100 Selected Problems

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Last updated December 31, 2024

Preface

There's a saying: *"Tell me who your friends are, and I will tell you who you are."* When talking about mathematicians, this saying might be reformulated as: *"Tell me what your favorite problems are, and I will tell you who you are."*

With my career as a participant soon coming to a logical conclusion, I wanted to take a moment to reminisce about my journey and compile a collection of my favorite problems.

Each of these problems made the list for different reasons. Some were ones I couldn't solve despite high effort during the contest, leaving me stunned while listening to the solution from a teammate afterward. Others, in contrast, earned me a desired gold medal through last-minute flashes of inspiration and hurriedly written solutions that I proudly said to have gotten 5 minutes after a read to tease others. Some problems were highly instructive, and reflecting on them later enabled me to solve many others of a similar nature. Others were fun to solve, offering multiple perspectives to explore and leaving me mind-blown upon reading their official solutions. What unites all these problems is the emotion they evoke whenever I revisit their statements. That's what I find truly remarkable: how a problem statement can transport me back to the exam hall or a camp where I spent late nights solving it, pausing only for poker games and debates about Steins;Gate's ending with friends. With all this in mind, I want to share that same joy with you!

Despite it being a write up of miscellaneous problems, I aimed to provide textbook value as well. I avoided repeating the same techniques across problems (Had I not, the selection would probably be different), ensuring the handout spans a wide range of approaches relevant to modern olympiad problems — from the popular to the more niche. In some cases, I have included multiple solutions to the same problem, remembering Bruce Lee's famous words: *"I fear not the man who has practiced 10,000 kicks once, but I fear the man who has practiced one kick 10,000 times."* I also tried to present complete proofs and give the motivation behind every step of the problem, as I have always been frustrated by solutions that omit steps — even the "obvious" ones. [1]

To further aid readers, I have included 2^{10} hints throughout the collection, designed as step-by-step guides through solutions; I know how important it's for self-confidence to be able to finish the solution on your own, even if it took 30 hints to do it. Where multiple solutions exist, I have provided separate lines of hints to reflect it.

Even though I tried to prove every step, this handout is not beginner-friendly. It's intended for those who are confident in their skills and seek to tackle challenging, high-quality problems to deepen their understanding and, I hope, learn something new.

To conclude, the problems are not ordered by their difficulty, so make sure to take your time with each one and don't read the solutions too fast. Shoutout to everyone whose influence can be seen in this book: Without you, I wouldn't be who I am today! Enjoy this selection from my journey! May God forgive me for not using directed angles in geo section.

[1] Some people might be having a bad day and not get these things fast, or they might be NT/Algebra/Combi pros but think it's hard to solve an IMO P4 geo. While filling in gaps is a useful exercise, I believe solutions, if they are claimed to be solutions and not outlines, should be thorough and accessible

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Part I.
Algebra

§1 Problems

Problem 1 (IMO 2021/2). Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \dots x_n$.

Hints: 419 981 766 424

Hints: 688 196 846 104

Problem 2 (ISL 2019 A5). Let $x_1, x_2, \dots, x_n$ be different real numbers. Prove that

$$\sum_{1\leqslant i\leqslant n}\prod_{j\neq i}\frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Hints: 371 528 77 391

Hints: 811 845 995 818 209

Problem 3 (ISL 2022 A7). For a positive integer n we denote by $s(n)$ the sum of the digits of n. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial, where $n \geqslant 2$ and a_i is a positive integer for all $0 \leqslant i \leqslant n-1$. Could it be the case that, for all positive integers k, $s(k)$ and $s(P(k))$ have the same parity?

Hints: 1000 958 912 291

Hints: 1015 558 508 708 344 959

Problem 4 (ISL 2021 A5). Let $n \geqslant 2$ be an integer and let $a_1, a_2, \dots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n}\frac{a_k}{1 - a_k}(a_1 + a_2 + \cdots + a_{k-1})^2 < \frac{1}{3}.$$

Hints: 905 916

Hints: 929 1002

Hints: 625 736

Problem 5 (ISL 2020 A8). Let R^+ be the set of positive real numbers. Determine all functions $f : R^+ \to R^+$ such that for all positive real numbers x and y,

$$f(x + f(xy)) + y = f(x)f(y) + 1.$$

Hints: 56 12 85 396 1017

Problem 6 (ISL 2018 A6). Let $m, n \geq 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \cdots + x_n}{m} \right\rfloor \text{ for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

Hints: 287 890 50 1010

Hints: 474 1013 1006 680 86

Problem 7 (USAMO 2020/6). Let $n \geq 2$ be an integer. Let $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$ be $2n$ real numbers such that

$$0 = x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n$$

$$\text{and } 1 = x_1^2 + x_2^2 + \ldots + x_n^2 = y_1^2 + y_2^2 + \ldots + y_n^2.$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+1-i}) \geq \frac{2}{\sqrt{n-1}}.$$

Hints: 883 476 336 851 386

Problem 8 (USAMO 2018/2). Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

Hints: 701 974 103 295 342 713 267

Problem 9 (IZHO 2022/6). Do there exist two bounded sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that for each positive integers n and $m > n$ at least one of the two inequalities $|a_m - a_n| > 1/\sqrt{n}$, and $|b_m - b_n| > 1/\sqrt{n}$ holds?

Hints: 375 924 733 539 608 722

Problem 10 (RMM 2018/2). Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

Hints: 227 777 592 519

Hints: 467 163 411 769

Problem 11 (IZHO 2019/6). On a polynomial of degree three it is allowed to perform the following two operations arbitrarily many times:

- Reverse the order of its coefficients including zeroes (for instance, from the polynomial $x^3 - 2x^2 - 3$ we can obtain $-3x^3 - 2x + 1$);
- Change polynomial $P(x)$ to the polynomial $P(x + 1)$.

Is it possible to obtain the polynomial $x^3 - 3x^2 + 3x - 3$ from the polynomial $x^3 - 2$?

Hints: 69 186 436

Hints: 709 841 537 412 201

Problem 12 (SRMC 2023/4). Let $\mathcal{M} = \mathbb{Q}[x, y, z]$ be the set of three-variable polynomials with rational coefficients. Prove that for any non-zero polynomial $P \in \mathcal{M}$ there exists non-zero polynomials $Q, R \in \mathcal{M}$ such that

$$R(x^2y, y^2z, z^2x) = P(x, y, z)Q(x, y, z).$$

Hints: 977 889 224 721 65

Hints: 469 5 109 761 93 910

Problem 13 (USA TST 2024/6). Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(xf(y)) + f(y) = f(x + y) + f(xy).$$

Hints: 626 1022 472 216 992 870 867 543

Problem 14 (ISL 2010 A6). Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

Hints: 458 740 672 852 123 511 100 1014 838

Problem 15 (ISL 2018 A7). Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a + b + c + d = 100$.

Hints: 200 771 628 447 716 373 46 595

Hints: 815 745 744 262

Problem 16 (ISL 2007 A4). Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f(x + f(y)) = f(x + y) + f(y)$ for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

Hints: 911 576 58 435 331 79

Hints: 309 942 499

Problem 17 (Tuymaada Senior league 2021/8). In a sequence P_n of quadratic trinomials each trinomial, starting with the third, is the sum of the two preceding trinomials. The first two trinomials do not have common roots. Is it possible that P_n has an integral root for each n?

Hints: 17 691 899 238 820 1024

Problem 18. Given an integer $n \geq 2$. Suppose that $a_1 + a_2 + ... + a_n = b_1 b_2 ... b_n = 1$. Prove that

$$\frac{\sum_{i=1}^{n}(1 - a_i)b_i}{n - 1} \geq \sqrt[n-1]{\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right)}.$$

Hints: 217 339 872 394

Problem 19 (2018 China TST 1/6). Let $A_1, A_2, \cdots, A_m$ be m subsets of a set of size n. Prove that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |A_i| \cdot |A_i \cap A_j| \geq \frac{1}{mn} \left(\sum_{i=1}^{m} |A_i| \right)^3.$$

Hints: 409 482 994 538 676 914

Hints: 438 1 491 132 542 117

Problem 20 (ISL 2007 A5). Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m + n) \leq 2 \cdot a(m) + 2 \cdot a(n) \text{ for all } m, n \geq 1,$$

and $a\left(2^k\right) \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence $a(n)$ is bounded.

Hints: 925 686 413 276 687 102 83 710

Problem 21 (USEMO 2021/5). Given a polynomial $p(x)$ with real coefficients, we denote by $S(p)$ the sum of the squares of its coefficients. For example $S(20x + 21) = 20^2 + 21^2 = 841$. Prove that if $f(x)$, $g(x)$, and $h(x)$ are polynomials with real coefficients satisfying the indentity $f(x) \cdot g(x) = h(x)^2$, then

$$S(f) \cdot S(g) \geq S(h)^2.$$

Hints: 180 290 158 809

Hints: 749 933 233 277

Problem 22 (IOM 2021/3). Let $a_1, a_2, \ldots, a_n$ ($n \geq 2$) be nonnegative real numbers whose sum is $\frac{n}{2}$. For every $i = 1, \ldots, n$ define

$$b_i = a_i + a_i a_{i+1} + a_i a_{i+1} a_{i+2} + \ldots + a_i a_{i+1} \ldots a_{i+n-2} + 2a_i a_{i+1} \ldots a_{i+n-1}$$

where $a_{j+n} = a_j$ for every j. Prove that $b_i \geq 1$ holds for at least one index i.

Hints: 743 461 340

Hints: 193 162 168 326 752

Problem 23 (USAMO 2000/6). Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{i,j=1}^{n} \min(a_i a_j, b_i b_j) \leq \sum_{i,j=1}^{n} \min(a_i b_j, a_j b_i).$$

Hints: 588 587 192 134 507 285

Problem 24 (RMM 2024/6). A polynomial P with integer coefficients is square-free if it is not expressible in the form $P = Q^2 R$, where Q and R are polynomials with integer coefficients and Q is not constant. For a positive integer n, let P_n be the set of polynomials of the form

$$1 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

with $a_1, a_2, \ldots, a_n \in \{0, 1\}$. Prove that there exists an integer N such that for all integers $n \geq N$, more than 99% of the polynomials in P_n are square-free.

Hints: 75 727 397 343 4 759 210 965 248 980

Problem 25 (2020 Cyberspace Mathematical Competition/8). Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers such that for each positive integer n we have

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_{n+1}^2}{n+1}}.$$

Prove that the sequence $a_1, a_2, \ldots$ is constant.

Hints: 754 836 534 505 553

§2 Solutions

§2.1 IMO 2021/2, proposed by Calvin Deng

Problem 1 (IMO 2021/2)
Show that the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j|}$$

holds for all real numbers $x_1, \ldots x_n$.

¶ First solution (Shifting and analyzing convexity) First, we notice a trick often used when dealing with pairwise differences: We can shift every variable by a constant without changing the LHS. More formally, $x_i \to x_i + c$ transformation makes the inequality look like

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i - x_j|} \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{|x_i + x_j + 2c|}.$$

Now, that the RHS is a function of c, natural question is for which value of c does it attain the minimum value.

If we try to imagine the graph of this function, we can see that it's piecewise concave on the intervals $(-\infty; r_1)$, $(r_1; r_2)$, ... , $(r_{k-1}; r_k)$, $(r_k; +\infty)$, where each $r_i = -\frac{x_j + x_k}{2}$ for some $1 \leq j, k \leq n$, and all r in non-decreasing order. The reason for this is that $f(x) = \sqrt{|2x + c|}$ is concave for $2x + c \neq 0$ because the second derivative of $\sqrt{|x|}$ is $-\frac{x^2}{4|x|^{\frac{7}{2}}} < 0$ if $x \neq 0$. So the RHS is just a sum of piecewise concave functions, which is concave everywhere except for the points at which some term is zero.

It's easy to notice that the RHS is $+\infty$ for both $x \to +\infty$ and $x \to -\infty$. Now, we can consider the smallest value of r. Suppose it's r_{min}. We know that the function decreases on $(-\infty; r_1)$, increases on $(r_k, +\infty)$, so on both these intervals $RHS \geq min(r_1, r_k) \geq r_{min}$. Similarly, RHS $\geq min(r_s, r_{s+1}) \geq r_{min}$ for $c \in (r_s; r_{s+1})$ because of the concavity. And, of course, for every c in the set $\{r_1, r_2, \ldots, r_k\}$, RHS $\geq r_{min}$.

Hence, we can say that the minimum value of RHS occurs when $2c = -x_i - x_j$. For this value of c, $\sqrt{|2c + x_i + x_k|} = \sqrt{|x_k - x_j|}$ and $\sqrt{|2c + x_j + x_k|} = \sqrt{|x_k - x_i|}$. Using these equations we are only left to show the initial inequality for $x_1 + c$, $\ldots$, $x_{i-1} + c$, $x_{i+1} + c$, $\ldots$, $x_{j-1} + c$, $x_{j+1} + c$, $\ldots$, $x_n + c$ case, which holds by induction hypothesis because terms with i and j cancel (there are $n - 2$ variables if $i \neq j$ and $n - 1$ if $i = j$).

¶ Second solution (Integrals) If we set $f(x) = \sqrt{|x|}$, we could rewrite the problem in the following way:

$$\sum_{1 \leq i,j \leq n} (f(x_i + x_j) - f(x_i - x_j)) \geq 0$$

This form is more general than the $f(x) = \sqrt{|x|}$ given by the problem statement. We can immediately recall that for $f(x) = x^2$ it's just $4(\sum x)^2 \geq 0$, or we can remember a standard product-to-sum trigonometric formula, and get that for $f(x) = -\cos x$, the sum is $2(\sum \sin x)^2 \geq 0$.

Of course, we can't mimic this approach for $f(x) = \sqrt{|x|}$. Nevertheless, we are motivated to call the functions for which the inequality holds *good*. Sum of two good functions is good, which is a tool to obtain new good functions from what we already have. It also allows to integrate a good function with any non-negative weight $w(t)$. This way we can get a ton of good functions even from the two trivial ones that I stated at the beginning. Wishful thinking suggests us that we can obtain $f(x) = \sqrt{|x|}$.

What will follow is a clever way to isolate a power function from a good function by considering a function $g(xt)$. Formally, given a good function $g(xt)$ (multiplying by t doesn't change anything), we will multiply it by $w(t) = \frac{1}{t^p}$ and integrate with respect to t to isolate the power of x:

$$\int_0^{\infty} \frac{g(xt)}{t^p} dt = x^{p-1} \int_0^{\infty} \frac{g(xt)}{(xt)^p} d(xt) = x^{p-1} \int_0^{\infty} \frac{g(t)}{t^p} dt$$

Now, can vary x and the RHS is a power of x multiplied by a constant $I = \int_0^{\infty} \frac{g(t)}{t^p} dt$, given that the integral converges. Choose $p = \frac{3}{2}$. We also need to take care of an absolute sign. To do this, we just ensure that g is even, then the integration result is the same for $g(xt)$ and $g(-xt)$.

Now, we have to choose the right function g. It has to satisfy three conditions:

- It's good.
- The integral $\int_0^{\infty} \frac{g(t)}{t^{\frac{3}{2}}} d(t)$ converges and is positive.
- It's even.

Choices containing x^2 don't work because $\frac{x^2}{x^{\frac{3}{2}}}$ diverges on infinity. So, we are only left with $-\cos x$ which is, luckily, an even function. $-\cos x$ itself doesn't work because $\frac{-\cos x}{x^{\frac{3}{2}}}$ diverges when x approaches zero. Fixing this divergence, we might try $g(x) = 1 - \cos x$. It's good (constant are also good), and $1 - \cos x = 2\sin^2 \frac{x}{2} \leq \min(2, \frac{x^2}{2})$ ($x - \sin x$ is increasing, so it's greater than 0 for $x \geq 0$). $\int_0^{\infty} \frac{1-cos(t)}{t^{\frac{3}{2}}} dt$ converges because from 0 to 1, we can bound it with $\int_0^1 \frac{t^{\frac{1}{2}}}{2} = \frac{1}{4} dt$, and from 1 to $+\infty$, we bound with $\int_1^{+\infty} \frac{2}{t^{\frac{3}{2}}} dt = 4$. Function is positive everywhere, so the integral is positive.

All the above was the motivation. The actual solution can be written as follows:

$$\sum_{1\leq i,j\leq n} \sqrt{|x_i + x_j|} - \sqrt{|x_i - x_j|} = \sum_{1\leq i,j\leq n} \int_0^{\infty} \frac{cos((x_i - x_j)t) - cos((x_i + x_j)t)}{t^{\frac{3}{2}} I} dt = \int_0^{\infty} \frac{2(\sum_{1\leq i\leq n} \sin x_i)^2}{t^{\frac{3}{2}} I} dt$$

which is clearly non-negative.

> **Remark.** The problem is true in the case of $|x|^p$ for $p \in (0;2)$. The first solutions also holds for $|x|^p$ where $0 < p \leq 1$, then concavity breaks. The second solution holds for $0 < p \leq 1$ by the argument above, but to prove that the integral $\int_0^{\infty} \frac{1-\cos t}{t^{1+p}} dt$ converges for $1 < p < 2$ one has to say that
>
> $$\int_0^{\infty} \frac{1 - \cos(t)}{t^{1+p}} dt = \frac{\pi}{2\Gamma(p+1)\sin(p\pi/2)}.$$

§2.2 ISL 2019 A5, proposed by Grigoriy Chelnokov

Problem 2 (ISL 2019 A5)

Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1\leq i\leq n} \prod_{j\neq i} \frac{1-x_ix_j}{x_i-x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

¶ First solution (Clever Lagrange interpolation) We are motivated to use *Lagrange Interpolation* because the product of pairwise differences in the denominator looks suspiciously similar to it. We need to find a suitable polynomial. One symmetric option is $P(x) = \prod(1-xx_1)\ldots(1-xx_n)$, except, when we substitute $x = x_i$, we find an extra $(1-x_i^2) = (1-x_i)(1+x_i)$ term, which we are motivated to cancel out by adding two additional points – 1 and -1.

We can only pull off this trick if no variable equals to 1 or -1. We will handle it separately. For $n = 2$ the sum is indeed 0. Now induct. It's easy to check that in the case $x_i = 1$ the expression is equal to $1 - E_{n-1}$, where E_{n-1} is the answer for n variables, which is 0 if n is even and 1 if n is odd. If $x_i = -1$ then the expression equal $(-1)^{n-1} + E_{n-1}$, which is 0 if n is even and 1 if n is odd. *Alternatively,* we could just say that if we nudge a variable that's equal to a one or a minus one by a small amount then the result won't change, a.k.a "continuity", so we can shift variables by a bit so they are still different and none of them equal to one, and the result is still the same as we will prove further.

Now suppose that no variable equals to 1 or -1. Consider the interpolation applied to $P(x) = \prod(1-xx_1)\ldots(1-xx_n)$ and points x_1, x_2, ... , x_n, 1, -1. As $\deg P \leq n$, we can write

$$P(x) = \sum_{i=1}^{n} P(x_i)\frac{(x^2-1)}{(x_i^2-1)}\prod_{j\neq i}\frac{x-x_j}{x_i-x_j} + \frac{P(1)(x+1)}{2}\prod_{j=1}^{n}\frac{x-x_j}{1-x_j} + \frac{(-1)^{n+1}P(-1)(x-1)}{2}\prod_{j=1}^{n}\frac{x-x_j}{1+x_j}$$

Comparing the x^{n+1} coefficient, we get that

$$0 = -\sum_{1\leq i\leq n}\prod_{j\neq i}\frac{1-x_ix_j}{x_i-x_j} + \frac{1}{2} + \frac{(-1)^{n+1}}{2},$$

so, if n is even, then $\sum_{1\leq i\leq n}\prod_{j\neq i}\frac{1-x_ix_j}{x_i-x_j} = 0$. If n is odd then $\sum_{1\leq i\leq n}\prod_{j\neq i}\frac{1-x_ix_j}{x_i-x_j} = 1$. Exactly what was asked.

¶ Second solution (Alon's Combinatorial Nullstellensatz) Imagine multiplying expression in the statement by $\prod_{i<j}(x_i - x_j)$ to clear the denominators. We will prove that this expression is zero whenever $x_a = x_b$. We only care about

$$\prod_{j\neq a}\frac{1-x_ax_j}{x_a-x_j} + \prod_{j\neq b}\frac{1-x_bx_j}{x_b-x_j}$$

because other terms have a $(x_a - x_b)$ factor (after multiplication).

$$\prod_{j\neq a}\frac{1-x_ax_j}{x_a-x_j} + \prod_{j\neq b}\frac{1-x_bx_j}{x_b-x_j} = \frac{1-x_ax_b}{x_a-x_b}\Big(\prod_{j\neq a,b}\frac{1-x_ax_j}{x_a-x_j} - \prod_{j\neq a,b}\frac{1-x_bx_j}{x_b-x_j}\Big).$$

The latter two terms in the brackets are symmetric in x_a and x_b, so the sum is zero if $x_a = x_b$ (of course, when multiplied by $\prod_{i<j}(x_i - x_j)$, any issues with $(x_a - x_b)$ in the denominator are avoided and the result is simply zero). Now consider a new polynomial

$$P(x_1, x_2, ..., x_n) = \prod_{i<j}(x_i - x_j)\Big(\sum_{1\leq i\leq n}\prod_{j\neq i}\frac{1 - x_i x_j}{x_i - x_j} - (n \bmod 2)\Big).$$

We know that it's zero for $x_i = x_j$ from what we've just proved. From the first solution we also know that it vanishes whenever some variable is equal to 1. Direct expansion shows that in every summand, the degree of each x_i not exceeding $n - 1$. Suppose that P is not identically zero, then it has some monomial $Ax_1^{d_1} ... \, x_n^{d_n}$ of highest total degree and $A \neq 0$, and we know that $d_1, \ldots, d_n \leq n - 1$.

Now use *Alon's combinatorial Nullstellensatz*: There's at least one $(x_1, x_2, \ldots, x_n) \in S \times S \times \ldots \times S$ such that $P(x_1, x_2, \ldots, x_n) \neq 0$, given that $|S| \geq max(d_1 + 1, \ldots, d_n + 1)$. Suppose that $S = \{1, a_1, \ldots, a_{n-1}\}$, where all a_i's are distinct are differ from 1. First, $|S| = n - 1 + 1 \geq max(d_1 + 1, \ldots, d_n + 1)$. Now, by the theorem above, we know that $P(x_1, x_2, \ldots, x_n) \neq 0$ for some choice of x_is. But if some x is 1, then the value of P is zero. So no variable equals to 1; it follows that there are $n - 1$ choices of each x (namely, $a_1, a_2, \ldots, a_{n-1}$), but there are n numbers $x_1, x_2, \ldots, x_n$, so some two of them are equal, which also induces the value to be zero. So the polynomial is identically zero, and we are done.

Remark 1. Second solution can be finished by noting that P is divisible by $(x_i - x_j)$ because it vanishes when $x_i = x_j$ (which is not completely obvious, so try to prove it yourself!). Consequently, it's divisible by $\prod_{i<j}(x_i - x_j)$. We can also say that it's divisible by $x_i - 1$ by the same reason, so it's divisible by $\prod_{i<j}(x_i - x_j)\prod_{1\leq i\leq n}(x_i - 1)$. The degree of this expression is $\frac{n(n-1)}{2} + n$. And $\deg P$ doesn't exceed $\frac{n(n-1)}{2} + 2(n-1) - (n-1) = \frac{n(n-1)}{2} + n - 1$, which means that $P = 0$.

Remark 2. There are also algebraic solutions using induction and direct computation after noting that a substitution $x_i = \coth\theta_i$ (where $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$) gives

$$\prod_{j\neq i}\frac{1 - x_i x_j}{x_i - x_j} = \prod_{j\neq i}\coth(\theta_i - \theta_j),$$

and then one can spam $\coth(\alpha - \beta) = \frac{1 - \coth\alpha\coth\beta}{\coth\alpha - \coth\beta}$ identity to get that the value of the expression for n variables is the same as if we reduced it to $n - 2$ variables.

§2.3 ISL 2022 A7, proposed by Belarus

Problem 3 (ISL 2022 A7)
For a positive integer n we denote by $s(n)$ the sum of the digits of n. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial, where $n \geqslant 2$ and a_i is a positive integer for all $0 \leqslant i \leqslant n-1$. Could it be the case that, for all positive integers k, $s(k)$ and $s(P(k))$ have the same parity?

¶ First solution (Powers of ten forward differences) We don't need the polynomial to be monic for this solution, so suppose that it starts with a_nx^n . Of course, when dealing with sums of digits, we want to have some control over at least some digits of the number. This motivates us to shift or multiply by 10^k for big enough k whenever possible. Fix x. Consider

$$P(x+10^k) = P(x) + 10^kP_1(x) + 10^{2k}P_2(x) + \ldots + 10^{kn}P_n(x),$$

where each polynomial $P_i(x)$ is determined by $P(x)$ and can be easily reconstructed using binomial expansion (in fact, $P_1(x) = P'(x)$, $P_2(x) = \frac{P''(x)}{2}$, $P_3(x) = \frac{P'''(x)}{3!}$, $\ldots$, which is a discrete analogue of Taylor series expansion). Now we can say that

$$s(x) + 1 = s(x+10^k) \equiv s(P(x+10^k)) \equiv s(P(x)) + s(P_1(x)) + + \ldots + s(P_n(x)) \pmod 2.$$

Note that $s(x)$ and $s(P(x))$ cancel out, leaving us with a new congruence involving sums of digits of polynomial outputs, but the highest degree now is $n-1$ and is unique. If we repeat the same argument, we will get a new sum with a highest degree term having degree $n-2$ and being unique. In the end, we will be left with

$$s(ax+b) + c \equiv 0 \pmod 2.$$

Now consider $x = 10^k - 10^m$ and for big enough k and m, and assume that a doesn't end with zero.

$$ax + b = \overline{(a-1)99...9a_1...a_l000...0b_1\ldots b_k},$$

where $\overline{a_1...a_k} = 10^l - a$ for the smallest number l such that $10^l \geq a$, the number of 9's is $k-l-m$, and $b = \overline{b_1\ldots b_k}$. Now, if we increase k by one, an additional 9 will change the parity of the digit sum, which is a contradiction.

¶ Second solution (Auxiliary polynomial) When given a certain polynomial $P(x)$, we can transform it by plugging another polynomial $Q(x)$ instead of x. This easy idea allows to solve problems about polynomials digit sum.

Consider the polynomials of degree n with a positive leading coefficient. Suppose that it has a positive integer coefficient following (except for, maybe, the last coefficient) and preceding every negative integer coefficient. Call this situation a *sign change*.

Note that for such polynomial, if we consider its values for 10^m and 10^{m+1}, for a big enough m, it has n *gaps* of the same digit (either all zeros or all nines) in its decimal representation with "all nines" contributing to changes of sign. And 10^{m+1} value differs from 10^m by exactly $n-\#$(changes of sign) 0s and #(changes of sign) 9s.

This way, we can just try to find a polynomial Q such that $Q(x)$ has odd number of changes in sign and $P(Q(x))$ has an even number of them. One way to do so is a zero for $P(Q(x))$, which is the same as making all the coefficients to be positive, and one for $Q(x)$.

We will make $Q(x)^n$'s coefficients positive, where n is the degree of P. Then, multiplying all the coefficients of Q by a large constant C, we will have that each coefficient of $P(Q(x))$ is a polynomial in C with leading coefficient being the coefficient from $Q(x)^n$. Thus, setting C to be large, all the coefficients will be positive.Note that we can as well multiply by a constant to make coefficients integer, so we can search for rational polynomials.

It makes sense to find Q such that it has at least one zero coefficient and then tweak the value of this zero coefficient a little bit. It won't affect any coefficient much, but will get us the solution. For example, $Q(x) = x^4 + x^3 + x + 1$ (many other options work too, just ensuring high enough degree and not missing the second term is sufficient), because we can find $4a + 3b + c = m$ for $1 \le m \le 4n$ with $a + b + c \le n$. For example, $a = \lfloor \frac{m}{4} \rfloor$, and $(b, c) = (0, 0)$, $(1, 0)$, $(0, 2)$, $(0, 1)$ for different values of $m \pmod 4$; one can check that $a + b + c \le n$. This means that the coefficient of x^m is positive.

> **Remark.** Second solution originated from the idea from USA TSTST 2019/6. Another problem with similar flavor is RMMSL 2023 N2.

§2.4 ISL 2021 A5

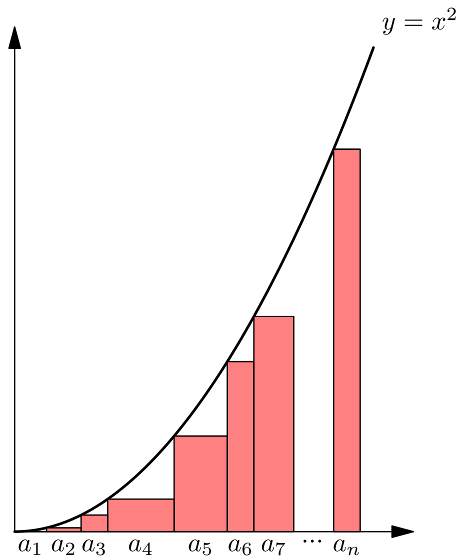
Problem 4 (ISL 2021 A5)
Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers with sum 1. Prove that

$$\sum_{k=1}^{n} \frac{a_k}{1-a_k}(a_1 + a_2 + \ldots + a_{k-1})^2 < \frac{1}{3}.$$

¶ First solution (Area under the x^2 curve) If we just had

$$\sum_{k=1}^{n} a_k(a_1 + a_2 + \ldots + a_{k-1})^2 < \frac{1}{3},$$

then the problem is just the standard *Riemann Sums* exercise because we can plot the segments with lengths a_1, a_2, ... , a_n that partition the segment of length 1 on x-axis and draw a graph of $y = x^2$.



The areas of shaded rectangles are the terms of the sum and $\int_0^1 x^2 dx = \frac{1}{3}$ is the area under the portion of graph from 0 to 1.

Coming back to original inequality and noting that the inequality we just obtained is not strict by any means, we might hope that the bound of the area under the curve on the given segment still holds. In other words, we want to show that

$$\int_{a_1+\ldots+a_{k-1}}^{a_1+\ldots+a_k} x^2 dx \geq \frac{a_k}{1-a_k}(a_1 + a_2 + \ldots + a_{k-1})^2.$$

This is equivalent to

$$\frac{a_k}{1-a_k}(a_1 + \ldots + a_{k-1})^2 \leq \frac{(a_1 + \ldots + a_k)^3 - (a_1 + \ldots + a_{k-1})^3}{3},$$

which is the same as

$$\frac{xy^2}{1-x} \le \frac{(x+y)^3 - y^3}{3}.$$

It is equivalent to

$$\frac{y^2}{1-x} \le \frac{x^3 + 3xy + 3y^2}{3},$$

which is the same as $(1-x)x^3 + 3xy(1-x) + 3y^2 - 3y^2x \ge 3y^2$, which follows from $1 - x \ge y \Leftrightarrow 3xy(1-x) - 3y^2x \ge 0$, and $x^3(1-x) > 0$.

¶ Second solution (Generalized induction) Note that most of the terms of the sum remain the same when we go from n to $n+1$. The only constraint that holds us from doing a successful induction is the condition on the sum to be one. One work around would be to come up with a variant of the problem for arbitrary $1 > S = a_1 + \ldots + a_n$ (because all the partial sums are $\in (0;1)$). For this, we apply the original inequality to $b_i = \frac{a_i}{S}$. We get that

$$\sum_{k=1}^{n} \frac{a_k}{S^2(S - Sa_k)}(a_1 + \ldots + a_{k-1})^2 < \sum_{k=1}^{n} \frac{a_k}{S^2(S - a_k)}(a_1 + \ldots + a_{k-1})^2 < \frac{1}{3}\ (\blacklozenge),$$

where the first inequality follows from $S \le 1$.

Now it's not hard to finish. $n = 1$ case is just $0 < \frac{1}{3}$, $n = 2$ case is $a_2(1 - a_2) \le \frac{1}{4} < \frac{1}{3}$. Now $n \ge 3$. For the induction step, suppose that $a_1 + a_2 + \ldots + a_{n-1} = S < 1$. Then

$$\sum_{k=1}^{n-1} \frac{a_k}{1-a_k}(a_1 + a_2 + \ldots + a_{k-1})^2 + \frac{a_n}{1-a_n}(a_1 + \ldots + a_{n-1})^2 < \frac{S^3}{3} + (1-S)S \le \frac{1}{3},$$

where we have used $\blacklozenge$. The last inequality is just $(1-S)^3 \ge 0$.

¶ Third solution (Smoothing and Riemann sums) Improving the accuracy of the answer when approximating the integral using Riemann sums can be done by using smaller segments. As we already know that the problem is strongly related to integrals, let's try to incorporate this reasoning here.

Let's divide one of the numbers $a_1, a_2, \ldots, a_n$ in two parts. Now, we have the following numbers: $a_1, \ldots, \frac{a_k}{2}, \frac{a_k}{2}, a_{k+1}, \ldots, a_n$. To prove that we increased the sum, we need to show that (other terms are the same for both cases)

$$\frac{\frac{a_k}{2}}{1 - \frac{a_k}{2}}(a_1 + \ldots + a_{k-1})^2 + \frac{\frac{a_k}{2}}{1 - \frac{a_k}{2}}(a_1 + \ldots + a_{k-1} + \frac{a_k}{2})^2 > \frac{a_k}{1-a_k}(a_1 + a_2 + \ldots + a_{k-1})^2.$$

Denote a_k as $2x$ and $a_1 + a_2 + \ldots + a_{k-1}$ as y. The inequality is just

$$\frac{x}{1-x}(y^2 + (y+x)^2) \ge \frac{2x}{1-2x}y^2.$$

After writing common denominator and expanding the brackets, it's the same as

$$2yx(1-y) + x^2(1-2x) > 4yx^2,$$

which follows from $y+2x=a_1+\ldots+a_{k-1}+a_k\le 1$ and that the equality case is either $y=0$ and $2x=1$, which is impossible for $n\ge 2$, or $x=0$, which is impossible in general.

Every time we perform this operation, we increase the sum. Using it, we can make the maximum number among $a_1, a_2, \ldots, a_n$ small. Now we are only left to say that

$$\sum_{k=1}^{n}\frac{a_k}{1-a_k}(a_1+a_2+\ldots+a_{k-1})^2\le\frac{1}{1-a_{\max}}\Big(\sum_{k=1}^{n}a_k(a_1+a_2+\ldots+a_{k-1})^2\Big)<\frac{1}{3}\Big(\frac{1}{1-a_{\max}}\Big).$$

And we can make this number arbitrarily close to $\frac{1}{3}$. This proves that the sum cannot be larger than $\frac{1}{3}$. If we had an equality, we can say that the value for $a_1, a_2, \ldots, a_n$ is strictly smaller than the one for $a_1, \ldots, \frac{a_k}{2}, \frac{a_k}{2}, a_{k+1}, \ldots, a_n$, but the latter is $\le\frac{1}{3}$ by what we had just proved.

> **Remark.** The integral method is also useful when dealing with sums of consecutive values of some function. Like bounding $f(a)+f(a+1)+\ldots+f(b)$. This reasoning was very useful for IZhO 2023/3

§2.5 ISL 2020 A8, proposed by Ukraine

Problem 5 (ISL 2020 A8)
Let R^{+} be the set of positive real numbers. Determine all functions $f : R^{+} \to R^{+}$ such that for all positive real numbers x and y,

$$f(x + f(xy)) + y = f(x)f(y) + 1$$

¶ Solution (Increasing and limits) As always, $P(x, y)$ is the assertion. $P(1, y)$ gives

$$f(1 + f(y)) + y = f(1)f(y) + 1.$$

Now, it's easy to get the injectivity by assuming that $f(a) = f(b)$ for $a \neq b$. We want to exploit injectivity. $P(a, y)$ gives that

$$f(a + f(ay)) = f(a)f(y) + 1 - y.$$

So, for $a \neq b$, $f(a + f(ay)) \neq f(b + f(by))$. Let's try to work backwards and find suitable y, a, b such that $a + f(ay) = b + f(by)$. We want $x_1 - x_2 = f(y_2) - f(y_1)$ with $y_1 = x_1 z$ and $y_2 = x_2 z$. We can find z as

$$z = \frac{f(y_2) - f(y_1)}{y_1 - y_2}.$$

For z to not exist, we must have that $f(y_1) \geq f(y_2)$ for $y_1 > y_2$, which is the same as f - nondecreasing. Combined with injectivity, this implies that f is increasing.

When we have the $R^{+} \to R^{+}$ equation and managed to prove that f is increasing, it's usually beneficial to consider limits because increasingness and boundedness implies that $\lim_{x \to a^+} f(x)$ exists for $a \geq 0$.

Consider $\lim_{x \to 0^+} f(x) = c$. Let's approach x to 0^+ in the original equation. The first bracket of LHS approaches c^+ ($x \to 0^+$, $xy \to 0^+$, $f(xy) \to c^+$), so $f(x + f(xy)) = \lim_{x \to c^+} f(x) = d$. Then the limiting equation reads as $y + d = cf(y) + 1$, which proves that f is linear. Substituting back into the original equation, it's not hard to derive that the only answer is $f(y) = y + 1$, which evidently works.

Remark. This idea of proving that the function is increasing could be used in Kazakhstan MO 2024 grade 11/3. The idea of limits could be used in Kazakhstan MO 2021 grade 10/3 (Russian website because it features mentioned solutions).

§2.6 ISL 2018 A6, proposed by Brazil

Problem 6 (ISL 2018 A6)
Let $m, n \geq 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \cdots + x_n}{m} \right\rfloor \text{ for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

¶ First solution (Multivariable polynomial equation, forward differences and Nullstellensatz) The reason why this polynomial seems to have a high degree is that when increasing some variable by a one, for many points it doesn't change the value, so if we consider some forward difference polynomial it must have a lot of roots which will imply the problem. Let's formalize this argument.

We first notice that the RHS is a function of $x_1 + \ldots + x_n$. To exploit it, we might find a polynomial, which exists due to *Lagrange interpolation*, that takes the values of $\lfloor \frac{x}{m} \rfloor$ for every $0 \leq x \leq n(m-1)$, and we ensure that it's of the smallest degree for convenience, which means that the $\deg P \leq n(m-1)$. This way we obtain a polynomial equation that holds for every $(x_1, x_2, \ldots, x_n) \in S \times S \times \ldots \times S$, where $S = \{0, 1, \ldots, m-1\}$, which cries for *Nullstellensatz* application. But before it, we need to make a claim about the degree of P.

We notice that consecutive values of $P(x)$ for $0 \leq x \leq n(m-1)$ are almost always the same, which is not typical for a polynomial. This suggests to consider $Q(x) = P(x+1) - P(x)$ which is zero for $x \neq mk - 1$ for $k \in \mathbb{Z}$, from which it follows that the number of roots of $Q(x)$ is at least

$$\lfloor \frac{n(m-1)}{m} \rfloor (m-1) \geq \frac{n(m-1)^2}{m} - m + 1 \geq n - 1.$$

Where the last inequality is equivalent to $n \geq \frac{m^2-2m}{m^2-3m+1}$, which is true for $m \geq 4$ as $\frac{m^2-2m}{m^2-3m+1} \leq 2$, for $m = 3$, last inequality is true for $n \geq 3$, for $n = 2$ it can be checked directly that the number of roots is at least 1. This means that for $m \geq 3$, the $\deg P = \deg Q + 1 \geq n$, because Q is non-zero and has $\geq n-1$ roots. For $m = 2$, we can say that there's at least one $Q(x) = \frac{1}{2}$ for $x \in (k, k+1)$ for each $k \in \{0, 1, \ldots, n-2\}$, because of the continuity. So $Q(x) - \frac{1}{2}$ is non-zero and has $n-1$ roots. Hence, $\deg P = \deg Q + 1 \geq n$. Thus, $\deg P \geq n$ in either case.

Denote the degree of P as d.

$$f_1(x_1, x_2, \ldots, x_n) \stackrel{\text{def}}{=} P(x_1 + x_2 + \ldots + x_n) - f(x_1, x_2, \ldots, x_n) = 0$$

for every $(x_1, x_2, \ldots, x_n) \in S \times S \times \ldots \times S$. Suppose that the $\deg f < n$. This means that no term from $(x_1 + \ldots + x_n)^d$ in P vanishes, and so there will be a term of the form $x_1^{m-1} \ldots x_{k-1}^{m-1} x_k^r$, where $k = \lfloor \frac{d}{m-1} \rfloor \leq n$ and $r = d - (k-1)(m-1) \leq (m-1)$. The existance of this term contradicts the statement of Nullstellensatz because $|S| = m - 1 + 1$ and f_1 vanishes on the $S \times \ldots \times S$ lattice.

¶ Second solution (Residues $\pmod m$ polynomial and induction) Consider the following polynomial:

$$Q(x_1,\ldots,x_n) \stackrel{\text{def}}{=} x_1 + x_2 + \ldots + x_n - mf(x_1,\ldots,x_n).$$

For $x_1, x_2, \ldots, x_n \in \{0,1,\ldots,m-1\}$ this polynomial will have a value of $x_1 + x_2 + \ldots + x_n \pmod m$. We claim that under such conditions it has a power of at least n, which, as $n \geq 2$, will prove that $\deg f$ is at least n.

We will prove this statement by induction on n. It's easy to check that for the $n = 1$ case, the degree of Q is at least 1. Suppose that $Q(x_1, x_2, \ldots, x_n) \stackrel{\text{def}}{=} x_1^d Q_d(x_2,\ldots,x_n) + \ldots + x_1 Q_1(x_2,\ldots,x_n) + Q_0(x_2,\ldots,x_n)$. Note that $S = \{Q(x_1, x_2, \ldots, x_{n-1}, i) \mid 0 \leq i \leq m-1\}$ contains all the residues $\pmod m$ once. So

$$\sum_{i=0}^{i=m-1} Q(i, x_2, \ldots, x_n) = \frac{m(m-1)}{2}.$$

This means that for $x_2, \ldots, x_n \in \{0,1,\ldots,m-1\}$,

$$Q_0(x_2,\ldots,x_n) = \frac{m-1}{2} - \sum_{i=1}^{i=d} \sum_{j=0}^{j=m-1} j^i Q_i(x_2,\ldots,x_n).$$

We know that $Q_0(x_2,\ldots,x_n)$ is just $Q(0,x_2,\ldots,x_n)$ that has a value of $x_2+\ldots+x_n \pmod m$ for $x_2,\ldots,x_n \in \{0,1,\ldots,m-1\}$. So, by induction hypothesis, its degree is at least $n-1$. Now if we assume that $\deg Q \leq n-1$ then $\deg Q_i \leq n-1-i$. Which shows that the degree is not greater than $n-1-1$, which is a contradiction.

> **Remark.** Another cool problem that uses forward differences is Kazakhstan MO 2016 grade 9/5.

§2.7 USAMO 2020/6, proposed by David Speyer and Kiran Kedlaya

Problem 7 (USAMO 2020/6)

Let $n \geq 2$ be an integer. Let $x_1 \geq x_2 \geq \ldots \geq x_n$ and $y_1 \geq y_2 \geq \ldots \geq y_n$ be $2n$ real numbers such that

$$0 = x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n$$
$$\text{and } 1 = x_1^2 + x_2^2 + \ldots + x_n^2 = y_1^2 + y_2^2 + \ldots + y_n^2.$$

Prove that

$$\sum_{i=1}^{n} (x_i y_i - x_i y_{n+1-i}) \geq \frac{2}{\sqrt{n-1}}.$$

¶ Solution (Expected value) Note that, given that the variables are sorted, $\sum_{i=1}^{n} x_i y_i$ is the maximum and $\sum_{i=1}^{n} x_i y_i$ is the minimum of $\sum_{i=1}^{n} x_i y_{\sigma(i)}$ over all the possible permutations σ of $\{1, \ldots, n\}$. This motivates us to examine a random variable $S = \sum_{i=1}^{n} x_i y_{\sigma(i)}$.

First, we claim that the range of a random variable X is at least $2\sqrt{\mathrm{Var}(X)}$. Shift the variables so that they start from zero (shift doesn't change the range and Var). And scale them by the value of the range (this is possible because Var scales by the square of this value and $\mathbb{E}$ by first power). Now they are in $[0;1]$. Range is 1 and

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \leq \mathbb{E}[X] - \mathbb{E}[X]^2 \leq \frac{1}{4},$$

where the first inequality follows from $X^2 \leq X$ and the second is just the maximum of the quadratic.

Now we will be calculating the $\mathrm{Var}(S)$:

$$\mathbb{E}[S] = \mathbb{E}[\sum_{i=1}^{n} x_i y_{\sigma(i)}] = \sum_{i=1}^{n} x_i \mathbb{E}[y_{\sigma(i)}] = 0.$$

$$\mathbb{E}[S^2] = \sum_{i=1}^{n} \mathbb{E}[x_i^2 {y_{\sigma(i)}}^2] + \sum_{1 \leq i < j \leq n} 2E[x_i x_j y_{\sigma(i)} y_{\sigma(j)}] = \sum_{i=1}^{n} x_i^2 E[{y_{\sigma(i)}}^2] +$$

$$2 \sum_{1 \leq i < j \leq n} x_i x_j E[y_{\sigma(i)} y_{\sigma(j)}] = \frac{(\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i^2)}{n} + \frac{2(\sum_{1 \leq i < j \leq n} x_i x_j)(2 \sum_{1 \leq i < j \leq n} y_i y_j)}{n(n-1)} = \frac{1}{n} + \frac{1}{n(n-1)}$$

$$= \frac{1}{n-1}$$

Now $\mathrm{Var}(S) = \mathbb{E}[S^2] - \mathbb{E}[S]^2 = \frac{1}{n-1} - 0^2 = \frac{1}{n-1}$. So $\sum_{i=1}^{n} (x_i y_i - x_i y_{n+1-i}) = \max(S) - \min(S) \geq 2\sqrt{\mathrm{Var}(X)} = \frac{2}{\sqrt{n-1}}$, which is what we want.

Remark. Other algebra problems where expected value proves to be useful: USAMO 2012/6, ISL 2019 A2, China TST 2023 Test 4/2 (Last one is rather tough).

§2.8 USAMO 2018/2, proposed by Titu Andreescu and Nikolai Nikolov

Problem 8 (USAMO 2018/2)
Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(x + \frac{1}{y}\right) + f\left(y + \frac{1}{z}\right) + f\left(z + \frac{1}{x}\right) = 1$$

for all $x, y, z > 0$ with $xyz = 1$.

¶ Solution (Cauchy FE for a bounded function on the interval). First of all, we perform the standard substitution: Plug in $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$. Now the equation reads as

$$f\left(\frac{b+c}{a}\right) + f\left(\frac{c+a}{b}\right) + f\left(\frac{a+b}{c}\right) = 1 \text{ for all } a,\ b,\ c > 0.$$

Now we need to derive the equation that connects $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$. Luckily, it's not that hard, because $\frac{b+c}{a} + 1 = \frac{a+b+c}{a}$ and $\frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1$. So it makes sense to write the equation as

$$f\left(\frac{1}{x} - 1\right) + f\left(\frac{1}{y} - 1\right) + f\left(\frac{1}{z} - 1\right) = 1$$

for $x + y + z = 1$ and x, y, z - positive. Now suppose that $g : (0, 1) \to (0, 1)$ is such that $g(x) = f\left(\frac{1}{x} - 1\right)$ for simplicity.

We have come to the standard yet technical part of the problem. Now, we have the following equation:

$$g(x) + g(y) + g(z) = 1 \text{ for all } x,\ y,\ z > 0 \text{ such that } x + y + z = 1. \quad (\heartsuit)$$

We can say that $g(x) + g(y) = 1 - g(z) = 1 - g(1 - x - y) = 2g(\frac{x+y}{2})$ for all $x + y < 1, x, y > 0$ (The last equation follows from substituting $(\frac{x+y}{2}, \frac{x+y}{2}, 1 - x - y)$ into $\heartsuit$). This is exactly *Jensen's Functional Equation* for an interval. To establish *Cauchy's Functional Equation*, we need to be able to plug in a zero, which is not in our domain. Workaround here is to shift the function so it still satisfies Jensen's FE, but now 0 is in the domain.

Suppose that $h : (-\frac{1}{2}, \frac{1}{2}) \to (0, 1)$ is such that $h(x) = g(x+1/2)$. Now, $h(x)+h(y)+h(z) = 1$ for all $-\frac{1}{2} < x,\ y,\ z < \frac{1}{2}$ with $x+y+z = -\frac{1}{2}$. We can now write that $h(x)+h(y) = 2h\left(\frac{x+y}{2}\right)$ for all $-\frac{1}{2} < x,\ y < \frac{1}{2}$ and $x + y < 0$. We can say that $h(x) + h(0) = 2h\left(\frac{x}{2}\right)$ for $-\frac{1}{2} < x < 0$. So $h(x) + h(y) = h(x + y) + h(0)$ for all $-\frac{1}{2} < x, y < 0$.

The last substitution is $h_0 : (-\frac{1}{2}, \frac{1}{2}) \to (-1, 1)$ such that $h_0(x) = h(x) - h(0)$, where the codomain follows from $0 < h(0) < 1$. h_0 satisfies Cauchy's FE, $h_0(x) + h_0(y) = h_0(x + y)$, for $-\frac{1}{2} < x, y < 0$ and the function is bounded. It's known that in this case, $h_0(x) = cx$ for $-\frac{1}{2} < x < 0$. Reversing substitutions, we have that $g(x) = cx + d$ for $0 < x < \frac{1}{2}$. $3g(\frac{1}{3}) = 1$ so $d = \frac{1-c}{3}$.

Suppose that, in ♥, we have a z in the interval $[\frac{1}{2};1)$. We find $x+y=1-z$, where both $x,y<\frac{1}{2}$. It follows that $g(z)=1-g(x)-g(y)=1-c(x+y)-\frac{2(1-c)}{3}=1-c+cz-\frac{2(1-c)}{3}=cz+\frac{1-c}{3}$. Now, we can conclude that $g(x)=cx+\frac{1-c}{3}$ for all $0<x<1$.

Now we remember that $f(x)=g(\frac{1}{x+1})=\frac{c}{x+1}+\frac{1-c}{3}$. This quantitiy has to be positive for all $x>0$, which is the same as

$$c>\frac{(c-1)(x+1)}{3}.$$

We see that $c\leq 1$, otherwise LHS can be made arbitrarly large. Now

$$x+1>\frac{3c}{c-1}.$$

Setting $x\to 0^+$, we can deduce that $1\geq\frac{3c}{c-1}$, which is equivalent to $c\geq -\frac{1}{2}$. All such functions trivially work.

§2.9 IZHO 2022/6, proposed by Nurtas Shyntas

Problem 9 (IZHO 2022/6)
Do there exist two bounded sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that for each positive integers n and $m > n$ at least one of the two inequalities $|a_m - a_n| > 1/\sqrt{n}$, and $|b_m - b_n| > 1/\sqrt{n}$ holds?

¶ Solution (Pairs of numbers = points on the coordinate plane). Whenever seeing problems that involve two (perhaps, more) sequences of numbers and some further constraints on them, one immediate neuron activation has to be plotting them on the coordinate plane. The next step is to note that the given condition rewrites nicely after we square absolute values and sum two inequalities to use the condition that **at least one of them** holds. We get $(a_m - a_n)^2 + (b_m - b_n)^2 > \frac{1}{n}$. This is obvious to imply that the distance between points (a_m, b_m) and (a_n, b_n) on the coordinate plane is at least $\frac{1}{\sqrt{min(m,n)}}$.

If the sequences are bounded then all the points are contained within some square centered at the origin. Now, for each point, let's draw a circle centered at it because we deal with distances it's easier to visualise inequalities on distances as non-intersecting circles. Our objective is to have no two circles overlap and their total area to be unbounded, which will cause the contradiction to appear as all the circles are contained within some bounded shape with constant area. This sounds doable because our lower bound on distances is $\frac{1}{\sqrt{n}}$, so we can make radius to be $\frac{c}{\sqrt{n}}$ for some constant c and the areas will be $\frac{\pi c^2}{n}$, which diverges if we sum across all points, because the *Harmonic Series* diverges. More formally,

$$c^2\pi(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots) \text{ is unbounded.}$$

It's not hard to check that $c = \frac{1}{2}$ works, because $\frac{1}{2\sqrt{m}} + \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{min(m,n)}}$ and so the sum of the radii is smaller than the distance between the centers, which shows that they don't overlap. Hence, concludes the proof.

Remark. Other problem that uses this idea is Serbia MO 2024/3.

§2.10 RMM 2018/2, proposed by Ilya Bogdanov

Problem 10 (RMM 2018/2)
Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying
$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

¶ First solution (Derivatives and degree counting) Solution is fairly short but a bit unmotivated. The only thing I can say is that there are a lot of cases when differentiating a polynomial equation results in a new useful one, so it's never harmful to try.

After we differentiate, we get
$$P'(x)P(x)^8(10P(x) + 9) = Q'(x)Q(x)^{19}(21Q(x) + 20).$$
Now, we essentially have two equations that we can work with. We can use that
$$\frac{Q(x)^{20}}{P(x)^9} = \frac{P(x) + 1}{Q(x) + 1}$$
in order to get $P'(x)(Q(x) + 1)Q(x)(10P(x) + 9) = Q'(x)P(x)(P(x) + 1)(21Q(x) + 20)$, which avoids big powers of polynomials and feels easier to set up degree counting arguments.

Indeed, from the initial equation $10 \deg P = 21 \deg Q$ and $(10P(x) + 9, P(x)) = 1$, $(10P(x)+9, P(x)+1) = 1$ in $\mathbb{C}[x]$, which essentially means they have no common roots. Using this, we can say that $10P(x)+9 | Q'(x)(21Q(x)+20)$, but $\deg Q'(21Q+20) = 2\deg Q-1 < \deg P = \deg 10P + 9$. Contradicting the divisibility, so we are done.

¶ Second solution (The roots lemma from Putnam). We will use the following very useful fact that appeared in the solution to an old Putnam problem:

Lemma
Let $P \in \mathbb{C}[x]$ be a non-constant polynomial and c be a non-zero constant. Let p_0, p_1 be the number of distinct roots of P and $P + c$, respectively. Then $p_0 + p_1 \geq \deg P + 1$.

Proof. Suppose that the root a in either P or $P+c$ is counted with multiplicity α, then this root will have multiplicity $\geq \alpha - 1$ in P'. Then, since sets of roots of P and $P+c$ are disjoint, we can say that P' is divisible by roots with total multiplicity $\deg P - p_0 + \deg P - p_1$. On the other hand this value is smaller than $\deg P' = \deg P - 1$. Rearranging, we get that $p_0 + p_1 \geq \deg P + 1$. □

Now in the main problem we see $P(x)^9(P(x) + 1) = Q(x)^{20}(Q(x) + 1)$. We can try to make claims about the roots of $Q, Q + 1$ and $P, P + 1$. Here, our lemma comes in play. Suppose that p_0, p_1 is the number of roots of P and $P + 1$, same definition for q_0, q_1. Obviously, $p_0 + p_1 = q_0 + q_1$, because the sets of roots are disjoint. We also know that $p_0 + p_1 \geq \deg P + 1 = \frac{21}{10}\deg Q + 1 > 2\deg Q$. So $2\deg Q < q_0 + q_1 \leq \deg Q + \deg Q$, which is a clear contradiction.

Remark. Other polynomial problems where differentiation helps are RMMSL 2018 A1, USA TST 2017/3.

§2.11 IZHO 2019/6, proposed by Alexander Golovanov

Problem 11 (IZHO 2019/6)

On a polynomial of degree three it is allowed to perform the following two operations arbitrarily many times:

i. Reverse the order of its coefficients including zeroes (for instance, from the polynomial $x^3 - 2x^2 - 3$ we can obtain $-3x^3 - 2x + 1$);

ii. Change polynomial $P(x)$ to the polynomial $P(x+1)$.

Is it possible to obtain the polynomial $x^3 - 3x^2 + 3x - 3$ from the polynomial $x^3 - 2$?

¶ First solution (Complex roots). Note that the polynomial can be restored if we know its roots. The second condition is easily restated in terms of the roots: After applying ii, each root α of $P(x)$ becomes $\alpha - 1$. The first condition is known as changing $P(x)$ by $x^d P\left(\frac{1}{x}\right)$, where d is the $\deg P$. Each non-zero root α of $P(x)$ will transform into $\frac{1}{\alpha}$ (easy to check that the roots never become zero).

The initial polynomial $x^3 - 2$ has roots $\sqrt[3]{2}$, $\sqrt[3]{2}(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$, $\sqrt[3]{2}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i)$. While the final polynomial, $(x-1)^3 - 2$, has roots $\sqrt[3]{2} + 1$, $\sqrt[3]{2}(-\frac{1}{2} + \frac{\sqrt{3}}{2}i) + 1$, $\sqrt[3]{2}(-\frac{1}{2} - \frac{\sqrt{3}}{2}i) + 1$. Now we have the initial set and the final set, and all the operations that can be applied to it. We are left to find an invariant or a monovariant. One crucial observation here is that we could easily obtain $(x-1)^3 - 2$ from $x^3 - 2$ by applying the $P(x) \to P(x-1)$ transformation. So the fact that we cannot add a one to the root seems to be crucial. We see that the complex roots of the first polynomial have a negative real part, while the complex roots of the final polynomial have a positive real part, $\frac{1}{x}$ seems to preserve the sign too. We are only left to formalize this argument.

Suppose that we have a root $a + bi$ with $a < 0$. If we apply the first operation, a clearly stays negative. The second operation turns $a + bi$ into $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$ with a changing into $\frac{a}{a^2+b^2}$, which preserves the sign. Note that a complex root stays complex and a real root stays real, and so we will never be able to get two positive-real-part roots from two negative-real-part roots. Contradiction proves that no such polynomial exists.

¶ Second solution (Linear independence). We start with the same setup as in the first solution. But this time, we proceed with paying attention to what happens to the real root, namely $\sqrt[3]{2}$. One can check that, given the operations $x \to \frac{1}{x}$, $x \to (x-1)$, the initial number stays irrational, and so, it's never equal to zero. Further, we claim that from a number n, we can only achieve numbers of the form $\frac{an+b}{cn+d}$ for some integers a, b, c, d. Claim is easily checked by induction, because initially we have $\frac{1 \cdot n + 0}{0 \cdot n + 1}$.

Supposing that the condition is true, we get that $\frac{\sqrt[3]{2}a+b}{\sqrt[3]{2}c+d} = \sqrt[3]{2} + 1$ for some integers a, b, c, d. Which is equivalent to $c\sqrt[3]{2^2} + (d + c - a)\sqrt[3]{2} + d - b = 0$. It's known that $\sqrt[3]{2^2}$, $\sqrt[3]{2}$ and 1 are *linearly independent* over $\mathbb{Z}$. We know that $x^3 - 2$ is the minimal integer polynomial of $\sqrt[3]{2}$, because if $Q(x)$ is the polynomial with the minimal degree such that $\sqrt[3]{2}$ is its root, then $Q | x^3 - 2$ as polynomials in $\mathbb{Z}$, otherwise it has a residue with a smaller degree. The former is impossible as $x^3 - 2$ is irreducible in $\mathbb{Z}$.

This means that $c = 0$, $d = a - c = a$, $d = b$, so $c = 0$ and $a = b = d = m$ for a non-zero m. Now, if we reverse the operations and represent the number as $\frac{an+b}{cn+d}$ each time, then, as we now add a one instead of subtraction, we either always increase or always decrease $a + b + c + d$ in the case of $x + 1$ (depending on the sign of m), or don't change it in the case of $\frac{1}{x}$. Therefore, we can't go from $3m$ to $1 + 0 + 0 + 1 = 2$ using these operations if we performed the $x + 1$ operation at least once; but it's obvious, otherwise we can't get $\frac{1 \cdot n+0}{0 \cdot n+1}$ from $\frac{m \cdot n+m}{0 \cdot n+m}$.

§2.12 SRMC 2023/4, proposed by Navid Safaei

Problem 12 (SRMC 2023/4)
Let $\mathcal{M} = \mathbb{Q}[x, y, z]$ be the set of three-variable polynomials with rational coefficients. Prove that for any non-zero polynomial $P \in \mathcal{M}$ there exists non-zero polynomials $Q, R \in \mathcal{M}$ such that

$$R(x^2y, y^2z, z^2x) = P(x, y, z)Q(x, y, z).$$

¶ First solution (Cubing a variable) We first characterize the monomials of $R(x^2y, y^2z, z^2x)$:

Claim — Polynomial $T(x, y, z)$ can be written as $R(x^2y, y^2z, z^2x)$ if for each its monomial $x^iy^jz^k$, $9|4i - 2j + k$.

Proof. We essentially need the existence of positive integer a,b,c so that $i = 2a + b$, $j = 2b + c$, $k = 2c + a$. We can set $a = \frac{4i-2j+k}{9}$, $b = \frac{4j-2k+i}{9}$, $c = \frac{4k-2i+j}{9}$. We can check that one of the divisibilities implies others. We also need that all of them are non-negative, which we will handle later. □

Now we will be done if $P(x, y, z)Q(x, y, z) = T(x^9, y^9, z^9)$, because the condition of the last claim holds. For this, we use the following variation of the problem as a lemma:

Claim — For any non-zero polynomial $P \in \mathcal{M}$ there exists a non-zero polynomials $Q, R \in \mathcal{M}$ such that

$$R(x^3, y, z) = P(x, y, z)Q(x, y, z).$$

Proof. This looks like something doable. We are motivated to distinguish monomials of P by the residue of the power of x (mod 3). For a polynomial $P(x, y, z)$ and $i = 0, 1, 2$, let $P_i(x, y, z)$ be the sum of all its monomials which have their x power congruent to i (mod 3). Then $P(x, y, z) = P_0(x, y, z) + P_1(x, y, z) + P_2(x, y, z)$. Now we want to multiply monomials from the same P_i three times or from every P_i one time each. So we want a polynomial that has $P_i(x, y, z)^3$ for every $i = 0, 1, 2$, and $P_1(x, y, z)P_2(x, y, z)P_3(x, y, z)$. Now we recall a standard factorization.

Set

$$Q(x, y, z) = P_0(x, y, z)^2 + P_1(x, y, z)^2 + P_2(x, y, z)^2 -$$
$$-P_0(x, y, z)P_1(x, y, z) - P_1(x, y, z)P_2(x, y, z) - P_2(x, y, z)P_0(x, y, z),$$

then

$$P(x, y, z)Q(x, y, z) = P_0(x, y, z)^3 + P_1(x, y, z)^3 + P_2(x, y, z)^3 - 3P_0(x, y, z)P_1(x, y, z)P_2(x, y, z).$$

Each monomial is composed either from a product of three monomials from the same P_i or three from different, both of which have their x power divisible by three. So every monomial of PQ has x power divisible by three and we can say that $P(x, y, z)Q(x, y, z) = R(x^3, y, z)$. □

Now, just spam the claim, interchanging variables, to get $R_1(x^3, y, z)$, then $R_2(x^9, y, z)$, then $R_3(x^9, y^3, z)$, ..., $T(x^9, y^9, z^9)$. Every time we multiplied by polynomials from $\mathcal{M}$; so their product, namely $\frac{T(x^9,y^9,z^9)}{P(x,y,z)}$, is also in that set, and it works. Now, we can multiply both sides by $(xyz)^{3N}$ for a big enough N to ensure that all the degrees are non-negative.

¶ Second solution (Grasp of linear algebra) Using the notation from the last solution, let $P_i(x, y, z)$ for $i = 0, 1, \ldots, 8$ be the sum of monomials $x^a y^b z^c$ of P that have $4a - 2b + c \equiv i \pmod 9$. Let's say that the monomials $x^a y^b z^c$ for which $4a - 2b + c \equiv 0 \pmod 9$ form a subset $\mathbb{S}$ and say that $\mathbb{S}[x, y, z]$ are the polynomials, the monomials of which are in $\mathbb{S}$ and that have rational coefficients. For convinience, $\mathbb{S}[x, y, z] = \mathcal{S}$.

We want to choose $Q(x, y, z) \in \mathcal{M}$ so that $P(x, y, z)Q(x, y, z) \in \mathcal{S}$. Suppose that $Q(x, y, z) = Q_0(x, y, z) + Q_1(x, y, z) + \cdots + Q_8(x, y, z)$, where monomials $x^a y^b z^c$ in Q_i are such that $4a - 2b + c \equiv i \pmod 9$. Same definition for P_i. What the condition asks us can be rewritten as:

$$\forall 1 \le r \le 8, \sum_{i+j \equiv r \pmod 9} P_i(x, y, z)Q_j(x, y, z) = 0.$$

Which is the same as saying that

$$\forall 1 \le r \le 8, \sum_{i+j \equiv r \pmod 9} P_i(x, y, z)z^k Q_j(x, y, z)z^s = 0,$$

where $k \equiv -i \pmod 9), s \equiv -j \pmod 9$, $0 \le s \le 8$ and $0 < k + s < 9$. Then $k + s$ is fixed and the equation holds. Note that k can be negative.

For convenience, multiply each equation by z^{9m} for large enough m so that $P_i(x, y, z)z^{k+9m}$ is in $\mathcal{S}$ too. The advantage of this is that now we can substitute $Q_i(x, y, z)z^s = Q_i'(x, y, z)$ and $P_i(x, y, z)z^{k+9m} = P_i'(x, y, z)$, where each P' and Q' are in $\mathcal{S}$. Now, we have a homogeneous system of 8 linear equations with 9 variables, in which coefficients and variables are polynomials in $\mathcal{S}$. Now, as with usual systems of equations, using *Gaussian Elimination*, we can get a non-trivial solution that can be represented as a rational function in polynomials from $\mathcal{S}$. But we can multiply the solutions by any polynomial. So, just multiply it so that every Q' is, in fact, a polynomial.

One remarkable property of $\mathcal{S}$ is that whenever we multiply two elements of it, we get an element of it, same with summation, i.e it forms a **ring**. After we found a solution in polynomials for Q', we know that they all are from $\mathcal{S}$. To get a solution in Qs, we just need to divide by some z in powers less than 9, we can ensure that we get polynomials by multiplying the solution for Q' by a large power of z^9 first and then performing the division. This way we get the desired $Q(x, y, z)$. As in the previous solution, we are left to multiply both sides by $(xyz)^{3N}$ for a big enough N to sort out any negative powers.

§2.13 USA TST 2024/6, proposed by Milan Haiman

Problem 13 (USA TST 2024/6)
Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all real numbers x and y,

$$f(xf(y)) + f(y) = f(x+y) + f(xy).$$

¶ Solution (Finding periodicities) As always, $P(x,y)$ denotes the assertion. Constant functions trivially work, so let's assume that f is not constant. We start with a trivial $P(x,0)$ that gives $f(xf(0)) = f(x)$, and $P(1,x)$ that gives $f(f(x)) = f(x+1)$. Now we want to use it by setting $y = f(0)$ in the initial. This gives

$$f(xf(f(0))) + f(f(0)) = f(x+f(0)) + f(xf(0)) = f(x+f(0)) + f(x).$$

But we know that $f(1 \cdot f(0)) = f(1)$. So,

$$f(x+1) + f(x) = f(xf(1)) + f(1) = f(x+f(0)) + f(x),$$

where we have used $P(x,1)$. It follows that $f(x+1) = f(x+f(0))$, which is essentially $f(x) = f(x+f(0)-1)$.

Claim — f is not periodic and $f(0) = 1$.

Proof. Denote the period as $p \neq 0$. If we substitute $y+p$ in y then $f(xy) = f(xy+xp)$ and so $f(0) = f(xp)$ which means that the function is constant, contradiction. Second part now follows. □

Now, we substitute $x = f(x)$, because we will be able to rewrite the $f(f(x)y)$ term in the RHS using $P(y,x)$ and exploit symmetry.

$$f(f(x)f(y)) + f(y) = f(f(x)+y) + f(f(x)y) = f(f(x)+y) + f(x+y) - f(x) + f(xy).$$

This gives that $f(f(x)+y)$ is symmetric in x and y so $f(f(x)+y) = f(f(y)+x)$, we call this $Q(x,y)$

Now, we exploit symmetry here. $Q(f(x),y)$ and $f(f(x)) = f(x+1)$ gives $f(f(x+1)+y) = f(f(f(x))+y) = f(f(x)+f(y))$. And so, by symmetry and $Q(y+1,x)$, $f(f(x+1)+y) = f(f(y+1)+x) = f(f(x)+y+1)$. Now we can conclude that $f(x+1) = f(x)+1$ by the above claim; $f(-1) = 0$ is clear.

We can find a lot of equations of the form $f(a) = f(b)$, so we want to find some sort of injectivity. Our next claim is the final result of this search.

Claim — f is injective at zero

Proof. Suppose that $f(c) = 0$ for $c \neq -1$, then $P(x,c)$ gives that $f(xc) + f(x+c) = 1$. $P(xc,-1)$ gives $f(-xc-1) + f(xc) = 1$, from which it's easy to get that $f(-xc-1) = f(x+c)$, but we can set $x = \frac{-2-c}{c+1}$ to have $-xc-1 = x+c+1$, and then $f(-xc-1) - f(x+c) = 1$, but it has to be zero, contradiction. □

To finish, we just substitute $Q(x, 1-f(x))$ to get $f(f(1-f(x))+x) = f(-1-f(x)+f(x)) = 0$ and so $f(-1-f(x))+x = -1$. If we take $x = x-1$ here then $f(-f(x)) = -x$. Now, we take $x = -1$ in the initial to see that $-y+f(y) = f(-f(y))+f(y) = f(-1+y)+f(-y) = f(y)-1+f(-y)$. And so, $f(-y) = 1-y$, which is the same as $f(x) = x+1$.

§2.14 ISL 2010 A6, proposed by Alex Schreiber

Problem 14 (ISL 2010 A6)
Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

¶ Solution (Drawing arrows and the image analysis) We start with some simple observations. If $a \in \operatorname{Im} f$, then $a + 1 \in \operatorname{Im} f$, same with g. Suppose that $\operatorname{Im} f = \{a, a + 1, ...\}$ and $\operatorname{Im} g = \{b, b + 1, ...\}$ with, WLOG, $a > b$.

Claim — $g(n)$ is injective for $n \geq a$.

Proof. If $g(a + k) = g(a + m)$ for $k, m \geq 0$ then there exist positive integer numbers s and t such that $f(s) = a + k$ and $f(t) = a + m$. Then $g(f(s)) = g(f(t))$ and so $g(s) + 1 = g(f(s)) = g(f(t)) = g(t) + 1$. Now we can say that $f(s) + 1 = f(g(s)) = f(g(t)) = f(t) + 1$. And so, $k = m$. □

The set of all values of $g(a + k) - 1$ for non-negative integer ks is exactly the values of $g(f(n)) - 1$ for all positive integer n, but we know that it's the set of all values of $g(n)$ which is just $\{b, b + 1, ...\}$. We also know that all these values are diffrent by the claim above, so each appears exactly once.

$g(f(n) + 1) = g(f(g(n))) = g(g(n)) + 1$. If we repeatedly plug $g(n)$ instead of n in this relation and use the original assertion, then we can further get $g(f(n) + k) = g^{k+1}(n) + 1$ for all non negative integers k.

Now, let $f(m) = a$ for some positive integer m. If $m \geq b$, then $g(s) = m$ for some positive integer s, and then $a = f(m) = f(g(s)) = f(s) + 1$, so $f(s) = a - 1$, but it's not in the $\operatorname{Im} f$. So $m \leq b - 1$. We know that $g^{k+1}(m) = g(f(m) + k) - 1 = g(a + k) - 1$ (As always, we denote $\underbrace{g(g(\dots g(x)\dots))}_{k} = g^k(x)$), which we know, takes every value from $\{b, b + 1, ...\}$ exactly once.

As it always happens to be useful when dealing with integer functional equations and their iterations, we will represent the function g as a directed graph where we draw an arrow from every positive integer x to $g(x)$. There's an edge from b to some number c in $\{b+1, b+2, ...\}$, because it's in the path that starts from m and traverses every integer in $\{b, b+1, ...\}$ exactly once, and trivially b cannot go to b, otherwise it would terminate. We also know that there's an edge from some number $\geq a$ to c (because the image of $g(a + k)$ is $\{b + 1, b + 2, ...\}$). So there are two arrows to c from numbers that are $\geq b$. But if we start from m, we will be traversing a path that covers each number in $\{b, b + 1, ...\}$ exactly once, so if $d \geq b$ and $e \geq b$ both go into c with, WLOG, d going first, then we will have a path $d \to c \to ... \to e \to c$ and it cycles from here, contradiction. This proves that $a = b$.

Now, it's not hard to finish. If the arrow from m goes to a number $> a$, then it won't ever go to a, because the image of $g(a + k)$ is $\{b + 1, b + 2, ...\}$ which is $\{a + 1, a + 2, ...\}$ in our case, so the arrow from m goes to a, and so $g(m) = a$.

Now, $f(a) = f(g(m)) = f(m)+1 = a+1$ and $g(a) = a+1$. If $f(a+k) = g(a+k) = a+k+1$, then $f(a + k + 1) = f(g(a + k)) = f(a + k) + 1 = a + k + 2$ and, in the same manner, $g(a+k+1) = a+k+2$. This sets up the induction that shows that $f(a+k) = g(a+k) = a+k+1$ for all non-negative integers k. If $f(s) = a + p$ and $g(s) = a + q$ with non-negative p and q, $a + p + 1 = f(s) + 1 = f(g(s)) = f(a + q) = a + q + 1$, so $p = q$ and $g(s) = f(s)$ for all positive integer s, exactly what we wanted.

> **Remark.** The idea of drawing a graph comes in handy quite often, not only in $\mathbb{N} \to \mathbb{N}$ FEs. My other favorite examples are Vietnam TST 1990/3 and ISL 2013 A5.

§2.15 ISL 2018 A7, proposed by Evan Chen

Problem 15 (ISL 2018 A7)
Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a+b+c+d=100$.

¶ First solution (Hölder's and the two variable version) As it is always crucial for the problems of this kind, we first need to find the equality case. Variables appear cyclically and it's natural to assume that a maximum one is followed by a minimum one (because x and $\frac{1}{x+7}$ are oppositely monotonic and we can use the rearrangement), this prompts us to try (a,b,a,b) 4-tuple for a maximum value. This naturally leads us to the two variable version of the inequality, a method that comes in handy very often when dealing with multivariable inequalities to boil them down to something easier.

How do we find the maximum of $\sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{a+7}}$ while knowing the sum? This screams *Hölder's Inequality*, but we first need to break the fractions down cleverly. It's obvious that we have to separate a and $\frac{1}{b+7}$, then we can add any constant multiple in. In this case it will be 7 because then, we will be able to somehow cancel the $\frac{1}{b+7}$ and free the linear variable to have the sum $a+b$. This argument can be written down neatly as the claim:

Claim — $\sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{a+7}} \le \sqrt[3]{\frac{a+b+14}{7}}$

Proof. From Hölder's Inequality, we get

$$\sqrt[3]{7\cdot a\cdot\frac{1}{b+7}} + \sqrt[3]{7\cdot b\cdot\frac{1}{a+7}} \le \sqrt[3]{7+b}\sqrt[3]{a+7}\sqrt[3]{\frac{1}{b+7}+\frac{1}{a+7}} = \sqrt[3]{a+b+14},$$

dividing by $\sqrt[3]{7}$, we reach the claimed. □

Examining the equality case, we can see that it holds if and only if the triples $(7, a, \frac{1}{b+7})$ and $(b, 7, \frac{1}{a+7})$ are proportional. So, $ab = 7\cdot 7 = 49$ and $a+b=50$, from which $a=49, b=1$ or vice versa, which gives the value of $\sqrt[3]{\frac{512}{7}}$.

It's not hard to finish from here because we essentially developed all the needed ideas. By rearrangement, we can say that

$$\sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}} \le \sqrt[3]{\frac{x}{t+7}} + \sqrt[3]{\frac{y}{z+7}} + \sqrt[3]{\frac{z}{y+7}} + \sqrt[3]{\frac{t}{x+7}},$$

where (x,y,z,t) is the decreasing permutation of (a,b,c,d).

Now, applying the claim and Power Mean inequality,

$$\sqrt[3]{\frac{x}{t+7}} + \sqrt[3]{\frac{y}{z+7}} + \sqrt[3]{\frac{z}{y+7}} + \sqrt[3]{\frac{t}{x+7}} \le \sqrt[3]{\frac{x+t+14}{7}} + \sqrt[3]{\frac{y+z+14}{7}} \le \sqrt[3]{\frac{4(x+y+z+t+28)}{7}} = \sqrt[3]{\frac{512}{7}}.$$

¶ Second solution (Magic AM-GM) Most steps in this problem can be proven either via *AM-GM* or *Hölder's Inequality*, because they both allow to transform products under roots. This solution will only use one AM-GM, applied to every root.

Motivational remark is that we have $\frac{1}{b+7}$ with some constant multiple and b among the possible multiples in the roots (we don't care that they belong to different roots because we transform everything to a cyclic sum in the end via AM-GM). In order for the cyclic sum to be a constant number, we need to complete $\frac{1}{b+7}$; the easiest way to do so is to have $\frac{7}{b+7}$ and $\frac{b}{b+7}$, but for this to be true, we need to divide and multiply the root with b by $b+7$, this doesn't cause any problem because we can separate the resulting $b+7$ term and it won't harm the cyclic sum.

The only trouble in this approach is that we need to figure out the coefficients in AM-GM so that the equality holds. Usually it's done by knowing the equality case beforehand. In our case, the other way is to assume that coefficients before $\frac{a}{a+7}$ and $\frac{7}{b+7}$ are 1 in order for them to cancel each other successfully. This demands $\frac{a}{a+7}$ to be equal to $\frac{7}{b+7}$. Writing down the cyclic equivalents and remembering that the sum is 100, we can also find the equailty case without the arguments of the above solution. This will lead us to a $\frac{1}{64}$ coefficient before $a+7$.

With all above being said, we can just write down the magic one-liner:

$$\sqrt[3]{\frac{7a}{64(b+7)}} = \sqrt[3]{\frac{7a(a+7)}{64(b+7)(a+7)}} \leq \frac{1}{3}\left(\frac{a+7}{64} + \frac{a}{a+7} + \frac{7}{b+7}\right).$$

Summing cyclically and multiplying by $\sqrt[3]{\frac{64}{7}}$, we reach the conclusion.

§2.16 ISL 2007 A4, proposed by Paisan Nakmahachalasint

Problem 16 (ISL 2007 A4)
Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $f(x + f(y)) = f(x + y) + f(y)$ for all pairs of positive reals x and y. Here, $\mathbb{R}^+$ denotes the set of all positive reals.

¶ First solution ($g(x)$ trick and injectivity) $P(x, y)$ is the assertion. First we do the trivial substitutions for $\mathbb{R}^+ \to \mathbb{R}^+$ FEs in which we want to force some terms to cancel leaving one term that equals to zero, which contradicts the codomain of the function.

More formally, If $c > f(c)$ for some positive real c. We substitute $P(c - f(c), c)$ to get $f(c) = f(c - f(c) + c) + f(c)$, but $f(c - f(c) + c)$ cannot be equal to zero. So $f(x) \geq x$ for all positive real x.

Now we perform a common trick that sometimes [1] (still worth trying) works once we proven $f(x) \geq cx$ (or, perhaps, any other function of x other than cx, but i have not seen any examples of that) for some c. Set $g(x) = f(x) - x \geq 0$.

Equation can be rewritten as $g(x + y + g(y)) = g(x + y) + y$, or, in a more succinct form, $g(x + g(y)) = g(x) + y$ for $x > y$; we call it $Q(x, y)$.

Now it's easy to prove that g is injective. If $g(a) = g(b)$ then take $c > a, b$ and compare $Q(c, a)$ and $Q(c, b)$, $g(c) + a = g(c + g(a)) = g(c + g(b)) = g(c) + b$, and so, $a = b$, proving injectivity.

Consider some positive real $z < y < x$. $g(x + g(y)) = g(x) + y = g(x + g(z)) + y - z = g(x + g(z) + g(y - z))$ so $x + g(y) = x + g(z) + g(y - z)$, by injectivity. This proves that g satisfies the *Cauchy's FE* for the R^+ interval. Of course, g is bounded from below by a zero, so it has to be of the form cx. Direct substitution shows that $g(x) = x$ and $f(x) = 2x$, as a result.

¶ Second solution (Three variables method) We see that the equation allows us to write some $f(f(x) + y)$ as a sum of two fs that don't have fs in their arguments. In general, we can break down any f that has an independent summand and some fs in the argument into a sum that has no fs in arguments. This is good, because we can write $f(f(x + f(y)) + z) = f(f(x + y) + f(y) + z)$ and rewrite the two sides, which sounds promising to arrive at some Cauchy-like equation.

$$\begin{aligned} f(x + f(y)) + z &= f(x + y) + f(y) + z \\ f(f(x + f(y)) + z) &= f(f(x + y) + f(y) + z) \\ f(x + f(y) + z) + f(x + f(y)) &= f(x + y + f(y) + z) + f(x + y) \\ f(x + y + z) + f(y) + f(x + y) + f(y) &= f(x + 2y + z) + f(y) + f(x + y) \\ f(x + y + z) + f(y) &= f(x + 2y + z), \end{aligned}$$

which is just Cauchy's FE because the above holds for all positive real x, y, z. Finish is the same as in the first solution.

[1] It's especially useful if we aim to show that cx is the solution, because then we seek to prove that $g(x) = 0$, which sounds strong enough for some strange results involving g to start appearing that only zero functions satisfy.

Remark. The three variables method and $g(x)$ trick are both very common when solving $\mathbb{R}^+ \to \mathbb{R}^+$ functions. Some famous examples: Iran TST 2017/3, ISL 2005 A2, BMO 2024/4.

§2.17 Tuymaada Senior league 2021/8, proposed by Alexander Golovanov

Problem 17 (Tuymaada Senior league 2021/8)
In a sequence P_n of quadratic trinomials each trinomial, starting with the third, is the sum of the two preceding trinomials. The first two trinomials do not have common roots. Is it possible that P_n has an integral root for each n?

¶ Solution (Exploring a rational function) Define the ith fibonacci number F_i as $F_0 = 0, F_1 = 1$ and $F_{i+2} = F_{i+1} + F_i$ for all $i \geq 0$. We start with an obvious fact:

Claim — $P_n = F_{n-2}P_1 + F_{n-1}P_2$ for $n \geq 2$

Proof. Follows from the definition of F_i and the condition on the sum of the two preceeding equals the next. □

Assume that each P_i has a root r_i. We know that $F_{n-2}P_1(r_n)+F_{n-1}P_2(r_n) = P_n(r_n) = 0$, so $\frac{P_1(r_n)}{P_2(r_n)} = -\frac{F_{n-1}}{F_{n-2}}$ for $n \geq 3$. $P_2(r_n) \neq 0$, otherwise $P_1(r_n) = 0$ too, contradicting the hypothesis that they don't have common roots.

Now we will introduce a function $f(x) = \frac{P_1(x)}{P_2(x)}$, we will call it a Quadratic Rational Function.

Claim — Quadratic Rational Function f is determined by 5 points $(x_i, f(x_i))$

Proof. Suppose that there are two functions $f_1 = \frac{P_1(x)}{Q_1(x)}$ and $f_2 = \frac{P_2(x)}{Q_2(x)}$ and 5 points x_1, x_2, ..., x_5 such that $f_1(x_i) = f_2(x_i)$ for $1 \leq i \leq 5$. Then $\frac{P_1(x_i)}{Q_1(x_i)} = \frac{P_2(x_i)}{Q_2(x_i)}$, therefore $P_1(x_i)Q_2(x_i) - P_2(x_i)Q_1(x_i) = 0$, but the LHS is a polynomial of degree at most 4 that attains zero in 5 points, which can only happen for an identically zero polynomial. This proves that $f_1(x) = \frac{P_1(x)}{Q_1(x)} = \frac{P_2(x)}{Q_2(x)} = f_2(x)$ □

By definition, there exists a Quadratic Rational Function f such that $f(r_n) = -\frac{F_{n-1}}{F_{n-2}}$ for all $n \geq 3$. One can show that a polynomial that produces rational outputs for rational inputs in at least $deg + 1$ points must have rational coefficients (for example, *Lagrange's Interpolation* suffices). The claim on its own is obvious, but serves as motivation to make a similar statement about the rational functions.

Suppose that $P_1(x) = a_2x^2 + a_1x + a_0$ and $P_2(x) = b_2x^2 + b_1x + b_0$.

Claim — $\frac{a_2}{b_2} \in \mathbb{Q}$.

Proof. By the above claim, we only need to define f for r_3, r_4, r_5, r_6, r_7 (its proved further that these are different), the rest is defined itself. Suppose that $P_1(x) = a_2x^2 + a_1x + a_0$ and $P_2(x) = b_2x^2+b_1x+b_0$. We need $a_2r_i^2F_{i-2}+a_1r_iF_{i-2}+a_0F_{i-2}+b_2r_i^2F_{i-1}+b_1r_iF_{i-1}+b_0r_iF_{i-1} = 0$ for $3 \leq i \leq 7$, which is a system of only 5 homogeneous equations with 6 variables, it has a solution due to *Gaussian Elimination*, and since the coefficients are integers, the solution is rational. Therefore, $a_2 \in \mathbb{Q}$ and $b_2 \in \mathbb{Q}$, forcing $\frac{a_2}{b_2} \in \mathbb{Q}$. □

Now we will prove a claim that will let us calculate the limits. Since we are dealing with fractions of polynomials and consecutive terms of Fibonacci sequence, it will be useful.

> **Claim —** No two r_i are equal for $i \geq 3$

Proof. If $r_i = r_j$ with $i, j \geq 3$ then $-\frac{F_{i-1}}{F_{i-2}} = \frac{P_1(r_i)}{P_2(r_i)} = \frac{P_1(r_j)}{P_2(r_j)} = -\frac{F_{j-1}}{F_{j-2}}$, but two consecutive Fibonacci numbers are coprime (otherwise, if $d|F_i$ and $d|F_{i+1}$, then $d|F_{i-1}$ as their difference, and we can repeat this argument, eventually reaching F_1, F_2, which are coprime). Therefore, $F_{i-1} = F_{j-1}$ (because they are both the numerators of equal irreducible fractions), and so, $i = j$. □

From this claim we see that the sequence $|r_i|$ is unbounded. So we can choose big enough n such that $|r_n|$ is big. For this value $f(r_n) = \frac{P_1(r_n)}{P_2(r_n)}$ approaches $\frac{a_2}{b_2}$, but $f(r_n) = -\frac{F_{n-1}}{F_{n-2}}$ and the latter approaches $-\frac{\sqrt{5}+1}{2}$, because $F_n = \frac{(\frac{\sqrt{5}+1}{2})^n + (\frac{\sqrt{5}-1}{2})^n}{\sqrt{5}}$ and the second summand approaches zero for big enough n. So $\frac{a_2}{b_2} = -\frac{\sqrt{5}+1}{2}$, contradicting the fact that it's rational by the argument above.

