

PowerNumbers.jl: a fast approach to automatic asymptotics

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We develop a scheme for quick arithmetic on asymptotic series, evaluated to just one or two terms. We give a full description of the algebraic rules for these "Power Numbers", with justification of sufficient equivalence to asymptotic algebra. Some example applications follow, including evaluation of rational functions at infinity.

CCS Concepts: • **Mathematics of computing** → **Solvers**.

Additional Key Words and Phrases: none

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1 INTRODUCTION

2 DESCRIPTION

Definition 2.1. Let K be a field. The set of functions of $\epsilon \in [0, \infty)$,

$$\mathbf{PN}_\epsilon^K = \{a\epsilon^\alpha + b\epsilon^\beta : a, b \in K; \alpha, \beta \in \mathbb{R} \cup \{\infty\}; \alpha \leq \beta\}$$

is called the set of **Power Numbers**.

In the case $\alpha = \beta$, we write only one term $(a + b)\epsilon^\beta$. Notationally, ϵ^0 is omitted. We enforce the equality $0\epsilon^\alpha + b\epsilon^\beta = b\epsilon^\beta$. However, in general, $a\epsilon^\alpha + 0\epsilon^\beta \neq a\epsilon^\alpha$. By defining $(+, *)$ below, we acquire the double monoid

$$\mathbf{PN}_\epsilon^K = (\mathbf{PN}_\epsilon^K, +, *)$$

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2.1 Addition

$+$: $\text{PN}_\epsilon^K \times \text{PN}_\epsilon^K \rightarrow \text{PN}_\epsilon^K$ is described by an algorithm.

Data: $a\epsilon^\alpha + b\epsilon^\beta, c\epsilon^\gamma + d\epsilon^\delta \in \text{PN}_\epsilon^K$; assume WLOG that $\beta \leq \delta$

Result: $(a\epsilon^\alpha + b\epsilon^\beta) + (c\epsilon^\gamma + d\epsilon^\delta) = p\epsilon^\zeta + q\epsilon^\eta \in \text{PN}_\epsilon^K$

if $\beta = \delta$ **then**

if $\gamma = \beta$ **then**

$p = a, q = b + c + d, \zeta = \alpha, \eta = \beta$

else if $\alpha < \gamma < \beta$ **then**

$p = a, q = c, \zeta = \alpha, \eta = \gamma$

else if $\gamma = \alpha$ **then**

$p = a + c, q = b + d, \zeta = \alpha, \eta = \beta$

else

$p = c, q = a, \zeta = \gamma, \eta = \alpha$

end

else

if $\beta < \gamma$ **then**

$p = a, q = b, \zeta = \alpha, \eta = \beta$

else if $\gamma = \beta$ **then**

$p = a, q = b + c, \zeta = \alpha, \eta = \beta$

else if $\alpha < \gamma < \beta$ **then**

$p = a, q = c, \zeta = \alpha, \eta = \gamma$

else if $\gamma = \alpha$ **then**

$p = a + c, q = b, \zeta = \alpha, \eta = \beta$

else

$p = c, q = a, \zeta = \gamma, \eta = \alpha$

end

end

Algorithm 1: Summing Power Numbers

The additive identity for Power Numbers is $0\epsilon^\infty$.

2.2 Multiplication

$*$: $\text{PN}_\epsilon^K \times \text{PN}_\epsilon^K \rightarrow \text{PN}_\epsilon^K$ can be most simply expressed as addition of Power Numbers:

$$(a\epsilon^\alpha + b\epsilon^\beta) * (c\epsilon^\gamma + d\epsilon^\delta) = (ac\epsilon^{\alpha+\gamma} + ad\epsilon^{\alpha+\delta}) + (bc\epsilon^{\beta+\gamma} + bd\epsilon^{\beta+\delta}) = p\epsilon^\zeta + q\epsilon^\eta \in \text{PN}_\epsilon^K$$

The multiplicative identity for Power Numbers is $1 + 0\epsilon^\infty$.

2.3 Pseudo-Negation

While PN_ϵ^K is a monoid under both addition and multiplication, we can define two operations that have some of the properties we expect for subtraction and division.

