NOTES

IDLE

Hello!

This note is mainly used to record some trivial but interesting problems that I encounter in daily life.

For as the heavens are higher than the earth, so are my ways higher than your ways and my thoughts than your thoughts.—Isaiah 55:9

Figure 1. Night City

Contents

1. 2023.12.22

1.1. Identification of co-character groups with Lie algebras. Settings: G complex connected reductive algebraic group, fix a splitting datum $(B, H, \{X_\alpha\})$.

There is an identification

$$\begin{array}{rccc} I: & X_*(H)\otimes\mathbb{C} & \longrightarrow & \mathfrak{h} \\ & \phi & \longmapsto & d\phi(1) \end{array}$$

The weight lattice for G is $P = \{\lambda \in X^*(H)\otimes\mathbb{C} \mid \langle\lambda, \check{\alpha}\rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$

The co-weight lattice is $\check{P} = \{\check{\lambda} \in X_*(H)\otimes\mathbb{C} \mid \langle\alpha, \check{\lambda}\rangle \in \mathbb{Z} \text{ for all } \ \check{\alpha} \in \check{\Delta}\}$

Under this identification we have

$$\begin{aligned} &\{\check{\lambda} \in \mathfrak{h} \mid exp(\check{\lambda}) = 0\} \\ &\quad = \{\check{\lambda} \in X_*(H)\otimes\mathbb{C} \mid \phi(exp(d\check{\lambda}(1))) = 1 \text{ for all } \phi \in X^*(H)\} \\ &\quad = \{\check{\lambda} \in X_*(H)\otimes\mathbb{C} \mid exp(\langle\phi, \check{\lambda}\rangle) = 1 \text{ for all } \ \phi \in X^*(H)\} \\ &\quad = \{\check{\lambda} \in X_*(H)\otimes\mathbb{C} \mid \langle\phi, \check{\lambda}\rangle \in 2\pi i\mathbb{Z} \text{ for all } \phi \in X^*(H)\} \\ &\quad = 2\pi i X_*(H). \end{aligned}$$

And similarly

$$\check{P} = \{\check{\lambda} \in \mathfrak{h} \mid exp(2\pi i\check{\lambda}) \in Z(G)\}$$

There is also an identification

$$\begin{array}{rccc} J: & X^*(H)\otimes\mathbb{C} & \longrightarrow & \mathfrak{h}^* \\ & \varphi & \longmapsto & d\varphi \end{array}$$

Under this identification ... I forget what I want to say.

1.2. Pinnings of algebraic groups.

Theorem 1.1 (see AdC18)**.** *Inner automorphism group* Inn(G) *of G is equal to* G/Z(G)*,* Inn(G) *acts freely and transitively on the set of pinnings.*

I know that there is an analogy in compact connected groups, and it has been proved.

Theorem 1.2. *If* G *is a compact connected group, then its inner automorphism group* Inn(G) = G/Z(G) *acts freely and transitively on the set of pinnings. (where pinnings for it is a bit different from the complex case, that $(T, B_{\mathbb{C}}, \mathcal{X})$ is a pinning if T is a maximal torus, $B_{\mathbb{C}}$ a Borel of $G_{\mathbb{C}}$ containing T, $\mathcal{X}$ is a set of real rays in simple root space.)*

Proof. Here's proof from mathoverflow Lspise's answer:

To prove this, I'll use a few pieces of structure theory:

1. All maximal tori in G are G°-conjugate.
2. All Borel subgroups of $G_{\mathbb{C}}$ are $G^\circ_{\mathbb{C}}$-conjugate.
3. For every maximal torus T in G, the map $\mathrm{W}(G^\circ, T) \to \mathrm{W}(G^\circ_{\mathbb{C}}, T_{\mathbb{C}})$ is an isomorphism.
4. If G_{sc} and $(G_{\mathbb{C}})_{\mathrm{sc}}$ are the simply connected covers of the derived groups of G° and $G^\circ_{\mathbb{C}}$, then $(G_{\mathrm{sc}})_{\mathbb{C}}$ equals $(G_{\mathbb{C}})_{\mathrm{sc}}$.
5. Every compact Lie group has a finite subgroup that meets every component.

