PowerNumbers.jl: a fast approach to automatic asymptotics

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We develop a scheme for quick arithmetic on asymptotic series, evaluated to just one or two terms. We give a full description of the algebraic rules for these "Power Numbers", with justification of sufficient equivalence to asymptotic algebra. Some example applications follow, including evaluation of rational functions at infinity.

CCS Concepts: • **Mathematics of computing** → **Solvers**.

Additional Key Words and Phrases: none

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1 INTRODUCTION

2 DESCRIPTION

Definition 2.1. Let K be a field. The set of functions of $\epsilon \in [0, \infty)$,

$$\mathbf{PN}_\epsilon^K = \{a\epsilon^\alpha + b\epsilon^\beta : a, b \in K; \alpha, \beta \in \mathbb{R} \cup \{\infty\}; \alpha \leq \beta\}$$

is called the set of **Power Numbers**.

In the case $\alpha = \beta$, we write only one term $(a + b)\epsilon^\beta$. Notationally, ϵ^0 is omitted. We enforce the equality $0\epsilon^\alpha + b\epsilon^\beta = b\epsilon^\beta$. However, in general, $a\epsilon^\alpha + 0\epsilon^\beta \neq a\epsilon^\alpha$. By defining $(+, *)$ below, we acquire the double monoid

$$\mathbb{PN}_\epsilon^K = (\mathbf{PN}_\epsilon^K, +, *)$$

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2.1 Basic Operations

Definition 2.2. **Addition** $+ : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is described by an algorithm.

Data: $a\epsilon^\alpha + b\epsilon^\beta, c\epsilon^\gamma + d\epsilon^\delta \in \mathbf{PN}_\epsilon^K$; assume WLOG that $\beta \leq \delta$
Result: $(a\epsilon^\alpha + b\epsilon^\beta) + (c\epsilon^\gamma + d\epsilon^\delta) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$

