

## NOTES

IDLE

Hello!

This note is mainly used to record some trivial but interesting problems that I encounter in daily life.



FIGURE 1. Night City

## CONTENTS

1.	2023.12.22	3
1.1.	identification of co-character groups with Lie algebras	3
1.2.	pinnings of algebraic groups	3
1.3.	coherent continuation representations	7
2.	2023.12.23	8
2.1.	review of regular parameter	8
2.2.	outer automorphism of algebraic groups	9
3.	2023.12.26	10
4.	2024.2.27	11
5.	2024.2.28	12
6.	2024.10.22	13
	References	13

## 1. 2023.12.22

**1.1. identification of co-character groups with Lie algebras.** Settings:  $G$  complex connected reductive algebraic group, fix a splitting datum  $(B, H, \{X_\alpha\})$ .

There is an identification

$$\begin{aligned} I : X_*(H) \otimes \mathbb{C} &\longrightarrow \mathfrak{h} \\ \phi &\longmapsto d\phi(1) \end{aligned}$$

The weight lattice for  $G$  is  $P = \{\lambda \in X^*(H) \otimes \mathbb{C} \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$

The co-weight lattice is  $\check{P} = \{\check{\lambda} \in X_*(H) \otimes \mathbb{C} \mid \langle \alpha, \check{\lambda} \rangle \in \mathbb{Z} \text{ for all } \alpha \in \check{\Delta}\}$

Under this identification we have

$$\begin{aligned} &\{\check{\lambda} \in \mathfrak{h} \mid \exp(\check{\lambda}) = 0\} \\ &= \{\check{\lambda} \in X_*(H) \otimes \mathbb{C} \mid \phi(\exp(d\check{\lambda}(1))) = 1 \text{ for all } \phi \in X^*(H)\} \\ &= \{\check{\lambda} \in X_*(H) \otimes \mathbb{C} \mid \exp(\langle \phi, \check{\lambda} \rangle) = 1 \text{ for all } \phi \in X^*(H)\} \\ &= \{\check{\lambda} \in X_*(H) \otimes \mathbb{C} \mid \langle \phi, \check{\lambda} \rangle \in 2\pi i\mathbb{Z} \text{ for all } \phi \in X^*(H)\} \\ &= 2\pi i X_*(H). \end{aligned}$$

And similarly

$$\check{P} = \{\check{\lambda} \in \mathfrak{h} \mid \exp(2\pi i \check{\lambda}) \in Z(G)\}$$

There is also an identification

$$\begin{aligned} J : X^*(H) \otimes \mathbb{C} &\longrightarrow \mathfrak{h}^* \\ \varphi &\longmapsto d\varphi \end{aligned}$$

Under this identification ... I forget what I want to say.

## 1.2. pinnings of algebraic groups.

**Theorem 1.1** (see AdC18). *Inner automorphism group  $\text{Inn}(G)$  of  $G$  is equal to  $G/Z(G)$ ,  $\text{Inn}(G)$  acts freely and transitively on the set of pinnings.*

I know that there is an analogy in compact connected groups, and it has been proved.

**Theorem 1.2.** *If  $G$  is a compact connected group, then its inner automorphism group  $\text{Inn}(G) = G/Z(G)$  acts freely and transitively on the set of pinnings. (where pinnings for it is a bit different from the complex case, that  $(T, B_{\mathbb{C}}, \mathcal{X})$  is a pinning if  $T$  is a maximal torus,  $B_{\mathbb{C}}$  a borel of  $G_{\mathbb{C}}$  containing  $T$ ,  $\mathcal{X}$  is a set of real rays in simple root space.)*

*Proof.* Here's proof from mathoverflow Lspise's answer:

To prove this, I'll use a few pieces of structure theory:

