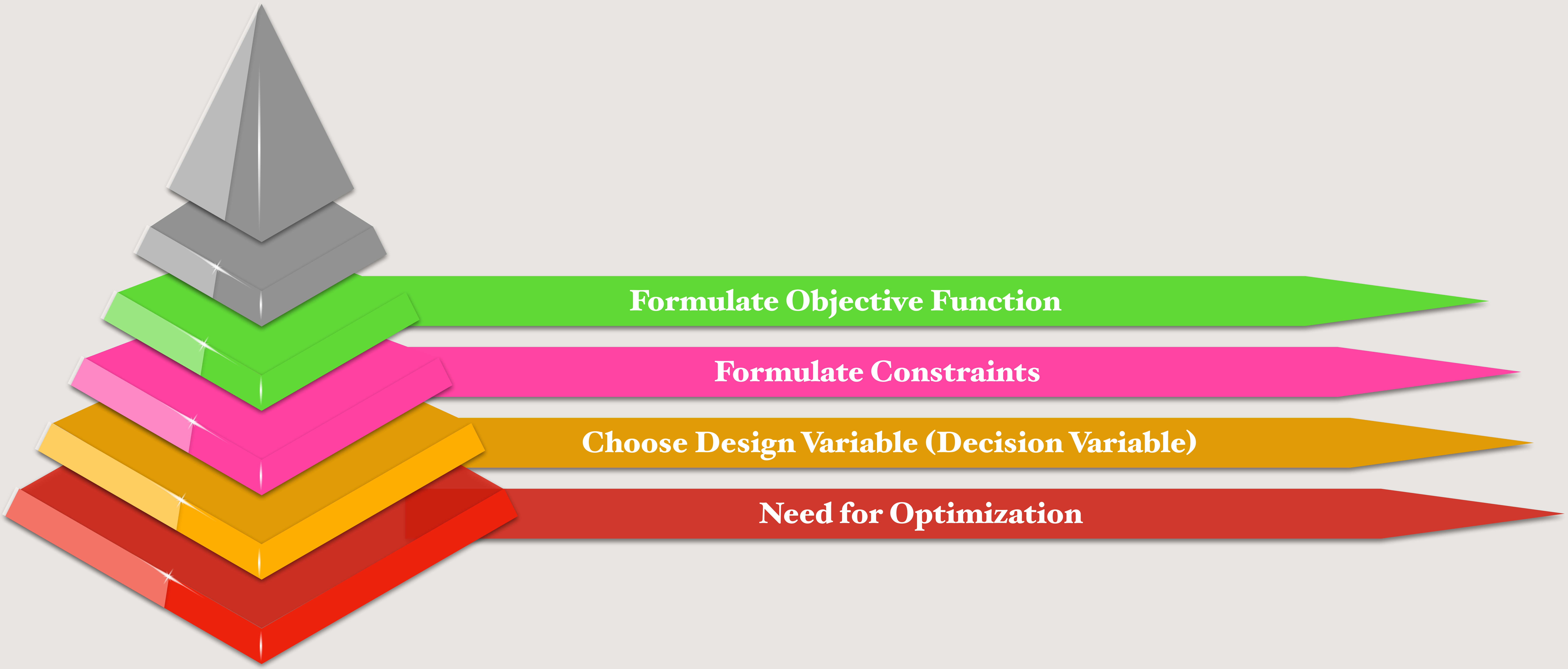
Flow Chart


Choose Design Variable (Decision Variable)
Need for Optimization

Flow Chart


Formulate Constraints
Choose Design Variable (Decision Variable)
Need for Optimization

