Problem 1

Prove that

$$1 + 4 + 7 + \cdots + (3n-2) = \frac{n(3n-1)}{2} \tag{1}$$

for all $n \geq 1$.

Proof. We consider this proof in two parts:

- For $n = 1$ note that $1 = 1 \cdot (3-1)/2 = 1$; thus the equality holds.
- For $n > 1$, we have the sum up to $3n - 2$:

  $$1 + 4 + 7 + \cdots + (3(n-1)-2) + (3n-2).$$

  By the induction hypothesis, we know that the statement holds true for all $1 \leq k < n$. Observe the statement for $n - 1$:

  $$1 + 4 + 7 + \cdots + (3(n-1)-2) = \frac{(n-1)(3(n-1)-1)}{2}.$$

  Adding $3n - 2$, we get:

  $$\frac{(n-1)(3(n-1)-1)}{2} + (3n-2).$$

  Simplifying the first term and expanding the sum gives us:

  $$\begin{aligned} &= \frac{(n-1)(3n-4)}{2} + \frac{2(3n-2)}{2} \\ &= \frac{(n-1)(3n-4)+6n-4}{2} \\ &= \frac{3n^2-n}{2} = \frac{n(3n-1)}{2} \\ &= \frac{3n^2-7n+4+6n-4}{2} = \frac{3n^2-n}{2} \end{aligned}$$

  Therefore the statement holds for all $n > 1$.

Bringing the two parts together, we observe that the statement holds for all n.

□

Problem 2

Prove that

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{2}$$

for all $n \geq 1$.

Proof. We consider this proof in two parts:

- For $n = 1$ note that $1 = 1 \cdot (1+1)(1+3)/6 = 1$; thus the equality holds.

- For $n > 1$, we have the sum of all positive integers up to n. Notice that:

$$\sum_{i-1}^{n} i = \sum_{i-1}^{n-1} i + n^2$$

By the induction hypothesis, we know that the statement holds true for all $1 \leq k < n$. Observe the statement for $n - 1$, which gives us:

$$\sum_{i-1}^{n-1} i = \frac{(n-1)(n)(2(n-1)+1)}{6} = \frac{(n-1)(n)(2n-1)}{6}$$

Adding this back to our original statement, we get:

$$\begin{aligned}
\sum_{i-1}^{n} i &= \frac{(n-1)(n)(2n-1)}{6} + n^2 \\
&= \frac{(n)(n-1)(2n-1) + 6n^2}{6} \\
&= \frac{(n)((n-1)(2n-1) + 6n)}{6} \\
&= \frac{(n)(2n^2 - 3n + 1 + 6n)}{6} \\
&= \frac{(n)(2n^2 + 3n + 1)}{6} \\
&= \frac{(n)(n+1)(2n+1)}{6}
\end{aligned}$$

Therefore the statement holds for all $n > 1$.

Bringing the two parts together, we observe that the statement holds for all n.

$\square$

Problem 3

Prove that

$$2\cos(2x) + 2\cos(4x) + \cdots + 2\cos(2kx) = \frac{\sin((2n+1)x)}{\sin(x)} - 1 \tag{3}$$

for all $n \geq 1$.

Proof. We consider this proof in two parts:

- For $n = 1$ note that

$$\frac{\sin((2+1)x)}{\sin(x)} - 1 = \frac{\sin(3x)}{\sin(x)} - 1 = \frac{\sin(2x)\cos(x) + \sin(x)\cos(2x) - \sin(x)}{\sin(x)}$$

Expanding further we get:

$$\begin{aligned}
\frac{\sin((2+1)x)}{\sin(x)} - 1 &= \frac{(2\sin(x)\cos(x) + \sin(x)\cos(2x)}{\sin(x)} - 1 \\
&= \frac{(2\sin(x)\cos(x))\cos(x)}{\sin(x)} + \cos(2x) - 1 \\
&= 2\cos(x)\cos(x) + \cos(2x) - 1
\end{aligned}$$

Now recall that $\cos(2x) = \cos(x)^2 - \sin(x)^2 \implies \cos(x)^2 = \cos(2x) + \sin(x)^2$ and also $\sin(x)^2 + \cos(x)^2 = 1$. Substituting these in gives us:

$$\begin{aligned}
&= \cos(x)\cos(x) + (\cos(2x) + \sin(x)^2) + \cos(2x) - \sin(x)^2 - \cos(x)^2 \\
&= \cos(2x) + \cos(2x) = 2\cos(2x)
\end{aligned}$$

thus the statement holds true for $n = 1$.

- For $n > 1$, notice that:

$$2\cos(2x)\cdots + 2\cos(2(n-1)x) + 2\cos(2nx)$$

By the induction hypothesis, we know that the statement holds true for all $1 \leq k < n$. Observe the statement for $n - 1$, which gives us:

$$2\cos(2x) + 2\cos(4x) + \cdots + 2\cos(2(n-1)x) = \frac{\sin((2(n-1)+1)x)}{\sin(x)} - 1$$

Adding this back to our original statement, we get:

$$2\cos(2x)\cdots + 2\cos(2(n-1)x) + 2\cos(2nx) = \frac{\sin((2(n-1)+1)x)}{\sin(x)} - 1 + 2\cos(2nx)$$

Simplifying this further and using the sine cosine addition formulas, we get:

$$\begin{aligned}
&= \frac{\sin((2n-1)x)}{\sin(x)} + 2\cos(2nx) - 1 \\
&= \frac{\sin(2nx)\cos(x) - \sin(x)\cos(2nx)}{\sin(x)} + 2\cos(2nx) - 1 \\
&= \frac{\sin(2nx)\cos(x)}{\sin(x)} - \cos(2nx) + 2\cos(2nx) - 1 \\
&= \frac{\sin(2nx)\cos(x)}{\sin(x)} + \cos(2nx) - 1 \\
&= \frac{\cos(x)\sin(2nx) + \sin(x)\cos(2nx)}{\sin(x)} - 1 \\
&= \frac{\sin((2n+1)x)}{\sin(x)} - 1.
\end{aligned}$$

Therefore the statement holds for all $n > 1$.

Bringing the two parts together, we observe that the statement holds for all n.

$\square$