1. All maximal tori in  $G$  are  $G^\circ$ -conjugate.
2. All Borel subgroups of  $G_{\mathbb{C}}$  are  $G_{\mathbb{C}}^\circ$ -conjugate.
3. For every maximal torus  $T$  in  $G$ , the map  $W(G^\circ, T) \rightarrow W(G_{\mathbb{C}}^\circ, T_{\mathbb{C}})$  is an isomorphism.
4. If  $G_{\text{sc}}$  and  $(G_{\mathbb{C}})_{\text{sc}}$  are the simply connected covers of the derived groups of  $G^\circ$  and  $G_{\mathbb{C}}^\circ$ , then  $(G_{\text{sc}})_{\mathbb{C}}$  equals  $(G_{\mathbb{C}})_{\text{sc}}$ .
5. [Every compact Lie group has a finite subgroup that meets every component.](#)

I only need (4) to prove that, for every maximal torus  $T$  in  $G$ , the map from  $T$  to the set of conjugation-fixed elements of  $T/Z(G^\circ)$  is surjective. This is probably a well known fact in its own right for real-group theorists.

Now consider triples  $(T, B_{\mathbb{C}}, \mathcal{X})$  as follows:  $T$  is a maximal torus in  $G$ ;  $B_{\mathbb{C}}$  is a Borel subgroup of  $G_{\mathbb{C}}^\circ$  containing  $T_{\mathbb{C}}$ , with a resulting set of simple roots  $\Delta(B_{\mathbb{C}}, T_{\mathbb{C}})$ ; and  $\mathcal{X}$  is a set consisting of a real ray in each complex simple root space (i.e., the set of positive real multiples of some fixed non-0 vector). (Sorry about the pair of modifiers "complex simple".) I will call these 'pinnings', although it doesn't agree with the usual terminology (where we pick individual root vectors, not rays). I claim that  $G^\circ/Z(G^\circ)$  acts simply transitively on the set of pinnings.

Once we have transitivity, freeness is clear: if  $g \in G^\circ$  stabilises some pair  $(T, B_C)$ , then it lies in  $T$ , and so stabilises every complex root space; but then, for it to stabilise some choice of rays  $\mathcal{X}$ , it has to have the property that  $\alpha(g)$  is positive and real for each simple root  $\alpha$ ; but also  $\alpha(g)$  is a norm-1 complex number, hence trivial, for each simple root  $\alpha$ , hence for each root  $\alpha$ , so that  $g$  is central.

For transitivity, since (1) all maximal tori in  $G$  are  $G^\circ$ -conjugate, so (2) for every maximal torus  $T$  in  $G$ , the Weyl group  $W(G_C^\circ, T_C)$  acts transitively on the Borel subgroups of  $G_C^\circ$  containing  $T_C$ , and (3)  $W(G^\circ, T) \rightarrow W(G_C^\circ, T_C)$  is an isomorphism, it suffices to show that all possible sets  $\mathcal{X}$  are conjugate. Here's the argument that I came up with to show that they are even  $T$ -conjugate; I think it can probably be made much less awkward. Fix a simple root  $\alpha$ , and two non-0 elements  $X_\alpha$  and  $X'_\alpha$  of the corresponding root space. Then there are a positive real number  $r$  and a norm-1 complex number  $z$  such that  $X'_\alpha = rzX_\alpha$ . Choose a norm-1 complex number  $w$  such that  $w^2 = z$ . There is then a unique element  $s_{\text{ad}}$  of  $T_C / Z(G_C^\circ)$  such that  $\alpha(s_{\text{ad}}) = w$ , and  $\beta(s_{\text{ad}}) = 1$  for all simple roots  $\beta \neq \alpha$ . By (4), we can choose a lift  $s_{\text{sc}}$  of  $s_{\text{ad}}$  to  $(G_{\text{sc}})_C = (G_C)_{\text{sc}}$ , which necessarily lies in the preimage  $(T_C)_{\text{sc}}$  of (the intersection with the derived subgroup of)  $T$ , and put  $t_{\text{sc}} = s_{\text{sc}} \cdot \overline{s_{\text{sc}}}$ . Then