§2.18 Problem 18

Problem 18

Given an integer $n \geq 2$. Real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are such that $a_1 + a_2 + \ldots + a_n = b_1 b_2 \ldots b_n = 1$ and b_i are different. Prove that

$$\frac{\sum_{i=1}^{n}(1-a_i)b_i}{n-1} > \sqrt[n-1]{\left(\sum_{i=1}^{n} \frac{a_i}{b_i}\right)}.$$

¶ Solution (Finding suitable numbers for AM-GM without explicitly stating them!) As spoiled in the headline, we will be adjusting the inequality to be a simple AM-GM, because the LHS looks similar to an arithmetic mean and the RHS looks similar to a geometric mean. In order to succeed, we only need to show the existence of real numbers $x_1, x_2, \ldots, x_n$ such that

$$x_1 + x_2 + \ldots + x_{n-1} = \sum_{i=1}^{n}(1-a_i)b_i \text{ and } x_1 x_2 \ldots x_{n-1} = \sum_{i=1}^{n} \frac{a_i}{b_i}.$$

After homogenization

$$\sum_{i=1}^{n}(1-a_i)b_i = \sum_{1 \leq i,j \leq n, i \neq j} a_i b_j = \sum_{i=1}^{n} a_i \sum_{1 \leq j \leq n, j \neq i} b_j \text{ and } \sum_{i=1}^{n} \frac{a_i}{b_i} = \sum_{i=1}^{n} a_i \prod_{1 \leq j \leq n, j \neq i} b_j.$$

These are the x^0 and x^{n-2} coefficients of a polynomial, namely

$$P(x) = \sum_{i=1}^{n} a_i \prod_{1 \leq j \leq n, j \neq i} (x - b_j).$$

Which is what we wanted, because then, if $x_1, x_2, \ldots, x_{n-1}$ are its roots,

$$x_1 + x_2 + \ldots + x_{n-1} = \sum_{i=1}^{n} a_i \sum_{1 \leq j \leq n, j \neq i} b_j \text{ and } x_1 x_2 \ldots x_{n-1} = \sum_{i=1}^{n} a_i \prod_{1 \leq j \leq n, j \neq i} b_j.$$

It's only left to show that the roots are indeed real. For this, we can sort b_i in ascending order.

Now the signs of $P(b_i)$ and $P(b_{i+1})$ for $1 \leq i \leq n$ are different because the sign of $P(b_i) = (b_i - b_1) \ldots (b_i - b_{i-1})(b_i - b_{i+1}) \ldots (b_i - b_n)$ is just $(-1)^{n-i}$. So, by *Intermediate Value Theorem*, a.k.a continuity, we get $n-1$ different roots on intervals $(b_i; b_{i+1})$, which are all different, therefore the equality case of AM-GM is not satisfied, so the inequality is strict.

§2.19 2018 China TST 1/6

Problem 19 (2018 China TST 1/6)
Let A_1, A_2, …, A_m be m subsets of a set of size n. Prove that

$$\sum_{i=1}^{m}\sum_{j=1}^{m}|A_i|\cdot|A_i\cap A_j| \geq \frac{1}{mn}\left(\sum_{i=1}^{m}|A_i|\right)^3.$$

¶ First solution (Indicator function) When dealing with inequalities about cardinalities, we cannot tackle them directly, instead we need to rewrite the condition in a more algebraic form using variables that we can then manipulate. Indicator function is a way of doing it. Set $a_{ij}=1$ if $i\in A_j$ and $a_{ij}=0$ if not. Suppose that the set from the statement is $\{1,2,\dots,n\}$.

We first note that the LHS counts the number of ordered 4-tuples (i,j,x,y) such that $x\in A_i$, $y\in A_i, A_j$. For this, we essentially need $a_{xi}=1$, $a_{yi}=1$, $a_{yj}=1$ or just $a_{xi}a_{yi}a_{yj}=1$.

We can now write the LHS as

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{x=1}^{n}\sum_{y=1}^{n}a_{xi}a_{yi}a_{yj}.$$

Now we will be changing the order of summation to make it look more pleasant. But before that, we denote $\sum_{i=1}^{m}a_{xi}=c_x$; essentially, it counts the number of sets that contain x.

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{x=1}^{n}\sum_{y=1}^{n}a_{xi}a_{yi}a_{yj}=\sum_{i=1}^{m}\sum_{x=1}^{n}\sum_{y=1}^{n}a_{xi}a_{yi}\sum_{j=1}^{m}a_{yj}=\sum_{i=1}^{m}\sum_{x=1}^{n}\sum_{y=1}^{n}a_{xi}a_{yi}c_y=\sum_{i=1}^{m}\sum_{y=1}^{n}a_{yi}c_y\sum_{x=1}^{n}a_{xi}=$$

$$=\sum_{i=1}^{m}\sum_{y=1}^{n}a_{yi}c_y|A_i|$$

We know that the variables in the last form also depend on n and m, we will try to use it to rewrite the mn multiple that appears in the RHS, and then pray that we can use Hölder's to clear the product of three sums in brackets that appear.

$$\sum_{i=1}^{m}\sum_{y=1}^{n}\frac{a_{yi}}{|A_i|}=m \text{ and } \sum_{i=1}^{m}\sum_{y=1}^{n}\frac{a_{yi}}{c_y}=n,$$

which both follow from the definition.

From all the above, we can write

$$mn\left(\sum_{i=1}^{m}\sum_{j=1}^{m}|A_i|\cdot|A_i\cap A_j|\right)=mn\left(\sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{x=1}^{n}\sum_{y=1}^{n}a_{xi}a_{yi}a_{yj}\right)=$$

$$=\left(\sum_{i=1}^{m}\sum_{y=1}^{n}\frac{a_{yi}}{|A_i|}\right)\left(\sum_{i=1}^{m}\sum_{y=1}^{n}\frac{a_{yi}}{c_y}\right)\left(\sum_{i=1}^{m}\sum_{y=1}^{n}a_{yi}c_y|A_i|\right)\geq\left(\sum_{i=1}^{m}\sum_{y=1}^{n}a_{yi}\right)^3=\left(\sum_{i=1}^{m}|A_i|\right)^3.$$

¶ Second solution (Tensor power trick) This solution is very slick and, perhaps, a bit combinatorial in nature [1]. First, we represent the elements and sets as two parts of a bipartite graph: One contains elements $\{1, 2, \ldots, n\}$, the other contains elements $\{A_1, A_2, \ldots, A_m\}$. It's not hard to understand that the 4-tuples (i, j, x, y) such that $x \in A_i$, $y \in A_i, A_j$ form a one-to-one correspondance to 3-edge paths in the graph. $\sum_{i=1}^{m} |A_i|$ is essentially the sum of degrees of the second part, which is the same as the total number of edges in the graph. With this setup, we can derive a weaker but asymptotically correct result:

Claim — In a bipartite graph G with parts $|G_1| = n$ and $|G_2| = m$ there are e edges. Then the number of paths of lenght 3 in G is at least $\frac{e^3}{27mn}$.

Proof. Suppose that A is the set of vertices of G_1 with degree at least $\frac{e}{3n}$, and suppose that B is the set of vertices of G_2 with degree at least $\frac{e}{3m}$. There are at most $\frac{e}{3n} \cdot (|G_1| - |A|) + \frac{e}{3m} \cdot (|G_2| - |B|) \leq \frac{e}{3n} \cdot n + \frac{e}{3m} \cdot m = \frac{2e}{3}$ edges that have at least one of their ends not in A or B, so there are at least $e - \frac{2e}{3} = \frac{e}{3}$ edges that have both of their ends in A and B. For them, they are the middle edges of at least $\frac{e}{3} \cdot \frac{e}{3m} \cdot \frac{e}{3n} = \frac{e^3}{27mn}$ paths of lenght three. □

When we are asked to calculate some bound and we can establish some result that is asymptotically correct, we can try to adjust it to work by scaling substantially so that the error is *too small*. In the case of graphs, it's reasonable to consider a new graph that has vertices as k-tuples of the vertices of the initial graph that are in the same part with the considered vertex, where two vertices $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ are connected if there's an edge between a_i and b_i for every i. This graph raises a lot of graph-related quantities to the kth power, which is suitable for scaling.

In our case, if we construct a new graph G' from G as discussed above, obviously $|G_1'| = n^k$ and $|G_2'| = m^k$. The number of edges is e^k because we can take any edge (a, b) instead of any (a_i, b_i) to construct an edge $((a_1, a_2, \ldots, a_k), (b_1, b_2, \ldots, b_k))$. We can also show that this graph has p^k 3-paths where p is the number of 3-paths in G, because, as with edges, we can choose any 3-path as a component of the new 3-path in the new graph, and there are k of them. If we use our lemma for this graph, then $p^k \geq \frac{e^{3k}}{27m^k n^k} = \frac{1}{27} \cdot (\frac{e^3}{mn})^k$. Taking the kth root and making k large enough, we see that $\sqrt[k]{\frac{1}{27}}$ approaches 1, so $p \geq \frac{e^3}{mn}$, and the problem follows.

Remark. Another, rather new, example that showcases usefulness of the indicator function is USAMO 2024/6

[1] Well, the problem itself can be classified as combinatorics.

§2.20 ISL 2007 A5, proposed by Vjekoslav Kovač

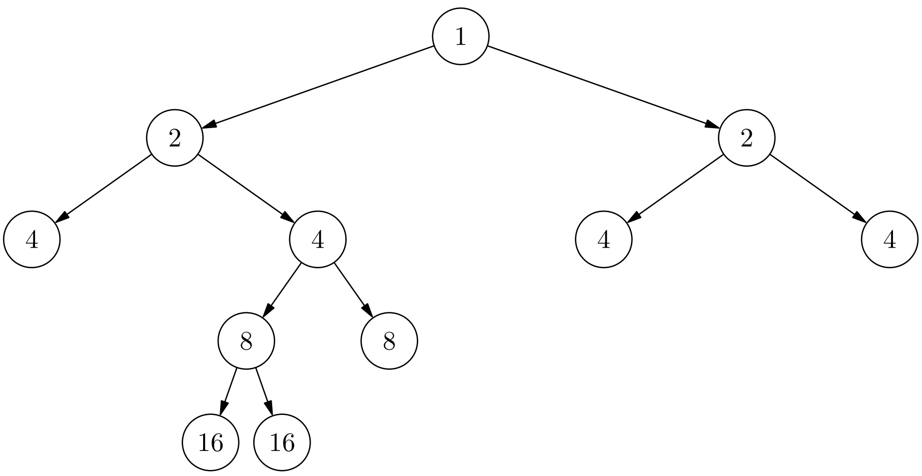
Problem 20 (ISL 2007 A5)
Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2 \cdot a(m) + 2 \cdot a(n) \text{ for all } m, n \geq 1,$$

and $a\left(2^k\right) \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence $a(n)$ is bounded.

¶ Solution (Binary representations and special choice of powers) The problem screams us what to do: Take n, start writing it as the sum of two numbers and use the inequality. At the end you must have the binary representation with some coefficients (because these are the only terms of the sequence that we have information about), use the bound and conclude that every term of the sequence is bounded. The only degree of freedom is choosing the coefficients, which turns out to be not as easy, but still manageable.

We should start with questioning ourselves **"which coefficients can we choose?"**. To answer it, we will build a binary tree in which the root is labeled 1 and a node labeled 2^x either has two children both of which are labeled 2^{x+1} or none.



If we set the leaves to each have a power of two that is in the binary representation of n and a number written in each node is the sum of its two children, then we can track down the movement of the decomposition of n using the inequality in the problem statement from the root to the leaves that are exactly the binary representation. The labels will be the coefficients of the numbers.

Suppose that $n = 2^{d_1} + 2^{d_2} + \ldots + 2^{d_m}$ is the binary representation of n, we want to choose coefficients 2^{c_1}, 2^{c_2}, $\ldots$, 2^{c_m}, so that it would be possible to construct the aforementioned tree with them as the leaves, then we can write the 2^{d_i} in corresponding nodes. Condition that is needed to us is that $\sum_{i=1}^{m} \frac{1}{2^{c_i}} = 1$. It's not hard to see that it's true in one direction, because this value doesn't change with each branching and it was true for the root vertex. In other direction, we can just combine two that are equal. At any moment, this operation doesn't change the sum of inverses of powers of two, so it's always 1. We do it before we no longer can. In that case, if there was no 1 among the numbers, the sum is less than

$\frac{1}{2} + \frac{1}{4} + \ldots < 1$, which is a contradiction. Or if there's a 1 then it's just a single one, which is just the root of the tree, and we constructed it.

Now, we have $a(n) \le \sum_{i=1}^{m} 2^{c_i} a(2^{d_i})$ for any choice of c_i such that $\sum_{i=1}^{m} \frac{1}{2^{c_i}} = 1$. We can further relax this condition so that $\sum_{i=1}^{m} \frac{1}{2^{c_i}} < 1$, because we have a simple claim:

> **Claim** — Given positive integers c_i such that $\sum_{i=1}^{m} \frac{1}{2^{c_i}} < 1$ then we can decrease some of them so that $\sum_{i=1}^{m} \frac{1}{2^{c_i}} = 1$.

Proof. The proof is by induction on the number of variables. If there's only one then we can make $c_1 = 0$ and it works. If there are two equal variables, then we can combine them into one, which already decreases the variable, and then decrease further by the induction hypothesis. If no two are equal then we can make it equal to $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{m-1}} + \frac{1}{2^{m-1}}$, because if c_i are sorted in ascending order then $c_1 \ge 1$, $c_2 \ge 2$, $\ldots$, $c_m \ge m$ (because they are different). □

The claim tells us that if $\sum_{i=1}^{m} \frac{1}{2^{c_i}} < 1$, then we can find new c_i' such that $\sum_{i=1}^{m} \frac{1}{2^{c_i'}} = 1$ and $c_i' \le c_i$. Then $a(n) \le \sum_{i=1}^{m} 2^{c_i'} a(2^{d_i}) \le \sum_{i=1}^{m} 2^{c_i} a(2^{d_i})$.

Now we are ready to choose the powers c_i in

$$a(n) \le \sum_{i=1}^{m} 2^{c_i} a(2^{d_i}) \le \sum_{i=1}^{m} \frac{2^{c_i}}{(d_i+1)^c} \text{ such that } \sum_{i=1}^{m} \frac{1}{2^{c_i}} \le 1$$

We want both of these sums to converge. If the second one converges, we can manually set it at most 1. Since d_i are different, one should remember that $\frac{1}{1^c} + \frac{1}{2^c} + \frac{1}{3^c} + \ldots$ converges only if $c > 1$, because this sum is less than the area under the graph of $\frac{1}{x^c}$ from 1 to $+\infty$ plus one (same Riemann sums idea as in this problem), which is

$$1 + \lim_{a\to\infty} \int_1^a \frac{1}{x^c}\, dx = 1 + 1 - \lim_{a\to\infty} \frac{1}{a^{c-1}} = 2,$$

so it converges, indeed.

Now, we take $N(d_i+1)^{\frac{c}{2}} < 2^{c_i} < 2N(d_i+1)^{\frac{c}{2}}$ (there exists at least one integer power of two in this interval), where $N = \frac{1}{2^{\frac{c}{2}}} + \frac{1}{3^{\frac{c}{2}}} + \ldots$, which converges by what we have explained above. We can clearly see that $\sum_{i=1}^{m} \frac{1}{2^{c_i}} < \frac{1}{N}(\sum_{i=1}^{m} \frac{1}{(d_i+1)^{\frac{c}{2}}}) < 1$, and so, $a(n) \le \sum_{i=1}^{m} \frac{2^{c_i}}{(d_i+1)^c} < (N+1)\sum_{i=1}^{m} \frac{1}{(d_i+1)^{\frac{c}{2}}} < 2N^2$. Hence, we are done.

§2.21 USEMO 2022/5, proposed by Bhavya Tiwari

Problem 21

Given a polynomial $p(x)$ with real coefficients, we denote by $S(p)$ the sum of the squares of its coefficients. For example $S(20x + 21) = 20^2 + 21^2 = 841$. Prove that if $f(x)$, $g(x)$, and $h(x)$ are polynomials with real coefficients satisfying the indentity $f(x) \cdot g(x) = h(x)^2$, then

$$S(f) \cdot S(g) \geq S(h)^2.$$

¶ First solution (Roots of unity filter and Cauchy-Schwartz) Note that the inequality that we are asked to prove in the problem looks a lot like *Cauchy-Schwartz*. We first need to write $S(p)$ in terms of the sum of some monomials depending on values of p, after that we will just use Cauchy-Schwartz.

We know that we can represent each coefficient of the polynomial using the sums of its values by using the *Roots Of Unity Filter* with a substantially large power of the root so that this coefficient is the only one that appears in the sum. We aim to do the same thing but with the squared coefficient.

Here's the proof with more rigor. First, choose ω as the nth root of unity, for $n > \deg f, \deg g, \deg h$. Using the root of unity filter,

$$a_k = \frac{1}{n}\sum_{i=0}^{n-1} \omega^{-ki} f(\omega^i) \text{ for } 0 \leq k \leq n-1.$$

The above is true because $n|k-m$ holds only if $k = m$ for the values in the range. f might not have some of the coefficients, but then we just set them to zero and it still holds.

$$a_k^2 = \left(\frac{1}{n}\sum_{i=0}^{n-1} \omega^{-ki} f(\omega^i)\right)^2 = \frac{1}{n^2}\sum_{0\leq i,j\leq n-1} \omega^{-k(i+j)} f(\omega^i) f(\omega^j).$$

Now we sum it over all k from 0 to $n-1$:

$$S(p) = \sum_k \left(\frac{1}{n^2}\sum_{0\leq i,j\leq n-1} \omega^{-k(i+j)} f(\omega^i) f(\omega^j)\right) = \frac{1}{n^2}\sum_{0\leq i,j\leq n-1}\left(f(\omega^i)f(\omega^j)\sum_k \omega^{-k(i+j)}\right) =$$

$$= \frac{1}{n^2}\sum_{n|i+j} n f(\omega^i) f(\omega^j) = \frac{1}{n}\sum_{i=0}^{n-1} f(\omega^i) f((\overline{\omega})^i) = \frac{1}{n}\sum_{i=0}^{n-1} f(\omega^i)\overline{f(\omega^i)} = \frac{1}{n}\sum_{i=0}^{n-1} |f(\omega^i)|^2,$$

where we have used the fact that $\omega^{n-i} = (\frac{1}{\omega})^i = (\overline{\omega})^i$ and $f(\overline{\omega}) = \overline{f(\omega)}$, that follows from $(\overline{\omega})^k = \overline{\omega^k}$.

Of course, right now it's easy to write down the desired Cauchy-Schwartz:

$$S(f) \cdot S(g) = \frac{1}{n^2}\left(\sum_{i=0}^{n-1} |f(\omega^i)|^2\right) \cdot \left(\sum_{i=0}^{n-1} |g(\omega^i)|^2\right) \geq \frac{1}{n^2}\left(\sum_{i=0}^{n-1} |f(\omega^i)| \cdot |g(\omega^i)|\right)^2 =$$

$$= \frac{1}{n^2}\left(\sum_{i=0}^{n-1} |h(\omega^i)|^2\right)^2 = S(h)^2.$$

¶ Second solution (Integrals and continuous Cauchy-Schwartz) This solution can be viewed as the continuous analogue of the first solution. To cancel out unwanted variables, roots of unity filter uses the fact that if $\omega = e^{\frac{2\pi i}{n}}$, then, for $n \nmid k$, $1 + \omega^k + \omega^{2k} + \ldots + \omega^{(n-1)k} = 0$ and is equal to n otherwise; more general version holds and it's going to be the main idea of the solution.

Claim — $\int_{-\pi}^{\pi} e^{in\theta} d\theta = 0$ for a non-zero integer, and $\int_{-\pi}^{\pi} e^{in\theta} d\theta = 2\pi$ for $n = 0$.

Proof. The second part is obvious as we are just integrating a function identical to 1 over an interval of length 2π. The first part follows from $e^{in\theta} = \cos n\theta + i\sin n\theta$, and we integrate the real and imaginary parts separately to get $\int_{-\pi}^{\pi} e^{in\theta} d\theta = \frac{1}{n}\left[\sin(n\theta) - \cos(n\theta)i\right]_{-\pi}^{\pi} = \frac{1}{n}(\sin(n\pi) - \cos(n\pi)i - \sin(-n\pi) + \cos(-n\pi)i) = 0$. □

Now, we can note that $S(p)$ is precisely the x^0 coefficient of a polynomial (allowing negative powers) $f(x) = p(x)p(\frac{1}{x})$. Our goal is still the same, to write $S(p)$ as the sum of some values of p. But in this case we will use a continuous version that also works and is easier to write. The claim allows us to easily separate the free coefficient of any polynomial (even with negative powers):

$$S(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(e^{i\theta})p(e^{-i\theta})d\theta$$

Now we can write down the continious version of Cauchy-Schwartz; it's easy to think about it as of the usual Cauchy-Schwartz but with infinitely many summands.

$$S(f)S(g) = \int_{-\pi}^{\pi} f(e^{i\theta})f(e^{-i\theta})d\theta \int_{-\pi}^{\pi} g(e^{i\theta})g(e^{-i\theta})d\theta \geq$$

$$\geq \left(\int_{-\pi}^{\pi} \sqrt{f(e^{i\theta})f(e^{-i\theta})g(e^{i\theta})g(e^{-i\theta})}d\theta\right)^2 = \left(\int_{-\pi}^{\pi} h(e^{i\theta})h(e^{-i\theta})d\theta\right)^2 = S(h)^2$$

§2.22 IOM 2021/3, proposed by Dušan Djukić

Problem 22 (IOM 2021/3)
Let $a_1, a_2, \dots, a_n$ $(n \geq 2)$ be nonnegative real numbers whose sum is $\frac{n}{2}$. For every $i = 1, \dots, n$ define

$$b_i = a_i + a_i a_{i+1} + a_i a_{i+1} a_{i+2} + \dots + a_i a_{i+1} \dots a_{i+n-2} + 2 a_i a_{i+1} \dots a_{i+n-1}$$

where $a_{j+n} = a_j$ for every j. Prove that $b_i \geq 1$ holds for at least one index i.

¶ First solution (Cyclic summation) We can express b_i using b_{i-1} because we can almost get b_{i-1} from b_i by multiplying by a_{i-1}.

More formally, one can easily check that $b_{i-1} = a_{i-1} + a_{i-1} b_i + a_1 a_2 \dots a_n - 2 a_1 a_2 \dots a_n a_{i-1} = a_{i-1} + a_{i-1} b_i + a_1 a_2 \dots a_n (1 - 2a_{i-1})$, but we know that the sum $1 - 2a_i$ over all i is equal to zero. It suggests to sum over all i with indices taken modulo n.

$$\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + a_{i-1} b_i) = \frac{n}{2} + \sum_{i=1}^{n} a_{i-1} b_i \Longrightarrow \frac{n}{2} = \sum_{i=1}^{n} (1 - a_{i-1}) b_i.$$

Now we suppose that $b_i < 1$ for all i. But we know that $a_i \leq b_i < 1$, so $(1 - a_{i-1}) b_i < 1 - a_{i-1}$, and so, $\frac{n}{2} = \sum_{i=1}^{n} (1 - a_{i-1}) b_i < \sum_{i=1}^{n} 1 - a_{i-1} = n - \frac{n}{2} = \frac{n}{2}$. Contradiction.

¶ Second solution (Gas station problem and generalized induction) The sum involves separated terms a_i, then a_i and a_{i+1}, then a_i,a_{i+1},a_{i+2}, and so on. This suggests us that the value we can bound can be achieved for an index for which all the prefix sums starting from it are bounded below. This will certainly make our job easier. This can be achieved by remembering the gas station problem and deriving a similar result.

Claim — There exists an index i such that

$$a_i + a_{i+1} + \dots + a_{i+j-1} \geq \frac{j}{2}$$

for all $1 \leq j \leq n$, where indices are taken mod n.

Proof. This is equivalent to the Gas Station Problem. For the sake of completeness, put the numbers on the circle and draw an arrow from index i to index $j + 1$ that is the closest to i if going in clockwise direction, and such that

$$a_i + a_{i+1} + \dots + a_j < \frac{j - i + 1}{2}.$$

We can clearly see that the resulting directed graph doesn't have loops and every vertex has an outdegree at least 1 (if it doesn't, then we found the needed index). So there exists a cycle in this graph. If it makes k loops around the circle then if we sum across every edge, then the sum is $\frac{kn}{2}$, but on the other hand it's $< \frac{kn}{2}$, because on every edge the sum of a is smaller than their quantity (how many of them are passed by the arrow) over two. □

Now, we claim that the index we found is the desired. Reindex so that this index is 1. As in the previous solution we can assume that $a_i < 1$. The problem follows from a bit more general claim:

> **Claim —** If $0 \le a_i \le 1$ for all i, $a_1 + \ldots + a_j \ge \frac{j}{2}$ for $1 \le j \le n-1$, and $a_1 + \ldots + a_n \ge \frac{n}{2} + \alpha$ with $\frac{1}{2} \ge \alpha \ge 0$. Then $a_1 + a_1a_2 + a_1a_2a_3 + \ldots + a_1a_2\ldots a_{n-1} + (2-2\alpha)a_1a_2\ldots a_n \ge 1$.

Proof. One can clearly that our problem follows from $\alpha = 0$ in the above. The proof goes by induction on n. For $n = 1$, $(2-2\alpha)a_1 \ge 2(1-\alpha)(\frac{1}{2}+\alpha) = 1 + \alpha(1-2\alpha) \ge 1$. For $n > 1$, $a_1 + \ldots + a_{n-1} \ge \frac{n}{2} + \alpha - a_n \ge \frac{n}{2} + \alpha a_n - a_n = \frac{n}{2} - (1-\alpha)a_n = \frac{n-1}{2} + \frac{1}{2} - (1-\alpha)a_n$. Obviously $\frac{1}{2} - (1-\alpha)a_n \le \frac{1}{2}$. If it's < 0 then $1 + 2(1-\alpha)a_n > 1 + 1$, and the problem would follow from $a_1 + a_1a_2 + a_1a_2a_3 + \ldots + a_1a_2\ldots a_{n-2} + 2a_1a_2\ldots a_{n-1} \ge 1$ (This is less than the sum for n variables), which is true by the induction hypothesis. Now $0 \le \frac{1}{2} - (1-\alpha)a_n \le \frac{1}{2}$, so we can use the induction hypothesis:

$$a_1 + a_1a_2 + a_1a_2a_3 + \ldots + a_1a_2\ldots a_{n-2} + (2 - 2(\frac{1}{2} - (1-\alpha)a_n))a_1a_2\ldots a_{n-1} \ge 1.$$

If we open the brackets, it's equivalent to the inductive step. □

The claim was motivated by the fact that the conditions are pleasant for induction, because the condition on sums doesn't change if we remove one varaible. But if we are trying the trivial induction, we will se that problems occur in the case $a_n < \frac{1}{2}$, then we try to write $a_n = \frac{1}{2} - \alpha$, and this leads to the more general claim.

> **Remark.** Other classical gas station problem example is the ISL 2021 C5. The idea itself is very useful for ISL 2017 N3.

§2.23 USAMO 2000/6

Problem 23 (USAMO 2000/6)
Let $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be nonnegative real numbers. Prove that

$$\sum_{1\le i,j\le n} \min(a_ia_j, b_ib_j) \le \sum_{1\le i,j\le n} \min(a_ib_j, a_jb_i).$$

¶ Solution (Integrals and an unexpected identity) For a person that hasn't seen problems of this kind [1] before it might sound very surprising, but the inequalities that involve minimum values can usually be solved using integrals. This is because $\min(r_i, r_j) = \int_0^\infty f_i(x)f_j(x)dx$, where

$$f_i(x) = \begin{cases} 1 & \text{if } x \le r_i \\ 0 & \text{otherwise.} \end{cases}$$

So we can write down a general inequality:

Claim — Let $r_1,\ldots,r_n$ be nonnegative reals. Let $x_1,\ldots, x_n$ be real numbers.

$$\sum_{1\le i,j\le n} \min(r_i, r_j)x_ix_j \ge 0.$$

Proof. If we remember what we have written before, then we have

$$\sum_{1\le i,j\le n} \min(r_i, r_j)x_ix_j = \sum_{1\le i,j\le n} x_ix_j \int_0^\infty f_i(x)f_j(x)dx = \int_0^\infty \sum_{1\le i,j\le n} x_if_i(x)x_jf_j(x)dx =$$

$$= \int_0^\infty \Big(\sum_{1\le i\le n} x_if_i(x)\Big)^2 \ge 0$$

□

Now we want to choose such real numbers $r_1, \ldots, r_n > 0$ and $x_1, \ldots, x_n$ such that $\min(a_ib_j, a_jb_i) - \min(a_ia_j, b_ib_j) = \min(r_i, r_j)x_ix_j$, which is the most unmotivated part of the problem. It takes a lot of faith into the approach to complete it.

The only way I see how to exlplain it is that one of the $\min(a_ib_j, a_jb_i)$ or $\min(a_ia_j, b_ib_j)$ is just $\min(a_i, b_i)\min(a_j, b_j)$. And it's reasonable to say that we want to choose $x_i = \min(a_i, b_i)$ and $x_j = \min(a_j, b_j)$. Then, we see that we either have 1-(something of a and b) or (something of a and b)-1, so we also need to choose the right signs. The choice of $\operatorname{sgn}(a_i - b_i)$ seems to work well. Set $x_i = \operatorname{sgn}(a_i - b_i)\min(a_i, b_i)$. We have that, if $a_i \ge b_i$ and $a_j \ge b_j$, $\min(a_ib_j, a_jb_i) - \min(a_ia_j, b_ib_j) = x_ix_j((\text{smth of a and b}) - 1)$, when both signs are reversed it's still true.

If one of the signs is reversed, then the thing in the brackets reverses the order, but, since we have a negative sign from the sign function, it still stays the same. Of course, if $a_i = b_i$ or $a_j = b_j$ then both sides are simply zero. This suggests to choose r_i as $\frac{\text{smth of } a_i \text{ and } b_i}{\text{smth of } a_i \text{ and } b_i} - 1$. By

[1] See, for example, Polish MO 1999/5.

considering some cases, we might come to a conclusion $r_i = \frac{\max(a_i,b_i)}{\min(a_i,b_i)} - 1$ that works, because we will definitely need to have some inequality of the form $a_ib_j \ ? \ a_jb_i$ or $a_ib_i \ ? \ a_jb_j$ to decide which by which number to multiply the $\min(a_i, b_i)\min(a_j, b_j)$. These inequalities are easily rewritten as $\frac{a_i}{b_i} \ ? \ \frac{a_j}{b_j}$ or $\frac{b_i}{a_i} \ ? \ \frac{a_j}{b_j}$.

We are left to prove that

$$\min(a_ib_j, a_jb_i) - \min(a_ia_j, b_ib_j) =$$

$$= \operatorname{sgn}(a_i - b_i)\min(a_i, b_i)\operatorname{sgn}(a_j - b_j)\min(a_j, b_j)\min\left(\left(\frac{\max(a_i, b_i)}{\min(a_i, b_i)} - 1\right), \left(\frac{\max(a_j, b_j)}{\min(a_j, b_j)} - 1\right)\right).$$

First, if we swap a_i and b_i or a_j and b_j then both sides just negate, so we can assume $a_i \geq b_i$ and $a_j \geq b_j$, then both sides become more definite:

$$\min(a_ib_j, a_jb_i) - b_ib_j = b_ib_j\min\left(\left(\frac{a_i}{b_i} - 1\right), \left(\frac{a_j}{b_j} - 1\right)\right).$$

This is easy to check by considering two cases. Hence, we are done by the inequality we proved before.

§2.24 RMM 2024/6, proposed by Navid Safaei

Problem 24 (RMM 2024/6)

A polynomial P with integer coefficients is square-free if it is not expressible in the form $P = Q^2R$, where Q and R are polynomials with integer coefficients and Q is not constant. For a positive integer n, let P_n be the set of polynomials of the form

$$1 + a_1x + a_2x^2 + \cdots + a_nx^n$$

with $a_1, a_2, \ldots, a_n \in \{0, 1\}$. Prove that there exists an integer N such that for all integers $n \geq N$, more than 99% of the polynomials in P_n are square-free.

¶ Solution (Reduction mod 2 and Hamming distance argument) We first start with reducing mod 2. It can be assumed that the leading coefficient of Q is 1, because it divides the leading coefficient of P, which is also going to be 1. Now, working in $\mathbb{F}_2$, $P = Q^2R$. $\deg R \leq \deg P - 2\deg Q$, so there are at most $2^{n-2\deg Q}$ (1 less in the power because the free coefficient is already set to 1) choices for the polynomial R, and $2^{\deg Q}$ choices for Q (same reason for 1 less in the power), they all correspond to possible polynomials P because after the reduction it stays the same. If the degree of Q is d then we have $\leq 2^{n-d}$ choices of P for that degree, from here it's not hard to understand that if we start summing starting from some big degree of Q, the proportion to the total of 2^n is going to be small. Summing over all degrees, in total we get

$$\sum_{N>d>20} 2^{n-d} < 2^{n-20}.$$

And the proportion to the total is $< \frac{1}{1000}$. Now we only have to take care of the smaller degree polynomials.

A useful claim, that already appeared in ISL 2005 A2, is that for every complex root of $a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + x^n$ with $a_1, a_2, \ldots, a_n \in \{0, 1\}$ and $n \geq 0$, we actually have $|r| < 2$. If we suppose that $a_0 + a_1r + \ldots + r^n = 0$ with $|r| \geq 2$, then $|-a_0| + |-a_1r^1| + \ldots + |-a_{n-1}r^{n-1}| = |a_0 + a_1r + \ldots + r^n| + |-a_0| + |-a_1r^1| + \ldots + |-a_{n-1}r^{n-1}| \geq |a_0 + a_1r + \ldots + a_{n-1}r^{n-1} + r^n - a_0 - \ldots - a_{n-1}r^{n-1}| = |r^n| = |r|^n \geq 2^n$, by triangle inequality. But $|-a_0| + |-a_1r^1| + \ldots + |-a_{n-1}r^{n-1}| \leq |a_0| + |a_1r^1| + \ldots + |a_{n-1}r^{n-1}| = |a_0| + 2|a_1| + \ldots + 2^{n-1}|a_{n-1}| \leq 1 + 2 + \ldots + 2^{n-1} < 2^n$.

Now, since P is of the above form (suppose that it's monic with degree not necessarily N), we can bound each of its roots. Hence, we can bound the roots of Q^2s that divide it. Further, by *Vieta's relations* and triangle inequality we can bound the coefficients of Q. The degree of Q is bounded too. Since the coefficients of Q are integers, a finite number of them can divide P. So we have a finite amount of roots that divide some of these Q.

Now, we can predict that, for big enough n, a fixed complex number cannot be a root of $1 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + x^n$ with all $a_i \in \{0, 1\}$ too often, otherwise it would spawn too many equations with it which can be manipulated to reach the contradiction. This prediction is true and can be formulated as the following claim:

Claim — Fix a choice of Q and consider all polynomials $P = 1 + a_1x + a_2x^2 + \cdots + a_nx^n$ with $a_1, a_2, \ldots, a_n \in \{0, 1\}$. The number N of them that satisfy $Q^2 \mid P$ is $\leq \frac{2^n}{n+1}$.

Proof. For the proof, we use a *Hamming distance* argument. Consider a choice of $(a_1, \ldots, a_n)$ that makes a polynomial P. This choice is a binary sequence. Flipping exactly one of the a, we will obtain n more choices (the ones for which the Hamming distance is 1). The claim is that none of them are divisible by Q. That is because r is the root of Q, suppose that it divides some P' obtained by flipping. Then r is a root of $P - P'$, but the latter is of the form $\pm x^a$ so the only way is $r = 0$, but then the x^0 coefficient of P is zero. Now suppose we have done the above operation for every binary sequence that makes a polynomial divisible by Q^2. And suppose that some two polynomials written coincide. The Hamming distance of polynomials they were obtained from is $\leq 1 + 1$. Call the polynomials P_1 and P_2. P_1 - P_2 has a double root r (because divisible by Q^2). Note that this difference is of the form $\pm x^a \pm x^b$ with $a > b$ or $\pm x^a$. We have dealt with the latter previously, now onto the former. r is a root of $x^b(\pm x^{a-b} \pm 1)$, and since it's non-zero, it has to be a root of $\pm x^{a-b} \pm 1$, moreover it has to be a double root. This means that r is a root of $(\pm x^{a-b} \pm 1)' = (a - b)x^{a-b-1}$, this cannot be zero for a non-zero r. This proves that all the polynomials that we got by the above flipping are actually different, and there's a total of $\leq 2^n$ of them. On the other hand there are $N(n + 1)$ of them, because each choice corresponds to n more. So $N \leq \frac{2^n}{n+1}$, from which we get the bound. $\square$

Thus, we can conclude. Consider all non-square free polynomials for a big enough n and choose a Q for each of them. The proportion of polynomials which correspond to Q with degree > 20 is $< \frac{1}{1000}$. And the proportion of polynomials that correspond to Q with degree ≤ 20 is $< 2^{-n}C\frac{2^n}{n+1} = \frac{C}{n+1}$, where C is the number of possible polynomials Q (we have proved that it's finite above). This quantity can be made arbitrarily small. Summing it up with $< \frac{1}{1000}$ won't make it greater than $\frac{1}{100}$ either. Thus, we are done.

§2.25 2020 Cyberspace Mathematical Competition/8, proposed by Alex Zhai

Problem 25

Let $a_1, a_2, \dots$ be an infinite sequence of positive real numbers such that for each positive integer n we have

$$\frac{a_1 + a_2 + \dots + a_n}{n} \geq \sqrt{\frac{a_1^2 + a_2^2 + \dots + a_{n+1}^2}{n+1}}.$$

Prove that the sequence $a_1, a_2, \dots$ is constant.

¶ Solution (Stronger QM-AM) Suppose that A_n and Q_n is the arithmetic and quadratic means of the first n a_i. We have that $Q_i \geq A_i \geq Q_{i+1}$ by the problem condition and $QM - AM$. Intuition tells us that if $|a_m - a_{m+1}| = d \neq 0$ for some m, then we will be able to establish a stronger bound on $Q_i - A_i$ in terms of d for $i \geq m+1$, because the equality case is when they are all equal. It's true and follows from one possible proof of the QM-AM inequality where we express the difference of squares of QM and AM as the sum of squares of all possible differences between two numbers. For $n \geq m+1$.

$$Q_n^2 - A_n^2 = \frac{1}{n^2}\left(n\sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i^2 - 2\sum_{1\leq i<j\leq n} a_i a_j\right) = \frac{1}{n^2}\left((n-1)\sum_{i=1}^{n} a_i^2 - 2\sum_{1\leq i<j\leq n} a_i a_j\right) =$$

$$= \frac{1}{n^2}\left(\sum_{1\leq i<j\leq n}(a_i - a_j)^2\right) \geq \frac{1}{n^2}\left((a_m - a_{m+1})^2 + \sum_{1\leq i<m, m+1<i\leq n}\left((a_i - a_m)^2 + (a_{m+1} - a_i)^2\right)\right) \geq$$

$$\geq \frac{1}{n^2}\left(d^2 + (n-2)\frac{d^2}{2}\right) = \frac{d^2}{2n}\text{, where we have used } 2(a^2 + b^2) \geq (a+b)^2.$$

Which is enough to imply that $Q_i^2 - Q_{i+1}^2 = Q_i^2 - A_i^2 + A_i^2 - Q_{i+1}^2 \geq Q_i^2 - A_i^2 \geq \frac{d^2}{2i}$ for $i \geq m+1$. So, $Q_m^2 \geq Q_{m+1}^2 + \frac{d^2}{2(m+1)} \geq Q_{m+2}^2 + \frac{d^2}{2(m+1)} + \frac{d^2}{2(m+2)} \geq \dots \geq Q_{m+k}^2 + \frac{d^2}{2(m+1)} + \frac{d^2}{2(m+2)} + \dots + \frac{d^2}{2(m+k)}$. The LHS is constant; the RHS diverges because Harmonic series diverges. Hence, we are done.

Remark. A problem similar in spirit is CAPS 2023/2.

Part II. Combinatorics

§3 Problems

Problem 1 (SRMC 2022/4). In a language, an alphabet with 25 letters is used; words are exactly all sequences of (not necessarily different) letters of length 17. Two ends of a paper strip are glued so that the strip forms a ring; the strip bears a sequence of 5^{18} letters. Say that a word is singular if one can cut a piece bearing exactly that word from the strip, but one cannot cut out two such non-overlapping pieces. It is known that one can cut out 5^{16} non-overlapping pieces each containing the same word. Determine the largest possible number of singular words.

Hints: 266 351 36 71 260 362 206 886

Problem 2 (USAMO 2022/1). Let a and b be positive integers. The cells of an $(a+b+1) \times (a+b+1)$ grid are colored amber and bronze such that there are at least $a^2 + ab - b$ amber cells and at least $b^2 + ab - a$ bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a + b$ chosen cells lie in the same row or column.

Hints: 615 923 741 509 489

Hints: 399 788 707 73 1008 2 840

Problem 3 (ISL 2014 C1). Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n + 1$ smaller rectangles.

Hints: 37 167 480 27

Hints: 88 624 738

Problem 4 (2022 Canada summer camp Difficult Problems handout/1). A $2^n \times n$ matrix of 1s and -1s is such that its 2^n rows are pairwise distinct. An arbitrary subset of the entries of the matrix are changed to 0. Prove that there is a nonempty subset of the rows of the altered matrix that sum to the zero vector.

Hints: 571 679 94 246

Problem 5 (IZHO 2021/3). Let $n \geq 2$ be an integer. Elwyn is given an $n \times n$ table filled with real numbers (each cell of the table contains exactly one number). We define a rook set as a set of n cells of the table situated in n distinct rows as well as in n distinct columns. Assume that, for every rook set, the sum of n numbers in the cells forming the set is nonnegative.

By a move, Elwyn chooses a row, a column, and a real number a, and then he adds a to each number in the chosen row, and subtracts a from each number in the chosen column (thus, the number at the intersection of the chosen row and column does not change). Prove that Elwyn can perform a sequence of moves so that all numbers in the table become nonnegative.

Hints: 837 569 338 68 48 484 969 296 354

Problem 6 (ISL 2015 C6). Let S be a nonempty set of positive integers. We say that a positive integer n is clean if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

Hints: 107 497 116 10 760 60 191 952 142

Hints: 657 172 7 323 717 575 858 389 629 984 742

Problem 7 (Balkan MO 2016/4). The plane is divided into squares by two sets of parallel lines, forming an infinite grid. Each unit square is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

Hints: 711 622 231 877 183 849

Problem 8 (ISL 2019 C6). Let $n > 1$ be an integer. Suppose we are given $2n$ points in the plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \ldots, A_{2n}$ in some order. We then consider the following angles: $\angle A_1A_2A_3, \angle A_2A_3A_4, \ldots, \angle A_{2n-2}A_{2n-1}A_{2n}$ $\angle A_{2n-1}A_{2n}A_1$ $\angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

Hints: 885 229 51 673 245 425 892

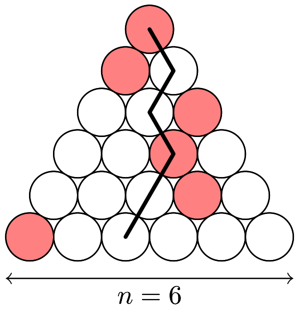
Problem 9 (RMM 2020/3). Let $n \geq 3$ be an integer. In a country there are n airports and n airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and m distinct airports $c_1, \ldots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: c_1 and c_2; c_2 and c_3; $\ldots$; c_{m-1} and c_m; c_m and c_1.

Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.

Hints: 780 330 485 294 584 767

Hints: 525 762 1016 996 465 496 431 848 993

Problem 10 (IMO 2023/5). Let n be a positive integer. A Japanese triangle consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1$, $2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Hints: 998 269 896 434 554 369

Hints: 349 934 835 666 1005 151

Problem 11 (All-Russian MO 2021 grade 11/8). Each of the 100 girls has 100 balls; there is a total of 10000 balls in 100 colors. 100 balls of each colour. In one move, two girls can exchange a ball (the first gives the second one of her balls, and vice versa). Prove that it's possible to execute such a sequence of exchanges such that each girl would have 100 balls of each colour in the end and each ball was exchanged at most once.

Hints: 616 502 999 774 976

Problem 12 (USAMO 2018/6). Let a_n be the number of permutations $(x_1, x_2, \ldots, x_n)$ of the numbers $(1, 2, \ldots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \le k \le n$ are all distinct. Prove that a_n is odd for all $n \ge 1$.

Hints: 494 140 254 700 463 297 982 568

Problem 13 (RMM 2023/6). et r, g, b be non negative integers and Γ be a connected graph with $r + g + b + 1$ vertices. Its edges are colored in red green and blue. It turned out that Γ contains

- A spanning tree with exactly r red edges.
- A spanning tree with exactly g green edges.
- A spanning tree with exactly b blue edges.

Prove that Γ contains a spanning tree with exactly r red edges, g green edges and b blue edges.

Hints: 795 782 61 945 422 639

Hints: 133 581 153 208 187 314 957

Problem 14 (All-Russian MO 2023 grade 10/4). There is a queue of n girls on one side of a tennis table, and a queue of n boys on the other side. Both the girls and the boys are numbered from 1 to n in the order they stand. The first game is played by the girl and the boy with the number 1 and then, after each game, the loser goes to the end of their queue, and the winner remains at the table. After a while, it turned out that each girl played exactly one game with each boy. Prove that if n is odd, then a girl and a boy with odd numbers played in the last game.

Hints: 220 316 678 451 940 38 703 182 526 566 194 826 630 714 671

Problem 15 (RMM 2021/3). A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

Hints: 31 623 357 387 960 490 882 551 421

Problem 16 (USAJMO 2018/6). Karl starts with n cards labeled $1, 2, 3, \ldots, n$ lined up in a random order on his desk. He calls a pair (a, b) of these cards swapped if $a > b$ and the card labeled a is to the left of the card labeled b. For instance, in the sequence of cards $3, 1, 4, 2$, there are three swapped pairs of cards, $(3, 1)$, $(3, 2)$, and $(4, 2)$.

He picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i card to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards $1, 2, \ldots, n$ exactly once in that order. (For example, the process starting at $3, 1, 4, 2$ would be $3, 1, 4, 2 \to 3, 4, 1, 2 \to 2, 3, 4, 1 \to 2, 4, 3, 1 \to 2, 3, 4, 1$.)

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

Hints: 55 157 346 492

Problem 17 (SRMC 2009/3). A tourist is going to visit the Complant. He has found that:

- Complant has 1024 cities, numbered by integers from 0 to 1023.
- Two cities with numbers m and n are connected if and only if the binary entries of numbers m and n they differ exactly in one digit.
- During the stay of a tourist in that country 8 roads will be closed for scheduled repairs.

Prove that the tourist can make a closed route along the existing roads of Complant, passing through each of its cities exactly once.

Hints: 651 567 869 696 355 653 861

Problem 18 (ISL 2023 C7). The Imomi archipelago consists of $n \geq 2$ islands. Between each pair of distinct islands is a unique ferry line that runs in both directions, and each ferry line is operated by one of k companies. It is known that if any one of the k companies closes all its ferry lines, then it becomes impossible for a traveller, no matter where the traveller starts at, to visit all the islands exactly once (in particular, not returning to the island the traveller started at). Determine the maximal possible value of k in terms of n.

Hints: 315 442 793 515 541 812 159 334 385 865

Problem 19 (ISL 2018 C7). Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice- once for each of the two circle that cross at that point. If the two colours agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

Hints: 646 893 81 501 1011 78 1001

Problem 20 (USEMO 2019/5). Let $\mathcal{P}$ be a regular polygon, and let $\mathcal{V}$ be its set of vertices. Each point in $\mathcal{V}$ is colored red, white, or blue. A subset of $\mathcal{V}$ is patriotic if it contains an equal number of points of each color, and a side of $\mathcal{P}$ is dazzling if its endpoints are of different colors.

Suppose that $\mathcal{V}$ is patriotic and the number of dazzling edges of $\mathcal{P}$ is even. Prove that there exists a line, not passing through any point in $\mathcal{V}$, dividing $\mathcal{V}$ into two nonempty patriotic subsets.

Hints: 1023 228 895 154 937 586

Problem 21 (ISL 2019 C7). There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game. In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps: (a) Bob chooses an integer k with $1 \leq k \leq 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$. (b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group. Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

Hints: 430 302 62 322 457 796 98 930

Problem 22 (ELMO 2017/3). Nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

Hints: 305 300 753 731 22 92

Problem 23 (ISL 1999 C6). Suppose that every integer has been given one of the colours red, blue, green or yellow. Let x and y be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x + y$ or $x - y$.

Hints: 856 564 176 401 223 52 941 684

Hints: 897 724 599 105 119 138

Problem 24 (ISL 2018 C6). Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

- If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.
- If no such pair exists, we write two times the number 0.

Prove that, no matter how we make the choices in the first type of operation, there will be only finitely many times when we perform the second type operation.

Hints: 935 35 814 298 378 503 423

Problem 25 (Kazakhstan MO 2023 grade 11/4). Let G be a graph whose vertices are 2000 points in the plane, no three of which are collinear, that are coloured in red and blue. Given that there exist 100 red points that form a convex polygon with every other point of G lying inside of it. Prove that one can connect some points of the same colour such that segments connecting vertices of different colours don't intersect, and one can move from a vertex to any vertex of the same colour using these segments.

Hints: 751 928 577 404 955

§4 Solutions

§4.1 SRMC 2022/4, proposed by Ilya Bogdanov

Problem 1 (SRMC 2022/4)
In a language, an alphabet with 25 letters is used; words are exactly all sequences of (not necessarily different) letters of length 17. Two ends of a paper strip are glued so that the strip forms a ring; the strip bears a sequence of 5^{18} letters. Say that a word is singular if one can cut a piece bearing exactly that word from the strip, but one cannot cut out two such non-overlapping pieces. It is known that one can cut out 5^{16} non-overlapping pieces each containing the same word. Determine the largest possible number of singular words.

¶ **Solution (Finding non-singular words and creating more unique ones)** **Example:** We will first provide an example that, in my opinion, is quite intuitive. Call a word that repeats 5^{16} times a *copypaste*. We will space copypastes evenly, i.e. there's a space of $\frac{5^{18}}{5^{16}} - 17 = 8$ letters between every two consecutive copypastes. In order to have more unique words, we will make all of these 8 letter words different, because if some of them are the same then we will have that none of the words that contain at least a part of it are singular. Here, we also get lucky with numbers because 25^8, i.e. the total number of 8 letter words, equals to 5^{16}, the total number of spots. This should reassure us that it's the right path. Now, we know for a fact that if a word contains only a part of the 8 letter portion, it's not singular (because there's another place in the sequence that contains the same part of the 8 letter portion, and we can add the remaining part of the copypaste, which is always the same). So, we want to make the words that contain the 8 letter portion entirely to be singular, there's a $10 \cdot 5^{16} = 2 \cdot 5^{17}$ of them, and we will be done.

This can be done by making the copypaste contain all different letters. Suppose that in this case two words that contain an entire 8 letter part are the same. They cannot have a different starting letter in the copypaste (it would then contain two coinciding letters), and cannot have a different ending letter in the copypaste. This either means they both start or end on the same letter in copypaste, and then it would mean that the 8 letter portions are the same. Or this means that the first word has only a starting point in the copypaste and the second has only a finish. It can happen only if the first word starts with 9 letters of the copypaste and the second word starts with an 8 letter portion and finishes with a copypaste. In this case, we have that the 17th letter of the copypaste, which is the 9th letter of the word if we consider the first word, coincides with the 1st letter of the copypaste if we consider the second word. This is a contradiction, so we are done.

Bound: To prove the bound, we will keep in mind the equality case we have provided earlier. Consider sequences $T_1, T_2, \ldots, T_{5^{16}}$ that consist of the first 8 letters before (anticlockwise) each copypaste. Equality case suggests us to consider their longest tail that appears in some other T, because the equality case is when all T_i are differnt. What I mean is that we take the length a_i of the first suffix of T_i that doesn't appear as a suffix of any other T_i. We can choose any $1 \le l \le a_i - 1$ and consider a suffix of length l, then add remaining letters from the copypaste and, by the minimality of a_i, there exist some other T for which we can do the same and obtain the same word. This proves that for each T, there are at least $a_i - 1$

non-singular words, which are different for different i (different in terms of the position in our circular strip). If this suffix doesn't exist, then we have two instances of the same 8 letter portions T_i and T_j. We can generate non-singular words from it by considering any suffix and adding remaining letters from the corresponding copypaste. For each T without an a there are 8 non-singular words, and, of course, they are different for different T (in terms of the position in the circular strip).

Now, we can count the number of non-singular words. Suppose that we chose a_{i_1}, a_{i_2}, ..., a_{i_t}. The number of non-singular words that contain at least one letter of some T is at least

$$\sum_{j=1}^{t}(a_{i_j} - 1) + 8(5^{16} - t).$$

Now, we will obtain another bound. For each a_i, write all the possible 25^{8-a_i} sequences after the chosen suffix. All the resulting 8 letter sequences are different because they have different suffixes between different i, and of course, they are different for the same i. So, we can bound

$$25^8 \geq \sum_{i=j}^{t} 25^{8-a_{i_j}} \geq \sum_{j=1}^{t}(9 - a_{i_j}) = 9t - \sum_{j=1}^{t} a_{i_j} = 8 \cdot 5^{16} - (\sum_{j=1}^{t}(a_{i_j} - 1) + 8(5^{16} - t)),$$

where we have used the trivial $25^n \geq n + 1$. Thus, the number of non-singular words that contain at least one letter of some T is at least $8 \cdot 5^{16} - 25^8 = 7 \cdot 5^{16}$. We can do the same thing for 8 letter sequences T' that are appearing after (anticlockwise) every copypaste and obtain $7 \cdot 5^{16}$ non-singular words that contain at least one letter in T's, we know that they don't coincide with the ones that contain at least one letter in Ts. We can also add 5^{16} copypastes, and we will only have at most $5^{18} - 5^{16} - 2 \cdot 7 \cdot 5^{16} = 10 \cdot 5^{16} = 2 \cdot 5^{17}$ possible singular words.

> **Remark.** Problem with similar idea of the bound: All-Russian 2021 grade 11/3.

§4.2 USAMO 2022/1, proposed by Ankan Bhattacharya

Problem 2 (USAMO 2022/1)
Let a and b be positive integers. The cells of an $(a+b+1)\times(a+b+1)$ grid are colored amber and bronze such that there are at least a^2+ab-b amber cells and at least b^2+ab-a bronze cells. Prove that it is possible to choose a amber cells and b bronze cells such that no two of the $a+b$ chosen cells lie in the same row or column.

¶ First solution (Discrete continuity and expected value) Call a set in which no two cells lie in the same row or column a *rook* set. The question asks us for a rook set with either a or $a+1$ amber cells. We start off by proving that there is a rook set that contains at least a amber cells and the one that contains at least b bronze cells, because the problem explicitly asks to find it. We know that the expected value of one cell is $\frac{a^2+ab-b}{(a+b+1)^2}$ if it's 1 for an amber cell and 0 otherwise. Thus, the expected value of the sum of the rook set is $(a+b+1)\cdot\frac{a^2+ab-b}{(a+b+1)^2}$, because all the cells are equally distributed and there are $a+b+1$ of them in the rook set. Hence, the expected value of the sum of the rook set is $a-1+\frac{1}{a+b+1}$. This proves that there exists a rook set S_a that has at least a amber cells. In the same way we can prove that there's a rook set S_b that has at least b bronze cells.

Now let's assume that there doesn't exist a set with a or $a+1$ amber cells. Then S_a contains $\geq a+2$ amber cells and S_b contains $\leq a-1$ amber cells. Represent a rook set as a permutation p of $\{1,2,\dots,a+b+1\}$ - $\{p_1,p_2,\dots,p_{a+b+1}\}$, where p_i is the number of the column of the cell in row i. We know that two permutations can be achieved from each other by performing a series of transpositions. Every transposition changes only two cells, so the number of as changes by at most two with each transposition. This means that if we start at S_a with $\geq a+2$ and go to S_b with $\leq a-1$, we will definitely hit one of a, $a+1$, otherwise we would change the value by at least three at some step.

¶ Second solution (Kőnig's theorem and smoothing process) First of all, we implement a basic idea in grid problems. Represented a grid as the bipartite graph where the first part is the rows, the second part is the columns, and two vertices are connected by an edge if the cell corresponding to the intersection of this row and this column is amber. We see that a rook set is essentially a matching. We seek to find a maximal matching in this graph. By *Kőnig's theorem*, the size of it is equal to the size of the minimal vertex cover. Essentially, it's the total amount of rows and columns we need to take to cover the whole set. We have $(a+b+1)(a-1)+1$ amber cells and each chosen row covers $a+b+1$ cells, so there has to be at least $\lceil\frac{(a+b+1)(a-1)+1}{a+b+1}\rceil = a$ rows chosen. Thus, we have at least a vertices in a minimal vertex cover, therefore we have at least a edges in a maximal matching. In the same way we can prove that there is a matching with b bronze edges.

Now we will present an alternative proof of how to move from these two sets to a set that is asked in the problem. Put these two sets on the board, we will just decrease the number of conflicting pairs, i.e. the pairs of different colors that are either in the same row or column. Suppose that there are two that, WLOG, are in the same column, then we know that there's a total of $a+b$ cells on the board, so at least one column and at least one row are free. Consider the cell that's in their intersection. Take one cell from the conflicting pair that has the same color as this cell and just replace it by this cell in the intersection. We will resolve one conflict and won't create another. By repeatedly applying this operation, we will reach the state in which no cells conflict and there are a amber cells and b bronze cells.

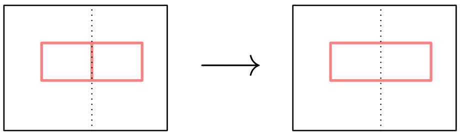
> **Remark 1.** Of course, the two solutions essentially present two ideas for each of the two steps of the problem; we can combine them in any order.

> **Remark 2.** Another problem that asked a similar bound on rook sets is 2021 China TST 1/2

§4.3 ISL 2014 C1, proposed by Serbia

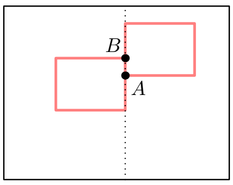
Problem 3 (ISL 2014 C1)
Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n + 1$ smaller rectangles.

¶ First solution (Induction and counting sides) It's easier to think about the problem if we reformulate the statement a bit: We essentially have n different horizontal or vertical lines (each corresponds to a side of the smaller rectangle that contains this point, and if it's the corner then we just take any of the two sides) inside the rectangle (not on the sides), and a partition of the rectangle into smaller rectangles for which each of the aforementioned lines contains at least one side of them. In this formulation, we want to prove that there are at least $n + 1$ smaller rectangles in the partition. First, suppose that some line contains only two sides (it must contain at least two if it contains one, because there's one from one side of it and one from the other). Easy to see that in this case, there will be two rectangles that intersect exactly on this line.



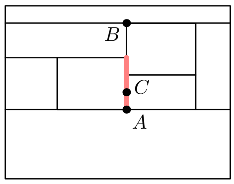
In this case we can join them to form one rectangle that will be a part of the partition, and we will have at least $n - 1$ lines, each of which contains a side of the rectangle. So we will need at least n, but when we disjoint the rectangle, we will have $n + 1$.

Now, assuming that we don't have a situation illustrated above, we obtain that the side of the second rectangle is misaligned with that of the first. In this case, at least 2 more sides of the rectangles belong to this line, one in each of the corners A and B.



Now, it's easy to finish, because we have found $4n$ different sides of the rectangles. Formally, suppose v and h is the number of vertical and horizontal lines, respectively. Then, we have $\geq 4v + 4h + 4 = 4n + 4$ sides of the small rectangles, where the last 4 comes from 4 sides of the big rectangle (we didn't include them in any lines inside of it). On the other hand, the number of sides is at most $4N$, where N is the number of rectangles of the partition. Thus, we get $4N \geq 4n + 4 \Longrightarrow N \geq n + 1$.

¶ Second solution (Counting corners) For each side of the rectangle of the partition, we will consider a maximal contiguous segment that contains it and call it a *stripe*. We know that for every point inside the rectangle there is a side of the smaller rectangle that contains it. Consider a stripe of this point (AB is the stripe of point C in the picture). We know that each of the endpoints of it is a corner of at least two small rectangles (one for each side). There will be a total of $2 \cdot 2 \cdot n$ corners inside the rectangle because none of them repeat (if it's an endpoint of the stripe, then, in perpendicular direction, this won't be the endpoint). We have $4 \cdot N - 4$ corners in total (subtracting 4 because these are the corners of the big rectangle, which cannot be the endpoints of stripes). So, we get $4N - 4 \geq 4n$, from which $N \geq n + 1$.



§4.4 Canada 2022 summer camp Difficult Problems handout/1

Problem 4 (Canada 2022 summer camp Difficult Problems handout/1)
A $2^n \times n$ matrix of 1s and -1s is such that its 2^n rows are pairwise distinct. An arbitrary subset of the entries of the matrix are changed to 0. Prove that there is a nonempty subset of the rows of the altered matrix that sum to the zero vector.

¶ Solution (Finding a cycle!) Treat the rows as 2^n vectors with entries in $\{1, 0, -1\}$. We first make the objects more uniform. Note that any vector v with entries in $\{1, 0, -1\}$ can be written as an arithmetic mean of two vectors u and w with entries in $\{1, -1\}$, where u is the original vector of 1s and -1s. Associate each of the 2^n vectors v_i with an ordered pair (u_i, w_i) mentioned above. We will construct a directed graph with pairs of vectors (u, w) as vertices, in which we will draw an arrow from (a, b) to (c, d) if $b + c$ is the zero vector. Consider an arbitrary pair (u_i, w_i) and consider $-w_i$. Note that this vector existed in the original matrix (before changing some entries to 0) as some v_j, and so that will be $u_j = -w_j$, so there's an arrow from (u_i, w_i) to (u_j, w_j). Thus, in this directed graph, outdegree of every vertex is at least one. It means that there must be a cycle (just start from arbitrary vertex and move by edges, and we will eventually come back to a vertex we have already visited). Suppose that the cycle is $i_1 \to i_2 \to \ldots \to i_k \to i_1$. Then $v_{i_1} + \ldots + v_{i_k} = (w_{i_1} + u_{i_2}) + (w_{i_2} + u_{i_3}) + \ldots + (w_{i_k} + u_{i_1}) = 0$.

Remark. The idea of finding a sum in which all terms cancel out by considering a directed cycle is not new. See Codeforces 1270G.

§4.5 IZHO 2021/3, proposed by Ilya Bogdanov

Problem 5 (IZHO 2021/3)
Let $n \geq 2$ be an integer. Elwyn is given an $n \times n$ table filled with real numbers (each cell of the table contains exactly one number). We define a rook set as a set of n cells of the table situated in n distinct rows as well as in n distinct columns. Assume that, for every rook set, the sum of n numbers in the cells forming the set is nonnegative.

By a move, Elwyn chooses a row, a column, and a real number a, and then he adds a to each number in the chosen row, and subtracts a from each number in the chosen column (thus, the number at the intersection of the chosen row and column does not change). Prove that Elwyn can perform a sequence of moves so that all numbers in the table become nonnegative.

¶ Solution (Decreasing numbers by Hall's lemma and making them all zeros) Let's start by proving that it is possible to decrease some numbers in the table so that in every rook set the sum equals zero. We start a process in which we identify a number in the table for which all the rook sets containing it have positive sums. We then reduce this number so that at least one rook set has a sum of zero, while the sums of the other rook sets remain non-negative. This process will eventually stop. At this point, consider any rook set with a positive sum. For each number in this rook set, we can select another rook set, containing this number, with a sum of zero. We will do this for each number in the rook set, and some cells may be selected multiple times.

Now, we utilize the idea of imagining a board as a bipartite graph, because we have a lot of rook sets with sum zero, and every rook set is a perfect matching in this graph. Consider a bipartite graph G where the vertices of the first part correspond to columns, and the vertices of the second part correspond to rows. We draw an edge in this graph if the cell at the intersection of the corresponding row and column is included in our selection (the graph may have multiple edges). It is easy to see that the degree of each vertex equals to n, because it consists of n perfect matchings. Now, we remove all edges corresponding to the initial set of vertices. Of course, our collection contains all of them. What remains is a graph in which the degree of each vertex equals $n-1$, so the graph remains regular (i.e. each vertex has the same degree). Now we need a claim about regular bipartite graphs:

Claim — Edges of a regular bipartite graph with degree k can be split into k perfect matchings.

Proof. Proof uses *Hall's lemma*. We will find one prefect matching by checking if the lemma condition is satisfied. Consider a set S_1 of vertices of the first half that consists of m vertices. There will be km edges that has an endpoint in the set S. Suppose that S_2 is the set of endpoints of these edges in the second half. If $|S_2| < m$, there must be $< km$ edges that have an endpoint in S_2, contradiction. So Hall's lemma condition is satisfied and we have a perfect matching. We then delete it and we still have a regular graph, then we repeat the operation. □

By the claim, we can split the selection above into $n-1$ perfect matchings. However, we know that each perfect matching corresponds to a rook set, so the total sum in our selection should be positive, because it consists of $n-1$ with non-negative sum (we were ensuring that at every step all the rook sets remain nonnegative) and one with a positive sum that we excluded at first. On the other hand, this was a selection of rook sets with a sum of zero, which leads to a contradiction. Therefore, at the end of this process, the sum in all rook sets must be zero.

It should be intuitive that in the case when all of the rook sets give a zero sum, we should be able to make all numbers zero. Now, let's prove it. Number the columns from left to right and the rows from top to bottom. First, it is easy to check that rook sums do not change under a specified operation (denoted by ♥). Second, if we consider two pairs of numbers that form the opposite corners of a rectangle with sides parallel to the sides of the big rectangle in the table, their sums are equal, because the sums of the corresponding rook sets, which differ only by these two pairs, are both equal to zero (it correspond to a transposition that we imposed in the solution to this problem), call it a $\star$. It is straightforward to make the first column to contain only zeros by applying the operation to the numbers in the second column, each time subtracting the number located at the intersection of the corresponding row and the first column. After this, by $\star$, the numbers in one column are equal.

Consider the first column in which all numbers are equal and non-zero. If it's the nth column, then we can find a rook set for each of its cells, and, by assumption, all the numbers in the rook set are zeros (because they are all not in the nth row), except the one in the last row, which is not possible because they all have a sum of zero. If it's not the nth column, then suppose it's the ith column and the numbers are equal to a. Consider a cell in the intersection of nth row and ith column, apply ♥ to it and decrease the column by a, and increase the row by a. Then, the first i cells of the nth row will have a number a written in them. Now, apply ♥ to the cell in the intersection of nth column and nth row, and decrease the numbers in the row by a, and increase the numbers in the column by a. We've just made the numbers from the first i columns equal to zero, but before it, only $i-1$ columns had all their numbers equal to zero. By repeating this process, we will sooner or later have that all numbers are equal zero.

		a		y
		a		y
		a		y
		a		y
0	0	a	x	y

$\longrightarrow$

		0		y
		0		y
		0		y
		0		y
a	a	a	$x+a$	$y+a$

$\longrightarrow$

		0		$y+a$
		0		$y+a$
		0		$y+a$
		0		$y+a$
0	0	0	x	$y+a$

Thus, by initially reducing some numbers, we have ensured that all numbers are zeros, which means that before the reduction, all numbers were non-negative, as required.

§4.6 ISL 2015 C6, proposed by USA

Problem 6 (ISL 2015 C6)
Let S be a nonempty set of positive integers. We say that a positive integer n is clean if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

¶ First solution (Even representations and inequalities) $S = \{s_1, s_2, \ldots\}$, where the elements are in ascending order. Suppose that all sufficiently large integers are clean. If we can somehow track odd representations, we must also know something about even representations. We will prove the following claim:

Claim — Every sufficiently large integer has a unique even representation.

Proof. First of all, every sufficiently large number has at most one even representation, because if there are two different ones for a number n, we can add an s_i that doesn't appear in either of the two representations. That way we obtain a large number $n + s_i$ that has two odd representations, contradiction. Now, consider s_t for a large enough t. Suppose that k has no even representation. $k + s_t$ has an odd representation that doesn't contain s_t. Otherwise, by cancelling it we would obtain an even representation for k. Consider now $k + 2s_t$. It has an even representation, because we can add s_t to an odd representation of $k+s_t$. Suppose that $k+3s_t$ has a term s_t in its odd representation, then, by cancelling it, we obtain an even representation of $k+2s_t$ without s_t, but we already know that there's an even representation of it that doesn't have s_t, but we know that there cannot exist two different even representations. Of course, in the same way we can induct to show that every $k+2ms_t$ has a unique even representation with s_t. And $k + (2m + 1)s_t$ has an odd representation without s_t. For every residue mod $2s_t$, we know that there exists only finitely many integers that are equivalent to it mod $2s_t$ and have no even representation. So, there exists only finitely many integers with no unique even representations. This finishes the proof. □

Claim — For every integer M, there exists an integer N such that $s_{i+1} - s_i \geq M$ for every $i \geq N$.

Proof. if $s_{i+1}-s_i = s_{j+1}-s_j$ for $j > i+1$ with i, j large enough, then $s_{j+1}+s_i = s_j+s_{i+1}$, which corresponds to two different even representations. Therefore, only finitely many differences can be equal to some d (for large enough indices, only two). Therefore, we can choose a moment in the sequence from which all differences are $\geq M$. Hence, proved. □

Consider now $s_{i+1} - s_i$ for big enough i. By the above claim, we can make it sufficiently large. Consider its even representation. If it doesn't contain s_i, then add it to the even representation, and we will get another (other than just taking s_{i+1}) odd representation of s_i. Otherwise, $s_{i+1} - s_i \geq s_i + s_1$, from which $s_{i+1} \geq 2s_i + s_1 \geq 2s_i + 1$ for all big enough i.

Consider $a_i = s_{i+1} - s_i - \ldots - s_1 - 1$ for some big enough i. Then $a_{i+1} = s_{i+2} - s_{i+1} - \ldots - s_1 - 1 = a_i - 2s_{i+1} + s_{i+2} \geq a_i + 1$. By repeatedly applying this argument, we can prove that $a_i \geq 1$ for all large enough i. Then, $s_i + s_{i-1} + \ldots s_1 + 1$ is not representable, because it's greater than sum of any subset of $\{s_1, s_2, \ldots, s_i\}$, but it's still less than s_{i+1}, because $s_{i+1} = s_i + \ldots + s_1 + 1 + a_i \geq s_i + \ldots + s_1 + 2$.

¶ Second solution (Generating functions and multiplicity of 1) As in the first solution, we can try to consider even representations, but from slightly different angle. If we wanted to count the number of representations of number n as a sum of different integers from the set $\{s_1, s_2, \ldots\}$, we would need to calculate the coefficient $[x^n]$ in

$$\prod_{i=1}^{\infty}(1 + x^{s_i}).$$

To differentiate between odd and even representations, we will use

$$g(x) = \prod_{i=1}^{\infty}(1 - x^{s_i}).$$

Now, $[x^n] = \#$(even representations of n) - $\#$(odd representations of n). Throughout the solution, $0 < x < 1$ to avoid convergence problems.

As in the previous solution, we can prove that there's at most one even representation. Thus, every coefficients of x^n for large n is 0 or -1. So, the coefficients are bounded.

Now, we will do a trick. First, suppose that there are infinitely many -1 among the coefficients. Consider

$$\frac{g(x)}{1-x} = \frac{1 - x^{s_1}}{1 - x}\prod_{i=2}^{\infty}(1 - x^{s_i}) = (1 + x + \ldots + x^{s_1 - 1})\prod_{i=2}^{\infty}(1 - x^{s_i}).$$

The coefficients of the latter product are bounded because we just restrict s_1 from the set, but the number of representations is still at most 1. Multiplying by some finite polynomial also doesn't create any unbounded coefficients. Thus,

$$\left|\frac{g(x)}{1-x}\right| \leq |M(1 + x + x^2 + \ldots)| = \left|\frac{M}{1-x}\right|$$

for some constant M that bounds the absolute values of coefficients. From this, we see that $|g(x)|$ is bounded. But if we substitute $x \to 1^-$ in $g(x)$ (it converges for this value, so we can do it), then it will contain the sum of infinitely many -1, which cannot be bounded.

Thus, in $g(x)$ only finitely many coefficients are non-zero. Now, we seek to extend the above argument. We could divide by $(1 - x)$ in any power k by separating first k brackets. Thus,

$$\left|\frac{g(x)}{(1-x)^{k+1}}\right| \leq \left|\frac{M}{1-x}\right| \Rightarrow |g(x)| \leq |M(1-x)^k|.$$

Limiting $x \to 1^-$, we get that the RHS approaches 0, so LHS has to be zero as well. But the value of $g(x)$ in the limit is simply $g(1)$. Thus, $g(1) = 0$. Since $g(x)$ is a polynomial, this means that $g(x) = (1 - x)h(x)$. Now, $|h(x)| \leq |M(1 - x)^{k-1}|$. Here, we can repeat the same procedure to conclude that $h(1) = 0$. Since k is unlimited, we can prove that the multiplicity of 1 is infinite in $g(x)$, which is a contradiction to finiteness of g.

Remark. Another problem of similar taste is APMO 2020/3.

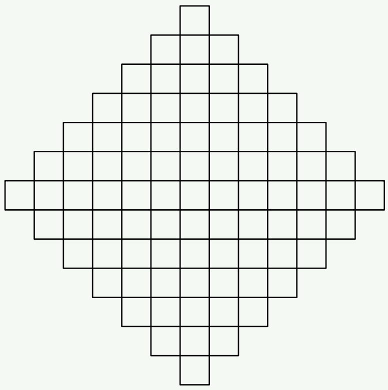
§4.7 Balkan MO 2016/4, proposed by Nikolai Beluhov

Problem 7 (Balkan MO 2016/4)
The plane is divided into squares by two sets of parallel lines, forming an infinite grid. Each unit square is coloured with one of 1201 colours so that no rectangle with perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 or 1201×1 contains two squares of the same colour.

¶ Solution (Star figure and distances argument) From the condition, squares that are situated close to each other have different colours. The problem asks us to derive rather concrete form of colouring from this, because every colouring in n colours where no $1 \times n$ rectangle contains two cells of the same colour is periodic in nature, and we only need to paint a 1201×1201 square and repeat it periodically. That's a strong conclusion to derive from the given statement, and it suggests that the property in the problem is strong itself.

We want to combine the statement about 25 different rectangles into something more exact. By experimenting with values smaller than 100, we obtain the following claim:

Claim — For every cell in the star figure drawn below, with the height and width equal to 49 and total area equal to $25^2 + 25^2 - 49 = 1201$, all cells have different colours.



Example for perimeter = 16.

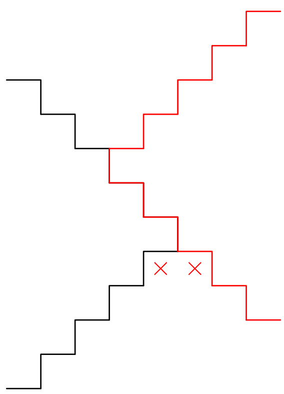
Proof. We will establish vector notation by introducing the coordinate system. Suppose that the center of the star is (0,0), then the cells of the star are exactly those that have integer coordinates (a, b) with $|a| + |b| \leq 24$. Consider some vector that connects (a, b) to (c, d), the cells of the star. The vector has coordinates $(c - a, d - b)$ and $|c - a| + |d - b| \leq |c| + |-a| + |d| + |-b| = (|c| + |a|) + (|d| + |b|) \leq 24 + 24 = 48$. But, for every vector (x, y) with this property, the perimeter of the rectangle with one corner $(0, 0)$ and opposite corner (x, y) has the perimeter $2(|x| + |y| + 2) \leq 100$, so they must be of different colours by the problem condition. Since every vector between the two cells in the star has this property, all cells are of different colours. Because there are exactly 1201 cells in the star, we can say that it contains the cell of each colour exactly once. $\square$

We will now use a simple fact: If the cell a is contained in the start of b, then b is contained in the star of a, because reversing a vector doesn't change its absolute coordinates. Now, consider all cells of colour c. Draw the star for each of them. If two stars intersect, then, for this cell, both the cells of the color c are contained it its star, which is a contradiction to the fact that all cells of the star are of different colours. Suppose that some cell is not in some star. Then, consider the star for this cell, it won't have any cells of colour c, which is a contradiction to the fact that each star must contain a cell of every color. From these two properties, we see that the stars partition the plane.

By experimenting with small cases, we can prove that there are only two different ways to partition the plane into stars.

Claim — If there's a star at (0,0), then the only way to partition the plane in the stars if the stars have centers at $(25i + 24j, 24i - 25j)$ or at $(24i + 25j, 25i - 24j)$ for integers i, j.

Proof. Consider a star that contains the cell with coordinates (24, 1), and suppose it's not (24, 25) or (25, 24). Then, it's not hard to see that there must occur the picture below (black star is the one at (0,0))



It's not hard to see that the two cells labeled with a cross cannot be covered with stars that don't intersect with other two stars and with each other. Suppose that the star that contains (24,1) is the one at (25, 24), i.e. we chose the first option in the claim (proofs are interchangeable). We can then prove that the star that contains the cell (24,-1) is centered either at (24, -25) or at (25, -24). Second is not possible because (25,0) is taken. So, it's (24, -25). We can then prove that there exists a star at (49,-1) similarly. It's straightforward to continue the same pattern in all directions.

□

By using this claim, we see that cells of colour c are either in the $(25i+24j, 24i-25j)$ cells or in the $(24i + 25j, 25i - 24j)$. Suppose the first case holds and $25i_1 + 24j_1 = 25i_2 + 24j_2$ with $i_1 \neq i_2, j_1 \neq j_2$, then $i_1 - i_2 = 24k$ and $j_2 - j_1 = 25k$. Then $|24i_1 - 25j_1 - 24i_2 + 25j_2| = |(24^2 + 25^2)k| \geq 24^2 + 25^2 = 1201$. Thus, the absolute difference between the y coordinates of the cells of the same colour in the same column is at least 1201. In the same way, we can prove this for x coordinates in the same row. This is equivalent to what the problem asks.

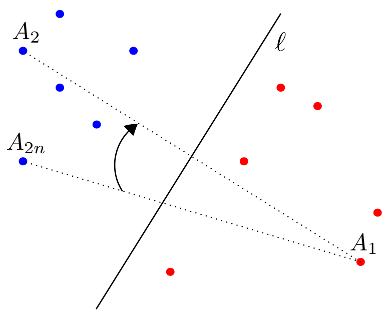
§4.8 ISL 2019 C6, proposed by USA

Problem 8 (ISL 2019 C6)

Let $n > 1$ be an integer. Suppose we are given $2n$ points in the plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \ldots, A_{2n}$ in some order. We then consider the following angles: $\angle A_1A_2A_3, \angle A_2A_3A_4, \ldots, \angle A_{2n-2}A_{2n-1}A_{2n}, \angle A_{2n-1}A_{2n}A_1$, $\angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

¶ Solution (Line rotation) The problem asks us to show that we can choose signs in the sum $\pm\angle A_1A_2A_3 \pm \angle A_2A_3A_4 \pm \ldots \pm \angle A_{2n-2}A_{2n-1}A_{2n} \pm \angle A_{2n-1}A_{2n}A_1 \pm \angle A_{2n}A_1A_2$ so that it equals to zero. We can think about the $\angle A_1A_2A_3$ as the angle of rotation around the point A_2 that maps the line A_1A_2 into A_2A_3. We will choose clockwise direction as positive and anticlockwise as negative. The sum of all angles with $\pm$ signs then will be the total angle the line rotated when we performed successive rotations around A_2, A_3, …, A_{2n}, A_1. Line will be transformed in the following way: $A_1A_2 \to A_2A_3 \to \ldots \to A_{2n}A_1 \to A_1A_2$. Therefore, the total angle of rotation is a multiple of 180°. If the angle is $\leq -180^\circ$ or $\geq 180^\circ$ then, since the rotation is continuous, at some point our line is parallel to any predetermined line, i.e. spans all the possible directions.

Now, we will use the condition that we are always rotating between the lines that connect odd-even indices. This suggests somehow splitting points in two groups and label one group even and other odd. We also want the lines that connect points between groups to not spawn all the possible directions. This suggests to split the points into two groups with non-intersecting convex hulls using a line ℓ, that exists because we can implement a coordinate system such that no two points have the same y-coordinate, then just "scan" the points using $y = c$ lines. Easy to see that some choice of c will leave exactly n points on top of it and n points below the line, because after passing a point, the difference between these quantities changes exactly by one, since no two of the given points can lie on the line $y = c$ at the same time, otherwise they have the same y-coordinate.



We can see that we are always restricted by the lines between odd and even indices and there can never be a line passing through one of the given points that is parallel to ℓ, otherwise we somehow escaped the odd-even lines.

Remark. Looking at the line rotation, direction, number of points on each of the half-planes of the rotated line is a very popular technique. Other notable examples are the infamous IMO 2011/2 and more recent All-Ukrainian 2024 11.7

§4.9 RMM 2020/3, proposed by Ron Aharoni

Problem 9 (RMM 2020/3)
Let $n \geq 3$ be an integer. In a country there are n airports and n airlines operating two-way flights. For each airline, there is an odd integer $m \geq 3$, and m distinct airports $c_1, \ldots, c_m$, where the flights offered by the airline are exactly those between the following pairs of airports: c_1 and c_2; c_2 and c_3; $\ldots$; c_{m-1} and c_m; c_m and c_1.

Prove that there is a closed route consisting of an odd number of flights where no two flights are operated by the same airline.

¶ First solution (Finding a maximal forest) We are given a collection of n monochromatic odd cycles in the graph with n vertices. We wish to take at most one edge from each cycle, union of which forms an odd cycle. Take any collection of edges such that there's at most one edge from each air company. Call the resulting subgraph a *selection*. If we assume the contrary to the problem condition, we see that any selection is bipartite. Every bipartite graph can be coloured in two colours such that no two adjacent vertices are coloured the same.

Odd cycles are good because whatever the colouring of the graph in two colours, there will exist two neighbouring cells of the same colour. So, when adding this edge to the selection, we either decrease the number of components (when connecting two vertices from different components), or, if added an edge to the component, it forms an odd cycle (consider a path with an even number of edges from one of the endpoints of the added edge to other endpoint). This odd cycle is what we need because we have used at most one edge from each company.

In the context of our problem, if we have a selection, and we have an odd cycle of an air company a that's not in this selection, but every vertex of it is in the selection, then, after colouring in two colours, we can add the edge of a that connects two vertices of the same colour. It would still form a selection, but it would decrease the number of connected components.

We only need to select at most one edge from each cycle so that there would still exist a cycle that has all vertices inside the induced subgraph. For this, we want to consider a maximal acyclic selection (by the number of edges). There's at least one such selection, because we may not take any edges at all. Since it's acyclic, it is a forest, and so, it has at most $n - 1$ edges, so there exist some air company that is not in the forest. Consider an odd cycle that contains edges of this air company. If there's at least one vertex of this cycle that is outside of the vertices of the selection, then choose an edge that contains this vertex and add it to the selection, the resulting graph is still acyclic and is a selection, which is a contradiction. So, every vertex is inside the selection, and we can decrease the number of components but it will still be a selection and a forest with more edges, contradicting maximality.

¶ Second solution (Hall's lemma and induction) We will be done if some $n - 1$ of the given cycles don't cover all vertices, because then there exist $n - 1$ odd cycles of different air companies that are placed on $n - 1$ vertices, and we are done by induction hypothesis (easy to check the base for $n = 3$, because we just have three triangles on same three vertices). Suppose that we cannot find $k \le n - 1$ cycles such that they only cover k vertices.

Let's try to select one edge of each air company such that all the chosen edges are different. This would give us a cycle because the number of edges is the same as the number of vertices, so it cannot be a tree. Moreover, all the edges of this cycle are of different colours. Then, we will try to rebuild it somehow to be odd.

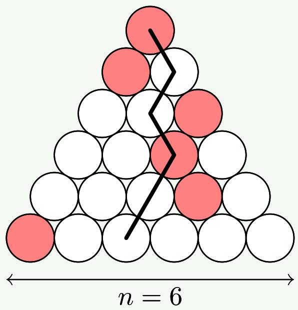
To prove that the selection exists, consider a bipartite graph where one side is the colours and the other is the edges. If there's no perfect matching of the first side to second, then there exists a set of k colours such that the union of their cycles contains at most $k - 1$ different edges. Delete all repeating edges from the subgraph induced by these k colours, then each connected component contains a cycle, so the number of vertices is at most the number of edges (because the number of edges of every connected graph is at least one less then the number of vertices with equality only for acyclic graphs), so the total number of vertices is at most the number of edges, which is at most $k-1$. So, there exist some $k-1 \le n-1$ (even k) of these colours that cover $\le k - 1$ vertices. Here, we are done by induction hypothesis.

Therefore, there's a perfect matching. Hence, we can choose one edge of each colour such that no edges repeat, i.e. the graph is simple. Suppose that it forms a subgraph G. Since G has more edges than the number of vertices - 1, there's a cycle. By the assumption, there's no odd cycles, so we can colour the vertices in two colours such that two adjacent are of different colours. Consider some edge of the cycle of G and the cycle of this company in the initial graph; we can now repeat the same approach to reach the contradiction as in the first solution. Replace the edge that we considered (It's in the cycle, so the components stay the same) by the one that connects two vertices of the same colour. The graph still has no repeating edges (because if the chosen edge already appeared, then the two-colouring would be violated), and the number of connected components decreased, because, otherwise, this edge would form an odd cycle (this is because if we take an edge uv with u, v of the same colour, there was a path from u to v in which the colours alternate, and so it has an even number of edges, adding a new edge uv results in the desired cycle). Restart this procedure because the graph still has to be bipartite.

§4.10 IMO 2023/5, proposed by Merlijn Staps and Daniël Kroes

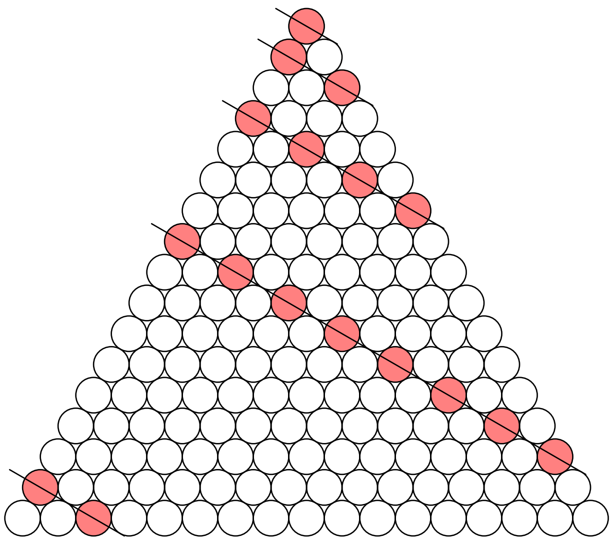
Problem 10 (IMO 2023/5)

Let n be a positive integer. A Japanese triangle consists of $1+2+\cdots+n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Here is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles.