I only need (4) to prove that, for every maximal torus T in G, the map from T to the set of conjugation-fixed elements of $T/\,\mathrm{Z}(G^\circ)$ is surjective. This is probably a well known fact in its own right for real-group theorists.

Now consider triples $(T, B_{\mathbb{C}}, \mathcal{X})$ as follows: T is a maximal torus in G; $B_{\mathbb{C}}$ is a Borel subgroup of $G^\circ_{\mathbb{C}}$ containing $T_{\mathbb{C}}$, with a resulting set of simple roots $\Delta(B_{\mathbb{C}}, T_{\mathbb{C}})$; and $\mathcal{X}$ is a set consisting of a real ray in each complex simple root space (i.e., the set of positive real multiples of some fixed non-0 vector). (Sorry about the pair of modifiers "complex simple".) I will call these 'pinnings', although it doesn't agree with the usual terminology (where we pick individual root vectors, not rays). I claim that $G^\circ/\,\mathrm{Z}(G^\circ)$ acts simply transitively on the set of pinnings.

Once we have transitivity, freeness is clear: if $g \in G^\circ$ stabilises some pair $(T, B_\mathbb{C})$, then it lies in T, and so stabilises every complex root space; but then, for it to stabilise some choice of rays $\mathcal{X}$, it has to have the property that $\alpha(g)$ is positive and real for each simple root α; but also $\alpha(g)$ is a norm-1 complex number, hence trivial, for each simple root α, hence for each root α, so that g is central.

For transitivity, since (1) all maximal tori in G are G°-conjugate, so (2) for every maximal torus T in G, the Weyl group $\mathrm{W}(G^\circ_\mathbb{C}, T_\mathbb{C})$ acts transitively on the Borel subgroups of $G^\circ_\mathbb{C}$ containing $T_\mathbb{C}$, and (3) $\mathrm{W}(G^\circ, T) \to \mathrm{W}(G^\circ_\mathbb{C}, T_\mathbb{C})$ is an isomorphism, it suffices to show that all possible sets $\mathcal{X}$ are conjugate. Here's the argument that I came up with to show that they are even T-conjugate; I think it can probably be made much less awkward. Fix a simple root α, and two non-0 elements X_α and X'_α of the corresponding root space. Then there are a positive real number r and a norm-1 complex number z such that $X'_\alpha = rzX_\alpha$. Choose a norm-1 complex number w such that $w^2 = z$. There is then a unique element s_ad of $T_\mathbb{C}/\,\mathrm{Z}(G^\circ_\mathbb{C})$ such that $\alpha(s_\mathrm{ad}) = w$, and $\beta(s_\mathrm{ad}) = 1$ for all simple roots $\beta \neq \alpha$. By (4), we can choose a lift s_sc of s_ad to $(G_\mathrm{sc})_\mathbb{C} = (G_\mathbb{C})_\mathrm{sc}$, which necessarily lies in the preimage $(T_\mathbb{C})_\mathrm{sc}$ of (the intersection with the derived subgroup of) T, and put $t_\mathrm{sc} = s_\mathrm{sc} \cdot \overline{s_\mathrm{sc}}$. Then

$$\alpha(t_\mathrm{sc}) = \alpha(s_\mathrm{sc})\overline{\overline{\alpha}(s_\mathrm{sc})} = \alpha(s_\mathrm{sc})\overline{\alpha(s_\mathrm{sc})^{-1}} = w \cdot \overline{w^{-1}} = z,$$

and, similarly, $\beta(t_\mathrm{sc}) = 1$ for all simple roots $\beta \neq \alpha$. Now the image t of t_sc in $G^\circ_\mathbb{C}$ lies in $T_\mathbb{C}$ and is fixed by conjugation, hence lies in T; and $\mathrm{Ad}(t)X_\alpha = zX_\alpha$ lies on the ray through X'_α.

Since G also acts on the set of pinnings, we have a well defined map $p : G \to G^\circ/\,\mathrm{Z}(G^\circ)$ that restricts to the natural projection on G°. Now $\ker(p)$ meets every component, but it contains $\mathrm{Z}(G^\circ)$, so it need not be finite. Applying (5) to the Lie group $\ker(p)$ yields the desired subgroup H. Note that, as requested in your improved classification, conjugation by any element of H fixes a pinning, hence, if inner, must be trivial.