$$\alpha(t_{\text{sc}}) = \alpha(s_{\text{sc}})\overline{\alpha(s_{\text{sc}})} = \alpha(s_{\text{sc}})\overline{\alpha(s_{\text{sc}})^{-1}} = w \cdot \overline{w^{-1}} = z,$$

and, similarly,  $\beta(t_{\text{sc}}) = 1$  for all simple roots  $\beta \neq \alpha$ . Now the image  $t$  of  $t_{\text{sc}}$  in  $G_C^\circ$  lies in  $T_C$  and is fixed by conjugation, hence lies in  $T$ ; and  $\text{Ad}(t)X_\alpha = zX_\alpha$  lies on the ray through  $X'_\alpha$ .

Since  $G$  also acts on the set of pinnings, we have a well defined map  $p : G \rightarrow G^\circ / Z(G^\circ)$  that restricts to the natural projection on  $G^\circ$ . Now  $\ker(p)$  meets every component, but it contains  $Z(G^\circ)$ , so it need not be finite. Applying (5) to the Lie group  $\ker(p)$  yields the desired subgroup  $H$ . Note that, as requested in your [improved classification](#), conjugation by any element of  $H$  fixes a pinning, hence, if inner, must be trivial.

□

**Remark 1.3.** *The pinnings show that split reductive algebraic groups look like butterflies!*

According to Milne, Grothendieck used the following analogy to talk about the structure theory of algebraic groups. An algebraic group is like a butterfly. The body of the butterfly is the maximal torus. The wings are two opposite Borel subgroups. And the pins are the pinning maps.



FIGURE 2. pinning of butterfly

**1.3. coherent continuation representations.** Until the end of today,  $G$  denotes a real reductive group in Harish-Chandra class.

Let's recall some basic definitions of coherent continuation representations, in order for me to do calculations in the case of real classical groups.

Fix a connected reductive complex Lie group  $G_{\mathbb{C}}$ , together with a Lie group homomorphism  $\iota : G \rightarrow G_{\mathbb{C}}$  such that its differential  $d\iota : \text{Lie}(G) \rightarrow \text{Lie}(G_{\mathbb{C}})$  has the following two properties:

- (1) the kernel of  $d\iota$  is contained in the center of  $\text{Lie}(G)$ ;
- (2) the image of  $d\iota$  is a real form of  $\text{Lie}(G)$ .

The analytical weight lattice of  $G_{\mathbb{C}}$  is identified with a subgroup of  ${}^a\mathfrak{h}^*$  via  $d\iota$ . We write  $Q_{\iota} \subset {}^a\mathfrak{h}^*$  for this subgroup. Denote root lattice by  $Q_{\mathfrak{g}}$ , weight lattice by  $Q^{\mathfrak{g}}$ , then

$$Q_{\mathfrak{g}} \subset Q_{\iota} \subset Q^{\mathfrak{g}}.$$

they are all  $W$  stable subgroups. And we have some categories:

- (1)  $\text{Rep}(\mathfrak{g}, Q_{\iota})$  the category of all finite-dimensional representations of  $\mathfrak{g}$  whose weight are contained in  $Q_{\mathfrak{g}}$ , **it can be identified with  $\text{Rep}(\text{Lie}(G_{\mathbb{C}}), Q_{\iota})$  and hence  $\text{Rep}_{\text{finite}}(G_{\mathbb{C}})$** ;
- (2)  $\text{Rep}(G)$  the category of all Casselman-Wallach representations of  $G$ , and its full subcategories  $\text{Rep}_{\mu}(G)$ ,  $\text{Rep}_{\mathfrak{s}}(G)$ .

To denote their Grothendieck groups, we replace  $\text{Rep}$  by  $\mathcal{K}$ .

Fix a  $Q_{\iota}$ -coset  $\Lambda = \lambda + Q_{\iota} \subset {}^a\mathfrak{h}^*$ .

