$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots,$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots.$$

These functions are studied in §3.6. Again, by comparison with exp, sin and cos are absolutely convergent series on all of $\mathbf{R}$. However, unlike exp, cosh, and sinh, we do not as yet know $\sin'$ and $\cos'$.

It turns out that functions constructed from power series are smooth in their interval of convergence. They have derivatives of all orders.

Theorem 3.4.4. *Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence $R > 0$. Then*

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots \tag{3.4.3}$$

has radius of convergence R, f is differentiable on $(-R, R)$, and $f'(x)$ equals (3.4.3) *for all x in $(-R, R)$.*

In other words, to obtain the derivative of a power series, one needs only to differentiate the series term by term. To see this, we first show that the radius of the power series $\sum(n+1)a_{n+1}x^n$ is R. Here the nth coefficient is $b_n = (n+1)a_{n+1}$, so

$$|b_n|^{1/n} = (n+1)^{1/n}|a_{n+1}|^{1/n} = (n+1)^{1/n}\left[|a_{n+1}|^{1/(n+1)}\right]^{(n+1)/n},$$

so the upper limit of $(|b_n|^{1/n})$ equals the upper limit of $(|a_n|^{1/n})$ since $(n+1)^{1/n} \to 1$ (§3.3) and $(n+1)/n \to 1$.

Now we show that $f'(c)$ exists and equals $\sum n a_n c^{n-1}$, where $-R < c < R$ is fixed. To do this, let us consider only a single term in the series, i.e., let us consider x^n with n fixed, and pick c real. Then by the binomial theorem (§3.3),

$$\begin{aligned} x^n = [c + (x-c)]^n &= \sum_{j=0}^{n} \binom{n}{j} c^{n-j}(x-c)^j \\ &= c^n + nc^{n-1}(x-c) + \sum_{j=2}^{n} \binom{n}{j} c^{n-j}(x-c)^j. \end{aligned}$$

Thus,

$$\frac{x^n - c^n}{x - c} - nc^{n-1} = \sum_{j=2}^{n} \binom{n}{j} c^{n-j}(x-c)^{j-1}, \qquad x \neq c.$$

Now choose any $d > 0$; then for x satisfying $0 < |x - c| < d$,

$$\begin{aligned}\left|\frac{x^n - c^n}{x - c} - nc^{n-1}\right| &= \left|\sum_{j=2}^{n} \binom{n}{j} c^{n-j}(x-c)^{j-1}\right| \\ &\le |x-c| \sum_{j=2}^{n} \binom{n}{j} |c|^{n-j}|x-c|^{j-2} \\ &\le |x-c| \sum_{j=2}^{n} \binom{n}{j} |c|^{n-j} d^{j-2} \\ &= \frac{|x-c|}{d^2} \sum_{j=2}^{n} \binom{n}{j} |c|^{n-j} d^{j} \le \frac{|x-c|}{d^2}(|c|+d)^n,\end{aligned}$$

where we have used the binomial theorem again. To summarize,

$$\left|\frac{x^n - c^n}{x - c} - nc^{n-1}\right| \le \frac{|x-c|}{d^2}(|c|+d)^n, \qquad 0 < |x-c| < d. \tag{3.4.4}$$

Now assume $|c| < R$ and choose $d < R - |c|$. Then by the triangle inequality, $|x - c| < d$ implies $|x| < R$ since $|x| \le |x - c| + |c| < d + |c| \le R$. Assume also temporarily, that the coefficients a_n are nonnegative, $a_n \ge 0$, $n \ge 0$. Multiplying (3.4.4) by a_n and summing over $n \ge 0$ yields

$$\left|\frac{f(x) - f(c)}{x - c} - \sum_{n=1}^{\infty} n a_n c^{n-1}\right| \le \frac{|x-c|}{d^2} f(|c|+d), \qquad 0 < |x-c| < d.$$

Letting $x \to c$ in the last inequality establishes the result when $a_n \ge 0$, $n \ge 0$. If this is not so we obtain instead

$$\left|\frac{f(x) - f(c)}{x - c} - \sum_{n=1}^{\infty} n a_n c^{n-1}\right| \le \frac{|x-c|}{d^2} g(|c|+d), \qquad 0 < |x-c| < d.$$

where $g(x) = \sum_{n=0}^{\infty} |a_n| x^n$, so the same argument works. $\square$

Since this theorem can be applied repeatedly to f, f', f'', …, *every power series with radius of convergence R determines a smooth function f on $(-R, R)$*. For example, $f(x) = \sum a_n x^n$ implies

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

on the interval of convergence. More generally, on $(-R, R)$,

$$f^{(j)}(x) = \sum_{n=j}^{\infty} n(n-1)\dots(n-j+1) a_n x^{n-j}, \qquad j \ge 0.$$