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if β = δ then
    if γ = β then
        p = a, q = b + c + d, ζ = α, η = β
    else if α < γ < β then
        p = a, q = c, ζ = α, η = γ
    else if γ = α then
        p = a + c, q = b + d, ζ = α, η = β
    else
        p = c, q = a, ζ = γ, η = α
    end
else
    if β < γ then
        p = a, q = b, ζ = α, η = β
    else if γ = β then
        p = a, q = b + c, ζ = α, η = β
    else if α < γ < β then
        p = a, q = c, ζ = α, η = γ
    else if γ = α then
        p = a + c, q = b, ζ = α, η = β
    else
        p = c, q = a, ζ = γ, η = α
    end
end
```

Algorithm 1: Summing Power Numbers

The additive identity for Power Numbers is $0\epsilon^\infty$.

Definition 2.3. **Multiplication** $* : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ can be most simply expressed as addition of Power Numbers:

$$(a\epsilon^\alpha + b\epsilon^\beta) * (c\epsilon^\gamma + d\epsilon^\delta) = (ac\epsilon^{\alpha+\gamma} + ad\epsilon^{\alpha+\delta}) + (bc\epsilon^{\beta+\gamma} + bd\epsilon^{\beta+\delta}) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$$

The multiplicative identity for Power Numbers is $1 + 0\epsilon^\infty$.

While $\mathbb{PN}_\epsilon^K$ is a monoid under both addition and multiplication, we can define two operations that have some of the properties we expect for subtraction and division.

Definition 2.4. **Subtraction** $- : \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is defined as:

$$-(a\epsilon^\alpha + b\epsilon^\beta) = (-a)\epsilon^\alpha + (-b)\epsilon^\beta$$

Naturally, $- : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is defined as $A + (-(B))$, where $A, B \in \mathbf{PN}_\epsilon^K$.

Definition 2.5. The **multiplicative pseudo-inverse** $inv : \mathbf{PN}_\epsilon^K \rightarrow \mathbf{PN}_\epsilon^K$ is slightly more complicated.

Data: $a\epsilon^\alpha + b\epsilon^\beta \in \mathbf{PN}_\epsilon^K$
Result:

$$\frac{1}{a\epsilon^\alpha + b\epsilon^\beta} = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$$

if $\alpha = \infty$ **then**
  Not defined.
else if $\alpha = \beta$ **then**
  $p = 0, q = \frac{1}{a+b}, \zeta = -\alpha, \eta = -\alpha$
else
  $p = \frac{1}{a+b}, q = -\frac{b}{a^2}, \zeta = -\alpha, \eta = \beta - 2\alpha$
end

Algorithm 2: Multiplicative Inversion

Division $\div : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \rightarrow \mathbf{PN}_\epsilon^K$ is defined as $A \div B = A * (inv(B))$, where $A, B \in \mathbf{PN}_\epsilon^K$.

This matches the Dual Numbers case, where $\alpha = 0$, $\beta = 1$.

There is a special case wherein the pseudo-negative and -inverse are the true negative and inverse, which motivates the following

Proposition 2.6. *For $A = a\epsilon^\alpha + b\epsilon^\beta \in \mathbf{PN}_\epsilon^K$, we have that $A - A = 0\epsilon^\infty$ and $A \div A = 1 + 0\epsilon^\infty$ if and only if $\beta = \infty$.*

2.2 Exponentiation, Analytic Functions & Asymptotic Series

The definition of A^m for $A \in \mathbf{PN}_\epsilon^K$, $m \in \mathbb{Z}$ follows naturally from the above definitions of multiplication and division.

Since we will generally consider ϵ to be small, we can extend any analytic function $h : K \rightarrow K$ to $h : \mathbf{PN}_\epsilon^K \rightarrow \mathbf{PN}_\epsilon^K$ for $\alpha \geq 0$. This is via Taylor series:

$$h(a\epsilon^0 + b\epsilon^\beta) = h(a + b\epsilon^\beta) = h(a) + b\epsilon^\beta h'(a)$$
$$h(a\epsilon^\alpha + b\epsilon^\beta) = h(0) + a\epsilon^\alpha h'(0) \quad where \quad \alpha \neq 0$$

Now, consider an asymptotic series that is also a Hahn series; that is,

$$S(z) = \sum_{n=1}^{\infty} a_n z^{\alpha_n} \quad as \quad z \rightarrow 0 \tag{1}$$

where the $\alpha_n \in \mathbb{R}$ are strictly increasing. The primary significance of Power Numbers comes from the following

Proposition 2.7. *Let $f(z) = az^\alpha + bz^\beta + o(z^\beta)$, $g(z) = cz^\gamma + dz^\delta + o(z^\delta)$ be series of the form in (1). Define the associated Power Numbers, $F = a\epsilon^\alpha + b\epsilon^\beta$, $G = c\epsilon^\gamma + d\epsilon^\delta$. Then, where $\cdot$ is one of $(+, -, *, /)$ we have that*

$$f(z) \cdot g(z) = pz^\zeta + qz^\eta + o(z^\eta)$$
$$\Leftrightarrow$$
$$F \cdot G = p\epsilon^\zeta + q\epsilon^\eta$$

Furthermore, for any analytic function $h : K \rightarrow K$ we have

$$h(f(z)) = pz^\zeta + qz^\eta + o(z^\eta)$$
$$\Leftrightarrow$$
$$h(F) = p\epsilon^\zeta + q\epsilon^\eta$$

3 EXAMPLES

3.1 Rational Polynomials at Infinity

We can get a simple system for evaluating a rational polynomial in the infinite limit. For example, consider:

$$f(z) = \frac{z^3 + z^2 + 1}{z^3 + z}$$

Taking the Power Number $z = \epsilon^{-1}$. By the arithmetic described above we have:

$$\frac{(\epsilon^{-3}) + (\epsilon^{-2}) + 1}{(\epsilon^{-3}) + (\epsilon^{-1})} = \frac{\epsilon^{-3}}{\epsilon^{-3}} = \epsilon^{0}$$

Which gives the correct limit when considering $\epsilon \longrightarrow 0$.

4 CONCLUSION