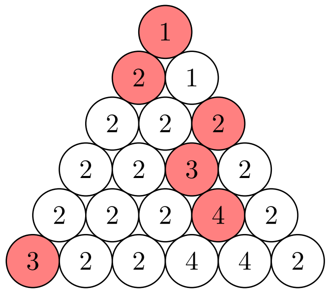
In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

¶ **First solution (Direct dynamic programming style recursion)** **Example:**



We see that we can take at most one red circle from each group that has a line passing through the circles of this group. The number of groups is $k = \lfloor \log_2(n) \rfloor + 1$. This example is the same for all solutions.

Bound: The idea of most of the solutions is to write a certain number in each of the circles, related to the maximum number of red circles in some path that finishes at this circle. For this solution, we will write the maximum number of red circles in the path starting on top and finishing in this circle. For example, if we do it in the example case, then we will get the following:



The advantage of it is that the recursion is not hard: If we denote the number in jth circle of ith row as $f(i, j)$, then

$$f(i, j) = max(f(i-1, j-1), f(i-1, j)) + (1 \text{ if the } (i, j) \text{ circle is red, otherwise } 0).$$

We seek to find the maximum number in the triangle. Since we don't know the possible position of it (it might be in the middle or on the sides), we will use an averaging approach. For this, consider the sum $S_i = \sum_{j=1}^{i} f(i, j)$. We seek to find a recursion that bounds S_{i+1} using the sum S_i. This seems possible because for every number in the following row, we can find a number on top of it that is at most this number. We also have a $+1$ from the red circle and more circles than in the previous row. More formally, consider any number $f(i, j)$ in the ith row.

Then, for $1 \leq k \leq j$,

$$f(i+1, k) \geq f(i, k) + (1 \text{ if the } (i+1, k) \text{ circle is red, otherwise } 0).$$

For $j + 1 \leq k \leq i + 1$,

$$f(i+1, k) \geq f(i, k-1) + (1 \text{ if the } (i+1, k) \text{ circle is red, otherwise } 0).$$

If we sum over all $1 \leq k \leq i + 1$,

$$S_{i+1} \geq S_i + f(i, j) + 1.$$

Since we can choose any $f(i, j)$, we can just choose the maximal number in ith row, which satisfies $\geq \lceil \frac{S_i}{i} \rceil$. Now we only need to calculate this recursion starting from $S_1 = 1$. The problem follows from the following claim:

Claim — $S_{2^n} \geq n \cdot 2^n + 1$ for $n \geq 0$.

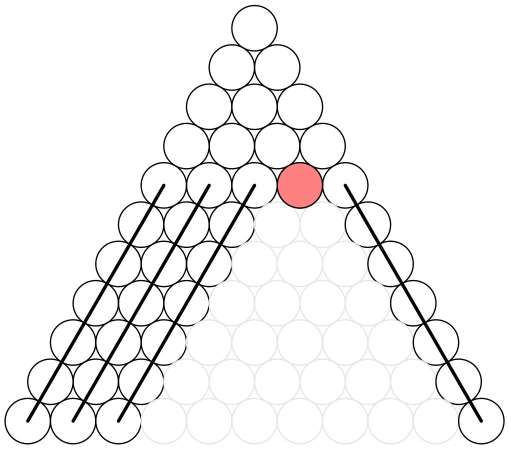
Proof. The proof is by induction on n. For $n = 0$, it's obvious. $iS_{i+1} \geq iS_i + i\lceil \frac{S_i}{i} \rceil + 1 > iS_i + S_i = (i+1)S_i \implies \frac{S_{i+1}}{i+1} \geq \frac{S_i}{i} \implies \lceil \frac{S_{i+1}}{i+1} \rceil \geq \lceil \frac{S_i}{i} \rceil$.

$\lceil \frac{S_{2^n+r}}{2^n+r} \rceil \geq \lceil \frac{S_{2^n}}{2^n} \rceil = n+1$ for $0 \leq r \leq 2^n - 1$. This means that $S_{2^n+r} \geq S_{2^n+r-1} + \lceil \frac{S_{2^n+r-1}}{2^n+r-1} \rceil + 1 \geq S_{2^n+r-1} + n + 2$ for $1 \leq 2^n$. Therefore, we can say that $S_{2^{n+1}} \geq S_{2^n} + (n+2)2^n \geq n \cdot 2^n + 1 + n \cdot 2^n + 2^{n+1} = (n+1)2^{n+1} + 1$, which proves the induction step. □

Hence, we see that in 2^nth row there must be a number that is at least $\lceil \frac{S_{2^n}}{2^n} \rceil = n + 1$. So the answer is indeed at least $\lfloor \log_2(n) \rfloor + 1$.

¶ Second solution (Mirsky's theorem) It's natural to make a partial order on the set of circles. Let $c_2 \prec c_1$, if c_2 can be reached from c_1, i.e. c_2 is contained in the subtriangle of c_1. We can further restrict the problem to only considering red circles. We are interested in the longest chain. It's known that the length of the maximal chain is equal to the minimal number of antichains needed to cover the whole set, by *Mirsky's theorem.*

Suppose that the highest number of the antichain is located in the kth row. This antichain contains at most k numbers. At most one for each skewed line and the one in the middle (its subtriangle also doesn't contain any circles of the antichain), example is the picture below:



Now, suppose that we have 2^n red circles and only n antichains that cover the set. Suppose their highest points are $1 = k_1 < k_2 < \ldots < k_n$. There are at most $k_1 + k_2 + \ldots k_{i-1}$ circles in the first $k_i - 1$ rows. Because each of the red circles in those rows is contained within some of the first $i-1$ antichains. This shows that $k_i \le k_1 + \ldots k_{i-1} + 1$. It's easy to show that $k_i \le 2^{i-1}$ by induction [1]. So, we will have a total of $\le k_1 + k_2 + \ldots + k_n = 1 + 2 + \ldots + 2^{n-1} = 2^n - 1$ red circles, but we assumed that we have 2^n of them. So, there must be at least $n+1$ antichains, and so, there must exist a chain of length $n + 1$.

> **Remark.** One could guess the answer asymptotically. For 2^n, we can consider 2^n ninja paths constructed explicitly. On first level, we branch paths in two directions. On second level, for each of the 2 circles, we also branch in two directions and so on, for each level that is a power of two less than 2^n. On levels from $2^{n-1}+1$ to 2^n, every circle is in some path. On levels from $2^{n-2}+1$ to 2^{n-1}, every circle is in at least two paths (because branching happens at 2^{n-1}). Thus, the total sums on all the paths is something like $2^n + 2^{n-1} + 2^{n-1} + 2^{n-1} + \ldots + 2^{n-1}$, where we have $n+1$ summands. Thus, some of these paths contains $\frac{n}{2} + 1$, which is asymptotically correct. Constructing the example is also a good way to reassure that the answer is logarithmic.

[1] Whenever we have statements involving sum of preceding terms, it makes sense to relate them to powers of two

§4.11 All-Russian MO 2021 grade 11/8, proposed by Ilya Bogdanov and Fedor Petrov

Problem 11 (All-Russian MO 2021 grade 11/8)

Each of the 100 girls has 100 balls; there is a total of 10000 balls in 100 colors. 100 balls of each colour. In one move, two girls can exchange a ball (the first gives the second one of her balls, and vice versa). Prove that it's possible to execute such a sequence of exchanges such that each girl would have 100 balls of each colour in the end and each ball was exchanged at most once.

¶ Solution (Reflecting a classic Hall's lemma problem) We start with a lemma that is a 38th problem in advanced section of 102 combinatorial problems by Titu Andreescu and Zuming Feng.

Claim — An $m \times n$ array is filled with numbers from the set $\{1, 2, \ldots, n\}$, each used exactly m times. Show that one can always permute the numbers within the columns to make each row contain every number from the set $\{1, 2, \ldots, n\}$ exactly once.

Proof. Note that we just have to prove that we can choose one number from each column such that all the numbers from 1 to n are chosen. Then, we can make the chosen numbers to be one row and then induct. Let's map n columns to n numbers. Consider a set of any k columns. They contain km numbers. If the columns are mapped to less than k numbers, then there can only be $< km$ cells in these columns, which is a contradiction. So, by *Hall's lemma*, we are done. □

Imagine each of the girls occupying one column of the 100×100 table and putting each of their 100 balls inside this column. We want to exchange the balls in some pairs of cells such that each cell appears in at most one pair, and, in the end, each column contains 100 balls of different colours. Note that, by the lemma, we can permute the columns such that each row contains balls of different colours, and it's not hard to reflect the board across the diagonal using swaps, then rows become columns and we are done.

§4.12 USAMO 2018/6, proposed by Richard Stong

> **Problem 12** (USAMO 2018/6)
> Let a_n be the number of permutations $(x_1, x_2, \ldots, x_n)$ of the numbers $(1, 2, \ldots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \le k \le n$ are all distinct. Prove that a_n is odd for all $n \ge 1$.

¶ Solution (Bijections and perfect matchings) In the problems where we are asked to prove the result about the parity [1], we tend to pair up the objects and see what's left. This is called establishing the bijection. There will either be nothing left or it will give more structure to what we have to consider.

Here the possibility for bijection follows from the fact that if $\frac{x_k}{k}$ are different, then so are $\frac{k}{x_k}$. So, we can reverse all edges of the corresponding permutation graph, i.e. consider a permutation y in which $y_{x_k} = k$. This pairs up all the permutations for which $y \neq x$. The ones that are left are permutations that consist of 2-cycles, i.e. involutions. We can represent them as perfect matchings in bipartite graph where each part consists of numbers $1, 2, \ldots, n$ and i, j are adjacent if $x_i = j$ and, as a result, $x_j = i$.

So, we are only left to consider the perfect matchings. Call a matching good if it satisfies the condition and bad otherwise. First of all, if we have a bad matching, then we have some $a \sim b$ and $c \sim d$ such that $\frac{a}{b} = \frac{c}{d}$, from which we can say that $\frac{a}{c} = \frac{b}{d}$, then we can find another bad matching if we instead connected $a \sim c$ and $b \sim d$. This works if $a \neq b$ and, as a result, $c \neq d$.

This suggests that we should think about the graph where vertices are bad perfect matchings with at most 1 self-edge, and two vertices x and y are connected if we can select some pairs of edges of matching x such that the respective ratios that are formed by them are equal, and reconnect them in a way described above, forming y. The phrase "respective ratios" is confusing because we can take it as $\frac{a}{b}$ as well as $\frac{b}{a}$, so we will use $\frac{a}{b}$ with $a \ge b$ for definiteness.

Now, consider some bad perfect matching. Suppose that $r_1, r_2, \ldots, r_s$ are all the possible ratios that are written on its edges. Suppose now that n_i is the numbers of edges with r_i written on it. When we choose some k pairs edges out of these, there are actually more ways to reconnect them. So, we just consider $2k$ edges and need to calculate the number of ways in which we pair them. Here's the claim that allows to calculate this quantity:

> **Claim** — The number of perfect matchings on the bipartite graph from vertices numbered $1, 2, \ldots, n$ to vertices numbered $1, 2, \ldots, n$, such that it doesn't contain two edges of the form $i \sim i$ (call them *self edges*), is $(2\lfloor \frac{n}{2} \rfloor + 1)!!$, i.e. the product of all odd numbers that are at most $2\lfloor \frac{n}{2} \rfloor + 1$.

Proof. The proof follows by induction. Suppose that $f(n)$ is the answer for n vertices. We see that if we have $2k$ vertices, then there shouldn't be any self edges (Otherwise, among the remaining $2k - 1$ vertices there must a self edge, since we cannot pair them all up). So, for

[1] In general, when asked about mod n, we want to make a correspondence of one instance to some other $n - 1$ instances, i.e. divide in groups of n

$2k$ vertices, $f(2k) = (2k-1)f(2k-2)$, because there are $2k-1$ possible neighbours for an arbitrary vertex, and we will have to pair up $2k-2$ vertices after choosing it. For $2k+1$, we see that there must be a self edge, but the other $2k$ vertices must be paired. It gives a total of $(2k+1)f(2k)$, which aligns with the $(2\lfloor \frac{n}{2} \rfloor + 1)!!$ assumption. Hence, the induction is completed. $\square$

Using this claim, we can finish the computation above. We have chosen $2k$ edges, there are $(2k-1)!!$ ways of pairing them up without self edges. So, we have a total of

$$\prod_{1\le i\le s} \sum_{0\le k\le \lfloor \frac{n_i}{2} \rfloor} \binom{n_i}{2k}(2k-1)!! \equiv \prod_{1\le i\le s} \sum_{0\le k\le \lfloor \frac{n_i}{2} \rfloor} \binom{n_i}{2k} =$$

$$= \prod_{1\le i\le s} \frac{(1-1)^{n_i} + (1+1)^{n_i}}{2} = \prod_{1\le i\le s} 2^{n_i-1} \pmod 2,$$

where we have used the binomial expansion in the middle. If $n_i \ge 2$ for some i, this number is even. But we know that for each bad matching, at least one i makes $n_i \ge 2$. We also need to subtract a one because we may have not reconnected any edges. So, the number of incident vertices is odd. Now, obviously, this graph has an even number of vertices, so there's an even number of bad perfect matchings with at most 1 self edge. The claim gives us the answer for the question about the total number of perfect matchings with at most 1 self edge: It's equal to $(2\lfloor \frac{n}{2} \rfloor + 1)!!$, which is odd. So, we have an odd number of good perfect matching with at most one self edge. But we cannot have any good matching with 2 self edges, because two ratios would be equal to 1, which is not possible.

§4.13 RMM 2023/6, proposed by Vasily Mokin

Problem 13 (RMM 2023/6)
Let r, g, b be non negative integers and Γ be a connected graph with $r+g+b+1$ vertices. Its edges are colored in red, green and blue. It turned out that Γ contains

- A spanning tree with exactly r red edges.
- A spanning tree with exactly g green edges.
- A spanning tree with exactly b blue edges.

Prove that Γ contains a spanning tree with exactly r red edges, g green edges and b blue edges.

¶ First solution (Adding bridges) Suppose that T_r, T_b, T_g are the spanning trees from the problem statement. We will consider a graph with the same $r+g+b+1$ vertices, but with the r edges taken from T_r, g edges taken from T_g, and b edges taken from T_b. We want to rebuild it so that it still contains $r+g+b$ edges, but is connected or contains no cycle (both will imply that the graph is a tree). Call this graph G.

G has a property that it has no monochromatic cycle. Since G is not connected, we can partition it into two vertex sets V_1 and V_2 such that there's no edge in G between them. For any colour $c \in \{r, g, b\}$, T_c contains an edge between V_1 and V_2, and it's not of the colour c (otherwise, it would be in G). G has a cycle (otherwise, it's a forest, but by considering the number of edges, we see that it's a tree). WLOG, this cycle has edges of colours r and b. Then, consider an edge between V_1 and V_2 from the tree T_g; it will be of the colour r or b. Add this edge to G and delete the edge of the same colour from the cycle considered before. We see that the graph still has r red edges, g greed edges, b blue edges. No connected component disconnected, because we deleted an edge from the cycle. No cycles were added, because we added a bridge (if we assume it's in the cycle, then we already had a path that connected two components in G before we added this edge, which is a contradiction). So, we still have no monochromatic cycle. Suppose that the edge connecting two components that we take from T_c is of the colour c at some point, then this c was in the graph in the beginning, and it connects two components (because every time we perform the operation, we just unite two components and don't alter the vertices of other) that were in the beginning, which is a contradiction. Therefore, with each operation the number of connected components decreases and all the properties that allowed to perform the operation remain true, and graph always has a correct number of edges of each colour. By repeating this operation enough times we can reach the desired tree.

¶ Second solution (Induction and merging vertices) Here's a slight alternative to the above proof. Take definitions from it. Suppose that G has two components C_1 and C_2 such that there's a colour c edge between them in Γ, and there's a cycle in G such that it has an edge of the colour c. Then, delete the edge from the cycle and connect two components using the bridge of the same colour as the deleted edge. The number of components decreased and G still has a correct number of edges of each colour. By performing this operation, we will see that there must be a situation in which every bridge in Γ between the components of G is of the colour such that there's no edge in the cycle in G of the same colour. We know that there must be a cycle in G (otherwise, it would be a tree) and it's not monochromatic, because there were no monochromatic cycles in the beginning and the operation doesn't add any new cycles. So, the cycle has at least two colours, WLOG, red and green. Then, every bridge is blue, otherwise we would perform the operation again. Now, partition the vertices of the graph in two sets V_1 and V_2 such that there's a blue bridge e between them. If we add it to any T (do nothing if it's already in some T), it will form a cycle (in this T) and there must be another edge between V_1 and V_2 in this cycle (we need to return from V_2 to V_1). Replace the existing edge by e, we still have the right red, blue, green edges count, but the graph now contains e. As we can do it for every T, we have that each tree contains the edge e. Now, merge this edge uv into one vertex w, i.e. if some edge has an endpoint in one of u or v, then we keep the colour, but the new endpoint is w. In a new graph Γ', if a subgraph was a tree, it's still acyclic (cycle could only appear from other cycle). If there was a path between two vertices, there still exists a path between them (and w is also connected to every vertex). Thus, if there was a tree, it's still a tree. Therefore, in Γ', there exists a spanning tree T_r' with r red edges, a spanning tree T_b' with $b-1$ blue edges, and a spanning tree T_g' with g green edges. Then, the problem follows from the induction on $r+g+b$ ($r+g+b=0$ is trivially true), because we can reconstruct the tree for Γ from the tree for Γ' by unmerging the edge. Connectedness remains true and there will still be no cycle, so the graph is a tree with exactly r red edges, g green edges and b blue edges.

§4.14 All-Russian MO 2023 grade 10/4, proposed by Alexander Gribalko

Problem 14 (All-Russian MO 2023 grade 10/4)

There is a queue of n girls on one side of a tennis table, and a queue of n boys on the other side. Both the girls and the boys are numbered from 1 to n in the order they stand. The first game is played by the girl and the boy with the number 1 and then, after each game, the loser goes to the end of their queue, and the winner remains at the table. After a while, it turned out that each girl played exactly one game with each boy. Prove that, if n is odd, then a girl and a boy with odd numbers played in the last game.

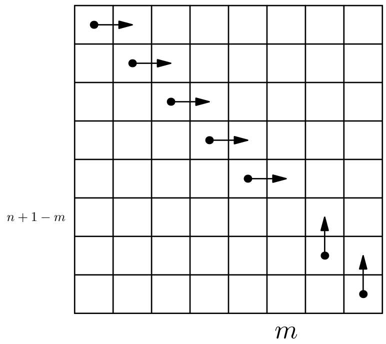
¶ Solution (Paths in a periodic board and parity of cycles) Note that we can represent every moment in games as a pair (a, b) where a is the number of the boy that is playing at this moment and b is the number of the girl. If the girl wins, we go to $(a + 1, b)$, and if the boy wins, then we go to $(a, b + 1)$. This suggest a very natural representation of the problem as a path in the table, because we either go right or up. But we need to make sure that a 1 follows n. We restrict the board to $n \times n$, and when we move upwards from the top edge, we appear in the respective column of the bottom row, and when we try to move right from right edge, we appear in the respective row of the leftmost column.

Place a pebble at the $(1, 1)$ cell and consider a board with rules of travelling described above. Paint the down-right diagonals in n colours as described in the picture.

n	n	1						$n-1$
		n	1					
			n	1				
				n	1			
					n	1		
						n	1	
2	2						n	1
1	1	2						n
	1	2						n

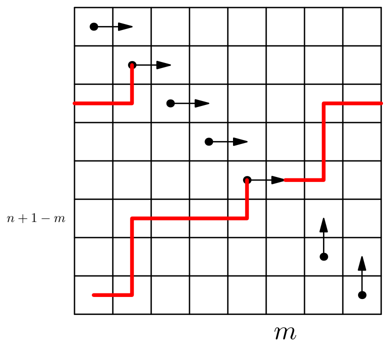
The condition of the problem tells that we have visited each cell exactly once. We have travelled exactly $n^2 - 1$ times, so in the end we will be in a diagonal labeled as $n^2 \pmod n$, which is the main diagonal. The cells on the main diagonal are exactly (a, b) with $a+b = n+1$, so we just have to show that one of the coordinates is odd, the other is going to be odd automatically, because $n + 1$ is even. For the contradiction, suppose that the finishing point is $(m, n + 1 - m)$ with m even.

From the point $(1, n)$ we cannot go upwards, because we have already been to $(1, 1)$. Therefore, after $(1, n)$, we have to go rightwards. After that, we can say that we cannot go upwards after $(2, n - 1)$, so we go rightwards again (because, if we were to go upwards and this cell is still free, then we wouldn't be able to move from $(1, n)$ that we have to visit later). Analogously, we can show that we go rightwards up until $(m-1, n+2-m)$. In the same way, we can show that we go upwards after cells from $(n, 1)$ to $(m+1, n-m)$.



Now, consider two cells that share a corner, i.e. $(a-1, b)$ and $(a, b-1)$ (both coordinates taken mod n). We see that if we performed different moves after landing on these cells, then if it was up and right, respectively, then we will have no chance to visit (a, b) (except when $a = b = 1$, but this case is handled because we know moves on the main diagonal). If it is right and up, then we visited (a, b) two times. So, if both $(a-1, b)$ and $(a, b-1)$ are not final points, then we have to perform the same move after them. This allows to only fill the direction of the path for only one cell on the diagonal, everything else will be filled automatically. Thus, if we consider a path from $(1, 1)$ to the first cell in the main diagonal, we will have a predetermined board. Suppose that we landed on the cell in kth column, i.e. $(k, n+1-k)$. Suppose that $k < m$, i.e. this cell is to the left of the final cell, otherwise just flip the board across the bottom to top main diagonal, which doesn't change the conclusion.

Since we visited the main diagonal cell, we will then go to a cell coloured as 1. We know that the path from it is the same as from $(1, 1)$ to $(k, n+1-k)$. Number cells according to their column. Using the path argument, we can conclude that from the cell numbered as $x < m$, that are to the left of the final cell, we go to the cell numbered as $x + k$, and from the cell numbered as $x > m$, we go to a cell numbered as $x + k - 1$; everything is mod n.



It's now natural to consider the main diagonal cells as vertices and draw an arrow from the vertex i to the vertex that it goes to after visiting the ith cell. We know that the resulting directed graph must be a path from k to m. Add an edge between m and k to obtain a cycle.

Note that this cycle can represent a permutation π. Now, we use a fact that a transposition, i.e. a swap of two elements of the permutation, changes its parity (parity of the number of inversions, i.e. pairs of elements i,j such that $i > j$ and i is to the right of j in the permutation). Every cycle of length l can be written as a composition of $l-1$ transpositions, so the permutation π is even, because the cycle has an odd length. Note that the permutation we obtained is almost the one that has arrows from x to $x+k \pmod{n}$, except that we need to change $m+1+k$ to $m+k$, $m+k+2$ to $m+k+1$, ..., $n+k$ to $n+k-1$, and $m+k$ to k, which can be written more neatly as a cycle π_2: $n+k \to n+k-1 \to n+k-2 \to \ldots \to m+k \to k \pmod{n}$, where the last is the same as $n+k$. The permutation π_1 defined as $x \to x+k$ is just a bunch of cycles of length $\frac{\operatorname{lcm}(n,k)}{k}$ (that's such a number that k times this number is divisible by n), but it's equal to $\frac{n}{\gcd(k,n)}$, which is odd, because n is odd, and the equality follows from the fact that $\gcd(n,k)\operatorname{lcm}(n,k) = nk$. So, each of these cycles is an even permutation, and the composition is even too. π_2 is a cycle of length $m+1$, and if m is even, then the permutation is odd. So their composition, which is π, is odd. But we have derived that it has to be even before. Hence, contradiction.

§4.15 RMM 2021/3, proposed by Mikhail Antipov

Problem 15 (RMM 2021/3)
A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

¶ Solution (Generating function and representing a complex number as an integer mod 17)
Denote the workers as $x_1, \ldots, x_{17}$. Note that choosing a leader of the brigade is the same as choosing one of the numbers from the set $\{x_i, x_{i+1}, \ldots, x_j\}$. And we are interested in the number of times each x_i was chosen. Note that it's the same as multiplying all the possible $(x_i + \ldots + x_j)$ with $j > i$, and then, every individual monomial in the sum represents the number of times each x_i was chosen by looking at its degree. We want to separate those monomials in which every degree is divisible by 4.

If we were dealing with a one variable version, then the answer would be to just use the *Roots of Unity Filter*. In the case of 17 variables, we just use the filter 17 times. Suppose that ω is the 4th root of unity; it's not hard to see that it's just i. Consider the set S of all the possible 17-tuples $(c_1, c_2, \ldots, c_{17})$ such that $c_i \in \{1, i, -1, -i\}$. Then we just need to calculate

$$N = \frac{1}{4^{17}} \sum_{(c_1,\ldots,c_{17}) \in S} f(c_1, \ldots, c_{17}), \text{ where } f(x_1, x_2, \ldots, x_{17}) = \prod_{1 \le i < j \le 17} (x_i + x_{i+1} + \ldots + x_j),$$

where we divide by 4^{17}, because each monomial that has all the powers divisible by 4 will be counted 4^{17} times (1 for each 17-tuple).

Now, we do a trick. We know that N is an integer, so all the summands containing i will cancel out. Thus, we could write an equivalent integer instead of i. By equivalent, i mean that it has to satisfy $i^2 = -1$, because that's the only thing that affects our integer part. But since we are only interested in mod 17, we can just take $i = 4$ (an easy way to think about it is performing all the same calculations, but in the from $a + b \cdot 4$, i.e. treating 4 as a variable, the result is going to be the same, because what will be in front of 4 in the end is just zero).

Now, the problem is classic and rather easy, because for every choice of $(c_1, c_2, \ldots, c_{17})$ such that $c_i \in \{1, 4, -1, -4\}$, we actually have that some two of c_1, $c_1 + c_2$, ..., $c_1 + c_2 + \ldots + c_{17}$ are equal mod 17, otherwise there would be a sum that gives a remainder zero, and it's not the c_1, so the respective product is 0. If some two are equal, then we can subtract them and obtain a contiguous sum (not of one summand again) that is equal to zero, and the product is zero again. In all cases every product is just zero mod 17. Hence, we are done.

Remark 1. Some will say that the only combinatorial part of this problem is its statement and a small argument with divisibility and prefix sums. While I agree that the solution looks purely algebraic (and maybe number theoretic in some sense), I still think that the technique is very useful and requires come combinatorial insights to think about, so I consider this problem appropriate for its place in this section of the book.

Remark 2. Other classical examples with the same idea of setting up a generating function and using the roots of unity filter are IMO 1995/6 and problem 10 from the advanced section in 102 combinatorial problems. But these problems are more computation based, while the discussed one has some insight involved. A recent example is USA TST 2024/3.

§4.16 USAJMO 2018/6, proposed by Maria Monks Gillespie

Problem 16 (USAJMO 2018/6)

Karl starts with n cards labeled $1, 2, 3, \ldots, n$ lined up in a random order on his desk. He calls a pair (a, b) of these cards swapped if $a > b$ and the card labeled a is to the left of the card labeled b. For instance, in the sequence of cards $3, 1, 4, 2$, there are three swapped pairs of cards, $(3, 1)$, $(3, 2)$, and $(4, 2)$.

He picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i card to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards $1, 2, \ldots, n$ exactly once in that order. (For example, the process starting at $3, 1, 4, 2$ would be $3, 1, 4, 2 \to 3, 4, 1, 2 \to 2, 3, 4, 1 \to 2, 4, 3, 1 \to 2, 3, 4, 1$.)

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.

¶ Solution (Modifying the process and finding the invariant) There's almost no way we can control where the cards land and how inversions change after the first turn. Only the inversions involving the smallest and largest number are easy to control. The turns are made in successive order, from smallest to largest, and the number that is written plays role. This suggests [1] to modify the number written on the card at each step. Since we are interested in the order of the cards and want to work with largest/smallest, we will just make this card have the largest label, so after each step we will be moving the smallest card and altering it to the largest card, which makes the process more definite. We will alter it by adding n. If we will be able to prove something in this case, then we can repeat it at each step, because all the steps are the same. It turns out that we can, and the main claim is:

Claim — In the modified process, the number of inversions doesn't change at each step.

Proof. We just have to show that for the first step, because, how I already said, all the steps are the same: We take the smallest number, reflect it and make it the largest. Inversions involving the smallest number are exactly the pairs with the numbers to the left of it. After we reflect it and make it largest, only inversions will be the ones that already existed and the ones with the new largest number with numbers to the right of it, but this quantity is equal to the inversions involving the unmoved smallest number, since we reflected the position. □

Spamming the claim, we see that in the end of the modified process, we will have the same number of inversions as in the beginning, and the numbers will be the initial numbers incremented by n, so if we decrease them by n, we will not change any of the inversions, so the number of inversions is the same even for the unmodified process.

[1] Honestly, not really, the problem is rather unmotivated

§4.17 SRMC 2009/3, proposed by Erzhan Baisalov

Problem 17 (SRMC 2009/3)

A tourist is going to visit the Complant. He has found that:

- Complant has 1024 cities, numbered by integers from 0 to 1023.
- Two cities with numbers m and n are connected if and only if the binary entries of numbers m and n they differ exactly in one digit.
- During the stay of a tourist in that country 8 roads will be closed for scheduled repairs.

Prove that the tourist can make a closed route along the existing roads of Complant, passing through each of its cities exactly once.

¶ Solution (Induction and Hamiltonian paths) It makes sense to generalize the problem for an arbitrary power of two, because the graph in the problem statement is easy to construct recursively. Denote the graph specified in the problem statement on 2^n as H_n. It's easy to see that H_{i+1} can be made from taking two copies of H_i, one where the $i+1$th bit is 0, the other is one for which this bit is 1, then connecting the respective vertices in these graphs. We see that the degree of each vertex in H_n is n, so if we just close all edges but one starting at some vertex, the route won't exist, because we need to come back to this vertex. But the problem suggests us that the route will exist if we delete $n-2$ edges. The route we are trying to find is called a *Hamiltonian cycle*.

The problem is trivially true for $n = 2$. Now, to exploit the recursive nature of this graph, we need to perform induction by merging two Hamiltonian cycles for smaller graphs. Note that we can move between groups only if the two corresponding vertices in two graphs are connected. This suggests that we want to find identical routes in both parts to easily connect them at their endpoints.

The other problem that we need to find a roundabout for is that one of the parts might have all the $n-2$ possible edges deleted, which can forbid a cycle appearing there. But to connect the two parts we actually don't need a cycle, we just need two paths, so we are interested in generalizing problem in a way that if we delete $n-1$ edges there will exist a path passing through each vertex exactly once, i.e. a *Hamiltonian path*, but it follows from the assumption about the cycle. Indeed, if there exists a cycle if we delete $n-2$ edges then, when we delete $n-1$ edges, just add one edge back, find a cycle, and then when we delete the edge again, it will either break a cycle and leave us with a path, or will leave the cycle that contains a path.

Now, we only need to make sure that the same paths appear in both parts, and that we can connect them. Connection part is easy, we just need every connecting edge between the two parts to be present. Note that there are n ways to split the vertices in two parts (changing one bit corresponds to each of those). If for each of the n ways one edge is deleted, then there will be a total of n edges deleted. Therefore, there exists at least one way to split the vertices into two identical graphs such that no edges in between are deleted.

To make sure that the paths are the same, we perform a trick. We make the two parts to have identical edges deleted. We just take the union of the deleted edges (regarding two edges the same if they connect the corresponding vertices of two parts). Since there were $n-2$ edges deleted in total, then the cardinality of the union is at most the sum of cardinalities of two sets, and so, it is at most $n-2$. Now, we have two identical graphs H_{n-1} with at most $n-2$ edges deleted. By assumption, there exists a Hamiltonian path in both of them. Let the path be $v_1 \to v_2 \to \ldots \to v_{2^{n-1}}$. And let the same exact path in the other part be $u_1 \to u_2 \to \ldots \to u_{2^{n-1}}$ with u_i connected to v_i. $v_1 \to v_2 \to \ldots \to v_{2^{n-1}} \to u_{2^{n-1}} \to u_{2^{n-1}-1} \to \ldots \to u_1 \to v_1$ is the working cycle for H_n, and we are done.

> **Remark.** The graph H_n is called a hypercube graph, because its structure is the same as Hypercube's. There's a well known result about hypercube graphs, that they allow a Hamiltonian cycle. This problem proves that, even if we restrict some edges, the cycle still exists.

§4.18 ISL 2023 C7, proposed by Anton Trygub

Problem 18 (ISL 2023 C7)

The Imomi archipelago consists of $n \geq 2$ islands. Between each pair of distinct islands is a unique ferry line that runs in both directions, and each ferry line is operated by one of k companies. It is known that if any one of the k companies closes all its ferry lines, then it becomes impossible for a traveller, no matter where the traveller starts at, to visit all the islands exactly once (in particular, not returning to the island the traveller started at). Determine the maximal possible value of k in terms of n.

¶ Solution (Repainting the graph using Bondy-Chvatal theorem) **Bound:** Equivalent graph formulation asks about the maximum number of colours we can use to paint the edges of the graph G such that each Hamiltonian path contains the edge of every color. Denote the number of edges of colour i that have an endpoint at the vertex v as $d_i(v)$. Consider some good colouring of the graph.

We want to repaint the edges while keeping the colouring good to give it more structure. *Bondy-Chvatal* helps out here. Even though the theorem is about Hamiltonian cycles, we can still alter it a bit to work for paths. From this theorem, we know that whenever we are given a graph G and non-adjacent u,v are its vertices with $\deg(v) + \deg(u) \geq n$, then, if G with the edge uv has a Hamiltonian cycle, then so does the graph G without this edge. For paths we can relax the condition to just $\deg(v) + \deg(u) \geq n - 1$, proofs are similar.

Claim — We can repaint some edge uv colored c in any other color if $d_c(v) + d_c(u) \leq n - 1$.

Proof. Consider graph G/c, which is a graph G without the edges of colour c, we know that it doesn't have a path. Let's calculate $\deg(u) + \deg(v)$ in this graph, it's equal to $n - 1 - d_c(u) + n - 1 - d_c(v) = 2n - 2 - (d_c(v) + d_c(u)) \geq n - 1$. If we repaint the edge uv in some colour different from c, then we will essentially add it to G/c, and it still won't form the path by Bondy-Chvatal for paths. The condition wasn't violated for any other colour as well. □

Now, we repaint the edges so that the graph has more definite structure. We can say that if we have a lot of edges of colour i coming from some vertex A, then the sum of $d_j(A)$ and $d_j(B)$, for any other colour j, will be small and we will be able to repaint AB into colour i. More formally, we can formulate the following claim:

Claim — Consider two vertices A, B. Suppose that i is the most popular colour among the edges with endpoint at A, and j is the most popular colour among the edges with endpoint at B. If $d_i(A) \geq d_j(B)$, then we can paint the edge AB in colour i.

Proof. Suppose that $k_1 \neq i$ is the colour of the edge AB. We just need to check that $d_{k_1}(A) + d_{k_1}(B) \leq n - 1$, but $d_{k_1}(A) \leq n - 1 - d_i(A)$ and $d_{k_1}(B) \leq d_j(B) \leq d_i(A)$. Summing up, we see that the conclusion follows. □

Now, we can choose a vertex v and a colour c such that $d_c(v)$ is maximal among all choices of c and v. It's easy to repaint all edges with an endpoint at v in colour c using the above claim, because $d_c(v)$ increases by 1 with each repainting and other increase by at most 1. After all edges from v are of colour c, we can find a new maximal $d_{c'}(v)$ from the remaining set, and so on. In the end, structure of the graph is that we have n vertices v_1, v_2, …, v_n, such that all edges v_iv_j for $n \geq j \geq i+1$ are of colour c_i. Colour vertex v_i in colour c_i then. Some of the c_i may be equal.

Now, we try to construct a Hamiltonian path that doesn't have an edge coloured in c. It will be possible if there's few vertices of this colour and they are far from the beginning. Formally, suppose that $i_1 < i_2 < \ldots < i_s$ are the indices of vertices coloured in c and $j_1 < j_2 < \ldots < j_t$ are the indices of vertices coloured not in c. Consider a Hamiltonian path that goes from v_{i_r} to v_{j_r}, then to $v_{i_{r+1}}$, when there are no more i left, just traverse the remaining js. It is possible if $t \geq s$. And it will not contain edges of colour c if $i_r > j_r$ for all r. This means that for all p, among the first p vertices there will be k vertices of colour c and $\geq k$ vertices of not colour c. So, in order for this not to happen, we can say that for each colour there exists some p such that among the first p vertices there will be more vertices of colour c than of any other colour.

Now, sort the first such prefixes for every colour, $1 = p_1 < p_2 < \ldots < p_k$. We will prove that $p_i \geq 2^i - 1$ by induction. Base case is trivially true. First p_i vertices must contain ≥ 1 vertex of colour 1, $\lceil \frac{2^2-1+1}{2} \rceil = 2$ (more than a half of the first p_2) vertices of colour 2, $\lceil \frac{2^3-1+1}{2} \rceil = 4$ vertices of colour 3, and so on. We will have at least $1+2+\ldots+2^{i-2} = 2^{i-1}-1$ vertices of colours from 1 to $i-1$. And since it has more than a half of vertices coloured i, we will have a total of $2^i - 1$. Therefore, we have $n \geq p_k \geq 2^k - 1$. But the equality doesn't hold, because otherwise we would have a path without the colour k. Example for $k = 3$ is

$$3 \to 1 \to 3 \to 2 \to 3 \to 2 \to 3 \text{ with vertices coloured as } 1\ 2\ 2\ 3\ 3\ 3\ 3.$$

This proves that $n \geq 2^k$, and the maximum of k is $\lfloor \log_2(n) \rfloor$.

Construction: It's not hard to provide a construction for $\lfloor \log_2(n) \rfloor$ from what we have found above. Keep the structure the same, i.e. we have vertices v_1, v_2, …, v_n coloured in c_1, c_2, …, c_n and edge v_iv_j with $j > i$ coloured in c_i. Just colour first vertex in colour 1, next two vertices in colour 2, next 4 in colour 3, …, 2^{k-2} vertices in colour $k-1$, and finally, $n - 2^{k-1} + 1$ vertices in colour k. Inequality $n \geq 2^k$ ensures that there will be at least $2^{k-1}+1$ vertices of colour k. Suppose that we can make a Hamiltonian path without edges of the colour i for $i \leq k$. We know that no vertices of colour $\geq i$ precede vertices of colour i, as well as no vertices with colour $\leq i$ follow vertices of colour i. This means that there is at least 1 number $< i$ between every two is, and we can improve to two numbers if the length of the gap is ≥ 2. Therefore, if some gap is ≥ 2, then we have at least 2^{i-1} vertices of colours $< i$, but there are exactly $1 + 2 + \ldots + 2^{i-2} = 2^{i-1} - 1$ vertices of colours $< i$. We can also show that there will be at least 2^{i-1} vertices with colour $< i$ if one of the v_1 or v_n is not of colour i (take the one preceding the first appearing, or the one following the last appearing). Hence, our path looks like

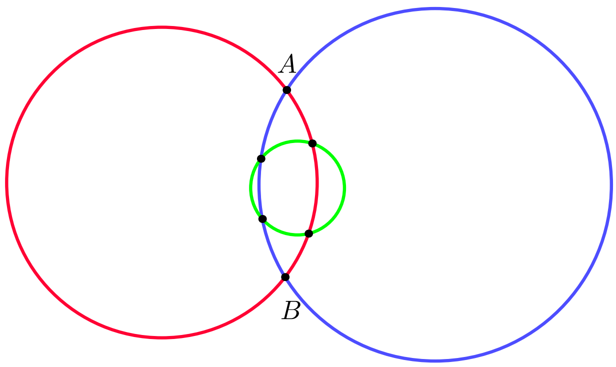
$$i \to< i \to i \to< i \to \ldots \to< i \to i.$$

If $i \neq k$ then we won't have colour k in this path, which is not possible. But if $i = k$ then there will be $2^{k-1} + 1 - 1$ numbers of colour $< k$, but we already established that there must be $2^{k-1} - 1$ of them. Contradiction, so every path contains edges of all colours.

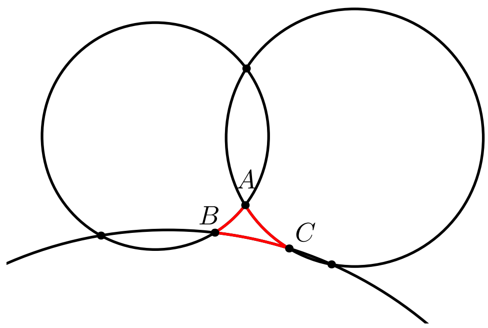
§4.19 ISL 2018 C7, proposed by Tejaswi Navilarekallu

Problem 19 (ISL 2018 C7)
Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice- once for each of the two circle that cross at that point. If the two colours agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

¶ Solution (Two clique graph and bounding the possible number of yellow points in regions)
We start with a simple observation: Any two circles intersect in an even number of yellow points. It's obviously true when no other circle intersects them. When we add one circle, since it's a cyclic and continuous path, it must intersect the border of the middle region even number of times (every time we "go in", we must escape it after), this means that the parities of the number of points inside the two inner arcs is the same (because their sum is even), this means, if both points had the same state (either both yellow or both non-yellow), then after adding another circle, they are still of the same state. Picture below shows the example where the added circle adds an even number of vertices to smaller arcs.



By experimenting, we see that among intersections of three circles, we must have one or three yellow points. More formally, if we have a curvilinear triangle formed by intersections of three circles (bounded by three circular edges), as the red one in the picture below, we must have an odd number of yellow points among its vertices.



Proof is similar to the previous observation's one. By repeating the argument from the previous observation's proof, we see that when we add a circle, it intersects this region's boundary at an even number of points. If there's an even number of points on the arc, we will say that it's even, and odd otherwise. Represent blue and red as 0 and 1, respectively. Then on the even arc the sum of numbers in the endpoints is odd, and on every odd arc this sum is even. But the parity of the total number of yellow points is the same as the sum of all written numbers (each yellow point gives an odd contribution, and non-yellow is even). This is just the sum of reversed parities of three arcs. In the beginning, this number is odd because all the edges are even. And every new circle gives an even contribution, because it changes the parity of an even number of edges.

Now, we are incentivised to consider a graph G in which the vertices are circles and two vertices are connected if the corresponding circles intersect in two yellow points. This graph has no anti-triangles, i.e. triples of vertices with no edges between them, because if there were any, then there would be a curvilinear triangle formed by them that has no yellow vertices, which is a contradiction. We can also check that this graph has no triples of edges u, v, w such that $u \sim v$, $u \sim w$ and $v \nsim w$, otherwise there would be a curvilinear triangle with two yellow points as vertices. Of course, now we can establish some nice properties of this graph.

First, it contains at most two connected components. Otherwise, by taking three vertices from different components, we could establish an anti-triangle. Consider one component and suppose that it has two vertices, u and v, that are not connected. Consider a path from u to v, and consider the first vertex w of this path that is not connected to u, it exists because v is such a vertex. Then, u is connected to the vertex that precedes w, w'. Then $w \sim w'$, $w \sim u$, but $u \nsim w'$, which is a contradiction. Therefore, our graph is either a clique (in which the conclusion of the problem is obvious) or it has two cliques.

Of course the graph with two cliques has too many edges for the original picture to not have a single region with all the vertices yellow. But to make a formal argument, we will still need one more claim:

Claim — Every region has an even number of non-yellow vertices.

Proof. This follows from the convenient notation that I established in the proof of the observation about intersections of three circles. Parity of the number of yellow vertices equals to the sum of reversed parities of edges. Every edge of the region is even (it's not intersected by any circle, because otherwise, it wouldn't be a separated region), so the parity of the total number of yellow vertices is equivalent to the number of edges mod 2. Thus, the total number of non-yellow vertices is the total number of vertices - total number of yellow vertices $\equiv$ total number of edges - total number of yellow vertices $\equiv 0 \pmod 2$. □

Thus, if we assume that every region has > 0 non-yellow vertices, then it has at least two non-yellow vertices. Note that the total number of non-yellow vertices is twice the number of non-edges in the graph G. The total number of regions can be found using the *Euler's Formula for Planar Graphs*, because the graph in the problem statement is planar (no edges intersect). $V - E + F = 2$, where V are the vertices, of which there are $2 \cdot \binom{2018}{2}$ of (2 for each intersection). Each vertex has degree 4, so $E = \frac{1}{2} \cdot 4 \cdot 2 \cdot \binom{2018}{2}$. In total, this gives F, the number of regions, equal to $2018(2018-1)+2$. Each region contributes to 2 non-yellow vertices, and each vertex is counted in 4 regions. So there's a total of at least $\frac{1}{2}\cdot(2018(2018-1)+2)$ non-yellow vertices in this graph. On the other hand, it's easy to bound the number of edges between the two cliques. If m is the largest number of vertices in one of the cliques, then, since there exists a circle with 2061 yellow points on it, its vertex is incident to at least 1031 other vertices in the graph G, so $m \geq 1032$. Thus, the number of edges between the cliques is $2m(2018-m)$, which attains its maximum, with a given constraint on m, at $m = 1032$. One can check that the value of the maximum is 2035104 and the minimum we have established before is 2035154, which is a clear contradiction.

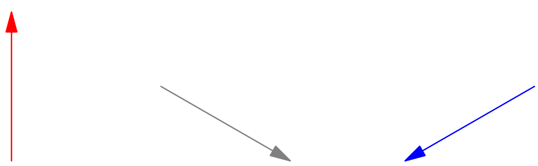
§4.20 USEMO 2019/5, proposed by Ankan Bhattacharya

Problem 20 (USEMO 2019/5)

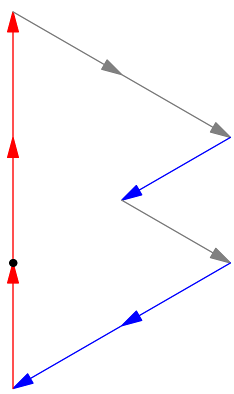
Let $\mathcal{P}$ be a regular polygon, and let $\mathcal{V}$ be its set of vertices. Each point in $\mathcal{V}$ is colored red, white, or blue. A subset of $\mathcal{V}$ is patriotic if it contains an equal number of points of each color, and a side of $\mathcal{P}$ is dazzling if its endpoints are of different colors.

Suppose that $\mathcal{V}$ is patriotic and the number of dazzling edges of $\mathcal{P}$ is even. Prove that there exists a line, not passing through any point in $\mathcal{V}$, dividing $\mathcal{V}$ into two nonempty patriotic subsets.

¶ Solution (Interpreting colours as vectors and finding an easier-to-work-with condition) This problem is the eight wonder of the world. The following motivation is far-fetched, yet with only a bit of luck it does lead to a solution. Of course, the first thing that comes to mind after reading the statement is to set up a counter. We start at $(0, 0, 0)$, and increase the corresponding coordinate by one with each encounter of r, w or b. We know that before coming to the vertex we started from, we had (n, n, n), and we want to show that this moment had occurred before that too. We want to restate this counter in the way that doesn't depend which point to start from. This suggests trying vectors, because then we would have a big cycle and will want to find a smaller one inside. Now, we need to choose right vectors to obtain the contradiction. These vectors must add up to a zero vector to form a cycle, and no two of them supposed to be linearly dependent (i.e. it's impossible to choose integer coefficients a, b so that $a \cdot \vec{v} + b \cdot \vec{u} = 0$). Linear independence is needed so that if their sum is zero, then we could make a conclusion that there's an equal number of each of them. A possible option that is the easiest to work with in terms of the resulting shape, is taking the three vectors below where we replace white by grey (formally, the vectors that correspond to the roots of $x^3 - 1 = 0$):



Now, we will construct a path that corresponds to the polygon in the problem statement. Picture below corresponds to $RRWWBWBBR$ colouring:



All we need to check is that it's self intersecting, then we will find a sub-loop that we can choose for the cut. Now, every dazzling edge of the original polygon is a vertex of the new polygon (points where we don't change the colour form the sides of it). All we need to show is that when the polygon is not self intersecting, it has an odd number of vertices. In this types of problems it's usually important to consider the sum of angles of the polygon (chosen in the order of traversing edges), which is always invariant and equal to $180^\circ(n-2)$. In this case all the possible angles are 60° and reflexes of them - 300°. So, we get that $300^\circ a + 60^\circ b = 180^\circ(n-2) = 180(a+b-2)$, from which $a-b=3$, and so, $n = a+b = 2b+3$, which is odd, contradiction.

§4.21 ISL 2019 C7, proposed by Chzech Republic

Problem 21 (ISL 2019 C7)

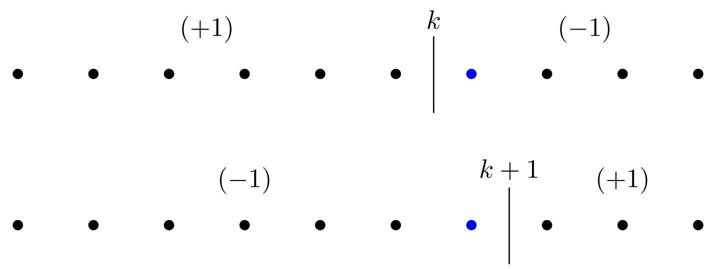
There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game. In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps:

- Bob chooses an integer k with $1 \leq k \leq 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$.
- Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.

Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

¶ Solution (Induction and forcing to pick the same side by examining cancellations) We will call an operation described in the problem statement for some k as a k-cut.

Strategy for Bob: Suppose that n is the minimal number of pebbles such that Bob doesn't have a winning strategy for some distribution of pebbles. A very useful observation here is that if we have a k-cut and a $k+1$-cut with left and right sides chosen at these steps, respectively, then we added 1 to each number and subtracted 1 from each box except the B_{k+1} from which 1 was subtracted two times, leaving us with 2 less stones than we had before.



One might want to say that we cannot make such a step on behalf of Alice, because the number of stones would decrease, and we can obviously win as Bob then. That doesn't really work, because while it surely decreased from the state we had without the combination of these two steps, it might have not decreased from the initial number. But, getting rid of two stones is not a very sane idea, because Bob can still perform the same actions as before, but Alice has less stones available. Formal argument is quite tedious, though.

Suppose that Bob's first cut is at k and that Alice chose right. We perform the following process on array of size k initially filled with zeros, starting in its kth number and moving in the left direction. If we are at ith number, make an i cut as Bob. If we are moving to the left and Alice chooses the right part, then add 1 to ith number and move one number to the left. If we were moving to the left, and Alice chooses the part to the left, then we change the direction and move one number to the right to $i+1$, then add 1 to this number. If we were moving to the right, then just reflect the same process in other direction. But each time, write numbers modulo 2, i.e. we only have 1s and 0s in the array.

We can visualize it as going left, and changing zeros to ones, then we retract a one and, essentially, move back; then going right, changing ones to zeros and, eventually, retracting last step again, etc. This is convenient, because 1 at ith position corresponds to Alice choosing a group to the right of the i-cut. 0 at ith position corresponds to Alice choosing a group to the left of the $(i-1)$-cut. By what we have noted above, a "cancellation" of the step at number i correspond to a cancellation with loss of two points in the box B_i.

Note that this process doesn't go back to kth number. Otherwise, there will be a moment when every number in the array is 0, meaning that we only had cancellations (because we had a one in the beginning), which means that the total amount is $n-2c$, where c is the number of cancellations, but we assumed that n is the minimal for which Bob is losing, so we are winning as Bob for the new distribution. Therefore, we are always within the array. Thus, the process can only finish if we reach the first number and Alice chooses part to the right of the cut. Suppose it's not the case, then we see that we should be cancelling infinitely often, because we are limited on both edges. Note that the additions to the sum $B_1 + B_2 + \ldots + B_k$ can only come when a 1 appears and there's at most one for each of the indices (because they appear, then cancel, then appear again). Thus, there's a finite number of ways to possibly add something to the boxes, but infinitely many cancellations happening, so, eventually, the sum is negative, which is a contradiction. This means that at some point, we will come to the first number and finish. Not hard to see that in this case every number is 1, even if we had some cancellations. If the ith is 1, then we have subtracted from boxes with indices from 1 to i, which means that we have subtracted from box B_i exactly $k-i+1$ times before starting adding to it (means that there will be a point at which we subtracted from it $k-i+1$ times without additions, and, perhaps, with some cancellations), and since the number of stones in this box is still ≥ 1, then initially it had at least $k-i+2$ stone.

The same approach can be reflected to estimate the sum of boxes to the right of the box with index $k+1$, for which Alice chooses part to the left of the cut if we perform it in the beginning. We just need to split our boxes into two parts $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$ such that Alice chooses right for the k-cut, and left for the $(k+1)$-cut, or vice versa, then we will be able to lower bound the sum. It's easy to control what Alice chooses at first step for k-cuts with $k \neq 30$, because one of the parts has more elements and we must choose it to add to, otherwise we will decrease the sum in the first step and Bob wins by induction. It's only left to understand what's happening to the 30-cut. WLOG, Alice chose part to the right of the cut (We can think of it in the way that at least one of the parts, whether to the left or to the right, contains at least that many stones). We know that for 31-cut, Alice must choose part to the left, and we have found the desired partition.

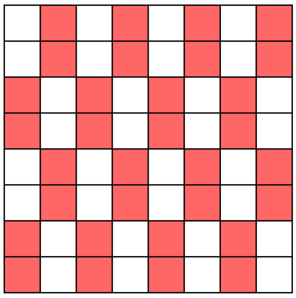
Now, it's easy to lower bound the sum as $(30-30+2)+(30-29+2)+\ldots+(30-1+2)+(29-29+2)+\ldots+(29-1+2)+1 = \frac{31\cdot 32}{2} - 1 + \frac{30\cdot 31}{2} - 1 + 1 = 960$, where the first is the sum from B_1 to B_{30}, the second is the sum from B_{60} to B_{32}, and the last one is from B_{31}.

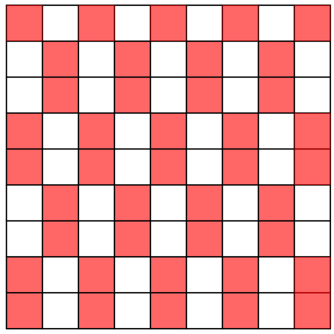
Strategy for Alice: From the above argument we see that the only (up to a reflection) example that achieves 960 is 31, 30, …, 2, 1, 2, …, 30. For k-cut with $k \leq 30$, Alice chooses the right side. For $k \geq 31$, Alice chooses the left side. If Bob performs the same cut two times then we can cancel it by choosing opposite sides. Easy to check that if each cut applied at most one time, then every number stays positive.

§4.22 ELMO 2017/3, proposed by Mihir Singhal and Michael Kural

Problem 22 (ELMO 2017/3)
Nicκy is drawing kappas in the cells of a square grid. However, he does not want to draw kappas in three consecutive cells (horizontally, vertically, or diagonally). Find all real numbers $d > 0$ such that for every positive integer n, nicκy can label at least dn^2 cells of an $n \times n$ square.

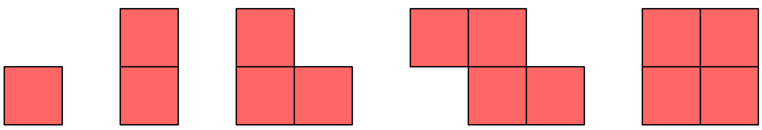
¶ Solution (Enlarging the connected components) **Example:** Note that we can place non-neighbouring dominoes made of kappas on the board, and they will not form three cells in the diagonal, column or row. If we assume that they exist, then, by trying to map each of the three cells to its domino neighbour, we can come to the conclusion that some two dominoes must share an edge. So we just need to place a lot of not connected dominoes. We can easily easily cover $n \times n$ as shown below for 8×8 and 9×9:



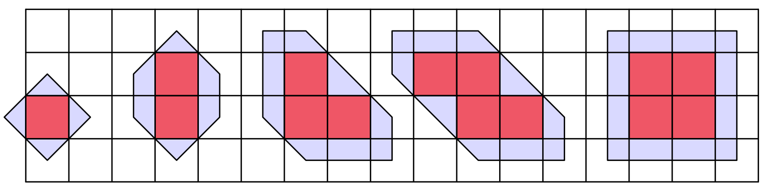


For the n - even case, kappas take up exactly half of the area. For the odd case, we can repaint all uncoloured cells in red and it will satisfy the condition, so we may assume that kappas take up at least half too. This shows that all $d \leq \frac{1}{2}$ are possible.

Bound: Note that our three-cell restrictions limit the connected components (with two cells considered neighbouring if they share a side) to the following five shapes (easy to see that they are the only ones if we just start connecting cells and restricting the third cells in the row, column, diagonal before we cannot mark any other cell).



Note that because of the three cells restriction, there will be a lot of free space in between the shapes. We can formalize it by trying to extend every shape such that they still don't intersect. To prove that $d \leq \frac{1}{2}$ must hold, we will increase the area of each shape in at least two times. By playing with possible positions on the board, we see that the following extensions works:



Explanation behind this enlarging is that the new borders contain points, for which the perpendicular distance from them to any other cell that can be placed on the board without violating the three cells rule, is at least the distance from them to cells of the figure that we enlarged. They trivially don't intersect, except for boundaries, because of that inequality.

One can check that the ratio of area of a new figure above to the one of the old figure is equal to two, except for the 2×2 square, for which it's $\frac{9}{4}$. Enlarged figures don't necessarily lie inside the square; they might go beyond the grid, but they will still be contained in an enlarged square with the additional length of 0.5 added to its edges. The area of such square is $(n+1)^2$. If we suppose that $d > \frac{1}{2}$ is possible, then $2 \cdot dn^2 \leq (n+1)^2$, but the LHS is a quadratic polynomial with a leading coefficient of > 1, while the RHS is a quadratic polynomial with leading coefficient equal to 1, which is a contradiction for all large enough n.

> **Remark.** The method of enlargement to make the packing more dense appears quite often. A recent example was USAJMO 2021/3. And the problem itself was really inspired by All-Russian MO 2013 grade 9/8 (Russian website, because the official thread on AoPS didn't have a solution using enlargement).

§4.23 ISL 1999 C6

Problem 23 (ISL 1999 C6)
Suppose that every integer has been given one of the colours red, blue, green or yellow. Let x and y be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x+y$ or $x-y$.

¶ First solution (Discrete Fourier Transform) Even if you have no idea what *Discrete Fourier Transform* is, this problem is a classical first example to start with. This is a type of problem in which we colour some objects (usually, represented as numbers) and want to prove that some two objects of the same colour satisfy some relation with each other. The problems of this type can be rewritten using indicator functions. We do that in this problem too. If we assume the opposite to the condition, then for every n, there are numbers of each of the four colours among n, $n+|x|$, $n+|y|$, $n+|x|+|y|$ (because if we have a difference of k, then we have a difference of $-k$). Suppose that f is an indicator function of some colour, i.e. $f(m)=1$ if m is of this colour, and $f(m)=0$ otherwise. The condition implies that

$$f(n)+f(n+|x|)+f(n+|y|)+f(n+|x|+|y|)=1 \ (\star)$$

for every integer n.

A segment of numbers $f(n)$, $f(n+1)$, ..., $f(n+|x|+|y|)$ determines the whole sequence in both sides, and this segment trivially repeats, because the sequence is made of 0 and 1. This proves the existence of the period ℓ of the function f, which is going to be equal to the difference between indices of the first terms of the sequence of $|x|+|y|$ elements that repeat.

For a periodic function, we can transform the equation of the form "sum of fs equals something" into a new equation. We will be doing this using the DFT, short for Discrete Fourier Transform.

All the theory we need is that the discrete Fourier transform of the periodic function f with period ℓ is denoted as $\hat{f}(n)$ is defined as

$$\hat{f}(n)=\frac{1}{\ell}\sum_{k=0}^{\ell-1} f(k)e^{-\frac{2\pi i n k}{\ell}}.$$

We can see that this function is also periodic with period ℓ. We can also invert it to get

$$f(n)=\sum_{k=0}^{\ell-1}\hat{f}(k)e^{\frac{2\pi i n k}{\ell}}.$$

The proof is to substitute the definition of $\hat{f}(n)$ and change the order of summation so that the sum is just

$$\frac{1}{\ell}\sum_{j=0}^{\ell-1} f(j)\sum_{k=0}^{\ell-1} e^{\frac{2\pi k(j-n)i}{\ell}},$$

which has only one non-zero summand for $j=n$, that will result into $f(n)$. The last identity we use is the obvious $f\widehat{(n+a)}=\hat{f}(n)e^{-\frac{2\pi i a n}{\ell}}$.

Now, we will apply the discrete Fourier transform to $\star$, but we first need to have a function of n in the RHS. For this reason, suppose that $g(n) = 1$ for all integer n. We get $\hat{f}(n)+\hat{f}(n+|x|)+\hat{f}(n+|y|)+\hat{f}(n+|x|+|y|) = \hat{g}(n)$, because DFT is a linear combination of fs. Which is the same as

$$\hat{f}(n) + \hat{f}(n)e^{-\frac{2\pi i|x|n}{\ell}} + \hat{f}(n)e^{-\frac{2\pi i|y|n}{\ell}} + \hat{f}(n)e^{-\frac{2\pi i(|x|+|y|)n}{\ell}} = \hat{g}(n).$$

We need to calculate $\hat{g}(n)$. If $n \neq 0$, then

$$\hat{g}(n) = \frac{1}{\ell}\sum_{k=0}^{\ell-1} e^{-\frac{2\pi ink}{\ell}} = \frac{e^{-2\pi i}-1}{e^{-\frac{2\pi ink}{l}}-1} = 0.$$

We can check that $\hat{g}(0) = 1$. The function satisfying this relation is called Dirac delta function and is denoted as $\delta(n)$.

Therefore,

$$\hat{f}(n)(1+e^{-\frac{2\pi i|y|n}{\ell}})(1+e^{-\frac{2\pi i|x|n}{\ell}}) = \delta(n).$$

If $e^{-\frac{2\pi i|x|n}{\ell}} = -1$, then $e^{-\frac{2\pi i|y||x|n}{\ell}} = -1$ (using that they are both odd). Same, if $e^{-\frac{2\pi i|y|n}{\ell}} = -1$. In the case when none of the above equals to -1, $\hat{f}(n) = 0$ for $n \neq 0$, and $\hat{f}(0)\cdot 2\cdot 2 = \delta(0) = 1 \Rightarrow \hat{f}(0) = \frac{1}{4}$.

Now, to exploit the $e^{-\frac{2\pi i|y||x|n}{\ell}} = -1$, we calculate

$$f(|x||y|) = \hat{f}(0) - \hat{f}(a_1) - \hat{f}(a_2) - \ldots - \hat{f}(a_t),$$

where $a_i \neq 0$ are the numbers from 0 to $l-1$ such that $\hat{f}(a_i) \neq 0$, we know that $e^{-\frac{2\pi i|y||x|a_i}{\ell}} = -1$ for those. We can also calculate

$$f(2|x||y|) = \hat{f}(0) + \hat{f}(a_1) + \hat{f}(a_2) + \ldots + \hat{f}(a_t),$$

same approach, except $e^{-\frac{2\pi i2|y||x|a_i}{\ell}} = (-1)^2 = 1$.

Thus, $f(|x||y|) + f(2|x||y|) = 2\hat{f}(0) = \frac{1}{2}$, but all f are integers, contradiction.

¶ Second solution (Grid colouring and periodicity) Consider an infinite grid in which the cell (a, b) has the same colour as the number $ax + by$. The condition tells us that every 2×2 square has cells of all 4 colours.

Claim — Either the rows periodic with period 2 or the columns.

Proof. If some row is not periodic with period 2, then it must have three consecutive cells of different colours - 1, 2, 3. Then, the cell directly under the cell coloured in 2 must be coloured in 4 (otherwise, some two cells that share a corner coincide). Then, we can say that the three cells directly under the 123 must be 341, we can then repeat the same argument and derive that the three cells under 341 must be 123 again. Repeating this in both upwards and downwards directions, we see that at least one column is periodic with period two. After that it's easy to see that columns neighbouring the periodic ones must be periodic themselves. This means that all columns are periodic with period two. □

WLOG, all rows are periodic with period 2. This means that differences of x coordinates of cells of the same colour is even. But (a, b) and $(a + y, b - x)$ must have the same colour (they correspond to the same number), which is a contradiction due to odd y condition.

> **Remark.** The existence of an easy purely combinatoric solution makes the problem a bit uninteresting (Even though, I believe that finding this exact neat argument is not trivial). I only added the problem for the DFT part, which I think is extremely nice. Another trivial-by-DFT combo with a less obvious non-DFT solution is 2009 China TST Quiz 2/2.

§4.24 ISL 2018 C6, proposed by Serbia

Problem 24 (ISL 2018 C6)
Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

- If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.
- If no such pair exists, we write two times the number 0.

Prove that, no matter how we make the choices in the first type of operation, there will be only finitely many times when we perform the second type operation.

¶ Solution (Stabilizing the configuration using Chip-firing lemma and permuting moves) If $\gcd(a, b) = d > 1$, then every number on the board is divisible by d, so let's divide by d and assume $\gcd(a, b) = 1$. We first assume that for every integer N, we will have to make at least N moves of type 2. Consider an infinite directed graph that has vertices numbered from 0 to infinity, where x is connected to $x + a$ and $x + b$. Place chips in the vertices, such that each vertex i contains as many chips as the number of times it appears on the board. Whenever we choose x in the problem statement, the number of chips at x must be at least two, and then we "fire" some two chips to $x + a$, $x + b$ from x.

The key motivation in this problem is that, by Chip-Firing Lemma on infinite graphs, whatever the moves we make in the configuration where we have to fire in each of the neighbours (not considering the direction), then after adding the last two zeros, we will eventually land on a configuration where "firing" is impossible. We cannot use this argument, and the problem tells us this is incorrect in the case of an infinite directed graph, but it gives the motivation to explore stable configurations and assume that no matter how many times we are adding a zero, we will arrive to a stable configuration. By experimenting with smaller cases, we state the result:

Claim — Suppose that we reached a stable configuration after applying the type 2 operation k times. Then, if we started at $2k$ zeros initially, whatever the sequence of moves of type 1 we perform, we will land on the same configuration.

Proof. We first prove the result about the uniqueness of a stable configuration if we don't perform any type 2 operations. Suppose that we can land on a stable configuration with some sequence m_1, m_2, …, m_k of type 1. Suppose that there's also a sequence n_1, n_2, …, n_s of type 1 moves from the initial configuration that leads to a stable configuration. If n_1 fires from vertex n, then we see that we must fire from this vertex during some m_i. We can assume that it's the first time we did it, and we can further assume that it was m_1 (we can perform it because we did it as n_1 for the same configuration). It doesn't change the final configuration, and we can still make all the moves. The only moves we might have troubles at are the ones where we fire from n, but all of them were made after the original m_i anyway. Thus, we can truncate two sequences from the beginning and do the same. Moves made are the same and $k = s$, otherwise we will have moves to make to reach the stable configuration from the stable configuration, which is a contradiction.

Now, we are ready to prove the claim. Perform the operations on a graph that has $2k$ chips in the zero vertex, and every time we need to perform type 2 operation, fire from the zero vertex. Then, as the stable configuration is unique in this case, because we don't perform type 2 operations, it was unique in the beginning, because we didn't change the final configuration by modifying the process with adding $2k$ chips in the zero vertex. □

Consider some big N such that we made the $N + 1$st type 2 move. Then, we had a stable configuration with N type 2 moves made. We know that every sufficiently large integer can be represented as $ax + by$ (formally, quote *Chicken McNugget Theorem*). Thus, we can make $ax + by$ using some big number of zeros (if we can make $ax + by$ from $2z$ zeros, then we can obtain $a(x + 1) + bx$ from $2z + 2z$ zeros, by obtaining $ax + by$ two times, and only after adding a). Thus, from a very big number of zeros initially, we can make all sufficiently large numbers. Just make x, $x + 1$, $\ldots$, $x + a - 1$ and x, $x + 1$ $\ldots$, $x + b - 1$, then we can always take the two smallest numbers and shift these two sequences by 1 to the right. We can repeat that and never land on a stable configuration, which is a contradiction, because if we landed on a stable configuration using $2N$

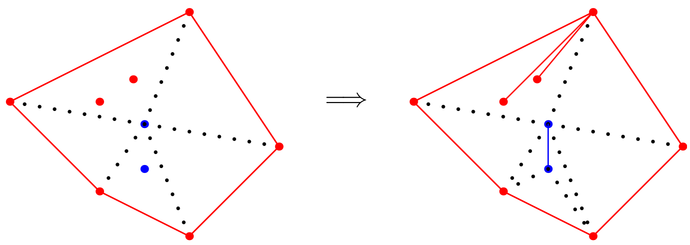
> **Remark.** Problems in which Chip Firing actually applies: ISL 2022 C4, 2018 China TST 2/3.

§4.25 Kazakhstan MO 2023/4, proposed by Adilet Zauytkhan

Problem 25 (Kazakhstan MO grade 11 2023/4)
Let G be a graph whose vertices are 2000 points in the plane, no three of which are collinear, that are coloured in red and blue. Given that there exist 100 red points that form a convex polygon with every other point of G lying inside of it. Prove that one can connect some points of the same colour such that segments connecting vertices of different colours don't intersect, and one can move from a vertex to any vertex of the same colour using these segments.

¶ Solution (Triangulations with non-monochromatic triangles) Having non-intersecting segments suggests to use triangulations in which we draw segments corresponding to sides of the triangles of triangulation that connect vertices of the same colour. We are then interested in triangles of triangulation being non-monochromatic, because if we have a monochromatic triangle of red points and there's a blue point inside of it, then it's impossible to connect this blue point to the points outside of the red triangle without intersecting its sides. If every triangle contains vertices of different colours, then we can connect the point inside of it to a suitable coloured vertex of the triangle to maintain the connectedness.

This suggests the following sequential algorithm: Consider a triangle with 2 red vertices and 1 blue vertex. If it only contains red vertices inside of it, just connect all of them to one of the red vertices and consider a different triangle. If there's a vertex of blue colour then split the triangle into three triangles, 2 triangles with 2 blue vertices and 1 triangle with 2 red vertices. Same for 2 blue 1 red triangles with flipped colours. We do this as long as we have points inside the triangles. One step of this algorithm is shown below:



We need to perform the first step when we triangulate the polygon into non-monochromatic triangles. It's already done in the above picture; we take a blue point inside the full red convex hull, and draw all triangles with this point as a vertex and two neighbouring red points on the convex hull as two other vertices. We see that in the beginning of the step of the algorithm, all the points that were involved in steps before form a graph that is desired - with no intersecting edges and two connected components of different colours. When we make another step of the algorithm, we only add vertices to connected components, and we only add edges between points of the same colour. At some step, we will exhaust all the vertices.

Part III. Geometry

§5 Problems

Problem 1. Quadrilateral $ABCD$ is inscribed in a circle with center O. Rays BA and CD intersect in a point P, rays AD and BC intersect in a point Q. Suppose that I and J are the centers of circles inscribed in triangles ΔADP and ΔCPQ, respectively. It turned out that $PD = DQ$. Prove that $OI = OJ$.

Hints: 97 775 705 533 226

Problem 2 (ISL 2022 G7). Two triangles ABC, $A'B'C'$ have the same orthocenter H and the same circumcircle with center O. Letting PQR be the triangle formed by AA', BB', CC', prove that the circumcenter of PQR lies on OH.

Hints: 524 621 14 307 1007 758 311 839

Hints: 825 634 589 765 635 797 250 975

Problem 3 (All-Russian MO grade 11 2021/4). In triangle ABC angle bisectors AA_1 and CC_1 intersect at I. Line through B parallel to AC intersects rays AA_1 and CC_1 at points A_2 and C_2 respectively. Let O_a and O_c be the circumcenters of triangles AC_1C_2 and CA_1A_2 respectively. Prove that $\angle O_aBO_c = \angle AIC$.

Hints: 907 433 215 560

Hints: 645 540 460 367

Problem 4 (IOM 2019/3). In a non-equilateral triangle ABC point I is the incenter and point O is the circumcenter. A line s through I is perpendicular to IO. Line ℓ symmetric to like BC with respect to s meets the segments AB and AC at points K and L, respectively (K and L are different from A). Prove that the circumcenter of triangle AKL lies on the line IO.

Hints: 243 921 823 255 699 444 871

Hints: 764 1020 732 417 450 495

Hints: 518 218 405 282 617 734

Problem 5. Given $n \geq 4$ convex polygons in the plane. It's know that their intersection (can be an empty set) can be covered with a strip of width 1. Prove that we can choose 4 of these polygons such that their intersection (can be an empty set) can be covered with a strip of width 1.

Hints: 712 234 203 145 936 468

Problem 6 (IMO 2019/6). Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

Hints: 908 563 146 149 76 26 572 383

Hints: 604 720 258 609 561 478

Hints: 443 286 456 926 265

Problem 7 (USA TST 2023/2). Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

Hints: 654 189 600 618

Hints: 529 902 365 384 439

Problem 8. Given a quadrilateral $ABCD$ inscribed in a circle ω. Its diagonals AC and BD intersect in point P. Suppose that ω_1 and ω_2 are the circles inscribed in angles $\angle APD$ and $\angle BPC$, respectively, that touch ω internally. Prove that one of the common external tangents of w_1 and w_2 is parallel to CD.

Hints: 379 310 894 403 891 674 978

Hints: 34 962 335 690 327

Problem 9 (USA TSTST 2020/6). Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that the lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

Hints: 155 280 101 898 932 178

Hints: 96 33 454 735

Problem 10. Vertices of an equilateral triangle Δ_1 lie on three distinct internal angle bisectors of the triangle Δ_2. Prove that the center of Δ_1 lies on the line connecting incenter and the circumcenter of Δ_2.

Hints: 440 416 516 324 585 853 493 931 111 692

Hints: 786 602 706

Problem 11 (IMO 2021/3). Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

Hints: 776 500 531 24 114 395 884 683 598 748

Hints: 973 658 773 259 580 288 418

Problem 12 (USA TST 2016/6). Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

Hints: 284 913 320 177 611 131 321

Hints: 521 778 89 278 512 437

Problem 13 (Kvant M2717). In an acute triangle ABC the heights AD, BE and CF intersecting at H. Let O be the circumcenter of the triangle ABC. The tangents to the circle (ABC) drawn at B and C intersect at T. Let K and L be symmetric to O with respect to AB and AC respectively. The circles (DFK) and (DEL) intersect at a point P different from D. Prove that P, D and T lie on the same line.

Hints: 173 552 806 426 802 614 649 135 536

Problem 14 (Sharygin Final Round 2022 grade 10/4). Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D$. Prove that the midpoint of BD lies on the common internal tangent to the incircles of triangles ABC and ACD.

Hints: 372 739 376 301 161 918 844 9 337

Problem 15. Given an integer $n \geq 3$. We call a 3×3 square *good* if it's central cell is of different colour than all the other 8 cells of the square. Colour some infinite grid (side of one cell is 1) in black and white. Suppose that some $a \times b$ rectangle, with sides not necessarily parallel to lines of the grid, contains at least $n^2 - n$ good squares. Find the minimum value of $a + b$, given that a and b are positive integers.

Hints: 232 768 641 770 887 253 582

Problem 16 (ISL 2021 G8). Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

Hints: 640 565 950 530 557 556 570 968 67

Hints: 522 144 127 689 273 510 757 970 382 794 800 652 791

Problem 17 (Kazakhstan MO 2021 grade 11/3). Point M is chosen inside a triangle ABC such that $\max(\angle MAB, \angle MBC, \angle MCA) = \angle MCA$. Prove that $\sin\angle MAB + \sin\angle MBC \leq 1$.

Hints: 446 317 483 971 160 147 54 87 828 128 728 345 991

Problem 18 (RMM 2020/5). A lattice point in the Cartesian plane is a point whose coordinates are both integers. A lattice polygon is a polygon all of whose vertices are lattice points.

Let Γ be a convex lattice polygon. Prove that Γ is contained in a convex lattice polygon Ω such that the vertices of Γ all lie on the boundary of Ω, and exactly one vertex of Ω is not a vertex of Γ.

Hints: 47 632 415

Hints: 506 70 118 6 141 514 49

Problem 19 (USA TSTST 2023/6). Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of PQ lies on the Euler line of ABC.

Hints: 293 668 670 792 462 25 175 414 535 862 805 363

Hints: 875 268 545 904 702 445

Problem 20 (IMO 2011/6). Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

Hints: 620 3 903 16 579 222 650 319 667 920 719

Hints: 366 961 868 900 169 829 590 464 967

Problem 21 (USEMO 2021/3). Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let O_1 and H_1 denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let O_2 and H_2 denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines O_1O_2 and H_1H_2 are either parallel or coincide.

Hints: 605 603 8 195 488 352 171 325 927

Hints: 198 636 18 642 532 350 596 573 271 593 129 377

Problem 22 (RMM 2019 G5). A quadrilateral $ABCD$ is circumscribed about a circle with center I. A point $P \neq I$ is chosen inside $ABCD$ so that the triangles PAB, PBC, PCD and PDA have equal perimeters. A circle Γ centered at P meets the rays PA, PB, PC, and PD at A_1, B_1, C_1, and D_1, respectively. Prove that the lines PI, A_1C_1, and B_1D_1 are concurrent.

Hints: 263 612 360 695 520 473 647 1003 956 810 406

Problem 23 (China TST 2023 1/5). Let $\triangle ABC$ be a triangle, and let $P_1, \cdots, P_n$ be points inside where no three given points are collinear. Prove that we can partition $\triangle ABC$ into $2n+1$ triangles such that their vertices are among $A, B, C, P_1, \cdots, P_n$, and at least $n + \sqrt{n} + 1$ of them contain at least one of A, B, C.

Hints: 504 398 20 19 631 804 251 938

Problem 24. Suppose that $\triangle ABC$ has a circumcenter O. External tangents of incircles of triangles $\triangle BAO$ and $\triangle CAO$ intersect BC at points X and Y. Prove that $\angle XAC = \angle YAB$.

Hints: 156 204 130 750 747 370 64 244 972 847 546 59 181 966 106

Problem 25 (All-Russian MO grade 11 2022/8). From each vertex of triangle ABC we draw two rays, red and blue, symmetric about the angle bisector of the corresponding angle. The circumcircles of triangles formed by the intersection of rays of the same color. Prove that if the circumcircle of triangle ABC touches one of these circles then it also touches to the other one.

Hints: 555 11 400 471 697

Hints: 470 477 578 392 174 408 143 807 833

Hints: 95 627 664 13 252

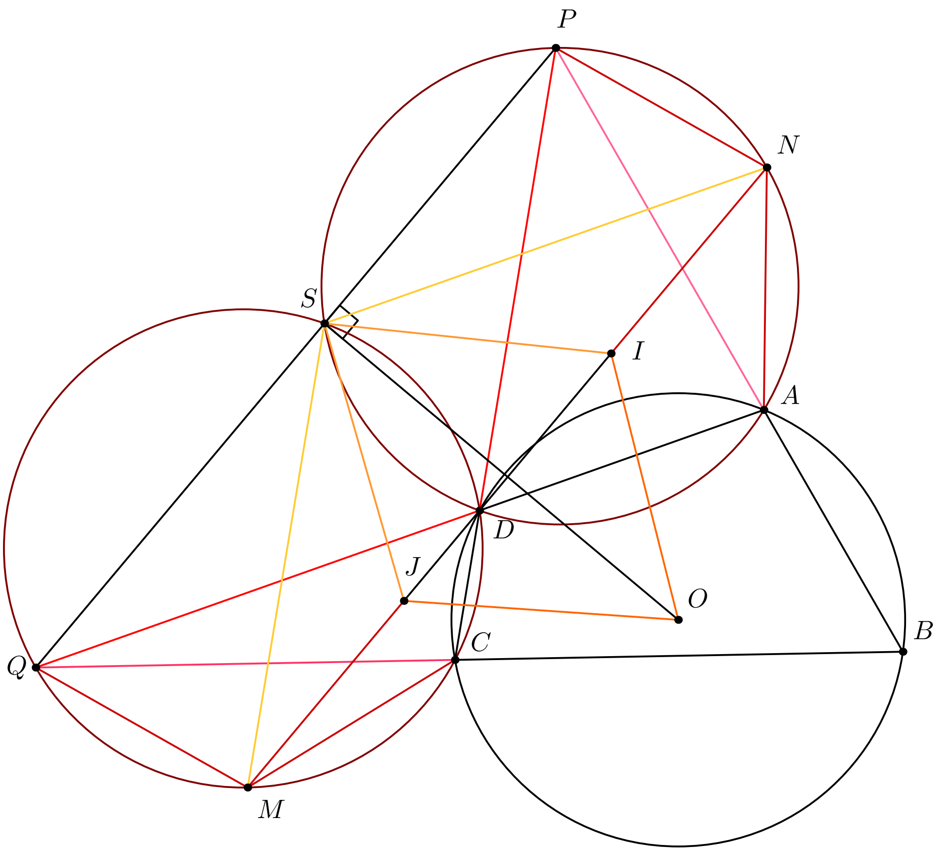
§6 Solutions

§6.1 Problem 1

Problem 1

Quadrilateral $ABCD$ is inscribed in a circle with center O. Rays BA and CD intersect in a point P, rays AD and BC intersect in a point Q. Suppose that I and J are the centers of circles inscribed in triangles ΔADP and ΔCPQ, respectively. It turned out that $PD = DQ$. Prove that $OI = OJ$.

¶ Solution (Rotation and incenter-excenter lemma)



We start with dealing with the $PD = DQ$ condition. Note that $\sin\angle PDA = \sin\angle QDC$ and $\sin\angle DAP = \sin\angle DCQ$. Therefore, by the *Law of Sines*,

$$\frac{DP}{PA} = \frac{\sin\angle DAP}{\sin\angle PDA} = \frac{\sin\angle QDC}{\sin\angle DCQ} = \frac{DQ}{QC},$$

so $QC = PA$.

Now, it seems useful and rather standard to consider the rotation that maps PA to QC centered at the Miquel point of $ABCD$ - call it S. This rotation takes the midpoint of arc PA not containing D to the midpoint of arc QC not containing D. Call these midpoints N and M, respectively. The last fact is that $\Delta ANP = \Delta CMQ$, because they have the same angles and $AP = CQ$. We now see that $SN = SM$.

The reason why we introduced midpoints of arcs is to deal with incenters using the *Incenter-Excenter Lemma*. The lemma tells us that $IN = NP = CM = JM$, so we can say that $\Delta SNI = \Delta SMJ$, which shows that $SI = SJ$.

We are now left to show that $OS \perp MN$, this follows from the fact that $OS \perp IJ$, which is a well know consequence of *Brokard's theorem*. Combined with $PQ \parallel IJ$, because IJ is an external angle bisector in isosceles triangle and is parallel to the base, we reach the disered conclusion because OS is a perpendicular bisector of IJ. Hence, $OI = OJ$.

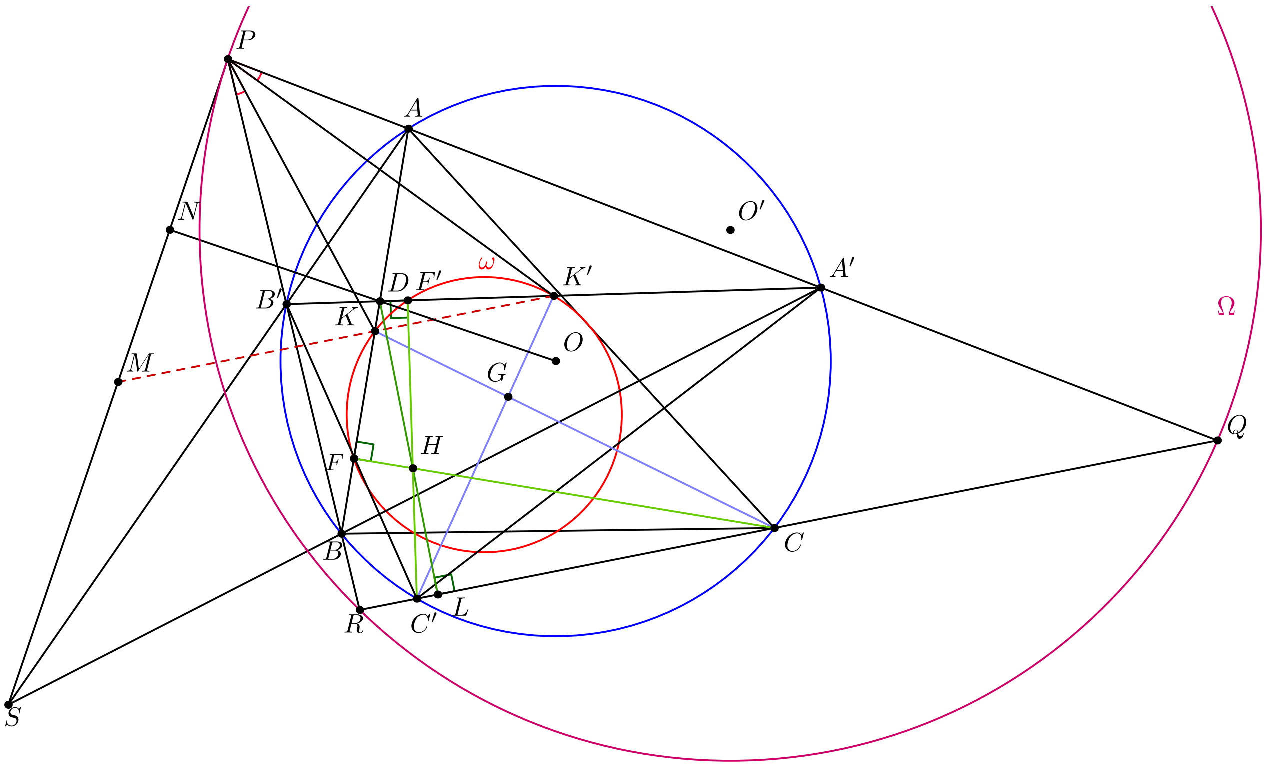
Remark. The technique is routine. Another closely related problem is IZhO 2018/2.

§6.2 ISL 2022 G7, proposed by Denmark, originally discovered by Makhamadkhan Ishmatov

Problem 2 (ISL 2022 G7)

Two triangles ABC, $A'B'C'$ have the same orthocenter H and the same circumcircle with center O. Let Ω be the circumcircle of the triangle determined by the lines AA', BB', CC'. Prove that the circumcenter of Ω lies on OH.

