PowerNumbers.jl: a fast approach to automatic asymptotics

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We develop a scheme for quick arithmetic on asymptotic series, evaluated to just one or two terms. This generalizes the concept of dual numbers in automatic differentiation to support general algebraic powers, including negative powers to capture behaviour at infinity. We give a full description of the algebraic rules for these newly defined *power numbers*, with justification of guaranteed equivalence to asymptotic algebra. Some example applications follow, including evaluation of complicated rational functions at infinity defined in terms of linear algebra, hypergeometric functions, and Stieltjes transforms over intervals for functions with power-like singularities and the endpoints.

CCS Concepts: • **Mathematics of computing → Solvers**.

Additional Key Words and Phrases: none

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1 INTRODUCTION

Dual numbers are a simple algebraic structure that underlies forward mode automatic-differentiation, with a convenient representation in terms of 2×2 matrices:

Definition 1.1. Let K be a field. The set of matrices

$$\mathbf{D}^K = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in K \right\}$$

are called the set of **dual numbers**. The elements are typically denoted by

$$a + b\epsilon \qquad \text{where} \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}.$$

The property that $\epsilon^2 = 0$ suffices to guarantee that general analytic matrix functions applied to dual numbers encode the derivative at the point a, namely:

$$f(a + b\epsilon) = f(a) + bf'(b)\epsilon$$

Thus computations using dual numbers preserve the most important information encoded in the Taylor Expansion.

In analogy to dual numbers, we seek a simple algebraic structure that encodes the leading-order information of Hahn-form asymptotic expansions. Asymptotic expansions are important

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computational tools in applied mathematics, that can reduce complicated functions to simple forms with specified accuracy. Many such expansions are of the Hahn form:

$$S(z) = \sum_{n=1}^{\infty} a_n z^{\alpha_n} \quad as \quad z \to 0 \tag{1}$$

where the $\alpha_n \in \mathbb{R}$ are strictly increasing. We we call the set of all such expansions $\mathbb{S}$.

By restricting these series to the terms most significant in the limit, we can define the set of **power numbers**. Specifically, we will be restricting to the first two terms with $\alpha \leq \beta$, so that we are essentially considering:

$$S(z) = az^{\alpha} + bz^{\beta} + o(z^{\beta})$$

By giving algebraic rules that are consistent with their counterparts in $\mathbb{S}$, we will allow for fast manipulation without compromising on accuracy or losing important order information.

In **Section 2** we will rigorously describe the algorithms that govern the basic operations of addition, subtraction, multiplication and division on power numbers. These have been defined to correspond with what we expect from similar operations on $\mathbb{S}$, and we provide a theorem guaranteeing this symmetry. Furthermore, this guarantee extends to exponentiation and analytic functions of power numbers.

In **Section 3** we will give some of the possible applications of the construction, including the evaluation of rational polynomials at infinity and finding limiting expressions of hypergeometric functions(?). We also provide an implementation written in the Julia language, "PowerNumbers.jl".

2 FORMAL DESCRIPTION

Definition 2.1. Let K be a field. The set of functions of $\epsilon \in [0, \infty)$,

$$\mathbf{PN}^K = \{a\epsilon^{\alpha} + b\epsilon^{\beta} : a, b \in K; \alpha, \beta \in \mathbb{R} \cup \{\infty\}; \alpha \leq \beta\}$$

is called the set of **power numbers**.

In the case $\alpha = \beta$, we write only one term $(a + b)\epsilon^{\beta}$. Notationally, ϵ^0 is omitted. While we do not have a one-to-one correspondence between $\mathbf{PN}^K$ and $\mathbb{S}$ (there is usually a choice between one or two terms), this is not a major concern. Power numbers are not considered to be equal unless the values (a, b, α, β) are all equal.

We enforce the equality $0\epsilon^{\alpha} + b\epsilon^{\beta} = b\epsilon^{\beta}$. However, in general, $a\epsilon^{\alpha} + 0\epsilon^{\beta} \neq a\epsilon^{\alpha}$. By defining $(+, *)$ below, we acquire the double monoid

$$\mathbb{PN}^K = (\mathbf{PN}^K, +, *)$$

2.1 Basic Operations

Definition 2.2. **Addition** $+ : \mathbf{PN}^K \times \mathbf{PN}^K \to \mathbf{PN}^K$ is described by an algorithm.

Data: $a\epsilon^\alpha + b\epsilon^\beta, c\epsilon^\gamma + d\epsilon^\delta \in \mathbf{PN}^K$; assume WLOG that $\beta \le \delta$
Result: $(a\epsilon^\alpha + b\epsilon^\beta) + (c\epsilon^\gamma + d\epsilon^\delta) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}^K$
if $\beta = \delta$ **then**
  if $\gamma = \beta$ **then**
    $p = a, q = b + c + d, \zeta = \alpha, \eta = \beta$
  else if $\alpha < \gamma < \beta$ **then**
    $p = a, q = c, \zeta = \alpha, \eta = \gamma$
  else if $\gamma = \alpha$ **then**
    $p = a + c, q = b + d, \zeta = \alpha, \eta = \beta$
  else
    $p = c, q = a, \zeta = \gamma, \eta = \alpha$
  end
else
  if $\beta < \gamma$ **then**
    $p = a, q = b, \zeta = \alpha, \eta = \beta$
  else if $\gamma = \beta$ **then**
    $p = a, q = b + c, \zeta = \alpha, \eta = \beta$
  else if $\alpha < \gamma < \beta$ **then**
    $p = a, q = c, \zeta = \alpha, \eta = \gamma$
  else if $\gamma = \alpha$ **then**
    $p = a + c, q = b, \zeta = \alpha, \eta = \beta$
  else
    $p = c, q = a, \zeta = \gamma, \eta = \alpha$
  end
end

Algorithm 1: Summing Power Numbers

The additive identity for Power Numbers is $0\epsilon^\infty$.

