PowerNumbers.jl: a fast approach to automatic asymptotics

SHEEHAN OLVER, Imperial College
MATTHEW REES, University College London

We develop a scheme for quick arithmetic on asymptotic series, evaluated to just one or two terms. We give a full description of the algebraic rules for these "Power Numbers", with justification of sufficient equivalence to asymptotic algebra. Some example applications follow, including evaluation of rational functions at infinity.

CCS Concepts: • **Mathematics of computing** → **Solvers**.

Additional Key Words and Phrases: none

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1 INTRODUCTION

2 DESCRIPTION

Definition 2.1. Let K be a field. The set of functions of $\epsilon \in [0, \infty)$,

$$\mathbf{PN}_\epsilon^K = \{a\epsilon^\alpha + b\epsilon^\beta : a, b \in K; \alpha, \beta \in \mathbb{R} \cup \{\infty\}; \alpha \leq \beta\}$$

is called the set of **Power Numbers**.

In the case $\alpha = \beta$, we write only one term $(a + b)\epsilon^\beta$. Notationally, ϵ^0 is omitted. We enforce the equality $0\epsilon^\alpha + b\epsilon^\beta = b\epsilon^\beta$. However, in general, $a\epsilon^\alpha + 0\epsilon^\beta \neq a\epsilon^\alpha$. By defining $(+, *)$ below, we acquire the double monoid

$$\mathbb{PN}_\epsilon^K = (\mathbf{PN}_\epsilon^K, +, *)$$

Authors' addresses: Sheehan Olver, s.olver@imperial.ac.uk, Imperial College, London, England, SW7 2AZ; Matthew Rees, matthew.rees.16@ucl.ac.uk, University College London, London, England, WC1E 6BT.

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2.1 Addition

$+ : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is described by an algorithm.

Data: $a\epsilon^\alpha + b\epsilon^\beta, c\epsilon^\gamma + d\epsilon^\delta \in \mathbf{PN}_\epsilon^K$; assume WLOG that $\beta \leq \delta$
Result: $(a\epsilon^\alpha + b\epsilon^\beta) + (c\epsilon^\gamma + d\epsilon^\delta) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$
if $\beta = \delta$ **then**
  if $\gamma = \beta$ **then**
    $p = a, q = b + c + d, \zeta = \alpha, \eta = \beta$
  else if $\alpha < \gamma < \beta$ **then**
    $p = a, q = c, \zeta = \alpha, \eta = \gamma$
  else if $\gamma = \alpha$ **then**
    $p = a + c, q = b + d, \zeta = \alpha, \eta = \beta$
  else
    $p = c, q = a, \zeta = \gamma, \eta = \alpha$
  end
else
  if $\beta < \gamma$ **then**
    $p = a, q = b, \zeta = \alpha, \eta = \beta$
  else if $\gamma = \beta$ **then**
    $p = a, q = b + c, \zeta = \alpha, \eta = \beta$
  else if $\alpha < \gamma < \beta$ **then**
    $p = a, q = c, \zeta = \alpha, \eta = \gamma$
  else if $\gamma = \alpha$ **then**
    $p = a + c, q = b, \zeta = \alpha, \eta = \beta$
  else
    $p = c, q = a, \zeta = \gamma, \eta = \alpha$
  end
end

Algorithm 1: Summing Power Numbers

The additive identity for Power Numbers is $0\epsilon^\infty$.

2.2 Multiplication

$* : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ can be most simply expressed as addition of Power Numbers:

$$(a\epsilon^\alpha + b\epsilon^\beta) * (c\epsilon^\gamma + d\epsilon^\delta) = (ac\epsilon^{\alpha+\gamma} + ad\epsilon^{\alpha+\delta}) + (bc\epsilon^{\beta+\gamma} + bd\epsilon^{\beta+\delta}) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$$

The multiplicative identity for Power Numbers is $1 + 0\epsilon^\infty$.

