STONE-WEIERSTRASS THEOREM

Theorem 1 (Berstein's Theorem)**.** *Let $\epsilon > 0$ and let f be a continuous function on $[a.b]$. There exists a polynomial p satisfying $\|f - p\| < \epsilon$.*

Proof. For simplicity, take $[a, b] = [0, 1]$. The general case can be reduced to $[0, 1]$ by translation and dilation.

For $n \geq 1$ and $0 \leq k \leq n$ let

$$r_k = r_k(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Then

$$\sum_{n=0}^{n} r_k = 1.$$

Differentiating $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ with respect to x and multiplying by x and inserting $y = 1 - x$ yields

$$\sum_{k=0}^{n} k r_k = nx.$$

Differentiating twice $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ with respect to x and multiplying by x^2 and inserting $y = 1 - x$ yields

$$\sum_{k=0}^{n} k(k-1) r_k = n(n-1)x^2$$

which yields

$$\sum_{k=0}^{n} k^2 r_k = n(n-1)x^2 + nx.$$

The above equations imply

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 r_k(x) = \frac{x(1-x)}{n}. \tag{1}$$

Let

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) r_k(x).$$

Then p_n is a polynomial and

$$|f(x) - p_n(x)| \leq \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| r_k(x).$$

Since f is uniformly continuous, there is a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Break this last sum into two parts depending on whether $|k/n - x| < \delta$ or $|k/n - x| \geq \delta$. The first part is less than

$$\sum_{|k/n-x|<\delta} \epsilon r_k \leq \epsilon \sum_{k=0}^{n} r_k = \epsilon.$$

Let M be the sup norm of f. Since in the second case $|k/n - x|/\delta \geq 1$, by (1) the second part is no greater than

$$2M \sum_{|k/n-x|\geq\delta} r_k \leq \frac{2M}{\delta^2} \cdot \sum_{|k/n-x|\geq\delta} \left(\frac{k}{n} - x\right)^2 r_k \leq \frac{2Mx(1-x)}{\delta^2 n}.$$

Since x in $[0,1]$, we can choose $n \geq 1$ large enough so that this is less than ϵ. Hence the total sum is less than 2ϵ. □

Let X be a compact metric space and let $C(X)$ denote the **real-valued** continuous functions on X endowed with the sup norm. Then $C(X)$ is a complete metric space.

An *algebra* is a collection $\mathcal{B} \subset C(X)$ of functions containing the constant functions and closed under multiplication and addition. An algebra $\mathcal{B}$ *separates points* if given $x \neq y$ in X there is an $f \in \mathcal{B}$ satisfying $f(x) \neq f(y)$.

Theorem 2 (Stone-Weierstrass)**.** *An algebra $\mathcal{B}$ that separates points is dense in $C(X)$. This means given $f \in C(X)$ and $\epsilon > 0$ there is a $g \in \mathcal{B}$ satisfying $\|f - g\| < \epsilon$.*

If $cl(\mathcal{B})$ denote the closure of $\mathcal{B}$, then $cl(\mathcal{B})$ is an algebra.

Let f be in the closure of $\mathcal{B}$ and let M be the sup norm of f. By Berstein's theorem there is a polynomial p such that $|p(t) - |t|| < \epsilon$ on $[-M, M]$. Therefore, $\|p \circ f - |f|\| < \epsilon$ on X. Since $p \circ f$ is in the closure of $\mathcal{B}$, and ϵ is arbitrary, we have $|f|$ is in the closure of $\mathcal{B}$.

Now for any real numbers a and b we have

$$a \vee b = \max(a,b) = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|, \qquad a \wedge b = \min(a,b) = \frac{1}{2}(a+b) - \frac{1}{2}|a-b|$$

thus we conclude if f and g are in $cl(\mathcal{B})$, then $f \vee g$ and $f \wedge g$ are in $cl(\mathcal{B})$.

Now fix $f \in C(X)$ and $\epsilon > 0$.

Given $a \neq b$ in X select $g \in \mathcal{B}$ with $g(a) \neq g(b)$. By subtracting a constant, we may assume $g(b) = 0$. By dividing by a constant, we may assume $g(a) = 1$. Define

$$f_{ab}(x) = f(a)g(x) + f(b)(1 - g(x)).$$

Then $f_{ab} \in \mathcal{B}$ and f and f_{ab} have the same values at a and b. So we found a function in $\mathcal{B}$ which is close to f [equal actually] at two points. We now make combinations of these functions f_{ab}.

Since $f_{ab}(b) = f(b)$, by continuity $f_{ab}(x) - f(x) > -\epsilon$ for x in some neighborhood $B(b)$. Since X is compact, there is a finite cover $B(b_1), \ldots, B(b_N)$ by these neighborhoods. Define

$$f_a(x) = \max(f_{ab_1}(x), \ldots, f_{ab_N}(x)).$$

Then $f_a \in cl(\mathcal{B})$ and

$$f_a(x) - f(x) > -\epsilon \text{ on } X, \qquad \text{and } f_a(a) = f(a)$$

for all a in X.

Now by continuity, $f_a(x) - f(a) < \epsilon$ for x in a neighborhood $B(a)$. Since X is compact, there is a finite cover $B(a_1), \ldots, B(a_N)$ by these neighborhoods. Define

$$g(x) = \min(f_{a_1}(x), \ldots, f_{a_N}(x)).$$

Then $g \in cl(\mathcal{B})$ and $g(x) - f(x) > -\epsilon$ on X and $g(x) - f(x) < \epsilon$ on X. Thus given $f \in C(X)$ and $\epsilon > 0$ we found $g \in cl(\mathcal{B})$ satisfying $\|g - f\| < \epsilon$. Thus $cl(\mathcal{B})$ is dense in $C(X)$ which implies $\mathcal{B}$ is dense in $C(X)$. $\square$