**Definition 1.4** (coherent family). *A  $\mathcal{K}(G)$ -valued  $\Lambda$  coherent family is a map*

$$\Phi : \Lambda \rightarrow \mathcal{K}(G)$$

*satisfies the following two conditions:*

- (1) *for all  $\nu \in \Lambda$ ,  $\Phi(\nu) \in \mathcal{K}_{\nu}(G)$ ;*
- (2) *for all representations  $F \in \text{Rep}(\mathfrak{g}, Q_{\iota})$  and all  $\nu \in \Lambda$ ,*

$$F \cdot (\Phi(\nu)) = \sum_{\mu} \Phi(\nu + \mu),$$

*where  $\mu$  runs over all weights of  $F$ , counted with multiplicities.*

*The set of coherent families is denoted by  $\text{Coh}_{\Lambda}(\mathcal{K}(G))$*

We have the integral weyl group  $W_{\Lambda} = \{w \in W \mid w\lambda - \lambda \in Q_{\iota}\}$  there is a representation of  $W_{\Lambda}$  on  $\text{Coh}_{\Lambda}(\mathcal{K}(G))$  by

$$(w \cdot \Phi)(\nu) = \Phi(w^{-1}\nu).$$

This is called a coherent continuation representation.



## 2. 2023.12.23

**2.1. review of regular parameter.** Following yesterday's discussion of coherent continuation representations, we define two basis for it, use langlands classification.

Suppose  $H$  is a Cartan subgroup of  $G$ , by which we mean a subgroup that is the centralizer of a Cartan subalgebra of  $\text{Lie}(G)$  (In the case of linear groups, it's just a real algebraic torus.).  $H$  has a unique maximal compact subgroup (since  $G$  is in Harish-Chandra's class, adjoint representation of  $H$  is trivial,  $H_0$  lies in the center of  $H$ , as we know, all maximal subgroups of Lie groups are conjugate to each other, use Cartan decomposition of  $H$ , we win. In fact,  $H$  has a unique maximal compact torus and a unique maximal split torus) We denote by  $\Delta_{\mathfrak{h}} \subset \mathfrak{h}^*$  the root system of  $\mathfrak{g}$ . A root is called imaginary if  $\check{\alpha} \in \mathfrak{t}$  (i.e. roots which take pure imaginary values on  $\text{Lie}(H)$ ), an imaginary root  $\alpha$  is called compact if  $\mathfrak{g}_{\alpha}$  and  $\mathfrak{g}_{-\alpha}$  is contained in the complexified Lie algebra of a common compact subgroup of  $G$ . (this is equivariant to  $\mathfrak{g}_{\alpha}$  belong to a Lie subalgebra of a compact subgroup, since  $\mathfrak{g}_{\alpha}$  lies in a Lie subalgebra of maximal compact subgroup is equivalent to  $\mathfrak{g}_{-\alpha}$  Lies in the same subgroup)

There are two important facts about the representation theory of  $H$ :

- (1) Every Casselman-Wallach representation of  $H$  is finite-dimensional, it follows directly from that  $H$  is an extension of an abelian group and a finite group;
- (2) For every  $\Gamma \in \text{Irr}(H)$  differential of  $\Gamma$  is a direct sum of one-dimensional representations attached to a unique  $d\Gamma \in \mathfrak{h}^*$ . (it is true because  $H$  is in Harish-Chandra's class).

For every Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{g}$  containing  $\mathfrak{h}$ , write

$$\xi_{\mathfrak{b}} : \mathfrak{h} \rightarrow {}^a\mathfrak{h}$$

for the linear isomorphism attached to  $\mathfrak{b}$  defined by

$$\mathfrak{h} \subset \mathfrak{b} \rightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = {}^a\mathfrak{h}.$$

the transpose inverse of this map is still denoted by  $\xi_{\mathfrak{b}} : {}^a\mathfrak{h}^* \rightarrow \mathfrak{h}^*$ .