$-$: $\text{PN}_\epsilon^K \rightarrow \text{PN}_\epsilon^K$ is defined as:

$$-(a\epsilon^\alpha + b\epsilon^\beta) = (-a)\epsilon^\alpha + (-b)\epsilon^\beta$$

Naturally, $-$: $\text{PN}_\epsilon^K \times \text{PN}_\epsilon^K \rightarrow \text{PN}_\epsilon^K$ is defined as $A + (-B)$, where $A, B \in \text{PN}_\epsilon^K$. Note that for $\beta \neq \infty$, we have $(a\epsilon^\alpha + b\epsilon^\beta) - (a\epsilon^\alpha + b\epsilon^\beta) = 0\epsilon^\beta \neq 0\epsilon^\infty$.

2.4 Pseudo-Inverse

The multiplicative pseudo-inverse is slightly more complicated.

Data: $a\epsilon^\alpha + b\epsilon^\beta \in \text{PN}_\epsilon^K$

Result:

$$\frac{1}{a\epsilon^\alpha + b\epsilon^\beta} = p\epsilon^\zeta + q\epsilon^\eta \in \text{PN}_\epsilon^K$$

if $\alpha = \beta$ **then**

$p = 0, q = \frac{1}{a+b}, \zeta = -\alpha, \eta = -\alpha$

else

$p = \frac{1}{a+b}, q = -\frac{b}{a^2}, \zeta = -\alpha, \eta = \beta - 2\alpha$

end

Algorithm 2: Multiplicative Inversion

This matches the Dual Numbers case, where $\alpha = 0, \beta = 1$. Similarly to subtraction,

$$\frac{a\epsilon^\alpha + b\epsilon^\beta}{a\epsilon^\alpha + b\epsilon^\beta} \neq 1 + 0\epsilon^\infty$$

2.5 Exponentiation & Analytic Functions

The definition of A^m for $A \in \text{PN}_\epsilon^K, m \in \mathbb{Z}$ follows naturally from the above definitions of multiplication and division.

Since we will generally consider ϵ to be small, we can extend any analytic function $h : K \rightarrow K$ to $h : \text{PN}_\epsilon^K \rightarrow \text{PN}_\epsilon^K$ for $\alpha \geq 0$. This is via Taylor series:

$$\begin{aligned} h(a\epsilon^0 + b\epsilon^\beta) &= h(a + b\epsilon^\beta) = h(a) + b\epsilon^\beta h'(a) \\ h(a\epsilon^\alpha + b\epsilon^\beta) &= h(0) + a\epsilon^\alpha h'(0) \quad \text{where } \alpha \neq 0 \end{aligned}$$

2.6 Asymptotic Series

Consider an asymptotic series that is also a Hahn series; that is,

$$S(z) = \sum_{n=1}^{\infty} a_n z^{\alpha_n} \quad \text{as } z \rightarrow 0 \quad (1)$$

where the $\alpha_n \in \mathbb{R}$ are strictly increasing. The primary significance of Power Numbers comes from the following

PROPOSITION 2.2. *Let $f(z) = az^\alpha + bz^\beta + o(z^\beta), g(z) = cz^\gamma + dz^\delta + o(z^\delta)$ be series of the form in (1). Define the associated Power Numbers, $F = a\epsilon^\alpha + b\epsilon^\beta, G = c\epsilon^\gamma + d\epsilon^\delta$. Then, where \cdot is one of $(+, -, *, /)$ we have that*

$$f(z) \cdot g(z) = pz^\zeta + qz^\eta + o(z^\eta)$$

$$\Leftrightarrow$$

$$F \cdot G = p\epsilon^\zeta + q\epsilon^\eta$$

Furthermore, for any analytic function $h : K \rightarrow K$ we have

$$h(f(z)) = pz^\zeta + qz^\eta + o(z^\eta)$$

$$\Leftrightarrow$$

$$h(F) = p\epsilon^\zeta + q\epsilon^\eta$$

3 EXAMPLES

3.1 Rational Polynomials at Infinity

We can get a simple system for evaluating a rational polynomial in the infinite limit. For example, consider:

$$f(z) = \frac{z^3 + z^2 + 1}{z^3 + z}$$

Taking the Power Number $z = \epsilon^{-1}$. By the arithmetic described above we have:

$$\frac{(\epsilon^{-3}) + (\epsilon^{-2}) + 1}{(\epsilon^{-3}) + (\epsilon^{-1})} = \frac{\epsilon^{-3}}{\epsilon^{-3}} = \epsilon^0$$

Which gives the correct limit when considering $\epsilon \rightarrow 0$.

4 CONCLUSION