¶ First solution (Coaxiality and finding the radical axis explicitly)



ΔABC and $\Delta A'B'C'$ have the same centroid, because they have the same orthocenter and circumcenter and $HG : GO = 2 : 1$. We know that, since nine-point circle of the triangle is the image of its circumcircle after the homothety centered at the centroid with factor $-\frac{1}{2}$, ΔABC and $\Delta A'B'C'$ have the same nine-point circle denoted as ω. Suppose that ΔPQR is the triangle formed by AA', BB', CC', as shown in the picture. Let $AB' \cap A'B = S$.

Now, we need a leap of faith here. We need to show that the centers of (PQR), (ABC) and ω are collinear because the second two form the *Euler's line*. We will instead try to show that (PQR), (ABC) and ω are coaxial. It turns out that the midpoint of the segment SP, point M, has equal power with respect to (PQR), (ABC) and ω.

It's well known that KK' passes through M, because they form a *Gauss line* of $AB'BA'$. It's also well known that for cyclic $(AB'BA')$, $MP^2 = MK \cdot MK'$. This is the same as ISL 2009 G4.

S lies on the polar of P with respect to (ABC), which is well-known to imply that circle with diameter SP is orthogonal to (ABC), which is the same as $\text{Pow}(M, (ABC)) = MP^2$.

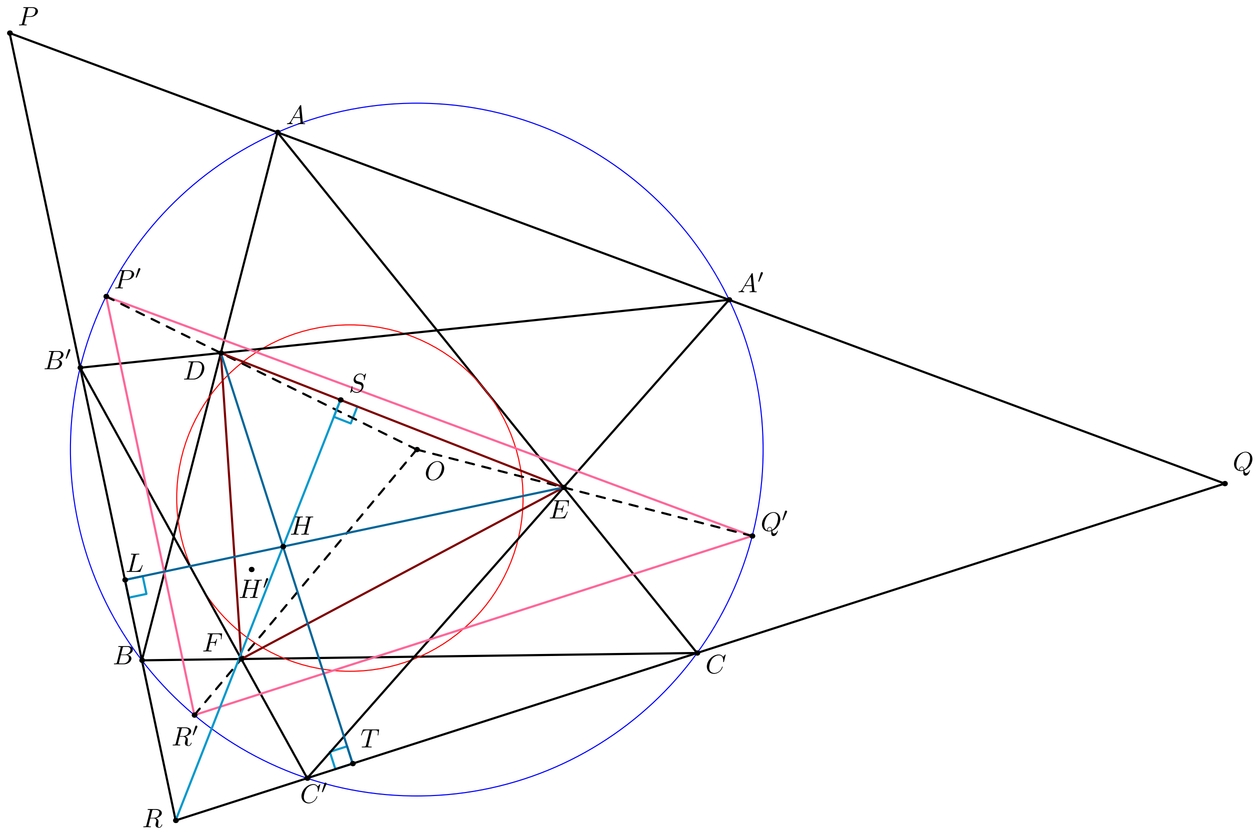
Combining with the result from the last paragraph, $\text{Pow}(M, (ABC)) = MP^2 = MK \cdot MK' = \text{Pow}(M, \omega)$.

It's only left to establish the equality to the power of M with respect to (PQR). This is done by noting that SP is, in fact, tangent to (PQR). After that, $\text{Pow}(M, (PQR))$ is trivially equal to MP^2. We can establish it in two ways.

First way is to note that $\angle SPR = \angle SPK - \angle RPK = \angle SPK - \angle K'PA' = \angle(KK', PQ)$, where the second equality follows from KP and $K'P$ being the corresponding elements in $\Delta PBA \sim \Delta K'PA$. Hence, we only need to prove that $\angle(KK', PQ) = \angle RQP$, which is the same as $KK' \parallel CC'$. This is obvious, because homothety at G with factor $-\frac{1}{2}$ takes $C \to K$ and $C \to K'$.

Second way is more tedious. We need $PO' \perp PS$, where O' is the circumcenter of (PQR). We also know that $OD \perp PS$, where D is $AB \cap A'B'$. Since perpendiculars from O and H onto lines TA' and TB are concylic, it's well known that OD and DH are isogonal in $\angle BDA'$ (Follows from the fact that isogonal conjugates have the same *Pedal Circle* and phantom points argument). Since we know that OD has to be parallel to $O'P$, we will draw lines parallel to PQ and PR through D, call them ℓ_1 and ℓ_2. We see that these lines are themselves isogonal with respect to $\angle BDA'$, because $\angle(\ell_1, DA') = \angle DA'A = \angle DBB' = \angle(\ell_2, BD)$. In $\angle RPQ$, PO' is isogonal to the perpendicular from P to RQ. So, $PO' \parallel DO \Leftrightarrow DH \perp RQ$. It's not hard to see that there exists a negative inversion with radius r^2 centered at H that maps (ABC) to ω. Suppose that $C \to F$ and $C' \to F'$ after this inversion. F and F' are the feet of altitudes in ABC and $A'B'C'$, respectively. We know that $HF' \cdot HC' = r^2 = HF \cdot HC$. Therefore, $(FF'C'F)$. Suppose that $DH \cap CC'$ is L. $\angle FDH = \angle F'FH = \angle F'C'C$, because of the cyclic quadrilaterals $(F'DFH)$ and $(FFC'C)$. Thus, $(DF'LC')$ and $\angle HLC = 90^\circ$, proving that $DH \perp CC'$.

¶ Second solution (Alternative finish using Sondat's theorem and homothety) After establishing that $PO' \parallel DO$ and defining E and F similarly, we can try to consider the homothety (In the diagram, it has a positive factor, but it doesn't matter) that takes Ω to (ABC). It will map ΔPQR to $\Delta P'Q'R'$ such that lines $P'D$, $R'F$, $Q'E$ intersect in O. We also know that $DH \perp CC'$, $EH \perp BB'$, and $EH \cdot HL = DH \cdot HT$, where $L = BB' \cap EH$ and $T = DH \cap CC'$. Thus, we can repeat the argument from the last solution and derive that $RH \perp DE$. Now we have a lot of perpendiculars that intersect in the same point, this suggests to use *Sondat's theorem*. Suppose that the image of H after the homothety is H'. We know that $R'H' \parallel RH \perp DE$; same for cyclic variants. Thus, H' is the first center of orthology of ΔDEF and $\Delta P'Q'R'$. And $DH \perp CC' \parallel R'Q'$, same for cyclic variants. Hence, H is the second center of orthology. By Sondat's theorem, we know that H', H, O are collinear. We also know that $H'H$ passes through the center of homothety, as well as OO'. Conclusion easily follows if $H \neq H'$, but in the case of $H = H'$, we know that H must be the center of the homothety and it lies on OO', as a result.


P
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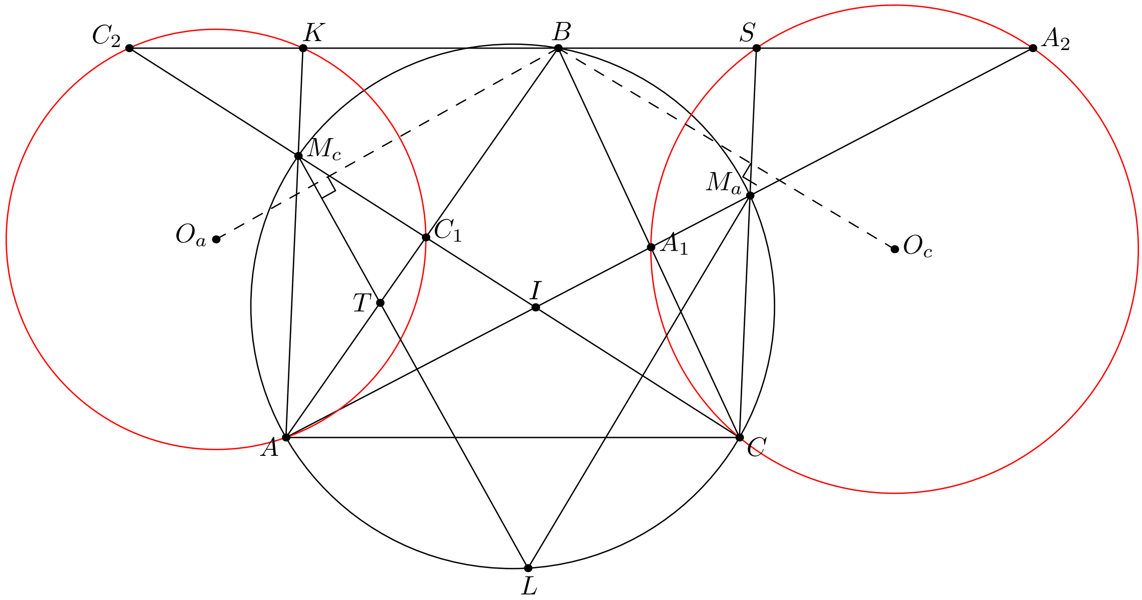
§6.3 All-Russian MO grade 11 2021/4, proposed by Alexander Kuznetsov

Problem 3 (All-Russian MO grade 11 2021/4)

In triangle ABC angle bisectors AA_1 and CC_1 intersect at I. Line through B parallel to AC intersects rays AA_1 and CC_1 at points A_2 and C_2, respectively. Let O_a and O_c be the circumcenters of triangles AC_1C_2 and CA_1A_2, respectively. Prove that $\angle O_aBO_c = \angle AIC$.



¶ First solution (Completing cyclic quadrilaterals and Brokard's theorem) We start with marking the midpoints of arcs AB and BC not containing C and A, respectively. Call them M_c and M_a. Good thing about these points is that $M_cA \cap C_2B = K \in (AC_1C_2)$, because $\angle M_cAB = \angle M_cCB = \angle C_1CA = = \angle BC_2C_1$. In the same way, for $M_aC \cap BA_2 = S$, (CA_1SA_2).

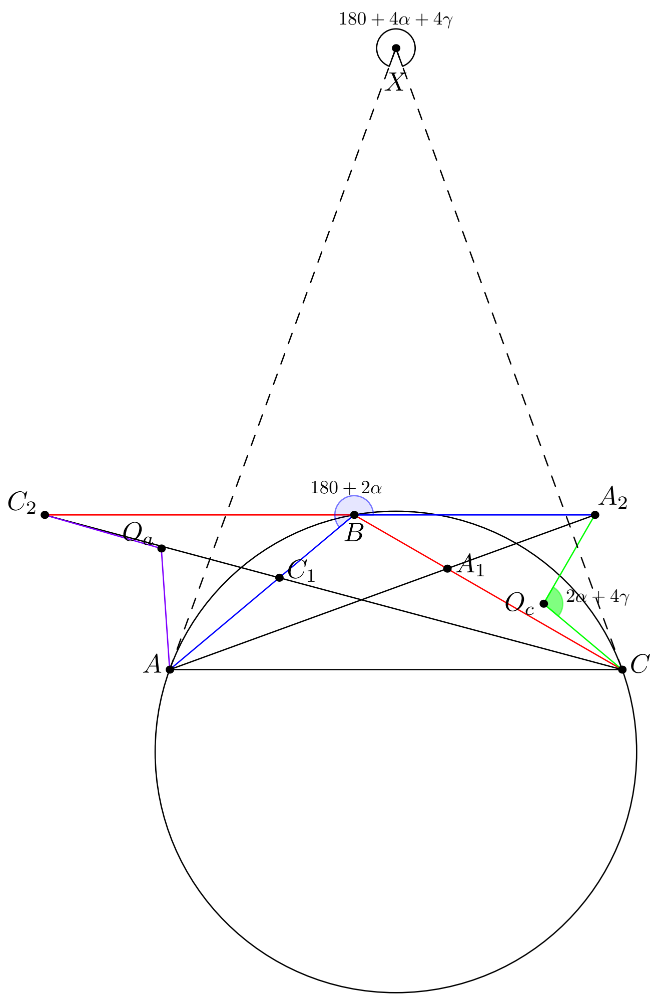
Now we want to find the angle $\angle O_bBO_c$. Since we have a complete cyclic quadrilateral (AC_1KC_2), it's useful to notice that the direction of the line BO_b is known to be perpendicular to the polar of B with respect to (AC_1KC_2), which also passes through M_c, by Brokard's theorem. We also know that, from the definition of polar, that it intersects AB in point T such that $(AC_1; BT) = -1$. Suppose that $M_cT \cap (ABC) = L$, then $-1 = (AC_1; BT) \overset{M_c}{=} (AC; BL)$. Analogously, $M_aL \perp BO_c$. Thus, $\angle O_bBO_c = 180^\circ - \angle M_cLM_a = 180^\circ - \frac{\angle BAC}{2} - \frac{\angle ACB}{2} = 90^\circ + \frac{\angle ABC}{2} = \angle AIC$.

¶ Second solution (Compositions of rotations) Note that all the segments that are coloured with the same colour in the diagram are trivially equal. This amount of isosceles triangles in which we know the angles suggests to use compositions of rotations.

Now, we need to do some angle chase to know the angles of rotations. Suppose that $\angle A = 2\alpha$, $\angle B = 2\beta$ and $\angle C = 2\gamma$. $\angle CO_cA_2 = 2\angle CA_1A_2 = 2\alpha + 4\gamma$, and in the same way $\angle C_2O_aA = 2\angle C_2C_1A = 4\alpha + 2\gamma$. $\angle ABA_2 = 180^\circ - 2\alpha$ and $\angle CBC_2 = 180^\circ - 2\gamma$.

Consider the composition of rotations R_{O_c} and R_B with centers O_c and B by angles $\angle CO_cA_2$ and $360 - \angle A_2BA$, both in anticlockwise direction. It's well known to be equivalent to an anticlockwise rotation by the angle equal to the sum of the angles of the rotations and centered at point X, intersection of line ℓ_1, that is an anticlockwise rotation of BO_c by angle $\frac{360-\angle A_2BA}{2}$, and line ℓ_2, that is a clockwise rotation of O_cB around O_c by angle $\frac{\angle CO_cA_2}{2}$. It's not hard to check this statement, because the rotation centered at O_c is a composition of reflections in XO_c, then BC. Rotation at B is the composition of reflections in BC and XB. Their composition is a reflection in XC, followed by reflection in XB, which is an anticlockwise rotation by an angle equal to the sum of aforementioned angles. In general, point X is constructed in such a way that it would be fixed after performing the composition of rotations.

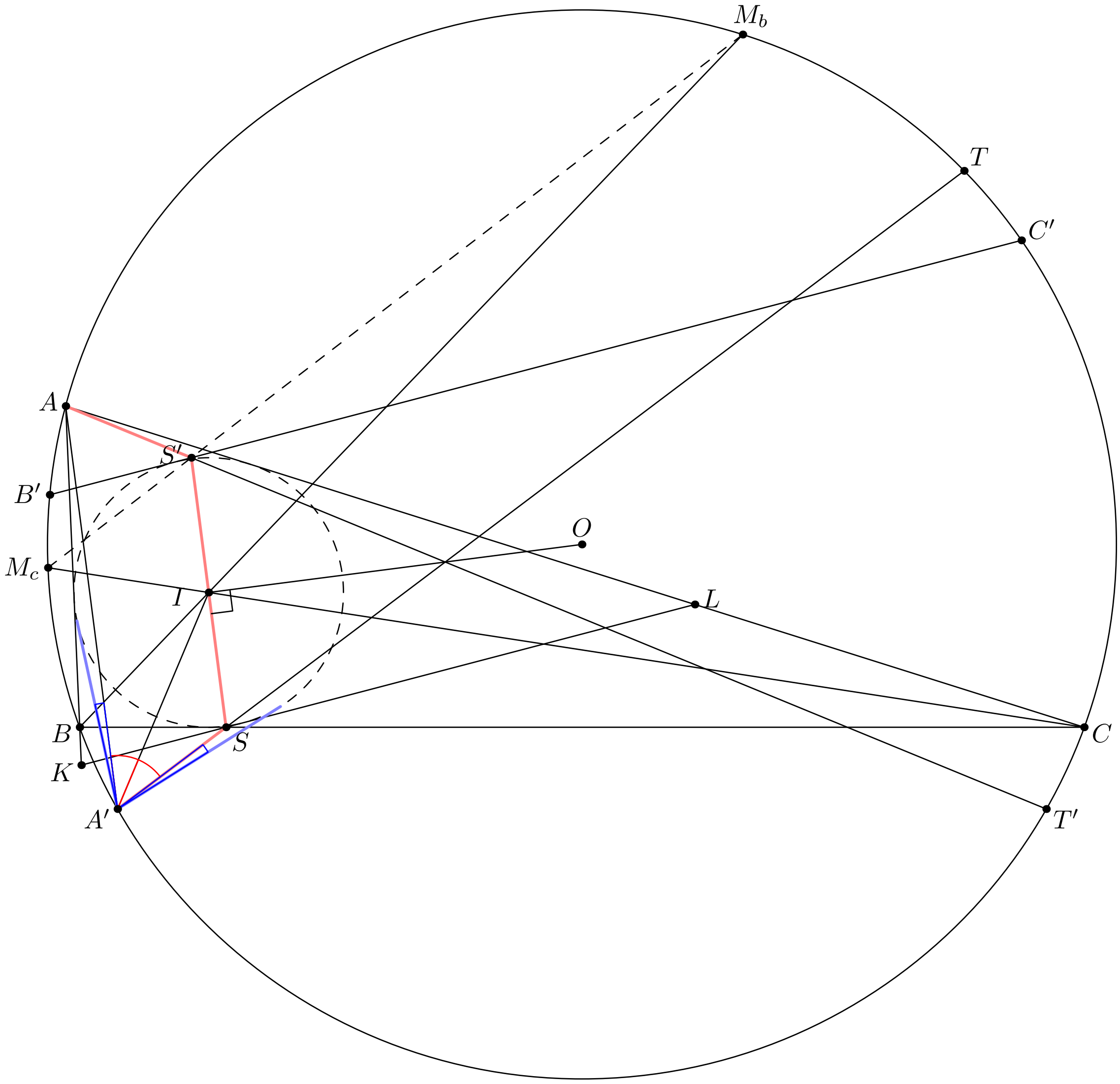
This rotation maps $C \to A_2 \to A$ and the angle is $\angle CO_cA_2 + 360 - \angle A_2BA = 2\alpha + 4\gamma + 360 - 180 + 2\alpha = 4\alpha + 4\gamma + 180$. In the same way, we can consider the composition $R'_B \circ R_{O_a}$ on angles $360 - \angle C_2BC$ and $\angle C_2O_aA$, both in anticlockwise direction. This takes $C \to C_2 \to A$ and the angle is $\angle C_2O_aA + 360 - \angle C_2BC = 4\alpha + 2\gamma + 360 - 180 + 2\gamma = 4\alpha + 4\gamma + 180$. So, these two compositions must be the same (because there can be only one point X that satisfies $XA = XC$ and anticlockwise angle CXA equals to $4\alpha + 4\gamma + 180$). Since the centers of the compositions are the same, we can say that $\angle O_cBX = \frac{360-\angle A_2BA}{2}$ and $O_aBX = \frac{360-\angle C_2BC}{2}$. Therefore $\angle O_aBO_c = 360 - \frac{360-\angle A_2BA}{2} - \frac{360-\angle C_2BC}{2} = \frac{\angle A_2BA+\angle C_2BC}{2} = 180 - \alpha - \gamma = \angle AIC$.



§6.4 IOM 2019/3, proposed by Dušan Djukić

Problem 4 (IOM 2019/3)
In a non-equilateral triangle ABC point I is the incenter and point O is the circumcenter. A line s through I is perpendicular to IO. Line ℓ symmetric to like BC with respect to s meets the segments AB and AC at points K and L, respectively (K and L are different from A). Prove that the circumcenter of triangle AKL lies on the line IO.

¶ First solution (Butterfly theorem and reflections)



Suppose that $s \cap BC = S$. We first implement a standard idea, that is well known by All-Russian MO grade 10 2017/8, to deal with lines perpendicular to line through center: Reflect a point on this line and find a quadrilateral for which the statement of *Butterfly Theorem* holds. In this case, it turns out extremely nice, because, by aforementioned Butterfly theorem, S', the reflection of S in IO, lies on M_cM_b, where M_c and M_b are the midpoints of arcs AB and AC not containing B and C, respectively. M_bM_c is known to be a perpendicular bisector of AI, as a consequence of the *Incenter-Excenter Lemma*, because $AM_b = M_bI$ and $AM_c = M_cI$. We can then say that $AS' = S'I$.

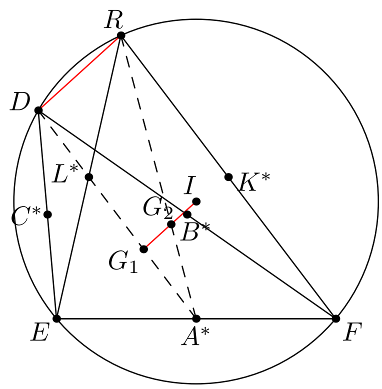
To use this result in full, we will mark another important point, A', that is a reflection of A across IO. $A'S = SI$ as a result of the reflection. Importance of A' lies in the fact that if the conclusion is correct, then AA' has to be the common chord of (ABC) and (AKL) (because common chord is perpendicular to the line of centers, which is just IO in this case), and A' is a Miquel point of $ABSL$ as a result. We also know that $ABSL$ is circumscribed, because $\angle BIS = \angle ISL$ from the reflection. We know that $\angle AA'I = \angle A'IS = \angle IA'S$.

Suppose that B' and C' are the reflections of B and C with respect to IO. $B'BSS'$ is an isosceles trapezoid and $S' \in B'C'$, so $\angle B'S'S = \angle BSS' = \angle S'SL$, so $KL \parallel B'C'$. Now, since $\angle AA'I = \angle IA'S$, then $\angle A'AI = \angle IAS'$, therefore, AA' and AS' are isogonal in $\angle BAC$. Hence, $AS' \cap (ABC)$ is a point T' such that $A'T' \parallel BC$. So, if we reflect T' across IO in point T, then $AT \parallel B'C' \parallel KL$. Thus, by *Reim's theorem*, $(A'SLC)$. As a result, A' is a Miquel point of $ABSL$, as desired.

¶ Second solution (Alternative Dual of Desargues Involution Theorem finish) Unlike the involved angle chase above, this solution requires zero thinking except knowing *Dual of Desargues Involution Theorem*. Draw tangents t_1 and t_2 from point A' (take the notation from the last solution) to the incircle, they are reflections of each other with respect to $A'I$, as well as $A'A$ and $A'S$. By DDIT, there exists an involution swapping $(AA'; A'S)$, $(t_1; t_2)$, $(A'K; A'C)$, $(A'B, A'L)$. First two pairs also swapped by the reflection across AI, which is an involution itself. It's well known that if two involutions intersect in two pairs, then they coincide. So $A'K$ is a reflection of $A'C$ across AI, as well as BA' and LA'. Thus, $\angle KA'B = \angle LA'B$ and $\angle A'BK = \angle A'CL$, hence $\angle BKA' = \angle CLA'$, which shows that $(A'KAL)$, as desired.

¶ Third solution (Incircle inversion) Since the problem asks us to prove that the line IO passes through some other center, inversion at I seems a great choice. The line through the center is fixed under the inversion. The center of the image lies on the line through center of the original circle and center of the inversion. Those simple facts allow us to get rid of all the information except the images and their centers.

$ABSL$ is circumscribed. Suppose the incircle touches AB at F, BC at D, SL at R, AC at E, it's not hard to check that problem becomes the following, denote images with a star symbol:



IO is the *Euler line* of ΔDEF because it passes through its nine-point circle's center and its circumcenter of it. It passes through the nine-point circle's center because nine-point circle is an image of (ABC) (they go to respective midpoints of sides ΔDEF).

In the picture above, DR is parallel to IO, which is perpendicular to SI, but SI is perpendicular to DS due to obvious reasons. IO is the Euler line of ΔDEF, by what we have explained.

We want to show that centers of nine-point circles in ΔREF and ΔDEF are collinear with I, but that is the same as proving that Euler lines coincide. Suppose that centroids of ΔDEF and ΔREF are G_1 and G_2, respectively. Then $G_1G_2 \parallel DR$, because $DG_1 : G_1A^* = 2 : 1 = RG_2 : G_2A^*$. G_1I, as the Euler line of ΔDEF, is also parallel to DR. It follows that G_1, G_2, I are collinear, as desired.

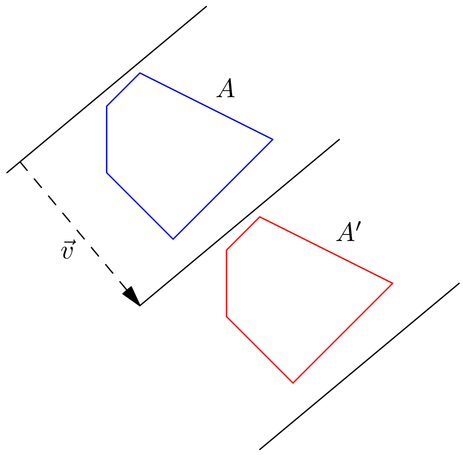
> **Remark.** In the first two solutions, we proved the converse of Serbia 2019/3.

§6.5 Problem 5

Problem 5

Call the set of all points in between a pair of parallel lines with distance 1 a *strip*. Given $n \geq 3$ convex polygons in the plane. It's know that their intersection (can be an empty set) can be covered with a strip. Prove that we can choose 3 of these polygons such that their intersection (can be an empty set) can be covered with a strip of width 1.

¶ Solution (Helly's theorem and translations) Our condition looks similar to Helly's theorem, where the condition on any three sets intersecting make all sets intersect. Here we want to get that every three sets not satisfying some condition makes intersection of all sets to not satisfy it. For this, we want to rewrite the condition in terms of intersecting. We see that if the polygon can be contained in the strip, then we can translate it a little bit and it wouldn't intersect itself. More formally, note that if a polygon X can be covered with a strip of width 1, then we can translate the polygon by vector $\vec{v}$ corresponding to the direction of the distance of the strip, call the resulting polygon A'. The polygons end up being non-intersecting.

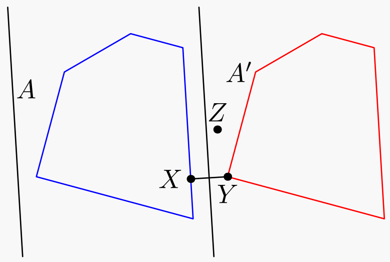


We can actually prove the converse, that polygon can be covered if for some $|\vec{v}| = 1$, $A \cap A + \vec{v} = \varnothing$. Again, $A + \vec{v}$ is A'. Take the line separating A and A' and shift it by vector $\vec{v}$ to form the strip covering A. The fact that separating line exists is also called *Hyperplane separation theorem*, and is true for higher dimensions.

Consider any unit vector $\vec{v}$. Suppose that polygons of our set are $A_1, A_2, \ldots, A_n$. If we suppose that for any A_i, A_j, A_k in our set of polygons, they cannot be covered by a strip, then $(A_i \cap A_j \cap A_k) \cap ((A_i \cap A_j \cap A_k) + \vec{v})) \neq \varnothing$, which is the same as saying any three of $(A_i \cap A_i + \vec{v})$ intersect. These intersections are convex polygons themselves, as they are the intersections of convex polygons. Now we can use Helly's theorem that tells us that if any sets of the family of convex sets in $\mathbb{R}^2$ intersect, then all sets intersect. Which means that for for any unit vector $\vec{v}$, $(A_1 \cap A_2 \cap \ldots \cap A_n) \cap ((A_1 \cap A_2 \cap \ldots \cap A_n) + \vec{v})) \neq \varnothing$, this means that the intersection cannot be covered with a strip, contradiction.

Remark. The proof of the plane case of the Hyperplane separation theorem is not hard.

Suppose that we have two non-intersecting convex polygons A and A'. Consider a pair of points $X \in A$, $Y \in A'$ that minimizes the distance XY across all choices of X and Y. Now we draw the perpendicular bisector of XY, we claim that it separates two polygons. Assume not, then, WLOG, XZ intersects it for some vertex Z of A. Then $\angle ZXY < 90^\circ$, so we can consider the perpendicular from Y to ZX, which lies inside A, contradicting the minimality of the distance.



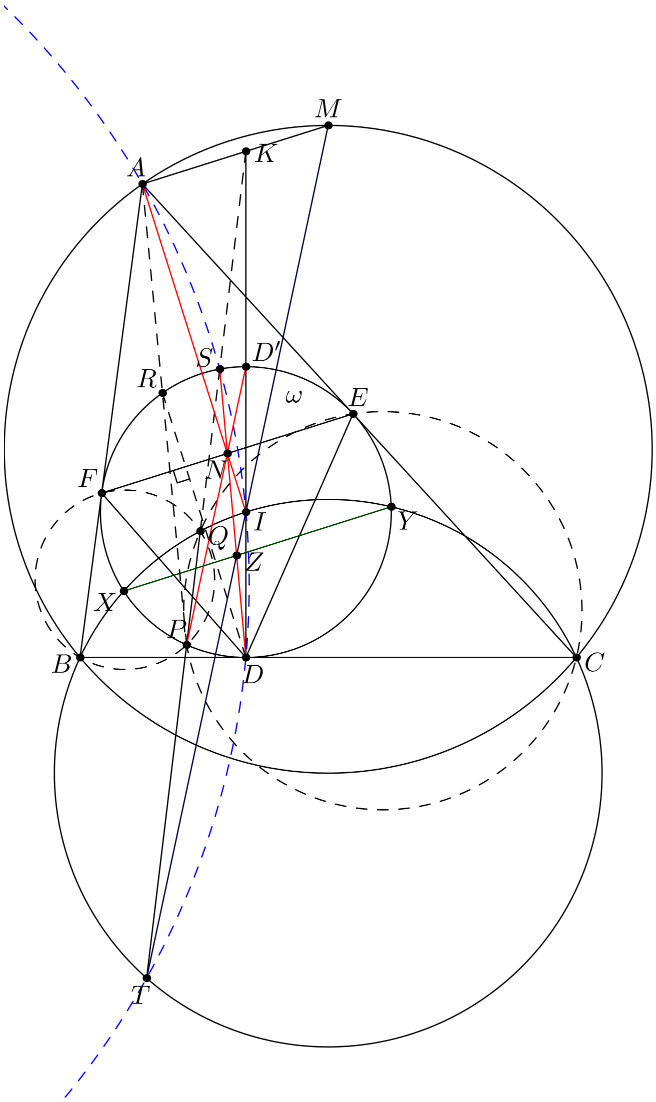
It's almost obvious that such a pair exists, but a formal proof is as follows. The sets A and A' are compact (closed and bounded), hence $A \times A'$ (set of all pairs) is compact. We can define the distance function from $A \times A'$ to $\mathbb{R}$ that represents the distance between two points from the two polygons. The distance function is continuous and positive for this set (since polygons have no common points), so it reaches the minimum positive value, because the domain is compact. We choose points X and Y for which this minimum occurs.

§6.6 IMO 2019/6, proposed by Anant Mudgal

Problem 6 (IMO 2019/6)
Let I be the incenter of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω at R. Line AR meets ω again at P. The circumcircles of triangle PCE and PBF meet again at Q.

Prove that lines DI and PQ meet on the line through A perpendicular to AI.

¶ First solution (Similar quadrilaterals, radical axis, polars, incenter config)



Let M be a midpoint of arc BC that contains A. $AM \perp AI$, so we want to show that PQ, ID and AM intersect in one point. We first observe that AM is the polar of the midpoint of EF with respect to ω. Therefore, the polar of K passes through N. On the other hand, PR is a *symmedian* in ΔFPE, so PN has to pass through D', because $RD' \parallel FE$, because ID and DR are isogonal in $\angle FDE$. If $DN \cap \omega = S$, then $PS \cap DD'$ lies on polar of N with respect to ω, which is AM, by *Brokard's theorem.* Thus, it's enough to show that PQ passes through S.

We start with some radical axes, because a lot of lines pass through N. $IN \cdot NA = = FN \cdot NE = DN \cdot NS$. Therefore, $(ASID)$.

Now, we will try to characterize the point Q. We know that $\angle BQP = \angle BFP = \angle PRF$; in the same way, $\angle PQC = \angle PRE$. So $\angle BQC = \angle FRE = 180^\circ - \angle FDE = \angle BIC$, where the last is because $IC \perp DE$ and $BI \perp DF$. Hence, $(BQIC)$.

$\angle ICB = \angle IDE = \angle RDF = \angle REF$; analogously, $\angle RFE = \angle IBC$. This proves $\Delta FRE \sim \Delta BIC$. Suppose that $QP \cap (BIC) = T$, then $\angle BCT = \angle BQC = \angle FRP = \angle FEP$, and, similarly, $\angle TBC = \angle PFE$. Thus, $IBTC \sim RFPE$, so the latter is harmonic too. But it's well known that MB and MC are tangents to (BIC). Thus, TI passes through M. It's well known that $(AIDT)$. To prove it, suppose that $MA \cap BC = W$, then it's easy to check that $MA \cdot MW = MB^2$. We also have that $MB^2 = MI \cdot MT$. Hence, $(AWTI)$. And, obviously, $(WAID)$ is a circle with diameter IW, so $(AWTID)$. We also know that S lies on this circle by one of the steps above.

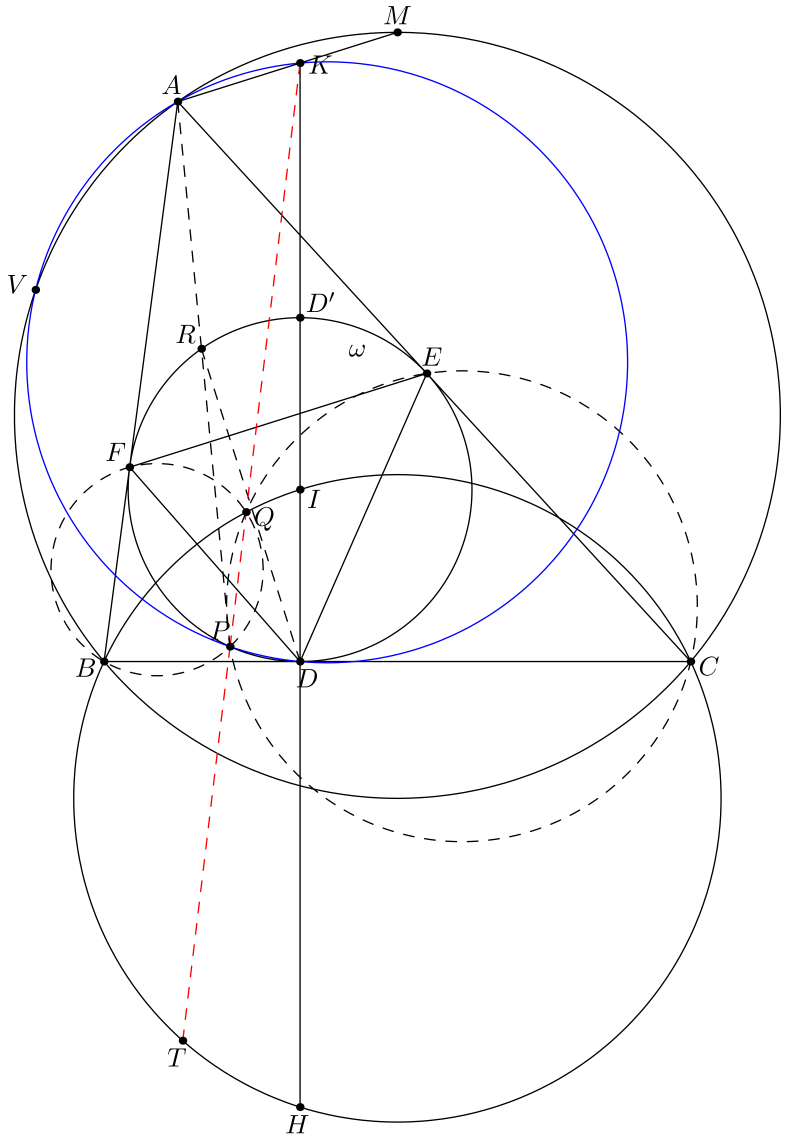
Now, we have a one free application of radical axes theorem. Apply it to ω, $(BICT)$, $(TDIS)$ to get that $XY \cap DS \cap IT = Z$, where XY is the radical axis of (BIC) and ω. It's well known that it's actually a midline of ΔDEF parallel to FE. The reason for this is that by radical axis on $(IDBF)$, ω, (BIC), we know that $IB \cap DF$ lies on XY, but $IB \cap DF$ is just the midpoint of DF; analogously, the midpoint of DE lies on XY. Thus, IT passes through the midpoint of DN and DD', so it's a midline of $\Delta DD'N$ parallel to DN.

Now we can write $\angle SPD' = \angle SDD' = \angle STI$, but since $D'N \parallel IT$, we know that it implies that PT passes through S, as well as PQ does.

¶ Second solution (Spiral similarity) Take the notation from the last solution. After noting two completely similar figures, $AFREP \sim MBICT$, we can try to consider the spiral similarity that takes one of them to another. Happily, this is a well known point of intersection of (AFE) and (ABC), which is known as *Sharky-Devil* point[1]. Call this spiral similarity centered at V as f. We know that $\Delta VPT \sim \Delta VAM$, because $f(P) = T$ and $f(A) = M$. So, we only need to show that $\angle VPT = 180^\circ - \angle VPK$, which is the same as $\angle VAK + \angle VPK = 180^\circ$. Therefore, we will be showing that $(VAKP)$.

$RD' \parallel FE \parallel AM$, so, by *Reim's theorem*, $(AKDP)$. We are left to show that $\angle VAM = 180^\circ - \angle VDK$. But this is obvious, because if $ID \cap (BIC) = H$, then $f(D) = H$, so $\angle VDH = \angle VAM$, which rewrites as the needed.

[1] Even though this solution doesn't really use the theory behind it



¶ Third solution (Linearity of power of a Point) Linearity of power of a Point refers to a method of setting up a function $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $f(X) = \text{Pow}(X, \omega_1) - \text{Pow}(X, \omega_2)$. This function is linear in X. What this means is that if we are given collinear points A, B, C in this order, then $f(C) - f(A) = \frac{CA}{BA}(f(B) - f(A))$. The proof is simple. We have $\text{Pow}((x, y), \omega_1) = (x - a_1)^2 + (y - b_1)^2 - R_1^2$, where (a_1, b_1) are the coordinates of the center of ω_1. Same for $\text{Pow}((x, y), \omega_1)$. Subtracting them, we will have a linear function in x and y, and it will satisfy the property above.

It's usually the most convenient to work with differences equal to zero, because if two points A and B have zero values for a linear function, then every point on the line AB has a value of zero. We want to prove that $\text{Pow}(K, (BPF)) - \text{Pow}(K, (CPE)) = 0 = \text{Pow}(K, (IDBF)) - \text{Pow}(K, (IDCE))$, where K is the intersection of ID with the external angle bisector of $\angle BAC$. Here, we used that K lies on the radical axis of $(IDBF)$ and $(IDCE)$. We also know that $\text{Pow}(A, (BPF)) - \text{Pow}(A, (CPE)) = 0 = \text{Pow}(A, (IDBF)) - \text{Pow}(A, (IDCE))$, because A is the intersection of two radical axes of pairs of circles again (From here, it can be seen that the approach is powerful when we have intersection points of two pairs of radical axes). We are now motivated to consider the following linear (difference of two linear functions is linear too) function:

$$f(X) = \text{Pow}(X, (BPF)) - \text{Pow}(X, (CPE)) - \text{Pow}(X, (IDBF)) + \text{Pow}(X, (IDCE)).$$

Because it has to be zero along the external bisector of $\angle BAC$. One more point for which we will be proving that it's zero is going to be the intersection of AK with BC, point Z, motivated by the fact that we will only need to consider the line BC and powers with respect to $(IDBF)$ and $(IDCE)$ are known. Then, we will only need to define $(BPF) \cap BC = X$ and $(CPE) \cap BC = Y$.

We need to show that $ZB \cdot ZX - ZY \cdot ZC - ZB \cdot ZD + ZC \cdot ZD = 0$, which is the same as $ZX \cdot DX = ZY \cdot DY$.

Now, we will focus on finding $\frac{YD}{DX}$. By an easy application of Law of Sines, in ΔPYD and ΔPDX, we get that

$$\frac{YD}{DX} = \frac{PY}{PX} \cdot \frac{\sin \angle YPD}{\sin \angle XPD}.$$

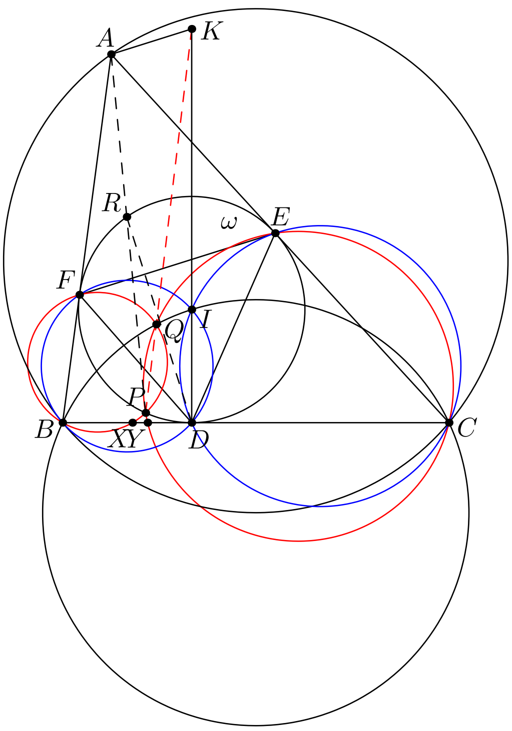
$\frac{PY}{PX} = \frac{\sin \angle PXY}{\sin \angle PYX}$, by Law of Sines. And $\frac{\sin \angle PXY}{\sin \angle PYX} = \frac{\sin \angle PFB}{\sin \angle PEC} = \frac{\sin \angle PEF}{\sin \angle PFE} = \frac{PF}{PE}$, by trivial tangencies and cyclicities, and Law of Sines for the last equality.

$\angle XPD = 360^\circ - \angle XPF - \angle FPD = \angle XBF + \angle FED = 90^\circ + \frac{\angle B}{2}$; similar computations give $\angle YPD = 90^\circ - \frac{\angle C}{2}$. Anyway, we have $\frac{\sin \angle YPD}{\sin \angle XPD} = \frac{\cos \frac{\angle C}{2}}{\cos \frac{\angle B}{2}}$.

$\frac{PF}{PE} = \frac{RF}{RE} = \frac{\sin \frac{\angle C}{2}}{\sin \frac{\angle B}{2}}$, because $PFRE$ is harmonic, and the second equality is Law of Sines and some trivial angle equalities. After that, we just multiply them to get

$$\frac{YD}{DX} = \frac{\sin \frac{\angle C}{2} \cdot \cos \frac{\angle C}{2}}{\sin \frac{\angle B}{2} \cdot \cos \frac{\angle B}{2}} = \frac{\sin C}{\sin B} = \frac{AB}{AC} = \frac{ZX}{ZY},$$

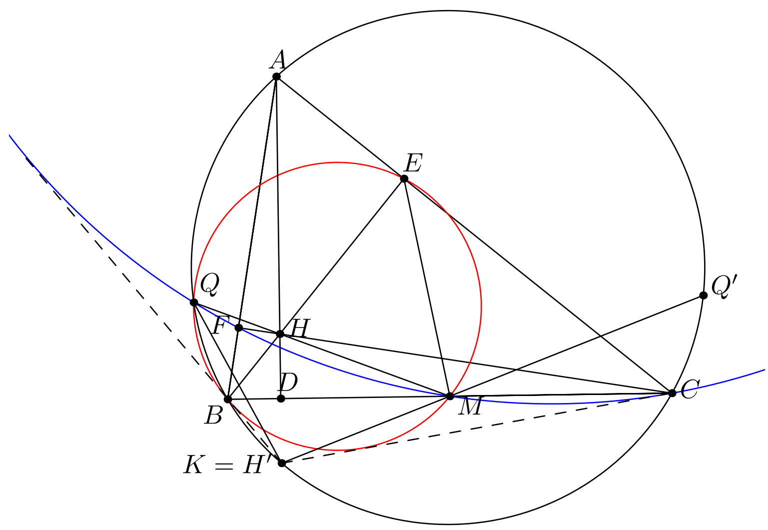
where we have used the external angle bisector theorem; this rewrites into a needed equation, so we are done.



§6.7 USA TST 2023/2. proposed by Kevin Cong

Problem 7 (USA TST 2023/2)
Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

¶ First solution (Queue-point and homothety angle chase)

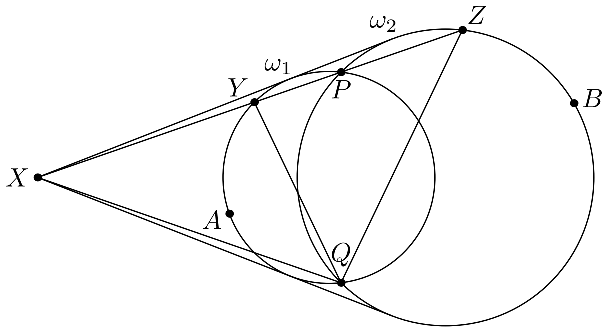


When dealing with conditional problems, it's important to make the most of the observations without sticking to the condition. Once we established some general facts, it's easier to arrive at special facts granted by the condition.

It's well known that $(BME) \cap (CMF) = Q$ is actually the intersection of HM with (ABC). For the proof, we can reflect H across M to get H'. Easy to show that the resulting point is diametrically opposite point to A on (ABC). This proves $\angle MQA = 90^\circ$. It follows that $(MDQA)$ with diameter AM. Then $QH \cdot HM = AH \cdot HD = BH \cdot HE$, which proves that $(BMEQ)$; analogously, $(CMFQ)$. Point Q is known as the *Queue-point*.

Now, we will use another useful well-known fact about intersecting circles and the point of intersection of their external common tangents.

Claim — Circles ω_1 and ω_2 intersect in two points, P and Q. X is the intersection of their common external tangents. A and B are arbitrary points on arcs PQ in ω_1 and ω_2, as shown in the picture. Then $\angle PXQ = \angle PAQ - \angle PBQ$.

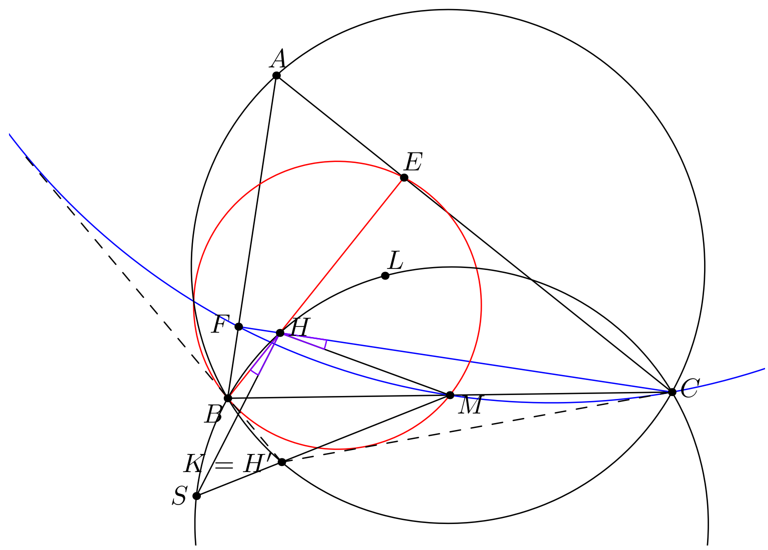


Proof. Suppose that the line XP intersects ω_1 and ω_2 for the second time in points Y and Z, respectively. Considering the homothety that takes $\omega_1 \to \omega_2$, $\frac{XY}{XP} = \frac{XP}{XZ}$. It can be rewritten as $XY \cdot XZ = XP^2 = XQ^2$, where the last equality follows from the obvious symmetry. This proves that $\angle PBQ = \angle PZQ = \angle YQX$, so $\angle PAQ = \angle PYQ = \angle YXQ + \angle XQY = \angle PXQ + \angle PBQ$; hence the conclusion. $\square$

Using this claim in the problem, we can calculate $\angle QKM = \angle QBM - \angle QCM$. The latter is an angle related to just the triangle ΔABC. If we can prove that this difference always equals to the angle $\angle QH'M$, where H' is $AD \cap (ABC)$, then we will be done as then, K lies on $(QH'M)$ and on (ABC), which means that $K = H'$, because, obviously, $K \neq Q$.

The latter is true because if $H'M$ intersects (ABC) the second time in Q', then $\angle QMB = = \angle BMH' = \angle Q'MC$, where the first angle equality is because H' is the reflection of orthocenter in BC. From this angle equality it's evident that Q and Q' are reflections across the perpendicular bisector of BC. So $\angle QH'M = \angle QCQ' = \angle Q'CB - \angle QCB = \angle QBM - \angle QCM$. Therefore, we are done.

¶ Second solution (Diameter inversion) Since the two main circles in the problem pass through M, we are motivated to perform the inversion centered at M. The presence of feet of perpendiculars from B and C suggests that it will be more convenient to choose the radius as MB. [1]



[1] But, of course, only the center of inversion actually matters

This inversion fixes B, C, F, E. (MFC) goes to FC, (MBC) goes to BE. Common tangents become circles through M that are tangent to CH and BH. K becomes their second point of intersection S. We know that these two circles are symmetric with respect to angle bisector of $\angle BHC$, so S is a point symmetric to M with respect to the angle bisector of $\angle BHC$ (because it trivially also lies on both of these circles). We also know that K is on (ABC), so S lies on the image of (ABC), which is (BHC), because A goes into a point L [1] such that $\angle MAB = \angle LBC$ and $\angle MAC = \angle LCB$. So, $180^\circ - \angle BHC = \angle BAC = \angle BAM + \angle MAC = \angle LBC + \angle LCB = 180^\circ - \angle BLC$. Therefore, (ABC) goes into $(BLHC)$. Since HS is a symmedian in ΔBHC, $(BHCS)$ is harmonic, so HS is a symmedian in ΔBSC. $\angle MHC = \angle SHB = \angle MCS$, $\angle MSB = \angle HSB = \angle HCM$. Therefore, $\angle HMB = \angle BMS = \angle BMK$. But $\angle HMB = \angle BMH'$ for H' the reflection of H in BC, which is an intersection of the A-altitude with a circumcircle. Hence, $K = H'$, as desired.

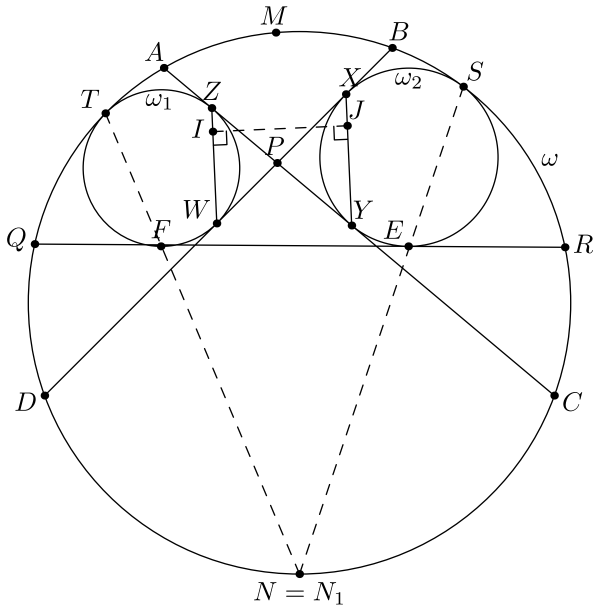
[1] L is also called an *A-Humpty point*

§6.8 Problem 8

Problem 8

Given a quadrilateral $ABCD$ inscribed in a circle ω. Its diagonals AC and BD intersect in point P. Suppose that ω_1 and ω_2 are the circles inscribed in angles $\angle APD$ and $\angle BPC$, respectively, that touch ω internally. Prove that one of the common external tangents of w_1 and w_2 is parallel to CD.

¶ First solution (Sawayama's lemma, Shooting lemma, radical axis)



Since we are dealing with circles inscribed in segments, we recognize *Sawayama's lemma* usefulness. Suppose that X,Y are the contact points of ω_1 with the diagonals. Same for ω_2 and Z, W. If I and J are the incenters of ΔABD and ΔACB, respectively, then, by the aforementioned lemma, $I \in ZW$ and $J \in XY$. It's known that IJ is parallel to the angle bisector of $\angle APD$. Neat proof of it is to take the circle inscribed in the segment of the circle formed by the angle $\angle APB$, then I and J both lie on the line through the contact points of this circle with the diagonals, by Sawayama's lemma; parallelness follows. Now note that $ZW \parallel XY$ and both perpendicular to the angle bisector of $\angle APD$, so $IJ \perp ZW$ and $IJ \perp XY$.

We will now try to connect the external tangent of ω_1 and ω_2 to the picture. If we intersect it with ω, we will have two circles, namely ω_1 and ω_2, inscribed in the segment. We now recognize the *Shooting lemma.* [1] If N is the midpoint of arc QR not containing A, then $NF \cdot NT = NQ^2 = NE \cdot NS$, so N is on the radical axis of ω_1 and ω_2.The condition on $QR \parallel CD$ can be restated conveniently as midpoints of arcs CD and QR not containing A are the same.

Therefore, we just have to show that the radical axis of ω_1 and ω_2 passes through the midpoint of arc CD not containing A, call it N_1 for now. We already know that the radical axis of ω_1 and ω_2, that has to be the midline of the isosceles trapezoid $XYWZ$, is actually

[1] Also known as *Archemedes' lemma*

a perpendicular bisector of IJ. Mark the midpoint of arc AB not containing D as M. $\angle DMN_1 = \angle CMN_1$ and $MI = MA = MB = MJ$ from the *Incenter-Excenter lemma*, so MN_1 is indeed the perpendicular bisector of IJ, which concludes the proof.

¶ Second solution (Casey's theorem) When dealing with tangent circles, it's useful to think of *Casey's theorem*. Take the notation from the last solution. We will try to show that $QD = RC$ or $QC = DR$. Apply Casey's theorem to Q, R, D (circles of zero-radius) and ω_1:

$$QF \cdot RD + RF \cdot QD = DW \cdot QR.$$

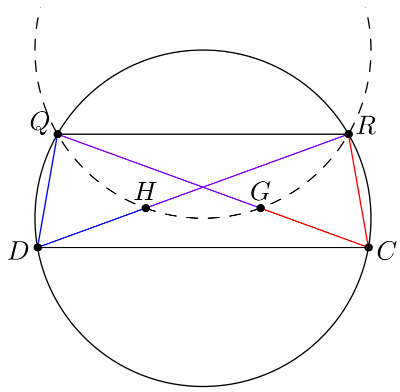
Now apply to Q, R, D and ω_2:

$$QD \cdot RE + DR \cdot QE = DX \cdot QR.$$

If we subtract the former from the latter, we get

$$DR \cdot EF - QD \cdot EF = WX \cdot QR.$$

Analogously, by subtracting the equation for Q, R, C, ω_2 from the one for Q, R, C, ω_1, we get that $QC \cdot EF - RC \cdot EF = ZY \cdot QR$. Therefore, since $ZY = WX$, $QC - CR = DR - QD$. From this, it should be clear that $CR = QD$ and $QC = DR$. One possible way of doing it is to take points G and H on QC and DR such that $CG = CR$ and $DH = DQ$, then $\angle QHR = 90^\circ + \frac{\angle QDR}{2} = 90^\circ + \frac{\angle QCR}{2} = \angle QGR$, so $(QGHR)$. We also know that $QG = RH$ from the metric condition, so $QHGR$ is an isosceles trapezoid, so $\angle QRH = \angle RQG$, and so, $RC = QD$, as desired.

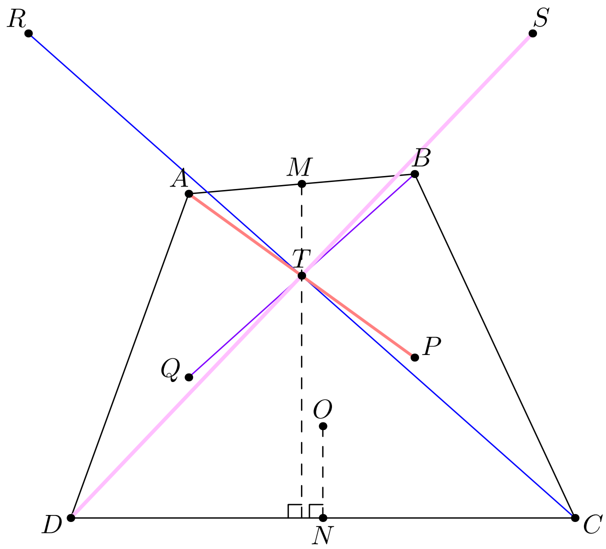


> **Remark.** I really like EMC Senior league 2022/4 and wanted to include it in the book, but the statements of this problem and the one from EMC are very similar, so I decided not to include it due to variety reasons. The solutions are rather different, so I still recommend solving it.

§6.9 USA TSTST 2020/6, proposed by Andrew Gu

Problem 9 (USA TSTST 2020/6)
Let A, B, C, D be four points such that no three are collinear and D is not the orthocenter of ABC. Let P, Q, R be the orthocenters of $\triangle BCD$, $\triangle CAD$, $\triangle ABD$, respectively. Suppose that the lines AP, BQ, CR are pairwise distinct and are concurrent. Show that the four points A, B, C, D lie on a circle.

¶ First solution (Symmetries and anticenter)



We immediately see that this problem is a converse of the well-known problem about the *Anticenter of the quadrilateral*. We are inclined to adjust the given configuration to fit the statement of the converse problem.

Suppose that the lines intersect at point T. In the end, T has to be a midpoint of all the segments connecting a vertex of the quadrilateral to the orthocenter of the triangle formed by three other vertices. This motivates us to calculate the ratios in which the point T divides these segments and hope that something of the from $\frac{x}{y} = \frac{y}{x}$ pops up. Our tool here will be the parallel lines formed by the altitudes to the same sides.

More formally, $BR \perp AD$ and $CQ \perp AD$, so $CQ \parallel BR$; hence, $\frac{TR}{TC} = \frac{TB}{TQ}$. In the same way, we can prove that $BP \parallel AQ$ and $AR \parallel CP$. Now, we can calculate $\frac{TR}{TC} = \frac{TB}{TQ} = \frac{TP}{TA} = \frac{TC}{TR}$. This proves that $TC = TR$, $TA = TP$, $TB = TQ$.

Now, we have a symmetry to work with. $DQ \perp AC$ and $ARPC$ is a parallelogram, so $DQ \perp RP$. In the same way we can show that $DR \perp PQ$. Therefore, D is the orthocenter of ΔPQR. But, by reverting the symmetry, a point symmetrical to D with respect to T is an orthocenter of ΔABC, call it S.

Now we have the same symmetric setup as in the converse problem. We know that T is the Anticenter of $ABCD$, because if M is the midpoint of AB, then $MT \parallel BP$, the latter is perpendicular to CD, same for other midpoints.

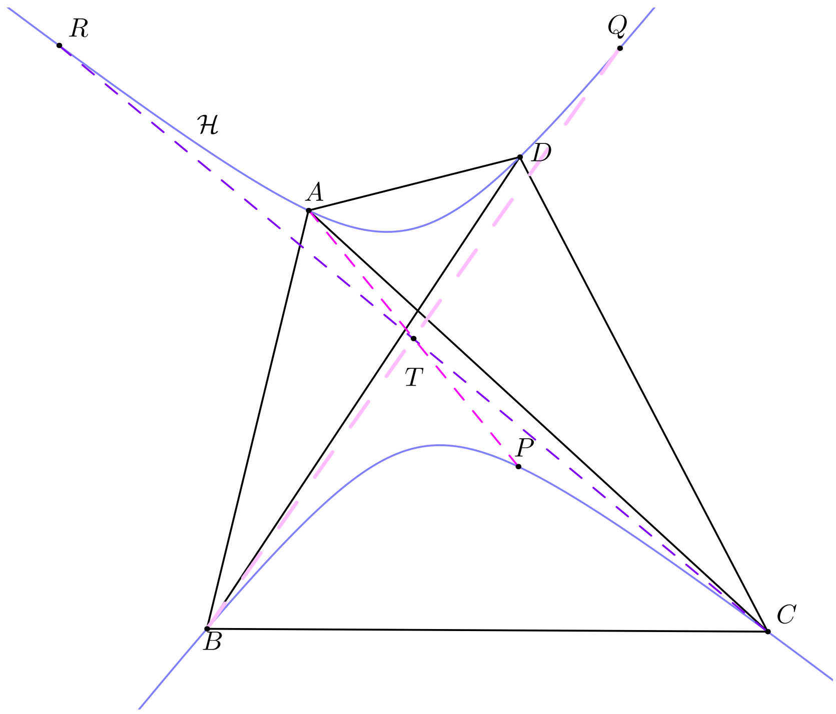
It's not hard to prove that the Anticenter exists only if $ABCD$ is cyclic. Consider the reflection of T in the center of mass of $ABCD$, point O. M goes to the midpoint of CD, point N. And we know that $NO \perp CD$. Hence, $OC = OD$. Analogously, O is equidistant from every other vertex.

Sketch of the alternative finish. After proving that $AQ = BP$ from the parallelogram, one can use $|\cot \angle DBC| \cdot CD = BP = AQ = |\cot \angle DAC| \cdot CD$, which follows from similar triangles in orthocentric configuration. Thus, $|\cot \angle DBC| = |\cot \angle DAC|$, so the angles $\angle DBC$ and $\angle DAC$ are either equal or sum to 180°. Depending on the configuration (whether A and B on the same side of CD or not), it's not hard to exclude one of the cases, and the one that will be left shows $(ABCD)$.

¶ Second solution (Rectangular circumhyperbola, Pascal's theorem, Simson's line) When dealing with configurations including orthocenter (or multiple orthocenters as in this case) we can try to include rectangular circumhyperbolas in the picture.

Suppose that $\mathcal{H}$ is a conic passing through $ABCDP$. It has to be a hyperbola, because one of the points is contained inside the convex hull. Suppose that ∞_{ℓ_1} and ∞_{ℓ_2} are the points on the infinity along its asymptotes ℓ_1 and ℓ_2. Suppose that the line parallel to ℓ_1 passing through B intersects CD in the point X, and line parallel to ℓ_2 passing through P intersects BH in the point Y. Pascal's theorem on $B\infty_{\ell_1}\infty_{\ell_2}CDP$ shows that XY is parallel to CH, i.e. perpendicular to AB, so $PY \perp AX$, which shows that the asymptotes are perpendicular. [1] By reversing the argument, we can also show that this hyperbola passes through Q and R.

Now, we have a lot of points that lie on the same conic. This suggests using Pascal's theorem to find concurrencies. By Pascal's theorem on $DRCBAP$, we know that $DR \cap BA$, $RC \cap AP = T$, $CB \cap PD$ lie on the same line. Similarly, by Pascal's theorem on $DRCABQ$, we get that $DQ \cap CA$ also lie on this line. But $DR \cap BA$, $CB \cap PD$, $DQ \cap CA$ are exactly the feet of perpendiculars from D onto the sides of the triangle ΔABC. By the converse of Simson's line, we get that D is on the (ABC), as desired.

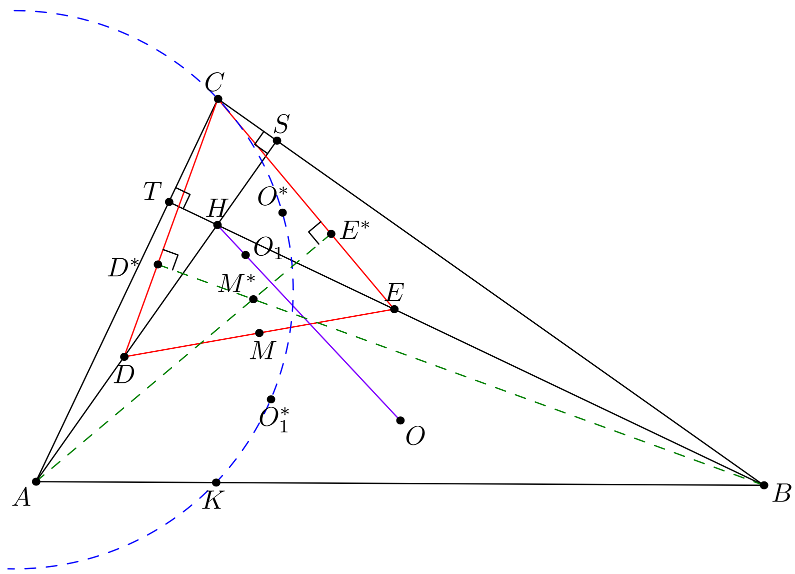


[1] That's why it's a rectangular circumhyperbola

§6.10 Problem 10

Problem 10

Vertices of an equilateral triangle Δ_1 lie on three distinct internal angle bisectors of the triangle Δ_2. Prove that the center of Δ_1 lies on the line connecting incenter and the circumcenter of Δ_2.



¶ First solution (Incenter-orthocenter duality, inversions and Pascal's) Of course, almost every problem about the incenter can be restated in terms of the orthocenter after performing the inversion around the incenter.[1] In this case we can state the problem in terms of the triangle with vertices being the midpoints of arcs of Δ_2, call this triangle ΔABC. Now vertices of Δ_1 lie on altitudes, and we want to prove that the center of Δ_1 lies on the Euler line of ΔABC. We then perform one more simplification and consider the homothety centered at H that takes Δ_1 to a triangle with one vertex coinciding with C, call two other points lying on A and B altitudes as D and E, respectively. It remains to show that the center of ΔCDE lies on the Euler line of ΔABC.

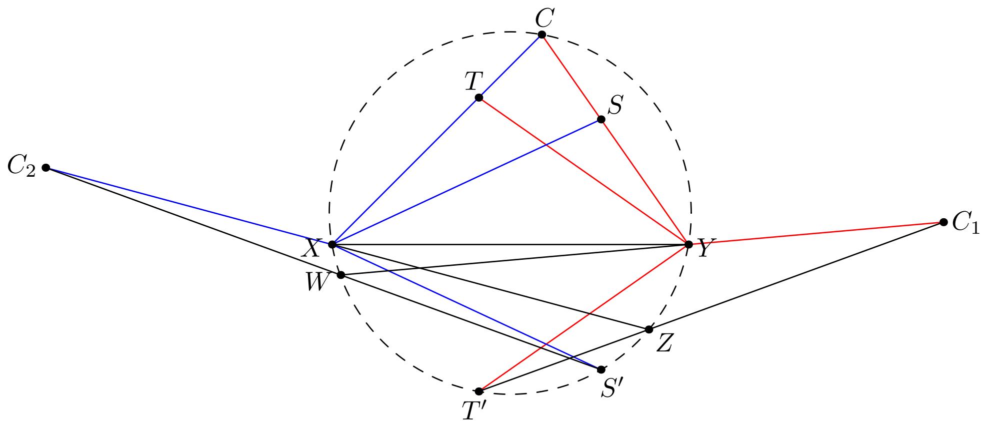
S, T, K are feet of altitudes from A, B, C, respectively. H and O are the orthocenter and circumcenter of ΔABC, and O_1 is the circumcenter of ΔABC. Consider the inversion centered at C with radius $\sqrt{CT \cdot CA} = \sqrt{CS \cdot CB} = \sqrt{CH \cdot CK}$. Note that H goes into K. O goes into a point O^* such that (SO^*OB), (AOO^*T), from which $CS = SO^*$ and $CT = TO^*$. Therefore, ST is a perpendicular bisector of CO^*. Now we are left to understand where the O_1 is going. D goes to D^* such that D^* lies on the image of AS, which is a (CSA), so $BD^*C = 90^\circ$; analogously, $\angle AE^*C = 90^\circ$, so the midpoint of DE goes into $AM \cap (CD^*E^*) = M^*$, which is a point diametrically opposite to C on (CD^*E^*). It's easy to prove that the distance from C to the reflelction of C across the side D^*E^* is $\frac{3}{2}$ times the distance CM^*, but since $CO_1 = \frac{2}{3}CM$, then this reflection coincides with the inverse of O_1 - O_1^*.

[1] Sometimes it's easier to perform inversions when dealing with orthocenter, but in some cases it's convenient to go back to incenter.

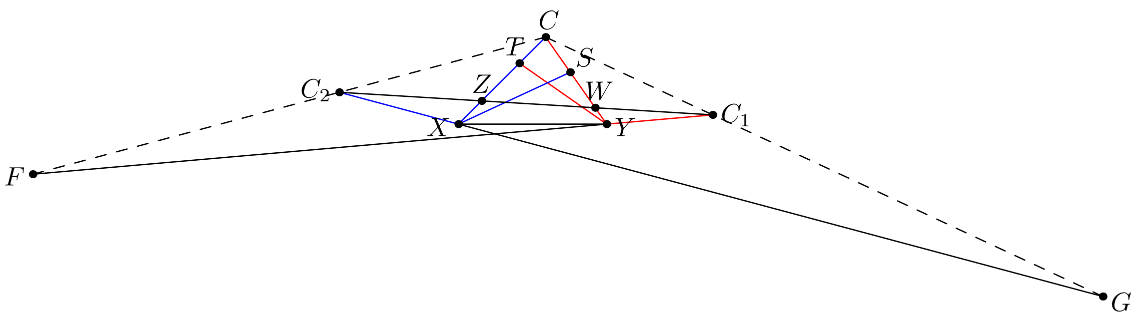
Now that we know all the inverses, we are left to prove that $(CO^*O_1^*K)$. Note that perpendicular bisectors of all the segments CO^*, CO_1^*, CK are well-defined. This suggests to prove instead that they are concurrent.

It's now easier to restate the problem in terms of the triangle ΔCXY, where X and Y are the midpoints of AC and BC. Because the problem is pivoted around two circles centered at X and Y passing through C. Note that D^*E^* passes through a point C_1 such that $CX = XC_1$, $\angle CXC_1 = 120^\circ$, because $\angle CD^*E^* = 60^\circ$. In the same way, D^*E^* passes through a point C_2 such that $CX = XC_2$, $\angle CXC_2 = 120^\circ$. Now, we can reflect T and S across XY to points T' and S'. $\angle XT'Y = \angle XTY = 180^\circ - \angle XCY$, so $(T'XCY)$. Analogously, $(S'YCX)$. We also know that TS and $T'S'$ intersect on XY, so we can instead prove that $T'S'$, XY, C_1C_2 are concurrent.

Now note that $\angle CT'C_1 = \frac{\angle CYC_1}{2} = 60^\circ$. Then, since $\angle CXC_2 = 120^\circ$, C_2X intersects C_1T' in a point Z that lies on $(CXYT'S')$. Analogously, C_2S' intersects C_1Y in a point W that lies on the same circle. Now, we have too many points on one circle, and the problem asks for some concurrency, so it's reasonable to try Pascal's theorem. Indeed, Pascal's theorem for $WYXZT'S'$ gives that C_2, C_1 and $XY \cap T'S'$ are collinear, which is what we wanted.



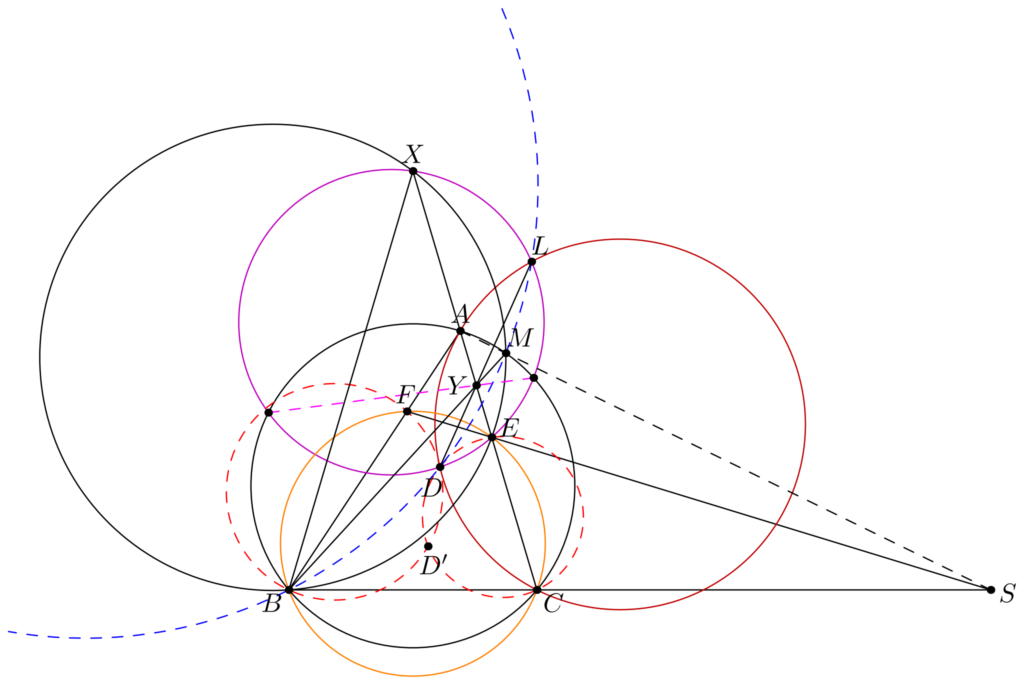
¶ Second solution (Alternative finish using cross ratios) First of all, $\angle CTC_1 = \frac{\angle CTC_1}{2} = 60^\circ$. And since $\angle CXC_2 = 120^\circ$, then $C_1T \parallel C_2X$. Analogously, $C_2S \parallel C_1Y$. Suppose that $C_2C_1 \cap CX = Z$ and $C_2C_1 \cap CY = W$, then we want to show that $(CZ;TX) = (CW;SY)$. Now, we want to use our parallel lines, so we are going to project onto one of them. But, before that, we define $C_1C \cap XC_2 = F$ and $C_2C \cap YC_1 = G$. $(CZ;TX) \overset{C_1}{=} (C_2G;\infty_{C_2X};X) = \frac{XG}{C_2X} = \frac{XG}{CX}$. Analogously, $(CW;SY) = \frac{YF}{CY}$. But now, the conclusion follows from the similarity $\Delta XCG \sim \Delta FYC$, which is true because corresponding angles are trivially equal.



§6.11 IMO 2021/3, proposed by Mykhailo Shtandenko

Problem 11 (IMO 2021/3)
Let D be an interior point of the acute triangle ABC with $AB > AC$ so that $\angle DAB = \angle CAD$. The point E on the segment AC satisfies $\angle ADE = \angle BCD$, the point F on the segment AB satisfies $\angle FDA = \angle DBC$, and the point X on the line AC satisfies $CX = BX$. Let O_1 and O_2 be the circumcenters of the triangles ADC and EXD, respectively. Prove that the lines BC, EF, and O_1O_2 are concurrent.

¶ First solution (Isogonal conjugates, Isogonality lemma, radical axes, inversion)



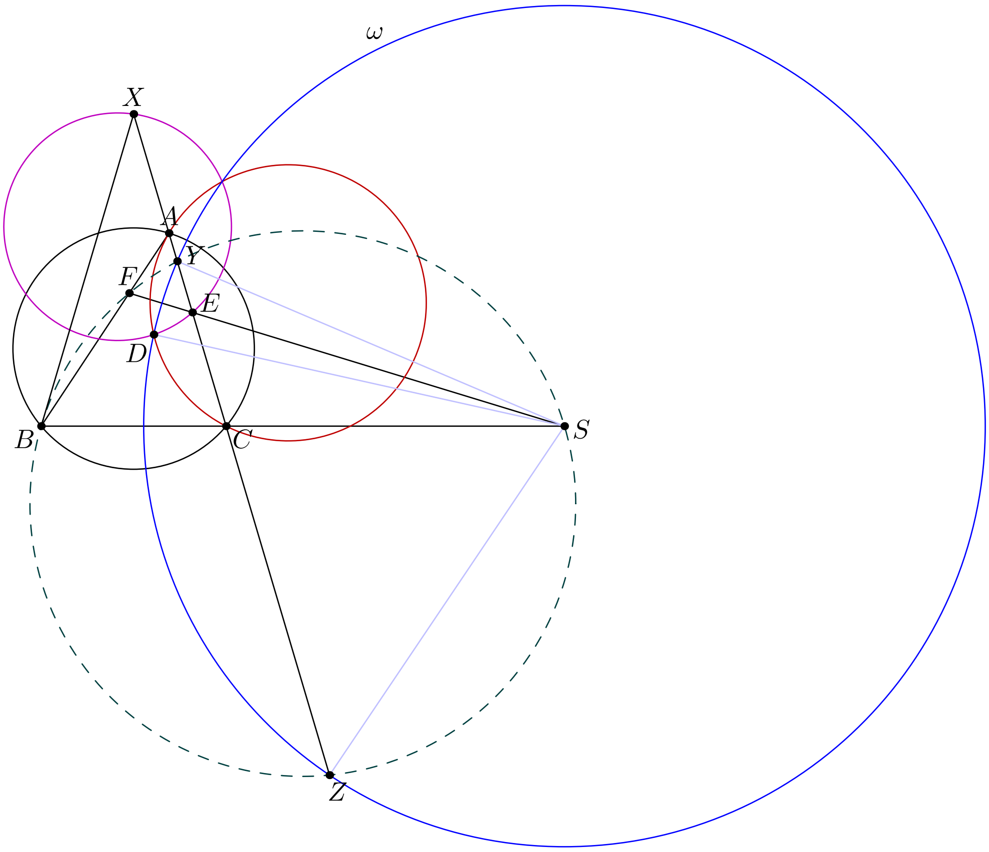
We first implement the first step in every problem with a weird angle conition: Rewrite it. One of the techniques that works often is to look at the reflected angles or isogonal conjugates. Note that, if D' is the isogonal conjugate of D in triangle ΔABC, [1] then D' also lies on the angle bisector of $\angle BAC$. We also have $\angle D'CE = \angle DCB = \angle ADE$, so $(D'DEC)$. Analogously, $(BD'DF)$. $AE \cdot AC = AD \cdot AD' = AF \cdot AB$. Hence, $(BFEC)$. Now, we intersect EF and BC in point S. Now, $\angle FDE = \angle FDA + \angle ADE = \angle CBD + \angle BCD = 180^\circ - \angle BDC$. Thus, BD and DE are isogonal with respect to $\angle FDC$. Hence, by *Isogonality lemma*, $\angle CDS = \angle ADF = \angle DBC$, so $SD^2 = SC \cdot SB = SE \cdot SF$.

Now, we can restate the problem to just prove that (ADC) and (XED) intersect in point L such that $SL = SD$. This suggest to consider inversion centered at S with radius SD and prove that it fixes L. (ADC) goes into (MDB), where M is the Miquel point of $BFEC$, so we just have to show that (MDB) passes through L. There's a lot of circles to apply radical axes theorem to. If our assumption about (MDB) is correct, then DL, MB, AC are concurrent in point Y. But, by radical axes on (DEX), (ADC), (ABC), Y has to lie on radical axis of (XDE) and (ABC), then $BY \cdot YM = YE \cdot YX$, so (BEM) has to pass

[1] By $\angle FDE + \angle BDC = 180^\circ$, we can even say that D and D' are isogonal conjugates in $BFEC$

through X. This should be angle chasable. It turns out it is: $\angle BME = \angle AME - \angle AMB = 180^\circ - \angle AFE - \angle ACB = 180^\circ - 2\angle ACB = \angle BXC$. Hence, we are done.

¶ Second solution (Alternative finish using DIT)



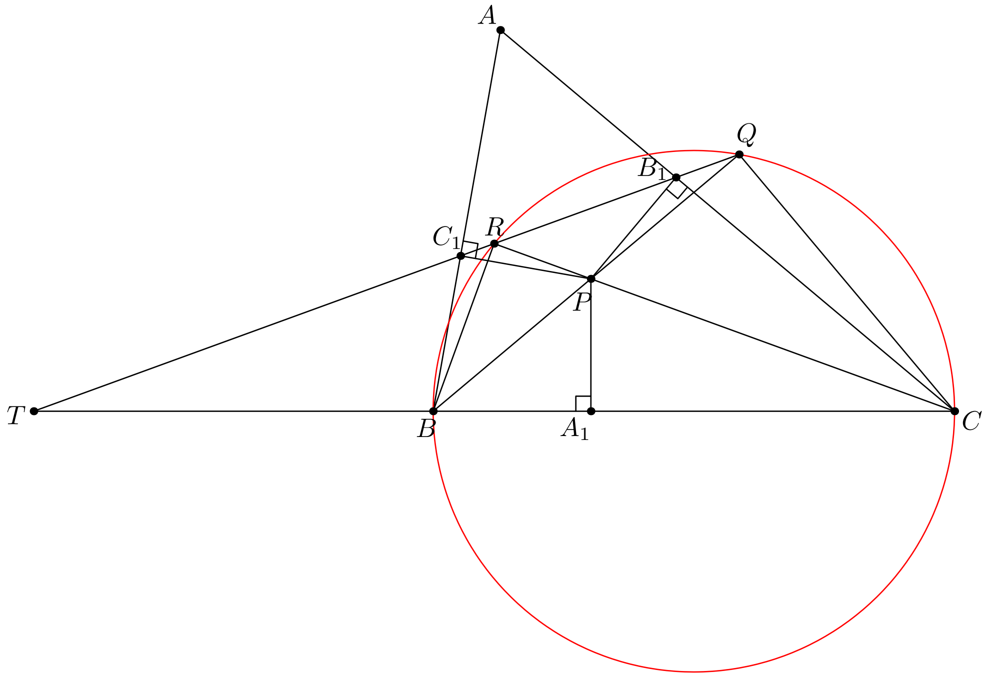
After proving $(BFEC)$ and $SD^2 = SE \cdot SF$, we could finish differently. The problem asks to prove that (DEX), (ADC) and the circle with center S and radius SD are coaxial; call the last circle ω. This is equivalent to involution on line AC that swaps $(X; E)$, $(A; C)$, $(Y; Z)$, where Y and Z are the points of intersection of ω and the line AC.

Since $SY^2 = SZ^2 = SE \cdot SF = SC \cdot SB$, by the converse of *Shooting Lemma*, $(BFYSZ)$. $\angle XBA = \angle XBC - \angle ABC = \angle ACB - \angle CES = \angle FSB$. This means that X is the intersection of tangent to (BFS) at B and line AC. The desired involution can be found using *Desargues Involution Theorem* applied to $BBFS$, inscribed in (BFS), and line AC. Then there's an involution on line AC that swaps $(AC \cap BF; AC \cap BS)$, $(AC \cap FS; AC \cap BB)$, $(AC \cap FB; AC \cap BS)$, as desired.

§6.12 USA TST 2016/6, proposed by Ivan Borsenco

Problem 12 (USA TST 2016/6)
Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90^\circ$.

¶ First solution (Adjusting to Iran Lemma, cross ratios and polars)



Note that the circumcenter, orthocenter and incenter satisfy this condition. We will prove that these are the only possibilities.

We first rewrite the "weird angle condition":

$$\angle C_1B_1A = \angle C_1PA = 90^\circ - \angle C_1AP = \angle PCA + \angle PBC.$$

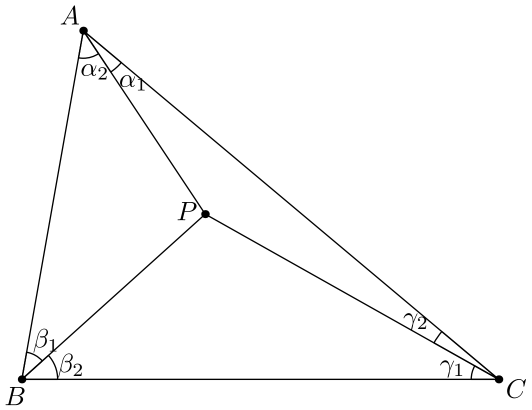
Now, if $CP \cap C_1B_1 = R$ and $BP \cap C_1B_1 = Q$, then $\angle PBC = \angle PRB_1$, so $(BRQC)$.

Concurrency condition can be rewritten as $(TA_1; BC) = -1$ for $T = C_1B_1 \cap BC$. Note that the case of P circumcenter is different because $C_1B_1 \parallel BC$. Thus, we assume that A_1 is not a midpoint (one of B_1, C_1, A_1 is not a midpoint of respective side, WLOG, assume it's A_1). Note that for P incenter, the resulting picture is the *Iran Lemma*, so we want to show that $\angle BRP = 90^\circ = \angle PQC$.

It's actually well known that in this picture, $(BRQC)$ has diameter BC. For example, one may note that A_1 lies on the polar of T with respect to $(BRQC)$, as well as P, by Brokard's theorem. Therefore, PA_1 is a polar of T. Hence, circumcenter of $(BRQC)$ lies on TA_1 because it's perpendicular to PA_1.

Thus, $\angle BRP = 90^\circ = \angle PQC$. Now, we need to deal with the orthocenter, in its case $R = C_1$ and $Q = B_1$. If $R = C_1$ and $Q \neq B_1$, then (PB_1QC), and so, $\angle BCP = \angle PQB_1 = \angle PCB_1$ and $PC \perp AB$. Hence, $AC = CB$, which is a contradiction. If $R \neq C_1$ and $Q \neq B_1$, then $\angle BCP = \angle PCA$, $\angle CBP = \angle PBA$, so P is the incenter.