Write

$$W({}^a\mathfrak{h}^*, \mathfrak{h}^*) := \{\xi_{\mathfrak{b}} : {}^a\mathfrak{h}^* \rightarrow \mathfrak{h}^* \mid \mathfrak{b} \text{ is a Borel subalgebra containing } \mathfrak{h}\}.$$

**Definition 2.1.** For every element  $\xi \in W({}^a\mathfrak{h}^*, \mathfrak{h}^*)$ , put

$$\delta(\xi) := \frac{1}{2} \cdot \sum_{\alpha \text{ is an imaginary root in } \xi\Delta^+} \alpha - \sum_{\beta \text{ is an compact imaginary root in } \xi\Delta^+} \beta \in \mathfrak{h}^*.$$

Write  $\mathcal{P}_{\Lambda}(G)$  for the set of all triples  $\gamma = (H, \xi, \Gamma)$ , where  $H$  is a Cartan subgroup of  $G$ ,  $\xi \in W({}^a\mathfrak{h}^*, \mathfrak{h}^*)$ , and

$$\begin{array}{ccc} \Gamma : & \Lambda & \longrightarrow \text{Irr}(H) \\ & \nu & \longmapsto \Gamma_{\nu} \end{array}$$

is a map with the following properties:

- (1)  $\Gamma_{\nu+\beta} = \Gamma_{\nu} \otimes \xi(\beta)$  for all  $\beta \in Q_{\iota}$  and  $\nu \in \Lambda$ ;
- (2)  $d\Gamma_{\nu} = \xi(\nu) + \delta(\xi)$  for all  $\nu \in \Lambda$ . (note that Cartan subgroups not always have irr repn with this differential!)

Here  $\xi(\beta)$  is naturally viewed as a character of  $H$  by using the homomorphism  $\iota : H \rightarrow H_{\mathbb{C}}$ , and  $H_{\mathbb{C}}$  is a Cartan subgroup of  $G_{\mathbb{C}}$  containing  $\iota(H)$ .



## 2.2. outer automorphism of algebraic groups.

**Lemma 2.2** (a simple discover of abstract algebra).  *$G$  an abstract group,  $H$  an normal subgroup of  $G$ , if  $G$  acts transitively on a set  $X$  such that the restriction of this action to  $H$  is free and transitive, then the short exact sequence of groups*

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$$

*Proof.* Actually choose an element  $x \in X$ , we have

$$G \simeq \text{Stab}_G(x) \ltimes H.$$

□

Using this lemma, we can see that for a complex connected reductive algebraic group  $G$ ,  $\text{Aut}(G)$  acts on the set of pinning transitivity, with its normal subgroup the  $\text{Inn}(G)$  acts freely and transitively, so every pinning gives us a split of the short exact sequence

$$1 \rightarrow \text{Inn}(G) \rightarrow \text{Aut}(G) \rightarrow \text{Out}(G) \rightarrow 1.$$

Some facts about split connected reductive algebraic groups over arbitrary field  $k$ :

- (1)  $\{\text{split connected reductive algebraic groups}\}/\sim = \{\text{root datums}\}/\sim$ ;
- (2) For any two automorphisms of  $G$ , if they induce the same automorphism of root datum, then they differ by an inner automorphism;
- (3) Fix a pinning, we get an isomorphism  $\text{Out}(G) \simeq \text{Aut}(D_b)$ .