□

Remark 1.3. *The pinnings show that split reductive algebraic groups look like butterflies!*

According to Milne, Grothendieck used the following analogy to talk about the structure theory of algebraic groups. An algebraic group is like a butterfly. The body of the butterfly is the maximal torus. The wings are two opposite Borel subgroups. And the pins are the pinning maps.

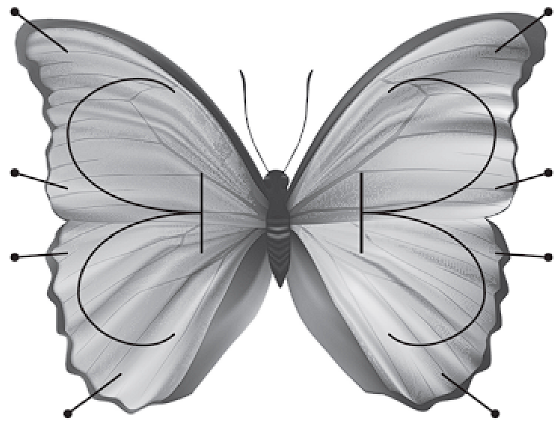


FIGURE 2. Pinning of butterfly

1.3. **Coherent continuation representations.** Until the end of today, G denotes a real reductive group in Harish-Chandra class.

Let's recall some basic definitions of coherent continuation representations, in order for me to do calculations in the case of real classical groups.

Fix a connected reductive complex Lie group $G_{\mathbb{C}}$, together with a Lie group homomorphism $\iota : \mathrm{G} \to \mathrm{G}_{\mathbb{C}}$ such that its differential $\mathrm{d}\iota : \mathrm{Lie}(\mathrm{G}) \to \mathrm{Lie}(\mathrm{G}_{\mathbb{C}})$ has the following two properties:

(1) the kernel of $\mathrm{d}\iota$ is contained in the center of $\mathrm{Lie}(\mathrm{G})$;
(2) the image of $\mathrm{d}\iota$ is a real form of $\mathrm{Lie}(\mathrm{G})$.

The analytical weight lattice of $\mathrm{G}_{\mathbb{C}}$ is identified with a subgroup of ${}^a\mathfrak{h}^*$ via $\mathrm{d}\iota$. We write $Q_\iota \subset {}^a\mathfrak{h}^*$ for this subgroup. Denote root lattice by $Q_{\mathfrak{g}}$, weight lattice by $Q^{\mathfrak{g}}$, then

$$Q_{\mathfrak{g}} \subset Q_\iota \subset Q^{\mathfrak{g}}.$$

They are all W stable subgroups. And we have some categories:

(1) $\mathrm{Rep}(\mathfrak{g}, Q_\iota)$ the category of all finite-dimensional representations of $\mathfrak{g}$ whose weight are contained in $Q_{\mathfrak{g}}$, it can be identified with $\mathrm{Rep}(\mathrm{Lie}(\mathrm{G}_{\mathbb{C}}), \mathrm{Q}_\iota)$ and hence $\mathrm{Rep}_{\mathrm{finite}}(\mathrm{G}_{\mathbb{C}})$;
(2) $\mathrm{Rep}(\mathrm{G})$ the category of all Casselman-Wallach representations of G, and its full subcategories $\mathrm{Rep}_\mu(\mathrm{G})$, $\mathrm{Rep}_{\mathrm{S}}(\mathrm{G})$.

To denote their Grothendieck groups, we replace Rep by $\mathcal{K}$.

Fix a Q_ι-coset $\Lambda = \lambda + Q_\iota \subset {}^a\mathfrak{h}^*$.

Definition 1.4 (coherent family)**.** *A $\mathcal{K}(G)$-valued Λ coherent family is a map*

$$\Phi : \Lambda \to \mathcal{K}(G)$$

satisfies the following two conditions:

(1) *for all $\nu \in \Lambda$, $\Phi(\nu) \in \mathcal{K}_\nu(G)$;*
(2) *for all representations $F \in \mathrm{Rep}(\mathfrak{g}, Q_\iota)$ and all $\nu \in \Lambda$,*

$$F \cdot (\Phi(\nu)) = \sum_{\mu} \Phi(\nu + \mu),$$

where μ runs over all weights of F, counted with multiplicities.