¶ Second solution (Trigonometry with complex numbers)



Our goal is to write both conditions as trigonometric equations. Denote $\angle PAC = \alpha_1$, $\angle PAB = \alpha_2$, $\angle PBA = \beta_1$, $\angle PBC = \beta_2$, $\angle PCB = \gamma_1$, $\angle PCA = \gamma_2$. First condition can be rewritten as $\frac{\cot\beta_2}{\cot\gamma_1} \cdot \frac{\cot\gamma_2}{\cot\alpha_1} \cdot \frac{\cot\alpha_2}{\cot\beta_1} = 1$, because of *Ceva's Theorem* and $\frac{BA_1}{A_1C} = \frac{\cot\beta_2}{\cot\gamma_1}$, similar for others. Second condition is just $\alpha_1 + \beta_1 + \gamma_1 = 90^\circ = \alpha_2 + \beta_2 + \gamma_2$. We also know that $\sin\alpha_1 \sin\beta_1 \sin\gamma_1 = \sin\alpha_2 \sin\beta_2 \sin\gamma_2$, from *Trigonometric Ceva's Theorem.*

Thus, for $\alpha_1 + \beta_1 + \gamma_1 = 90^\circ = \alpha_2 + \beta_2 + \gamma_2$, we know that

$$\sin\alpha_1 \sin\beta_1 \sin\gamma_1 = \sin\alpha_2 \sin\beta_2 \sin\gamma_2, \quad \cos\alpha_1 \cos\beta_1 \cos\gamma_1 = \cos\alpha_2 \cos\beta_2 \cos\gamma_2.$$

Easier way to manipulate trigonometric functions is to rewrite them in complex numbers, then operations will be the same as with ordinary numbers.

Let $a_1 = e^{i\alpha_1}$, $a_2 = e^{i\alpha_2}$, $b_1 = e^{i\beta_1}$, $b_2 = e^{i\beta_2}$, $c_1 = e^{i\gamma_1}$, $c_2 = e^{i\gamma_2}$. Then, since $e^{i\theta} = cos\theta + sin\theta i$, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Hence, we can rewrite the given as

$$\left(a_1 - \frac{1}{a_1}\right)\left(b_1 - \frac{1}{b_1}\right)\left(c_1 - \frac{1}{c_1}\right) = \left(a_2 - \frac{1}{a_2}\right)\left(b_2 - \frac{1}{b_2}\right)\left(c_2 - \frac{1}{c_2}\right),$$

$$\left(a_1 + \frac{1}{a_1}\right)\left(b_1 + \frac{1}{b_1}\right)\left(c_1 + \frac{1}{c_1}\right) = \left(a_2 + \frac{1}{a_2}\right)\left(b_2 + \frac{1}{b_2}\right)\left(c_2 + \frac{1}{c_2}\right),$$

$$a_1b_1c_1 = e^{(\alpha_1+\beta_1+\gamma_1)i} = e^{90^\circ i} = \cos(90^\circ) + \sin(90^\circ)i = i = a_2b_2c_2.$$

This suggests that $\{a_1, b_1, c_1\} = \{a_2, b_2, c_2\}$. Proof is actually simple. We multiply first two equations by $a_1b_1c_1 = a_2b_2c_2$:

$$(a_1^2 - 1)(b_1^2 - 1)(c_1^2 - 1) = (a_2^2 - 1)(b_2^2 - 1)(c_2^2 - 1),$$

$$(a_1^2 + 1)(b_1^2 + 1)(c_1^2 + 1) = (a_2^2 + 1)(b_2^2 + 1)(c_2^2 + 1).$$

From first, $a_1^2+b_1^2+c_1^2-(a_1b_1)^2-(a_1c_1)^2-(b_1c_1)^2 = a_2^2+b_2^2+c_2^2-(a_2b_2)^2-(a_2c_2)^2-(b_2c_2)^2$. From second, $a_1^2+b_1^2+c_1^2+(a_1b_1)^2+(a_1c_1)^2+(b_1c_1)^2 = a_2^2+b_2^2+c_2^2+(a_2b_2)^2+(a_2c_2)^2+(b_2c_2)^2$. So, $(a_1b_1)^2+(a_1c_1)^2+(b_1c_1)^2 = (a_2b_2)^2+(a_2c_2)^2+(b_2c_2)^2$. As a result, $a_1^2+b_1^2+c_1^2 = a_2^2+b_2^2+c_2^2$. Combining with $a_1^2b_1^2c_1^2 = a_2^2b_2^2c_2^2$, we get that $(x-a_1^2)(x-b_1^2)(x-c_1^2) = P(x) = (x-a_2^2)(x-b_2^2)(x-c_2^2)$, by Vieta's relations. Thus $\{a_1^2, b_1^2, c_1^2\}$ and $\{a_2^2, b_2^2, c_2^2\}$ are both sets of roots of a polynomial $P(x)$. Thus, they are equal.

We know that all considered complex numbers have both coordinates positive. Therefore, they cannot sum to 0, so $x^2 = y^2$ yields $x = y$. Thus, $\{a_1, b_1, c_1\} = \{a_2, b_2, c_2\}$. And so, $\{\alpha_1, \beta_1, \gamma_1\} = \{\alpha_2, \beta_2, \gamma_2\}$.

If $\alpha_1 = \alpha_2$ and $\beta_1 \neq \beta_2$, then $\angle ABC = \angle ACB$, contradiction. So, in this case, $\beta_1 = \beta_2$. Thus, P is the incenter.

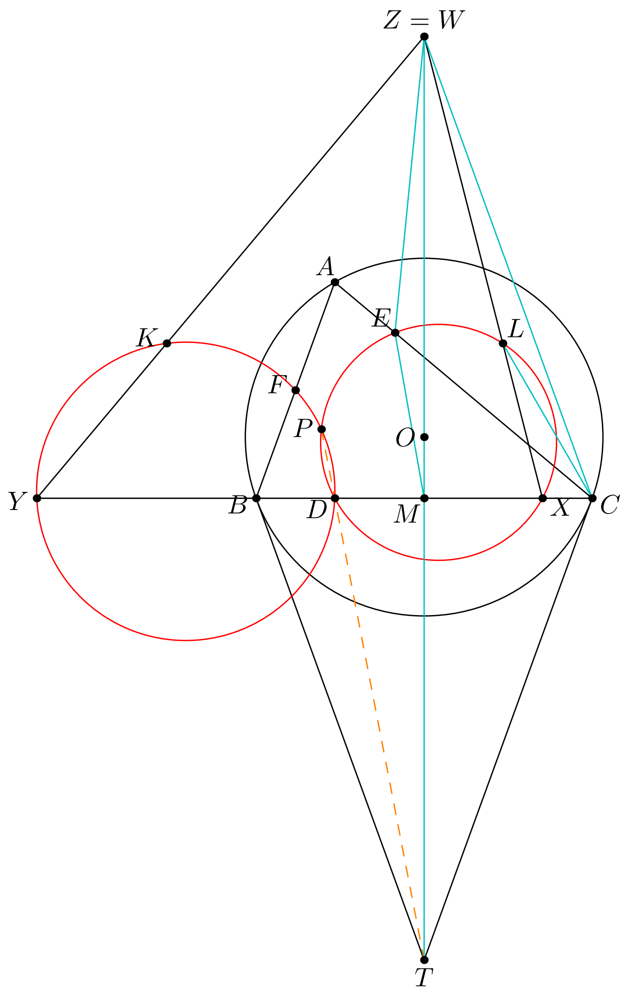
If $\alpha_1 = \gamma_2$ and $\gamma_1 \neq \beta_2$, then $\angle BAC = \angle BCA$, contradiction. So, in this case, $\gamma_1 = \beta_2$, and so P is the circumcenter.

Now, $\alpha_1 = \beta_2$. Then, $\angle PBC + \angle PBA + \angle PCB = \angle PAC + \angle PBA + \angle PCB = 90^\circ$. So $PC \perp AB$. Analogously for others. Thus, P is the orthocenter.

§6.13 Kvant M2717, proposed by Don Luu

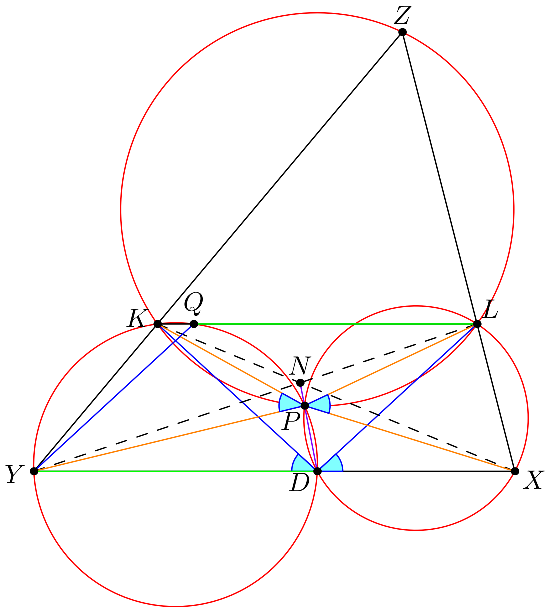
Problem 13 (Kvant M2717)
In an acute triangle ABC the heights AD, BE and CF intersecting at H. Let O be the circumcenter of the triangle ABC. The tangents to the circle (ABC) drawn at B and C intersect at T. Let K and L be symmetric to O with respect to AB and AC respectively. The circles (DFK) and (DEL) intersect at a point P different from D. Prove that P, D and T lie on the same line.

¶ First solution (Isogonal lemma, reflections, similar triangles)



We start with proving initial claims. $AK = AO = AL$, by symmetry. KL is parallel to the line through midpoints of OK and OL, which is the line through midpoints of AB and AC. Therefore, $KL \parallel BC$. This means that $AD \perp KL$. Thus, AD is the perpendicular bisector of KL. So, $DK = DL$. In this configuration we are inclined to mark $(DPK) \cap BC = Y$, $(DPL) \cap BC = X$, and $XL \cap YK = Z$, because there's an interesting claim about this configuration:

Claim — In the configuration above, $\angle PDY = \angle ZDX$.



Proof. $\angle PKZ = \angle PDY = \angle PLX$, so $(ZKPL)$. $\angle LPX = \angle LDX = \angle DLK = \angle DKL = \angle KDY = \angle KPY$. Thus, by *Isogonality Lemma*, PZ and PN are isogonal in $\angle KPL$. So, $\angle NPK = \angle ZPL = \angle ZKL = \angle KYD = 180^\circ - \angle KPD$. Therefore, N, P, D are collinear. DK and DL are isogonal with respect to $\angle YDX$, so, by *Isogonality Lemma* again, DN and DZ are isogonal with respect to the same angle. So, $\angle PDY = \angle NDY = \angle ZDX$, as desired. □

Now, we just need to prove that the reflection of T with respect to BC, point W, lies on DZ. Luckily, it just coincides with Z. To prove it, we show that W lies on XL; analogously, we will be able to show it for YK.

Let M be the midpoint of BC.

$$\frac{CL}{CW} = \frac{CO}{CT} = \frac{CM}{MT} = \frac{ME}{MW},$$

where the second equality follows from similarity of right triangles ΔTMC and ΔTCO. And $\angle EMW = 90^\circ - \angle EMB = 90^\circ - 2\angle ACB = \angle ABC + \angle BCA + \angle BAC - 2\angle ACB - 90^\circ = \angle ABC + \angle BAC - \angle ACB - 90^\circ = \angle BAC - \angle ACB - (90^\circ - \angle ABC) = \angle BAC - \angle ACB - \angle OCA = \angle BAC - \angle ACB - \angle ACL = \angle BCT - \angle ACB - \angle ACL = \angle BCW - \angle ACB - \angle ACL = \angle LCW$. Hence, $\Delta LCW \sim \Delta EMW$. From this, it's easy to obtain that $\Delta WEL \sim \Delta WMC$ (From the existence of spiral similarity centered at W). Thus, $\angle ELW = \angle MCW = \angle MCT = \angle BAC = \angle EDX = 180^\circ - \angle ELX$, so W lies on XL. Hence, the conclusion.

¶ Alternative proof of the claim using linearity of PoP A different proof of the lemma is based on *Linearity of Power of a Point.* As in the previous solution, we need to prove that N lies on PD, which is the same as N lies on the radical axis of (DKY) and (DLX). As in the proof of IMO 2019/6, we define the function:

$$f(X) = \text{Pow}(X, (DKY)) - \text{Pow}(X, (DLX)),$$

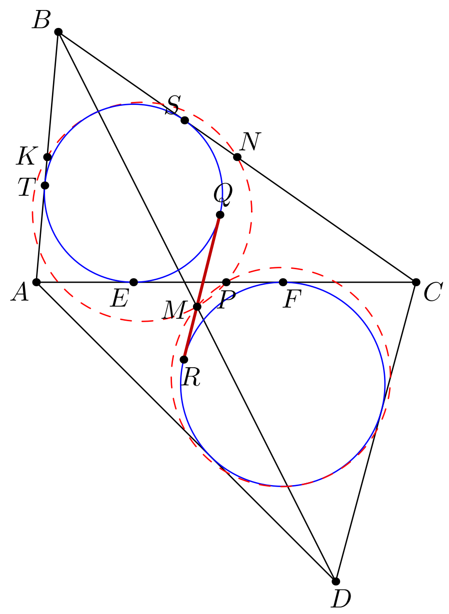
which we know is linear.

We want to show that $f(N) = 0$. We know that $f(L) - f(Y) = \frac{YL}{YN}(f(N) - f(Y))$, so $f(N) = \frac{YN}{YL} \cdot f(L) + \frac{NL}{YL} \cdot f(Y)$. Since we are showing that the latter is zero, we can prove that $\frac{YN}{NL}f(L) + f(Y) = 0$ Suppose that $(PDY) \cap KL = Q$, then $f(L) = LQ \cdot LK - 0$ and $f(Y) = 0 - YD \cdot YC$. Then, by replacing $\frac{YN}{NL} = \frac{YX}{KL}$, we will be left to show that $LQ = DY$. Which is true because $KL \parallel YD$, so $YKQD$ is an isosceles trapezoid, so $\angle QYD = \angle QKD = \angle KLD = \angle LDX$, so $DL \parallel YQ$ and $YQ = DK = DL$, so $YQLD$ is a parallelogram and $LQ = YD$.

§6.14 Sharygin Final 2022 grade 10/4, proposed by Ivan Frolov and Andrey Matveev

Problem 14 (Sharygin Final 2022, grade 10/4)
Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D$. Prove that the midpoint of BD lies on the common internal tangent to the incircles of triangles ABC and ACD.

¶ Solution (Fuerbach's and Casey's theorems, lengths chasing)



Easy way to connect the angle condition to the midpoint is to note that the midpoint of BD, point M, lies on both nine-point circles of ΔABC and ΔACD. This is because, if K and N are the midpoints of AB and AC, then $\angle KMN = \angle ADC = \angle ABC = \angle KPN$, where P is the midpoint of AC.

Now, we can approach the tangent from M. We can find the length of the tangent from M to the incircle of ΔABC, the length MQ, by using Casey's theorem on point-circles M, N, K and the incircle of ΔABC, that are all tangent to the nine-point circle due to Fuerbach's theorem.

$$KT \cdot MN + NS \cdot KM = KN \cdot MQ.$$

By substituting $KT = \frac{BC-AC}{2}$ and $SN = \frac{AC-AB}{2}$ and $KM = \frac{AD}{2}$, $MN = \frac{CD}{2}$, $KN = \frac{AC}{2}$, we get

$$MQ = \frac{(BC - AC)CD}{2AC} + \frac{(AC - AB)AD}{2AC}.$$

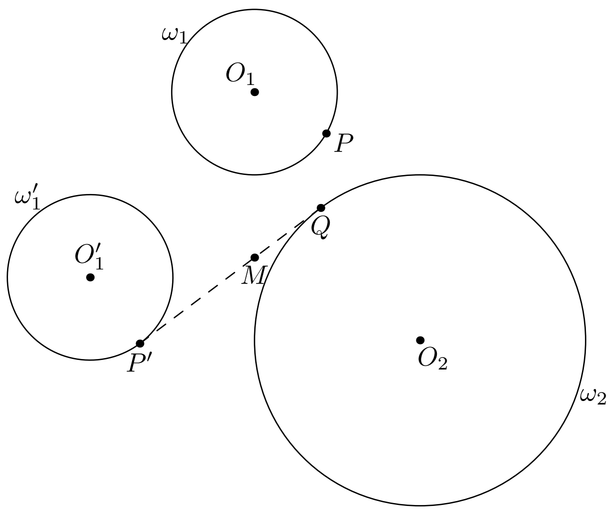
Analogously, if MR is the tangent to the incircle of ΔACD, then

$$MR = \frac{(AC - CD)BC}{2AC} + \frac{(AD - AC)BA}{2AC}.$$

Lengths may vary depending on the positions of points, but other cases are done similarly.

Now, we calculate $MR+MQ$. We want it to be equal to EF, which is equal to $CE-CF = \frac{AC+BC-AB}{2} - \frac{AC+DC-AD}{2} = \frac{BC+AD-AB-DC}{2}$. This can be verified using the direct expansion, all the products in the numerators not containing AC will cancel.

Now, we will prove the converse of the above statement. That is, given the two circles ω_1 and ω_2, and a point M such that $MP+MQ = d$, where MP and MQ are the tangents from M to ω_1 and ω_2, respectively, and d is the length of the common internal tangent, then, in this configuration, M lies on the internal tangent.



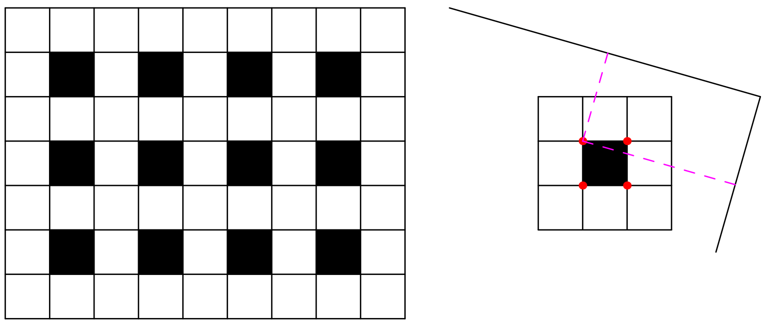
Suppose that MP and MQ are the tangents from M to ω_1 and ω_2 such that the circles lie on different sides of the tangents (as in the picture). Suppose that MO_1 and MO_2 are both not the external angle bisector of $\angle QMP$. If MO_1, for example, is the external angle bisector of $\angle QMP$, then M, Q, R are collinear, where MR is the second tangent to ω_1, which is needed. Now, we reflect MP and ω_1 with respect to the external angle bisector of $\angle QMP$ to get P' and ω_1'. We will then obtain that QP' is the internal common tangent to ω_2 and ω_1', but $QP' = MQ + MP' = MQ + MP = d$. Thus, $O_2O_1' = \sqrt{QP'^2 + (O_1'P + P'O_2)^2} = \sqrt{d^2 + (O_1P + O_2Q)^2} = O_2O_1$. So, $O_1 \neq O_1'$, $MO_1 = MO_1'$ and $O_2O_1 = O_2O_1'$. Therefore MO_2 is bisecting $\angle O_1MO_1'$, and, as a result, it bisects $\angle PMP'$, which means that O_2 lies on the external angle bisector of $\angle PMQ$.

§6.15 Problem 15

Problem 15

Given an integer $n \geq 3$. We call a 3×3 square *good* if it's central cell is of different colour than all the other 8 cells of the square. Colour some infinite grid (side of one cell is 1) in black and white. Suppose that some $a \times b$ rectangle, with sides not necessarily parallel to lines of the grid, fully contains at least $n^2 - n$ good squares. Find the minimum value of $a + b$, given that a and b are positive integers.

¶ Solution (Finding lattice points and using Pick's Theorem) **Example:** Of course, we can use the trivial example with $(2n-1) \times (2n+1)$ board and mark the cells as shown in the picture:



Then, the black cells are the centers of all the good squares; there are exactly $n^2 - n$ of them and $a + b = 4n$. We will prove that it's, in fact, optimal. Usually, such bounding problems in geometry involve finding some non-intersecting objects and then estimate some quantities such as areas, perimeters, etc.

Bound: Note that any good square gives at least 4 lattice points that don't intersect with other points, namely we can mark 4 points in the center. They don't intersect, because vertices of centers of good squares don't touch, otherwise not all squares surrounding them are of different colours than they are. We can also estimate the distance from marked points to the sides. Draw a perpendicular from a marked point to the side of the $a \times b$ rectangle, it intersects some side of the good square that has this marked point (formally, because the foot of perpendicular lies outside or on the boundary of this square, so the segment connecting the point to the foot has to intersect some side of the square), but each marked point is on the distance ≥ 1 from the boundary of the good square.

Thus, all the lattice points we marked are contained within the $(a-2) \times (b-2)$ rectangle. Lattice points are great, because we can estimate the area of any polygon with these points as vertices using *Pick's Theorem*. For this reason, we consider the convex hull of these points. Suppose it contains k points. Its area is at least $\frac{k}{2} + 4n^2 - 4n - k - 1$, where the number of lattice points on the boundary is at least k, the number of points in the interior is at least $4n^2 - 4n - k$. We can also estimate that the perimeter is at least k, because each side of the convex hull is the distance between two lattice points, which is ≥ 1.

On the other hand, the area has to be $\leq (a-2)(b-2)$. And there's a well known fact that the perimeter of the convex shape lying inside other convex shape is smaller for the former. Proof is just prolonging one side of the convex polygon inside; it will cut a part of the outer polygon. The perimeter of the part that is left is smaller, because any side of the polygon is smaller than the sum of the other sides. We can also say that the polygon that is left is convex, because it's the intersection of the line and other convex polygon. Repeating this step, we will soon be left with just the inner polygon.

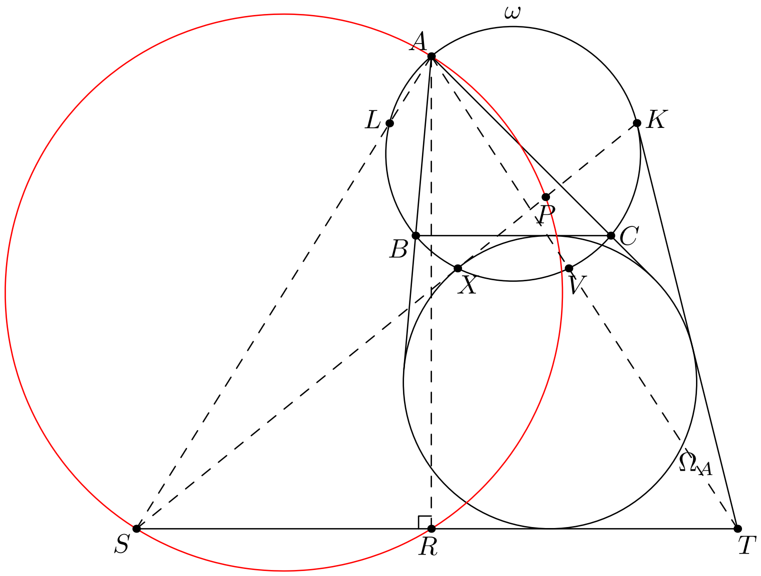
Therefore, we have two inequalities $(a-2)(b-2) \geq 4n^2 - 4n - \frac{k}{2} - 1$ and $2a + 2b - 8 \geq k$. We seek to eliminate k. Thus, we will sum the first inequality with the second divided by 2. $ab - a - b \geq 4n^2 - 4n - 1$. So, $(\frac{a+b-2}{2})^2 \geq (a-1)(b-1) \geq 4n^2 - 4n$. If $a + b \leq 4n - 1$, then $16n^2 - 24n + 9 \geq 16n^2 - 16n$, which is a contradiction due to $n \geq 3$. Thus, $a + b \geq 4n$.

Remark. Similar problems are USEMO 2022/1 and RMM 2008/4.

§6.16 ISL 2021 G8, proposed by Dominik Burek

Problem 16 (ISL 2021 G8)
Let ABC be a triangle with circumcircle ω and let Ω_A be the A-excircle. Let X and Y be the intersection points of ω and Ω_A. Let P and Q be the projections of A onto the tangent lines to Ω_A at X and Y respectively. The tangent line at P to the circumcircle of the triangle APX intersects the tangent line at Q to the circumcircle of the triangle AQY at a point R. Prove that $\overline{AR} \perp \overline{BC}$.

¶ First solution (Poncelet's porism, DDIT, Reim's theorem) The only unmotivated step in this problem is to guess the point of intersection. By carefully examining the picture, we can guess that it's the intersection of the A-altitude and the tangent to excircle parallel to BC, call this point R. We will only prove that RP is tangent to (APX).

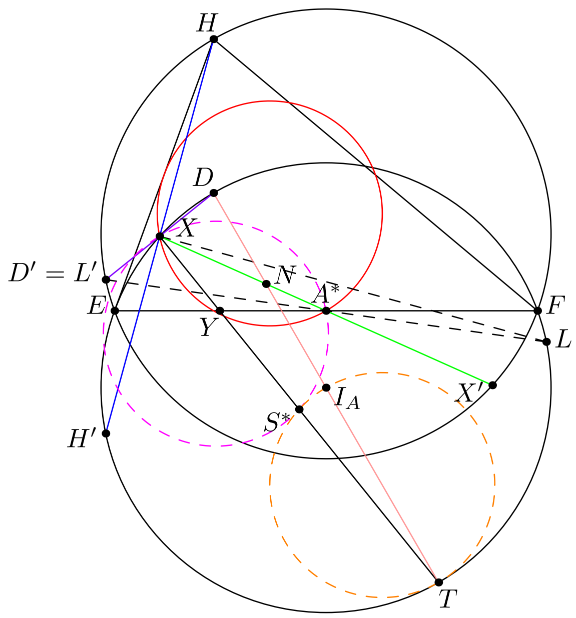


It's hard to establish any angle congruencies, but the picture is approachable to projective techniques because it involves many intersections and tangencies. Hence, we want to change the statement from angle equality to some intersection or parallelism. First, we note that $\angle XAP = \angle XPR$ is equivalent to $\angle XAS = \angle RAP = \angle PSR$, where we used the $(APRS)$ and S is the intersection of XP and the line through R parallel to BC. Now, it's easy to spot *Reim's theorem*. If we intersect XS and AS with (ABC) in points K and L, then LK is parallel to the tangent to (SAX) in point S, which is the same as ST. So, we have to prove that $LK \parallel BC \parallel SR$.

The point K is known. By *Poncelet's Porism* with point X on ω and circle Ω_A, we know that KK has to be tangent to Ω_A. Suppose that T is the intersection of this common tangent with SR.

We also know that $(AS; AK)$, $(AB; AC)$, and $(AX; AT)$ are the pairs of involution derived from applying *Dual of Desargues Involution Theorem* on $KXST$, it's incircle Ω_A and the point A. Thus, we want to mark the point of intersection of AT and ω, call it V. Now, we know that if we project the aforementioned involution on ω, then KL, BC, XV are concurrent. Now, to use the abundance of intersections, the finishing step is to apply Pascal's theorem on $AVXKKL$. It tells that S, T and the intersection of XV and LK are collinear. Hence, BC, ST, XV, KL are concurrent, but the first two are parallel, so all of them are parallel. Therefore, $KL \parallel ST$, and we are done.

¶ Second solution (Excircle inversion and nine-point circle reflections) After noting that the problem is equivalent to SR tangent to (AXS), we are inclined to apply the inversion about the excircle. Mark the inverses with a star. Let D, E, F be the points of tangency of the excircle with sides BC, AC, AB. The inversion maps S to the midpoint of XT, where T is the antipode of D on the excircle that also lies on SR. A goes to the midpoint of EF. X doesn't move, but it was on (ABC), so now it's on the nine-point circle of ΔDEF. We need to show that (S^*XA^*) is tangent to the inverse of ST, which is just circle with diameter TI_A, where I_A is the A-excenter. Now, we will restate the problem with respect to ΔDEF.



We first want to rewrite the tangency condition. It's equivalent to $\angle A^*SI_A = \angle A^*XS^* + \angle I_ATS^*$ (then, we would draw the tangent to one circle at S^* and easily show that it's a tangent to the other circle). The appearance of the latter sum prompts us to intersect TD and A^*X at point N, then $\angle A^*NT = \angle A^*XS^* + \angle I_ATS^*$, so we just want to show that $(A^*NS^*I_A)$. We also know that $XT \cap EF = Y$ [1] lies on this circle, because $\angle I_AA^*Y = 90^\circ$ and $\angle I_AS^*Y = 90^\circ$. So, we instead will be showing that $\angle YND = 90^\circ$, which is the same as $(XDNY)$.

We will prove that $\angle YDN = \angle YXN$. Now, we want to use the fact that X lies on the nine-point circle. Points on the nine-point circle are remarkable because we can use a lot of homotheties, especially in combination with circle ΔDEF on which X also lies. We will also complete the configuration and mark the orthocenter H of ΔDEF. Now, the nine-point circle is homothetic to (HEF) with center at D and coefficient 2. It's also homothetic to (EDF) with center H and coefficient 2. Both are clear from the definition. We will mark the point symmetric to D with respect to X. As $\angle DXT = 90^\circ$, we have $TD = TD'$. Now, we want to show that $\angle YD'T = \angle A^*XT$. This is equivalent to showing that the intersection

[1] Forgot about the one in the problem statement, oops

of $YD' \cap A^*X$ lies on $(D'XT)$. By radical axes, if we intersect $D'Y$ with (HEF), then the point of intersection lies on $(D'XT)$. That point also has to lie on A^*X. Which is great, because (HEF) and (DEF) are symmetric with respect to A^*. So, we are proving $(D'XX'T)$, where X' is symmetric A^*. As $\angle D'XT = 90^\circ$, we are only left to show that $\angle D'X'T = 90^\circ$. Two points of this angle have a good reflection across A^*. Thus, for a point symmetric to D' with respect to A^*, we have to show that H, X, and this point form a right angle.

Mark a point L on (DEF) such that $\angle HXL = 90^\circ$. And mark the point symmetric to X with respect to H; we know that it lies on (DEF). $\angle H'XL = 90^\circ$, so $H'L$ is the diameter of (DEF). We also know that DT is the diameter of this circle. Thus, $HL = H'L = DT$. Reflect L across the midpoint of EF to the point L'. $DT = HL = TD'$, so $D' = L'$. Hence, proved.

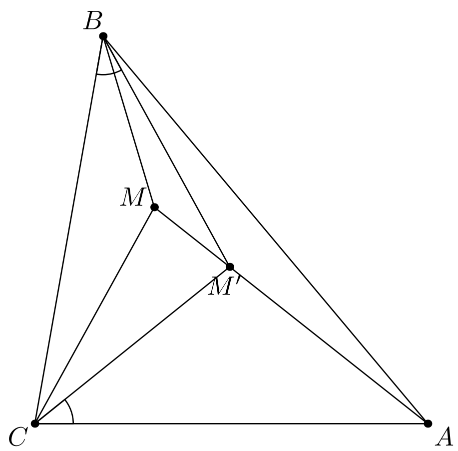
§6.17 Kazakhstan MO 2021 grade 11/3, proposed by Nairi Sedrakyan

Problem 17 (Kazakhstan MO 2021 grade 11/3)
Point M is chosen inside a triangle ΔABC such that $\max(\angle MAB, \angle MBC, \angle MCA) = \angle MCA$. Prove that $\sin\angle MAB + \sin\angle MBC \le 1$.

¶ Solution (Smoothing to a Humpty point inequality) It's known that the less general the construction is, the less possible ways of advancing you have. It makes it easier to find the right path. Here, we are dealing with a very general construction, so we can try to move the points in a way that the sum of sines increases, but we have some more definiteness; then, if we prove that the sum is at most 1 in this case, we would be done.

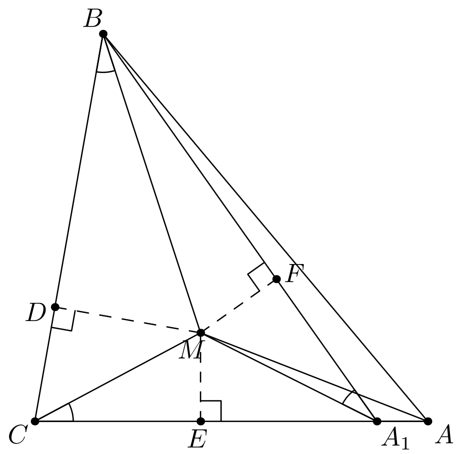
We first note that $180^\circ > \angle MAB + \angle MCA \ge 2\angle MAB$ and $180^\circ > \angle MBC + \angle MCA \ge 2\angle MBC$, so both considered angles are acute. It's convenient, because the sine is monotonically increasing from 0° to 90°, which gives the direction in which we have to animate the points. We just want both angles to increase during the transformations. Suppose that, WLOG, $\angle MBC \ge \angle MAB$.

First transformation will be moving the point M on the line AM towards A. This doesn't change $\angle MAB$, increases $\angle MBC$ and decreases $\angle MCA$ with the difference $\angle MBC - \angle MCA$ increasing continuously. Since for $M = A$ it's equal to $\angle ABC > 0$ and it starts with a non-positive value, there's a point M' on the segment AM such that $\angle M'AB = \angle MAB \le MBC \le \angle M'BC = \angle M'CA$. Thus, we can reduce the problem to the case $\angle MCA = \angle MBC \ge \angle MAB$. Note that now all the angles are acute.



We now fix M, B, C and find the point A' on the line AC such that A' and C are still on different sides of BM, and the angle $\angle MA'B$ is maximised. It's known that this position is the one where $(MA'B)$ is tangent to the line $A'C$. The reason for this is that $(MA'B)$, for this choice of A', intersects the interior of every other triangle ΔMXB for any point X on AC lying on the same side of MB as X. In this case, we can mark a point Y on $(MA'B)$ such that it lies in the interior of ΔMXB, then $\angle MA'B = \angle MYB = \angle MXB + \angle YBX + \angle YMX > \angle MXB$.

If $\angle MA'B > \angle MBC = \angle MCA$, then we can mark the point A_1 on the segment AA' such that $\angle MA_1C = \angle MBC = \angle MCA$. This is because if we vary A_1 from A to A', then the angle $\angle MA_1B$ changes continuously, and since it's $> MBC$ for $A_1 = A'$ and $\leq MBC$ for $A_1 = A$, it will be equal to $\angle MBC$ for some A_1. But it's well known that such an angle in the triangle is $\leq 30^\circ$. [1] The proof is due to *Erdős–Mordell inequality*: if D, E, F are the feet of perpendiculars from M to BC, CA_1, BA_1, then $BM + MC + MA_1 \geq 2(MD + ME + MF)$. Then, if $\angle MBC = \angle MCA_1 = \angle MA_1B > 30^\circ$, then $\frac{MD}{BM} = \sin\angle MBD > \sin 30^\circ = \frac{1}{2}$, analogously for others. Sum of these contradicts the Erdős–Mordell inequality. So, in this case, $\sin\angle MAB + \sin\angle MBC \leq 2\sin\angle MBC \leq 2\sin(30^\circ) = 1$.

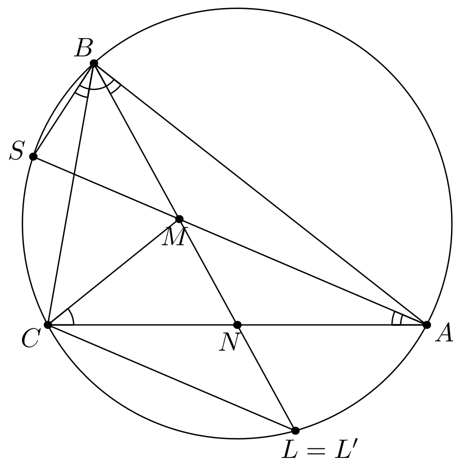


In the other case, we can successfully reduce the problem to only considering M as the *A-Humpty point* of ΔABC. $\angle MBC = \angle MCA \geq \angle MAB$ and $\angle MAC = \angle MBA$. We want to prove that $\sin\angle MAB + \sin\angle MBC \leq 1$. Note that we can rewrite the sine inequality into segments by intersecting BM and AM with (ABC) and using the Law of Sines. More formally, from the Law of Sines

$$\frac{BS}{\sin\angle SAB} = 2R = \frac{CL}{\sin\angle MBC},$$

where $S = MA \cap (ABC)$, $L = BM \cap (ABC)$ and R is the radius of (ABC). So, we only have to show that $BS + CL \leq 2R$. We know that $CN^2 = NM \cdot NB = NA^2$, and $\angle CMA = 180^\circ - \angle MCA - \angle MAC = 180^\circ - \angle CBA$. Therefore, if we mark L' symmetric to M with respect to N, then $CMAL'$ is a parallelogram, and $\angle CL'A + \angle CBA = 180^\circ$, so L' coincides with L. Thus, $CL = AM$. We can also rewrite $\angle MBC = \angle MCA \geq \angle MAB$ as $\angle SBM = \angle SBC + \angle CBM = \angle SAC + \angle CBM = \angle MBA + \angle CBM \geq \angle MBA + \angle MAB = \angle SMB$. Hence, $SB + CL \leq SM + MA = SA \leq 2R$, because diameter is the longest chord.

[1] It's called the Brocard angle and the point is called the Brocard point; there are two of these points



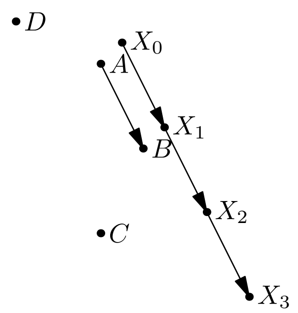
B
S
M
C
N
A
$L = L'$

§6.18 RMM 2020/5, proposed by Maxim Didin

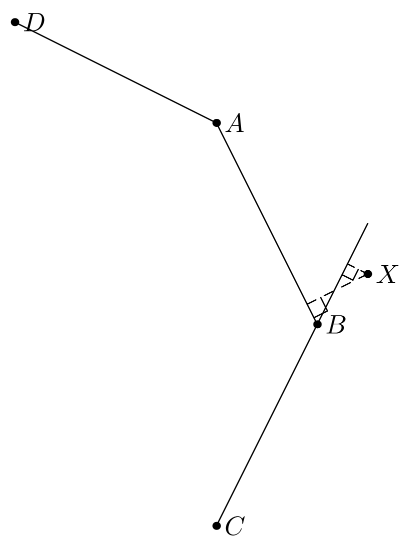
Problem 18 (RMM 2020/5)
A lattice point in the Cartesian plane is a point whose coordinates are both integers. A lattice polygon is a polygon all of whose vertices are lattice points.

Let Γ be a convex lattice polygon. Prove that Γ is contained in a convex lattice polygon Ω such that the vertices of Γ all lie on the boundary of Ω, and exactly one vertex of Ω is not a vertex of Γ.

¶ First solution (Minimal distance and translation) Our goal is to tweak the polygon a little bit. It's natural to try to find a lattice point outside the polygon the "closest" to it in some sense. Note that for a point X that is a particular distance from the side AB, we can translate it by the vector $\overrightarrow{AB}$ such that the projection of this point on the line AB lies on the segment AB. We can also ensure that it lies outside the polygon by reflecting it across AB. To ensure the "closeness", take the pair of a lattice point and a side of Ω that give the minimal distance.



Suppose that $\Omega \cup X$ is not convex. Then, if vertices are in the order D, A, B, C (where C might coincide with D), either X and A are on different sides of BC, or X and B are on different sides of AD. Suppose the former. Then the perpendicular from X to AB intersects the line BC, so the distance from X to AB is less than the distance from X to BC, contradiction to the minimality of the distance.



¶ Second solution (Farey sequence) Our conclusion is equivalent to proving that some region formed by the intersection of three consecutive sides of Γ contains a lattice point. Then, adding this point to the vertices of Γ will result in a convex polygon that contains Γ. If one of formed regions is infinite, suppose that it is formed by the rays AB and DC and the segment BC, then the point symmetric to B with respect to A lies within the region and is obviously lattice. So, we can assume that every region is finite.

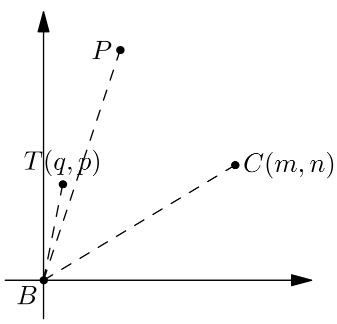
Now, we consider one region formed by the sides AB, BC, CD. Suppose that AB and CD intersect in point P such that P and the polygon lie on different sides of AB. Translate (and maybe reflect) the coordinate plane so that B is the origin. Suppose that C has coordinates (m, n) and consider the case when m, n are positive integers first. We know that the triangle ΔBPC doesn't contain a lattice point. We can suppose that $(m, n) = 1$, otherwise we can just consider a sublattice consisting of points which have both coordinates divisible by (m, n).

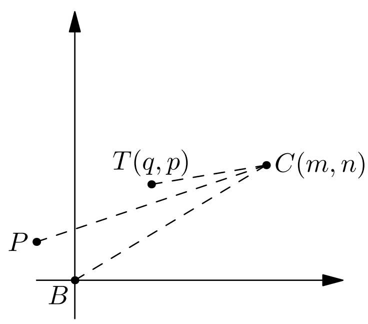
One way of generating a lattice point close to the segment BC is to take the fraction of the *Farey sequence* next to $\frac{n}{m}$ or $\frac{m}{n}$, depending on what is greater, m or n. We also need to make sure that it's in the top half-plane about the line BC. If $m > n$ (we will treat the other case later), then we just take the first fraction greater than $\frac{n}{m}$ in the Farey sequence of order m, suppose it's $\frac{p}{q}$, and consider a point T with coordinates (q, p).

Now we will prove a well-known fact about Farey sequences:

> **Claim** — If $\frac{a}{b} < \frac{c}{d} < 1$ are the consecutive fractions in the Farey sequence, then $bc - ad = 1$.

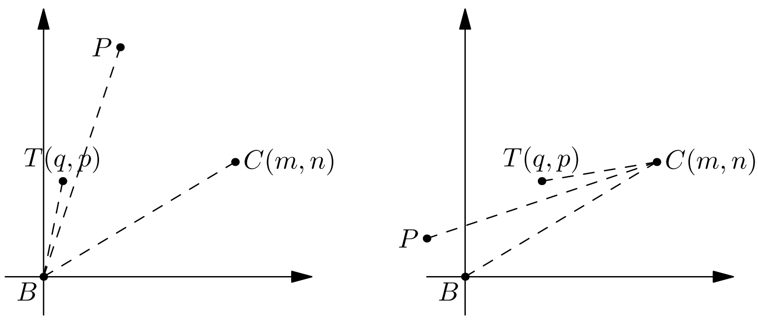
Proof. That fact allows many different proofs. Here's the slick one. Consider a triangle ΔOXY, where O is the origin, X is (b, a), Y is (d, c). Suppose that the triangle contains a lattice point $Z = (x, y)$ inside or on the border excluding the vertices. Then $\frac{y}{x}$ is the slope of the line OZ. We also know that its slope is in-between the slopes of OX and OY, so $\frac{a}{b} < \frac{y}{x} < \frac{c}{d} < 1$, which means that x is out of the order of the Farey sequence, otherwise $\frac{a}{b}$ and $\frac{c}{d}$ wouldn't be consecutive ($\frac{\frac{y}{(x,y)}}{\frac{x}{(x,y)}}$ is between them), so $x > \max(b, d)$, which is not possible for a point inside ΔOXY. Thus, no lattice point lies inside or on the border of ΔOXY, so, by Pick's formula, its area is just $0 + \frac{3}{2} - 1 = \frac{1}{2}$. On the other hand, the area of ΔOXY is known to be equal to $\frac{1}{2}|\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)| = \frac{bc-ad}{2}$ (simple calculation using right-angle triangles and a right-angle trapezoid suffices, for example), from this the conclusion follows.





$\square$

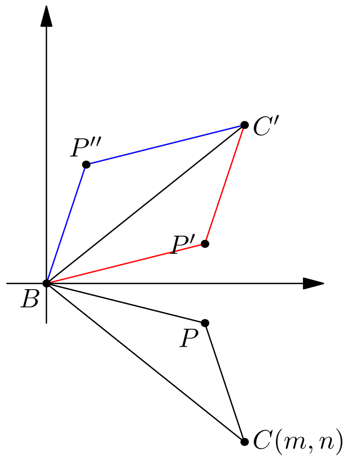
Now, back to the main problem. We know that $pm - nq = 1$. If $q = m$, then $m|1$, so $1 = m > n$, which is a contradiction. Thus, $q < m$ and $p \leq n$. Now, we have that P lies outside of ΔBPC. So, either BP is in between BC and BT, which means that the slope of BP is in between the slopes of BT and BC, so, by the argument similar to the one used in the proof of the claim to prove that no lattice points contained in the triangle ΔOXY, the denominator (in the simplified fraction) of the slope of BP is greater than the denominators (in the simplified fractions) of slopes of BT and BC. The case when CP is in between the CT and CB is similar. $\frac{n-p}{m-q}$ and $\frac{n}{m}$ are the slopes, respectively. We know that $n(m-q) - m(n-p) = 1$; it's easy to prove that $\frac{n}{m}$ and $\frac{n-p}{m-q}$ are consecutive in the Farey sequence of order m. Therefore, again, the denominator (in the simplified fraction) of the slope of CP is greater than m and $m - q$. It can be merged with the previous case in a form that the denominator (in the simplified fraction) of the slope of either BP and CP is greater than that of BC.



If every slope satisfied the assumptions that lead to the previous case, then we would just take the one with the maximal denominator (in the simplified fraction), and its region would contain a lattice point. Now we will go onto omitted cases for the rigorous proof.

If $m < n$, then we take the fraction $\frac{p}{q}$ that is the first one smaller than $\frac{m}{n}$ and consider a point T with coordinates (p, q). We can think about it as reflecting the previous situation across $y = x$, thus flipping coordinates of every point. All slopes will flip (become $\frac{1}{\text{slope}}$). In this case, after repeating the same argument as in the previous case, the numerator of the slope of either BP or CP is greater than the numerator of the slope of BC. So, we can merge it with the previous case by considering a quantity $f\left(\frac{x}{y}\right) = \max(x, y)$, then $\max(f(m(BP)), f(m(CP))) > f(m(BC))$, where m is a simplified fraction of the slope of the line. The only case we need to highlight is when $m = 1$, then $\frac{p}{q} = 0$, which is impossible to flip. In this case, consider T with coordinates $(0, 1)$. If BP is in between BT and BC, then $m(BP) > n$, which proves that the numerator of $m(BP)$ is greater than n. If CP is in between CT and CB. Then the y-intercept of CP is $\frac{p}{q} < 1$ (obviously, it's rational), then $m(CP)$ is $n - \frac{p}{q}$, that has a numerator $nq - p > (q-1)n \geq n$. This case is thus proved.

Now, we have to deal with different signs of m and n. Note that we were dealing only with positive slopes before, so, to account for negative slopes, we will set $f(-p) = f(p)$ for every rational number p. Now, whatever the signs of m and n, we can reflect the lines with respect to axes (that flips the sign of the slope, but by our definition of f, it doesn't matter), and reflect across the point (doesn't change the slope, but need to make sure that the coordinates are still integers). Example of turning $m > 0$, $n < 0$ into the case previously done is as follows: We first reflect P and C to P', C' across the x-axis, then we reflect P across the midpoint of BC' to the point P''. It's not hard to check that all the lattice points go into lattice points, and no f values of the slopes change.



The only case that we didn't cover is when one of the m or n is zero. However, if take the side that is not axis-aligned and has the maximum f-value, then one of the neighbouring sides must also not be axis-aligned and have a greater f-value, which is a contradiction. The only non-covered case is when every side is axis-aligned, which gives a rectangle, but then regions formed by three consecutive sides are infinite, which was covered above.

§6.19 USA TSTST 2023/6, proposed by Holden Mui

Problem 19 (USA TSTST 2023/6)
Let ABC be a scalene triangle and let P and Q be two distinct points in its interior. Suppose that the angle bisectors of $\angle PAQ$, $\angle PBQ$, and $\angle PCQ$ are the altitudes of triangle ABC. Prove that the midpoint of PQ lies on the Euler line of ABC.

¶ First solution (Rectangular circumhyperbolas and antigonal conjugates) As we have seen in problem 11, problems about incenters and orhtocenters are equivalent in some sense. In this problem, we want to show that some point lies on a line through H, that will be fixed after performing the orthocenter inversion. Thus, it's reasonable to try to solve the incenter problem after the inversion.

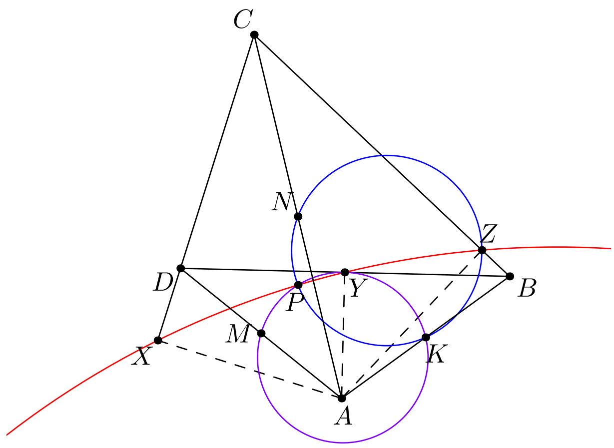
Keep the labels before and after inversion the same, but change H to I. $\angle PAH = \angle HAQ$ will now be $\angle IQA = \angle IPA$. It is equivalent to (PIA) and (QIA) symmetric with respect to IA (easy to check that P and Q are on different sides, because otherwise, P, Q, A would be collinear), same with others. Consider the triangle ΔBIC. $\angle BPC = \angle CPI - \angle BPI = \angle CQI - \angle BQI = \angle BQC$. Thus, (BQC) is symmetric to (BPC) with respect to BC. Such pair of pointsthat satisfy the symmetries of circles with respect to sides, is called the *Antigonal Conjugates* with respect to triangle. In the same way, we can prove that P and Q are antigonal conjugates in ΔABI, ΔAIC and ΔABC.

We will now develop some theory about antigonal conjugates and rectangular hyperbolas:

Claim — For any four point A, B, C, D that don't form an orthocentric quadruple (i.e. one of them is the orthocenter of a triangle formed by the three others), nine-point circles of triangles ΔABC, ΔABD, ΔBDC, ΔADC pass through one point. This point also lies on pedal circle of A with respect to ΔBCD, same for other vertices.

Proof. Intersect the pedal circle of A with respect to ΔBCD with the nine-point circle of ΔBCD in point P. Suppose that X, Y, Z are the feet of perpendicular from A to CD, BD, BC. Suppose also that M, N, K are the midpoints of AD, AC, AB. We will prove that P lies on (ZKN), which is a nine-point circle of ΔABC.

$\angle ZPK = \angle YPK - \angle YPZ = \angle YMK - \angle YXZ = \angle MYD + \angle CXY - \angle CXZ = \angle ADB + 90^\circ - \angle ADB - 90^\circ + \angle ACB = \angle ACB = \angle NZC = \angle ZNK$.



Thus, P lies on nine-point circle of ΔABC. Analogously, we can show that P lies on the nine-point circle of ΔACD. Hence, every three of the considered nine-point circles intersect in one point. Therefore, all of them do. And we have also showed that their point of intersection lies on every pedal circle of one of the vertices with respect to remaining triangle. □

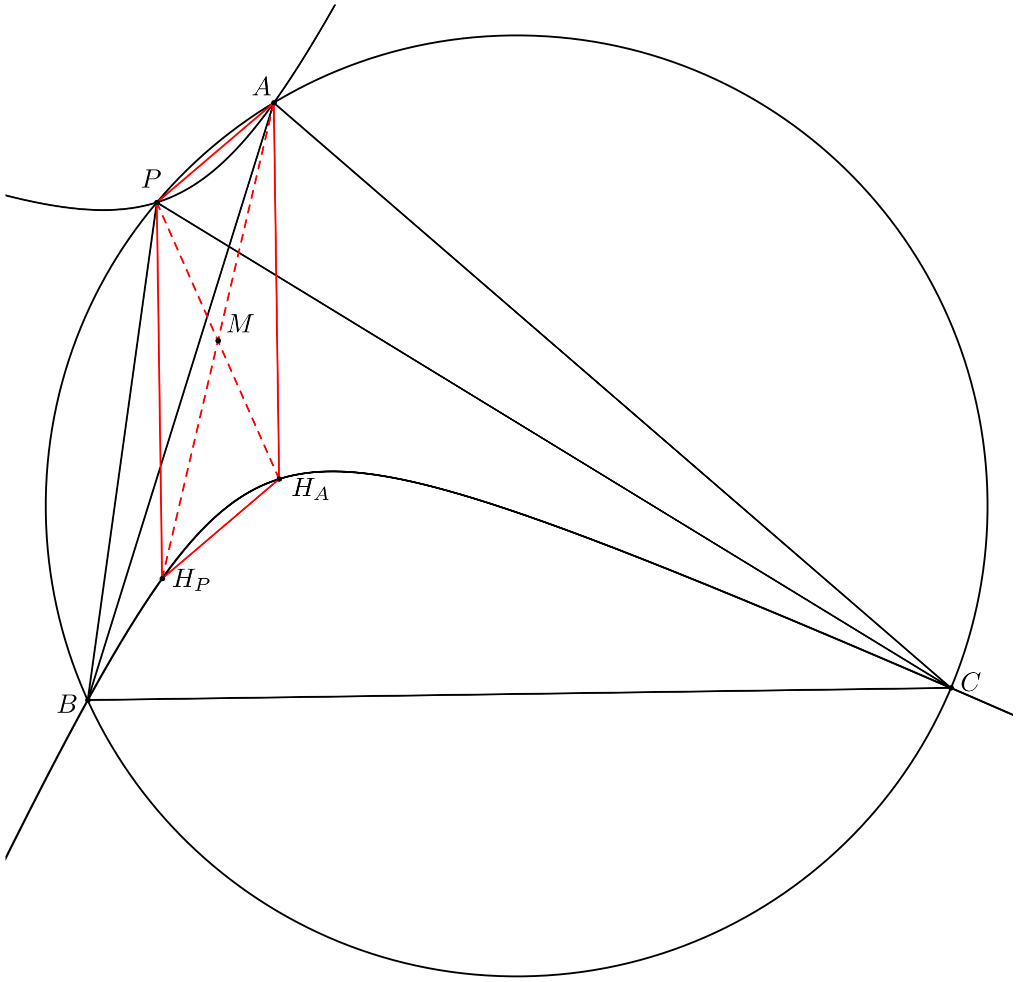
The point existence of which we proved in the last claim is called the *Poncelet's point*.

We will now prove an important fact about rectangular circumhyperbolas, besides the one proved in USA TSTST 2020/6:

Claim — Center of rectangular circumhyperbola $\mathcal{H}$ of triangle ΔABC lies on the nine-point circle of this triangle.

Proof. Intersect $\mathcal{H}$ with (ABC) in point P. For now, suppose that P differs from A. Since the hyperbola is rectangular, by USA TSTST 2020/6, orthocenter H_P of ΔABP lies on $\mathcal{H}$, as well as orthocenter H_A of ΔABC. We know that $AH_A = |\cot\angle BAC \cdot BC| = |\cot\angle BPC \cdot BC| = PH_P$ (same as in USA TSTST 2020/6) and both are perpendicular to BC, so APH_PH_A is a parallelogram. Suppose that the center of $\mathcal{H}$ is not the center of the parallelogram. If the center of $\mathcal{H}$ is M, then reflect A and P with respect to M to points A' and P'. $AA'H_PPP'H_A$ lie on one conic, so Pascal's theorem tells that $AA' \cap PP' = M$, $A'H_P \cap P'H_A$, $H_PP \cap H_AA$ are collinear, but it's not possible because the second and the third intersections form a line on infinity (given that no points coincide), while M doesn't lie on it. Thus, some of the six points coincide. Therefore, M is the center of APH_PH_A. But M is the midpoint of PH_A, which is known to lie on the nine-point circle of ΔABC, because it's homothetic to (ABC) with center H and a $\frac{1}{2}$ coefficient.

We can handle the case when $P = A$ by continuity (i.e. that all the points in the neighbourhood lie on the nine-point circle, and the center moves continuously). But here's the slick geometrical proof: We know that the tangent to $\mathcal{H}$ at A is also tangent to (ABC) (every line through A that intersects (ABC) for the second time must also intersect the hyperbola for the second time). Suppose that E and F are the feet of altitudes and $X = AH_A \cap EF$ from B and C in ΔABC. Suppose also that $X = AH_A \cap EF$. $(XT; FE) = (TX; EF) \overset{H_A}{=} (AH_A; BC) \overset{A}{=} (\infty_{EF}R; FE)$, because any conic preserves cross ratios and we can project them as we do with circles and lines (these are also special cases of conics), and we have also used that the tangent to (ABC) at A is parallel to EF. Thus, we have that the tangent at H_A to $\mathcal{H}$ is parallel to the tangent at A to the same hyperbola. But it's not hard to show that the parabola has only two points with the same slopes of tangents, and they must be symmetrical with respect to the center. Thus, center is the midpoint of AH_A, which obviously aligns with the desired statement about the nine-point circle.



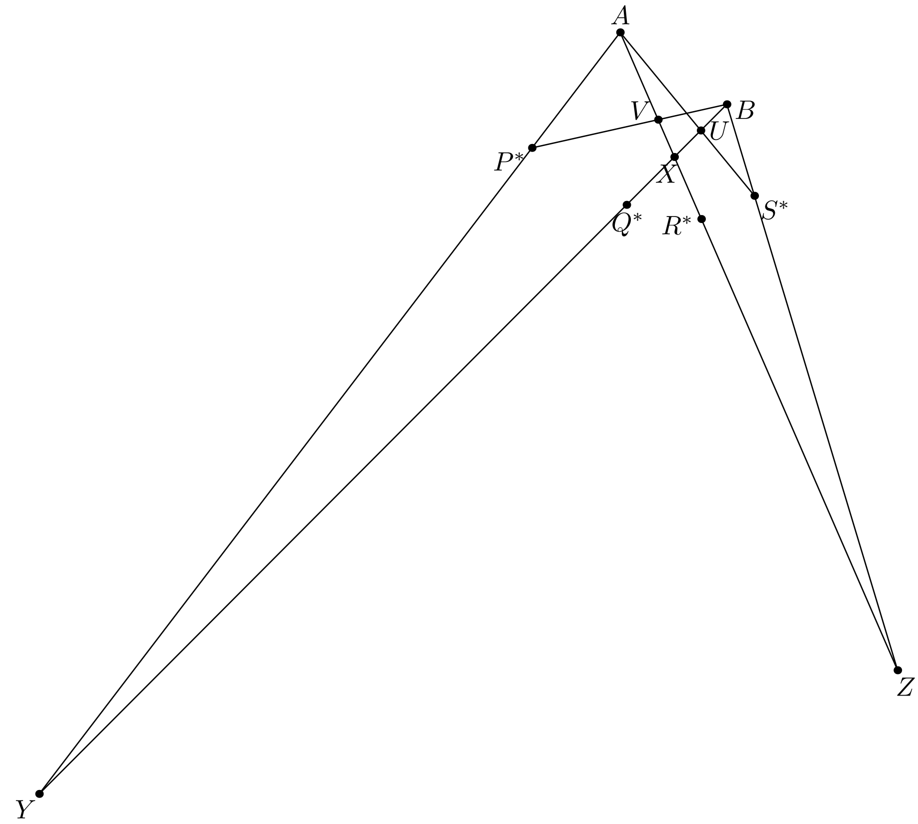
□

Now, we need another claim for understanding the nature of circumconics:

Claim — Image of a line ℓ under the isogonal conjugation with respect to ΔABC is a conic passing through A, B, C.

Proof. Usually this statement is proved by *Trilinear Coordinates*, but we will present a more geometric proof. Suppose that P, Q, R, $S \in \ell$, and suppose that P^*, Q^*, R^*, S^* are their isogonal conjugates. Suppose that P^*, Q^*, R^* are collinear, we will prove that R^* is on this line too. $(AP^*, AQ^*; AR^*, AS^*) = (AP, AQ; AR, AS) = (PQ; RS) = (BP, BQ; BR, BS) = (BP^*, BQ^*; BR^*, BS^*)$. If we project is onto line $\overline{P^* - Q^* - R^*}$, then BR^* and AR^* intersect it in the same point. Either R^* lies on AB or R^* is collinear with the. Thus, the image of ℓ is the line $\overline{P^* - Q^* - R^*}$ with some points on the sides AB, BC, AC, not necessarily collinear with the aforementioned points, but the isogonal conjugates of these point must be the vertices of the triangle (because of the continuity), contradicting the fact that they lie on ℓ. Thus, we can assume that no three points of the isogonal conjugate images are collinear.

We have showed that $(AP^*, AQ^*; AR^*, AS^*) = (BP^*, BQ^*; BR^*, BS^*)$; it's well-known to imply that $AP^*Q^*R^*S^*B$ lie on the same conic. The proof is as follows:



The easiest geometric way to check that six points lie on the same conic is the converse of the Pascal's theorem. We are given two equal cross-ratios, so it makes sense to try to establish collinearities by projecting. $(R^*Z;VX) = (VX;R^*Z) = (BP^*, BQ^*; BR^*, BS^*) = (AP^*, AQ^*; AR^*, AS^*) = (YQ^*; XU)$, so VU, Q^*R^*, YZ are concurrent. By Pappus' theorem on lines $\overline{A-V-Z}$ and $\overline{B-Y-U}$, $BV \cap AY$, $AU \cap ZB$ and $VU \cap YZ$ are collinear. Which shows that P^*S^* passes through the intersection point of Q^*R^*, VU and YZ. Thus, the converse Pascal's on $AS^*P^*BQ^*R^*$ tells that these six points lie on one conic. In the same way, the conic also passes through C.

Now we are left to show the converse statement. Take two points X and Y on the conic and find their isogonal conjugates X^* and Y^*. Then draw a conic that is the image of the line through X^* and Y^*. It passes through A, B, C, X, Y, so it coincides with the initial conic. Thus, it's image coincides with the line through X^* and Y^*.

Alternatively, to prove one direction of the theorem, we could quote the *Solleritinsky Lemma* that states that if we have two points, X and Y, and we animate two lines centered at X and Y projectively, then their intersection moves in a conic passing through X and Y.

To use it in the problem, it's trivial that the reflection with respect to angle bisector fixes cross-ratios, so just animate a point Z on the line ℓ projectively and reflect AZ and BZ with respect to corresponding angle bisectors. The intersections of obtained lines will be the isogonal image of ℓ. □

Now, we connect the rectangular hyperbolas and antigonal conjugates:

> **Claim —** If P and Q are antigonal conjugates with respect to ΔABC, then $ABCPQ$ lie on the rectangular hyperbola.

Proof. We first prove that the midpoint of PQ is the Poncelet's point of $\Delta ABCD$. Reflect the point P across AB to the point P'; $(ABQP')$ by the angle conditions. Homothety with coefficient $\frac{1}{2}$ centered at P takes (ABP') to the nine-point circle of ΔPAB, so the midpoint of PQ lies on the nine-point circle of ΔPAB, ΔPAC, and ΔPBC. The only point satisfying it is the Poncelet's point. Now, draw a rectangular hyperbola through points $ABCP$. It's center lies on nine-point circles of ΔABC, ΔABP, ΔAPC. ΔPBC by the above claim, so it coincides with the Poncelet's point of $ABCP$. Reflecting P across the center of the hyperbola takes it to a point on the hyperbola, but it also takes it Q, so Q lies on this hyperbola

□

All the facts above are actually known, but I promised to prove everything, so consider it a small handout on properties of rectangular hyperbolas. We are now ready to tackle the problem.

We stopped at proving that P and Q are antigonal conjugates with respect to ΔABC, ΔABI, ΔACI, ΔBCI, so $PQABCI$ lie on a rectangular hyperbola [1]. We have to show that OI is the symmedian of triangle ΔIPQ, then OH would pass through the midpoint of PQ before the inversion. The hyperbola is an isogonal conjugation image of OI (because H and I lie on the hyperbola). Suppose that OI intersects the hyperbola for the second time in point P, then the isogonal conjugation image of P lies on the hyperbola and the line OI, so it's either P itself or I. In the first case, P is either I or one of the excenters (as the only points that go to itself after the isogonal conjugation); if it's the excenter, then it cannot lie with O and I on the same line. Thus, $P = I$ in either case. Therefore, OI is tangent to the hyperbola. We also know that the center of the hyperbola is a midpoint of PQ. Now, the problem can be restated as follows with respect to triangle ΔPQI:

Lemma (Restated problem)
Given a triangle ΔABC. Suppose that $\mathcal{H}$ is a rectangular hyperbola passing through A, B, C, whose center is the midpoint of BC. Then the tangent to $\mathcal{H}$ at A is the symmedian of ΔABC.

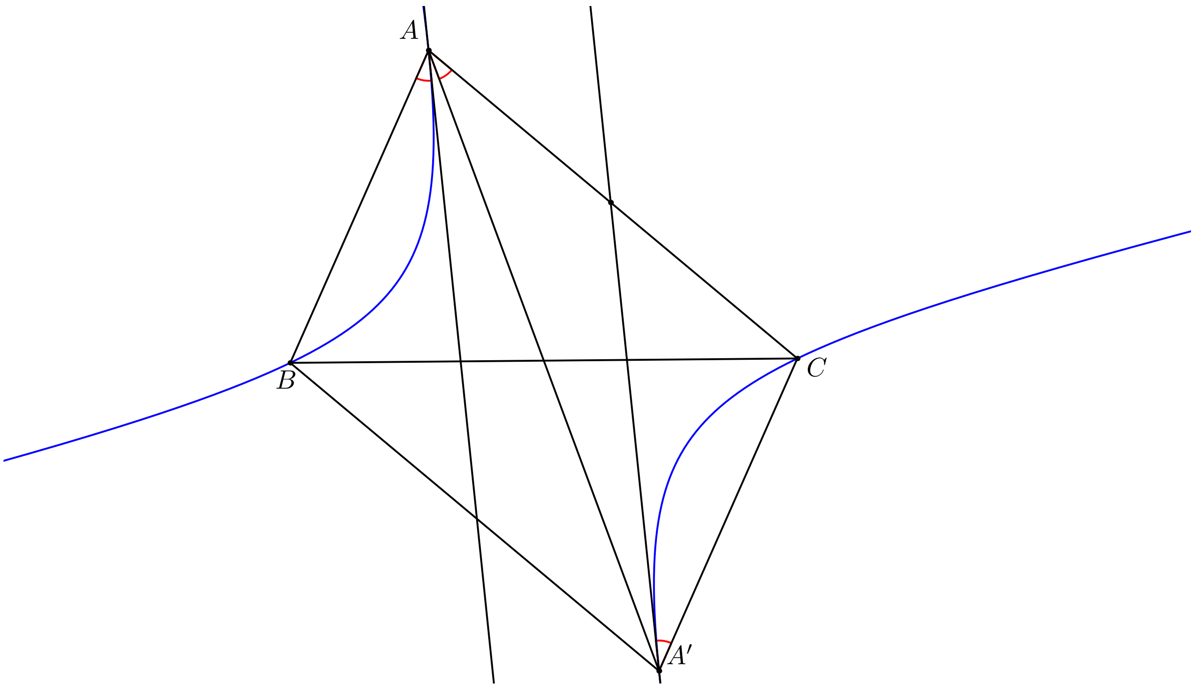
Proof. First, the condition of the center on BC is actually equivalent to the fact that $\mathcal{H}$ is the isogonal conjugation image of the bisector to BC. Reflect A across the midpoint of BC, the resulting point A' lies on the hyperbola. $\angle A'BA = \angle A'BC$, so it's not hard to establish that the isogonal conjugation image of A' lies on the perpendicualr bisector of BC. We also know that the image passes through circumcircle (because the orthocenter lies on the hyperbola). Thus, since the image is the line, it coincides with the perpendicular bisector to BC.

We now consider the triangle $\Delta ABA'$. The center of $\mathcal{H}$ is the midpoint of AA'. Therefore, it's the isogonal conjugation image (with respect to triangle $\Delta ABA'$) of the perpendicular bisector of AA'. From this, we can get that for every point X on $\mathcal{H}$, $\angle XA'B = \angle XAB$. It's reasonable that if $X = A'$, then the angle between the tangent at A' to $\mathcal{H}$ and $A'B$ is equal to $\angle A'AB$. The formal proof is as follows: suppose the line with such angle property is

[1] It's called the Feurbach's hyperbola

not tangent to $\mathcal{H}$, then consider it's second intersection point Y, then $\angle A'AB = \angle XA'B = \angle YA'B = \angle YAB$. Thus, $Y = A'$, which is a contradiction.

But that angle equality is what we wanted, because, by symmetry, angle between the tangent at A' and $A'B$ is equal to the angle between the tangent at A and AC, and we just proved that it's equal to $\angle A'AB$. Therefore, it's indeed the symmedian.



□

¶ Second solution (Complex bash) Set (ABC) as the unit circle. a, b, c, p, q are the complex coordinates of points A, B, C, P, Q, respectively. Then the condition of AH bisecting $\angle PAQ$ can be restated as

$$\frac{p-a}{b+c} : \frac{b+c}{q-a} \in \mathbb{R}.$$

Where $b+c = a+b+c-a$ is the coordinate for vector $\overrightarrow{AH}$. We can rewrite the real condition in an algebraic way by noting that:

$$\frac{(p-a)(q-a)}{(b+c)^2} = \overline{\left(\frac{(p-a)(q-a)}{(b+c)^2}\right)}.$$

This can be rewritten using linearity and multiplicativity of complex conjugation, as well as the fact that $a\overline{a} = |a|^2 = 1$, $b\overline{b} = 1$, $c\overline{c} = 1$.

$$\frac{(p-a)(q-a)}{(b+c)^2} = \frac{(\overline{p}-\frac{1}{a})(\overline{q}-\frac{1}{a})}{(\frac{1}{b}+\frac{1}{c})^2}.$$

$$(p-a)(q-a) = b^2c^2(\overline{p}-\frac{1}{a})(\overline{q}-\frac{1}{a}).$$

Which is the same as

$$pq - a(p+q) + a^2 - b^2c^2\overline{pq} + \frac{b^2c^2}{a}(\overline{p}+\overline{q}) + \frac{b^2c^2}{a^2}. \ (\heartsuit)$$

And, of course, its cyclic variants.

To simplify our job further, let's see what we actually need to show in terms of complex numbers. The midpoint of PQ lies on OH is the same as proving that $\frac{\frac{p+q}{2}}{a+b+c} \in \mathbb{R}$. Using the same tricks we have encountered before and multiplying by 2, it can be rewritten as:

$$\frac{\overline{p}+\overline{q}}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{p+q}{a+b+c} \iff \frac{p+q}{\overline{p}+\overline{q}} = \frac{ab+bc+ac}{abc(a+b+c)}.$$

We see that we have $p+q$ and $\overline{p}+\overline{q}$ with coefficients as polynomials in a, b, c in three equations derived from $\heartsuit$. We also have a zero in the RHS. It screams to try to find weights (in terms of a, b, c) for each equation, so that when we sum them, we will only have $f(a,b,c)(p+q) + g(a,b,c)(\overline{p}+\overline{q}) = 0$, and then it will be easy to finish. It's not the most certain way, but it still worth trying; it happens to work nicely here.

If weights assigned are α, β, γ, then, equating pq coefficient to zero, $\alpha+\beta+\gamma = 0$, so $\gamma = -\alpha-\beta$. Equating the $\overline{pq}$ term to zero and changing γ,

$$(b^2c^2 - a^2b^2)\alpha = (a^2b^2 - a^2c^2)\beta.$$

From this, we see that

$$\frac{\alpha}{\beta} = \frac{\frac{1}{c^2}-\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{c^2}}.$$

Note that, since our equations are equal to zero, we can scale (α,β,γ) by any constant. Thus, we can assume that $\alpha = \frac{1}{c^2}-\frac{1}{b^2}$, $\beta = \frac{1}{a^2}-\frac{1}{c^2}$, $\gamma = \frac{1}{b^2}-\frac{1}{a^2}$. We only need to check that the constant term is 0. It is true because

$$\sum_{\text{cyc}} \frac{(a^4-b^2c^2)(b^2-c^2)}{a^2b^2c^2} = 0,$$

which can be shown by direct expansion.

What is left is

$$(p+q)\left(\frac{a}{b^2}-\frac{a}{c^2}+\frac{b}{c^2}-\frac{b}{a^2}+\frac{c}{a^2}-\frac{c}{b^2}\right) = (\overline{p}+\overline{q})\left(\frac{c^2}{a}-\frac{b^2}{a}+\frac{a^2}{b}-\frac{c^2}{b}+\frac{b^2}{c}-\frac{a^2}{c}\right) \iff$$

$$(p+q)(a^3c^2-a^3b^2+a^2b^3-b^3c^2+b^2c^3-a^2c^3) = (\overline{p}+\overline{q})(ab^2c^4-ab^4c^2+a^4bc^2-a^2bc^4+a^2b^4c-a^4b^2c).$$

What is left is just factoring (or tedious expansion works too, if you don't happen to come across the factorization). Usually, the expressions like this are divisible by $(a-b)(b-c)(a-c)$ or some other symmetric options. It's easy to check divisibility by $a-b$ by simply plugging $a=b$ and observing if we have a zero or not. Both brackets equate to zero, so we can divide by $(a-b)(b-c)(a-c)$. What will be left, believe it or not, is just

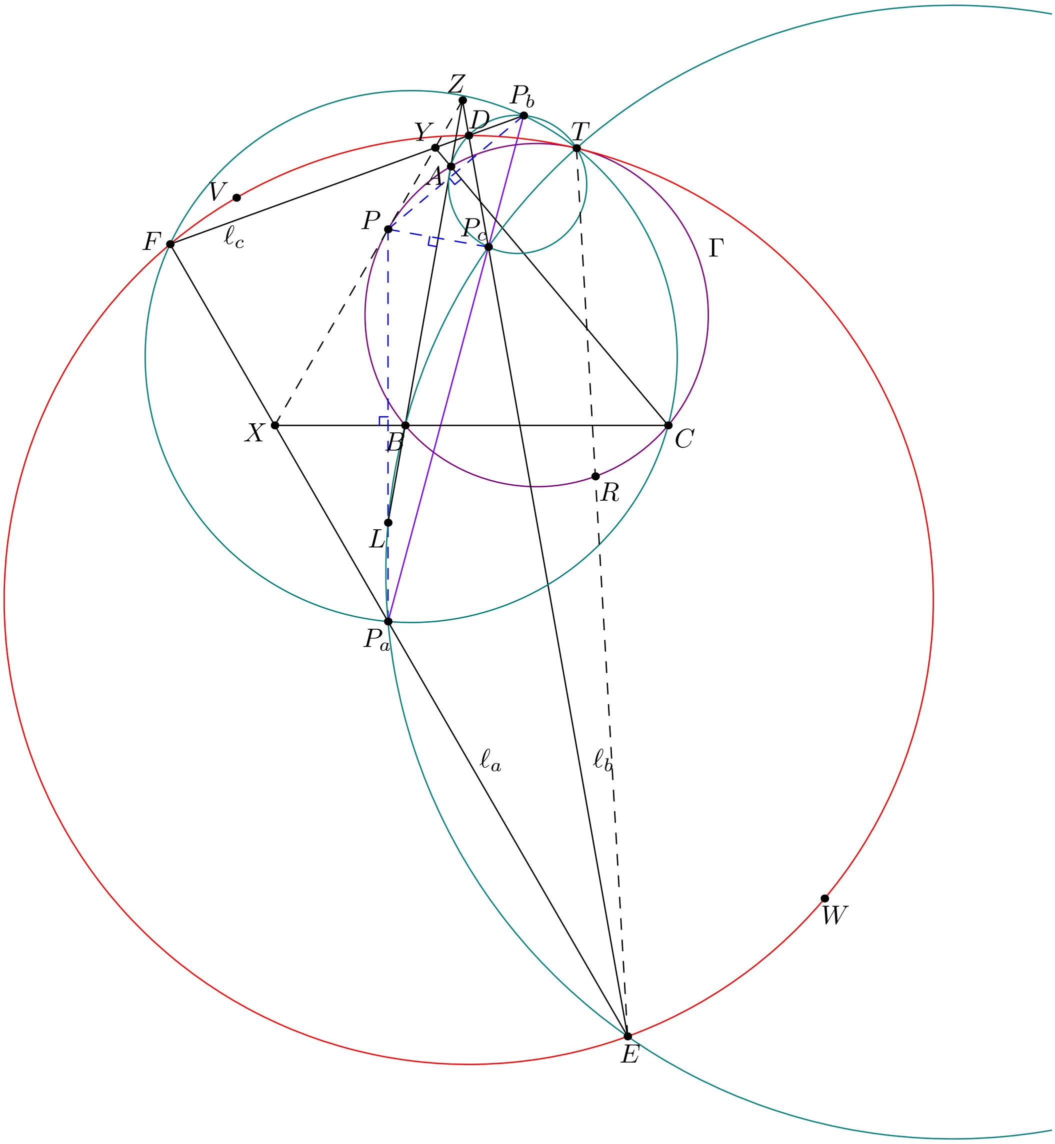
$$(p+q)(ab+bc+ca) = (\overline{p}+\overline{q})(abc(a+b+c)).$$

Which is exactly the same as what we wanted to show.

§6.20 IMO 2011/6, proposed by Japan

Problem 20 (IMO 2011/6)
Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

¶ First solution (Miquel point and angle chase) There are quite a few problems about intersecting three lines to form a triangle, then proving that its circumcircle is tangent to some other circle. Key to those problems is that the tangency point is usually the Miquel point of some quadrilateral formed by the three aforementioned lines and one arbitrary. After that we have enough cyclic quadrilaterals to angle chase.



In this problem, we already have ℓ_a, ℓ_b, ℓ_c. We need to find a fourth line that is symmetrically defined with respect to every side AB, BC, AC. Leap of faith would be to consider a line through points symmetrical to P with respect to the sides of the triangle (this line is also called the *Steiner line* of point P); call the points P_a, P_b, P_c for lines BC, AC, AB, respectively. The existence of this line easily follows from the existence of Simson line of point P. This line is natural to consider because the points of intersection with ℓ_a, ℓ_b, ℓ_c are easily defined; it's just P_a, P_b, P_c. And it's symmetrically defined with every side.

Suppose that $\ell_b \cap \ell_c = D$, $\ell_a \cap \ell_c = E$, $\ell_a \cap \ell_b = F$. Suppose also that T is the Miquel point of the quadrilateral DFP_aP_c. We will prove that T is the touching point.

We first note that (ADP_bP_c). It's true because $\angle P_cAP_b = 2\angle P_cPP_b = 2\angle BAC = 2(180^\circ - \angle AYZ - \angle AZX) = 360^\circ - 360^\circ + \angle PYD - \angle YZD = \angle YDZ = \angle P_cDP_b$, where we have used that $P_cA = PA = AP_b$ to establish the first equality. Analogously, (BEP_aP_c), (CFP_aP_b).

Now, we will show that $T \in \Gamma$. $\angle ATC = \angle ATP_c + \angle P_cTP_a + \angle P_aTC = \angle AP_bP_c + 180^\circ - \angle P_cBP_a + \angle P_aP_bC = 90^\circ - \angle BAC + 180^\circ - 2\angle ABC + 90^\circ - \angle BCA = 2(\angle BAC + \angle BCA + \angle ABC) - \angle BAC - 2\angle ABC - \angle BCA = \angle BAC + \angle BCA = 180^\circ - \angle BAC$, which is equivalent to T lying on Γ.

To prove that circles are tangent, we will use a strategy of proving that they are homothetic. For this, consider $TA \cap (DEF) = V$ and $TC \cap (DEF) = W$. We will prove that $AC \parallel VW$. Since we intersected some line with a circle and we have many intersections of circles, it's reasonable to try to find *Reim's theorem* somewhere. $AP_c \parallel EV$ by Reim's theorem on (DTP_cA) and $(DTEV)$. We need to show that $\angle ACT = \angle VWT$, but $\angle VWT = \angle VET = \angle(ET, AP_c)$. Thus, we need to show that $\angle ACT = \angle(ET, AP_c)$, which is equivalent to $ET \cap AP_c \in \Gamma$. Mark $ET \cap \Gamma = R$ and $AB \cap (ELBP_c) = L$. Reim's theorem for (EP_aBT) and $(ARBT)$ tells that $AR \parallel EL$. Now it's easy to finish. $\angle BAR = 180^\circ - \angle BLE = 180^\circ - \angle BP_aE = \angle BP_aX = \angle BPX = \angle BAP = \angle BAP_c$, so P_c lies on AR. Hence, proved

¶ Second solution (Miquel point inversion) With geometry progressing, this problem is no longer hard. The setup is well-known if you recognize it.

As in the previous solution, mark the triangle formed by three lines as DEF and mark points of intersection of ℓ with sides of it as X, Y, Z. C is the excenter of ΔFYX, A is the excenter of ΔYZD, B is the incenter of ΔZXE. We also know that $\angle YAD + \angle XBE = 90^\circ - \frac{\angle YZD}{2} + 90^\circ + \frac{\angle YZD}{2} = 180^\circ$, so $AD \cap BE$ also lies on Γ. So, Γ is just a circle through intersections of internal angle bisectors of angles of the quadrilateral $XYDE$. We are given that it's tangent to XY and we need to prove that it's also tangent to (FDE). Now, the problem will follow from the following fact, also used in the solution to Brazil National Olympiad 2016/6:

> **Claim** — Opposite sides of quadrilateral $ABCD$ intersect in points $E = AB \cap CD$ and $F = BC \cap AD$. M is the Miquel point of this $ABCD$. Consider the inversion with radius $\sqrt{MA \cdot MC} = \sqrt{MB \cdot MD} = \sqrt{ME \cdot MF}$ followed by the symmetry with respect to the common angle bisector of $\angle EMF$, $\angle AMC$, $\angle BMD$ (these products and angle bisectors are equal because of the spiral similarities). Then, the quadrilateral formed by intersections of angle bisectors of consecutive angles of $ABCD$ is cylic and it's circumcircle stays fixed after the inversion.

Proof. It's easy to show that the resulting quadrilateral is cyclic. Now, we will go to a more interesting part of the problem.

Write $*$ for images. We will prove that, if X^* is the inverse of X, then $\angle AX^*D = \angle ABX + \angle XCD$.

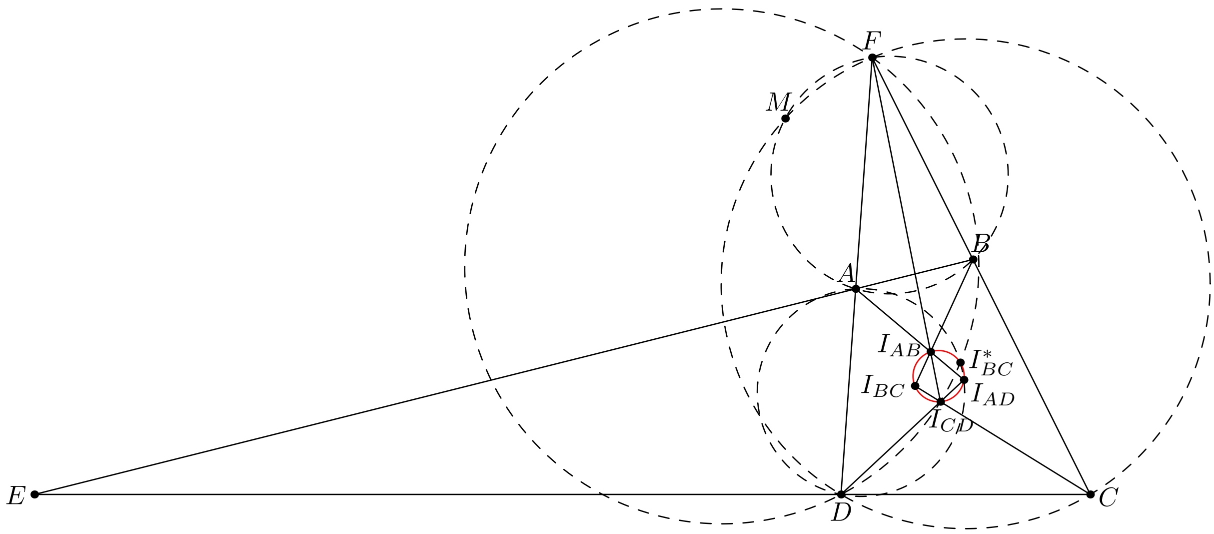
$\angle XBA + \angle XCD = 360^\circ - \angle XBE - \angle XCE = \angle AED + \angle BXC = \angle AMD + \angle BXM + \angle MXC = \angle MAX^* + \angle AMD + \angle X^*DM = \angle AX^*D$, where we have used $\Delta MBX \sim \Delta MX^*D$ and $\Delta MXC \sim \Delta MAX^*$ that follows from $MX \cdot MX^* = MA \cdot MC = MB \cdot MD$ and $\angle BMX = \angle X^*MD$, $\angle XMC = \angle XM^*A$.

Suppose that I_{AB} is the intersection point of angle bisectors of A and B. Same for I_{BC}, I_{CD}, I_{AD}. Consider I_{BC}^*. $\angle AI_{BC}^*D = \angle I_{BC}BA + \angle I_{BC}CD = \angle I_{BC}BC + \angle I_{BC}CB = 180^\circ - \angle CI_{BC}B = \angle AI_{AD}D$, so $(AI_{BC}^*I_{AD}D)$.

We are left to show that $(FI_{BC}^*I_{AD}D)$, then I_{BC}^* will be the Miquel point of $AI_{AB}I_{CD}D$, and it will on $(I_{AB}I_{CD}I_{AD})$.

We hope that it's done in a similar way as $(AI_{BC}^*I_{AD}D)$; it was possible to prove because we had a way to calculate $\angle AI_{BC}^*D$. Here, we need to calculate $\angle FI_{BC}^*D$. We can use the same fact that we used above, but for quadrilateral $FBED$, which has the same Miquel point as $ABCD$ but with minor changes — instead of the sum, we have a difference; proof is easily reconstructible. $\angle FI_{BC}^*D = \angle I_{BC}BF - \angle I_{BC}ED = 180^\circ - \angle I_{BC}BE - \angle I_{BC}EB = \angle BI_{BC}E = 90^\circ + \frac{\angle FCE}{2} = \angle FI_{CD}D$, so $(FI_{BC}^*I_{CD}D)$.

In the same way, we can prove that I_{AB}^*, I_{CD}^*, I_{AD}^* also lie on $(I_{AB}I_{CD}I_{AD}I_{BD})$, so the circle stays fixed.



