MATH 3046, Differential Equations with Computer Lab
Spring 2017

Lab 13

The focus of this lab is investigating a nonlinear predator-prey system.

In the 1920s, the Italian mathematician Umberto Volterra proposed the following mathematical model of a predator-prey situation to explain why during World War I, a larger percentage of the catch of Italian fisherman consisted of sharks and other fish-eating fish than was true both before and after the war. Let $x(t)$ denote the population of the prey, and let $y(t)$ denote the population of the predators.

In the absence of the predators, the prey population would have a birth rate greater than its death rate and would grow according to the exponential model of population growth, i.e. the growth rate of the population would be proportional to the population itself. The presence of the predator population has the effect of reducing the growth rate, and this reduction depends on the number of encounters between individuals of the two species. Since it is reasonable to assume that the number of such encounters is proportional to the number of individuals of each population, the reduction in the growth rate is proportional to the product of the two populations, so there are constants a and b such that

$$x' = ax - bxy.$$

Since the predator population depends on the prey population for its food supply, it is natural to assume that in the absence of the prey population, the predator population would decrease, i.e. the growth rate would be negative. Furthermore, the (negative) growth rate would be proportional to the population. The presence of the prey population provides a source of food, increasing the growth rate of the predator species. By the same reasoning used for the prey species, this increase is proportional to the product of the two population, so there are constants c and d such that

$$y' = -cy + dxy.$$

1. Compute the solution of the system over the interval [0, 100] with $a = 0.4, b = 0.01, c = 0.3,$ and $d = 0.005$, and initial conditions $x(0) = 50$ and $y(0) = 30$. Plot a phase portrait and a time plot for the solutions.

After Volterra had obtained his model of the predator-prey populations, he improved it to include the effect of fishing (more generally the removal of individuals of the two populations not discriminating between the two species). The effect reduces the growth rate for each population by an amount which is proportional to the individual populations with the same proportionality constant e in each case. Thus, the system with fishing included is

$$\begin{aligned} x' &= ax - bxy - ex, \\ y' &= -cy + dxy - ey. \end{aligned}$$

2. To see the effect of fishing, compute the solutions of the system for $e = 0, 0.02, 0.04,$ and 0.06 with all other parameters from problem 1 being the same. Plot a phase portrait and a time plot for the solutions.

3. Can you use the plots constructed in problem 2 to explain why the fishermen caught more sharks during World War I? Assume that there was less fishing because of the war.