The set of coherent families is denoted by $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$

We have the integral Weyl group $W_\Lambda = \{w \in W \mid w\lambda - \lambda \in Q_\iota\}$ there is a representation of W_Λ on $\mathrm{Coh}_\Lambda(\mathcal{K}(G))$ by

$$(w \cdot \Phi)(\nu) = \Phi(w^{-1}\nu).$$

This is called a coherent continuation representation.

2. 2023.12.23

2.1. **Review of regular parameter.** Following yesterday's discussion of coherent continuation representations, we define two basis for it, use Langland classification.

Suppose H is a Cartan subgroup of G, by which we mean a subgroup that is the centralizer of a Cartan subalgebra of $\mathrm{Lie}(G)$ (In the case of linear groups, it's just a real algebraic torus.). H has a unique maximal compact subgroup (since G is in Harish-Chandra's class, adjoint representation of H is trivial, H_0 lies in the center of H, as we know, all maximal subgroups of Lie groups are conjugate to each other, use Cartan decomposition of H, we win. In fact, H has a unique maximal compact torus and a unique maximal split torus) We denote by $\Delta_{\mathfrak{h}} \subset \mathfrak{h}^*$ the root system of $\mathfrak{g}$. A root is called imaginary if $\check{\alpha} \in \mathfrak{t}$ (i.e. roots which take pure imaginary values on $\mathrm{Lie}(H)$), an imaginary root α is called compact if $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ is contained in the complexified Lie algebra of a common compact subgroup of G.(this is equivalent to $\mathfrak{g}_\alpha$ belong to a Lie subalgebra of a compact subgroup, since $\mathfrak{g}_\alpha$ lies in a Lie subalgebra of maximal compact subgroup is equivalent to $\mathfrak{g}_{-\alpha}$ Lies in the same subgroup)

There are two important facts about the representation theory of H:

(1) Every Casselman-Wallach representation of H is finite-dimensional, it follows directly from that H is an extension of an abelian group and a finite group;
(2) For every $\Gamma \in \mathrm{Irr(H)}$ differential of Γ is a direct sum of one-dimensional representations attached to a unique $\mathrm{d} \in \mathfrak{h}^*$. (it is true because H is in Harish-Chandra's class).

For every Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$, write

$$\xi_{\mathfrak{b}} : \mathfrak{h} \to {}^a\mathfrak{h},$$

for the linear isomorphism attached to $\mathfrak{b}$ defined by

$$\mathfrak{h} \hookrightarrow \mathfrak{b} \to \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] = {}^a\mathfrak{h}.$$

The transpose inverse of this map is still denoted by $\xi_{\mathfrak{b}} : {}^a\mathfrak{h}^* \to \mathfrak{h}^*$.

Write

$$W({}^a\mathfrak{h}^*, \mathfrak{h}^*) := \{\xi_{\mathfrak{b}} : {}^a\mathfrak{h}^* \to \mathfrak{h}^* \mid \mathfrak{b} \text{ is a Borel subalgebra containing } \mathfrak{h}\}.$$

Definition 2.1. *For every element $\xi \in W({}^a\mathfrak{h}^*, \mathfrak{h}^*)$, put*

$$\delta(\xi) := \frac{1}{2} \cdot \sum_{\alpha \text{ is an imaginary root in } \xi\Delta^+} \alpha - \sum_{\beta \text{ is an compact imaginary root in } \xi\Delta^+} \beta \in \mathfrak{h}^*.$$

Write $\mathcal{P}_\Lambda(G)$ for the set of all triples $\gamma = (H, \xi, \Gamma)$, where H is a Cartan subgroup of G, $\xi \in W({}^a\mathfrak{h}^*, \mathfrak{h}^*)$, and

$$\begin{array}{rccc} \Gamma : & \Lambda & \longrightarrow & \mathrm{Irr(H)} \\ & \nu & \longmapsto & \Gamma_\nu \end{array}$$

is a map with the following properties:

(1) *$\Gamma_{\nu+\beta} = \Gamma_\nu \otimes \xi(\beta)$ for all $\beta \in Q_\iota$ and $\nu \in \Lambda$;*
(2) *$\mathrm{d}_{\,\nu} = \xi(\nu) + \delta(\xi)$ for all $\nu \in \Lambda$.(note that Cartan subgroups not always have irr repn with this differential!)*

Here $\xi(\beta)$ is naturally viewed as a character of H by using the homomorphism $\iota : H \to H_{\mathbb{C}}$, and $H_{\mathbb{C}}$ is a Cartan subgroup of $G_{\mathbb{C}}$ containing $\iota(H)$.

2.2. Outer automorphism of algebraic groups.

Lemma 2.2 (a simple discover of abstract algebra)**.** *G an abstract group, H a normal subgroup of G, if G acts transitively on a set X such that the restriction of this action to H is free and transitive, then the short exact sequence of groups*

$$1 \to H \to G \to G/H \to 1$$

Proof. Actually choose an element $x \in X$, we have

$$G \simeq \mathrm{Stab}_G(x) \ltimes H.$$

$\square$

Using this lemma, we can see that for a complex connected reductive algebraic group G, $\mathrm{Aut}(G)$ acts on the set of pinnings transitively, with its normal subgroup the $\mathrm{Inn}(G)$ acts freely and transitively, so every pinning gives us a split of the short exact sequence

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

Some facts about split connected reductive algebraic groups over arbitrary field k:

(1) $\{\text{split connected reductive algebraic groups}\}/_{\backsim} = \{\text{root datums}\}/_{\backsim}$;
(2) For any two automorphisms of G, if they induce the same automorphism of root datum, then they differ by an inner automorphism;
(3) Fix a pinning, we get an isomorphism $\mathrm{Out}(G) \simeq \mathrm{Aut}(D_b)$.

3. 2023.12.26

Theorem 3.1 (Harish-Chandra's parameter of discrete series repn). *G be a connected semisimple Lie group (maybe there is an analogy for real reductive group), K a maximal torus of G, suppose that* $\operatorname{rank}(K) = \operatorname{rank}(G)$, *fix a compact maximal torus $T \in K$ and positive system Δ^+. Then for any analytically integral weight $\lambda + \rho$, there exist a discrete series representation π_λ, such that:*

1. *The infinitesimal character of π_λ is λ;*
2. *$\nu = \lambda + \rho - 2\rho_c$ is the highest weight of a minimal K type of π_λ;*
3. *If μ is any highest weight of irreducible K representation exist in π_λ, then*

$$\mu = \nu + \sum_{\alpha \in \Delta^+} n_\alpha \alpha$$

.

Moreover, $\pi_\lambda \simeq \pi_{\lambda'}$ if and only if λ and λ' lies in the same W_c orbit.

In the above theorem, $\rho - 2\rho_c$ is canonically realized as a continuous character of double cover of T, called $\tilde{T}$, which can be realized as a pull-back diagram:
$$\begin{tikzcd} \tilde{T} \arrow[r] \arrow[d] & \mathbb{C}^\times \arrow[d, "z \mapsto z^2"] \\ T \arrow[r, "2\rho - 4\rho_c"] & \mathbb{C}^\times \end{tikzcd}$$

Similarly, λ is also a character of a double cover of T, with respect to -2λ, but the difference of -2λ and $2\rho - 4\rho_c$ is a square of a character of T, the two double covers are canonically isomorphic.

In fact, there is a one to one correspondence! Which I finally get it:

$$\{\text{genuie characters of } \tilde{T}_{\rho_{im}}\} \simeq \{(\Gamma, \nu) \in X^*(T) \times \mathfrak{t}^* \mid d\Gamma = \nu + \rho_{im} - 2\rho_{im,c}\}.$$

The correspondence is given by

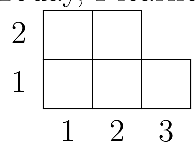
$$\chi \mapsto (\chi \otimes (\rho_{im} - 2\rho_{im,c}), d\chi)$$
$$(\Gamma, \nu) \mapsto \Gamma \otimes (2\rho_{im,c} - \rho_{im})$$