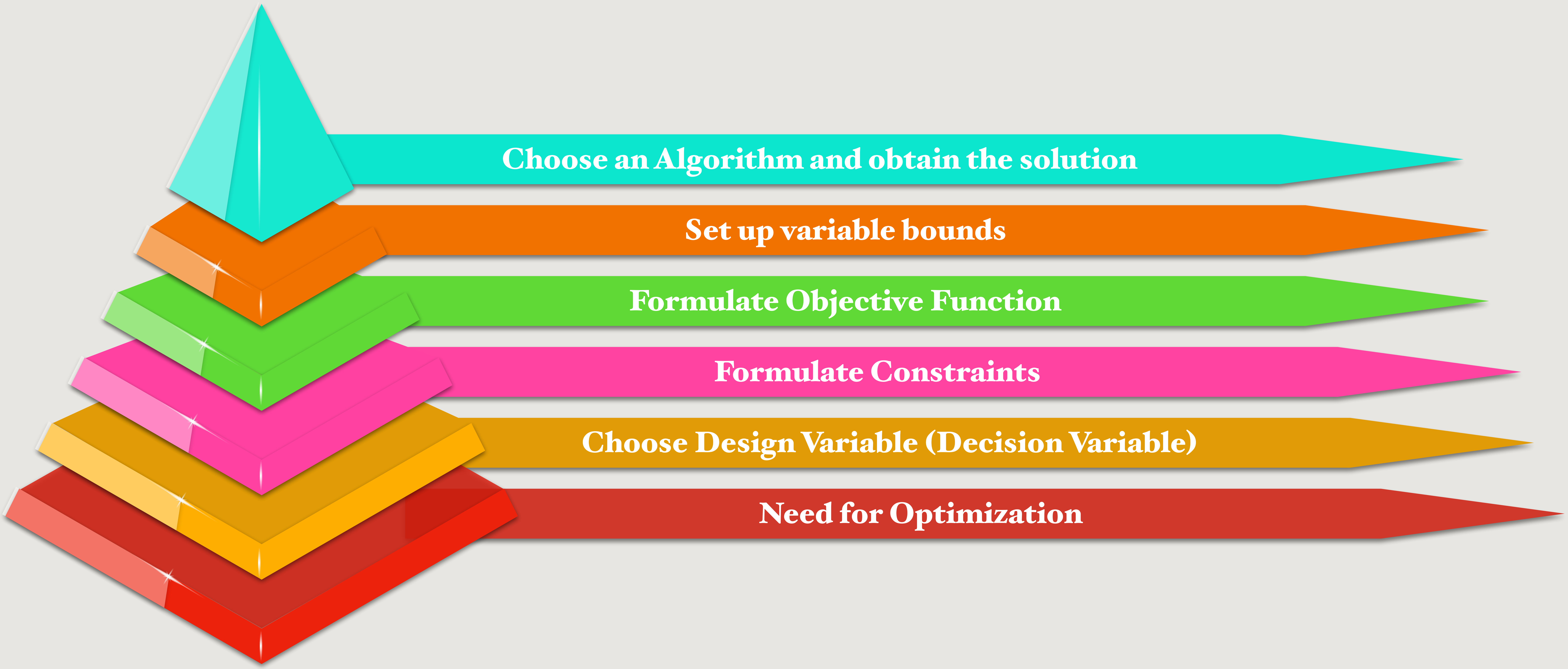
Flow Chart


Formulate Objective Function
Formulate Constraints
Choose Design Variable (Decision Variable)
Need for Optimization

Flow Chart


Set up variable bounds
Formulate Objective Function
Formulate Constraints
Choose Design Variable (Decision Variable)
Need for Optimization

Flow Chart


Choose an Algorithm and obtain the solution
Set up variable bounds
Formulate Objective Function
Formulate Constraints
Choose Design Variable (Decision Variable)
Need for Optimization

Challenges in Optimization

- **Complexity:**
  - High-dimensional problems.
- **Non-convexity:**
  - Multiple local optima.
- **Computation:**
  - Large-scale problems require significant computational resources.

Key Ideas to Design Constraints for Complex Problems

- **Understand the Problem Thoroughly:**
  - Clearly define the objective function and understand the nature of the constraints.
  - Identify all variables involved and how they interact with each other.
  - Gather all necessary data and understand the physical, economic, or logical restrictions that govern the problem.

Key Ideas to Design Constraints for Complex Problems

- **Formulate the Objective Function:**
  - Express the goal mathematically (e.g., maximize profit, minimize cost).
  - Ensure the objective function accurately reflects the real-world scenario.
- **Identify and Classify Constraints:**
  - **Equality Constraints:** Constraints that specify that two expressions must be equal (e.g., balance equations in chemical processes).
    - $g_i(x) = 0$
  - **Inequality Constraints:** Constraints that specify that one expression must be greater or less than another (e.g., capacity limits).
    - $h_i(x) \leq 0$
  - **Bound Constraints:** Constraints that specify the upper and lower bounds for variables

Key Ideas to Design Constraints for Complex Problems

- **Simplify When Possible:**
  - Reduce the number of variables and constraints through dimensional analysis or by exploiting problem symmetries.
  - Use approximations or relaxations for constraints to simplify the problem.
- **Hierarchical Approach:**
  - Decompose large problems into smaller, more manageable sub-problems.
  - Solve sub-problems individually and then integrate their solutions.
- **Iterative Refinement:**
  - Start with a simpler version of the problem and iteratively refine the constraints and objective function.
  - Use insights from initial solutions to improve problem formulation.

Constrained Optimization

Introduction to Constrained Optimization

- **Definition:**
  - Constrained optimization involves finding the best solution to a problem within a set of restrictions or constraints.
- **Key Components:**
  - **Objective Function:** The function to be optimized (maximized or minimized).
  - **Constraints:** Conditions that must be satisfied for the solution to be valid.

Categories of Optimization Problems

- **Linear vs. Non-linear:** Linear problems have linear constraints and objectives, while non-linear problems involve non-linear functions.
- **Discrete vs. Continuous:** Discrete involves variables that can only take on specific values, continuous involves variables that can take any value within a range.
- **Deterministic vs. Stochastic:** Deterministic problems have known parameters, while stochastic involve randomness and uncertainty.

Linear Optimization

- **Characteristics:** Constraints and objective functions are linear equations.
- **Applications:** Used in resource allocation, manufacturing plans, etc.

