Channel Flow Database Functions: interpolation and differentiation

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The channel flow database functions for the JHU Turbulence Database Cluster are discussed here. The functions discussed consist of spatial interpolation, spatial differentiation, and temporal interpolation. Other functionality such as spatial filtering and integration methods such as particle tracking will be add later.

1 Spatial interpolation inside the database

Spatial interpolation is applied using multivariate polynomial interpolation using the barycentric Lagrange interpolation of Ref. [1]. Using this approach, we are interested in interpolating the field f at point $\boldsymbol{x}'$. The point $\boldsymbol{x}'$ is known to exist within the grid cell at location (x_m, y_n, z_p) where (m, n, p) are the cell indices. The interpolation stencil also contains q points in each direction for a q order interpolant. The resulting interpolated value is expressed as:

$$f(\boldsymbol{x}') = \sum_{i=i_s}^{i_e} \sum_{j=j_s}^{j_e} \sum_{k=k_s}^{k_e} c_x^i(x') c_y^j(y') c_z^k(z') f(x_i, y_j, z_k) \tag{1}$$

where the starting and ending indices are given as

$$\begin{aligned} i_s &= m - q/2 + 1 \\ i_e &= i_s + q - 1 \\ j_s &= n - q/2 + 1 + j_o \\ j_e &= j_s + q - 1 \\ k_s &= p - q/2 + 1 \\ k_e &= k_s + q - 1 \end{aligned}$$

and j_o is the index offset for the y direction depending on the distance from the top and bottom walls (discussed below). The interpolation weights c_x, c_y, c_z are given as:

$$c_\theta^\xi(\theta') = \frac{\frac{w_\xi}{\theta' - \theta_\xi}}{\sum_{\eta=\eta_s}^{\eta_e} \frac{w_\eta}{\theta' - \theta_\eta}} \tag{2}$$

and θ may be x, y, or z. The barycentric weights, w_ξ, in (2) are given as

$$w_\xi = \frac{1}{\prod_{\eta=\eta_s, \eta \neq \xi}^{\eta_e} \theta_\xi - \theta_\eta} \tag{3}$$

The weights may be computed by applying a recursive update procedure as in Ref.[1], but slightly modified, resulting in the following algorithm:

for $\xi = \xi_s$ to ξ_e **do**
  $w_\xi = 1$
end for
for $\xi = \xi_s + 1$ to ξ_e **do**
  for $\eta = \xi_s$ to $\xi - 1$ **do**
    $w_\eta = (\theta_\eta - \theta_\xi) w_\eta$
    $w_\xi = (\theta_\xi - \theta_\eta) w_\xi$
  end for
end for
for $\xi = \xi_s$ to ξ_e **do**
  $w_\xi = 1/w_\xi$
end for

The index offset for the y direction may be evaluated based upon the y cell index and the interpolation order as

$$j_o = \begin{cases} \max(q/2 - n, 0) & \text{if } n \leq n_y/2 \\ \min(n_y - n - q/2, 0) & \text{otherwise} \end{cases} \tag{4}$$

where n_y are the number of grid points in the y direction.

To account for the periodic domain along the x and z directions we may adjust the i and k indices when referencing f in (1) to

$$f(\boldsymbol{x}') = \sum_{i=i_s}^{i_e} \sum_{j=j_s}^{j_e} \sum_{k=k_s}^{k_e} c_x^i(x') c_y^j(y') c_z^k(z') f(x_{(i-1)\%Nx+1}, y_j, z_{(k-1)\%Nz+1}) \tag{5}$$

and % is the modulus operator. The indices for the interpolation coeficients will remain the same, however, we may use the fact that the grid points are uniformly spaced such that (2) becomes

$$c_\theta^\xi(\theta') = \frac{\frac{w_\xi}{\theta' - (\xi - 1)d\theta}}{\sum_{\eta=\eta_s}^{\eta_e} \frac{w_\eta}{\theta' - (\eta - 1)d\theta}} \tag{6}$$

and similarly for the barycentric weights, (3) becomes

$$w_\xi = \frac{1}{\prod_{\eta=\eta_s, \eta \neq \xi}^{\eta_e} (\xi - \eta) d\theta} \tag{7}$$

along the x and z directions. The computation of the barycentric weights for the periodic weights may be carried out once for all interpolation cells using (7); for the y direction the barycentric weights will have to be computed for each interpolation cell using (3).

2 Spatial differentiation inside the database

References

[1] J.P. Berrut and L.N. Trefethen. Barycentric lagrange interpolation. *SIAM*, 46(3):501–517, 2004.