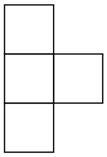
Definition 3.2 (Langlands parameter). *Let G be a linear real reductive group, the Langlands parameters for G is a triple $(H, \gamma, R^+_{i\mathbb{R}})$. Such that*

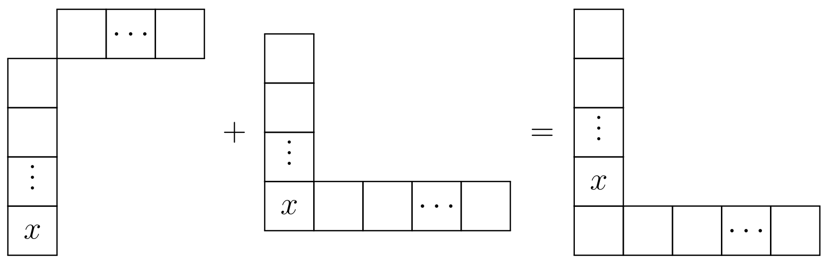
1. Hladky *á*

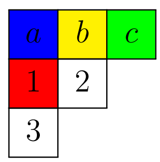
4. 2024.2.27

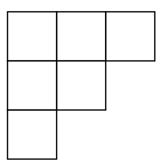
Today, I learned how to draw the Young tableau.

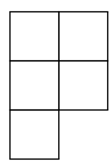


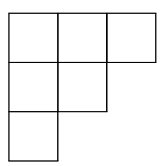


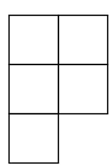


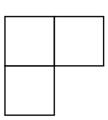


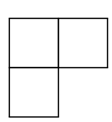


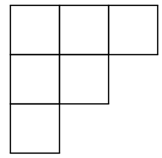


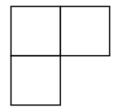




这是中文  这是英文

$a =$  $+ b$

杨图 1: 

杨图 2: 

For a compact connected Lie group G, choose a maximal torus T, there is a canonical 1-1 correspondence:

$$X^*(T)/W \longleftrightarrow (X^*(T)+\rho)/W \longleftrightarrow \mathrm{Irr}(G)$$

Moreover, if we choose a positive system Ψ^+, and then ρ_{Ψ^+}, the above sets also canonically isomorphic to the set of dominant analytically integral weights:

$$\{\lambda \in X^*(T) \,|\, \langle \lambda, \alpha \rangle \geq 0, \text{for all } \alpha \in \Psi^+\}$$

This can be generalized to parameter of discrete series representations for connected semisimple Lie groups G with compact Cartan subgroup T :

$$(X^*(T)+\rho)^{reg}/W_c$$

If a discrete series representation V has minimal K type which has maximal weight ν, then it's parameter is $\lambda = \nu - \rho + 2\rho_c$.(not sure)

Note that for a fixed infinitesimal character, there are exactly $|W/W_c|$ different discrete series representations.

5. 2024.2.28

Some facts about continuous complex characters of some common abelian topological groups:

field	character group	correspondence
$(\mathbb{R}_+^\times, \times)$	$\mathbb{C}$	$\xi \mapsto (t \mapsto t^\xi)$
$(\mathbb{C}^\times, \times)$	$\mathbb{C} \times \mathbb{Z}$	$(\xi, n) \mapsto (z \mapsto \lvert z\rvert^\xi (\frac{z}{\lvert z\rvert})^n)$
$(\mathbb{S}^1, \times)$	$\mathbb{Z}$	$n \mapsto (z \mapsto z^n)$
$(\mathbb{Q}_p, +)$	$\mathbb{Q}_p$	$\xi \mapsto (x \mapsto \exp(2\pi i \xi \{x\}_p))$

Remark 5.1. *Note that all continuous characters of $\mathbb{Q}_p$ are automatically locally constant (smooth), which is consistent with the case of real lie groups.*

6. 2024.10.22

Lemma 6.1. *The associated variety of an irreducible Casselman-Wallach representation of a real reductive group in Harish-Chandra class is the closure of a nilpotent orbit in* $\mathfrak{g}^*$.