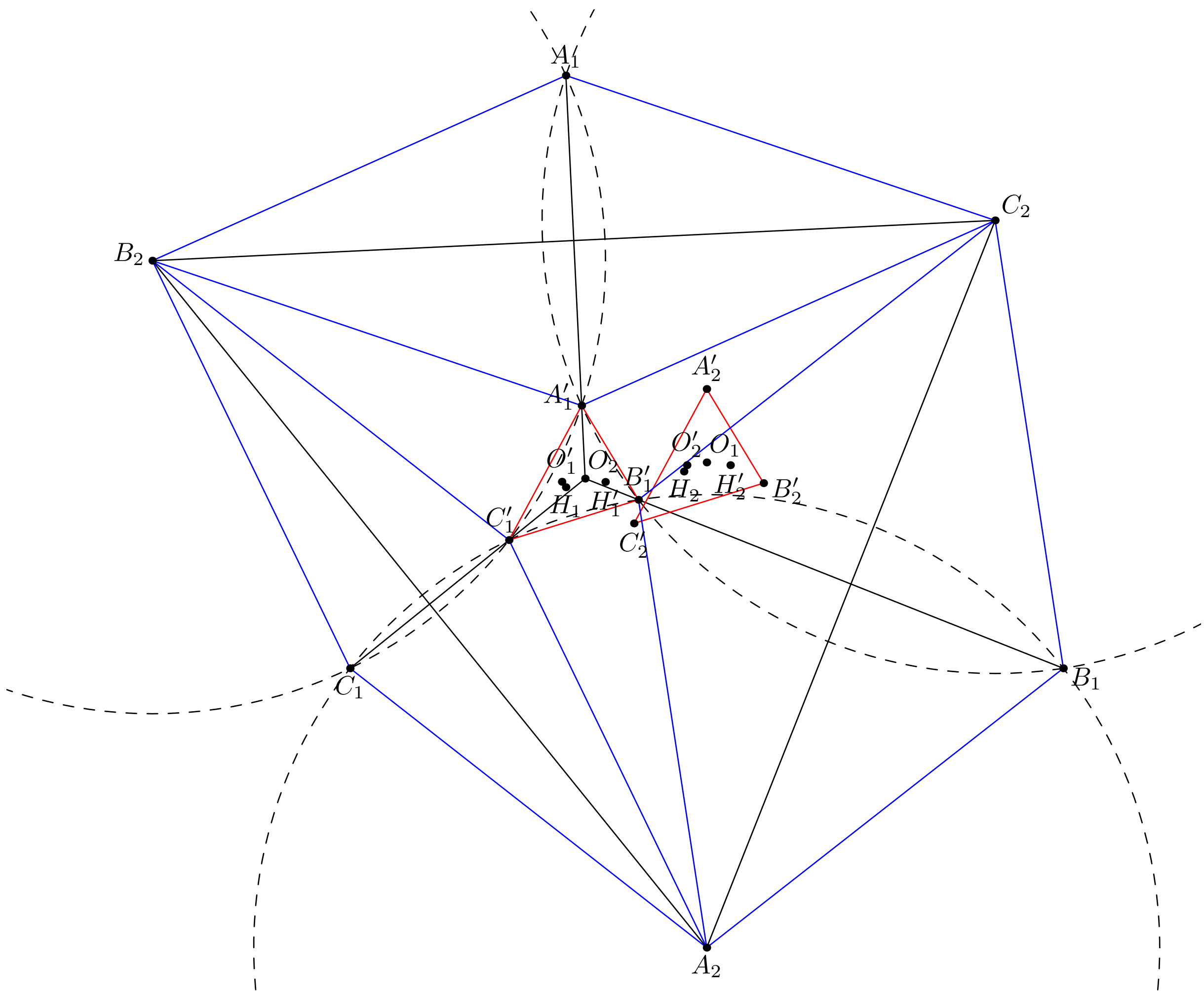
□

Now, the initial problem is obvious. Apply the claim for quadrilateral $XYDE$. Γ stays fixed and the line XY goes to (MDE), where M is the Miquel point of $XYDE$. But (MDE) is just (FDE), so Γ touches (FDE), because they are the images of tangent pbjects.

§6.21 USEMO 2021/3, proposed by Ankan Bhattacharya

Problem 21 (USEMO 2021/3)
Let $A_1C_2B_1A_2C_1B_2$ be an equilateral hexagon. Let O_1 and H_1 denote the circumcenter and orthocenter of $\triangle A_1B_1C_1$, and let O_2 and H_2 denote the circumcenter and orthocenter of $\triangle A_2B_2C_2$. Suppose that $O_1 \neq O_2$ and $H_1 \neq H_2$. Prove that the lines O_1O_2 and H_1H_2 are either parallel or coincide.

¶ First solution (Reflections and vectors) The main idea of this solution is that we can construct several more equilateral hexagons that share vertices with the one given; the statement has to be true for them too and we will extract some useful information from that. If we reflect A_1 across B_2C_2, B_1 across A_2C_2, C_1 across A_2B_2 to the points A_1', B_1', C_1', then $B_2A_1'C_2B_1'A_2C_1'$ is an equilateral hexagon too. Same with points A_2', B_2', C_2' and the hexagon $B_2'A_1C_2'B_1A_2'C_1$.



From $A_1'B_2 = A_1B_2 = B_2C_1 = B_2C_1'$, we get that $(A_1A_1'C_1'C_1)$. In the same way, $(A_1'A_1B_1B_1')$, $(C_1C_1'B_1'B_1)$. So, the inversion centered at O_2 with radius $\sqrt{O_2A_1' \cdot O_2A_1} = \sqrt{O_2C_1' \cdot O_2C_1} = \sqrt{O_2B_1' \cdot O_2B_1}$ swaps $\triangle A_1'B_1'C_1'$ with $\triangle A_1B_1C_1$. Hence, their circumcenters are collinear with O_2. Therefore, the circumcenter O_1' of $\triangle A_1'B_1'C_1'$ lies on the line

O_2O_1. Analogously, circumcenter O_2' of $\triangle A_2'B_2'C_2'$ lies on this line. Now, if the problem is true, then H_1H_2 is parallel to $\overline{O_1' - O_2 - O_2' - O_1}$, as well as $H_1'H_2$ and $H_2'H_1$ from the equilateral hexagons $B_2A_1'C_2B_1'A_2C_1'$ and $B_2'A_1C_2'B_1A_2'C_1$, where H_2' is the orthocenter of $\triangle A_2'B_2'C_2'$ and H_1' is the orthocenter of $\triangle A_1'B_1'C_1'$. It follows that, if the conclusion of the original problem is true, H_1, H_2, H_1', H_2' are collinear on a line parallel to $\overline{O_1' - O_2 - O_2' - O_1}$.

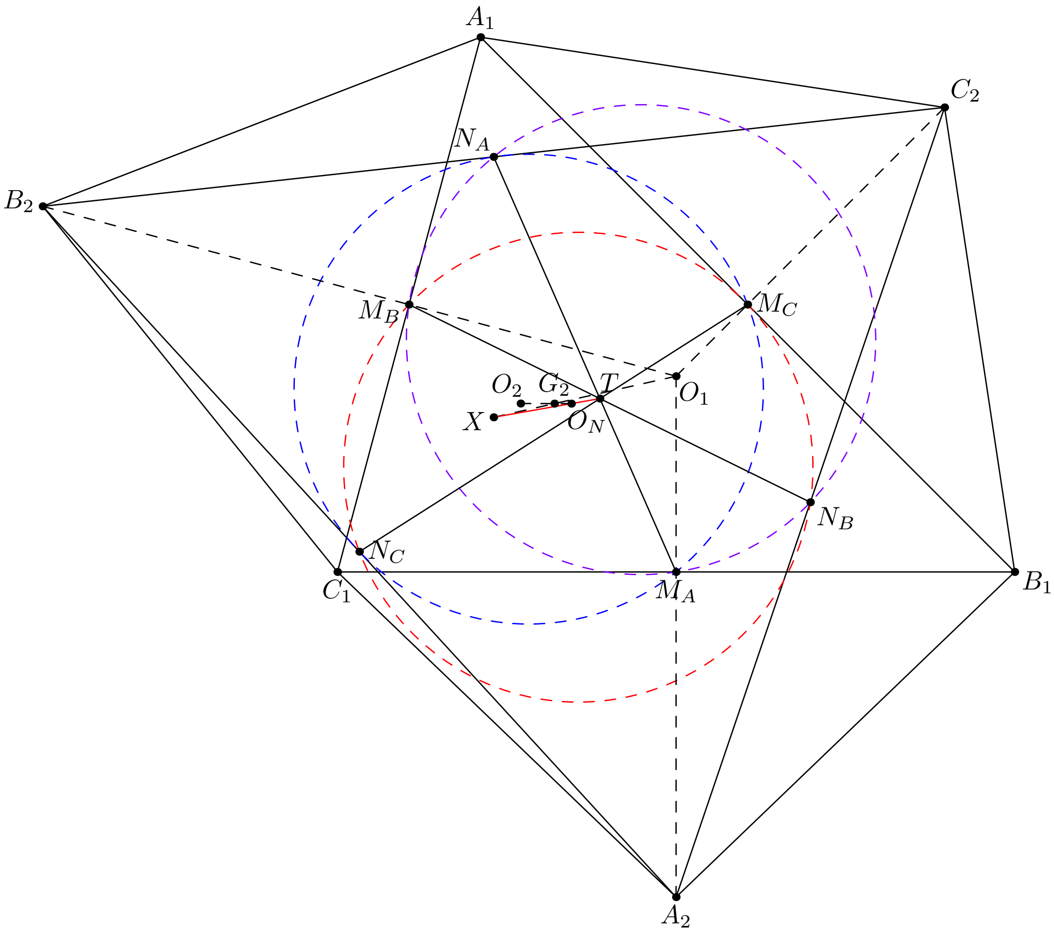
Now, from a nice drawing, it's evident that $\triangle A_1'B_1'C_1'$ and $\triangle A_2'B_2'C_2'$ differ by a translation. A way of proving it is simple too, A_1 and A_1' are not only reflections of each other in B_2C_2, they also form a parallelogram $A_1C_2A_1'B_2$, so it's easy to express A_1' in terms of A_1, B_2, C_2 when we consider vectors. We can do the same with other dashed vertices.

$\overrightarrow{A_1'A_2'} = A_2' - A_1' = B_1 + C_1 - A_2 - (B_2 + C_2 - A_1) = A_1 + B_1 + C_1 - A_2 - C_2 - B_2$. One can easily show that it's equal to $\overrightarrow{B_1'B_2'}$ and $\overrightarrow{C_1'C_2'}$, so $\triangle A_1'B_1'C_1'$ and $\triangle A_2'B_2'C_2'$ are indeed homothetic.

Since we can easily calculate vectors of vertices, we can also calculate expressions for centroids. If we prove that the centroids are collinear on a line parallel to $\overline{O_1' - O_2 - O_2' - O_1}$, then orthocenters will also be collinear on a line parallel to it. This is because in every triangle the centroid divides the segment between the orthocenter and the circumcenter in ratio $2 : 1$ (*Euler's line*).

Let G_1, G_2, G_1', G_2' be the centroids of $\triangle A_1B_1C_1$, $\triangle A_2B_2C_2$, $\triangle A_1'B_1'C_1'$, $\triangle A_2'B_2'C_2'$. $\overrightarrow{G_1G_2} = G_2 - G_1 = \frac{1}{3}(A_2 + B_2 + C_2 - A_1 - B_1 - C_1)$, $\overrightarrow{G_1'G_1} = G_1 - G_1' = \frac{1}{3}(A_1 + B_1 + C_1 - B_2 - C_2 + A_1 - C_2 - A_2 + B_1 - C_2 - B_2 + C_1) = \frac{2}{3}(A_2 + B_2 + C_2 - A_1 - B_1 - C_1)$, so G_1, G_2, G_1' are collinear. Analogously, G_2' also lies on G_1G_2. And we know that $\overrightarrow{G_2'G_1'} = \overrightarrow{O_2'O_1'}$, because the $\triangle A_1'B_1'C_1'$ and $\triangle A_2'B_2'C_2'$ are the translations, so $\overline{G_2' - G_1' - G_1 - G_2}$ is indeed parallel to $\overline{O_2' - O_1' - O_1 - O_2}$. Therefore, we are done.

¶ Second soluiton (Nine-point circles, radical axes, Sondat's theorem)



As in the previous solution, instead of dealing with orthocenters we can instead switch to other points on Euler line. This suggests considering medial triangles of $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$, because they give connection to centroids, nine-point circle centers and circumcenters. Suppose that M_A, M_B, M_C, N_A, N_B, N_C are the midpoints C_1B_1, C_1A_1, A_1B_1, C_2B_2, C_2A_2, B_2A_2.

First, we want to rewrite the equilateral condition using new midpoints. If K is the midpoint of A_1A_2, then $MN_C = MM_B = MM_C = MN_B$, all of which are equal to the half of the side of the hexagon. Thus, $(M_BM_CN_BN_C)$ with center K. In the same way, we can prove that $(N_AM_CM_AN_C)$, $(M_AM_BN_AN_B)$. Radical axes on these three circles give that M_AN_A, M_CN_C, M_BN_B intersect in T. We also know that the inversion centered at T with radius $\sqrt{M_CT \cdot TN_C} = \sqrt{M_AT \cdot N_AT} = \sqrt{M_BT \cdot N_BT}$ followed by the reflection in T swaps $(N_AN_BN_C)$ with $(M_AM_BM_C)$. Thus, if O_M is the center of $M_AM_BM_C$ and O_N is the center of $N_AN_BN_C$, then T lies on O_NO_M.

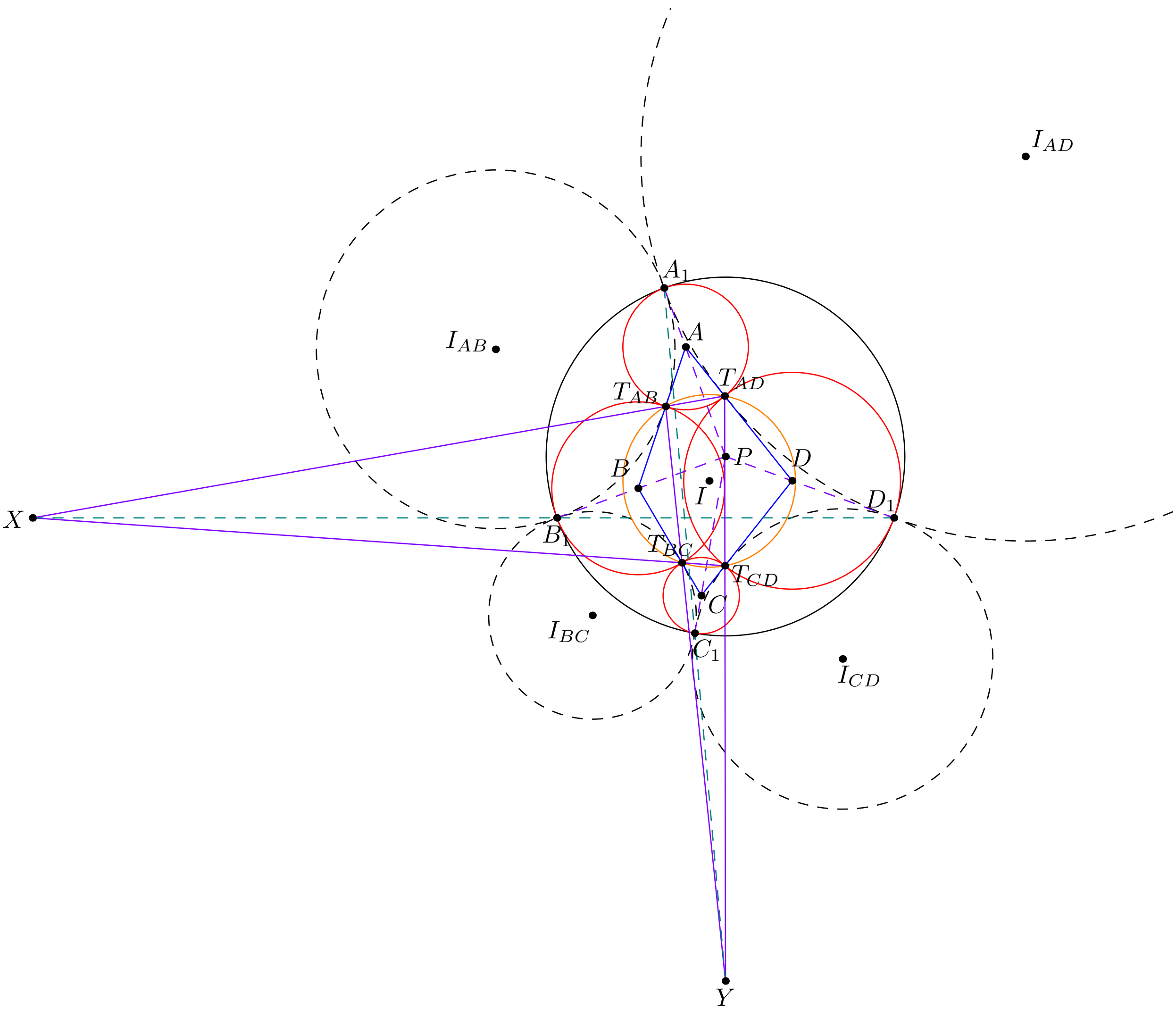
Now let's look at O_1. It lies on B_2M_B, C_2M_C, A_2M_A. If G_2 is the centroid of $\triangle A_2B_2C_2$, then G_2 lies O_2O_N with $O_2G_2 = 2G_2O_N$ (because the homothety centered at G_2 with coefficient $-\frac{1}{2}$ takes $\triangle N_AN_BN_C$ to $\triangle A_2B_2C_2$). So, if X is a point on the line G_2O_1 such that G_2 lies on O_1X with $O_1G_2 = 2G_2X$, then X has to lie on O_NO_M. This is because we want to prove that $O_NO_M \parallel O_1O_2$, and since O_N is the midpoint of O_2H_2 and O_M is the midpoint O_1H_1, it would be enough to imply that H_1H_2 is also parallel to O_1O_2.

$XN_B \parallel B_2O_1$, because $B_2G_2 = 2G_2N_B$. So, $XN_B \perp A_1C_1$, but the latter is parallel to M_CM_A. Analogously, $N_CX \perp M_AM_B$ and $N_AX \perp M_BM_C$, so X is the *center of orthology* of $\triangle M_BM_CM_A$ and $\triangle N_AN_BN_C$. By *Sondat's theorem*, since X is the center of perspective of the two aforementioned trianges, XT is perpendicular to the perspective axis of these two triangles. Perspective axis is the line through $M_BM_A \cap N_BN_A$, $M_BM_C \cap N_BN_C$, $M_AM_C \cap N_AN_C$ (it always exists for perspective triangles due to *Desargues' theorem*). In this case, by radical axes theorem for $(M_AM_BM_C)$, $(N_AN_BN_C)$, and $(M_BN_AN_BM_A)$, $M_BM_A \cap N_BN_A$ lies on the radical axis of $(M_AM_BM_C)$ and $(N_AN_BN_C)$. Analogously for other intersections. Thus, the perspective axis is just the radical axis of $(M_AM_BM_C)$ and $(N_AN_BN_C)$, so $XT \parallel O_NO_M$ (latter is perpendicular to the radical axis), but T lies on O_NO_M, so X lies on O_NO_M, as desired.

§6.22 RMM 2019 G5, proposed by Ankan Bhattacharya

Problem 22 (RMM 2019 G5)
A quadrilateral $ABCD$ is circumscribed about a circle with center I. A point $P \neq I$ is chosen inside $ABCD$ so that the triangles PAB, PBC, PCD, and PDA have equal perimeters. A circle Γ centered at P meets the rays PA, PB, PC, and PD at A_1, B_1, C_1, and D_1, respectively. Prove that the lines PI, A_1C_1, and B_1D_1 are concurrent.

¶ Solution (Drawing auxillary circles and finding similitude centers, polars and radical axes)



We first note that the conclusion doesn't depend on the choice of Γ. It's enough to prove the statement for some particular choice of Γ and the conclusion would be true for all Γ. Indeed, every two choices of $A_1B_1C_1D_1$ are homothetic with center P, so the line through points of intersection of diagonals passes through P, and only one case is sufficient to show that I lies on this line too.

Now, we need to use the strange condition of all $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\triangle PAD$ having the same perimeter. Whenever the perimeter is mentioned, it turns out to be useful to consider the excircle, because it adds the segment with this length into the picture. More formally, set the radius of Γ to be equal to half-perimeter, then A_1 is the point of tangency of the excircle of $\triangle PAD$ with side PA, as well as the touch point of excircle of $\triangle PAB$ with side PA. Thus, if we draw P-excircles of all the four aforementioned triangles, then the neighbouring pairs are mutually tangent in points A_1, B_1, C_1, D_1. Suppose that the excenters of $\triangle PAB$, $\triangle PBC$, $\triangle PCD$, $\triangle PAD$ are I_{AB}, I_{BC}, I_{CD}, I_{AD}.

We already have a lot of tangencies, but we can do more. Suppose that T_{AB}, T_{BC}, T_{CD}, T_{AD} are the touchpoints of respective excircles with sides AB, BC, CD, AD. Then, since $AT_{AD} = AA_1 = AT_{AB}$, we can draw a circle (A) centered at A through A_1, passing through T_{AB}, T_{AD}. Do the same for B, C, D with radii BB_1, CC_1, DD_1. Call these circles (B), (C), (D).

We are motivated to consider these circles because A_1, B_1, C_1, D_1 are now exsimilicenters of (A), (B), (C), (D) and Γ. Now, *Monge's theorem* applied to Γ, (B), (D), tells that the exsimilicenter X of (B) and (D) lies on B_1 and D_1. We can also find that X lies on $T_{BC}T_{CD}$, because T_{BC} is the insimilicenter of (B) and (C) and $T_{CD}T_{BC}$, and the exsimilicenter lies on the line through two insimilicenters.

We also know that B_1 and D_1 swap after the inversion centered at X that takes (B) to (D), as well as T_{BC} to T_{CD} and T_{AB} to T_{AD} (these points are called *antihomologous*). So, $XT_{BC} \cdot XT_{CD} = XT_{AB} \cdot XT_{AD} = XB_1 \cdot XD_1$. Thus, X is the radical center of (A), (C) and Γ, but the radical axis of (A) and Γ is the tangent to Γ at A, and same for (C). Therefore, X is the intersection of tangents in points A_1 and C_1 to the circle Γ. So, A_1C_1 is the polar of X; analogously, B_1D_1 is the polar of exsimilicenter of (A) and (C). Therefore, $A_1C_1 \cap B_1D_1$ is the pole of XY with respect to Γ. Hence, this point lies on the line perpendicular to XY and passing through P. We are left to show that I lies on this line too.

We know that $BT_{AB} = BT_{BC}$, so $IT_{AB} = IT_{BC}$. Analogously, I is equidistant from other Ts. So, I is the center of $(T_{AB}T_{BC}T_{CD}T_{AD})$. But we know that X and Y have equal powers with respect to the circle $(T_{AB}T_{BC}T_{CD}T_{AD})$ and Γ. Thus, XY is the radical axis of these two circles and it's perpendicular to the line of centers —PI. Which is what we wanted to show.

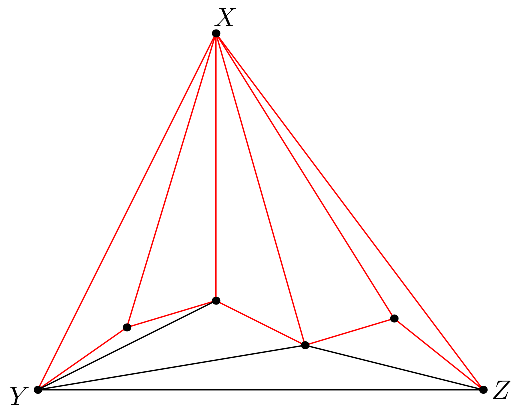
> **Remark.** Another problem where the excircle importance is masked behind the "perimeter" condition is All-Russian MO 2011 grade 10/4.

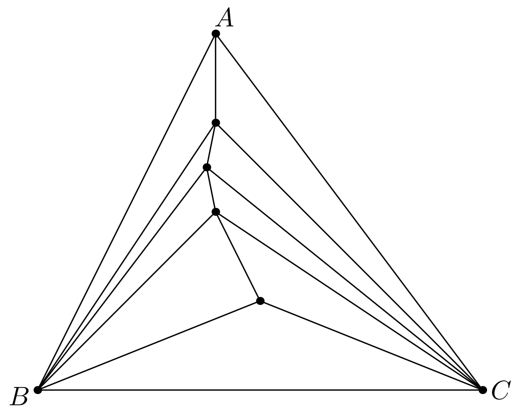
§6.23 China TST 2023 1/5

Problem 23 (China TST 2023, Test 1/5)
Let $\triangle ABC$ be a triangle, and let $P_1, \ldots, P_n$ be points inside where no three given points are collinear. Prove that we can partition $\triangle ABC$ into $2n+1$ triangles such that their vertices are among $A, B, C, P_1, \cdots, P_n$, and at least $n + \sqrt{n} + 1$ of them contain at least one of A, B, C.

¶ Solution (Finding good triangles with intersecting vertices by Erdos-Szekeres) By experimenting with small cases, we might find that every triangulation of a triangle with n vertices inside of it contains $2n+1$ triangles. For the proof, sum all the angles of triangles of the triangulation. The sum is $180^\circ k$, where k is the number of triangles in the triangulation. On the other hand, every vertex inside adds 360° because every part of the full angle around it will be included. Vertices of the initial triangle add 180° in total. The conclusion follows.

Call a triangle good if it contains one of the vertices of $\triangle ABC$. Note that given a triangle $\triangle XYZ$ and k points inside of it, we can construct a partition with at least $k+2$ triangles containing a vertex X by simply connecting X to all the points inside and draw triangles as in the picture, then arbitrarily triangulating the $(k+2)$-gon that is left.





If we find m points such that there's an entirely good triangulation using them, then we will be able to triangulate them again using the triangulation above that adds $k+1$ good triangles (note that it works for $k=0$ as well because we may just not triangulate). In total, there will be at least $(n-m)+2m+1 = n+m+1$ good triangles in total.

The easiest way to ensure this triangulation is to make sure that there are triangles $\triangle ABX_1$, $\triangle ABX_2$, $\cdot$, $\triangle ABX_m$, with $\triangle ABX_i$ inside $\triangle ABX_j$ for $j > i$ as in the picture (instead of A and B, it can be any two vertices of $\triangle ABC$). And we want to make $m = \lceil \sqrt{n} \rceil$.

Note that the condition of a sequence of triangles lying inside each other can be rewritten using inequalities on angles. This gives the idea of finding monotonic subsequences. *Erdős-Szekeres* gives such a subsequence of length $k = \lceil \sqrt{n} \rceil$ in any sequence of length n. Now, we formalize this idea.

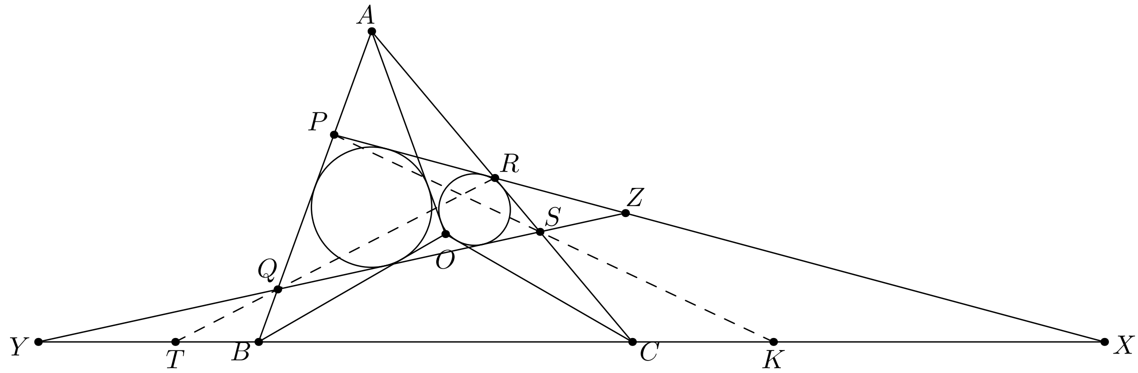
Sort the angles $\angle P_iBC$ in increasing order, suppose that P_1, P_2, P_3, $\ldots$, P_n is the resulting order. Consider a sequence $\angle P_iCB$. By *Erdős-Szekeres*, there exists an increasing or decreasing subsequence P_{i_1}, P_{i_2}, $\ldots$, P_{i_k}, where $k = \lceil \sqrt{n} \rceil$. If the sequence is increasing, then $\triangle BCP_j$ contains $\triangle BCP_i$ for $j > j$. If the sequence is decreasing, then $\triangle ABP_i$ contains $\triangle ABP_j$ for $j > i$. In both cases, we find the desired $\lceil\sqrt{n}\rceil$. In total, there will be $n + \lceil\sqrt{n}\rceil + 1 \geq n + \sqrt{n} + 1$.

§6.24 Problem 24

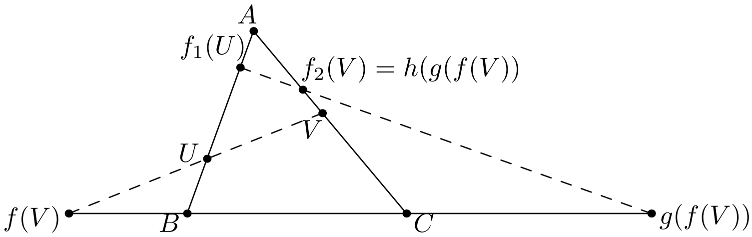
Problem 24

Suppose that $\triangle ABC$ has a circumcenter O. External tangents of incircles of triangles $\triangle BAO$ and $\triangle CAO$ intersect BC at points X and Y. Prove that $\angle XAC = \angle YAB$.

¶ Solution (Involutions and moving points)



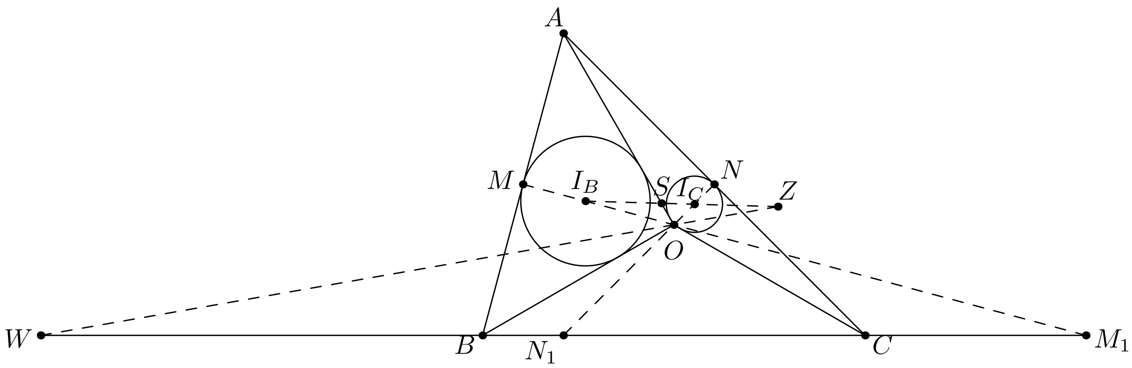
Suppose that the tangents meet AC and AB in P, Q and R, S. Suppose PS and QR meet BC at K and T. Suppose that AX and AY are isogonals with respect to $\angle BAC$. Applying DIT to the line BC and quadrilateral $PQSR$ and then moving to pencil of lines through A gives $(AX; AY)$, $(AK; AT)$, $(AB; AC)$ are the pairs of involution, and since the involution is given by its two pairs, it coincides with the reflection across the bisector of $\angle BAC$, so $\angle KAB = \angle TAC$. The problem is also true if one of the two points chosen on sides (instead P,Q and R, S) are A and C or A and B. Thus, it's natural to generalize the statement for any pairs U, $f_1(U)$ and V, $f_2(V)$ on sides AB and AC, respectively. Where f_1 and f_2 are involutions on lines AB and AC, such that f_1 swaps $(A; B)$, $(P; Q)$ and f_2 swaps $(A; C)$ and $(R; S)$. We are inclined to consider those two because we also know other pair of these involutions by *Dual of Desargues Involution Theorem*.



We will prove it by moving points, because all the maps are projective. First, fix U and triangle ABC and animate V projectively on AC. f is the projection from U to BC, which is a projective map. g is a reflection of pencil through A with respect to $\angle BAC$ and projection onto BC. h is a projection from $f_1(U)$ to AC. So, $h \circ g \circ f$ is a projective map, and we want to check that it coincides with f_2; we only need three positions for it. A and C are obvious and we need one "special" position. The trick here is to use moving points again for the special position.

Indeed, fix this special V and apply the same maps with interchanged sides. We will need to check three cases, two of which are A and B and one "special". So, in total we need to check only one case, apart from obvious ones that contain A, B or C.

Suppose that M and N are the midpoints of AB and AC. In the initial setup, we were talking about getting one more pair of f_1 and f_2 from DDIT. Namely, by DDIT on $AOCN$, its incircle and point Z, f_1 also swaps the midpoint of AB with $ZO \cap AB$ and, by DDIT on AOB f_2 swaps the midpoint of AC with $ZO \cap AC$. Take the two special points as M and N. We need to prove that $MN \cap BC = \infty_{BC}$ and $ZO \cap BC = W$ are two points such that $\angle \infty_{BC} AC = \angle WAB$, but the first angle is just $\angle ACB$, because of the parallel lines. Thus, we need to show that $\angle WAB = \angle BCA$.



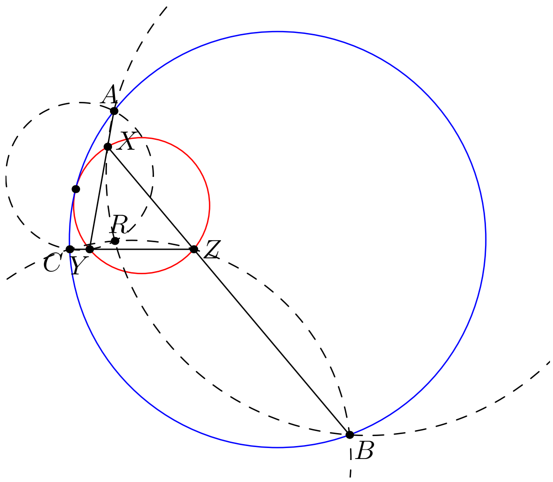
Let I_B and I_C be incenters of $\triangle AOB$ and $\triangle AOC$, and let S be the insimilicenter of incircles of $\triangle AOB$ and $\triangle AOC$, i.e. the point of intersection of I_BI_C and AO. It's known that $(I_BI_C; SZ) = -1$. Suppose that MO and NO intersect BC at M_1 and N_1, then $-1 = (I_BI_C; SZ) \overset{O}{=} (M_1N_1; (AO \cap BC)W)$. O is the incenter of $\triangle AN_1M_1$, because of perpendicular bisectors. Thus, AW has to be the external angle bisector of $\angle N_1AM_1$, because $-1 = (M_1N_1; (AO \cap BC)W) \overset{A}{=} (AW, AO; AN_1, AM_1)$. But then, $\angle WAO = 90^\circ$, so WA is the tangent at A to (BAC).

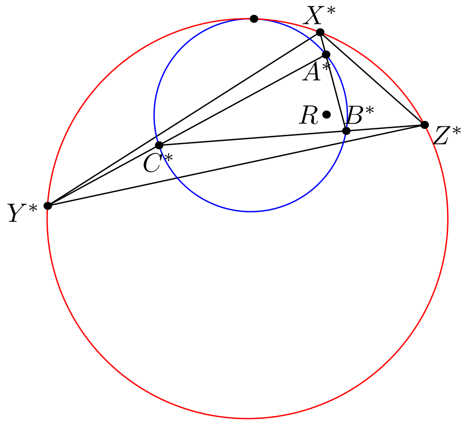
¶ Alternative moving points approach We will use untethered moving points. Animate U and V projectively on AB and AC. Let homogeneous coordinates of U be linear polynomials in t_1, and let homogeneous coordinates of V be linear polynomials in t_2. Line UV's coordinates are polynomials in with t_1 degree ≤ 1 and t_2 degree ≤ 1. Intersection $UV \cap BC$ has degree $t_1 \leq 1$ and degree $t_2 \leq 1$. Reflecting $(UV \cap BC)A$ across $\angle BAC$ and intersecting with BC yields a point G with degree $t_1 \leq 1$ and $t_2 \leq 1$, because the reflection is involution and it doesn't change the degrees. $f_1(U)$ and $f_2(V)$ have degree $t_1 \leq 1$ and $t_2 \leq 1$, respectively. Collinearity of G, $f_1(U)$, $f_2(V)$ can be written as a polynomial of degree $t_1 \leq 2$ and degree $t_2 \leq 2$. Thus, by *Combinatorial Nullstellensatz*, we need to check all the pairs from $X_1 \times X_2$, where X_1 and X_2 are the sets consisting of 2+1 points on AB and AC, respectively. We will choose $X_1 = \{B; ZO \cap AB; M\}$ and $X_2 = \{A; ZO \cap AC; N\}$ (we can't have B in X_1 and C in X_2, and both A in X_1 and X_2). We already checked the cases M, N and $ZO \cap AB$, $ZO \cap AC$. To check $ZO \cap AB$, N, we apply *Desargues Involution Theorem* as in the step of the above solution where we proved that $\angle KAB = \angle TAC$ assuming that $\angle XAB = \angle YAC$.

§6.25 All-Russian MO grade 11 2022/8, proposed by Alexander Kuznetsov

Problem 25 (All-Russian MO grade 11 2022/8)
From each vertex of $\triangle ABC$ we draw two rays, red and blue, symmetric about the angle bisector of the corresponding angle. The circumcircles of triangles formed by the intersection of rays of the same color. Prove that if the circumcircle of $\triangle ABC$ touches one of these circles then it also touches to the other one.

¶ First solution (Exploiting symmetry using inversion) When we are given the tangency of two objects and want to show tangency of two other circles, usage of inversion is more than justified. The usage of inversion is further incentivised by the fact that we can "flip angles" when using the inversion; usually, it results in creating isogonal conjugates. We only need to find the right center of inversion.





Suppose that rays intersect in X, Y, Z, as shown in the picture. Suppose further that (XYZ) and (ABC) are tangent. Suppose that the center is R. Mark images after inversion by $*$. After the inversion, circle $(A^*B^*C^*)$ will be inside $(X^*Y^*Z^*)$ [1], and we wish the picture is the same as if we intersected isogonal rays in the original picture. Thus, we want X^* to lie on the line through two of the A^*, B^*, C^*, so R lies on some circle through X and two of the A, B, C. Same for Y and Z. After experimenting for a bit, it's obvious that the most symmetric option is to use the intersection of (XAB), (ACY) and (BZC). It exists, because $\angle AYC + \angle AXB + \angle BZC = 3 \cdot 180^\circ - \angle XYZ - \angle YXZ - \angle YZX = 360^\circ$. So, if we intersect two of the circles, then this point will also lie on the third circle.

Now, we only need to check the angle conditions. $\angle Y^*X^*Z^* = \angle RZX + \angle RYX = \angle RCB + \angle RCA = \angle ACB$. In the same way, we can show that $\triangle X^*Y^*Z^* \sim \triangle CBA$. $\angle B^*X^*Z^* = \angle RX^*Z^* - \angle RX^*B^* = \angle RZX - \angle RBX = \angle RCB - \angle RCZ = \angle ZCB$. So, X^*B^* in $\triangle X^*Y^*Z^*$ is equivalent to ray symmetric to CZ with respect to the angle bisector of $\angle BCA$ in $\triangle CBA$. Analogously, we can prove other isogonalities. Hereby, tangency of

[1] It's actually not necessary that $\triangle XYZ$ is inside, but that's the easiest case to proceed with.

$(A^*B^*C^*)$ and $(X^*Y^*Z^*)$ is equivalent to tangency of the (ABC) and triangle formed by isogonal rays, as desired.

¶ Second solution (Casey's theorem, trigonometric bash and Combinatorial Nullstellensatz) Note that both conditions are equivalent to some Casey's theorem; we want to show that they are, in fact, equivalent. Let's rewrite the given one in terms of it:

$$\sqrt{AX \cdot AY} \cdot BC + \sqrt{CY \cdot CZ} \cdot AB = \sqrt{BZ \cdot BX} \cdot AC.$$

But the problem here is that the condition can be any in which sum of the two from $\sqrt{AX \cdot AY} \cdot BC$, $\sqrt{CY \cdot CZ} \cdot AB$, $\sqrt{BZ \cdot BX} \cdot AC$ equals to third. To get rid of the asymmetries, we will use the

$$x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2 = (x+y+z)(x+y-z)(x+z-y)(y+z-x)$$

identity, where $x = \sqrt{AX \cdot AY} \cdot BC$, $z = \sqrt{CY \cdot CZ} \cdot AB$, $y = \sqrt{BZ \cdot BX} \cdot AC$. We are given that the above symmetric relation is equal to zero.

Now we need to rewrite the condition in terms of angles that rays form with sides, because then it would be possible to show that it holds when we permute the angles to form isogonals. It's done by sine law. Using the same labeling as in the previous solution, $\angle YAC = \alpha$, $\angle YAB = \alpha_1$, $\angle XBA = \beta$, $\angle XBC = \beta_1$, $\angle ZCB = \gamma$, $\angle ZCA = \gamma_1$. By sine law, $AX = \frac{\sin\beta}{\sin(\alpha_1+\beta)} \cdot AB$, $AY = \frac{\sin\gamma_1}{\sin(\gamma_1+\alpha)} \cdot AC$, so $x^2 = AX \cdot AY \cdot BC^2 = \frac{\sin\beta}{\sin(\alpha_1+\beta)} \cdot \frac{\sin\gamma_1}{\sin(\gamma_1+\alpha)} \cdot 2R\sin(\alpha+\alpha_1) \cdot AB \cdot BC \cdot AC$, where R is the circumradius of $\triangle ABC$. y, z can be rewritten similarly. Of course, we can divide through $2R \cdot AB \cdot BC \cdot AC$. Now we multiply though common denominator—$\sin(\alpha_1+\beta)\sin(\gamma_1+\alpha)\sin(\beta_1+\gamma)$ —to get $x = \sin\beta\sin\gamma_1\sin(\beta_1+\gamma)\sin(\alpha+\alpha_1)$, $y = \sin\alpha_1\sin\gamma\sin(\alpha+\gamma_1)\sin(\beta+\alpha_1)$, $z = \sin\alpha\sin\beta_1\sin(\alpha_1+\beta)\sin(\gamma+\gamma_1)$, and $x^2+y^2+z^2-2xy-2yz-2xz = 0$.

Call this $f(\alpha,\alpha_1,\beta,\beta_1,\gamma,\gamma_1)$. We want to show that from $f(\alpha,\alpha_1,\beta,\beta_1,\gamma,\gamma_1) = 0$, it follows that $f(\alpha_1,\alpha,\beta_1,\beta,\gamma_1,\gamma) = 0$. We intend to rewrite the problem in the more algebraic form and get rid of sines; we will use the same idea as in USA TST 2016/6. Change $\sin\alpha = \frac{e^{i\alpha}-\frac{1}{e^{1\alpha}}}{2i}$, and similarly for other angles. Then we see that we can multiply each of x, y, z through $(2i)^4 e^{i(\alpha+\beta+\gamma+\alpha_1+\beta_1+\gamma_1)}$. Now we can change $e^{2i\alpha}$ to a. And the same for a_1, b, b_1, c, c_1. What is left if the polynomial $F(a,b,c,a_1,b_1,c_1) = x^2+y^2+z^2-2xy-2yz-2xz$, where

$$x = (b-1)(c_1-1)(b_1c-1)(aa_1-1),$$
$$y = (a_1-1)(c-1)(ac_1-1)(bb_1-1),$$
$$z = (a-1)(b_1-1)(a_1b-1)(cc_1-1).$$

We will actually show that $F(a,b,c,a_1,b_1,c_1) = F(a_1,b_1,c_1,a,b,c)$, which will imply the main problem. At this point one can just expand, but I'm not sure that it can be done withing a reasonable time without making a mistake somewhere, so we will stick to a more effective approach.

Call the flipped F F_1. Note that both F and F_1 have a power of each variable of at most 2. In order to show that $F - F_1 \equiv 0$, by *Combinatorial Nullstellensatz*, we only need to check the result for some $S_a \times S_b \times S_c \times S_{a_1} \times S_{b_1} \times S_{c_1}$, where each S has three elements. For the "check" values we want to choose the ones that will cancel some terms. It's, of course, zero. But we also see that we have brackets with elements decreases by one, so it makes sense to add one. We also see the products equal to one, so we add -1. Thus, we set each of our sets to be of the form $\{-1; 0; 1\}$.

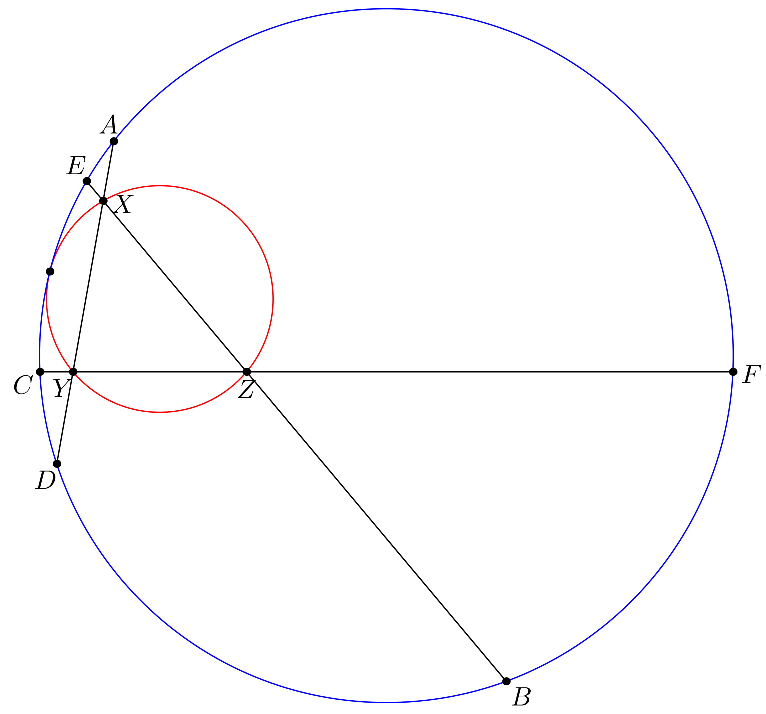
The remaining part is feasible, but still demands care. I will just outline the steps.

If $b = 1$, then $x = 0$ in F, so it is just $(y-z)^2 = ((a_1-1)(b_1-1)(c-a)(c_1-1))^2$. F_1 can be calculated in the same way, and it will be the same.

Now we will assume that no variable is equal to 1. If $a_1 = -1$ and $a = -1$, then $x = 0$ and $F = (y-z)^2 = (2(c-1)(c_1+1)(bb_1-1) - 2(b_1-1)(b+1)(cc_1-1))^2$. For F_1, x is also 0, and $y - z$ is the same, because its expression is symmetric with respect to swap of indexed and non-indexed variables.

Now we can say that aa_1, bb_1, cc_1 are all zeros. The rest can be bruteforced by hand. [1]

¶ Alternative solution (Hidden symmetry in Casey's) The above solution suggests that there must be some symmetry in Casey's theorem. The key to achieve it is taking a different reference triangle for Casey's theorem with the same inner triangle. A way of doing so is marking the second points of intersection of rays with the circumcircle. Suppose that $\overline{A-X-Y} \cap (ABC) = D$, $\overline{B-X-Z} \cap (ABC) = E$, $\overline{C-Y-Z} \cap (ABC) = F$.



Taking the definitions from the previous solution, $\angle DEZ = \angle DAB = \alpha_1$, $\angle ZEF = \angle BCY = \gamma$, $\angle ZFE = \angle CBX = \beta_1$, $\angle ZFD = \angle CAD = \alpha$, $\angle ADF = \angle ACF = \gamma_1$, $\angle ADE = \angle ABX = \beta$. So, if $f(\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1) = 0$, then $f(\alpha_1, \gamma, \beta_1, \alpha, \gamma_1, \beta) = 0$, which is the same as $x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ for $x = \sin\beta_1 \sin\beta \sin(\alpha+\gamma_1)\sin(\alpha_1+\gamma)$, $y = \sin\gamma\sin\gamma_1\sin(\alpha_1+\beta)\sin(\alpha+\beta_1)$, $z = \sin\alpha_1\sin\alpha\sin(\beta_1+\gamma)\sin(\beta+\gamma_1)$. We

[1] It's painful but rather straightforward

can see that if we swap indexed variables with non-indexed, x, y, z won't change, so $f(\alpha, \gamma_1, \beta, \alpha_1, \gamma, \beta_1) = f(\alpha_1, \gamma, \beta_1, \alpha, \gamma_1, \beta) = 0$. Now, applying the change of reference triangle, $f(\alpha_1, \alpha, \beta_1, \beta, \gamma_1, \gamma) = f(\alpha, \gamma_1, \beta, \alpha_1, \gamma, \beta_1) = 0$, which is what we had to show.

Part IV. Number Theory

§7 Problems

Problem 1 (ISL 2018 N5). Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

Hints: 23 523 947 72 63 448 313

Hints: 1021 822 997 669 665 939 420 949

Problem 2 (Kazakhstan MO grade 11 2015/5). Find all pairs (n, k) of positive integers such that $(n+1)\dots(n+k) - k$ is a perfect square.

Hints: 249 517 197 113 513 756 348 261

Problem 3 (ISL 1998 N7). Prove that for each positive integer n, there exists a positive integer with the following properties: It has exactly n non-zero digits, and it is divisible by the sum of its digits.

Hints: 21 633 790 824 613 257 850

Hints: 230 221 548 550 236 272

Problem 4 (USA TSTST 2013/5). Let p be a prime. Prove that any complete graph with $1000p$ vertices, whose edges are labelled with integers, has a cycle whose sum of labels is divisible by p.

Hints: 32 723 644 241 303

Problem 5 (China TST 2017 2/4). Given integer $d > 1, m$,prove that there exists integer $k > l > 0$, such that

$$(2^{2^k} + d, 2^{2^l} + d) > m.$$

Hints: 638 798 148 963 407 843 184 989 41 279

Problem 6 (Saint-Petersburg Math Olympiad 1998 grade 11/6). Suppose that $d(n)$ is the number of divisors of a number n. Prove that the sequence $d(n^2 + 1)$ doesn't become monotonic from any point. [1]

[1] Original problem asked for strict monotonicity, which is considerably easier.

Hints: 737 393 817 312 240 115

Hints: 873 944 677 547 607 789 888 597

Problem 7 (IZHO 2023/5). We call a positive integer n is *good* , if there exist integers a, b, c, x, y such that $n = ax^2 + bxy + cy^2$ and $b^2 - 4ac = -20$. Prove that the product of any two good numbers is also a good number.

Hints: 275 725 783 685 53 879 901

Hints: 329 219 878 353 834 863 943 983

Problem 8 (ISL 2018 N3). Define the sequence $a_0, a_1, a_2, \cdots$ by

$$a_n = 2^n + 2^{\lfloor n/2 \rfloor}.$$

Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

Hints: 864 264 205 239 328 166 390 772

Problem 9. Given numbers a_1, a_2, ..., a_n such that $1 \le a_i \le 2023$ and $lcm(a_i, a_j) > 2023$ for all $1 \le i < j \le n$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < 2.$$

Hints: 682 675

Problem 10 (USAMO 2020/3). Let p be an odd prime. Denote by A the set of all integers a such that $1 \le a < p$, and both a and $4 - a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

Hints: 318 821 827 74 212

Hints: 356 122 164 813 601 715 729 381 606 988 498 656

Problem 11 (USA TST 2017/6). Prove that there are infinitely many triples (a, b, p) of positive integers with p prime, $a < p$, and $b < p$, such that $(a + b)^p - a^p - b^p$ is a multiple of p^5. [2]

Hints: 830 831 915 726 637 619 746 948 874 866 427 559 922 755

[2] Original problem asked for divisibility by p^3.

Problem 12 (USA TST 2014/2). Let $a_1, a_2, a_3, \ldots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

Hints: 380 82 170 784 179 693 486 660 299 990

Problem 13 (Saint-Petersburg MO 2003 grade 10/7). Given a prime number p and an integer $n \geq p$. Consider an arbitrary array of positive integers a_1, a_2, $\ldots$, a_n. Let f_k be the number of k element subsets with sum of their elements divisible by p. Prove that

$$p \mid \sum_{k=0}^{n} (-1)^k f_k.$$

Here, we assume that $f_0 = 1$.

Hints: 449 787 358 429 274

Problem 14 (USAMO 2012/3). Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of nonzero integers such that the equality

$$a_k + 2a_{2k} + \ldots + na_{nk} = 0$$

holds for every positive integer k.

Hints: 459 84 1009 987 655 544 453 214 549 698 120

Problem 15 (Kazakhstan MO 2017 grade 11/6). Prove that there are infinitely many composite numbers n, such that $2^{\frac{n-1}{2}} - 1$ is divisible by n.

Hints: 1012 876 364 110 99 979 808 986 663 136

Problem 16 (ELMOSL 2011 N2). Let $p \geq 5$ be a prime. Show that

$$\sum_{k=0}^{(p-1)/2} \binom{p}{k} 3^k \equiv 2^p - 1 \pmod{p^2}.$$

Hints: 842 281 368 292 45 832

Problem 17 (IMO 2003/6). Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

Hints: 661 188 659 165 15 610 964 819 816

Problem 18. Prove that we can fill a grid with positive integers such that the sum of entries of any square sub-grid is a perfect square, and the sum of entries of a non-square sub-grid is not a square.

Hints: 946 137 402 859 718 341 730 481 112 91

Hints: 954 694 562 57 213 953 30

Problem 19 (Putnam 2012 B6). Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

Hints: 583 781 199 466 432 242

Hints: 125 801 108 270 594 917 40

Hints: 881 361 1018 185 308 211 42

Problem 20 (ISL 2018 N6). Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m+n) | f(m)+f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

Hints: 66 455 80 441 487 39 880 333 139 347 44

Problem 21 (ELMO 2022/2). Find all monic nonconstant polynomials P with integer coefficients for which there exist positive integers a and m such that for all positive integers $n \equiv a \pmod m$, $P(n)$ is nonzero and

$$2022 \cdot \frac{(n+1)^{n+1} - n^n}{P(n)}$$

is an integer.

Hints: 909 225 359 428 283 28 479 763 857 1019 1004 985 803 124 785 90 304 779

Problem 22. For an odd prime p, prove that

$$\sum_{i=1}^{p-1} 2^i \cdot i^{p-2} \equiv \sum_{i=1}^{\frac{p-1}{2}} i^{p-2} \pmod p.$$

Hints: 289 574 527 126 256 332 662

Problem 23 (USEMO 2022/5). Let $\tau(n)$ denote the number of positive integer divisors of a positive integer n (for example, $\tau(2022) = 8$). Given a polynomial $P(X)$ with integer coefficients, we define a sequence $a_1, a_2, \dots$ of nonnegative integers by setting

$$a_n = \begin{cases} \gcd(P(n), \tau(P(n))) & \text{if } P(n) > 0 \\ 0 & \text{if } P(n) \le 0 \end{cases}$$

for each positive integer n. We then say the sequence has limit infinity if every integer occurs in this sequence only finitely many times (possibly not at all).

Does there exist a choice of $P(X)$ for which the sequence a_1, a_2, ... has limit infinity?

Hints: 306 235 951 150 190 43 681

Problem 24 (USAMO 2018/3). For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \cdots + a_m^k$ is divisible by m for every positive integer k.

Hints: 121 410 202 374 799 643 860 237 854 475 152 247 704

Problem 25 (All-Russian MO grade 11 2024/8). Prove that there exists $c > 0$ such that for any odd prime $p = 2k + 1$, the numbers $1^0, 2^1, 3^2, \ldots, k^{k-1}$ give at least $c\sqrt{p}$ distinct residues modulo p.

Hints: 388 452 591 29 855 919 648 207 906

§8 Solutions

§8.1 ISL 2018 N5, proposed by Russia

Problem 1 (ISL 2018 N5)
Four positive integers x, y, z and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

¶ First solution (Finding a hidden identity and applying the Four Numbers Lemma) For the sake of contradiction, suppose that $xy = a^2$, $zt = b^2$. Note that we can write $t = x + y - z$ to eliminate a variable. So, $xy - zt = xy - z(x + y - z) = z^2 - zx - zy + xy = (z - x)(z - y)$. But we know that we can write $xy - zt$ as a product in another way, namely $xy - zt = a^2 - b^2 = (a - b)(a + b)$. Now we have to recognize a premise from the following fact:

Lemma (Four Numbers Lemma)
For integers a, b, c, d, satisfying $ab = cd$, there exist integers p, q, r, s such that $a = pq$, $b = rs$, $c = ps$, $d = qr$.

Proof. Condition can be rewritten as $\frac{a}{c} = \frac{d}{b}$. Write them both as a reduced fraction $\frac{q}{s}$, then $a = pq$, $c = ps$, and $d = qr$, $b = sr$. □

Applying it to $(a-b)(a+b) = (z-x)(z-y)$ in the original problem, we see the existence of integer a, b, c, d such that $z - x = pq$, $z - y = rs$, $a - b = ps$, $a + b = qr$. We know that $x + y = a^2 - b^2 = pqrs$, $xy = a^2 = (\frac{ps+qr}{2})^2$, and we have another equation that connects x and y, namely $x - y = (z - y) - (z - x) = rs - pq$. Now, by $(x + y)^2 = (x - y)^2 + 4xy$, we can form an equation connecting p, q, r, s:

$$(ps + qr)^2 + (pq - rs)^2 = (pqrs)^2.$$

But the LHS is $(p^2 + r^2)(s^2 + q^2)$, so we have

$$(p^2 + r^2)(s^2 + q^2) = (pqrs)^2.$$

Product in RHS feels too big compared to the sums in the LHS, so we bound it as

$$(p^2 + r^2)(s^2 + q^2) \le 4 \max(p^2, r^2) \max(q^2, s^2).$$

Therefore, $\min(p^2, r^2) \min(q^2, s^2) \le 4$. Note that no number from p, q, r, s is zero, otherwise $xy = pqrs = 0$. So, if $\min(p^2, r^2) \min(q^2, s^2) < 4$, then it has to be equal to 1, because it's a square and is not a zero. In this case, $p^2 + r^2 > p^2r^2$ because one of them is 1, and $q^2 + s^2 > q^2s^2$. Multiplying them, we reach the contradiction. If the equality holds in $\min(p^2, r^2) \min(q^2, s^2) \le 4$, then $p^2 + r^2 = 2 \max(p^2, r^2)$ and $q^2 + s^2 = 2 \max(q^2, s^2)$, from which we can say that $p = r$ and $q = s$, but then, $ps = qr$, so $a - b = a + b$, but then $zt = b^2 = 0$, which is a final contradiction.

¶ Second solution (Multiplying squares to get a square and neighbouring squares trick) We first try to describe all the quadruples (x, y, z, t) that satisfy these equations. The way it's done is similar to what we have done in the previous solution. Suppose that $s = xy - zt = x + y = z + t$. We know that $(z - x)(z + x - s) = (z - x)(z - y) = xy - zt = s$. Suppose that $z - x = a$, $z + x - s = b$. Then, $s = ab$ and $z = \frac{ab+a+b}{2}$, $x = \frac{ab-a+b}{2}$. We can also say that $y = \frac{ab+a-b}{2}$, $t = \frac{ab-a-b}{2}$. We see that all the possible combinations of signs are present; in this case, it's useful to multiply them, and it will usually result into a difference of squares.

$$16xyzt = (a^2b^2 - a^2 - b^2 + 2ab)(a^2b^2 - a^2 - b^2 - 2ab) = (a^2b^2 - a^2 - b^2)^2 - 4a^2b^2.$$

We know that $xyzt$ is the product of two squares, namely xy and zt. $a^2b^2 - a^2 - b^2 = (a^2 - 1)(b^2 - 1) - 1$. $ab = s = x + y > 0$, so $a \neq 0$ and $b \neq 0$. If $a = 1$, then $1 - 4b^2$ is a perfect square, which is only possible for $b = 0$, contradiction. Analogously, $b \neq 1$. So, $a \geq 2$, $b \geq 2$, then $a^2b^2 - a^2 - b^2 = (a^2 - 1)(b^2 - 1) - 1 > 0$. Then, for $(a^2b^2 - a^2 - b^2)^2 - 4a^2b^2$ to be a square, it has to be $\leq (a^2b^2 - a^2 - b^2 - 1)^2$ (it's a square and smaller than $(a^2b^2 - a^2 - b^2)^2$).

$4a^2b^2 \geq (a^2b^2 - a^2 - b^2)^2 - (a^2b^2 - a^2 - b^2 - 1)^2 = 2a^2b^2 - 2a^2 - 2b^2 - 1 < 2a^2b^2$, which is a clear contradiction.

§8.2 Kazakhstan MO grade 11 2015/5, proposed by Denis Ovchinnikov, Saken Ilyasov

Problem 2 (Kazakhstan MO grade 11 2015/5)
Find all pairs (n, k) of positive integers such that $(n+1)(n+2)\dots(n+k) - k$ is a perfect square.

¶ Solution (Divisors of a sum of squares) Suppose that $(n+1)(n+2)\dots(n+k) - k = m^2$. Products of consecutive numbers are good because they are divisible by the number of multiples. This suggests to add a number to both sides and use the divisibility to obtain a contradiction. Since we have a square, we will be adding another square because of the following result:

Lemma
If $p \equiv 3 \pmod 4$ is a prime that divides $a^2 + b^2$, then $p \mid a$ and $p \mid b$.

Proof. Suppose that a and b are not divisible by p (if one of them is divisible by p, then the other divisible too).

$a^2 \equiv -b^2 \pmod p \Longrightarrow \left(\frac{a}{b}\right)^2 \equiv -1 \pmod p$ and $\left(\frac{a}{b}\right)^4 \equiv 1 \pmod p$. Thus,

$$\left(\frac{a}{b}\right)^{p-3} \equiv \left(\left(\frac{a}{b}\right)^4\right)^{\frac{p-3}{4}} \equiv 1 \pmod p$$

and $\left(\frac{a}{b}\right)^{p-1} \equiv 1 \pmod p$, by Fermat's Little Theorem. Therefore, dividing by the previous congruence, $-1 \equiv \left(\frac{a}{b}\right)^2 \equiv 1 \pmod p$. Which is a contradiction to $p \equiv 3 \pmod 4$. □

We will use two special cases of this result: $a^2 + 1$ and $a^2 + 4$ have no prime divisors of the form $4t + 3$.

Applying the idea stated before, $k - 1 \mid (n+1)(n+2)\dots(n+k) - (k-1) = m^2 + 1$ and $k - 4 \mid (n+1)(n+2)\dots(n+k) - (k-4) = m^2 + 4$.

Suppose that k is even and $k \geq 4$. $k - 1$ is a divisor of $m^2 + 1$, so the prime factorization of $k - 1$ consists only of prime divisors of the form $4t + 1$ (it's not divisible by 2). Thus, $k - 1 \equiv 1 \pmod 4$. Since $k \geq 4$, $m^2 = (n+1)(n+2)\dots(n+k) - k \equiv -k \equiv 2 \pmod 4$, which is a contradiction.

Suppose that $k \equiv 1 \pmod 4$ and $k \geq 5$. $(n+1)(n+2)\dots(n+k) - k \equiv -k \equiv 3 \pmod 4$, which is not a residue of a square.

If $k \equiv 3 \pmod 4$ and $k > 4$, then $k - 4$ is a divisor of $m^2 + 4$. As in the previously considered case, prime factorization of $k - 4$ consists only of prime divisors of the form $4t + 1$ (it's not divisible by 2). Thus, $k - 4 \equiv 1 \pmod 4$. Contradiction.

Only cases that are left are $k \leq 3$.

If $k = 1$, we have a solution $(m^2, 1)$ for $m \neq 0$.

If $k = 2$, $n^2 + 3n$ has to be a perfect square. Of course, when $n = 1$, $1^2 + 3 = 4$ is a perfect square. For $n > 1$, $(n+1)^2 < n^2 + 3n < (n+2)^2$. So it cannot be a perfect square.

If $k = 3$, $m^2 + 3 = (n+1)(n+2)(n+3)$. RHS is even, so m is odd. Therefore, $m = 2s+1$. $4(s^2 + s + 1) = (n+1)(n+2)(n+3)$. Now we will state a lemma similar to the one proved before:

Lemma

If $p \equiv 2 \pmod 3$ is a prime that divides $a^2 + ab + b^2$, then $p \mid a$ and $p \mid b$.

Suppose that a, b are not divisible by p (if one is divisible, then the other is divisible too). Dividing the congruence by b, $\left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) + 1 \equiv 0 \pmod p$. Thus, $\left(\frac{a}{b}\right)^3 \equiv 1 \pmod p$.

$$\left(\frac{a}{b}\right)^{p-2} \equiv \left(\left(\frac{a}{b}\right)^3\right)^{\frac{p-2}{3}} \equiv 1 \pmod p$$

and $\left(\frac{a}{b}\right)^{p-1} \equiv 1 \pmod p$, by Fermat's Little Theorem. Therefore, dividing by the previous congruence, $\left(\frac{a}{b}\right) \equiv 1 \pmod p$. $3 \equiv \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) + 1 \equiv 0 \pmod p$, which is a contradiction to $p \equiv 2 \pmod 3$.

$4|(n+1)(n+2)(n+3)$. If n is even, then $4|n+2$. $\frac{n+2}{4} \equiv n+2 \pmod 3$, so among $n+1$, $\frac{n+2}{4}$, $n+3$, one of the numbers is $\equiv 2 \pmod 3$ (they are congruent to $n+1$, $n+2$, $n \pmod 3$, respectively) and all of them are odd, because they are the divisors of $s^2 + s + 1$ – an odd number. This number is not divisible by three, and if it's divisible by $p \equiv 2 \pmod 3$, then, by the lemma, $p \mid 1$, which is a contradiction. Thus, it contains only prime numbers $p \equiv 1 \pmod 3$ in its prime factorization, but if we multiply numbers of this kind, the result is still $\equiv 1 \pmod 3$, which is a contradiction.

If n is odd, then $(n+1)$ and $(n+3)$ are two consecutive even numbers. One of them is divisible by at least the second power of two. Thus, the product is divisible by 8, which means that $8 \mid 4(s^2 + s + 1)$, but $s(s+1) + 1$ is always odd. A final contradiction.

Remark. A standard way to show that $m^2 + 3$ is not divisible by a prime $p \equiv 2 \pmod 3$ other than two, would be to calculate $\left(\frac{-3}{p}\right)$, where $\left(\frac{a}{p}\right)$ is the *Legendre's Symbol*. By *Quadratic Reciprocity*, $\left(\frac{3}{p}\right) \cdot \left(\frac{p}{3}\right) = (-1)^{\frac{3-1}{2} \cdot \frac{p-1}{2}} = (-1)^{\frac{p-1}{2}} = \left(\frac{-1}{p}\right)$. If we multiply both sides by $\left(\frac{p}{3}\right) \cdot \left(\frac{-1}{p}\right)$, then $\left(\frac{-3}{p}\right) = \left(\frac{p}{3}\right)$, which is only one for $p \equiv 1 \pmod 3$ primes.

§8.3 ISL 1998 N7

Problem 3 (ISL 1998 N7)
Prove that for each positive integer n, there exists a positive integer with the following properties: It has exactly n digits. None of the digits is 0. It is divisible by the sum of its digits.

¶ First solution (Divisibility by 5^k) As the title of the solution suggests, we will find a number divisible by 5^k that consists of k non-zero digits. After that, we will just add more non-zero numbers to the left of it; it doesn't affect the divisibility condition, because we are essentially adding numbers of the form $a \cdot 10^m$, where $m \geq k$ and a is a non-zero digit. Using this approach, we will be able to make the digit sum to be equal to 5^k.

To find the desired number, we will quote the result of USAMO 2003/1, which tells that there exists a desired number made of odd digits, effectively avoiding zero. We can actually choose any subset of digits that covers all residues modulo 5. The proof goes by induction on the number of digits. If we found a desired $(i-1)$-digit number $X = x \cdot 5^{i-1}$, then we need to add $a \cdot 10^{i-1}$ for some odd digit a. The result is $5^{i-1}(x + a \cdot 2^{i-1})$, and we can find an a such that $5 \mid x + a \cdot 2^{i-1}$, because we assumed that we have a full residue system modulo 5 and we need $a \equiv -\frac{x}{2^{i-1}} \pmod 5$.

Now we need to make sure that the number of digits is n and that the sum is 5^k. Suppose that the k digit number has digit sum $k \leq s \leq 9k$, then we just need to make sure that $9(n-k) + s \geq 5^k \geq (n-k) + s$ (because the digits are between 1 and 9). We will be done if for every positive integer n we could find a positive integer k such that $9n \geq 5^k + 8k$ and $5^k - 8k \geq n$. If that's true then, $9(n-k) + s \geq 9n - 8k \geq 5^k \geq n + 8k \geq (n-k) + s$. Let's find the first positive integer k such that $5^k - 8k \geq n$, then the second inequality is fulfilled. Since it's the first integer, then $n > 5^{k-1} - 8k + 8$. $9n > 9 \cdot 5^{k-1} - 72k + 72 \geq 5^k + 8k$, where the last inequality is satisfied for $k \geq 4$. Thus, we have proved the problem for $n > 5^3 - 8 \cdot 3 = 101$.

Now we will optimize a rough estimate of $s = 9k$ and $s = k$ to cover smaller cases. Consider 125 for $k = 3$, then $s = 8$. We want $9(n-3) + 8 \geq 125 \geq (n-3) + 8$, which has a solution for $120 \geq n \geq 16$. If we consider 25 for $k = 2$, then $s = 7$. We want $9(n-2) + 7 \geq 25 \geq (n-2) + 7$, which gives a solution for $20 \geq n \geq 4$. For $n = 3$, 111 works. For $n = 2$, 12 works. For $n = 1$, 1 works. Hence, we are done.

¶ Second solution (Digit sum of multiples of $10^k - 1$ and LTE) We will denote the digit sum of N as $S(N)$. We will first note that

$$10^{3^k} - 1 = \underbrace{999\ldots9}_{3^k \text{ 1s}}$$

works as an example for $n = 3^k$. This is because, by *Lifting the Expnonent Lemma (LTE)*, $\nu_3(10^{3^k} - 1) = \nu_3(3^k) + \nu_3(10-1) = k + 2$. Therefore, $S(10^{3^k} - 1) = 9 \cdot 3^k = 3^{k+2} \mid 10^{3^k} - 1$. Now we need to modify this number to preserve the divisibility condition and get n digits. One can notice [1] that whenever we multiply $10^k - 1$ by small numbers, the digit sum doesn't

[1] I know that it's hard to "notice" it without already knowing

change (it's still $9k$), which enables us to do exactly what we wanted. We can state the following:

Claim — $S((10^k - 1)n) = 9k$ for $n \le 10^k$.

Proof. The proof is direct. Suppose that n has $m \le k$ digits (the statement is obvious for $n = 10^k$). $n = \overline{a_1 a_2 \dots a_m}$ and we can suppose that the last digit is non zero, because a zero in the end doesn't affect the digit sum.

$$(10^k - 1)n = 10^k n - n = \overline{a_1 a_2 \dots a_m \underbrace{00\dots0}_{k \text{ 0s}}} - \overline{a_1 a_2 \dots a_m}.$$

And we can evaluate it to be equal to

$$\overline{a_1 a_2 \dots (a_m - 1) \underbrace{9\dots9}_{k-m \text{ 9s}} (9 - a_1)(9 - a_2) \dots (10 - a_m)}.$$

The digit sum is obviously $9k$, and the number of digits is $k + m$ (except for $n = 1$). □

From the above claim we can now get a big chunk of solutions. Consider

$$\underbrace{11\dots1}_{m \text{ 1s}}2 \cdot (10^{3^k} - 1),$$

where $0 \le m \le 3^k - 1$. It's easy to check that the resulting number has no zero digits because we've established the general form in the proof above. This gives solutions to $3^k + 1 \le n \le 2 \cdot 3^k$. To cover the rest we will set the power of 10 to be equal to $2 \cdot 3^k$. By LTE, $10^{2 \cdot 3^k} - 1$ is still divisible by $9 \cdot 3^k$, but now we also need it to be divisible by 2. Considering

$$n = \underbrace{11\dots1}_{m \text{ 1s}}2 \cdot (10^{2 \cdot 3^k} - 1),$$

where $0 \le m \le 2 \cdot 3^k - 1$, suffices for a solution to $2 \cdot 3^k + 1 \le n \le 4 \cdot 3^k - 1$. Note that, combined with the former interval, $[3^k + 1; 2 \cdot 3^k]$, it covers all positive integer n.

§8.4 USA TSTST 2013/5

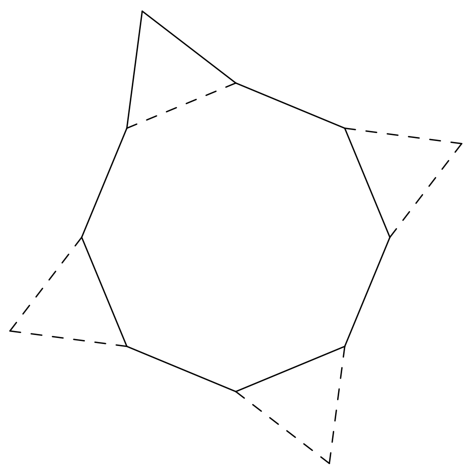
Problem 4 (USA TSTST 2013/5)
Let p be a prime. Prove that any complete graph with $1000p$ vertices, whose edges are labelled with integers, has a cycle whose sum of labels is divisible by p.

¶ Solution (Cauchy Davenport) Whenever we have a statement that involves finding some particular sum with a fixed remainder modulo prime p, it's useful to think about *Cauchy Davenport Theorem* that gives a bound on the cardinality of a *sumset* of A and B, i.e. elements of $A + B = \{a + b | a \in A, b \in B\}$, where A, $B \in \mathbb{Z}/p\mathbb{Z}$. Theorem tells that $|A + B| \geq min(p, |A| + |B| - 1)$.

Now, we need to construct the cycle using this theorem. We will consider some cycle and $p - 1$ disjoint pairs of vertices for which we can choose an alternative path (such that all main paths and alternative paths don't share edges between each other and with other pairs'). Call the set of two sums modulo p on the main path and the alternative path a pair set. If we consider a sumset of all pair sets. By repeatedly applying Cauchy Davenport, the cardinality of it is p, which means we can achieve any residue modulo p by combining some main paths and alternative paths. The only thing we need to ensure is that every pair set has cardinality 2.

Note that the main path and the alternative path form a cycle. We can split every cycle of odd length $2m + 1$ in two parts such that their sums are not equal modulo p. To do this, consider all the possible splits in two parts such that the size of the first part is m and the size of the second part is $m + 1$, and suppose that they are always equal. If we sum all the splits, then in one side we will have mS, and in the other side we will have $(m + 1)S$, where S is the total sum of the edges of the cycle. By assumption, these two sides are equal, which means that $p \mid S$, and we have found a cycle that we were asked about in the problem. Otherwise, there exists the desired split. All that's left is to connect these splits and choose the combination of "mains" and "alternatives" such that the sum is $0 \pmod{p}$.

A way to implement it that gives the lowest vertices bound is to use triangles as follows (example for $p = 5$):



That requires a total of $3p - 3$ vertices, which is a lot less than $1000p$.

§8.5 China TST 2017 2/4

Problem 5 (China TST 2017 2/4)
Given positive integers $d > 1$ and m. Prove that there exists integer $k > l > 0$, such that

$$(2^{2^k} + d, 2^{2^l} + d) > m.$$

For brevity $a_k = 2^{2^k} + d$. In this solution, given one number of the sequence that is divisible by a particular large p, we will try to construct several of them such that the gcd is large and examine the conditions under which it's not possible.

For the sake of contradiction, assume there exists an integer m such that $(a_k, a_l) < m$ for all $k \neq l$. Now, consider the set S of prime numbers that divide at least one a_i. For each $p \in S$, consider the sequence $v_p(a_i)$. Suppose that this sequence is unbounded; then, there exist indices $i < j$ such that $v_p(a_i), v_p(a_j) > X$ for some X such that $p^X \geq m$, otherwise we can bound the sequence by the first term that is greater than X. This leads to a contradiction, as $(a_i, a_j) > p^X$ and we can choose latter exponent to be greater than m. Suppose that $|S|$ is bounded, then for each $p \in S$, the exponent is bounded, hence the terms of $\{a\}$ are bounded, which is not true. Now, it makes sense to differentiate between $p \in S$ that are greater than m and $\leq m$, call the first subset S_1 and the second S_2.

The advantage of it is that the primes of S_1 divide only one element of the sequence. But it's known that the sequence 2^{2^k} has repetitive fashion mod p. Formalizing this argument, consider a sufficiently large n (we will determine how large shortly). Suppose that $p \mid 2^{2^n} + d$ for some $p \in S_1$. Note that if $p - 1 | 2^k - 2^n$ for $k > n$, then $2^{2^k} \equiv 2^{2^n} \pmod p$, and so $2^{2^k} + d$ is also divisible by p. But then, $(a_n, a_k) \geq p > m$. We have to examine when does this divisibility not hold for all k greater than n. Suppose that $p - 1 = 2^a b$ with $2 \nmid b$. Suppose that $a \leq n$, then we just need to find a k such that $b \mid 2^{k-n} - 1$. $k = n + \phi(b)$, where ϕ is *Euler's Totient Function*. Thus, it's the case that $2^{n+1} \mid p - 1$ for all $p \in S_1$ that divide $2^{2^n} + d$.

If we consider the prime decomposition of a_n, we can change all $p \in S_1$ to one mod 2^{n+1}, leaving only bounded products, which is useful because the sequence itself is unbounded. Formally,

$$d \equiv 2^{2^n} + d \equiv \prod_{p \in S_2} p^{v_p(a_n)} \pmod{2^{n+1}}.$$

The exponents on the RHS are bounded. Therefore, the differences is bounded and is divisible by 2^{n+1} which we could make large enough. So,

$$d = \prod_{p \in S_2} p^{v_p(a_n)},$$

from which it follows that $d \mid 2^{2^n} + d$, and consequently, $d = 2^t$.

Now, we are left to show that a sequence defined by $a_n = 2^t(2^{2^n - t} + 1)$ has two terms with unbounded gcd. It would work if we proved that the sequence with terms of the form $2^n - t$ has two terms that are both divisible by a large odd integer. Indeed, if $m \mid 2^n - t, 2^k - t$, where m is odd, then a_n and a_k are both divisible by $2^{v_2(t)m}+1$, but to ensure that, we have to make $n > v_2(t)$. The latter number can be made arbitrarily large.

Let $t = 2^c d$, where $2 \nmid d$. We will prove that in the sequence of odd numbers $b_n = 2^{n-c} - d$, for $n > c$, there will be two terms with gcd larger than any given number.

Similarly to the previous argument, by assuming the contrary, it can be shown that the set of primes dividing at least one element of $\{b\}$ is infinite, since otherwise a sequence of exponents for some prime in the set will be unbounded, and there will be two terms that are both divisible by a large power of this prime. This sequence is periodic modulo any prime p. Therefore, if $p \mid 2^{n-c} - d$, it's also true that $p \mid 2^{n+p-1-c} - d$, then the gcd of these numbers is at least p, which can be made arbitrarily large. Of course, since the terms are odd, the gcd is odd too.

Remark 1. Note that the step with infinitely many primes dividing a_n and $2^{n-c} - d$ can be treated by quoting *Kobayashi's Theorem* that states that if the number of primes dividing at least one term of unbounded sequence a_n is finite, then the same set is infinite for the sequence $a_n + c$ for non-zero constant c. You can see a "proof" of this fact here.

Remark 2. Other problems of similar sort are Iran Third Round 2018 N4 and ISL 2012 N6.

§8.6 Saint-Petersburg MO Selection Round 1998 grade 11/6, proposed by Alexander Golovanov

Problem 6 (Saint-Petersburg MO Selection Round 1998 grade 11/6)
Suppose that $d(n)$ is the number of divisors of a number n. Prove that the sequence $d(n^2+1)$ doesn't become monotonic from any point.

¶ First Solution (Bounding intervals of equal sequence members and bounding the number of divisors) It's known that $d(n)$ has sub-polynomial growth. For the sake of completeness, we will prove it as a separate lemma:

Lemma
For any $\epsilon > 0$, there exists a constant C such that $d(n) < Cn^\epsilon$.

Proof. Suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$. Then, since $d(n) = (\alpha_1 + 1) \dots (\alpha_k + 1)$, we need to prove that

$$\prod_{i=1}^{k} \frac{\alpha_i + 1}{p_i^{\epsilon \alpha_i}}$$

is bounded.

Consider a function $f(x) = \frac{x}{p^{kx}}$. For $x \to +\infty$, exponent is far greater than the linear function, so f approaches 0. For $x \to 0$, the denominator is 1, and the numerator is zero, so f also approaches zero. We can conclude that this function has the maximum value on the interval $(0; +\infty)$. Equating the derivative to a zero,

$$0 = f'(x) = \frac{p^{kx} - xk\ln(p)p^{kx}}{p^{2kx}}.$$

From which it follows that the maximum is attained at $x = \frac{1}{k\ln(p)}$, and the maximum value itself is $f(\frac{1}{k\ln(p)}) = \frac{1}{ek\ln(p)}$. Now we can manipulate the expression in the product as

$$\prod_{i=1}^{k} \frac{\alpha_i + 1}{p_i^{\epsilon\alpha_i}} \le \prod_{i=1}^{k} \frac{2\alpha_i}{p_i^{\epsilon\alpha_i}} \le \prod_{i=1}^{k} \frac{2}{e\epsilon\ln(p_i)}$$

. This is almost bounded, except it might have a very large number of terms. But this can be avoided, because for primes $p > e^{\frac{2}{e\epsilon}}$, the respective multiple is less than 1, and doesn't contribute to the growth of this bound. Therefore, we are only interested in the bounded number of primes. Consequently, the product is bounded.

> **Remark.** Note that we used $f(x) = \frac{x}{p^{kx}}$ because if we used $f(x) = \frac{x+1}{p^{kx}}$, then the upper bound we would get is $\frac{k^\epsilon}{e\epsilon\ln(k)}$, which cannot be bounded.

□

Suppose that the sequence becomes monotonic from some point. If it's non-increasing, then it becomes constant from some point (cannot decrease indefinitely). An easy way to prove it would be to find m and n such that $n^2+1 \mid m^2+1$, then m^2+1 has all the divisors of n^2+1 and at least one more, namely m^2+1. This divisibility is equivalent to $n^2+1 \mid (m-n)(m+n)$, which can be satisfied if m is set to n^2+n+1. This already settles the case of non-increasingness, but also, if we try to divide, $(n^2+n+1)^2+1$ by n^2+1, we will arrive at the miraculous identity:

$$(n^2+n+1)^2+1 = (n^2+1)((n+1)^2+1).$$

Which is going to be the key in proceeding with the problem. Essentially, this identity tells that we can multiply two consecutive terms of the sequence and get some other term.

Note that, if $(n^2+1,(n+1)^2+1)=1$, then $d((n^2+n+1)^2+1) = d(n^2+1)d((n+1)^2+1)$, and if $(n^2+1,(n+1)^2+1)>1$, then $d((n^2+n+1)^2+1) < d(n^2+1)d((n+1)^2+1)$ (because $a+b+1 < (a+1)(b+1)$ for $a>0$, $b>0$, if a is $\nu_p(n^2+1)$ and b is $\nu_p((n+1)^2+1)$.

Now we assume that the function is non-decreasing. If $d(n^2+1) = d((n+1)^2+1) = d((n+2)^2+1)$ for $(n^2+1,(n+1)^2+1)=1$ and $((n+1)^2+1,(n+2)^2+1)>1$, then

$$d((n^2+n+1)^2+1) = d(n^2+1)d((n+1)^2+1) = d((n+2)^2+1)d((n+1)^2+1) > d((n^2+3n+2)^2+1),$$

which contradicts non-decreasingness.

Now, we need to find when does $d=(n^2+1,(n+1)^2+1)>1$. Note that $d \mid (n+1)^2+1-n^2-1 = 2n+1$, and so, $n \equiv -\frac{1}{2} \pmod{d}$, where we can divide by 2 because d is odd (one of the n^2+1 or $(n+1)^2+1$ is odd). $0 \equiv n^2+1 \equiv \frac{5}{4} \pmod{d} \Rightarrow d=5$. And we can easily see that for $n=5k+2$, the gcd is actually equal to 5. Thus, if we have five consecutive terms of the sequence that have the same d value, then then there will be three that are of the form $5k+1$, $5k+2$, $5k+3$. For the first pair, the gcd of their respective n^2+1 values is 1, and the gcd of the respective n^2+1 values for the second pair is 5, which we established to give a contradiction earlier.

Thus, $d(n^2+1) > \frac{n}{5}$ for all n, which is a contradiction to sub-polynomial growth of d.

¶ Second solution (Pell's equation) Treat the non-increasing case the same as in the previous solution. In this solution, assuming non-decreasingness, we will establish an exponential growth of ds on a sequence of arguments that is not growing as fast, then we will use the sub-polynomial growth statement to obtain the contradiction. If it doesn't sound clear, you will understand in a moment. [1]

How can we establish the exponential growth? We want to find pairs (x,y) that satisfy $x^2+1 = c(y^2+1)$ that can be found using *Pell's Equation*. To avoid complications with $(c, y^2+1) \neq 1$, and that it might be a *Pell-like Equation* rather than an ordinary variant, we will set $c=2$. In this case, if $2 \mid y^2+1$, then $4 \mid x^2+1$, which is a contradiction. And this equation is equivalent to Pell's equation in the form $x^2-2y^2=1$.

General solution (x_n,y_n) to it can be found as $x_n+y_n\sqrt{2} = (x_0+y_0\sqrt{2})^{n+1}$, where (x_0,y_0) is the minimal solution, which in this case equals to (3,2). We can rewrite it as $x_n+y_n\sqrt{2} =$

[1] It's probably because I'm bad at writing :)

$(x_{n-1}+y_{n-1}\sqrt{2})(x_0+y_0\sqrt{2}) = (x_0x_{n-1}+2y_0y_{n-1)})+(x_0y_{n-1}+x_{n-1}y_0)\sqrt{2}$. Now, we substitute $x_0 = 3$, $y_0 = 2$ to find the recurrence $x_n = 3x_{n-1} + 4y_{n-1}$, $y_n = 3y_{n-1} + 2x_{n-1}$.

Now, we have $d(x_n^2 + 1) = 2d(y_n^2 + 1)$, which is a sign of a fast growth. It's also evident that $y_{n+1} > x_n$. So we can simplify the work to only one sequence:

$$d(y_{n+1}^2 + 1) \geq d(x_n^2 + 1) = 2d(y_n^2 + 1),$$

where we have used the non-decreasingness.