3. 2023.12.26

**Theorem 3.1** (Harish-Chandra's parameter of discrete series repn).  *$G$  be a connected semisimple Lie group (maybe there is an analogy for real reductive group),  $K$  a maximal torus of  $G$ , suppose that  $\text{rank}(K) = \text{rank}(G)$ , fix a compact maximal torus  $T \in K$  and positive system  $\Delta^+$ . Then for any analytically integral weight  $\lambda + \rho$ , there exist a discrete series representation  $\pi_\lambda$ , such that:*

- (1) *The infinitesimal character of  $\pi_\lambda$  is  $\lambda$ ;*
- (2)  *$\nu = \lambda + \rho - 2\rho_c$  is the highest weight of a minimal  $K$  type of  $\pi_\lambda$ ;*
- (3) *If  $\mu$  is any highest weight of irreducible  $K$  representation exist in  $\pi_\lambda$ , then*

$$\mu = \nu + \sum_{\alpha \in \Delta^+} n_\alpha \alpha$$

Moreover,  $\pi_\lambda \simeq \pi_{\lambda'}$  if and only if  $\lambda$  and  $\lambda'$  lies in the same  $W_c$  orbit.

In the above theorem,  $\rho - 2\rho_c$  is canonically realized as a continuous character of double

cover of  $T$ , called  $\tilde{T}$ , which can be realized as a pull-back diagram:

$$\begin{array}{ccc} \tilde{T} & \longrightarrow & \mathbb{C}^\times \\ \downarrow & & \downarrow z \mapsto z^2 \\ T & \xrightarrow{2\rho - 4\rho_c} & \mathbb{C}^\times \end{array}$$

Similarly,  $\lambda$  is also a character of a double cover of  $T$ , with respect to  $-2\lambda$ , but the difference of  $-2\lambda$  and  $2\rho - 4\rho_c$  is a square of a character of  $T$ , the two double covers are canonically isomorphic.

**In fact, there is an one to one correspondence! which I finally get it:**

$$\{\text{genuie characters of } \tilde{T}_{\rho_{im}}\} \simeq \{(\Gamma, \nu) \in X^*(T) \times \mathfrak{t}^* \mid d\Gamma = \nu + \rho_{im} - 2\rho_{im,c}\}.$$

the correspondence is given by

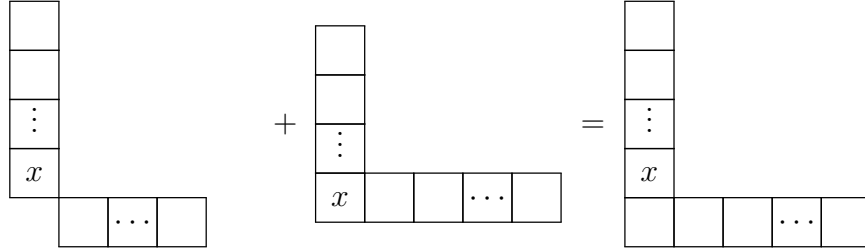
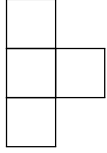
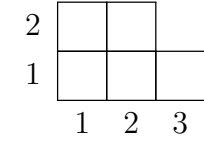
$$\begin{aligned} \chi &\mapsto (\chi \otimes (\rho_{im} - 2\rho_{im,c}), d\chi) \\ (\Gamma, \nu) &\mapsto \Gamma \otimes (2\rho_{im,c} - \rho_{im}) \end{aligned}$$

**Definition 3.2** (Langlands parameter). *Let  $G$  be a linear real reductive group, the Langlands parameters for  $G$  is a triple  $(H, \gamma, R_{i\mathbb{R}}^+)$ . Such that*

- (1) Hladký á

## 4. 2024.2.27

Today, I learned how to draw the Young tableau.



For a compact connected Lie group  $G$ , choose a maximal torus  $T$ , there is a canonical 1-1 correspondence:

$$X^*(T)/W \longleftrightarrow (X^*(T) + \rho)/W \longleftrightarrow \text{Irr}(G)$$

Moreover, if we choose a positive system  $\Psi^+$ , and then  $\rho_{\Psi^+}$ , the above sets also canonically isomorphic to the set of dominant analytically integral weights:

$$\{\lambda \in X^*(T) \mid \langle \lambda, \alpha \rangle \geq 0, \text{ for all } \alpha \in \Psi^+\}$$

This can be generalized to parameter of discrete series representations for connected semisimple Lie groups  $G$  with compact Cartan subgroup  $T$  :

$$(X^*(T) + \rho)^{reg}/W_c$$

If a discrete series representation  $V$  has minimal  $K$  type which has maximal weight  $\nu$ , then it's parameter is  $\lambda = \nu - \rho + 2\rho_c$ . (not sure)

Note that for a fixed infinitesimal character, there are exactly  $|W/W_c|$  different discrete series representations.