Non-linear Optimization

- **Characteristics:** At least one non-linear constraint or objective function.
- **Types:** Quadratic programming, convex and non-convex problems.
- **Example:** Path Problems

Stochastic Optimization

- **Usage:** Ideal for dealing with uncertainties in parameters or model.
- **Difference from Deterministic:** The role of probability and scenarios.
- **Example:** Portfolio optimization under uncertain market conditions.

Linear Programming

(LP)

Introduction to Linear Programming

- **Definition:** Linear Programming (LP) is a mathematical technique used for optimization by maximizing or minimizing a linear objective function, subject to a set of linear constraints.
- **History:** Developed during World War II to optimize resource allocation.

Key Concepts and Terminology

- **Objective Function:** The function to be optimized (e.g., maximize profit, minimize cost).
- **Decision Variables:** The variables that impact the objective function.
- **Constraints:** The limitations or requirements (e.g., resource limits).
- **Feasible Region:** The set of all possible solutions that satisfy the constraints.

Solving Linear Programming Problems

- **Graphical Method:** Used for problems with two variables.
  - Plot the constraints on a graph.
  - Identify the feasible region.
  - Determine the objective function's optimal value at the vertices of the feasible region.
- **Simplex Method:** Used for larger problems.

Simplex Method

- **Definition:** The Simplex Method is an algorithm used to solve linear programming problems. It efficiently finds the optimal solution by moving along the edges of the feasible region.
- **Invented by:** George Dantzig in 1947.

Formulating LP Problems in Standard Form

- **Objective Function:** The function you want to maximize or minimize. For example, Maximize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$.
- **Constraints:** These are linear inequalities that limit the values of the decision variables. For example,

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$\vdots$$

- $$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
- **Non-negativity Constraints:** All decision variables must be non-negative.

Problem Formulation

$$\text{Maximize} \quad Z = 3x_1 + 2x_2 + 4x_3$$

$$2x_1 + 3x_2 + x_3 \leq 4 \quad \text{(Constraint 1)}$$

$$x_1 + x_2 + 2x_3 \leq 3 \quad \text{(Constraint 2)}$$

$$x_1, x_2, x_3 \geq 0$$

Simplex Method Overview

- Convert constraints into equations.
- Set up the initial simplex tableau.
- Perform pivot operations to find the optimal solution.

Convert constraints into equations

$$\text{Maximize} \quad Z = 3x_1 + 2x_2 + 4x_3$$

$$2x_1 + 3x_2 + x_3 + s_1 = 4$$

$$x_1 + x_2 + 2x_3 + s_2 = 3$$

Mixed Integer Linear Programming

MILP

MILP

- **Definition:** MILP is a mathematical optimization or feasibility program where some variables are required to be integers.
- **Components:**
  - Linear objective function
  - Linear constraints
  - Integer constraints on some variables

Why MILP?

- **Flexibility:** Can model a wide range of problems.
- **Precision:** Provides exact solutions under given constraints.
- **Applicability:** Used in various fields like logistics, finance, manufacturing, etc.

Applications of MILP

- **Logistics:** Vehicle routing, supply chain management.
- **Finance:** Portfolio optimization, risk management.
- **Manufacturing:** Production planning, scheduling.
- **Energy:** Power generation, distribution optimization.

Formulating MILP Problems

- **Objective Function:** Maximize or Minimize a linear function.

  Maximize/Minimize $c^T X$

- **Constraints:** Subject to a set of linear inequalities/equalities.

  $Ax \leq b$

- **Integer Constraints:** Some variables must take integer values.

  $x_i \in \mathbb{Z} \quad \forall i \in I$

Example Problem Formulation

- **Objective:** Minimize cost
- **Constraints:** Budget limit, resource constraints
- **Integer Variables:** Number of items to produce, number of trucks, etc.

Solving MILP Problems

- **Exact Methods:**
  - Branch and Bound
  - Branch and Cut
  - Cutting Planes
- **Heuristic Methods:**
  - Genetic Algorithms
  - Simulated Annealing

Software Tools for MILP

- **Commercial:**
  - CPLEX
  - Gurobi
- **Open Source:**
  - CBC (COIN-OR)
  - GLPK
- **Interfaces:**
  - Python (PuLP, Pyomo)
  - MATLAB
  - R

Case Study: Warehouse Location Problem

- **Problem Description:**
  - The Warehouse Location Problem involves selecting locations for warehouses from a set of potential locations to minimize the total cost of building the warehouses and transporting goods to various demand points (customers). The problem takes into account fixed costs for opening warehouses and variable costs for transportation.

Decision Variables

- y_j : Binary variable that equals 1 if a warehouse is opened at location j is taken, otherwise 0
- $x_{i,j}$: Continuous variable representing the amount of goods shipped from warehouse j to customers i

Parameters

- f_j : Fixed cost of opening a warehouse at location j
- $c_{i,j}$: Variable cost of shipping one unit of goods from warehouse j to customers i
- d_i : Demand at customer i
- s_j : Supply capacity of warehouse j
- n : Number of potential warehouse locations.
- m : Number of customers.

Any Questions?