Definition 2.3. **Multiplication** $* : \mathbf{PN}^K \times \mathbf{PN}^K \to \mathbf{PN}^K$ can be most simply expressed as addition of Power Numbers:

$$(a\epsilon^\alpha + b\epsilon^\beta) * (c\epsilon^\gamma + d\epsilon^\delta) = (ac\epsilon^{\alpha+\gamma} + ad\epsilon^{\alpha+\delta}) + (bc\epsilon^{\beta+\gamma} + bd\epsilon^{\beta+\delta}) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}^K$$

The multiplicative identity for Power Numbers is $1 + 0\epsilon^\infty$.

While $\mathbb{PN}_\epsilon^K$ is a monoid under both addition and multiplication, we can define two operations that have some of the properties we expect for subtraction and division.

Definition 2.4. **Subtraction** $- : \mathbf{PN}^K \to \mathbf{PN}^K$ is defined as:

$$-(a\epsilon^\alpha + b\epsilon^\beta) = (-a)\epsilon^\alpha + (-b)\epsilon^\beta$$

Naturally, $- : \mathbf{PN}^K \times \mathbf{PN}^K \to \mathbf{PN}^K$ is defined as $A + (-(B))$, where $A, B \in \mathbf{PN}^K$.

Definition 2.5. The **multiplicative pseudo-inverse** $inv : \mathbf{PN}^K \longrightarrow \mathbf{PN}^K$ is slightly more complicated.

Data: $a\epsilon^\alpha + b\epsilon^\beta \in \mathbf{PN}^K$

Result:

$$\frac{1}{a\epsilon^\alpha + b\epsilon^\beta} = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}^K$$

if $\alpha = \infty$ **then**
  Not defined.
else if $\alpha = \beta$ **then**
  $p = 0, q = \frac{1}{a+b}, \zeta = -\alpha, \eta = -\alpha$
else
  $p = \frac{1}{a+b}, q = -\frac{b}{a^2}, \zeta = -\alpha, \eta = \beta - 2\alpha$
end

Algorithm 2: Multiplicative Inversion

Division $\div : \mathbf{PN}^K \times \mathbf{PN}^K \longrightarrow \mathbf{PN}^K$ is defined as $A \div B = A * (inv(B))$, where $A, B \in \mathbf{PN}^K$.

This matches the Dual Numbers case, where $\alpha = 0$, $\beta = 1$.

There is a special case wherein the pseudo-negative and -inverse are the true negative and inverse, which motivates the following

Proposition 2.6. *For $A = a\epsilon^\alpha + b\epsilon^\beta \in \mathbf{PN}^K$, we have that $A - A = 0\epsilon^\infty$ and $A \div A = 1 + 0\epsilon^\infty$ if and only if $\beta = \infty$.*

2.2 Exponentiation, Analytic Functions & Asymptotic Series

The definition of A^m for $A \in \mathbf{PN}^K$, $m \in \mathbb{Z}$ follows naturally from the above definitions of multiplication and division.

Since we will generally consider ϵ to be small, we can extend any analytic function $h : K \longrightarrow K$ to $h : \mathbf{PN}^K_\epsilon \longrightarrow \mathbf{PN}^K_\epsilon$ for $\alpha \geq 0$. This is via Taylor series:

$$h(a\epsilon^0 + b\epsilon^\beta) = h(a + b\epsilon^\beta) = h(a) + b\epsilon^\beta h'(a)$$

$$h(a\epsilon^\alpha + b\epsilon^\beta) = h(0) + a\epsilon^\alpha h'(0) \quad where \quad \alpha \neq 0$$

Now, consider an asymptotic series $S(z)$ that is also a Hahn series *ie* $S \in \mathbb{S}$ as in (1). The primary significance of Power Numbers comes from the following

Proposition 2.7. *Let $f(z) = az^\alpha + bz^\beta + o(z^\beta)$, $g(z) = cz^\gamma + dz^\delta + o(z^\delta)$ be series of the form in* (1). *Define the associated Power Numbers, $F = a\epsilon^\alpha + b\epsilon^\beta$, $G = c\epsilon^\gamma + d\epsilon^\delta$. Then, where $\cdot$ is one of $(+, -, *, /)$ we have that*

$$f(z) \cdot g(z) = pz^\zeta + qz^\eta + o(z^\eta)$$

$$\Leftrightarrow$$

$$F \cdot G = p\epsilon^\zeta + q\epsilon^\eta$$

Furthermore, for any analytic function $h : K \longrightarrow K$ we have

$$h(f(z)) = pz^\zeta + qz^\eta + o(z^\eta)$$

$$\Leftrightarrow$$

$$h(F) = p\epsilon^\zeta + q\epsilon^\eta$$

3 EXAMPLES

3.1 Rational Polynomials at Infinity

We can get a simple system for evaluating a rational polynomial in the infinite limit. For example, consider:

$$f(z) = \frac{z^3 + z^2 + 1}{z^3 + z}$$

Taking the Power Number $z = \epsilon^{-1}$. By the arithmetic described above we have:

$$\frac{(\epsilon^{-3}) + (\epsilon^{-2}) + 1}{(\epsilon^{-3}) + (\epsilon^{-1})} = \frac{\epsilon^{-3}}{\epsilon^{-3}} = \epsilon^{0}$$

Which gives the correct limit when considering $\epsilon \longrightarrow 0$.

4 CONCLUSION