PowerNumbers.jl: a fast approach to automatic asymptotics

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We develop a scheme for quick arithmetic on asymptotic series, evaluated to just one or two terms. We give a full description of the algebraic rules for these "Power Numbers", with justification of sufficient equivalence to asymptotic algebra. Some example applications follow, including evaluation of rational functions at infinity.

CCS Concepts: • **Mathematics of computing** → **Solvers**.

Additional Key Words and Phrases: none

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1 INTRODUCTION

2 DESCRIPTION

Let K be a field. The set of functions of $\epsilon \in [0, \infty)$,

$$\mathbf{PN}_\epsilon^K = \{a\epsilon^\alpha + b\epsilon^\beta : a, b \in K; \alpha, \beta \in \mathbb{R} \cup \{\infty\}; \alpha \leq \beta\}$$

is called the set of Power Numbers. In the case $\alpha = \beta$, we write only one term $(a+b)\epsilon^\beta$. Notationally, ϵ^0 is omitted. We enforce the equality $0\epsilon^\alpha + b\epsilon^\beta = b\epsilon^\beta$. However, in general, $a\epsilon^\alpha + 0\epsilon^\beta \neq a\epsilon^\alpha$. By defining $(+, *)$ below, we acquire the double monoid

$$\mathbb{PN}_\epsilon^K = (\mathbf{PN}_\epsilon^K, +, *)$$

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2.1 Addition

$+ : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is described by an algorithm.

Data: $a\epsilon^\alpha + b\epsilon^\beta, c\epsilon^\gamma + d\epsilon^\delta \in \mathbf{PN}_\epsilon^K$; assume WLOG that $\beta \leq \delta$
Result: $(a\epsilon^\alpha + b\epsilon^\beta) + (c\epsilon^\gamma + d\epsilon^\delta) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$

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if β = δ then
    if γ = β then
        p = a, q = b + c + d, ζ = α, η = β
    else if α < γ < β then
        p = a, q = c, ζ = α, η = γ
    else if γ = α then
        p = a + c, q = b + d, ζ = α, η = β
    else
        p = c, q = a, ζ = γ, η = α
    end
else
    if β < γ then
        p = a, q = b, ζ = α, η = β
    else if γ = β then
        p = a, q = b + c, ζ = α, η = β
    else if α < γ < β then
        p = a, q = c, ζ = α, η = γ
    else if γ = α then
        p = a + c, q = b, ζ = α, η = β
    else
        p = c, q = a, ζ = γ, η = α
    end
end
```

Algorithm 1: Summing Power Numbers

The additive identity for Power Numbers is $0\epsilon^\infty$.

2.2 Multiplication

$* : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ can be most simply expressed as addition of Power Numbers:

$$(a\epsilon^\alpha + b\epsilon^\beta) * (c\epsilon^\gamma + d\epsilon^\delta) = (ac\epsilon^{\alpha+\gamma} + ad\epsilon^{\alpha+\delta}) + (bc\epsilon^{\beta+\gamma} + bd\epsilon^{\beta+\delta}) = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$$

The multiplicative identity for Power Numbers is $1 + 0\epsilon^\infty$.

2.3 Pseudo-Negation

While $\mathbb{PN}_\epsilon^K$ is a monoid under both addition and multiplication, we can define two operations that have some of the properties we expect for subtraction and division.
$- : \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is defined as:

$$-(a\epsilon^\alpha + b\epsilon^\beta) = (-a)\epsilon^\alpha + (-b)\epsilon^\beta$$

Naturally, $- : \mathbf{PN}_\epsilon^K \times \mathbf{PN}_\epsilon^K \to \mathbf{PN}_\epsilon^K$ is defined as $A + (-(B))$, where $A, B \in \mathbf{PN}_\epsilon^K$. Note that for $\beta \neq \infty$, we have $(a\epsilon^\alpha + b\epsilon^\beta) - (a\epsilon^\alpha + b\epsilon^\beta) = 0\epsilon^\beta \neq 0\epsilon^\infty$.

2.4 Pseudo-Inverse

The multiplicative pseudo-inverse is slightly more complicated.

Data: $a\epsilon^\alpha + b\epsilon^\beta \in \mathbf{PN}_\epsilon^K$
Result:

$$\frac{1}{a\epsilon^\alpha + b\epsilon^\beta} = p\epsilon^\zeta + q\epsilon^\eta \in \mathbf{PN}_\epsilon^K$$

if $\alpha = \beta$ **then**
  $p = 0, q = \frac{1}{a+b}, \zeta = -\alpha, \eta = -\alpha$
else
  $p = \frac{1}{a+b}, q = -\frac{b}{a^2}, \zeta = -\alpha, \eta = \beta - 2\alpha$
end

Algorithm 2: Multiplicative Inversion

This matches the Dual Numbers case, where $\alpha = 0$, $\beta = 1$. Similarly to subtraction,

$$\frac{a\epsilon^\alpha + b\epsilon^\beta}{a\epsilon^\alpha + b\epsilon^\beta} \neq 1 + 0\epsilon^\infty$$

3 EXAMPLES

4 CONCLUSION