Math 56 Winter 2025 Midterm Exam

Student name (print): ______________________

Date: ______________________

Start/end time: ______________________

This exam contains 16 pages (including this cover page) and 8 questions. The total number of possible points is 63.

- **Take the exam in one six-hour sitting,** and mark your starting/ending times at the top of the exam in the space provided. The exam should not take the full six hours. You are allowed to take reasonably short breaks.
- **The exam is open-book and open-note, but NOT "open-internet" or "open-neighbor".** Discussion about the exam with other students (or other people) is not allowed. This includes chatbots such as ChatGPT. An exception to the internet rule are the YouTube videos I have prepared for the course, as well as the classroom lecture recordings on Panapto.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive less credit. *There may be more blank space provided than is actually needed to answer each question.*
- **Use of calculators/code is permitted.**
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations may still receive partial credit.
- **Provide exact answers** unless otherwise instructed.
- **Simplify all answers as much as possible.**
- **Clearly identify your final answer for each problem from any scratch work.**

Question	Points	Score
1	10	
2	12	
3	5	
4	5	
5	5	
6	8	
7	12	
8	6	
Total:	63	

Do not write in the table to the right. **Good luck!**

1. (10 points) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Prove that

$$\mathbf{A}^{\dagger} = \lim_{\delta \to 0^{+}} \left(\mathbf{A}^{T}\mathbf{A} + \delta \mathbf{I} \right)^{-1} \mathbf{A}^{T}. \tag{1}$$

Hint: use the SVD, and pick a convenient matrix norm.

2. (12 points) *Part (a):* Find an elementary lower triangular matrix $\mathbf{M}$ such that

$$\mathbf{M}\begin{bmatrix}2\\2\\1\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}. \tag{2}$$

Part (b): Find the Householder reflector $\mathbf{H} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T \in \mathbb{R}^{3\times 3}$ (with $\mathbf{u}$ a unit vector) such that

$$\mathbf{H}\begin{bmatrix}2\\2\\1\end{bmatrix} = \begin{bmatrix}\star\\0\\0\end{bmatrix}, \tag{3}$$

where "$\star$" denotes a generic nonzero entry. You can simply state the vector $\mathbf{u}$ and do not need to explicitly compute the entries of $\mathbf{H}$.

Part (c): Find (you do not need to give the entries explicitly) an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{5\times 5}$ such that

$$\mathbf{Q}\begin{bmatrix}5\\4\\2\\2\\1\end{bmatrix} = \begin{bmatrix}\star\\\star\\\star\\0\\0\end{bmatrix}. \tag{4}$$

3. (5 points) Let $\mathbf{A} = \hat{\mathbf{Q}}\hat{\mathbf{R}}$ be the economic QR decomposition where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Explain how the Cholesky factorization $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ can be obtained directly from the QR decomposition with no additional cost.

4. (5 points) Explain how the decomposition $\mathbf{PAQ} = \mathbf{LU}$ arising in Gaussian elimination with complete pivoting (GECP) can be used to solve a linear system $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is invertible.

5. (5 points) Let $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{n \times n}$ where is invertible. Suppose that $\|\mathbf{A}\mathbf{y} - \mathbf{b}\|_2$ is "small". Does this imply that $\|\mathbf{y} - \mathbf{x}\|_2$ is "small" as well? Why or why not? Explain.

6. (8 points) Suppose that we have computed the economic QR decomposition $\mathbf{A}_n = \hat{\mathbf{Q}}_n \hat{\mathbf{R}}_n$ where $\mathbf{A}_n = [\mathbf{a}_1, \ldots, \mathbf{a}_n] \in \mathbb{R}^{m \times n}$ $(m > n)$ has full column rank. Assuming that $\mathbf{a}_{n+1} \notin \text{col}(\mathbf{A}_n)$, explain how the economic QR decomposition $\mathbf{A}_{n+1} = [\mathbf{A}_n, \mathbf{a}_{n+1}] = \hat{\mathbf{Q}}_{n+1} \hat{\mathbf{R}}_{n+1}$ can be computed cheaply by making use of the previously computed factors $\hat{\mathbf{Q}}_n$ and $\hat{\mathbf{R}}_n$. *Bonus: how can this procedure be modified to still apply even when* $\mathbf{a}_{n+1} \in \text{col}(\mathbf{A}_n)$*?*

7. (12 points) Part (a): Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ be of the form $\mathbf{C} = \mathbf{I} + \mathbf{u}\mathbf{u}^T$ for some $\mathbf{u} \in \mathbb{R}^n$. Show that $\mathbf{C}$ is symmetric positive definite, and find an expression for $\mathbf{C}^{-1}$. *Hint:* $\mathbf{C}^{-1}$ *has the form* $\mathbf{C}^{-1} = \mathbf{I} + a\mathbf{u}\mathbf{u}^T$ *for some* $a \in \mathbb{R}$.

   Part (b): Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be of the form $\mathbf{A} = \mathbf{B} + \mathbf{u}\mathbf{u}^T$ where $\mathbf{B}$ is symmetric positive definite and $\mathbf{u} \in \mathbb{R}^n$. Show that $\mathbf{A}$ can be factorized as

$$\mathbf{A} = \mathbf{L}\left(\mathbf{I} + \mathbf{v}\mathbf{v}^T\right)\mathbf{L}^T \tag{5}$$

   where $\mathbf{L}$ is invertible lower-triangular and $\mathbf{v} \in \mathbb{R}^n$ is some vector (to be found).

   Part (c): Assume that the Cholesky factorization of $\mathbf{B}$ has already been computed. Outline an efficient procedure to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ without computing the Cholesky factorization of $\mathbf{A}$, where $\mathbf{A}$ is of the form in Part (b). How many flops does your procedure require? "Big-O" notation is fine.

8. (6 points) You have been hired as a computational scientist at `Pseudoinverse Inc.`, and your first task is to design a specialized method for computing matrix-vector products of the form $\mathbf{L}^\dagger \mathbf{x}$ given an input vector $\mathbf{x} \in \mathbb{R}^n$, where $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix} \in \mathbb{R}^{(n-1)\times n}. \tag{6}$$

Describe every numerical method that you know (number them individually, clearly) which could be used to compute these matrix-vector products. Then, choose the method you believe is the best for the job and write a short pitch (2-3 sentences) for your boss to read detailing why this is the best. *Hint: amongst other methods, you might consider the approximation* $\mathbf{L}^\dagger \approx (\mathbf{L}^T\mathbf{L} + \delta \mathbf{I})^{-1}\mathbf{L}^T$. *Also, depending on the argument you make, there may not be a single best method.*