Now, we need to establish the bound of y_{n+1} through y_n, it's possible to find a constant bound because the recurrences are linear. $2x_n = 6x_{n-1} + 8y_{n-1} < 6x_{n-1} + 9y_{n-1} = 3y_n$, so $y_{n+1} = 3y_n + 2x_n < 6y_n$.

$$d(6^{2k}y_n^2 + 1) > d(y_{n+k}^2 + 1) \geq 2^k d(y_n^2 + 1)$$

. Now we will use the statement about the sub-polynomial growth and find the right exponent. Assume that y_n is large.

$$C2^{\alpha}6^{2k\alpha}y_n^{2\alpha} > C(6^{2k}y_n^2 + 1)^{\alpha} > d(6^{2k}y_n^2 + 1) > d(y_{n+k}^2 + 1) \geq 2^k d(y_n^2 + 1) \geq 2^k.$$

Now it's enough to just make $6^{2\alpha} < 2$ because then the $(\frac{2}{6^{2\alpha}})^k$ can be made large enough to exceed the constant $C2^{\alpha}y_n^{2\alpha}$.

§8.7 IZHO 2023/5, proposed by Alexander Golovanov

Problem 7 (IZHO 2023/5)
We call a positive integer n is *good*, if there exist integers a, b, c, x, y such that $n = ax^2 + bxy + cy^2$ and $b^2 - 4ac = -20$. Prove that the product of any two good numbers is also a good number.

¶ First solution (Prime factorization of good numbers) First of all, we see a homogeneous quadratic equation and a condition on the discriminant. We are thus inclined to do stuff that is usually done to quadratic equations, i.e. completing the square.

$$4an = (2ax + by)^2 - b^2y^2 + 4acy^2 = (2ax + by)^2 + 20y^2 = (2ax + by)^2 + 5 \cdot (2y)^2.$$

When dealing with numbers of the form $x^2 + ny^2$, similar to *Sum of Two Squares Theorem*, we want to make statements about their prime factors. In this case, if a prime $p \mid n$, then $p \mid (2ax + by)^2 + 5(2y)^2$, and if $(2y, p) = 1$, then $-5 \equiv (\frac{2ax+by}{2y})^2 \pmod p$. Which means that -5 is a quadratic residue mod p. If p is 2, then -5 is still a quadratic residue mod p. If $(2y, p) \neq 1$ and $p \neq 2$, then $y \mid p$, this means that $p \mid ax^2$. If $p \mid a$, then $p \mid b^2 + 20$, and so, $-5 \equiv (\frac{b}{2})^2$. Hence, once again, -5 is a quadratic residue mod p. If $p \mid x$, then $p^2 \mid n$.

Thus, if some prime p for which -5 is not a quadratic residue mod p divides n in an odd power, then we find that $p \mid x$, y, and we can divide them by p, obtaining a good number $\frac{n}{p^2} = a(\frac{x}{p})^2 + b(\frac{x}{p} \cdot \frac{y}{p}) + c(\frac{y}{p})^2$, which is also divisible by p in the odd power. After that, we can just descent infinitely.

We have concluded a statement that the exponents of primes in the factorization of a good n can only be odd if -5 is a quadratic residue mod p. Now, we will prove that numbers n with this property are necessarily good.

The easiest way to construct good integers is to set y or x to zero. We can then represent every number of the form ax^2, where we can find b and c such that $b^2 - 4ac = -20$. Note that, of course, multiplying by x^2, we can take care of all the even powers and reduce the problem to only representing the *square-free* integers.

First observation is that b here is even, so we will immediately replace it by $b_1 = \frac{b}{2}$, and get that we want $b_1^2 + 5 = ac$. This condition implies that a must only have prime factors for which -5 is a quadratic residue. So far, it's equivalent to the condition we have set before. We are only left to construct integer b such that $b^2 + 5$ is divisible by a for $a = p_1p_2 \dots p_k$, where -5 is a quadratic residue mod p_i. This tells that we can find integers x_1, x_2, $\dots$, x_k such that $x_i^2 + 5 \equiv 0 \pmod{p_i}$. Now we somehow need to combine them into a single congruence. The theorem that clearly performs this task is the *Chinese Remainder Theorem*, which can be used because modulos are coprime.

In this form, of course, the product of two good numbers remains good.

¶ Second solution (Pure algebra with reduced quadratic forms complex factorizations) It would be great to just multiply two good numbers and do some algebraic manipulations to derive a product with discriminant -20. But it's not a feasible task if a, b, c are unknown. Hence, we need some restriction. Luckily, quadratic forms with fixed discriminants can be associated with a reduced form by applying a transformation $(x, y) \to (px' + ry', qx' + sy')$ with $ps - qr = \pm 1$. Assuming that the transformation is linear, it's exactly the condition that preserves the value of the discriminant (it will come out as $(b^2 - 4ac)(ps - qr)^2$). These are also great because we can reverse the representation and obtain an integer solution in x' and y' that gives the right x and y. So, if we are given a value of n that can be represented in the first form, then it can be represented in the second form as well. The forms that are obtained after transformation with $ps - qr = 1$ are called properly equivalent to the first one, and the ones that were obtained from $ps - qr = -1$ are called improperly equivalent (we won't need that chunk of theory but it's just for general knowledge).

Consider some $n = ax^2 + bxy + cy^2$. Among all the possible equivalent forms choose the one that has the smallest value of $|b|$. For any form with discriminant -20, a and c are positive, because $4an = (2ax + by)^2 + 5 \cdot (2y)^2 > 0$ (if it's zero, then easy to check that n is zero) and same for c. Applying a transformation $(x, y) \to (x + my, y)$ that generates an equivalent form (because $ps - qr = 1$), (a, b, c) goes into $(a, b - 2ma, c')$ and the discriminant stays the same. If $|b| > a$, then we can find m such that $-a \le b - 2am \le a$ and then $|b - 2am| \le a < |b|$, contradicting the assumption that it's the smallest value. Analogously, by considering $(x, y) \to (x, xm + y)$, we can prove that $|b| < c$. We can also assume that in this form, b is positive by swapping signs. By swapping variables, we can assume $a \le c$. Now, we can bound the variables as follows: $-3a^2 = a^2 - 4a^2 \ge b^2 - 4ac = -20$, so a is either 1 or 2 (cannot be zero). If a is 1, then b can be 0 or 1. It cannot be one so it's zero and we found the representation in the form $x^2 + 5y^2$. If a is 2, then b can only be 0, 1 or 2. It's not 0 and not 1, so we find a solution $2x^2 + 2xy + 3y^2$.

Now we just need to prove that the multiplication of numbers of the form $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$ is still of the same type. Now we remember the formula that is used to prove that the multiplication of sums of squares can be represented as a sum of squares too:

$$(x^2 + y^2)(z^2 + t^2) = (xz - yt)^2 + (xt + yz)^2.$$

We can actually come up with the same type of formula for one of the cases in our problem (general idea is too get rid of the $xyzt$):

$$(x^2 + 5y^2)(z^2 + 5t^2) = (xz + 5yt)^2 + 5(yz - xt)^2.$$

But other multiplications require some ingenuity. The trick here is to look at the complex factorization of brackets.

$$(2x^2+2xy+3y^2)(2z^2+2zt+3t^2) = 4\left(x-\frac{1+\sqrt{5}i}{2}y\right)\left(x-\frac{1-\sqrt{5}i}{2}y\right)\left(z-\frac{1+\sqrt{5}i}{2}t\right)\left(z-\frac{1-\sqrt{5}i}{2}t\right).$$

Now we will group them differently. First with third, second with fourth:

$$(2xz - 2yt + yz + xt - \sqrt{5}i(yz + xt + yt))(2xz - 2yt + yz + xt + \sqrt{5}i(yz + xt + yt)) = (2xz - 2yt + yz + xt)^2 + 5(yz + xt + yt)^2.$$

This grouping is motivated by the fact that multiplications that differ by a sign of $\sqrt{5}i$ will only differ by a sign of it, so in the end we will be able to get a single representation (and our conclusion tells that there will exist a representation in the form $x^2 + 5y^2$ or $2x^2 + 2xy + 3y^2$, so we had to try both).

One left. Here we will do the same grouping as in the former case:

$$(2x^2+2xy+3y^2)(z^2+5t^2)=2(x-\frac{1+\sqrt{5}i}{2}y)(x-\frac{1-\sqrt{5}i}{2}y)(z-\sqrt{5}it)(z+\sqrt{5}it)=$$
$$=2(xz+xt+3yt-\frac{1+\sqrt{5}i}{2}(yz-2xt-yt))(xz+xt+3yt-\frac{1-\sqrt{5}i}{2}(yz-2xt-yt))=$$
$$=2(xz+xt+3yt)^2+2(xz+xt+3yt)(yz-2xt-yt)+3(yz-2xt-yt)^2.$$

The $(x^2+5y^2)(z^2+5t^2)=(xz+5yt)^2+5(yz-xt)^2$ identity can be proved in the same way.

§8.8 ISL 2018 N3, proposed by Serbia

Problem 8 (ISL 2018 N3)
Define the sequence $a_0, a_1, a_2, \ldots$ by

$$a_n = 2^n + 2^{\lfloor n/2 \rfloor}.$$

Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

¶ Solution (Equivalence classes based on representability) The idea of the solution is to subtract representable terms from the sum of previous terms to get another representable integer. More formally:

$$a_0 + a_1 + \ldots + a_n = 2^{n+1} - 1 + 2^{\lfloor \frac{n}{2} \rfloor + 1} - 1 + 2^{\lfloor \frac{n-1}{2} \rfloor + 1} - 1 = a_{n+1} + 2^{\lfloor \frac{n+2}{2} \rfloor} - 3.$$

So, instead of representing a_{n+1}, we will focus on representing $2^{\lfloor \frac{n+2}{2} \rfloor} - 3$. It also works in the other direction: If $2^k - 3$ is not representable, then so are a_{2k} and a_{2k-1}.

To examine the representability of $2^k - 3$, we will also use the idea of complementary sums.

$$a_0 + a_1 + \ldots + a_{2k+1} = 2^{2k+2} + 2^{k+2} - 3,$$

from which we see that $2^{k+2} - 3$ is representable if and only if 2^{2k+2} is representable (they can only contain terms with indices less than $2k+2$ because they are both less than $a_{2k+2} = 2^{2k+2} + 2^{k+1}$). But we can also use the second power of two:

$$a_0 + a_1 + \ldots + a_{4k+1} = 2^{4k+2} + 2^{2k+2} - 3,$$

so 2^{2k+2} is representable if and only if $2^{4k+2} - 3$ is representable (again, because they are both smaller than a_{4k+2}). Now, we have a chain of equivalent numbers of the form $2^k - 3$, from which we can construct needed terms of the sequence. Now, we just need to find one representable number of this form and one non-representable. $2^3 - 3 = 5 = a_0 + a_1$ works. One can check that $2^7 - 3 = 125$ is non-representable because $a_0 + \ldots + a_6 = 149$ and $a_7 = 136$. So, if it's representable, then 24 is representable, but $a_5 = 36$ and $a_0 + a_1 + a_2 + a_3 = 21$, so $a_4 = 20$ is in the representation, which means that 4 is representable, but it's easy to see that it's not.

§8.9 Problem 9

Problem 9

Given numbers a_1, a_2, …, a_n such that $1 \le a_i \le 2023$ and $lcm(a_i, a_j) > 2023$ for all $1 \le i < j \le n$. Prove that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} < 2.$$

¶ Solution (Counting numbers divisible by a_i) Of course, all the numbers are different, so $n \le 2023$. The key to this problem is to consider the union of sets $S_i = \{a_i, 2a_i, \ldots\}$. There's no number ≤ 2023 such that it's contained in both S_i and S_j for $j \ne i$. That's because to be contained in both sets, the number has to be divisible by $lcm(a_i, a_j)$, which is greater than 2023. This provides us with a way to bound the number of members of sets in the integers from 1 to 2023. On one hand, their it is at most 2023. On the other hand, it's exactly (since no intersections)

$$\left\lfloor \frac{2023}{a_1} \right\rfloor + \left\lfloor \frac{2023}{a_2} \right\rfloor + \ldots + \left\lfloor \frac{2023}{a_n} \right\rfloor.$$

Thus, we can make the following rough estimate:

$$2023 \ge \left\lfloor \frac{2023}{a_1} \right\rfloor + \ldots + \left\lfloor \frac{2023}{a_n} \right\rfloor > \frac{2023}{a_1} - 1 + \ldots + \frac{2023}{a_n} - 1,$$

from which it follows that $2023(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}) < 2023 + n \le 4046$. And our bound follows.

§8.10 USAMO 2020/3, proposed by Richard Stong and Toni Bluher

Problem 10 (USAMO 2020/3)
Let p be an odd prime. Denote by A the set of all integers a such that $1 \le a < p$, and both a and $4-a$ are quadratic non-residues. Calculate the remainder when the product of the elements of A is divided by p.

¶ First solution (Double counting) The idea of the solution is to consider the set $S = \{x \in \mathbb{F}_p \mid x, 4-x \text{ - quadratic residues}, x \neq 0, 4\}$. It's connection to A is that they both produce elements x for which $x(4-x)$ is a quadratic residue, we can further rewrite this as $4-(2-x)^2$ being a quadratic residue, which is the same as saying that $(2-x)^2$ produces the element from the set S . This gives the motivation for the double counting, hoping that everything cancels. Let's formalize this argument.

x is in this set if and only if $4-x$ is also in the set, so we can pair every x with $4-x$ and double count

$$\prod_{x\in S} x = \prod_{x\in S} 4-x = \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is a qr}}} (4-y^2) = \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is a qr}}} (2-y)(2+y) =$$

$$= \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is a qr}}} (2-y) \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is a qr}}} (2+y) = \prod_{\substack{(p+1)/2\le y\le p-1 \\ y\neq p-2 \\ 4-y^2 \text{ is a qr}}} (2+y) \prod_{\substack{1\le y\le (p-1)/2 \\ y\neq 2 \\ 4-y^2 \text{ is a qr}}} (2+y) =$$

$$= \prod_{\substack{1\le y\le p-1 \\ y\neq 2, p-2 \\ 4-y^2 \text{ is a qr}}} (2+y)$$

.

Now we will substitute $z = y+2$ to rewrite the last expression as

$$\prod_{\substack{3\le z\le p+1 \\ z\neq 4, p \\ (4-z)z \text{ is a qr}}} z = \prod_{\substack{1\le z\le p-1 \\ z\neq 2, 4 \\ (4-z)z \text{ is a qr}}} z$$

(We changed the boundaries because are working in $\mathbb{F}_p$).

Now, we see that z for which $z(4-z)$ is a non-zero quadratic residue are exactly those in A and those that are in S, and we need to exclude the 2. So we can rewrite as

$$\prod_{x\in S} x = \prod_{\substack{1\le z\le p-1 \\ z\neq 2, 4 \\ (4-z)z \text{ is a qr}}} z = \frac{1}{2} \prod_{x\in S} x \prod_{y\in A} y.$$

We are only left with dividing through the non-zero $\prod_{x\in S} x$ to see that

$$\prod_{y\in A} y = 2.$$

¶ Second solution (Characterizing all the elements of A in $\mathbb{F}_{p^2}$) This solution will use the basics of fields algebra. Let n be any quadratic non-residue. Every other non-residue can be represented as $\frac{x^2}{n}$ for some $x \in \mathbb{F}_p$. Thus, we are interested in the $\mathbb{F}_p$ solutions of

$$\frac{(x^2 + y^2)}{n} = 4.$$

Solving these types of equations is routine if we expand the field. We will work in $\mathbb{F}_p[\omega]$, where $\omega^2 = n$, i.e. we will just consider the numbers of the form $a + b\omega$ with $a, b \in \mathbb{F}_p$; there are p^2 of them, so it's a $\mathbb{F}_p$. Benefit of this field is that every element of $\mathbb{F}_p$ now has a square root. Let $i \in \mathbb{F}_p$ be the square root of -1. We will use the following factorization:

$$(x + yi)(x - yi) = (2\omega)^2.$$

Suppose that one bracket is $2t\omega$ and the other is $\frac{2\omega}{t}$. Then $x = \omega\left(t + \frac{1}{t}\right)$ and $y = \frac{\omega}{i}\left(t - \frac{1}{t}\right)$. This is a general solution of this equation in $\mathbb{F}_{p^2}$, but now we need to find those $t \in \mathbb{F}_{p^2}$ such that x and y are in $\mathbb{F}_p$. This check can be made using the fact that the elements of $\mathbb{F}_p$ are exactly those x for which $x^p = x$, which is a consequence of *Lagrange's Theorem* and the fact that every polynomial of degree d with coefficients in the field has at most d roots. We will also use the fact that $(a + b)^p = a^p + b^p$ in this field because $p \mid \binom{p}{i}$ for $1 \le i \le p - 1$. And this field has characteristic p, i.e. $\underbrace{x + x + \ldots + x}_{p \text{ times}} = 0$ for x in this field.

Now, using the above fact, we can raise to the power painlessly. We want to check that $x^p = x$, which is the same as

$$\omega\left(t + \frac{1}{t}\right) = n^{\frac{p-1}{2}}\omega\left(t^p + \frac{1}{t^p}\right).$$

And $y^p = y$, which is the same as

$$\frac{\omega}{i}\left(t - \frac{1}{t}\right) = \frac{n^{\frac{p-1}{2}}\omega}{i^p}\left(t^p - \frac{1}{t^p}\right).$$

It's known that $n^{\frac{p-1}{2}} = -1$ for quadratic non-residue $n \in \mathbb{F}_p$ by *Euler's Criterion*. So, the first equation reads as $(t^{p+1} + 1)(t^{p-1} + 1) = 0$. The second equation is trickier because i^p is either i or $-i$ depending on $p \pmod 4$. This splits the problem into two cases:

If $p = 1 \pmod 4$, then the second equation reads as $(t^{p-1} + 1)(t^{p+1} - 1) = 0$, which gives the only possibility of $t^{p-1} + 1 = 0$. So

$$A = \{(t + 1/t)^2 : t \in \mathbb{F}_{p^2}, t^{p-1} + 1 = 0\}.$$

Some of the elements of this set will be equal. $(t + 1/t)^2 = (t' + 1/t')^2$ if and only if $t' \in \{t, -t, \frac{1}{t}, -\frac{1}{t}\}$. First of all,

$$t^{p-1} + 1 \mid t^{2(p-1)} - 1 \mid t^{p^2-1} - 1 \mid t^{p^2} - t.$$

Where the last polynomial's roots are precisely all the elements of F_{p^2} (Lagrange's Theorem), so they are all different. To split these roots, we will consider the equation $t^{\frac{p-1}{2}} - i = 0$. If t is the root of it, then $\frac{1}{t}$ and $-\frac{1}{t}$ are not roots. But $t' = -t$ is always a root and $t + \frac{1}{t} = -t' - \frac{1}{t'}$. If we consider

$$\prod_{\substack{t \in \mathbb{F}_{p^2} \\ t^{\frac{p-1}{2}} - i = 0}} (t + 1/t),$$

it is the product of distinct elements of A multiplied by $(-1)^{\frac{p-1}{4}}$. The roots split in pairs of t and $-\frac{1}{t}$ (which don't coincide as $\pm i$ is not the root), and we have to multiply by -1 in the power of the number of these pairs. So

$$\prod_{a \in A} a = (-1)^{\frac{p-1}{4}} \prod_{\substack{t \in \mathbb{F}_{p^2} \\ t^{\frac{p-1}{2}} - i = 0}} (t + 1/t).$$

This part is also routine. Set

$$f(x) = x^{\frac{p-1}{2}} - i = \prod_{\substack{t \in \mathbb{F}_{p^2} \\ t^{\frac{p-1}{2}} - i = 0}} (x - t) = \prod_{\substack{t \in \mathbb{F}_{p^2} \\ t^{\frac{p-1}{2}} - i = 0}} (t - x).$$

Where the last step is because there is an even number of brackets. So

$$(-1)^{\frac{p-1}{4}} \prod_{\substack{t \in \mathbb{F}_{p^2} \\ t^{\frac{p-1}{2}} - i = 0}} (t + 1/t) = (-1)^{\frac{p-1}{4}} \frac{f(i)f(-i)}{f(0)} = (-1)^{\frac{p-1}{4}} \frac{f(i)^2}{-i}.$$

The value of this depends on the residue modulo 8. If $p = 1 \pmod 8$, $(-1)^{\frac{p-1}{4}} \frac{f(i)^2}{-i} = \frac{(1-i)^2}{-i} = \frac{-2i}{-i} = 2$. If $p = 5 \pmod 8$, $(-1)^{\frac{p-1}{4}} \frac{f(i)^2}{-i} = -\frac{(-1-i)^2}{-i} = = -\frac{2i}{-i} = 2$.

If $p = 3 \pmod 4$, nothing changes much. The polynomial we consider is $t^{p+1} + 1 = 0$. We can evaluate, for $f(x) = x^{\frac{p-1}{2}} - i$,

$$\prod_{a \in A} a = (-1)^{\frac{p+1}{4}} \prod_{\substack{t \in \mathbb{F}_{p^2} \\ t^{\frac{p+1}{2}} - i = 0}} (t + 1/t) = (-1)^{\frac{p+1}{4}} \frac{f(i)f(-i)}{f(0)}.$$

Same modulo 8 check is sufficient to conclude that the value is 2 in this case too.

§8.11 USA TST 2017/6, proposed by Noam Elkies

Problem 11 (USA TST 2017/6)
Prove that there are infinitely many triples (a, b, p) of positive integers with p prime, $a < p$, and $b < p$, such that $(a+b)^p - a^p - b^p$ is a multiple of p^5.

¶ Solution (Thue's lemma and factorization) Key step here is to think about the factorization of $(a+b)^p - a^p - b^p$. Since all the binomial coefficients are divisible by p, we get one multiple of p for free, but it turns out that we can extract more:

Claim — For $p = 3k+1$, $p(a^2+ab+b^2)^2 \mid (a+b)^p - a^p - b^p$ as polynomials.

Proof. Standard thing to do with homogeneous polynomials is to let $a = bx$, then, after dividing by b^4, we need to prove that $p(x^2+x+1)^2 \mid (x+1)^p - x^p - 1$. We need to prove that the third root of unity is a double root of $(x+1)^p - x^p - 1$. First, check that it's a root:

$$(\omega+1)^p - \omega^p - 1 = -(\omega^2)^p - \omega^p - 1 = -\omega^2 - \omega - 1 = 0,$$

where we have used that $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$. Now, we find a derivative to check that it's a double root:

$$((x+1)^p - x^p - 1)' = p(x+1)^{p-1} - px^{p-1},$$

and when we substitute ω,

$$p(-\omega^2)^{p-1} - p(\omega)^{p-1} = 0,$$

because $6 \mid p-1$. □

Now the problem is considerably easier. We need to find $0 < a, b < p$ so that $p^2 \mid a^2+ab+b^2$. If we had a lenient bound of p^2 instead of p, then we could take $a = g^{\frac{p(p-1)}{3}} \pmod{p^2}$ and $b = 1$, where g is the primitive root of unity modulo p^2. But note that we have two numbers, so we modify the above example by applying *Thue's Lemma* that gives the bound of $\sqrt{p^2}$, exactly as we needed.

By Thue's lemma, for every x relatively prime to p, we can find $0 < |a|, |b| < \sqrt{p^2}$ such that $a = bx \pmod{p^2}$. Take $p^2 \mid x^2+x+1$ that we found above. For this x, $p^2 \mid a^2+ab+b^2$. We only need to take care of the possibility that a or b are negative. If both of them are negative, then consider $(-a, -b)$. If $a < 0$, $b > 0$, then consider $(-a, a+b)$ or $(b, -a-b)$, depending on the sign of $a+b$.

§8.12 USA TST 2014/2, proposed by Evan O'Dorney and Victor Wang

Problem 12 (USA TST 2014/2)

Let $a_1, a_2, a_3, \ldots$ be a sequence of integers, with the property that every consecutive group of a_i's averages to a perfect square. More precisely, for every positive integers n and k, the quantity

$$\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}$$

is always the square of an integer. Prove that the sequence must be constant (all a_i are equal to the same perfect square).

¶ Solution (Quadratic residues analysis) We will instead rewrite the condition in the form

$$\left(\frac{a_n + \ldots + a_{n+k-1}}{p}\right) = \left(\frac{k}{p}\right)$$

for every p relatively prime to both $a_n + \ldots + a_{n+k-1}$ and k. Consider any prime divisor p of a_1. Since $a_1 + a_2 + \ldots + a_k \equiv a_2 + \ldots + a_k \pmod p$, we can say that either $p \mid k$ or $p \mid a_2 + \ldots + a_k$, or $(\frac{k}{p}) = (\frac{k-1}{p})$. We can force the second condition if we consider the first quadratic non-residue k. $(\frac{k}{p}) \neq (\frac{k-1}{p})$ and $0 < k < p$, so $p \mid a_2 + \ldots + a_k$. Now, we extend the same argument to the other side. Since, $a_2 + \ldots + a_k + a_{k+1} \equiv a_{k+1} \pmod p$, then $(\frac{1}{p}) = (\frac{k}{p})$ or $p \mid a_{k+1}$, or $p \mid k$. We know that the last and first condition don't hold, so we are only left with $p \mid a_{k+1}$.

Note that we could perform this argument to both sides of a number divisible by p, so, in general, if $p \mid a_i$, then $p \mid a_{i-k}$ and $p \mid a_{i+k}$. It's also trivial that $p \mid a_i + \ldots + a_{i+p-1}$ and that $p \mid a_{i+1} + \ldots + a_{i+p}$, so, subtracting, $p \mid a_{i+p}$.

Thus, if $p \mid a_i$, then $p \mid a_{i+km+np}$ for integer m and n. But it's known that the latter can take every value because $(k, p) = 1$ (*Bezout's Theorem*), so $p \mid a_i$ for all i. Which, combined with the fact that they are all squares, forces $p^2 \mid a_i$, so we change the sequence to a' with $a'_i = \frac{a_i}{p^2}$. Let's check that the condition didn't change:

$$\frac{a'_n + \ldots + a'_{n+k-1}}{k} = \frac{\frac{a_n + a_{n+1} + \cdots + a_{n+k-1}}{k}}{p^2} = (\frac{t}{p})^2.$$

The latter is either a perfect square or not an integer, but the LHS is an integer. This way, we will keep dividing until we will be left with the sequence of ones, indicating that initially all numbers were the same too.

§8.13 Saint-Petersburg MO 2003 grade 10/7, proposed by Andrey Badzyan

Problem 13 (Saint-Petersburg MO 2003 grade 10/7)
Given a prime number p and an integer $n \geq p$. Consider an arbitrary array of positive integers $a_1, a_2, \ldots, a_n$. Let f_k be the number of k element subsets with sum of their elements divisible by p. Prove that

$$p \mid \sum_{k=0}^{n} (-1)^k f_k.$$

Here we assume that $f_0 = 1$.

¶ Solution (Erdos-Ginsburg-Ziv theorem proof style count) There's a well known result due to Erdos-Ginzburg-Ziv that states that in every array of $2p-1$ integers, there exists a subset of size p such that its sum is divisible by p. The proof is to consider the sum of $p-1$th powers of sums of all size-p subsets. The trick here is that, under assumption that every sum is non-zero modulo p, every summand is 1 modulo p, but we can double count it and reach the contradiction. This clever idea comes in handy in this problem too to calculate f_k modulo p.

$$\sum_{\substack{A \subseteq \{1;2;\ldots;n\} \\ |A|=k}} \left(\sum_{i \in A} a_i \right)^{p-1} \equiv \binom{n}{k} - f_k \pmod{p}.$$

Because the summands on the left are all 1 except for f_k of them that are 0. Thus, we need to calculate the following sum modulo p:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} + (-1)^{k+1} \sum_{\substack{A \subseteq \{1;2;\ldots;n\} \\ |A|=k}} \left(\sum_{i \in A} a_i \right)^{p-1}$$

Of course,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = (1-1)^n = 0.$$

So, we are left to show that (divided by -1 for convenience)

$$\sum_{k=0}^{n} (-1)^k \sum_{\substack{A \subseteq \{1;2;\ldots;n\} \\ |A|=k}} \left(\sum_{i \in A} a_i \right)^{p-1} \equiv 0 \pmod{p}.$$

Consider some monomial $a_1^{x_1} a_2^{x_2} \ldots a_n^{x_n}$ in this sum. x_i are non-negative and $x_1 + \ldots + x_n = p-1$. For each bracket that spawns it, the coefficient in expansion is $\binom{p-1}{x_1}\binom{p-1-x_1}{x_2}\ldots\binom{x_n}{x_n}$ (First choosing which x_1 brackets to choose a_1 from, then which x_2 from the $p-1-x_1$ that are left, and so on). Suppose that m is the number of non-zero x_i. In order to get this monomial, we only need these m a_i to appear. Thus, the total coefficient is,

$$\binom{p-1}{x_1}\binom{p-1-x_1}{x_2}\dots\binom{x_n}{x_n}\left((-1)^m\binom{n-m}{0}+(-1)^{m+1}\binom{n-m}{1}+\dots+(-1)^n\binom{n-m}{n-m}\right)=$$

$$=\binom{p-1}{x_1}\binom{p-1-x_1}{x_2}\dots\binom{x_n}{x_n}(-1)^m(1-1)^{n-m}=0.$$

So, every monomial has a zero coefficient, making the sum 0 modulo p.

§8.14 USAMO 2012/3, proposed by Gabriel Caroll

Problem 14 (USAMO 2012/3)
Determine which integers $n > 1$ have the property that there exists an infinite sequence $a_1, a_2, a_3, \ldots$ of nonzero integers such that the equality

$$a_k + 2a_{2k} + \ldots + na_{nk} = 0$$

holds for every positive integer k.

¶ Solution (Multiplicativity and Bertrand's postulate) First of all, let's deal with the $n = 2$ case:

$$a_k = -2a_{2k},$$

which means that $a_{2^k} = \left(-\frac{1}{2}\right)^k a_1$, which forces $a_1 = 0$, which is a contradiction.

After playing around with the condition, one might suppose that the construction exists for $n > 2$. k appears in every index, so it would be great to cancel it out. Turns out it's possible to do if the sequence is multiplicative, i.e. $a_{mn} = a_m a_n$ for all positive integers m and n. So, we are interested in multiplicative sequences such that

$$a_1 + 2a_2 + \ldots + na_n = 0.$$

Multiplicative sequences only depend on values in prime inputs, so it makes sense to set as much of them to ones as we can to not have too many things to control. Leaving two primes to be non-one seems like a great idea, because of *Bezout's Theorem*. We will have some equation of the form $ka_p + ma_q = t$ with some integer k, m, t, and if k and m are relatively prime, we can definitely find suitable a_p and a_q.

To further simplify our work, we will set p and q close to n so that there would be only a handful of indices at most n that are divisible by p or q. How to do it? *Bertrand's Postulate*!

We can find a prime q such that $\lceil \frac{n}{2} \rceil < q < 2\lceil \frac{n}{2} \rceil$ and p such that $\frac{q-1}{2} < p < q - 1$. In order for it to work, though, we need $\frac{q-1}{2} \geq 2$, which is the same as $q \geq 5$, so we will assume that $n \geq 5$. $2q > n$ and $4p \geq 2q > n$. So we only care about a_q, a_p, a_{2p}, a_{3p}. $q \neq p$, so no two of them coincide. To avoid terms like a_{p^2} (because they spawn degree two terms), we will also make sure that $p > 3$, which leads to $q \geq 7$, which needs $n \geq 9$. Now, we have three cases which are essentially the same:

If $n \geq 3p$, then we need a_p, a_q

$$6pa_p + qa_q = 6p + q - \frac{n(n+1)}{2},$$

which we established to exist due to Bezout's theorem and $(6p, q) = 1$.

If $3p > n \geq 2p$, then we need

$$3pa_p + qa_q = 3p + q - \frac{n(n+1)}{2},$$

which exists due to $(3p, q) = 1$.

If $2p > n \geq p$, then we need

$$pa_p + qa_q = p + q - \frac{n(n+1)}{2},$$

which exists due to $(p, q) = 1$.

We are left to come up with examples for $3 \leq n \leq 8$. It's not hard to come up with them using the heuristic we developed before by introducing a few adjustments.

For $n = 3$, we take $a_3 = -1$ and other prime points are 1.

For $n = 4$, we take $a_3 = -7$ and $a_2 = 2$.

For $n = 5$, we take $a_5 = -2$ and others are 1.

For $n = 6$, the above construction works with $p = 3$, $q = 5$.

For $n = 7$, we take $a_7 = -3$ and others are 1.

For $n = 8$, the above construction works with $p = 5$ and $q = 7$.

§8.15 Kazakhstan MO 2017 grade 11/6, proposed by Anuarbek Tynyshbekov

Problem 15 (Kazakhstan MO 2017 grade 11/6)
Prove that there are infinitely many composite numbers n, such that

$$n \mid 2^{\frac{n-1}{2}} + 1.$$

¶ Solution (Recursion) A standard method of constructing solutions to these types of divisibilities is the recursion. We will try to find a solution of the form $\frac{2^a+1}{3}$ because to check the divisibility

$$2^a + 1 \mid 2^{\frac{2^{a-1}-1}{3}} + 1,$$

we only need to check that $3a \mid 2^{a-1} - 1$, which looks like the divisibility we derived it from. If a is composite and odd, then $\frac{2^a+1}{3}$ is not a prime either (divisible by $\frac{2^p+1}{3} > 1$, where p is any prime divisor of a), so we can generate a solution from any composite a that satisfies $3a \mid 2^{a-1} - 1$. In order to generate the solutions to this divisibility, we will also use the same argument.

Suppose that a_1 is some composite solution (we will find one later). Now, given a composite solution a_i, let $a_{i+1} = 2^{a_i} - 1$. a_{i+1} is composite because it's divisible by $2^p - 1$, where p is any prime divisor of a_i. Now, we can get

$$3a_{i+1} = 3(2^{a_i} - 1) \mid 3(2^{2^{a_i-1}-1} - 1) \mid 2^{2^{a_i}-2} - 1 = 2^{a_{i+1}-1} - 1,$$

where the last divisibility holds because $3 \mid 2^{2^{a_i-1}-1} + 1$. $a_{i+1} > a_i$ (because $a_1 > 1$ as well as every other a_i).

What is left is to construct a_1. Write $a_1 = pq$, then we need $\operatorname{ord}_q(2) \mid p - 1$ and $\operatorname{ord}_p(2) \mid q - 1$ (because $\operatorname{ord}_q(2) \mid q - 1$ and the same for p). We also need p, q, 3 to all be different. Easiest way is to find the numbers such that $\operatorname{ord}_q(2) \mid p - 1 \mid q - 1$. $\operatorname{ord}_q(2)$ has to be relatively small, so we will try to find a big prime divisor of some $2^d - 1$. We note that $d = 5$ works with 31. And we can find 11 from here. $a_1 = 11 \cdot 31$ works.

Remark. I first saw the recursion trick in IZHO 2007/3. This problem also has another variant of constructing infinitely many n such that $n^2 \mid a^n - b^n$ given fixed $|a - b| > 1$. Notably, both of these problems allow for an easier construction using *Zsigmondy's Theorem* (I'm not sure if Zsigmondy is helpful for the original problem).

The only confusing step of applying this trick in the original problem is to come up with division by three. But it's quite natural because it gives a nice divisibility to work with unlike $2^a + 1$ option that gives $2^{2^{a-1}} + 1$. We still want $2^a + 1$, so we have to divide by some constant (multiplying by a constant only makes things worse).

§8.16 ELMOSL 2011 N2, proposed by Victor Wang

Problem 16 (ELMOSL 2011 N2)
Let $p \geq 5$ be a prime. Show that

$$\sum_{k=0}^{(p-1)/2} \binom{p}{k} 3^k \equiv 2^p - 1 \pmod{p^2}.$$

¶ Solution (Binomial coefficients manipulation and Roots of Unity Filter roll up) As it happens a lot in these types of problems, we need to manipulate the binomial coefficient. It turns out that the binomial coefficient is simply congruent to a fraction:

$$\binom{p}{k} = \frac{p}{k} \cdot \frac{k \cdot \ldots \cdot (p-1)}{1 \cdot 2 \cdot \ldots \cdot (p-k)} \equiv (-1)^{p-k} \cdot \frac{p}{k} \cdot \frac{(p-k) \cdot \ldots \cdot 1}{1 \cdot 2 \cdot \ldots \cdot (p-k)} = (-1)^{k+1} \frac{p}{k} \pmod{p^2}$$

for $p \leq k > 0$.

Now, since we only have half of the values from 0 to p, it makes sense to double them, because then we would be able to roll it into an easier expression using filter. Fraction representation allows to do it well:

$$\binom{p}{k} \equiv (-1)^{k+1} \frac{p}{k} \equiv 2 \cdot (-1)^{k+1} \frac{p}{2k} \equiv 2 \cdot (-1)^k \binom{p}{2k} \pmod{p^2}.$$

Summing up, we have this transformation:

$$\sum_{k=0}^{(p-1)/2} \binom{p}{k} 3^k \equiv \sum_{k=0}^{(p-1)/2} \binom{p}{2k} (-3)^k \pmod{p^2},$$

which is equivalent to calculating the doubled sum of coefficients of even degrees in the expansion of $x(x + \sqrt{-3})^p$. But it's clear that it's equal to (the simplest case of the filter):

$$(1 + \sqrt{-3})^p + (1 - \sqrt{-3})^p = (-2\omega^2)^p + (-2\omega)^p = -(-2)^p = 2^p$$

Where $\omega = \frac{1+\sqrt{-3}}{2}$ is the third root of unity. The identity follows from the fact that $\omega^{3p} = 1$ and $\omega^p \neq 1$, because $p > 3$, so $\omega^{2p} + \omega^p + 1 = 0$.

Remark. Other examples of this technique include ISL 2011 N7 and ELMO 2009/6.

§8.17 IMO 2003/6, proposed by Johan Yebbou

Problem 17 (IMO 2003/6)
Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

¶ Solution (Characterizing q, constructing it using cyclotomic polynomial and order finish)
Let g be a primitive root modulo q, then $p = g^m$ and if $p \nmid q-1$, then we can find k such that $kp \equiv m \pmod{q-1}$, then $n = g^k$ would work. So, we must have $p \mid q-1$.

Now, if n exists, then

$$1 \equiv n^{q-1} = (n^p)^{\frac{q-1}{p}} \equiv p^{\frac{q-1}{p}} \pmod{q}.$$

Thus, we need to find a q such that $p \mid q-1$ and $p^{\frac{q-1}{p}} \not\equiv 1 \pmod{q}$.

First condition is satisfied by taking a prime divisor of something of the form $n^p - 1$ because it ensures that the order is p or 1, but we can take care of the 1 case if we take the divisor of $\frac{n^p-1}{n-1} = n^{p-1} + \ldots + 1$ (This is called the pth *Cyclotomic Polynomial*, and it allows an easy way of constructing primes that are 1 modulo p). If $q \mid n-1$ in this case, then $q \mid p$, but we can take n to be not 1 modulo p. Thus, the order is p, and so, $p \mid q-1$, because $n^{q-1} \equiv 1 \pmod{q}$.

Now, we need to be able to control the order of p modulo q. But we already control the order of n, so why not to make $n = p$. Now, we only need to ensure that $q \not\equiv 1 \pmod{p^2}$. But if every prime factor of $p^{p-1} + \ldots + 1$ was $1 \pmod{p^2}$, then the number itself has to be $\equiv 1 \pmod{p^2}$, which is not true as it's equivalent to $1 + p$.

§8.18 Problem 18

Problem 18
Prove that we can fill a finite square grid with positive integers such that the sum of entries of any square sub-grid is a perfect square, and the sum of entries of a non-square sub-grid is not a square.

¶ Solution (Coordinate markup and Zsigmondy's theorem) Label squares with coordinates (i, j), where $(0, 0)$ is the bottom left cell. The idea is that we want to control the sum dependent on the coordinates, so it makes sense to write numbers related to the coordinates of the cell. Since we have a two dimensional sum, it makes sense to make a number in cell (a, b) to be $k^a m^b$ (because the sum then will roll up into two brackets easily). To minimize the degrees of freedom, we will set $k = m$. We are also given that each square has a square number, so let the number written be k^{2a+2b}. It all boils down to choosing a right integer k.

Now, the sum in the square with bottom left coordinate (a, b) and upper right coordinate $(a + l - 1, b + l - 1)$ is

$$k^{2a}k^{2b}(1 + k^2 + \ldots + k^{2l-2})(1 + k^2 + \ldots + k^{2l-2}),$$

which evidently is a square.

The sum in the non-square grid with sides l and s and bottom left coordinate (a, b) and upper right coordinate $(a + l - 1, b + s - 1)$ is

$$k^{2a}k^{2b}(1 + k^2 + \ldots + k^{2l-2})(1 + k^2 + \ldots + k^{2l-2}) = k^{2a+2b}\frac{(k^{2l} - 1)(k^{2s} - 1)}{(k^2 - 1)^2}.$$

The plan here sounds simple. Find a k such that for all $r > 1$, $k^{2r} - 1$ has a prime factor p such that $p \nmid k^{2i} - 1$ for $i < r$, and $v_p(k^{2r} - 1) = 1$. This would make the exponent of p to be 1, violating the square condition. This sounds a lot like *Zsigmondy's theorem*.

Suppose that we have a k such that this condition holds for $k^2 - 1$, $k^4 - 1$, $\ldots$, $k^{2r} - 1$, and we have to add another $k^{2r+2} - 1$. Suppose that $k^{2i} - 1$ is divisible by p_i (of course, $p_i \neq p_j$ for $i \neq j$). Note that we are free to add $\prod p_i^2$ to k, so we can make it not equal to 2 (exceptions, duh). Zsigmondy tells us that there exists some prime p_{r+1} (trivially not equal to other p) such that it divides $k^{2r+2} - 1$ and doesn't divide any of previous $k^{2i} - 1$. Now, suppose that $p_{r+1}^2 \mid k^{2r+2} - 1$. Consider some number k' such that $k' \equiv k \pmod{p_i^2}$ for $1 \leq i \leq r$ and $k' \equiv k + p_{r+1} \pmod{p_{r+1}^2}$, which exists due to CRT. Then, all divisibilities and non-divisibilities related to p_i with $1 \leq i \leq r$ still hold. And

$$k'^{2r+2} - 1 \equiv k^{2r+2} - 1 + (2r + 2)p_{r+1}k^{2r+1} \equiv (2r + 2)p_{r+1}k^{2r+1} \pmod{p_{r+1}^2}$$

by binomial expansion, in which all other terms have p_{r+1} raised to at least second power. Now, to ensure the exponent of 1, we need to ensure that $p_{r+1} \nmid 2r + 2$. But this was easy to set up initially.

Consider all primes p that divide $2r + 2$ that are not listed in p_1, p_2, $\ldots$, p_r (these are safe because they cannot be p_{r+1}). Also make sure to pick such divisors that $p - 1 \nmid 2r + 2$.

For them, take $k \equiv g \pmod{p}$, where g is the primitive root modulo p, then $p \mid k^{2r+2} - 1$ is equivalent to $p - 1 \mid 2r + 2$, which we assumed is not true. Since the modulos are coprime, by *Chinsese Remainder Theorem*, it's possible to find such a k that also satisfies equivalence to original k modulo all p_i^2. For this new k, all the divisibilities and non-divisibilities related to p_i with $1 < i \le r$ still hold. Consider p_{r+1} from Zsigmondy's theorem and suppose it divides $2r + 2$, then, since we eliminated the divisibility by all the prime factors p of $2r + 2$ for which $p - 1$ doesn't divide $2r + 2$, $p_{r+1} - 1 \mid 2r + 2$. $2r + 2 > p_{r+1} - 1$, because otherwise it couldn't be divisible by p_{r+1}. Therefore, $2r + 2 - p_{r+1} + 1$ is also even (except $p_{r+1} = 2$ but in this case, $k^2 - 1$ is also divisible by 2, contradicting the choice of p_{r+1}), and so $p_{r+1} \nmid k^{2r+2-p_{r+1}+1} - 1$, but the exponent is divisible by $p_{r+1} - 1$, so the divisibility holds. Contradiction.

Thus, adding another $k^{2t} - 1$, we can manipulate k by *Chinese Remainder Theorem* so that the condition with unique primes with exponent one would hold.

¶ Second solution (Alternative finish through irreducible polynomials factorization) To avoid the usage of Zsigmondy's theorem, one could use a standard polynomials approach. In the exact same way, suppose that we have a k such that the condition holds for $k^2 - 1$, $k^4 - 1$, $\ldots$, $k^{2r} - 1$ with primes $p_1, p_2, \ldots, p_r$. We want to find some prime p such that p only divides $k^{2r+2} - 1$, $p \nmid 2r + 2$ and $p \neq p_i$, then we will be able to manipulate it using the *Chinese Remainder Theorem* so that the exponents are one.

Consider $e^{\frac{2i\pi}{2r+2}}$. It's a root of $k^{2r+2} - 1$, and not a root of any $k^{2i} - 1$ for $1 \le i < k + 1$. Consider the minimal polynomial P of $e^{\frac{2i\pi}{2r+2}}$, i.e. a polynomial with the smallest degree and integer coefficients that has this number as a root. It's irreducible (otherwise, we would find a smaller degree polynomial), so either $\gcd(k^{2r+2} - 1, P) = P$ or 1. In the latter case, there exists integer polynomials A, B and an integer $C \neq 0$ such that $A(k^{2r+2} - 1) + B \cdot P = C$ (*Bezout's Identity* for polynomials). Substituting $e^{\frac{2i\pi}{2r+2}}$, we get that $C = 0$, contradiction. So it's the case that $P \mid k^{2r+2} - 1$. Since $e^{\frac{2i\pi}{2r+2}}$ is not a root of all the other $k^{2i} - 1$, P doesn't divide any of them. Now, we will consider the factorizations of $k^2 - 1$, $k^4 - 1$, $\ldots$, $k^{2r} - 1$ into irreducible polynomials (analogy with prime factorization). The idea here is that two irreducible polynomials only have bounded common factors. Because if the two irreducible polynomials P and Q are different, then $\gcd(P, Q) = 1$. By Bezout's identity, $A \cdot P + B \cdot Q = C$, so P and Q can only be divisible by a prime at most $|C|$. So, there exists some constant such that if P is divisible by some prime, then it doesn't divide any of the other irreducible polynomials we are considering, i.e. it won't divide any of the $k^2 - 1$, $k^4 - 1$, $\ldots$, $k^{2r} - 1$. Now, we only need to quote the *Schur's Theorem* that says that the set of primes dividing at least one integer point with non-zero value is infinite and take big enough such prime, so that it would be greater than $2r + 2$ and all the p_i. This way we will ensure the condition and then construct the needed k using CRT as we did in the previous solution.

§8.19 Putnam 2012 B6

Problem 19 (Putnam 2012 B6)
Let p be an odd prime number such that $p \equiv 2 \pmod 3$. Define a permutation π of the residue classes modulo p by $\pi(x) \equiv x^3 \pmod p$. Show that π is an even permutation if and only if $p \equiv 3 \pmod 4$.

¶ First solution (Primitive root transformation) It's easy to check that a swap of two neighbouring elements changes the number of inversions by 1 (the state of only this pair changes). Every transposition, i.e. swap of any two elements, can be represented as the composition of $2k+1$ swaps of neighbours: If there are k elements in between a and b and b is on the right of a, then we swap a to the right $k+1$ times, then swap b to the left k times. Thus, since every swap changes the parity of the permutation, the transposition also changes the parity. Every permutation can be represented as the union of cycles. Every cycle of length k can be represented as the union of $k-1$ transposition: Swapping the first element with the second, then second with third, and so on until the last element, then every element is in the place. Thus, the parity change of each cycle is equal to the parity of the number of its elements - 1. Thus, the total parity change is the number of elements in the permutation - #number of cycles. In the original problem, we need to show that the permutation contains an odd number of cycles if and only if $p \equiv 3 \pmod 4$. Cycles analysis is not needed for this solution, but we will use it in the second solution.

The main idea of this solution is to change the mapping $x \to x^3 \pmod p$ to $x \to 3x \pmod{p-1}$, which is doable if we first change the sequence $\{1, 2, \dots, p-1\}$ to $\{a_1, \dots, a_{p-1}\}$ by the rule $x \to a_x$ if $g^{a_x} \equiv x \pmod p$, where g is the primitive root modulo p. Before that, we exclude the element 0 that has a one element cycle; after this, we are interested in proving that the permutation of $\{1, 2, \dots, p-1\}$ is even. Note that, by the argument in the previous paragraph, if we apply a composition of two permutations, their signs multiply (imagine that we just perform transpositions of the first permutation and then of the second).

Now, we need to represent the permutation $x \to x^3 \pmod p$ through permutations $p_1 : x \to 3x \pmod{p-1}$ and $p_2 : x \to a_x$ if $g^{a_x} \equiv x \pmod p$. We first apply p_2, then change the exponents by p_1, and then convert back by p_2^{-1}. So, $\pi = p_2^{-1} \circ p_1 \circ p_2$. Calculating sign, $\operatorname{sign}(\pi) = \operatorname{sign}(p_2^{-1}) \cdot \operatorname{sign}(p_1) \cdot \operatorname{sign}(p_2) = \operatorname{sign}(p_1)$, where the last follows because sign of inverse permutation coincides with the sign of the permutation itself.

Now, we only need to calculate the sign of an easier permutation of non-zero residues modulo p defined as $x \to 3x \pmod{p-1}$ (note that it's a permutation because each element has an image and every element is an image of some element because we can divide by 3 modulo $p-1$ because $(3, p-1) = 1$). Suppose that $p = 3k+2$, then the permutation can be written explicitly:

$$(3, 6, 9, \dots, 3k, 2, 5, \dots, 3k-1, 1, 4, \dots, 3k+1).$$

Now, we can explicitly calculate the number of inversions as

$$\frac{1}{2}(2k + (k+1) + 2 + (2k-2) + (k+1) + 4 + \dots + 2 + (k+1) + 2k + 0),$$

where, starting with 1 until $3k+1$, we just calculate the number of inverted pairs such that they contain this number (that's why we divide by 2). Each three elements sum up to $(3k+3)$. So, we get an answer $\frac{(3k+3)k}{2}$. We know that k is odd, so this number will be even if and only if $k \equiv 3 \pmod 4$, which corresponds to $p \equiv 3 \pmod 4$.

¶ Second solution (Analyzing cycles by calculating the orders) Each cycle has a form $x \to x^3 \to x^9 \to \ldots \to x^{3^{k-1}} \to x^{3^k} = x$. Note that, if $\mathrm{ord}_p(x) = d$, then $\mathrm{ord}_p(x^3) = d$ because if $d \mid 3m$ for $0 < m < d$, then $(d, 3) \neq 1$, so $3 \mid d \mid p-1$, which contradicts $p \equiv 2 \pmod 3$. Thus, every element in the cycle has the same order. The total number of elements in the cycle is $\mathrm{ord}_d(3)$, because $x^{3^k} = x$ is equivalent to $d \mid 3^k - 1$.

For each $d \mid p-1$, there exists exactly $\phi(d)$ elements of order d among non-zero residues. It's equivalent to finding $1 \le k \le p-1$ such that $p-1 \mid kd$ and $p-1 \nmid km$ for $0 < m < d$. From the first, we can say that $k = \frac{t(p-1)}{d}$. If $(t,d) > 1$, then $p-1 \mid k\frac{d}{(t,d)}$, where the last multiple is less than d. Thus, t has to be relatively prime to d and less than d, so there are $\phi(d)$ choices. Thus, the number of cycles with elements of order d modulo p is $\frac{\phi(p)}{\mathrm{ord}_d(3)}$. Summing over all divisors of $p-1$, we get that the total number of cycles (excluding the trivial $0 \to 0$) is

$$\sum_{d \mid p-1} \frac{\phi(d)}{\mathrm{ord}_d(3)}.$$

Let's find which terms might be odd. The idea is that $\phi(d) = \phi(a)\phi(b)$ for coprime a and b (*Euler's totient function* is multiplicative). The order can be calculated as the least common multiple of orders of it's coprime factors (because it has to be divisible by both of them). But $\phi(a)$ is divisible by order modulo a, and the same for b. Formally, if $d = ab$ with $(a,b) = 1$ and a, $b > 2$ (for strict inequality). Then

$$v_2(\phi(a)) = v_2(\phi(a)) + v_2(\phi(b)) > v_2(\phi(a)) \ge v_2(\mathrm{ord}_a(3)) \ge v_2(\mathrm{ord}_d(3)),$$

where $\phi(b)$ is even because we can pair t with $b-t$ and $(\frac{b}{2}, b) > 1$ for $b > 2$. And we chose a as the one with greater power of two dividing $\mathrm{ord}_a(3)$. Thus, for this d, the fraction is even. Therefore, we only care about $d = 1$, $d = 2^k$, $d = q^k$, $d = 2q^k$. $\phi(q^k) = \phi(2q^k)$ and $\mathrm{ord}_{q^k}(3) = \mathrm{ord}_{2q^k}(3)$, so these two will sum up to an even number. The only ones left are the powers of two. By *LTE Lemma*,

$$v_2(3^m - 1) = 2 + v_2(m).$$

Thus, $\mathrm{ord}_{2^k}(3) = 2^{k-2}$ for $k \ge 3$, so $\frac{\phi(2^k)}{\mathrm{ord}_{2^k}(3)} = 2$ for $k \ge 3$. For other powers of two, it's easy to check that it's one. Now, it's easy to check that for $4 \mid p-1$, there will be three powers of two that give an odd summand. And for $4 \nmid p-1$, there will be only two, implying the result.

¶ Third solution ($\mathbb{F}_{p^2}$ calculation) We will calculate the number of inversions in a clever way. Suppose that we are given a permutation σ of numbers from 0 to $p-1$, then we can calculate the number of inversions as

$$\prod_{0 \le i < j < p} \frac{\sigma(j) - \sigma(i)}{j - i} = (-1)^{\#\text{inverted pairs}}.$$

This is true because the multiple is -1 if the corresponding pair is inverted and 1 otherwise. We can apply this to π in the form:

$$\prod_{0\le i<j<p} \frac{j^3 - i^3}{j - i} \equiv (-1)^{\#\text{inverted pairs of } \pi} \pmod{p}.$$

This will be enough to establish the sign of the permutation because $-1 \not\equiv 1 \pmod{p}$. Thus, we want to calculate

$$\prod_{0\le i<j<p} (j^2 + ij + i^2) \pmod{p}.$$

It's possible to calculate it directly but we will expand the field of $\mathbb{F}_p$ by adding a third root of unity - ω (similarly to what we did in USAMO 2020/3). The resulting field is $\mathbb{F}_{p^2}$ and all further calculations are in this field. Now, we will transform the product:

$$\prod_{0\le i<j<p} (j - \omega i)(j - \omega^2 i) = \frac{1}{(-\omega)^{\frac{p(p-1)}{2}}} \prod_{0\le i<j<p} (j - \omega i)(i - \omega j) =$$

$$= \frac{1}{(-\omega)^{\frac{p(p-1)}{2}}} \prod_{0\le i,j<p,(i,j)\neq(0,0)} (i + \omega j) \prod_{0<i<p} \frac{1}{(i - \omega i)} = \frac{1}{(-\omega)^{\frac{p(p-1)}{2}} (1-\omega)^{p-1}(p-1)!} \prod_{0\le i,j<p,(i,j)\neq(0,0)} (i + \omega j).$$

The last product can be calculated using the fact that the product of all units (invertible elements) of every finite field is -1. Proof is the same as the proof to *Wilson's Theorem* (which is a special case of this fact by itself), where we pair every number with its inverse except 1 and -1. After applying Wilson's theorem in its standard form, the product finally equals to

$$\frac{(1-\omega)}{(-\omega)^{\frac{p(p-1)}{2}} (1-\omega)^p}$$

Now we will use the fact about $(a + b)^p = a^p + b^p$ (see the USAMO 2020/3) and $\omega^3 = 1$ (which allows to change the power to a residue modulo 3), and $\omega^2 + \omega + 1 = 0$:

$$\frac{(1-\omega)}{(-\omega)^{\frac{p(p-1)}{2}} (1-\omega)^p} = \frac{(1-\omega)}{(-\omega)^{\frac{p(p-1)}{2}} 1 - (\omega)^p} = \frac{(1-\omega)}{(-\omega)^{\frac{p(p-1)}{2}} (1-\omega^2)} = \frac{1}{(-\omega)^{\frac{p(p-1)}{2}} (1+\omega)}$$

$$= \frac{1}{(-1)^{\frac{p(p-1)}{2}} (\omega)(-\omega^2)} = (-1)^{\frac{p(p-1)}{2}+1}.$$

It's easy to check that the condition holds based on residue modulo 4.

§8.20 ISL 2018 N6

Problem 20 (ISL 2018 N6)
Let $f : \{1, 2, 3, \dots\} \to \{2, 3, \dots\}$ be a function such that $f(m + n)|f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

¶ Solution (Blend of NT functional equations techniques) First thing we see is that if some value $f(n)$ is greater than the values at points smaller than n, then $f(n) \mid f(n-k) + f(k)$ implies $f(n) = f(n-k)+f(k)$ (because both of the summands are smaller than $f(n)$). There will be infinitely many "peaks" if the sequence is unbounded (if peaks stopped appearing then the sequence is bounded by the latest of them). Consider any number k and a large enough peak n such that $n > k + 1$.

$$f(n-k) \mid f(n-k-1) + f(1)$$

and

$$f(n-k) = f(n)-f(k) = f(n)-f(k+1)+f(k+1)-f(k) = f(n-k-1)+f(k+1)-f(k).$$

Thus,

$$f(n-k) \mid f(n-k) - f(k+1) + f(k) + f(1) \Longrightarrow f(n-k) \mid f(k+1) - f(k) - f(1),$$

Right hand side is fixed, and we can find infinitely many peaks n for the left hand side. We only need to ensure that the right hand side can be made infinitely large, which will imply that the left hand side 0. It's easy because $f(n-k) = f(n) - f(k)$, and values at peaks are increasing. Thus, $f(k+1) = f(k) + f(1)$ and $f(k) = kf(1)$, which proves the problem as $f(1) > 1$.

What's left is the bounded case. Bounded sequence and divisibility condition prompts us to try to consider prime numbers that are dividing the terms because there will be a finite number of them. $f(a) \mid f(a-b) + f(b)$, so, if $p \mid f(a), f(b)$ with $a > b$, then $p \mid f(a-b)$. This condition allows us to perform Euclidean algorithm. In other words, if $p \mid f(a), f(b)$, then $p \mid f(\gcd(a, b))$. It's now useful to consider the smallest n_p such that $p \mid f(n_p)$. If $p \mid f(a)$, then $p \mid f(\gcd(n_p, a))$, so $\gcd(n_p, a) \geq n_p$, which is only possible if $n_p \mid a$. Thus, for each prime p that divides at least one value of the function, we can match it with this value of n_p such that $p \mid f(a)$ is only possible if $n_p \mid a$. Since each value of the function is > 1, a prime divisor exists for each value. Now, it's easy to obtain that some value of n_p must be 1, because otherwise $f(\prod n_p + 1)$ has no prime divisors because it's argument is not divisible by any of n_p [1]. So, $n_p = 1$ for some prime p. Now we would want to conclude that all the values are divisible by p, because if $p \mid f(a)$, then $f(a-1), f(a-2), \dots, f(1)$ are all divisible by p by the argument above. But there's a problem if there are only finitely many values that are divisible by p. This prompts us to modify the argument slightly and consider primes that divide infinitely many values and match n_p to them. If we succeed in proving that some $n_p = 1$, then we would be done.

[1] We could also just take any prime divisor of $f(1)$, oops

Now, we just use the same argument with numbers of the form $f(k\prod n_p + 1)$. There are infinitely many of them and none of them are divisible by the primes that divide infinitely many values. Thus, they only have primes that divide finitely many values. And there's a finite amount of these primes. So it must be that some number is not divisible by any of these primes, which means it has no primes whatsoever, but this is a clear contradiction. Hence, we are done.

§8.21 ELMO 2022/2, proposed by Jaedon Whyte, Luke Robitaille, and Pitchayut Saengrungkongka

Problem 21 (ELMO 2022/2)
Find all monic nonconstant polynomials P with integer coefficients for which there exist positive integers a and m such that for all positive integers $n \equiv a \pmod m$, $P(n)$ is nonzero and
$$2022 \cdot \frac{(n+1)^{n+1} - n^n}{P(n)}$$
is an integer.

¶ Solution (Schur's theorem and polynomials gcd calculation) Fix some a and m. We only consider values of the form $P(a + mn)$. If we take $Q(n) = P(a + mn)$, then by *Schur's theorem*, there exists infinitely many primes p such that $p \mid Q(n)$ for some n. Take large enough such p, then
$$p \mid Q(n) = P(a + mn) \mid 2022((a + mn + 1)^{a+mn+1} - (a + mn)^{a+mn}),$$
and if we take $p > 2022$, then
$$p \mid (a + mn + 1)^{a+mn+1} - (a + mn)^{a+mn}. \qquad (\heartsuit)$$
Standard trick to get another divisibility is to consider $Q(n+p)$, which gives (after the same manipulations and canceling terms divisible by p)
$$p \mid (a + mn + 1)^{a+mn+1+mp} - (a + mn)^{a+mn+mp}.$$
Comparing this to the divisibility above and using Fermat's Little Theorem, we get
$$(a + mn + 1)^{mp} \equiv (a + mn)^{mp} \pmod p \implies (a + mn + 1)^m \equiv (a + mn)^m \pmod p$$

Substituting this equivalence back into $\heartsuit$ and using the equivalence of powers of m, we get
$$(a + mn + 1)^{a+1} \equiv (a + mn)^a \pmod p \implies (a + mn + 1)^{ma+m} \equiv (a + mn)^{ma},$$
so
$$(a + mn + 1)^m \equiv 1 \pmod p,$$
but before this, we needed to ensure that $p \nmid a + mn$, which is easy. If $p \mid a + mn$, then $p \mid P(a + mn)$, which means that p divides the free coefficient, but we could make p large enough. The only case left is when the free coefficient is 0, but in this case, $n \mid P(n)$, so take $n \equiv a \pmod m$ with $n > 2022$, then $n \mid 2022((n+1)^{n+1} - n^n)$, but $(n, (n+1)^{n+1} - n^n) = 1$, so $n \mid 2022$, for which we set n to be too large.

Thus, we see that there exists an n such that $(a + mn + 1)^m - 1$ and $(a + mn)^m - 1$ are both divisible by p for infinitely many values of p such that $p \mid P(a + mn)$. This prompts to calculate possible values of $\gcd((x + 1)^m - 1, x^m - 1)$ as polynomials and then compare it with $P(x)$. Suppose that ω is the complex root of both of these polynomials, then $(\omega+1)^m = 1 \Rightarrow |\omega+1|^m = 1 \Rightarrow |\omega+1| = 1$, and in the same way, $|\omega| = 1$. Write $\omega = e + fi$, then $e^2+f^2 = 1$ and $(e+1)^2+f^2 = 1$. $(e+1)^2 = e^2 \Rightarrow e = \frac{-1}{2}$ and $f = \pm\frac{\sqrt{3}}{2}$. Thus, the only roots these two polynomials might share is $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}$. We also know that $x^m - 1$ has no double roots because the double root would also be a root of the derivative, which is mx^{m-1}, but 0 is not a root of the original. If they share a root, then they are both divisible by the minimal polynomial of this root. For both of these roots, the minimal polynomial is x^2+x+1. If they don't have any common roots, then the gcd is constant. So the gcd is either $c(x^2+x+1)$ or just c, where c is a constant.

Now, we will set up a standard *Bezout's identity* argument. It grants the possibility of finding integer constants A, B, C, D such that $A(x^m-1)+B((x+1)^m-1)+CP(x) = D\gcd(x^m-1,(x+1)^m-1,P(x))$ (It's easy go from two polynomials to three). We know that there are infinitely many values of p such that the three of these polynomials are all divisible by p for some value $x = a+mn$. Thus, the gcd is non-constant, which means that $x^2+x+1 \mid P(x)$, as x^2+x+1 is the only non-constant value of the gcd of the first two polynomials. We could take P as any irreducible factor of the original P. Thus, any irreducible factor is of the form x^2+x+1. So $P(x) = (x^2+x+1)^n$ (no constant because it's monic).

Now, we need to find a highest power of n. First of all, we need to have x^m-1 divisible by x^2+x+1, which is only possible if $3 \mid m$ (because if $x^2+x+1 \mid x^m-1$, then $x^2+x+1 \mid x^{m-3k}-1$ and we will arive at 0 or $x-1$, or x^2-1). Suppose that m is odd, then $x^2+x+1 \mid (x+1)^m-1$ is equivalent to $x^2+x+1 \mid (-x^2)^m - 1 = -x^{2m}-1$, but $x^2+x+1 \mid -x^{2m}+1$ because $3 \mid m$, which cannot both hold. Thus, $6 \mid m$.

Suppose that $c = x^2+x+1$, then $c \mid 2022((x+1)^{x+1}-x^x)$. Suppose that $2 \mid x$, then $c \mid 2022((-x^2)^{x+1}-x^x) \Rightarrow c \mid 2022(-x^{2(x+1)}-x^x) = -2022x^x(x^{x+2}+1)$. $(c, x^x) = 1$, so $c \mid 2022(x^{x+2}+1)$. If $x+2 \equiv 1 \pmod 3$, then $c \mid 2022(x+1)$, which is not true for big enough x. If $x+2 \equiv 2 \pmod 3$, then $c \mid 2022(x^2+1) \Rightarrow c \mid -2022x$, which is not true for big enough x again. If $x+2 \equiv 0 \pmod 3$, then $c \mid 4044$, so no solutions either. If $2 \nmid x$, then doing the same manipulations, we only get a possible solution for $x \equiv 1 \pmod 3$. So $a \equiv 1 \pmod 6$.

Now, $x^2+x+1 \equiv 3 \equiv 0 \pmod 3$, so if $n \geq 3$, then $27 | 2022(x^{x+1}-x^x)$, so $9 \mid (x+1)^{x+1}-x^x$. x and $x+1$ are both relatively prime to 3, so $x^6 \equiv 1 \pmod 9$ as well as $(x+1)^6$, so $0 \equiv (x+1)^{x+1}-x^x \equiv (x+1)^2 - x = x^2+x+1 = 36k^2+18k+3 = 3 \pmod 9$ if $x = 6k+1$. This final contradiction proves $n \leq 2$.

Now, we will prove that $n = 2$ works. Again, $c = x^2+x+1$ and let $m = 6$ and $a = 1$ because the condition almost forces us. The idea is to try to use the variable c and binomial expansion to cancel terms with power of c greater than 1.

$$(x+1)^{x+1} - x^x = (x^2+2x+1)^{\frac{x+1}{2}} - x^x \equiv (c+x)^{\frac{x+1}{2}} - x^x \equiv \frac{x+1}{2}cx^{\frac{x-1}{2}} + x^{\frac{x+1}{2}} - x^x \pmod{c^2},$$

where the last equivalence follows from binomial expansion. Let's divide through $x^{\frac{x-1}{2}}$ because it's coprime to c^2. We need to show that

$$\frac{x+1}{2}c + x - x^{\frac{x+1}{2}} \equiv 0 \pmod{c^2}.$$

It's obvious that it's divisible by c, because one term is evidently divisible by it, and $x(1 - x^{\frac{x-1}{2}})$ is divisible because $3 \mid \frac{x-1}{2}$. But we need to raise the power to second. Again, we will be expressing through c^2 and using the binomial expansion to deal with $x^{\frac{x+1}{2}}$ term.

$$x^{\frac{x-1}{2}} = (x^3)^{\frac{x-1}{6}} = (1 + c(x-1))^{\frac{x-1}{6}} \equiv 1 + c\frac{(x-1)^2}{6} \pmod{c^2}.$$

Now, we only need to simplify.

$$\frac{x+1}{2}c + x - x^{\frac{x+1}{2}} \equiv \frac{x+1}{2}c - xc\frac{(x-1)^2}{6} \equiv c\frac{3x + 3 + 3x^2}{6} = \frac{c^2}{2} \equiv 0 \pmod{c^2},$$

where the last is true because $(2, x^2 + x + 1) = 1$. Hence, we are done.

§8.22 Problem 22

Problem 22

For an odd prime p, prove that

$$\sum_{i=1}^{p-1} 2^i \cdot i^{p-2} \equiv \sum_{i=1}^{\frac{p-1}{2}} i^{p-2} \pmod p.$$

¶ Solution (Representing a power of two as sum of polynomials) When we are given a sum over numbers from 0 to $p-1$ (1 to $p-1$ doesn't differ much), the best thing to have is polynomials because

Lemma

$$\sum_{i=0}^{p-1} f(i) \equiv 0 \pmod p$$

for prime p and polynomial f with degree at most $p-2$.

Proof. This fact follows from

$$1^d + \ldots + (p-1)^d \equiv 0 \pmod p$$

for $1 \leq d \leq p-2$.

This can be proven if we introduce g as a primitive root modulo p, then $\{1, 2, \ldots, p-1\} = \{g^1, g^2, \ldots, g^{p-1}.\}$ Thus,

$$1^d + \ldots + (p-1)^d \equiv g^d + g^{2d} + \ldots + g^{(p-1)d} = g^d \frac{g^{(p-1)d} - 1}{g^d - 1} \equiv 0 \pmod p.$$

Denominator of the last fraction is not zero because $0 < d < p-1$. □

But how to use polynomials in this problem if we have a power of two? It turns out that a power of two can be represented as a sum of polynomials. We remember the basic binomial coefficeints identity (follows from expanding $(1+1)^i$):

$$2^i = \binom{i}{0} + \ldots + \binom{i}{i} = 1 + i + \frac{i(i-1)}{2} + \frac{i(i-1)(i-2)}{3!} + \ldots + \frac{i(i-1)\ldots(i-(i-1))}{i!}.$$

You might say that the fourth term won't exist in the expansion if, for example, $i = 2$. But in this case, it's just zero, so we can still inculde it. In the end, we get a magical identity:

$$2^i = 1 + i + \frac{i(i-1)}{2} + \frac{i(i-1)(i-2)}{3!} + \ldots + \frac{i(i-1)\ldots(i-(p-2))}{(p-1)!}.$$

We could include further terms too, but they are trivially zero for given constraints. Now, we will rewrite the sum using the fact about polynomials established before:

$$\sum_{i=1}^{p-1} 2^i \cdot i^{p-2} \equiv \sum_{i=1}^{p-1} \frac{2^i}{i} = \sum_{i=1}^{p-1} \frac{1}{i} + 1 + \frac{i-1}{2} + \frac{(i-1)(i-2)}{3!} + \ldots + \frac{(i-1)\ldots(i-(p-2))}{(p-1)!} \equiv$$

$$\equiv \sum_{i=1}^{p-1} \frac{1}{i} - 1 - \frac{0-1}{2} - \frac{(0-1)(0-2)}{3!} - \ldots - \frac{(0-1)(0-2)\ldots(0-(p-2))}{i} =$$

$$= \sum_{i=1}^{\frac{p-1}{2}} \frac{2}{2i} = \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i} \equiv i^{p-2} \pmod{p}.$$

§8.23 USEMO 2022/5, proposed by Jovan Vuković

Problem 23 (USEMO 2022/5)
Let $\tau(n)$ denote the number of positive integer divisors of a positive integer n (for example, $\tau(2022) = 8$). Given a polynomial $P(X)$ with integer coefficients, we define a sequence $a_1, a_2, \ldots$ of nonnegative integers by setting

$$a_n = \begin{cases} \gcd(P(n), \tau(P(n))) & \text{if } P(n) > 0 \\ 0 & \text{if } P(n) \le 0 \end{cases}$$

for each positive integer n. We then say the sequence has limit infinity if every integer occurs in this sequence only finitely many times (possibly not at all).

Does there exist a choice of $P(X)$ for which the sequence $a_1,\ a_2,\ .\ .\ .$ has limit infinity?

¶ Solution (Ensuring small exponents for small primes and bounding the value of polynomial with prime numbers density analysis) Clearly, P is non-constant and has a positive leading coefficient (otherwise $a_n = 0$ for all large enough n). Our plan is to find an integer n such that $P(n)$ is divisible by primes up to some value m in small exponents, so, if a_n is large, then there has to be some prime p greater than m such that p divides a_n. Thus, it will divide $\tau(P(n))$, but this allows to create a big exponent, which also must correspond to a big prime, from this we will be able to get a contradiction with polynomial growth.

Fix some $c = P(k) > 0$ such that the polynomial is increasing on $[k; +\infty)$ and $P(k) > \frac{k}{2}$ (because the leading coefficient is ≥ 1). Choose large enough m. We will find n such that $v_p(P(n)) = v_p(c)$ for all primes $p \le m$. This is easily done by setting $n \equiv k \pmod{p^{v_p(c)+1}}$ for all p, which is possible by adding a product of all these exponents of primes to k with any coefficient. If a_n is not divisible by primes $> m$, then $a_n \mid P(n)$, which means that $v_p(P(n)) = v_p(c)$ for all p dividing a_n. Thus, $a_n \le c$. Suppose that there exists a prime $q > m$ that divides a_n, then some exponent of $P(n)$ increased by one is divisible by q, which means that this exponent is $\ge q - 1$. $q - 1 \ge m$, and we can make m greater than maximum exponent in prime decomposition of c. Thus, this exponent, that is at least $q - 1$, corresponds to the prime r that is greater m. In total, $r^{q-1} \mid P(n)$, so $P(n) > m^m$, which looks suspicious.

We need to bound n first. The smallest $n > k$ is

$$n = k + \prod_{p \le m} p^{v_p(c)+1} = k + c\prod_{p \le m} p < 2c + c\prod_{p \le m} p \le 2c\prod_{p \le m} p$$

works. Now the problem boils down to standard analytic NT asymptotic argument.

$P(n)$ of degree d is $O(n^d)$, so $m^m < P(n) < Cn^d < C \cdot 2^d c^d \prod_{p \le m} p^d < C' m^{d\pi(m)}$, where C' is some constant and $\pi(m)$ is the *Prime-Counting Function*. It's known that $\pi(x) \approx \frac{x}{\log(n)}$, from which it's trivial to conclude that $m^{m - d\pi(m)}$ can be greater than any constant for large enough m. This fact is called the *Prime Number Theorem* and the proof is quite involved. For the sake of completeness, below is the proof of a weaker statement that is enough to conclude the problem.

We will prove that $\frac{\pi(x)}{x} < D$ for any constant D and large enough x, i.e. density of prime numbers is not bounded below. Consider all primes $\leq N$: $p_1, p_2, \ldots, p_k$. There exists $\lfloor \frac{x}{p_1} \rfloor - 1$ numbers $\leq x$ that are not prime (because they are divisible by p_1 and greater than it). In a similar fashion, one can calculate how many numbers $\leq x$ that are divisible by $p_1 p_2$ - $\lfloor \frac{x}{p_1} \rfloor - 1$. By exclusion-inclusion. There are at least

$$\sum_{i=1}^{k} (\lfloor \frac{x}{p_i} \rfloor - 1) - \sum_{1 \leq i < j = k} \lfloor \frac{x}{p_i p_j} \rfloor + \ldots$$

numbers that are not prime and $\leq x$. There is only finite amount of terms in the above sum, so we can clear floor signs and add some constant (just take away 1 for terms that have a positive sign and don't do anything for negative terms). The sum is

$$x(1 - (1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k})) + C,$$

where C is some integer constant.

$\pi(x)$, thus, is at most $x(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k}) - C$.

$$\frac{x}{\pi(x)} \geq \frac{1}{(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k}) - \frac{C}{x}},$$

which is essentially $\frac{1}{(1-\frac{1}{p_1})(1-\frac{1}{p_2})\ldots(1-\frac{1}{p_k})}$ for large enough x. We are interested to prove that the latter can be made arbitrarily large. Standard trick here is that $\frac{1}{1-\frac{1}{p_i}} = 1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \frac{1}{p_i^3} + \ldots$ by the sum of infinite geometric series. Thus,

$$\frac{1}{(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_k})} = \prod_{i=1}^{k} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \ldots)$$

If we expand the latter, then we will have all the numbers whose prime decomposition only contains primes $p_1, p_2, \ldots, p_k$. Lastly, note that numbers $\leq N$ satisfy this, so

$$\prod_{i=1}^{k} (1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \ldots) \geq 1 + \frac{1}{2} + \ldots + \frac{1}{N},$$

and it's known that the harmonic series diverges. Which proves that $\frac{\pi(x)}{x}$ is not bounded below.

This finishes the problem because we proved that for all large enough m, we can take

$$n = k + c \prod_{p \leq m} p,$$

for which $a_n \leq c$. And n can be unbounded.

§8.24 USAMO 2018/3, proposed by Ivan Borsenco

Problem 24 (USAMO 2018/3)
For a given integer $n \geq 2$, let $\{a_1, a_2, \ldots, a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k + a_2^k + \ldots + a_m^k$ is divisible by m for every positive integer k.

¶ Solution (Adding primes one by one and spamming binomial expansion) Call n that satisfy the condition *good*. We first understand what might be the primes dividing m: $m = \phi(n) = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \ldots p_k^{\alpha_k - 1}(p_1 - 1)(p_2 - 1)\ldots(p_k - 1)$, where $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$. Condition on n is weird. First thing we can note about it is that we can only consider the square-free part of n for it to be true. Suppose that the smallest prime divisor of n is $p \geq 3$, then $p - 1 \mid \phi(n)$. $p - 1$ has a prime factor $q < p$, so $q \mid p - 1 \mid \phi(n)$, which means that $q \mid n$ by the condition. But q is less than the smallest divisor of n. Contradiction. Thus, we see that $2 \mid n$, and 2 is good. Consider the second smallest prime $p \mid n$. Suppose that $2p$ is not good, then $\phi(2p) = p - 1$ and there exists $q \mid p - 1$ such that $q \nmid 2p$, so $2 < q < p$, but $q \mid p - 1 \mid \phi(n)$, so $q \mid n$, but we assumed that n has no prime factors in between 2 and p. So $2p$ is also good. In the same way it's possible to prove that the product of first t primes in the expansion of n is also good. This prompts us to add primes one by one and prove the result for the resulting product before we arrive at the square free part of n. Then, we will add the needed exponents.

Note that the problem holds for $n = 2$, which is the only good prime number and the base case for the induction. Suppose that n is good and the divisibility holds. Consider np such that $p \nmid n$ and np is good. It turns out that it's possible to characterize all numbers less than np and coprime to np through numbers less than n and coprime to n. For brevity, suppose that the first set is $\{b_1, b_2, \ldots, b_{(p-1)m}\}$ and the second is $\{a_1, a_2, \ldots, a_m\}$. Note that each b_i is equivalent to some $a_j \pmod{n}$. Thus, we just consider the union of the sets $\{a_i, a_i + n, \ldots, a_i + n(p-1)\}$ for $1 \leq i \leq m$. But each of these sets contains a number divisible by p (because all the residues modulo p are different). These numbers have a form ps, where $s < n$ and $(s, n) = 1$, from which it follows that we have to exclude exactly the set $\{pa_1, pa_2 \ldots, pa_m\}$. Thus,

$$b_1^k + b_2^k + \ldots + b_{(p-1)m}^k = \sum_{i=1}^{m} \sum_{j=0}^{p-1} (a_i + jn)^k - p^k \sum_{i=1}^{m} a_i^k.$$

Upon expanding the brackets, we will have some sums of a in some powers multiplied by some coefficients. We know, by inductive assumption, that all of these sums of a are divisible by m. It makes sense to denote them as $s_d = a_1^d + a_2^d + \ldots a_m^d$. Now, we can rewrite the sum in a more convenient way:

$$S = \sum_{i=1}^{m} \sum_{j=0}^{p-1} (a_i + jn)^k - p^k s_k = \sum_{i=1}^{m} \sum_{j=1}^{p-1} (a_i + jn)^k - (p^k - 1)s_k =$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{p-1} (a_i + jn)^k - (p^k - 1)s_k = \sum_{i=1}^{m} \sum_{j=1}^{p-1} \left(a_i^k + \binom{k}{1} a_i^{k-1} jn + \cdots + \binom{k}{k} (jn)^k \right) - (p^k - 1)s_k =$$

$$= \sum_{j=1}^{p-1} \left(s_k + \binom{k}{1} s_{k-1} jn + \cdots + \binom{k}{k} s_0 (jn)^k \right) - (p^k - 1) s_k.$$

We need to prove that it's divisible by $(p-1)m$. Trivially, since $m \mid s_i$ for every i, the sum is divisible by m. If $q \nmid p-1$ and $q \mid m$, then $v_q(S) \geq v_q(m) = v_q(m(p-1))$. If $q \mid p-1$, then $q \mid np$ because np is good, but $q \nmid p$, so $q \mid n$. $(p^k - 1)s_k$ term is divisible by $(p-1)m$, so we only have to take care of the terms of the form

$$\binom{k}{r} s_{k-r} (1^r + \cdots + (p-1)^r) n^r.$$

Here we will use a result about sums of first n integers raised to kth powers. It's known (*Faulhaber's formula*) that

$$S(n,p) = \sum_{i=1}^{n} i^p$$

is a polynomial of degree $p+1$ such that if we multiply it by $(p+1)!$, then all of its coefficients are integers and it has no free coefficient. Faulhaber's formula is an overkill here. This result can be proven by noting that

$$(m+1)^{p+1} - 1 = \sum_{i=1}^{m} ((i+1)^{p+1} - i^{p+1}) = \sum_{t=0}^{p} \binom{p+1}{t} S(m,t),$$

from which we can express $S(m,p)$ through $S(m,d)$ with $0 \leq d \leq p-1$ and prove the needed conditions.

In our original problem, this fact allows us to prove that $v_q((1^r + \cdots + (p-1)^r)n^r) \geq v_q(p-1) - v_q((r+1)!) + r$ because the rth powers sum is essentially $\frac{p-1}{(r+1)!}$ times some integer and $q \mid n$. By *Legendre's formula*,

$$v_q((r+1)!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{q^i} \right\rfloor < \frac{r+1}{q} + \frac{r+1}{q^2} + \ldots = \frac{1}{q} \cdot \frac{(r+1)q}{q-1} = \frac{r+1}{q-1} \leq r+1,$$

from which it follows $v_q((r+1)!) \leq r$, so $v_q \binom{k}{r} s_{k-r} ((1^r + \cdots + (p-1)^r) n^r) \geq v_q(m) + v_q(p-1)$ because $m \mid s_{k-r}$. This is exactly what we wanted.

Now we need to expand the result to arbitrary exponents. Suppose that n is good and the divisibility holds. Consider now np with $p \mid n$. Inheriting the notation from the previous part, $\{b_1, b_2, \ldots, b_{pm}\}$ is just a union of $\{a_i, a_i + n, \ldots, a_i + n(p-1)\}$ for $1 \leq i \leq m$ because being relatively prime with n is the same as being relatively prime with np. Now we rewrite the sum in the same way as in the previous part using s_d:

$$S = \sum_{i=1}^{m} \sum_{j=0}^{p-1} (a_i + jn)^k = \sum_{j=0}^{p-1} \left(s_k + \binom{k}{1} s_{k-1} jn + \cdots + \binom{k}{k} s_0 (jn)^k \right).$$

s_k terms will sum up to ps_k, which is divisible by pm. Other terms have the form of

$$\binom{k}{r} s_{k-r} (1^r + \cdots + (p-1)^r) n^r,$$

which is divisible by m because of s_{k-r} and n gives at least one more factor of p. Thus, $pm \mid S$. And the induction is finished.

§8.25 All-Russian MO grade 11 2024/8, proposed by Maksim Turevskii and Ilya Bogdanov

Problem 25 (All-Russian MO grade 11 2024/8)
Prove that there exists $c > 0$ such that for any odd prime $p = 2k + 1$, the numbers $1^0, 2^1, 3^2, \ldots, k^{k-1}$ give at least $c\sqrt{p}$ distinct residues modulo p.

¶ Solution (Introducing new function using the $f(x) = x^{x-1}$ that doesn't repeat values too often) As it frequently happens in problems where we need to prove the $\sqrt{n}$ asymptotic bound, we need to calculate the number of pairs and prove the linear asymptotic bound. In this problem it's motivated because, if we try to experiment, we wouldn't be able to find anything reasonable related to x^{x-1}; its residues can repeat a lot even under restrictions. Thus, we want to create a new function using x^{x-1} that is easier to work with. Functions that are easier to work with are exponents of the form c^x or polynomials. After some tries, it's possible to arrive at $\frac{f(2x)}{f(x)^2} = 2^{2x-1}x$. If $f(x)$ only gives d residues for $1 \le x \le \frac{p-1}{2}$, then $2^{2x-1}x$ is determined by the pair of residues of $f(2x)$ and $f(x)$ modulo p ($f(x)$ is non-zero, so we can divide through it freely), so there are at most d^2 values for residues of this new function.

Now, we can't say that the new function is injective modulo p. But we can say that for each $2^{2a-1}a \equiv 2^{2b-1}b \pmod{p}$, $|a-b| \ne 0$ is unique, which will allow to make a good bound. The proof is easy, suppose that $a > b$ and $c > d$ give the same residue and $a - b = c - d = r \ne 0$, then $2^{2r} = 2^{2(a-b)} \equiv \frac{b}{a} = \frac{a-r}{a} \pmod{p}$ and $2^{2r} = 2^{2(c-d)} \equiv \frac{d}{c} = \frac{c-r}{c} \pmod{p}$. Easy to see that $a \equiv \frac{r}{1-2^{2r}} \equiv c \pmod{p}$ ($1 - 2^{2r}$ is not zero because both a and r are non-zero). This gives a contradiction, so for each pair that have the same residue, we can match them with a unique difference.

Now, we will do some double counting based on the result we got. Suppose that we have k residues of the function $2^{2x-1}x$ for $1 \le x \le \lfloor\frac{p-1}{4}\rfloor$ (4 because we needed $2x \le \frac{p-1}{2}$). Suppose that, for ith residue, there are a_i points that leave it. $a_1 + a_2 + \ldots + a_k = \lfloor\frac{p-1}{4}\rfloor$. Now, for each two points that leave the same residue, we calculate the difference between the smallest and largest. The resulting value is in between 1 and $\lfloor\frac{p-1}{4}\rfloor - 1$, and we know that all of them are different by the argument above.

Thus,

$$\frac{a_1^2 + \ldots + a_k^2 - \lfloor\frac{p-1}{4}\rfloor}{2} = \frac{a_1^2 + \ldots + a_k^2 - (a_1 + \ldots + a_k)}{2} = \binom{a_1}{2} + \ldots + \binom{a_k}{2} \le \lfloor\frac{p-1}{4}\rfloor - 1.$$

From this, we get $a_1^2 + \ldots + a_k^2 \le 3\lfloor\frac{p-1}{4}\rfloor - 2$. And, knowing the sum of values, we can bound the value of k. By Cauchy-Schwartz inequality, $(3\lfloor\frac{p-1}{4}\rfloor - 2)k \ge k(a_1^2 + \ldots + a_k^2) \ge (a_1 + \ldots + a_k)^2 = \lfloor\frac{p-1}{4}\rfloor^2$. Clearly we have a linear bound. Something like $\frac{3p}{4}k \le (3\lfloor\frac{p-1}{4}\rfloor - 2)k \ge \lfloor\frac{p-1}{4}\rfloor^2 \ge \frac{p^2}{64} \Rightarrow k \ge \frac{p}{48}$. We know that $k \le d^2$, so $d \ge \frac{1}{\sqrt{48}}\sqrt{p}$. Exactly what we wanted to show.