2.3 Pseudo-Negation

While $\mathbb{PN}_\epsilon^K$ is a monoid under both addition and multiplication, we can define two operations that have some of the properties we expect for subtraction and division.
$- : \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is defined as:

$$-(a\epsilon^\alpha + b\epsilon^\beta) = (-a)\epsilon^\alpha + (-b)\epsilon^\beta$$

Naturally, $- : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is defined as $A + (-(B))$, where $A, B \in \mathbf{PN}_\epsilon^K$. Note that for $\beta \neq \infty$, we have $(a\epsilon^\alpha + b\epsilon^\beta) - (a\epsilon^\alpha + b\epsilon^\beta) = 0\epsilon^\beta \neq 0\epsilon^\infty$.

2.4 Pseudo-Inverse

The multiplicative pseudo-inverse is slightly more complicated.

Data: $a\epsilon^\alpha + b\epsilon^\beta \in \mathbf{PN}_\epsilon^K$

Result:

$$\frac{1}{a\epsilon^\alpha + b\epsilon^\beta} = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$$

if $\alpha = \beta$ **then**

| $p = 0, q = \frac{1}{a+b}, \zeta = -\alpha, \eta = -\alpha$

else

| $p = \frac{1}{a+b}, q = -\frac{b}{a^2}, \zeta = -\alpha, \eta = \beta - 2\alpha$

end

Algorithm 2: Multiplicative Inversion

This matches the Dual Numbers case, where $\alpha = 0$, $\beta = 1$. Similarly to subtraction,

$$\frac{a\epsilon^\alpha + b\epsilon^\beta}{a\epsilon^\alpha + b\epsilon^\beta} \neq 1 + 0\epsilon^\infty$$

2.5 Exponentiation & Analytic Functions

The definition of A^m for $A \in \mathbf{PN}_\epsilon^K$, $m \in \mathbb{Z}$ follows naturally from the above definitions of multiplication and division.

Since we will generally consider ϵ to be small, we can extend any analytic function $h : K \longrightarrow K$ to $h : \mathbf{PN}_\epsilon^K \longrightarrow \mathbf{PN}_\epsilon^K$ for $\alpha \geq 0$. This is via Taylor series:

$$h(a\epsilon^0 + b\epsilon^\beta) = h(a + b\epsilon^\beta) = h(a) + b\epsilon^\beta h'(a)$$
$$h(a\epsilon^\alpha + b\epsilon^\beta) = h(0) + a\epsilon^\alpha h'(0) \quad where \quad \alpha \neq 0$$

2.6 Asymptotic Series

Consider an asymptotic series that is also a Hahn series; that is,

$$S(z) = \sum_{n=1}^{\infty} a_n z^{\alpha_n} \quad as \quad z \longrightarrow 0 \tag{1}$$

where the $\alpha_n \in \mathbb{R}$ are strictly increasing. The primary significance of Power Numbers comes from the following

Proposition 2.2. *Let $f(z) = az^\alpha + bz^\beta + o(z^\beta)$, $g(z) = cz^\gamma + dz^\delta + o(z^\delta)$ be series of the form in (1). Define the associated Power Numbers, $F = a\epsilon^\alpha + b\epsilon^\beta$, $G = c\epsilon^\gamma + d\epsilon^\delta$. Then, where $\cdot$ is one of $(+, -, *, /)$ we have that*

$$f(z) \cdot g(z) = pz^\zeta + qz^\eta + o(z^\eta)$$
$$\Leftrightarrow$$
$$F \cdot G = p\epsilon^\zeta + q\epsilon^\eta$$

Furthermore, for any analytic function $h : K \longrightarrow K$ we have

$$h(f(z)) = pz^\zeta + qz^\eta + o(z^\eta)$$
$$\Leftrightarrow$$
$$h(F) = p\epsilon^\zeta + q\epsilon^\eta$$

The proof is trivial.

3 EXAMPLES

3.1 Rational Functions At Infinity

4 CONCLUSION