Channel Flow Database Functions: interpolation and differentiation

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The channel flow database functions for the JHU Turbulence Database Cluster are discussed here. The functions discussed consist of spatial interpolation, spatial differentiation, and temporal interpolation. Other functionality such as spatial filtering and integration methods such as particle tracking will be add later.

1 Spatial interpolation inside the database

Spatial interpolation is applied using multivariate polynomial interpolation of the barycentric Lagrange form from Ref. [1]. Using this approach, we are interested in interpolating the field f at point $\boldsymbol{x}'$. The point $\boldsymbol{x}'$ is known to exist within the grid cell at location (x_m, y_n, z_p) where (m, n, p) are the cell indices. The cell indices are obtained for the x and z directions, which are uniformly distributed, according to

$$
\begin{aligned}
m &= \mathrm{floor}(x'/dx) \\
p &= \mathrm{floor}(z'/dz)\ .
\end{aligned}
$$

In the y direction the grid is formed by Marsden-Schoenberg collocation points which are not uniformly distributed. Along this direction we perform a binary search to obtain n such that $y_n \le y' < y_{n+1}$. The cell indices are also assured to obey the following:

$$
\begin{aligned}
0 \le m &\le N_x - 2 \\
0 \le n &\le N_y - 2 \\
0 \le p &\le N_z - 2
\end{aligned}
$$

where N_x, N_y, and N_z are the number of grid points along the x, y, and z directions, respectively. In the case that $x' = x_{N_x-1}$ the cell index set to be $m = N_x - 2$; likewise for the y and z directions.

The interpolation stencil also contains q points in each direction for an order q interpolant (with degree $q-1$). The resulting interpolated value is expressed as:

$$
f(\boldsymbol{x}') = \sum_{i=i_s}^{i_e} \sum_{j=j_s}^{j_e} \sum_{k=k_s}^{k_e} c_x^i(x') c_y^j(y') c_z^k(z') f(x_i, y_j, z_k) \tag{1}
$$

where the starting and ending indices are given as

$$
\begin{aligned}
i_s &= m - q/2 + 1 \\
i_e &= i_s + q - 1 \\
j_s &= n - q/2 + 1 + j_o \\
j_e &= j_s + q - 1 \\
k_s &= p - q/2 + 1 \\
k_e &= k_s + q - 1
\end{aligned}
$$

and j_o is the index offset for the y direction depending on the distance from the top and bottom walls. The value for j_o may be evaluated based upon the y cell index and the interpolation order as

$$j_o = \begin{cases} \max(q/2 - n, 0) & \text{if } n \leq (N_y - 1)/2 \\ \min(N_y - 1 - n - q/2, 0) & \text{otherwise} \end{cases} . \tag{2}$$

The interpolation weights, c_x, c_y, and c_z, are given as

$$c_\theta^\xi(\theta') = \frac{\frac{w_\xi}{\theta' - \theta_\xi}}{\sum_{\eta=\eta_s}^{\eta_e} \frac{w_\eta}{\theta' - \theta_\eta}} \tag{3}$$

where θ may either be x, y, or z. The barycentric weights, w_ξ, in (3) are given as

$$w_\xi = \frac{1}{\prod_{\eta=\eta_s, \eta \neq \xi}^{\eta_e} \theta_\xi - \theta_\eta} \tag{4}$$

The weights may be computed by applying a recursive update procedure as in Ref.[1]. A slightly modified version of the algorithm in Ref. [1] is given below:

for $\xi = \xi_s$ to ξ_e **do**
  $w_\xi = 1$
end for
for $\xi = \xi_s + 1$ to ξ_e **do**
  for $\eta = \xi_s$ to $\xi - 1$ **do**
    $w_\eta = (\theta_\eta - \theta_\xi) w_\eta$
    $w_\xi = (\theta_\xi - \theta_\eta) w_\xi$
  end for
end for
for $\xi = \xi_s$ to ξ_e **do**
  $w_\xi = 1/w_\xi$
end for

To account for the periodic domain along the x and z directions we may adjust the i and k indices when referencing f in (1) such that

$$f(\boldsymbol{x}') = \sum_{i=i_s}^{i_e} \sum_{j=j_s}^{j_e} \sum_{k=k_s}^{k_e} c_x^i(x') c_y^j(y') c_z^k(z') f(x_{i\%Nx}, y_j, z_{k\%Nz}) \tag{5}$$

and % is the modulus operator. The indices for the interpolation coefficients will remain the same, however, we may use the fact that the grid points are uniformly spaced such that (3) becomes

$$c_{\theta}^{\xi}(\theta') = \frac{\frac{w_{\xi}}{\theta' - \xi d\theta}}{\sum_{\eta=\eta_s}^{\eta_e} \frac{w_{\eta}}{\theta' - \eta d\theta}} \tag{6}$$

and similarly for the barycentric weights, (4) becomes

$$w_{\xi} = \frac{1}{\prod_{\eta=\eta_s, \eta \neq \xi}^{\eta_e} (\xi - \eta) d\theta} \tag{7}$$

for the x and z directions. The computation of the barycentric weights for the x and z directions may be carried out once (for a given interpolation order) for all grid points using (7); for the y direction the barycentric weights will have to be computed for each point using (4).

2 Spatial differentiation inside the database

References

[1] J.P. Berrut and L.N. Trefethen. Barycentric lagrange interpolation. *SIAM*, 46(3):501–517, 2004.