Remark. The same idea with treating $\sqrt{n}$ was used in IZHO 2024/3.

Part V. Hints

1. RHS of the inequality is the number of edges e in the graph. LHS calculates the number of paths with three edges (some edges might repeat).
2. Combine them and decrease the number of conflicting pairs, i.e. the ones in the same column or row (they must be of different colours).
3. The fourth line has to be symmetrically defined with respect to AB, BC and AC and somehow take P into account.
4. Need to take care of small degrees.
5. Write Q as the sum of Q_i where each Q_i is composed of monomials $x^a y^b z^c$ such that $4a - 2b + c \equiv i \pmod 9$. Do the same thing with P. Note that a monomial from Q_i multiplied by a monomial from P_j leaves the same remainder $\pmod 9$ as $i + j$. Now, write explicitly the equation with Q_i as variables that only leaves those monomials that are divisible by 9.
6. Translate the grid so that B is the origin and C has coordinates (m, n), where both of them are integers. Consider first the case when m and n are > 0. A way of constructing the desired point that is close to BC is to consider the next fraction to $\frac{n}{m}$ in the Farey sequence of order m (we can suppose that $m > n$ by reflecting the axes if needed).
7. It's coefficient before the x^i term is #(even representations of i) - # (odd representations of i).
8. Because of equal segments, there are many cyclic quadrilaterals.
9. Consider two circles ω_1 and ω_2 with centers O_1 and O_2 and a point M for which the sum is equal to the length of the tangent. The condition for M to lie on it is that one of MO_1 or MO_2 is the external tangent of the angle between the tangents from M.
10. $k + s_t$ has a unique odd representation without s_t.
11. Inversion keeps tangency.
12. Use injectivity in the form of finding $a + f(ay) = b + f(by)$ for $a \neq b$ and finding the contradiction.
13. In the latter we can swap variables and nothing changes.
14. They are coaxial.
15. This and former are actually sufficient.
16. Consider the Steiner line of P with respect to ΔABC.
17. Make a connection with Fibonacci numbers.
18. Find cyclic quadrilaterals.
19. The answer to the last question is $k + 2$.
20. Given a triangle and k points inside of it. How many triangles can you make in its triangulation that contain a specific vertex of it?

21. Choose such sum of digits so you could just add any digits to satisfy the digit sum condition without worrying about the divisibility condition.
22. Extend each figure so that they still don't intersect but the area increases in two times.
23. Express t in terms of x,y,z and rewrite the condition in this form.
24. There are more angle equalities involving D.
25. After the inversion, median goes into symmedian.
26. $(ASIDT)$
27. If there's no situation as described in the last hint, then consider how many different sides of rectangles are contained on every line passing through a side that contains a point.
28. Calculate possible values of $\gcd(x^m - 1, (x+1)^m - 1)$ as polynomials.
29. Exponents and polynomials.
30. Combine divisibilities using Chinese Remainder Theorem.
31. Choosing leaders of the brigade is the same as choosing one integer from $\{i, i+1, \ldots, j\}$ for every $17 \geq j > i \geq 1$.
32. Choose one cycle such that we could modify it to find every possible residue modulo p.
33. It's known that we can draw a hyperbola through all the points in the picture.
34. Extend the common tangent to intersect ω at Q and R. We want to show that $QD = RC$ or $QC = DR$.
35. Consider a directed graph in which x is connected to $x+a$ and $x+b$. At each step, imagine that there are exactly as many chips in the vertex x as the times it's written on the board. And when we choose x, two chips are fired in $x+a$ and $x+b$.
36. Ensure some condition on the copypaste such that there would be $2 \cdot 5^{17}$ unique words.
37. Induction.
38. We go rightwards from every $(k, n+1-k)$ for $k \leq m-1$ and go upwards for every $k \geq m+1$.
39. If $p \mid f(a)$ and $f(b)$ with $a > b$, then $p \mid f(a-b)$.
40. Pair up q^k and $2q^k$.
41. Conclude that d is a power of two.
42. The rest is just manipulating the root of unity. You should get $(-1)^{\frac{p(p-1)}{2}+1}$ in the end.
43. $P(n) \leq n^d$, where d is the degree of P.
44. Look at arguments that are not divisible by any of the numbers associated with these primes and obtain a contradiction.
45. How to get the sum with even indexed binomial coefficients?
46. To get this form in the original problem, apply rearrangement.
47. Find some minimal quantity.
48. Regular bipartite graphs can be split into perfect matchings.

49. By ugly case bash, where we have to reflect the axes so that m and n would be positive and keep track of axis-alignments, $\max(f(m(BP)), f(m(CP))) > f(m(BC))$ in the above scenario. Contradiction follows from taking the side with the biggest value.
50. Prove that $\deg P \geq n$.
51. Start rotating the line from A_1A_2 to A_2A_3, then to A_3A_4, ..., and back to A_1A_2. Take the negative sign, when the rotation is anticlockwise and positive otherwise.
52. For $\hat{f}(n) \neq 0$, $e^{\frac{-2\pi i|y||x|n}{\ell}} = -1$.
53. Set $y = 0$.
54. We either hit a point at which all three mentioned angles are equal or the maximum is less than $\angle MCA = \angle MBC$.
55. It's easy to control inversions involving largest and smallest number.
56. Prove injectivity.
57. For any two irreducible polynomials, they cannot be both divisible by large primes.
58. Write $f(x) - x = g(x)$.
59. Moving points.
60. Each large enough integer has an even representation.
61. Add a bridge (connects two components) of colour c to this graph and delete some edge of colour c from the graph so that no connected component changes. This operation decreases the number of connected components.
62. Note that if we just move by 1 to the left each time after Alice chose right for k cut, then we either have a cancellation at some point or we have subtracted from left for k cut, $k - 1$ cut, ..., 1 cut. This gives a bound.
63. Only work with x and y.
64. By Dual of Desargues Involution Theorem, there's an involution that swaps $(P; Q)$, $(A; B)$ and $((ZO \cap AB); M)$, where M is the midpoint of AB. Same for AC.
65. Think about $x^3 + y^3 + z^3 - 3xyz$.
66. What happens if $f(n)$ is greater than all $f(k)$ with $k < n$.
67. Finish with Pascal's.
68. Grids with some cells selected are bipartite graphs. Rook sets are perfect matchings.
69. Think about the roots of the polynomial and how they are transformed.
70. If one of the regions is infinite, then conclude.
71. Consider 8 letter portions before (anticlockwise) each copypaste.
72. Four Numbers Lemma.
73. Kőnig's theorem gives one matching with a lot of amber cells and one with a lot of bronze cells.
74. If the last thing is a quadratic residue, then either both z and $4 - z$ are quadratic residues or both are non-residues.
75. For each Q calculate how many P are divisible by Q^2.

76. PQ and line through I and midpoint of arc BAC intersect on (BIC), call it T.
77. Introduce points 1 and -1. Deal with the case when $x_i = 1$ or -1 for some i.
78. There's an even number of non-yellow points in each region.
79. g satisfies Cauchy's relation and bounded from below. Hence, linear.
80. $f(n) - f(k) \mid f(k+1) - f(k) - f(1)$ for the number n specified in the first hint.
81. Consider a graph in which vertices are circles and two vertices are connected if they intersect in two yellow points.
82. Suppose that $p \mid a_1$ and prove that every term of the sequence is divisible by p.
83. $\frac{1}{2^{\frac{c}{2}}} + \frac{1}{3^{\frac{c}{2}}} + \dots$ is convergent because $c > 2$.
84. How to deal with the repeating k?
85. If $f(y_1) < f(y_2)$ for some $y_1 > y_2$, then $y = \frac{y_1 - y_2}{f(y_2) - f(y_1)}$, $a = y_1$, $b = y_2$ works. From this, conclude that the function is increasing.
86. Express $Q(0, x_2, \dots, x_n)$ that has degree $n-1$. In your expression RHS will consist of polynomials with degree at most $n-2$ times constants if we assumed that $\deg Q \le n-1$. Conclude from here.
87. The first case is a well known property of the Brocard point.
88. Bound the number of corners.
89. Rewrite using $e^{i\theta} = cos\theta + isin\theta$.
90. Let $c = x^2 + x + 1$.
91. Since all special primes are different, we can combine divisibilities by Chinese Remainder Theorem. Take care of the exponent equal to one in each step.
92. Think about marking every point for which the perpendicular distance from them to this figure is at most the distance from them to any other point that can still be marked without violating the three cells rule.
93. To get this form, we multiply each equation by a large power of z (to not have troubles with negative powers) and change $P_i z^k$ and $Q_i z^s$ to P_i' and Q_i' such that their monomials are divisible by 9.
94. Associate each vector with v with a pair (u, w) such that u is the original vector (before changing some entries to zero) and w is a vector that makes v to be the arithmetic mean of u and w.
95. First five hints are the same as in the previous solution.
96. Think about rectangular circumhyperbola.
97. Rewrite $PD = DQ$ using the Law of Sines and cyclic quad.
98. We bounded the sum to the left assuming that Alice first chose right. Now, we need a cut for which Alice chose left to bound the sum to the right.
99. You should find infinitely many composite a such that $3a \mid 2^{a-1} - 1$.
100. Assuming that $a \ne b$, find a cycle in this graph.
101. $BP \parallel AQ$, $CP \parallel RA$ etc. Calculate ratios.

102. The problem now is choosing 2^{c_i} such that $\sum \frac{2^{c_i}}{(d_i+1)^c}$ is convergent and $\frac{1}{2^{c_i}} \leq 1$.
103. Now, write an equation that connects the new arguments and change the function again.
104. $g(x) = 1 - \cos x$ and $p = \frac{3}{2}$ work.
105. Either rows are periodic or columns with period 2.
106. We could take special U and V to be the midpoints initially.
107. Let $S = \{s_1, s_2, \ldots\}$. Examine the representations containing an even number of terms.
108. There are $\phi(d)$ elements of order $d \mid p-1$ modulo p.
109. We have 8 homogeneous equations and 9 variables. This system of equations always has a solution in a ring. We only need to ensure the ring
110. Substitute $\frac{2^a+1}{3}$.
111. Try to use Pascal's theorem.
112. If $p \nmid 2m$ and $p \mid k^{2m} - 1$, then one of $v_p(k^{2m} - 1)$ or $v_p((k+p)^{2m} - 1)$ is one.
113. Same holds for $x^2 + 4$.
114. Use isogonality lemma.
115. Prove that $d(n^2 + 1) \geq \frac{n}{5}$ (or something of this asymptotic) doesn't hold for large enough n.
116. Consider k without an even representation. What can you say about the representations of $k + s_t$, $k + 2s_t$ for large enough t?
117. Now, find a bipartite graph that has e^k edges, p^k three-edge paths, m^k and n^k nodes in each side.
118. Try to generate the condition on two consecutive sides AB and BC such that the triangle ΔBPC doesn't contain any lattice points with P lying on the ray AB further than B.
119. How else can we get two cells coloured in the same colour if not using the periodicity.
120. Bezout's theorem. But make sure to take care of small n.
121. Understand the condition on n. Call such n good.
122. Quadratic residues as the ones of the form $\frac{x^2}{n}$.
123. Let $f(m) = a$ for some m. Prove that $m \leq b - 1$.
124. Prove that $n \leq 2$.
125. How many elements are in the cycle $x \to x^3 \to x^3 \to \ldots \to x^{3^k} = x$?
126. They can be represented as a sum of polynomials.
127. Excircle inversion.
128. For the second case, think about what's the position of point A for fixed M and B such that $\angle MAB$ is maximised.
129. Sondat's theorem is applicable, and the fact that the center of orthology lies on this line will prove that the line through centers of nine-point circles is parallel to O_1O_2.

130. This is true because, by Desargues Involution Theorem, there's an involution that swaps $(X; Y)$, $(K; T)$ and $(B; C)$.
131. Prove that $\angle BRC$ is right.
132. Consider the set A of the "elements" part of the graph such that the degree of each node in A is $\geq \frac{e}{3n}$. And same for set B of the "subsets" part with degrees at least $\frac{e}{3m}$.
133. First three hints are the same as in the first solution.
134. Now, need to choose r_i and x_i such that $\min(r_i, r_j)x_i x_j = \min(a_i b_j, a_j b_i) - \min(a_i b_i, a_j b_j)$.
135. Use similar right triangles with T to find equal ratios.
136. Consider some d such that $2^d - 1$ is divisible by some relatively large prime q.
137. Think about some functions that are easy to sum in two dimensions (to obtain the sum in the grid). Also note that each cell is a square.
138. $ax + by = (a + y)x + (b - x)y$.
139. We also want infinitely many terms to be divisible by this p.
140. $\frac{x_k}{k}$ are distinct if and only if $\frac{k}{x_k}$ are.
141. The quantity in the condition we are seeking for has to be related to slopes.
142. Prove that, for large i, s_{i+1} is greater than the sum of previous s + 1. Conclude from here.
143. Use Combinatorial Nullstellensatz to reduce to finite case bash.
144. Problem is equivalent to SR tangent to (AXS), where S is the intersection of XP and the line through R parallel to BC.
145. Converse is true.
146. We have to show that PQ passes through $S = \omega \cap DN$.
147. Move A on AC from C so that the angle increases.
148. Conclude that there are infinitely many of them.
149. Q lies on (BIC).
150. Prove that $P(n) \geq m^m$.
151. $k_i \leq k_1 + \ldots + k_{i-1} + 1$.
152. $m \mid s_d = a_1^d + \ldots + a_m^d$.
153. Graph has no monochromatic cycles.
154. Use vectors corresponding to sides of the equilateral triangle.
155. There's a well known problem that if $ABCD$ are concyclic, then the concurrence holds.
156. Suppose that the tangents meet AC in R, S and AB in P, Q. Intersect PS and RQ with BC in K and T.
157. Modify the process such that in each step the number of inversions is the same.
158. Roots of unity filter.
159. In this structure, try to construct a path without a colour c.
160. Move something else now.

161. Apply to the midpoint of BD, midpoint of BC, midpoint AB and incircle of ΔABC. Calculate the tangent from M to this incircle.
162. Gas station problem
163. Differentiate.
164. We want to solve $x^2 + y^2 = 4n$ in $\mathbb{F}_p$.
165. Use Fermat's little theorem.
166. Repeat this argument again with a different sum.
167. If there exist two rectangles that align across the side containing one of the points, then we can delete the point and merge two rectangles.
168. Induction.
169. This is a standard setup to invert in the Miquel point M of $ABCD$ with radius $\sqrt{MB \cdot MD}$ and reflect in the common angle bisector of $\angle BMD$ and $\angle AMC$.
170. Find another prefix divisible by p.
171. $\Delta A_1' B_1' C_1$ differs from $\Delta A_2' B_2' C_2'$ by a translation.
172. Set up a generating function that counts the representations.
173. $DK = DL$.
174. You want to prove something like $f(\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1) = 0 \Rightarrow f(\alpha_1, \alpha, \beta_1, \beta, \gamma_1, \gamma) = 0$. Rewrite trigonometric functions using complex numbers.
175. This hyperbola is the isogonal conjugation image of the line OI.
176. Indicator function f that equals to 1 if the number is of some colour and 0 otherwise. Prove that it's periodic with period ℓ.
177. Intersect CP with B_1C_1 in point R.
178. T is the Anticenter, so it makes sense to reflect it and prove that it's a circumcenter. Alternatively, you can calculate BP using CD and some trigonometry and do the same for AQ that is equal to it.
179. Find k such that $p \mid a_{k+m}$ if and only if $p \mid a_m$.
180. This is just Cauchy-Schwartz.
181. Untethered moving points is possible but we may do it without it. The trick here is to fix U on AB first and perform moving points. We will need to check three positions of V. A and C are obvious. One more special case is needed.
182. Suppose that the first cell we visited on the main down-right diagonal is $(k, n+1-k)$. WLOG, it's to the left of $(m, n+1-m)$ (otherwise, reflect the board with respect to the main up-right diagonal).
183. Suppose that there's a star at (0,0), there exists only two different ways to partition the plane.
184. Consider $2^{2^n} + d$ modulo 2^{n+1}.
185. It's easier to calculate in $\mathbb{F}_p^2 = \mathbb{F}_p[\omega]$, where ω is the third root of unity.
186. If we could add a one to the root, then we would obviously be able to get $(x - 1^3 - 2)$. So we have to use the fact that we only subtract.

187. Consider one of these bridges e. Try to make all the trees from the condition to have this edge.
188. Use primitive root.
189. It lies on HM.
190. Take n to not be too large.
191. Differences between consecutive terms of s don't repeat often for large indices.
192. We can prove a general inequality:
$$\sum_{1 \leq i,j \leq n} \min(r_i, r_j) x_i x_j \geq 0$$
for nonnegative reals r_i and real numbers x_i.
193. It would be easier to bound the index for which all the prefix sums are bounded.
194. Move to a directed graph on numbers from 1 to n. Starting from k, draw an arrow to next number we go to on the diagonal. This should be a path from k to m.
195. Prove that the centers of $\Delta A_1'B_1'C_1'$ and $\Delta A_2'B_2'C_2'$ lie on the line through centers of $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$.
196. Sum of two good functions is good. This allows to integrate with any non-negative weight $w(t)$.
197. $x^2 + 1$ has no prime factor of the form $4k + 3$.
198. Add the midpoints of segments in the picture.
199. You also need a permutation that maps x to a with $g^a \equiv x \pmod p$.
200. Try to find the equality case.
201. Now, once you got the values, reverse to operations to see that it's impossible to go from them to $\frac{1 \cdot n + 0}{0 \cdot n + 1}$.
202. 2 divides n and 2 is good.
203. If the polygon is in the strip, then we can shift it by a unit vector of distance of the strip. The resulting shape doesn't intersect the original.
204. If the problem statement is true, then $\angle CAK = \angle BAT$.
205. We need to represent $2^{\lfloor \frac{n+2}{2} \rfloor} - 3$.
206. For each such suffix, we can write all the possible combinations of 25 letters in places unoccupied by the suffix. They will all be different.
207. $|a - b|$.
208. All bridges are of the same colour.
209. Use Combinatorial Nullstellensatz.
210. By Vieta's and triangle inequality, one can bound the coefficients.
211. Product of all elements $(i + \omega j)$ of $\mathbb{F}_p^2$ not including $0 + 0 \cdot \omega$ is 1.
212. Double count the product of elements of the set in the first hint in $\mathbb{F}_p$.
213. Schur's theorem.

214. Make sure that we don't have many terms $\le n$ divisible by p or q.
215. Polars of B with respect to these two circles intersect on (ABC).
216. $f(f(x)+y) = f(f(y)+x)$.
217. Looks like AM-GM
218. If $s \cap BC = S$, then $ABSL$ is circumscribed. Suppose that it touches AB at F, BC at D, SL at R, AC at E.
219. Try to linearly transform the quadratic form so that coefficients are small and the discriminant is the same.
220. Represent every moment in the game as a pair of numbers (a,b), where a is the number of the boy and b is the number of the girl. We go from (a,b) to $(a+1,b)$ or $(a,b+1)$ modulo n.
221. Lifting the exponent lemma.
222. $\ell_b \cap \ell_c = D$ and P_b, P_c are the reflections of P with respect to AC and AB. Prove (ADP_aP_c).
223. Denote the discrete Fourier transform image of f as $\hat{f}$. You should get $\hat{f}(n)(1+e^{\frac{-2\pi i|y|n}{\ell}})(1+e^{\frac{-2\pi i|x|n}{\ell}}) = \delta(n)$, where $\delta(n) = 0$ if $n \neq 0$ and 1 otherwise. δ is just $\hat{g}$.
224. Prove that for each P, there exists a Q such that $P(x,y,z)Q(x,y,z) = R(x^3,y,z)$.
225. $Q(n+p)$ is also divisible by p.
226. $OS \perp PQ \parallel$ external bisector of $\angle PDA$.
227. How to obtain other equation to work with from the given.
228. Use vectors.
229. Interpret the angles as rotation angles. We want a sum with some signs to be zero.
230. Think about the example for $n = 3^k$.
231. For each cell of colour c, draw this star. Can they intersect?
232. Associate each good square with some elements such that the elements between the pair don't intersect.
233. $\int_{-\pi}^{\pi} e^{in\theta} d\theta = 0$ for non-zero n. And 2π otherwise.
234. Rewrite the condition in terms of intersections.
235. If we make all small primes in x have small exponents, then the above hint ensures that x is large.
236. Easy to find the solution for $3^k + 1 \le n \le 2 \cdot 3^k$.
237. If a_1, a_2, …, a_m are the numbers smaller than n coprime to n, then characterize the same numbers for np.
238. Think about the fact that if a polynomial gives rational output in deg +1 points, then its coefficients are rational. Similar thing holds here.
239. Look at the expression of the sum of first $2k+2$ terms. See anything common?
240. Establish that there cannot be a large segment of numbers on which the function is constant.

241. Main and alternative paths should have a different residue.
242. If $p = 3k + 2$, then there are $\frac{(3k+3)k}{2}$ inversions.
243. Let $s \cap S = BC$. Reflect S in IO. This point lies on a good line other than the reflection of s in IO.
244. Prove the problem in the case of lines MN and ZO.
245. If it's $\leq -180^\circ$ or $\geq 180^\circ$, then think about the directions of this line during the rotations.
246. Note that the complementary vector to w (i.e. $-w$) is a u for some other v. Draw an arrow from the first v to this v.
247. $(r + 1)!(1^r + \ldots + (p - 1)^r)$ is divisible by $p - 1$.
248. For each $Q^2 \mid P$ with some $P = 1+a_1+a_2x^2+\ldots+a_nx^n$, we can find n more polynomials such that Q doesn't divide them. And they are all different from each other
249. The product of k consecutive numbers is divisible by all integers from 1 to k.
250. Sondat's theorem to triangles ΔDEF and $\Delta P'Q'R'$, where $F = BC \cap B'C'$.
251. Think about sorting some angles.
252. Use the change of reference triangle again.
253. We need two inequalities, one for the perimeter and one for the area.
254. Consider $y_{x_k} = k$. The only ones without a pair are those for which $y = x$.
255. Mark A' - the reflection of A across IO. This point should be the common chord of (ABC) and (AKL).
256. $2^i = \binom{i}{0} + \ldots + \binom{i}{i}$.
257. $9(n - k) + s \geq 5^k \geq (n - k) + s$, where s is the sum k digit number divisible by 5.
258. We can also add corresponding points P and $T = MI \cap (BIC)$.
259. Involution that swaps $(X;E)$, (A,C), $(Y;Z)$, where Y and Z are the intersections of the circle with radius SD with AC.
260. We can generate non-singular words by finding the suffix of this 8 letter portion that already appeared in some other 8 letter portion.
261. Examine which of the numbers from $n + 1$, $n + 2$, $n + 3$ are divisible by 2 and in which powers.
262. Need the correct coefficients. Obtain them by examining the equality case. Since $\frac{b}{b+7}$ and $\frac{7}{c+7}$ are terms of AM-GM, when equality holds, they need to be equal. Using this, we can get the 4-tuple for which the equality holds.
263. Conclusion is true for arbitrary Γ. Choose the most suitable one.
264. Use that for a_{n+1} and $a_0 + \ldots + a_n$.
265. Using the law of sines and some trivial length chasing, one can establish the result for the feet of external angle bisector of $\angle BAC$.
266. Call the word that appears 5^{16} times a copypaste. In the example, copypastes are spaced evenly on the strip.

267. Shift the function again so that 0 is in the domain.
268. The condition is $\frac{p-a}{b+c} : \frac{b+c}{q-a} \in \mathbb{R}$ and its cyclic variants.
269. Mark first circle of the 4th row, third circle of the 5th row, fifth circle of the 6th row and the seventh circle of the 7th row. Can you see it?
270. Calculate the total number of cycles (excluding $0 \to 0$).
271. Their perspective axis is just the radical axis of their circumcircles.
272. Change the power of ten.
273. Add the orthocenter of this triangle.
274. The number of ways to choose a bracket will sum to zero.
275. Complete the square.
276. Relax this condition to sum < 1 in a way that we can decrease some c_i so that the sum is 1.
277. Continuous Cauchy-Schwartz finishes.
278. If we substitute $a = e^{i\alpha}$ or something similar, you should be able to extract the equality of three symmetric sums for three variables a^2, b^2, c^2 (or something similar)
279. Similarly to what we did before, prove that there are m and n such that $\gcd(2^m - t, 2^n - t)$ is unbounded and odd.
280. They are concurrent in the Anticenter of the quadrilateral.
281. $(-1)^{k+1}\frac{p}{k}$.
282. We need to show that Euler lines of ΔERF and ΔDEF are the same.
283. You should get $p \mid (a + mn + 1)^m - 1$, $(a + mn)^m - 1$ and $P(a + mn)$.
284. Circumcenter, ortocenter and incenter satisfy this condition.
285. Experiment with different inequalities on a_ib_i, a_jb_i, a_jb_j, a_ib_j to find the value of r_i. It has to be something of the form $\frac{\text{smth of } a}{\text{smth of } b} - 1$.
286. It lies on the line through A that is also the intersection of pairs of radical axes of the circles from the previous hint.
287. Find a polynomial of smallest degree such that $P(x) = \lfloor \frac{x}{m} \rfloor$ for all $0 \leq x \leq n(m-1)$. Now, we have a multivariable polynomial equation.
288. BX is tangent to (BFS).
289. Summation across all integers from 1 to $p - 1$ is good if we have polynomials.
290. We need to write $S(p)$ as sum of some values of p.
291. Consider $x = 10^k - 10^m$ and $x = 10^{k+1} - 10^m$ for large enough k and m.
292. By using the congruence in the previous hint, change the binomial coefficients to $\binom{p}{2k}$.
293. Invert in H to get the incenter problem.
294. The number of connected components decreases.
295. You should arrive at $g(x) + g(y) + g(z) = 1$ for $g : (0, 1) \to (0, 1)$ for all positive x, y, z such that $x + y + z = 1$.

296. It's easy to make the first column to contain only zeros. Go from first $i-1$ to first i columns to only have zeros for $i < n$.
297. $\frac{a}{b} = \frac{c}{d}$ if and only if $\frac{a}{c} = \frac{b}{d}$, so we can reconnect some edges and leave the graph bad.
298. x, $x+1$, ..., $x+a-1$, x, $x+1$, ..., $x+b-1$ can be continued indefinitely.
299. Bezout's proves that every number is divisible by p.
300. We can take up at least a half of the area.
301. Casey's theorem.
302. Call a cut from the statement k cut. Bob's goal is to force the cancellations, i.e. when he chooses k and $(k+1)$ cuts and Alice chooses left and right, respectively.
303. Connect triangles and find the split such that their alternative path is different from the main.
304. Use binomial expansions and that $6 \mid x-1$.
305. Think about dominoes for the example.
306. If $p \mid \tau(x)$, then x is divisible by some prime in power at least $p-1$.
307. We can find a point that lies on the radical axis explicitly.
308. $j - \omega^2 i = \frac{1}{-\omega}(i - \omega j)$.
309. We can rewrite a function that has an f in its argument into a sum of functions without fs.
310. Line IJ through incenters of ΔABD and ΔABC is parallel to the angle bisector of $\angle APD$.
311. We need to show that SP is tangent to (PQR). This can be done in at least two ways. The first way is to note that $\Delta PBA \sim \Delta K'PA$ and calculate $\angle SPR$. The second way is longer and is based on the fact that OD and DH are isogonal in angle formed by intersection of AB and $A'B'$, followed by angle chase and inversion in H.
312. Suppose that three consecutive terms are equal. Think about gcds, assuming that d is monotonic.
313. You should get something like $(p^2+q^2)(s^2+r^2) = p^2q^2r^2s^2$. RHS is almost always too big.
314. If the tree doesn't contain it, add it and consider the resulting cycle.
315. Fix some good colouring of the graph in which there's no Hamiltonian path without an edge of some colour. Our goal is to repaint it to give it more structure.
316. This can be rewritten nicely if we consider the board $n \times n$.
317. Make sure that the maximum angle is still $\angle MCA$ during this animation.
318. Consider the set S of x that make both x and $4-x$ quadratic residues and $x \neq 0, 4$.
319. Spot couple of Reim's theorems.
320. Find some cyclic quadrilaterals.
321. This is exactly what happens in Iran Lemma. Assuming no corner cases that lead to circumcenter and orthocenter, prove that it has to be the incenter.

322. When we have a cancellation, we will start moving to the right and wait for Alice to choose right again.
323. $\prod(1 - x^{s_i})$ works.
324. The only point it makes sense to invert is C. Choose radius as $\sqrt{CH \cdot CK}$, where H is the orthocenter and K is the foot of perpendicular from C to AB.
325. Vectors.
326. It's hard to induct when we have the fixed value of the sum. Generalize the result for $a_1 + \ldots + a_n = \frac{n}{2} + \alpha$ with, perhaps, some more conditions.
327. You should get $QC - CR = DR - QD$. Find a way to write it as an equality of two segments.
328. $2^{k+2} - 3$ can be represented is equivalent to $2^{2^{k+2}}$ representable.
329. It would be great to just multiply these two quadratic forms and rearrange them to get another form with discriminant -20.
330. For every two-colour colouring of the graph, an odd cycle contains two adjacent vertices of the same colour.
331. $g(x + g(y)) = g(x) + y$ and rewrite $g(x)$ again using variable z.
332. You can add all p summands because some of them will definitely be zero.
333. Associate each prime with a smallest number m such that $p \mid f(m)$. We want this value to be equal to 1.
334. The condition for it to be impossible is that there's some prefix in which there's more vertices of colour c than those that are not of colour c.
335. Apply to Q, R, D and ω_1 and Q, R, D and ω_2.
336. Prove Range $\geq 2\sqrt{\text{Var}}$.
337. Reflect one of the tangents and the circle with respect to that bisector and prove that the other center lies on it then.
338. If we assume that there exists a "positive" rook set, then, for each of its cell, choose a rook set with a zero sum.
339. Homogenize and think about coefficients of some polynomial.
340. You should get $\frac{n}{2} = \sum_{i=1}^{n}(1 - a_{i-1})b_i$. Easy to finish.
341. If the exponent of a prime p in a number is 1, then it's not a square.
342. Easy to find that $g(x) + g(y) = 2g(\frac{x+y}{2})$ for all positive x, y such that $x + y < 1$. This is almost Jensen's FE.
343. In total, starting from large degree of Q, there will be a small fraction of these polynomials compared to total 2^n.
344. We can make $Q(x)^n$ (n is the degree of P) have all positive coefficients, then multiply them by a large C to make all coefficients of $P(Q(x))$ positive and integer (before, we could make them rational).
345. M is the Humpty point in ΔABC.

346. Number of inversions involving the smallest number is precisely the amount of numbers before it. Number of inversions involving the largest number is precisely the amount of numbers after it.
347. Consider such p that divide infinitely many terms.
348. $s^2 + s + 1$ has no prime factor of the form $3k + 2$.
349. Construct the example. First two hints from the first solution help.
350. Consider nine-point circles.
351. Every 8 letter portion between the copypastes is different.
352. We have a lot of parallelograms.
353. Consider the equivalent form with $|b|$ the smallest. We can prove that it's less than both a and c.
354. When the first $n - 1$ columns are zero, the nth is inevitably zero too.
355. To ensure that we can connect two parts, take such a partition into two H_{n-1} such that all the edges connecting them are not deleted.
356. Suppose that n is a quadratic non-residue $\mathbb{F}_p$.
357. Take $\prod_{1\le i<j\le 17}(x_i + x_{i+1} + \ldots + x_j)$. We need to calculate the sum of coefficients before terms in which every degree is divisible by 4.
358. Calculate the coefficient of every monomial $a_1^{x_1}a_2^{x_2}\ldots a_n^{x_n}$.
359. Substitute this new equivalence back into the original.
360. If we consider extouch points of exircles of triangles ΔPAB, ΔPBC, ΔPCD, ΔPAD with PA, PB, PC, PD, then they all lie on a circle centered at P.
361. Multiply them across all pairs of $j > i$.
362. For each 8 letter portion, consider the suffix of smallest length that hasn't appeared as a suffix in some other 8 letter portion (if there's no such suffix, then we trivially get 8 non-singular words).
363. Reflect I with respect to the midpoint of PQ. Now it should be easy to finish.
364. You should reduce to checking that $2^m + 1 \mid 2^n + 1$ for some odd m and n such that $m \mid n$.
365. (MFC) and (MEB) go to FC and BH. Common tangents will be the circles inscribed in the angle formed by these lines passing through M. Their second point of intersection is the image of K and can be characterized in a good way.
366. Mark the triangle formed by ℓ_a, ℓ_b, ℓ_c as DEF. Points of intersection of ℓ with sides are X, Y, Z.
367. Do the same for rotations centered at B and O_c that take C to C_2 and C_2 to A, respectively. The center of the equivalent rotation is X.
368. $\frac{p-1}{2}$ should spark attention.
369. Each number in row $i + 1$ has at least one number on top that is smaller than it. In this way, we can bound S_{i+1} through S_i. Don't forget about an additional 1 from a red circle.

370. Think about involutions on AB and AC. The pairs for generalization are pairs of these involutions.
371. Denominator looks like Lagrange interpolation.
372. We are given a midpoint and incircles.
373. $7 \cdot a \cdot \frac{1}{b+7}$.
374. Consider q the second smallest divisor of n.
375. This is a problem about points on plane.
376. We can calculate the lengths now using Fuerbach's theorem.
377. Homothety at G_2.
378. If we have a very large number of zeros initially, then we could get this configuration for large enough x.
379. Use Sawayama's lemma.
380. Rewrite the condition in the form that $\left(\frac{a_n+\ldots+a_{n+k-1}}{p}\right) = \left(\frac{k}{p}\right)$ for p relatively prime to both numerators.
381. A consists of $(t + \frac{1}{t})^2$ with $t \in \mathbb{F}_{p^2}$ and $t^{p-1} + 1 = 0$.
382. Add the orthocenter H of the extouch triangle.
383. $IT \parallel D'N$ finishes the problem.
384. The image of K is the reflection of M in the angle bisector of $\angle BHC$.
385. If we sort these prefixes as $p_1, p_2, \ldots, p_k$, then $p_i \le 2^i - 1$.
386. $\mathrm{Var} = \frac{1}{n-1}$ by direct computation.
387. If there was one variable, it would be easy with roots of unity filter.
388. $\sqrt{n}^2 = n$.
389. $(1-x)^{-1} \prod(1-x^{s_i}) = (1 + x + \ldots + x^{s_1-1}) \prod(1-x^{s_i})$ also has bounded coefficients.
390. 2^{2k+2} can be represented if and only if $2^{4k+2} - 3$ can be represented.
391. This case can be dealt using continuity or inducting down.
392. Rewrite it in terms of trigonometric functions of angles that the rays form with sides.
393. For each n, find m so that $n^2 + 1 \mid m^2 + 1$.
394. Conclude by Intermediate Value Theorem.
395. We want to prove that $SD = SL$, where L is the second intersection of (ADC) and (EXD) and $S = BC \cap EF$.
396. Use limit of $f(x)$ when x approaches zero
397. There are 2^d choices for Q with degree d. We also need to choose R.
398. The proof is just calculating the sum of their angles.
399. Same as the first hint from the first solution.
400. Find a suitable center of inversion that the picture after inversion is the same as the picture before inversion.

401. Set $g(n) = 1$ for all integer n and apply Discrete Fourier Transform to both sides. If you don't know what this is, it might worth to check the first page of the solution.
402. Write $k^{2a}m^{2b}$ in the cell with coordinates (a, b).
403. Extend the common tangent to intersect ω at Q and R and use the Shooting lemma (or Archimedes' lemma) to find one point on the radical axis of ω_1 and ω_2.
404. At each step, triangulate one triangle into three triangles.
405. IO is the Euler line of ΔDEF.
406. Mark the exsimilicenter of (A) and (C) as Y and finish.
407. Consider big primes p dividing $2^{2^n} + d$.
408. After changing $a = e^{i\alpha}$ or something similar, f of the first choice of variables is equal to f of the second choice of variables as polynomials.
409. LHS calculates the number of ordered 4-tuples (i, j, x, y) such that $x \in A_i$, $y \in A_i, A_j$.
410. We only need to check this condition for the square free part of n.
411. Use this lemma to P, $P + 1$, Q, $Q + 1$ in the original.
412. $\sqrt[3]{2^2}$, $\sqrt[3]{2}$ and 1 are linearly independent over $\mathbb{Z}$.
413. We can construct a tree with leaves $2^{c_1}, 2^{c_2}, \ldots, 2^{c_m}$ if and only if $sum_{i=1}^{m} \frac{1}{2^{c_i}} = 1$.
414. OI is tangent to this hyperbola (just prove it has no second point of intersection).
415. Ensure that the projection of this point lies on the side and the point itself lies outside the polygon. This point will work.
416. We can also take the homothety and make one of the vertices of the equilateral triangle to coincide with a vertex of the triangle. WLOG, it's C and the triangle is ΔABC.
417. Apply Dual of Desargues Involution Theorem to $ABSL$ and incircle.
418. Conclude with the degenerate case of Desargues Involution Theorem.
419. Think about shifting every x_i by c.
420. The result is the difference of two squares. One of them is too small compared to the other one (almost always) but the difference is a square.
421. Prefix sums.
422. You should use a fact that no cycle in the graph is monochromatic.
423. So, we could make moves that lead to this infinite configuration described above.
424. Try to induct down.
425. Restrict some direction by segregating odd indices from even.
426. Can be done either using the isogonal lemma or Linearity of PoP.
427. Now use both numbers to "average".
428. Raise to mth power.
429. The number of ways to choose it for each bracket is the same.
430. Suppose that n is the minimal number of pebbles such that Bob cannot win.

431. Colour in two colours such that the two adjacent vertices are of different colours. Every odd cycle in the original graph contains two vertices of the same colour that are adjacent.
432. You can write the latter explicitly.
433. Use Brokard's theorem to deal with directions of BO_c and BO_a.
434. In each circle, write a number that is equal to the maximum number of red circles on the path from the top circle to this circle.
435. The equation should read as $g(x + g(y)) = g(x) + y$ for $x > y$. It's easy to prove that g is injective.
436. Look at the sign of the real part.
437. Thus, among the six angles inside the triangle, we have two identical triples. The rest is case bash to conclude all the possibilities.
438. Draw a bipartite graph with n nodes in one side equivalent to elements, and m nodes in the other side corresponding to subsets. Two nodes are connected iff this element is contained in this subset.
439. $(BHCS)$ and easy to get $\angle HMB = \angle BMS$.
440. Invert at the incenter. You will get an equivalent problem in terms of the orthocenter.
441. Problem is solved in f unbounded case.
442. Bondy-Chvatal theorem helps. The original formulation is about a cycle but we can change it a bit to fit for the path.
443. You have to prove something about the intersection point K of radical axis of (PBF) and (PCE) and radical axis of $(BDIF)$ and $(DICE)$.
444. If B' and C' are reflection of B and C in IO, then $KL \parallel B'C'$.
445. Factor in the end or just expanding works too.
446. Move the points so that the sum of sines increases.
447. It can be done using Hölder's inequality.
448. Find $x + y$, $x - y$ and xy.
449. An easy way to control if a number is divisible by p or not is to check its $p - 1$th power.
450. Conclude that $A'K$ and $A'C$ and $A'L$ and $A'B$ are reflections in AI.
451. To prove it, colour down-right diagonals in n colours modulo n.
452. Define a new good function that is formed by the two values of x^{x-1} and y^{y-1}. Our objective is that this function doesn't take the same value too often.
453. Leave two to not equal to one.
454. Pascal's theorem.
455. Look at $f(n - k) \mid f(n - k - 1) + f(1)$ for all $n > k + 1$.
456. Linearity of power of a point.
457. If she keeps choosing left, then we will soon arrive at the k cut again, but at this point we only had cancellations, and so we are in the situation with less pebbles than in the beginning, and Bob can win by induction.

458. $\operatorname{Im} f = \{a, a+1, \ldots\}$, $\operatorname{Im} g = \{b, b+1, \ldots\}$ with $a > b$.
459. $n = 2$ doesn't work.
460. If X is the center of the rotation equivalent to anticlockwise rotations centered at O_c and B that take C to A_2 and A_2 to A, respectively, then $\angle O_cBX$ is equal to half of the angle of the rotation centered at B.
461. Sum cyclically.
462. $ABCIPQ$ are known to lie on a rectangular hyperbola with center at the midpoint of PQ. In general, if P and Q are antigonal conjugates with respect to ΔABC, then $ABCPQ$ are on rectangular hyperbola with center at the midpoint of PQ.
463. Consider all such bipartite graphs that there's at most one edge of the form $i \sim i$. We will calculate the parity of bad graphs, i.e. the ones in which there are two equal ratios. By ratio of $a \sim b$, we will mean $\frac{a}{b}$ for $a \geq b$.
464. The inverse of one of the vertices of the small quadrilateral will be the Miquel point of a quadrilateral formed by three of the bisectors and one side.
465. You can find a simple graph in which every edge is chosen and is bipartite. There's a cycle in this graph.
466. Parity of the permutation is the same for $x \to x^3 \pmod p$ and $x \to 3x \pmod{p-1}$.
467. Prove that, if $P \in \mathbb{C}[x]$ is a con-constant polynomial and c is some non-zero constant, p_0, p_1 are the numbers of distinct roots of P and $P + c$, respectively. Then $p_0 + p_1 \geq \deg P + 1$.
468. Assume contrary to the problem statement. Shift all polygons by a unit vector and use Helly's theorem.
469. Think about the condition for a monomial to be able to write it as $R(x^2y, y^2z, z^2x)$.
470. Both conditions are equivalent to some equation after applying Casey's theorem.
471. the center of the inversion has to be on some three circles such that after the inversion they become the reflected rays.
472. Substitute $x = f(x)$ to exploit symmetry.
473. Mark X radical center of (A), (C) and Γ.
474. Define $Q(x_1, \ldots, x_n)$ as $x_1 + x_2 + \ldots + x_n - mf(x_1, \ldots, x_n)$. This polynomial takes the value of $(x_1 + \ldots + x_n) \pmod m$ for $x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}$. Induct on n that this implies $\deg Q \geq n$.
475. Binomial expansion.
476. Note that $\sum x_i y_i$ is the smallest and $\sum x_i y_{n+1-i}$ of all possible sums of the form $\sum x_i y_{\sigma(i)}$. This prompts to consider uniform random variable $S = \sum_{i=1}^n x_i y_{\sigma(i)}$.
477. We don't know the choice of signs.
478. Draw $H = ID \cap (BIC)$ and use that H and D are corresponding.
479. The only common roots that they might have are the ones of $x^2 + x + 1$.
480. Bound the number of sides of small rectangles.
481. Zsigmondy's theorem and induction.

482. Let a_{ij} be an indicator function that is 1 if $i \in A_j$ and 0 otherwise.
483. Move M on the line AM towards A.
484. Now, when every sum is zero, it's possible to make all numbers zero by applying the operations.
485. If some company hasn't appeared in the selection, but every vertex of its cycle is contained in the selection, then add the edge connecting two same-coloured vertices. We either solve the problem or...
486. Find another.
487. If f is bounded, then look at the primes dividing it.
488. We need to prove the same statement for othocenters.
489. Represent each rook set as a permutation and start making transpositions to move from the one rook set to another and see how amber cells number change.
490. We only care about $i^2 = -1$ because imaginary part vanishes anyway. Try to get a problem only about integers.
491. Find a bound on the number of edges in the form $\frac{e^3}{mn}$ times some constant.
492. At each step, make the smallest number the largest number.
493. State the problem in terms of triangle ΔCXY, where X and Y are the midpoints of CA and CB in the original triangle.
494. Come up with some bijection between good permutations.
495. Prove that $(A'SLC)$.
496. Try to reconstruct its even cycles.
497. Every large enough integer has at most one even representation.
498. We have to calculate $(-1)^{\frac{p-1}{4}}$ times the product of all $(t + \frac{1}{t})$ across all the roots of the polynomial from previous hint.
499. $f(f(x + f(y)) + z) = f(f(x + y) + f(y) + z)$ works.
500. Use isogonal conjugates.
501. We can prove that the graph is just two disjoint cliques.
502. Prove that it's possible permute the colours in each column such that each row contains each colour.
503. Suppose that we get a stable configuration after using type 2 operation for N times. Imagine this as putting $2N$ chips in a zero vertex and firing when we use type 2. Prove that whatever the moves we do, from this configuration we will get the same stable configuration in the end.
504. Prove that whatever the partition we have, we will have $2n + 1$ triangles.
505. $Q_n^2 - A_n^2 \geq \frac{d^2}{2n}$ for $n \geq m + 1$.
506. Consider regions formed by three consecutive sides. We want to add one lattice point that lies within this region.
507. $x_i = \operatorname{sgn}(a_i - b_i)\min(a_i, b_i)$.

508. We need an odd number of sign changes for $Q(x)$ and an even for $P(Q(x))$.
509. Suppose that no set contains exactly a or $a+1$ amber cells. We already found the one with $\geq a+2$ and $\leq a-1$.
510. Rewrite the angle condition in terms of cyclicity.
511. Draw a directed graph on numbers from 1 to infinity with an edge from x to $g(x)$. Starting from m, we must get every number in $\{b, b+1, \ldots\}$ exactly once.
512. Thus, a^2, b^2, c^2 are the roots of some monic cubic polynomial. And the triples have to be equal, because the complex numbers had both real and imaginary parts positive.
513. Use mod 4 to the original equation and conclude for $k > 4$.
514. We can get that $f(m(\ell))$ works, where $m(\ell)$ is the simplified fraction of the slope of ℓ and $f(\frac{x}{y}) = \max(x, y)$. We will only consider the not axis-aligned sides to not run into infinity.
515. We can show that if there's a lot of edges of colour i coming from v, then we can repaint all other edges in colour i
516. Invert.
517. Add number to both sides to ensure the divisibility.
518. Invert in I.
519. Degree counting suffices to conclude.
520. Suppose that A_1 is the extouch point on PA, then add a circle centered at A through A_1. Call this circle (A). Do the same for other vertices and consider (B), (C), (D). They are all tangent to Γ.
521. Write all the conditions in terms of angles and their trigonometric functions.
522. First two hints are the same.
523. If we suppose that $xy = a^2$ and $zt = b^2$, then pay closer attention to $xy - zt$.
524. ΔABC and $\Delta A'B'C'$ have the same centroid and nine-point.
525. Induction.
526. We can determine the next cell on this diagonal.
527. What to do with powers of two?
528. Use $P(x) = (1 - xx_1)(1 - xx_2)\ldots(1 - xx_n)$ and think what to do with the excessive $(1 - x_i^2)$.
529. Main circles in the problem pass through M.
530. Reim's theorem. Reduce to some parallelism.
531. $(BFEC)$.
532. Since O_1O_2 has to be parallel to H_1H_2, we may consider a line through two corresponding points on the Euler line, and it will also be parallel to the two aforementioned lines.
533. Using all the equal segments, find a kite.
534. $2((a_i - a_m)^2 + (a_{m+1} - a_i)^2) \geq (a_{m+1} - a_m)^2 = d^2$.

535. Restate the problem in terms of ΔIPQ.
536. $\Delta LCW \sim \Delta EMW$.
537. Substitute $\sqrt[3]{2}$ here and equate it to $\sqrt[3]{2}+1$. Find all possible values of a, b, c, d.
538. If c_x is the number of subsets that contain x, then the LHS is $a_{yi}c_y|A_i|$ over all $1 \le i \le m$ and $1 \le y \le n$.
539. All points are contained within some square.
540. Compositions of rotations are rotations and we can construct their center.
541. Consider the largest $d_c(v)$ among all choices of c and v. Then consider the same for the graph with deleted vertex v.
542. $\frac{e}{3}$ edges will have both of their ends in sets A and B. This gives the bound of $p\frac{e^3}{27mn}$.
543. f is injective at zero.
544. Make almost all values of a_p for primes $p \le n$ equal to one.
545. The condition we need to prove rewrites as $\frac{p+q}{\overline{p}+\overline{q}} = \frac{ab+bc+ac}{abc(a+b+c)}$. The three conditions that we have all contain $p+q$ and $\overline{p}+\overline{q}$ with some coefficients in a, b, c and zeros in RHS.
546. All maps are projective.
547. Establish some bounds on neighbouring terms of x_n and y_n. They are parts of linear recurrences, so it's possible to establish linear bounds.
548. Try to multiply $10^k - 1$ by small numbers and see how the digit sum changes.
549. Bertrand's postulate.
550. It doesn't change for $n(10^k-1)$ for $n \le 10^k$.
551. For every 17-tuple made of $\{4,-4,1,-1\}$, we can find a contiguous segment that has sum divisible by 17.
552. Mark intersections of (DPL) and (DPK) with BC. And $XL \cap YK = Z$.
553. Harmonic series diverges.
554. Consider the sum S_i of all values in row i.
555. We have two tangent circles and want to prove the tangency of two other circles.
556. T is the intersection of common tangent to ω and Ω_A.
557. Point K is connected to Poncelet's Porism.
558. Suppose that some polynomial have a positive coefficients surrounding every negative coefficient. Define the number of negative coefficients as sign changes. Check that when we go from 10^m to 10^{m+1}, we only get addition to the number of zeros and nines by a fixed amount
559. Thue's lemma.
560. If polar of B with respect to (AC_1C_2) intersects AB in point T, then $(AC_1;TB) = -1$. Do some projections to get that $\angle O_aBO_c + \angle M_aLM_c = 180^\circ$. The latter is easy to calculate.
561. Reduce the problem to showing $(VAKP)$, where $K = ID \cap AM$.

562. There exists an irreducible polynomial dividing $k^{2^m} - 1$ such that it doesn't divide any $k^{2^s} - 1$ for $s < m$.
563. Polar duality gives that the polar of $ID \cap PQ$ passes through the midpoint. Complete the quadrilateral and use Brokard's theorem as well.
564. Indicator function.
565. R is the intersection of the altitude and tangent to the excircle parallel to BC.
566. Number the diagonal cells according to their columns. If we were at $x < m$, then we go to $x + k$. If we were at $x > m$, then we go to $x + k - 1$.
567. It makes sense to unite some subproblems for two smaller H_{n-1} graphs.
568. Consider a graph of bad matchings, where we connect two vertices if one can reconnect edges as described in previous hint to go from one to other. Conclude about the parity of bad matchings. From this, it's easy to get the parity of good matchings.
569. You will arrive at the situation where each cell is contained in some rook set which has a sum of zero.
570. You should spot a familiar Dual of Desargues Involution Theorem picture.
571. This is a problem about finding a cycle in a directed graph.
572. Radical axis of ω and (BIC) is a midline of ΔDEF.
573. Medial triangles are perspective. Find their perspective axis.
574. $f(0) + f(1) + \ldots + f(p-1) \equiv 0 \pmod p$ for a polynomial with degree at most $p - 2$.
575. If the coefficients of $g(x)$ are bounded, then $|g(x)| \le M|1 + x + x^2 + \ldots| = \frac{M}{1-x}$.
576. You should get $f(x) \ge x$.
577. Only draw sides that connect vertices of the same colour.
578. $x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2 = (x + y + z)(x + y - z)(x + z - y)(y + z - x)$.
579. Find cyclic quadrilaterals.
580. Shooting lemma.
581. We should delete a bridge from the cycle. What will happen when we can no longer use this operation?
582. Perimeter of a convex shape lying inside the other convex shape is at most the perimeter of the latter.
583. Mapping $x \to x^3$ can be written through to $x \to 3x$.
584. Think about the starting selection such that we always have at most $n - 1$ edges (so we could always find an air company to add) and exactly $n - 1$ edges only in the case of a connected graph.
585. In the inverted problem you should have an equilateral triangle, a reflection of one of its vertices in the side. This should be the image of the center of the original equilateral triangle.
586. Calculate the angles and conclude that, if it's not self-intersecting, then it has an odd number of vertices.
587. $\min(r_i, r_j) = \int_0^\infty f_i(x) f_j(x)$, where $f_i(x)$ is zero if $x > r_i$ and 1 otherwise.

588. This is a problem about integrals.
589. Consider the homothety that takes (PQR) to (ABC).
590. If X^* is the inverse of X after the aforementioned inversion, then $\angle AX^*D = \angle ABX + \angle XCD$ and $\angle FX^*D = \angle FBX - \angle XED$, where $F = AD \cap BC$ and $E = AB \cap CD$.
591. What are the easiest examples of functions for which it's easy to control the repetition of the residue?
592. You have the differentiated equation and the original one. Rewrite the differentiated one using the original to eliminate high powers.
593. If Sondat's theorem was applicable, then the centers of orthology would lie on the line through centers.
594. It's equal to the sum of $\frac{\phi(d)}{\mathrm{ord}_d(3)}$ across all divisors of $p-1$.
595. Finish with power mean.
596. Use inversion to establish that the line through them passes through other good point.
597. d is sub-polynomial.
598. Spam radical axes to find one circle that passes through X.
599. 2×2 square must contain cells of all 4 colours.
600. General claim is that if we are given two circles intersecting in points P and Q and the point of intersection of common tangents X, then $\angle PXQ$ can be represented in terms arcs PQ of those two circles (or in terms of inscribed angles subtending this arc).
601. Check that the solutions are exactly $x = \omega(t + \frac{1}{t})$ and $y = \frac{\omega}{i}(t - \frac{1}{t})$.
602. Find parallels.
603. Reflect A_1 with respect to B_2C_2 to get A_1'. Do the same to obtain dashed images of every original point.
604. Find similar figures.
605. Construct more equilateral hexagons that share vertices with the given ones. The problem statement would hold for the as well.
606. Some of the elements are the same.
607. $y_{n+1} > x_n$ and $y_{n+1} < 6y_n$.
608. Draw circles such that no two circles overlap and the total area is unbounded
609. Consider the spiral similarity centered at $(AEF) \cap (ABC) = V$ that takes one of these figures to other.
610. Force the order of n modulo q to be equal to p.
611. Use concurrence to establish cross ratios.
612. Perimeter is closely related to excircle.
613. Make sure that we can add enough non-zero digits to the right to make the sum 5^k.
614. Prove that W lies on XL.
615. Call a set in which no two cells lie in the same row or column a rook set. Find a rook set with a amber cells and a rook set that contains b bronze cells.

616. Consider a 100×100 table. Assign each girl its column and write the colours of her balls in this column.
617. Prove that the line through centroids passes through I.
618. Using the last hint, we can calculate $\angle QKM$ in terms of angles of the triangle, as well as $\angle QH'M$, where $H' = AH \cap BC$.
619. We can factor $x^2 + x + 1$ in second power for $6 \mid p - 1$.
620. You already have three lines. Mark a Miquel point.
621. Let ΔPQR be a triangle formed by AA', BB', CC'. We want to show that the centers of (PQR), (ABC) and ω are collinear. We need to prove something about these circles.
622. Consider a star figure with area 1201, height and width equal to 49.
623. This is a problem about polynomials and counting sum of some coefficients.
624. For each side of the rectangle of the partition, define a maximal contiguous segment that contains it.
625. Note that if maximum of a_1, a_2, ..., a_n is very small, then the fact that

$$\sum_{k=1}^{n} a_k(a_1 + \ldots + a_{k-1})^2 < \frac{1}{3}$$

by Riemann sums, would suffice. Need to manipulate the numbers without decreasing the sum such that the largest decreases.
626. $f(f(0)) = f(1)$. Use it by setting $y = f(0)$ and $y = 1$. You will get a periodicity.
627. Mark second points of intersection of rays with circumcircle. We can also write Casey's theorem for the new triangle.
628. Try (a, b, a, b). We will arrive at the two variable version of the problem.
629. From the above, you can say that $|g(x)| \leq M$ for $0 < x < 1$.
630. This permutation is almost $x \to x + k$. Call it π_1.
631. Just choose m of the points for which we can find the entirely good triangulation.
632. Try to find a point closest to the polygon in some sense.
633. 5^k.
634. Go to the last two hints of the previous solution.
635. $RH \perp DE$, where E is $A'C' \cap AC$.
636. Rewrite the condition of equal sides in terms of midpoints.
637. We can factor more.
638. Consider the set of primes such that they divide at least one term of the sequence.
639. Bridge from T_c is not of c colour.
640. Try to guess what R is.
641. These lattice points are on some distance from the sides of the $a \times b$ rectangle.
642. Radical axes.
643. We can add these primes one by one.

644. Think Cauchy Davenport to get the desired combination of mains and alternatives. It should span all the residues.
645. There are many isosceles triangles formed by bisectors and parallel line and circumcenters and their angles are easily calculated.
646. Every two circles intersect in an even number of yellow points.
647. Monge's theorem.
648. Something is unique in $2^{2^{2a-1}}a \equiv 2^{2^{2b-1}}b \pmod{p}$.
649. Find similar triangles.
650. After proving that the Miquel point T lies on Γ, mark intersections of TA and TC with the circumcircle of triangle formed by ℓ_a, ℓ_b, ℓ_c. Call first one V and the second one W. Prove that $VW \parallel AC$.
651. The graph H_10 in the problem condition can be constructed recursively from connecting two copies of H_9. Generalize the problem for H_n.
652. You should use the point L on the excircle such that $\angle HXL = 90^\circ$.
653. To connect, we need the respective vertices in the paths.
654. $(BME) \cap (CMF) = Q$ is a known point.
655. We only have to define for primes and prove $a_1 + 2a_2 + \ldots + na_n = 0$.
656. Use
$$f(x) = x^{\frac{p-1}{2}} - i = \prod_{t \in \mathbb{F}_{p^2}, t^{\frac{p-1}{2}} - i = 0} (x - t).$$
657. First 2 hints are the same as in the first solution.
658. We need to prove that (DEX), (ADC) and the circle with radius SD and center $S = BC \cap EF$ are coaxial.
659. We must have $p \mid q - 1$.
660. $k = p$ works.
661. Try to extract necessary conditions on q.
662. $i^{p-2} \equiv \frac{1}{i} \pmod{p}$.
663. Find p and q such that $\operatorname{ord}_q(2) \mid p - 1 \mid q - 1$.
664. $f(\alpha, \alpha_1, \beta, \beta_1, \gamma, \gamma_1) = 0 \Rightarrow f(\alpha_1, \gamma, \beta_1, \alpha, \gamma_1, \beta) = 0$.
665. all combinations of signs are present in x, y, z, t.
666. If k is the topmost row of a number in an antichain, then it contains at most k circles.
667. $AP_c \parallel EV$, where $E = \ell_a \cap \ell_c$.
668. Labels after inversion will be the same, but H is now I. Rewrite angle conditions.
669. Find all x, y, z, t in terms of a and b.
670. (BQC) is symmetric to (BPC) with respect to BC and other symmetries of circumcircles.
671. Every cycle of π_1 is of length $\frac{\operatorname{lcm}(n,k)}{k}$. This allows to calculate the parity of π_1.

672. $g(a+k)-1$ takes every value from $\{b, b+1, \ldots\}$ exactly once.
673. Total sum is a multiple of 180°.
674. Radical axis of ω_1 and ω_2 is the perpendicular bisector of IJ.
675. How many numbers ≤ 2023 are divisible by each of a_i?
676. Hölder's inequality.
677. $d(x_n^2+1) = 2d(y_n^2+1)$, where (x_n, y_n) is a solution to Pell's equation $x^2-2y^2=1$.
678. We finish the path at the cell on the diagonal.
679. Treat rows as vectors. Eliminate zeros and remember that our objective is the sum.
680. Sum $Q(i, x_2, \ldots, x_n)$ over all $0 \le i \le m-1$.
681. Finish with prime number theorem.
682. There's no number that is divisible by both a_i and a_j.
683. Mark the Miquel point M and prove $(MDBL)$.
684. Calculate $f(|x||y|)$ and $f(2|x||y|)$.
685. n can only be good if the aforementioned primes divide it in even exponent. It's possible to prove that these numbers must be good by finding the example.
686. Draw a binary tree with root 1 and two 2^{x+1} written under 2^x (or nothing). Its leaves will be the coefficients.
687. Proof is by induction and merging two equal integers in one when possible.
688. Rewrite the inequality in the form

$$\sum_{1\le i,j\le n} \left(f(x_i+x_j) - f(x_i-x_j)\right) \ge 0$$

for $f(x) = \sqrt{|x|}$. This is a general form that holds for $f(x) = x^2$ or $f(x) = -\cos x$ (verify this yourself). Call the functions that satisfy this inequality good.
689. Restate the problem with respect to extouch triangle of ΔABC, call it ΔDEF, where D lies on BC.
690. Do the same but for Q, R, C.
691. Let r_n be an integer root of P_n. One can show that $\frac{P_1(r_n)}{P_2(r_n)} = -\frac{F_{n-1}}{F_{n-2}}$. $f(x) = \frac{P_1(x)}{P_2(x)}$ is a quadratic rational function.
692. Reflect some perpendicular bisector (of these three) across XY.
693. One such k can be found from the prefix argument.
694. Think about polynomials dividing $k^{2m}-1$.
695. Add more tangencies.
696. Think about Hamiltonian paths (paths through every vertex) and when they exist.
697. The problem is just angle chasing after that.
698. You will have something of the form $ka_p + ma_q = t$ for fixed k, m, t, and you need to find suitable a_p and a_q.
699. Mark more reflections.

700. These are exactly the involutions. Represent those as bipartite graphs from $\{1, 2, \ldots, n\}$ to itself. i and j are adjacent if $x_i = j$ and, as a result, $x_j = i$.
701. First thing that comes to mind when dealing with $xyz = 1$.
702. $\alpha = \frac{1}{c^2} - \frac{1}{b^2}$ and cyclic variants work as weights.
703. Prove that we have to go in the same direction for $(a, b-1)$ and $(a-1, b)$ (not for $a = 1$, $b = 1$, but this case should be handled). This allows us to determine only one cell on each down-right diagonal.
704. Going from square-free to all n is similar.
705. Incenter-Excenter lemma.
706. Project using the parallel lines.
707. Rook set is a matching.
708. Make it 1 and 0.
709. Think about how the real root is transformed.
710. Makes sense to make $2^{c_i} \approx (i+1)^{\frac{c}{2}}$ with some constant to satisfy all the inequalities.
711. The statement as it is involves 25 rectangles. Try to combine them.
712. Condition looks like Helly's theorem.
713. Jensen's FE is equivalent to Cauchy's FE if we can substitute zero. Find a workaround here.
714. We need to take $m+k+1$ to $m+k$, $m+k+2$ to $m+k+1$, ..., $m+k$ to k.
715. To check that they are in $\mathbb{F}_p$, check $x^p = x$ and $y^p = y$.
716. Need to break down the fraction so that $\frac{1}{b+7}$ cancels. For this, we need an additional constant multiple.
717. By the above, it's coefficients are bounded.
718. The problem will reduce to making $(k^{2l} - 1)(k^{2s} - 1)$ not square for $l \neq s$.
719. Mark $AB \cap (EBP_cP_a) = L$.
720. If M is the midpoint of arc BAC, then $MBIC \sim AFRE$.
721. Distinguish terms with degree of x by modulo 3. We want the resulting product to have monomials that are the multiplication of three monomials from each or three from the same.
722. Harmonic series helps
723. A way to modify it is to be able to choose the alternative path for some pairs of the vertices.
724. Colour (a, b) in colour of $ax + by$.
725. Find which primes might divide n.
726. Substitute the third root of unity ω. Note that we are free to choose p.
727. Reduce to $\mathbb{F}_2$.
728. Think about the circle.

729. There are two cases treated similarly. Further I will assume that $p = 4k + 1$.
730. Find k such that for each m, a special prime divides $k^{2m} - 1$ with exponent 1.
731. There's a lot of free space in between these figures. Try to take it up.
732. Tangents from A' to incircle are reflection as well.
733. Given inequality bounds the distance between points.
734. Homothety.
735. Establish the converse of Simson's line.
736. Changing $(a_1, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_n)$ to $(a_1, \dots, a_{k-1}, \frac{a_k}{2}, \frac{a_k}{2}, a_{k+1}, \dots, a_n)$ works.
737. Prove that the function cannot be non-increasing.
738. Consider endpoints of these segments.
739. Mark all the other midpoints and interpret the angle condition.
740. Prove that $g(n)$ is injective for $n \geq a$.
741. Expected value of a cell is $\frac{a^2+ab-b}{(a+b+1)^2}$ if we give 1 for an amber cell and 0 otherwise. If we calculate it for the rook set, we will be done.
742. Note that we could divide by higher powers of $(1-x)$. Spam this to obtain contradiction.
743. Write b_i through b_{i-1} because everything in cyclic.
744. $\frac{b}{b+7}$ works. To obtain it, we need to multiply and divide the $\frac{b}{c+7}$ term by $b + 7$. The excessive $b + 7$ will go as a separate term.
745. We have a $\frac{1}{b+7}$ term. We need something to cancel it in the sum.
746. Choose a and b smaller than p such that $p^2 \mid a^2 + ab + b^2$.
747. Generalize it.
748. $(XMEB)$.
749. Consider the polynomial $p(x)p(\frac{1}{x})$ (allow negative powers). $S(p)$ is its value at "zero".
750. We found other pairs for which the problem is true.
751. Triangulate.
752. We cannot use trivial induction only in the case $a_n < \frac{1}{2}$, so, if we write $a_n = \frac{1}{2} - \alpha$, we will get a general claim that will be the statement of our induction,
753. To prove the bound, combine all squares that share a side into one figure.
754. If $|a_m - a_{m+1}| = d \neq 0$, then we can bound $Q_i^2 - A_i^2$ stronger than 0 for $i \geq m + 1$. Where A_i is the arithmetic mean of first i terms and Q_i is the quadratic mean.
755. Think about $a + b$.
756. In $m^2 + 3 = (n + 1)(n + 2)(n + 3)$ m is odd.
757. Let T be the antipode of D on the excircle. Let A^* be the midpoint of EF. Intersect XA^* and TD in N. We need to show that $Y = XT \cap EF$ lies on (XDN). This is not the same Y as in the problem statement!
758. If $AB' \cap A'B = S$ and $BB' \cap AA' = P$, then the midpoint of SP lies on the radical axis.
759. One can bound the magnitude of a root of a polynomial that has coefficients 1 and 0.

760. $k + 2ms_t$ has a unique even representation with s_t.
761. Polynomials for which every monomial of them is divisible by 9 form a ring (by divisibility by 9, I mean $9 \mid 4a - 2b + c$).
762. Suppose that for any $k \le n - 1$ colours, their edges are on $k + 1$ vertices. Bound the number of different edges.
763. Prove that $x^2 + x + 1 \mid P(x)$ as polynomials.
764. First 4 hints are the same as in the previous solution.
765. $P'D$ passes through O, where P is the image after the homothety.
766. Think of the graph. Concavity helps.
767. Forest works. Ensure that no vertex of the cycle is outside of it.
768. Each good square gives four lattice points.
769. Degree counting suffices to conclude.
770. Pick's theorem.
771. x and $\frac{1}{x+7}$ are oppositely monotonic. Makes sense to alternate a big number with small.
772. $2^7 - 3$ is non-representable and $2^3 - 3$ is representable.
773. Coaxiality is equivalent to the involution.
774. We have the desired situation but for the rows, not columns.
775. $QC = PA$ is a standard condition to use rotation in Miquel point of $ABCD$ - S.
776. Rewrite the angle condition.
777. Differentiate.
778. You should have the equality of products of sines and products of cosines. As well as the equality of sums of angles.
779. $(x+1)^{x+1} = (x^2 + 2x + 1)^{\frac{x+1}{2}} = (c + x)^{\frac{x+1}{2}}$.
780. Call selection a subgraph in which we take at most one edge from each airline. Every selection is bipartite, i.e. can be coloured in two colours such that no two vertices of the same colour are adjacent.
781. Use the primitive root. Exclude zero.
782. We want no cycles or connectedness. Assume there's always a cycle and at least two connected components.
783. Think about -5 quadratic non-residue.
784. If such prefix doesn't exist then every residue is quadratic.
785. We know m and a modulo 6.
786. First eight hints are the same.
787. Raise the sum of every k element subset in $p - 1$th and sum to express f_k.
788. Grid is a bipartite graph.
789. d is growing too fast.
790. Try to make a number made off k digits that is divisible by 5^k.

791. Let D' be the reflection of D in X. L should be the reflection of D' in A^*.
792. P and Q are Antigonal conjugates in all triangles formed by A, B, C, I.
793. Denote by $d_c(v)$ the number of edges of colour c coming out of vertex v. If $d_c(v) + d_c(u) \le n - 1$, then it doesn't matter which colour is the edge uv.
794. Homothety at H also helps.
795. Let T_c be a spanning tree from the condition for every colour c. Consider a graph with all vertices of Γ. Add all edges of colour c from T_c to this graph. You have the correct number of edges as needed for the tree. Need to change some edges to edges of the same colour to get a tree in the end.
796. Thus, we will just go left, then right, then left again. And we are limited by the edges (kth cut and 1 cut). Prove that the sum of the numbers will be negative at some point, because we make infinitely many cancellations.
797. Lots of perpendiculars and intersections. Need to prove collinearity. Familiar?
798. Exponents of each of them cannot be unbounded.
799. $2q$ is good.
800. Radical axes.
801. Think about orders.
802. Reflect T to get W.
803. Easy to show that $x^2 + x + 1 \mid (x + 1)^{x+1} - x^x$ only for $6 \mid x - 1$.
804. We need m to be $\approx \sqrt{n}$.
805. In this triangle, this hyperbola is the image of perpendicular bisector. From this, we can get the angle condition on every point on it.
806. Prove that $\angle PDY = \angle ZDX$.
807. Each degree in every monomial is at most 2.
808. If a is the solution, then $2^a - 1$ is the solution.
809. Take root of unity of large power n and express the free coefficient of $x^{-k}p(x)$ for each k from 0 to $n - 1$ using root of unity filter (this coefficient will be the only summand) (allow zero coefficients).
810. X has equal power to Γ and some circle that is centered at I.
811. Multiply by $\prod_{i<j}(x_i - x_j)$ to get a multivariable polynomial.
812. The structure of the graph should be that we have vertices $v_1, v_2, \ldots, v_n$ coloured in colours $c_1, c_2, \ldots, c_n$ (some cs might be are equal) and v_iv_j is of the colour c_i for $i < j$.
813. Move to $\mathbb{F}_{p^2} = \mathbb{F}_p[\omega]$, where $\omega^2 = n$. We will also need the square root of -1 that exists in $\mathbb{F}_{p^2}$.
814. There's a lemma that states that in an infinite undirected graph, whatever the moves we perform with any distribution of chips, we will land on a configuration where the firing is impossible. If it was true for directed graphs, then the opposite to the problem statement would hold. Try to come up with a distribution and an infinite sequence of operations.

815. We can solve this problem by a one-liner AM-GM.

816. We should check $q \not\equiv 1 \pmod{p^2}$. Suppose that it's true for every divisor q.

817. $(n^2+n+1)^2+1=(n^2+1)((n+1)^2+1)$. So, we can multiply two consecutive terms of the sequence to get some other term.

818. Prove that it vanishes if some variable is one.

819. Just choose $n = p$.

820. One can prove that the ratio of two leading coefficients is rational.

821. $4-y^2=(2-y)(2+y)$ has to be quadratic residue.

822. Eliminate t from the equation.

823. $IS' = S'A$.

824. Induction. Avoid zero.

825. Let ΔPQR be a triangle formed by AA', BB', CC'. Let $AB \cap A'B' = D$. Prove that $DO \parallel PO'$, where O' is the circumcenter of (PQR). This is equivalent to SP tangent to (PQR), where $S = AB' \cap A'B$.

826. Now, the problem is about the parity of a permutation. Add an edge from m to k. This will be a permutation π.

827. $(2-y)(2+y)=z(4-z)$ for $z=y+2$.

828. The proof is an application of Erdős–Mordell inequality.

829. This inversion fixes the circumcircle of the small quadrilateral.

830. Binomial coefficients are divisible by p so we get one p for free.

831. Try to factor $(a+b)^p-a^p-b^p$ as a polynomial.

832. Roots of unity filter.

833. Use $\{1; 0; -1\}$.

834. Consider $(x,y) \to (x+my,y)$ and $(x,y) \to (x,xm+y)$.

835. Mirsky's theorem.

836. $Q_n^2 - A_n^2$ is $\frac{1}{n^2}$ times the sum of squared differences between each pair of as.

837. Try to decrease some numbers such that the sum in every rook set is zero.

838. Thus, $a=b$. Prove that $g(m)=a$ and then conclude that $f(n)=g(n)$ for all n.

839. Finish of the first way is to reduce the angle chase to $KK' \parallel CC'$, which is just the homothety.

840. Find a free row and a free column and consider the cell in their intersection.

841. Note that, if we started with n, then, at each step, we will have $\frac{an+b}{cn+d}$ for some integer a, b, c, d,

842. What is $\binom{p}{k}$ congruent to modulo p^2?

843. $2^{n+1} \mid p-1$ because no other 2^{2^m} gives the same residue modulo p.

844. Prove the converse, that this implies that M lies on the internal tangent.

845. Prove that it vanishes if some $x_a = x_b$.

846. Try to integrate $\frac{g(xt)}{t^p}$ with respect to t with good g from 0 to infinity. Choose a power p and a function g such that the resulting function is $\sqrt{|x|}$ times some constant.
847. After proving that, we have too many special cases and we have to prove the general statement.
848. If we add such an edge to a connected component, then we will find the desired cycle.
849. Proof is just case bash.
850. Use rough estimate $k \geq s \geq 9k$ first and something more accurate for small cases.
851. We can scale and shift numbers.
852. $g^{k+1}(n) + 1 = g(f(n) + k)$ for all non-negative k.
853. You should be proving that four points, including C, are concyclic. Prove that the perpendicular bisectors to three segments with an endpoint at C are concurrent. These perpendicular bisectors are well defined.
854. They are exactly the $a_i + jn$ for $1 \leq j \leq p - 1$ excluding pa_i.
855. Turns out their products are good too.
856. n, $n + |x|$, $n + |y|$, $n + |x| + |y|$ are of different colours.
857. Why does it imply that $P(x) = (x^2 + x + 1)^n$?
858. The number of representations (of both parities) using $\{s_1, s_2, \dots\}$ is at most the number of representations using $\{s_2, s_3, \dots\}$.
859. Make $k = m$ to have less degrees of freedom.
860. Induct by adding one prime at a time resulting in a good number.
861. Delete more edges if needed, but ensure that in each H_{n-1} the edges deleted are the same (i.e. corresponding) and no more than $n - 2$ deleted in each part.
862. After the inversion, median goes into symmedian.
863. Two different forms are $x^2 + 5y^2$ and $2x^2 + 2xy + 3y^2$.
864. If some number is representable, then we can subtract it from the sum of first n terms of the sequence and obtain other representable number (if n is such that all the terms of representation are contained within them).
865. $n \geq 2^k - 1$ but the equality doesn't hold if we try to restore the example. Restoring the example for $n \geq 2^k$ is easy.
866. $a = g^{\frac{p(p-1}{3}}$ and $b = 1$.
867. Try to find **some** injectivity.
868. Γ is just a circle through intersections of angle bisectors of angles of quadrilateral $XYDE$ (for some choice of X, Y, D, E).
869. Note that if we deleted $n - 2$ edges from one vertex in smaller H_{n-1} graph, then there might be no Hamiltonian cycle (cycle through each vertex). We need to find some other element.
870. You should find $f(x + 1) = f(x) + 1$.

871. We have to show that $(A'SLC)$. By Reim's theorem, it's the same as $AT' \parallel KL$, where $T' = A'S \cap (ABC)$. Reflections help again.

872. $$P(x) = \sum_{i=1}^{n} a_i \prod_{1 \le j \le n, j \ne i} (x - b_j).$$

873. Find $n^2 + 1 \mid m^2 + 1$ with $\frac{m}{n}$ small.

874. Primitive root.

875. Set (ABC) as the unit circle. P and Q are p and q.

876. Try to establish the recursion by raising 2 to the power of the previous number and doing some other manipulations.

877. Prove that they must partition the plane.

878. The only linear transformations that preserve the value of the discriminant are the ones of the form $(x, y) \to (px + ry, qx + sy)$ with $ps - qr = \pm 1$. They are also good because we can reverse this transformation.

879. Find a suitable square-free a and construct b.

880. Euclidean algorithm.

881. Think about $\frac{\sigma(j)-\sigma(i)}{j-i}$ for a permutation σ.

882. $4^2 \equiv -1 \pmod{17}$.

883. This is a problem about expected values.

884. Inversion at S with radius SD.

885. Think about rotating the line.

886. Use $25^n \ge n + 1$.

887. Convex hull.

888. $d(6^{2k} y_n^2 + 1) > 2^k d(y_n^2 + 1)$.

889. Using it, check that $T(x^9, y^9, z^9)$ is representable.

890. Combinatorial Nullstellensatz. Need to show that the difference of these two polynomials has a term that has every degree of $x_i \le m - 1$.

891. Prove that the midpoint of arc QR not containing A is the same as the midpoint of arc CD. Conclusion will follow.

892. Scan them with a line of fixed direction.

893. Prove a similar result for a curvilinear triangle that is formed by intersections of three circles.

894. Think about the radical axis of ω_1 and ω_2.

895. Vectors should sum to zero, and if they sum to zero, then there should be an equal number of each of them.

896. Try to establish some recurrence that is related to the maximum number of red circle in a path.

897. Consider a grid.
898. T bisects each of the segments.
899. Quadratic rational function is determined by a small number of points.
900. Restate the problem in terms of $ABCD$ and the cyclic quadrilateral formed by intersections of its consecutive internal angle bisectors.
901. Chinese Remainder Theorem.
902. Invert at M. Choose a convenient radius.
903. We already have the reflections.
904. Find a suitable linear combination to cancel out all extra terms.
905. If it was
$$\sum_{k=1}^{n} a_k(a_1 + \ldots + a_{k-1})^2 < \frac{1}{3},$$
then the problem would be a simple application of Riemann sums.
906. Double count using all pairs that both have the same value.
907. Mark midpoints of arc AB and BC not containing C and A, respectively.
908. External bisector is the polar of midpoint N of EF.
909. Take large primes dividing $Q(n) = P(a + mn)$ that exist due to Schur's theorem.
910. The solution is a rational function but, since we have a ring, we can clear the denominators.
911. Try to cancel some two terms by making their arguments equal.
912. Note that we can do it again and decrease the degree by 1. In the end, it's going to be $s(ax + b) + c \equiv 0 \pmod 2$.
913. Rewrite the angle condition.
914. Rewrite m and n as sums of something related to c_y and $|A_i|$. Then straight Hölder's suffices.
915. By saying that $a = bx$, we can only factor $(x + 1)^p - x^p - 1$.
916. Bound of $a_k(a_1+\ldots+a_{k-1})^2 <$ portion under the graph from $a_1+\ldots+a_{k-1}$ to $a_1+\ldots+a_k$ is very loose. Makes sense to think that the same holds for $\frac{a_k}{1-a_k}(a_1 + \ldots + a_{k-1})^2$.
917. These fractions are even for $d \neq 1$, 2^k, q^k, $2q^k$.
918. Make sure that the sum of tangents from M to two incircles equals to the length of their common internal tangent segment.
919. $\frac{(2x)^{2x-1}}{(x^x)^2} = 2^{2x-1}x$.
920. We need to prove that $ET \cap AP_c \in \Gamma$.
921. Use Butterfly theorem.
922. Values might be negative, so we have to transform (a, b) without changing the value.
923. Expected values.
924. Draw all the points (a_i, b_i).

925. By splitting the number in two, we should arrive at the binary representation with some coefficients. Which coefficients can you get?
926. The function is a combination of powers to 4 circles (the ones I mentioned before) such that it's zero on the external bisector of $\angle BAC$.
927. Prove that centroids are on one line parallel to the line through centers.
928. Make sure that every triangle of triangulation is non-monochromatic.
929. Problem looks feasible for induction if we drop the unit sum condition. Need to generalize for arbitrary sum.
930. Easy to control which side Alice takes for the first cut with $\neq 30$ on the sides.
931. The problem is pivoted around the circles centered at X and Y passing through C. Mark the intersections of the line through images of two vertices of the equilateral triangle with these circles.
932. Prove that DS also passes through T using symmetry.
933. Complex integration helps to clear all the non-zero powers.
934. Introduce partial order of red circles if c_2 can be reached from c_1.
935. Divide by $\gcd(a, b)$ first.
936. It's true that two convex shapes that don't intersect can be separated by a line that doesn't intersect any of them. This is called a two dimension case of Hyperplane separation theorem.
937. Resulting shape is a loop. We need to prove that it's self intersecting, i.e. contains a subloop. Relate dazzling vertices to it.
938. Erdős–Szekeres theorem.
939. Multiply them.
940. Suppose that the finishing point is $(m, n+1-m)$. Consider, where we have to go from each $(1, n)$, $(2, n-1)$, $(3, n-2)$ and so on.
941. Which values can we calculate from the above hint.
942. Set up some three variable equation and rewrite both sides of it.
943. We can find three algebraic formulas that prove that their product is of the same form.
944. Pell's equation with 2.
945. Delete c from the cycle and add a bridge from a tree.
946. Label cells with coordinates. We want to control which number is written in the cell based on it coordinates.
947. $(a-b)(a+b) = (z-x)(z-y)$.
948. Do it first with $a, b < p^2$.
949. Squeeze the product between two neighbouring squares.
950. Change the angle equality statement to something more projective.
951. Fix some $c = P(k)$. Find n such that $v_p(P(n)) = v_p(P(k))$ for all primes up to some m.

952. Consider an even representation of $s_{i+1} - s_i$ to prove some bound.
953. If $p \nmid 2m$ and $p \mid k^{2m} - 1$, then one of $v_p(k^{2m} - 1)$ or $v_p((k+p)^{2m} - 1)$ is one.
954. First 7 hints are the same.
955. If there are only red vertices in a 2 red 1 blue triangle, then connect them to one of its red vertices. Same for only blues in a 2 blue 1 red triangle.
956. A_1C_1 is the polar of X.
957. When all trees contain it, just merge the endpoints of e and induct.
958. You will end up with a new equation involving sums of digits of polynomials such that the largest degree polynomial is unique and its degree decreased by 1 comparing to original.
959. $Q(x) = x^4 + x^3 + x + 1$ almost works except it has a zero coefficient. Think how we can adjust that.
960. Apply it 17 times.
961. Think about incenters or excenters of triangles formed in the picture.
962. Casey's theorem for circles of zero radius.
963. Big primes divide only one term of the sequence.
964. We should be able to control the order of p.
965. With bounded coefficients and finite degree, there's only finitely many polynomials.
966. Fix this special choice of V. And perform moving points again, moving the point on AB. A and B are obvious and we need one more special case.
967. Use the fact that circle is fixed to finish the original problem.
968. Mark $AT \cap \omega$.
969. Prove that the sum of two cells is equal to the sum of two cells that form a rectangle with first two cells (with sides parallel to sides of the original).
970. Homothety at D with coefficient 2 helps.
971. Now, $\angle MCA = \angle MBC \geq \angle MAB$.
972. Project through O.
973. First five hints are the same as in the first solution.
974. $x = \frac{a}{b}$, $y = \frac{b}{c}$, $z = \frac{c}{a}$.
975. Image of H' after the homothety is the first center of orthology and H is the second.
976. Flip rows and columns using the exchanges.
977. Think about the condition for a monomial to be able to write it as $R(x^2y, y^2z, z^2x)$.
978. Using the Excenter-Incenter lemma, prove that it's the same the line through midpoints of arcs AB and CD not containing C and A, respectively.
979. Use similar recursion.
980. Hamming distance = 1.
981. RHS now is a function of c. Find its minimum.

982. Consider the number of ways to do so for each bad matching. It should be odd.
983. Look at complex factorizations.
984. $g(x)$ is a polynomial.
985. $6 \mid m$.
986. Find one working a of the form pq.
987. Make a multiplicative, i.e. $a_{mn} = a_m a_n$.
988. Consider $t^{\frac{p-1}{2}} - i$.
989. It will be equal to product of small primes in bounded exponents.
990. Divide by p.
991. Intersect MB and MA with the circle and use sine law.
992. Use $f(f(x)) = x + 1$ here.
993. Thus, we always decrease the number of components. Ensure that this process can continue indefinitely.
994. LHS is the sum of $a_{xi}a_{yi}a_{yj}$ over all $1 \le i, j \le m$, $1 \le x, y \le n$.
995. Subtract $(n \pmod 2) \prod_{i<j}(x_i - x_j)$. We want to prove that the new polynomial is identically zero.
996. Use Hall's lemma.
997. $z - x = a$, $z - y = b$.
998. To construct the example, try to split the red circles in groups such that two circles from the same group cannot be chosen.
999. Hall's lemma.
1000. Fix x. Consider $P(x + 10^k)$ for large k. And see how it differs from $P(x)$
1001. Finish with Euler's theorem for the planar graph to calculate the number of regions. Each point is in four regions.
1002. Apply the original for $b_i = \frac{a_i}{S}$, where $S = a_1 + \ldots + a_n$.
1003. X is also the exsimilicenter of (B) and (D).
1004. $x^2 + x + 1 \mid x^m - 1$ and $(x+1)^{m+1} - 1$.
1005. Suppose that 2^n circles can be covered by n antichains. Suppose that $1 = k_1 < k_2 < \ldots < k_n$ are the top rows of circles in these antichains.
1006. Complete residue system modulo m.
1007. Draw a complete quadrilateral and think about the Gauss line.
1008. You can move to the 4th hint from the first solution. Next hints are about a different approach to the same step.
1009. We want to prove only one equality.
1010. Forward differences.
1011. We can bound the number of non-yellow points because one of the cliques has at least $\lceil \frac{2061}{2} \rceil$ vertices. There will be too few of them for not a single all-yellow region to appear. But we need a better bound on the number of non-yellow points in each region.

1012. Construct the solution recursively.

1013. Think about some equation that eliminates one of the variables.

1014. Look at the arrow from b to $c \geq b$. There also must be an arrow from $\geq a$ to this c because the image of $g(a+k)$ is exactly $\{b+1, b+2, \ldots\}$.

1015. We want to find a polynomial $Q(x)$ such that it's easy to control what changes in $Q(10^m)$ and $P(Q(10^m))$ for large enough m.

1016. This should sound like the condition of some famous lemma.

1017. Limit x to zero in the original. It gives f is linear.

1018. We want to calculate $\prod(j^2 + ij + i^2)$.

1019. If $P(x) = x^2 + x + 1$, what can you say about m?

1020. $A'A$ and $A'S$ are reflections in $A'I$.

1021. Try to describe all the possible quadruples

1022. Function is not periodic.

1023. We need to set some counter on red, blue and white vertices. Its remarkable property is that it doesn't matter where to start it from.

1024. On the other hand, all r_i are distinct (prove it). Thus, we can calculate the limit directly.