Proof. By the work of Borho, Brylinski [BB], and Joseph [Jos], the associated variety of any primitive ideal (annihilator of an irreducible $\mathfrak{g}$-module) is the closure of a single nilpotent orbit. So we only need to prove that $\mathrm{gr}(\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(\mathrm{V}))$ is a primitive ideal.

Since the K-finite part V_K is dense in V, $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V) = \mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V_K)$, we only need to consider the annihilator of the irreducible $(\mathfrak{g}, K)$-module V_K.

If G is connected, then K is also connected, so V_K is irreducible as $\mathcal{U}(\mathfrak{g})$-module, in this case, $\mathrm{Ann}_{\mathcal{U}(\mathfrak{g})}(V)$ is a primitive ideal.

If G is not connected, let G_0 be the identity component of G, $K_0 = G_0 \cap K$ is a maximal compact subgroup of G_0, then $V_K|_{G_0}$ is a finite length $(\mathfrak{g}, K_0)$-module, it has an irreducible quotient $V_K \to V_0$, then according to the theory of induction of $(\mathfrak{g}, K)$-modules [KV, Chapter 2] we have an injective morphism of $(\mathfrak{g}, K_0)$-modules:

$$V_K|_{K_0} \hookrightarrow (\mathrm{induced}_{K_0}^{K}(V_0))|_{K_0} \cong \bigoplus_{\text{double cosets } K_0 k K_0} kV_0, \tag{6.1}$$

where kV_0 be the $(\mathfrak{g}, K_0)$-module whose underlying space is V_0, whose action by $X \in \mathfrak{g}$ is the usual action by $\mathrm{Ad}(k)X$, whose action by $g \in K_0$ is the usual action by $k^{-1}gk$, it has the same annihilator in $\mathcal{U}(\mathfrak{g})$ as V_0 since G is in Harish-Chandra's class, and hence the annihilator of V_K is the same as the annihilator of V_0, which is a primitive ideal.

$\square$

7. 2024.11.28

Any split connected reductive group over arbitrary field can be uniquely defined and split over $\mathbb{Z}$. (Chevalley group)

Classification of split reductive group is the same over arbitrary non-empty schemes, which is equivalent to the classification of root data.(cf. [SGA3])

8. 2024.12.17

Geometric Pre-quantization

Let X be a smooth manifold. There are natural maps

$$\mathrm{Pic}(X) \longrightarrow \mathrm{H}^2_{dR}(X;\mathbb{R}) \simeq \mathrm{H}^2_{Cech}(X;\mathbb{R}).$$

Construction of the first map: For a (complex) line bundle $\mathbb{L} \in \mathrm{Pic}^h(X)$, let $\langle,\rangle$ be a Hermitian metric on it and ∇ be a unitary connection. (Every line bundle $\mathbb{L} \in \mathrm{Pic}(X)$ has a Hermitian metric and a unitary connection on it.)

- choose a locally trivialization $\{U_\alpha\}$ of $\mathbb{L}$, such that all U_α and $U_\alpha \cap U_\beta$ are contractible;
- choose Unitary local frames $\{s_\alpha\}$ for $\mathbb{L}$ on each U_α, then $s \leftrightarrow (f_\alpha)$ and $\nabla \leftrightarrow (\theta_\alpha)$ (unitary implies that θ_α are pure imaginary);
- define $\Omega := \mathrm{d}\theta_\alpha$, and $c_1(\mathbb{L}) := [\frac{1}{2\pi i}\Omega] \in \mathrm{H}^2_{dR}(X;\mathbb{R})$ ($\Omega = \mathrm{curv}\nabla$).

$c_1(\mathbb{L})$ is independent of the choice of $\langle,\rangle$ and ∇ by Chern-Weil theory. And we can see that $\theta_\alpha - \theta_\beta = \frac{\mathrm{d}g_{\alpha,\beta}}{g_{\alpha,\beta}} = \mathrm{dlog}(g_{\alpha,\beta})$ on $U_\alpha \cap U_\beta$, where $g_{\alpha,\beta}$ is the translation function ($s_\beta = g_{\alpha,\beta}s_\alpha$).