## 5. 2024.2.28

Some facts about continuous complex characters of some common abelian topological groups:

field	character group	correspondence
$(\mathbb{R}_+^\times, \times)$	$\mathbb{C}$	$\xi \mapsto (t \mapsto t^\xi)$
$(\mathbb{C}^\times, \times)$	$\mathbb{C} \times \mathbb{Z}$	$(\xi, n) \mapsto (z \mapsto  z ^\xi (\frac{z}{ z })^n)$
$(\mathbb{S}^1, \times)$	$\mathbb{Z}$	$n \mapsto (z \mapsto z^n)$
$(\mathbb{Q}_p, +)$	$\mathbb{Q}_p$	$\xi \mapsto (x \mapsto \exp(2\pi i \xi \{x\}_p))$

**Remark 5.1.** *Note that all continuous characters of  $\mathbb{Q}_p$  are automatically locally constant(smooth), which is consistent with the case of real lie groups.*

6. 2024.10.22

**Lemma 6.1.** *The associated variety of an irreducible Casselman-Wallach representation of a real reductive group in Harish-Chandra class is the closure of a nilpotent orbit in  $\mathfrak{g}^*$ .*

*Proof.* By the work of Borho, Brylinski [BB], and Joseph [Jos], the associated variety of any primitive ideal (annihilator of an irreducible  $\mathfrak{g}$ -module) is the closure of a single nilpotent orbit. So we only need to prove that  $\text{gr}(\text{Ann}_{\mathcal{U}(\mathfrak{g})}(V))$  is a primitive ideal.

Since the  $K$ -finite part  $V_K$  is dense in  $V$ ,  $\text{Ann}_{\mathcal{U}(\mathfrak{g})}(V) = \text{Ann}_{\mathcal{U}(\mathfrak{g})}(V_K)$ , we only need to consider the annihilator of the irreducible  $(\mathfrak{g}, K)$ -module  $V_K$ .

If  $G$  is connected, then  $K$  is also connected, so  $V_K$  is irreducible as  $\mathcal{U}(\mathfrak{g})$ -module, in this case,  $\text{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$  is a primitive ideal.

If  $G$  is not connected, let  $G_0$  be the identity component of  $G$ ,  $K_0 = G_0 \cap K$  is a maximal compact subgroup of  $G_0$ , then  $V_K|_{G_0}$  is a finite length  $(\mathfrak{g}, K_0)$ -module, it has an irreducible quotient  $V_K \rightarrow V_0$ , then according to the theory of induction of  $(\mathfrak{g}, K)$ -modules [KV, Chapter 2] we have an injective morphism of  $(\mathfrak{g}, K_0)$ -modules:

$$(6.1) \quad V_K|_{K_0} \hookrightarrow (\text{induced}_{K_0}^K(V_0))|_{K_0} \cong \bigoplus_{\text{double cosets } K_0 k K_0} kV_0,$$

where  $kV_0$  be the  $(\mathfrak{g}, K_0)$ -module whose underlying space is  $V_0$ , whose action by  $X \in \mathfrak{g}$  is the usual action by  $\text{Ad}(k)X$ , whose action by  $g \in K_0$  is the usual action by  $k^{-1}gk$ , it has the same annihilator in  $\mathcal{U}(\mathfrak{g})$  as  $V_0$  since  $G$  is in Harish-Chandra's class, and hence the annihilator of  $V_K$  is the same as the annihilator of  $V_0$ , which is a primitive ideal.

□

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