Construction of the second map (classical de-Rham isomorphism), choose a 2-form $\omega \in \mathrm{H}^2_{dR}(X;\mathbb{R})$

- choose a cover $\{U_\alpha\}$ of $\mathbb{L}$, such that all U_α and $U_\alpha \cap U_\beta$ are contractible;
- on each U_α, choose a 1-form θ_α such that $\mathrm{d}\theta_\alpha = \omega|_{U_\alpha}$;
- since $\mathrm{d}\theta_\alpha = \mathrm{d}\theta_\beta$ on $U_\alpha \cap U_\beta$, we can choose smooth function $c_{\alpha,\beta}$ such that $\mathrm{d}c_{\alpha,\beta} = \theta_\alpha - \theta_\beta$;
- let $c_{\alpha,\beta,\gamma} = c_{\alpha,\beta} + c_{\beta,\gamma} + c_{\gamma,\alpha}$, which is a constant, and we get the corresponding 2-cocycle $(c_{\alpha,\beta,\gamma}) \in \mathrm{H}^2_{Cech}(X;\mathbb{R})$.

If ω come from the first map, we can choose $c_{\alpha,\beta} = \frac{1}{2\pi i}\mathrm{log}g_{\alpha,\beta}$, and hence $c_{\alpha,\beta,\gamma} = \frac{1}{2\pi i}(\mathrm{log}g_{\alpha,\beta} + \mathrm{log}g_{\beta,\gamma} + \mathrm{log}g_{\gamma,\alpha}) \in \mathbb{Z}$. So the morphism is in fact

$$\mathrm{Pic}(X) \longrightarrow \mathrm{H}^2_{dR}(X;\mathbb{Z}) \simeq \mathrm{H}^2_{Cech}(X;\mathbb{Z}).$$

In fact the first map is also an isomorphism. (cf. prequantization) For a 2-form ω, choose θ_α and $c_{\alpha,\beta}$ as above, let $g_{\alpha,\beta} = \exp(2\pi i c_{\alpha,\beta})$, these are translation functions of a complex line bundle $\mathbb{L}$.

Corollary 8.1. *Assume that G is a connected Lie group. The following are equivalent:*

- $\Omega \subseteq \mathfrak{g}^*$ *is integral* ($[\sigma_\Omega] \in \mathrm{H}^2(X;\mathbb{Z})$)*;*
- *there exist a G-equivariant complex line bundle over Ω with a G-invariant Hermitian connection ∇ such that*

$$\frac{1}{2\pi i}\mathrm{curv}(\nabla) = \sigma_\Omega.$$

Theorem 2.6 (Weil). *Let M be a smooth manifold and ω a real, closed 2-from whose cohomology class $[c]$ is integral. Then there is a unique Hermitian line bundle $\mathbb{L}$ over M with unitary connection ∇ so that $c_1(\mathbb{L}) = [c]$.*

Sketch of Proof. Existence: Reverse the arguments above. Since $[c]$ is integral, $e^{2\pi c_{ijk}} = 1$. As a consequence, $g_{ij}g_{jk}g_{kl} = 1$. So g_{ij}'s are the transition function for some line bundle whose first Chern class is $[c]$.

Uniqueness: Suppose $c_1(\mathbb{L}) = c_1(\widetilde{\mathbb{L}})$. Let $h_{ij} = \frac{1}{2\pi i} g_{ij}$ and define $\tilde{h}_{ij}$ similarly. Then the functions $\hat{h}_{ij} = h_{ij} - \tilde{h}_{ij}$ satisfies the relation

$$\hat{h}_{ij} + \hat{h}_{jk} + \hat{h}_{ki} = 0.$$

We take a partition of unity ρ_k and let $\lambda_i = e^{2\pi i \sum \hat{h}_{ki}\rho_k}$. Then

$$\lambda_i g_{ij} \lambda_j^{-1} = e^{2\pi i (h_{ij} + \sum(\hat{h}_{ki} - \hat{h}_{kj})\rho_k)} = e^{2\pi i (h_{ij} - \hat{h}_{ij})} = \tilde{g}_{ij}.$$

So as line bundles $\mathbb{L} \simeq \widetilde{\mathbb{L}}$. □

Figure 3. Proof

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