Assignment: Theory of Games and Statistical Decisions

RAMIT NANDI (MD2211)

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Contents

m=1 for all problems.

Problem 1.

The matrix game A is given by:

```
m = 1
(A = rbind(c(10+m,5+m,0,0),
           c(5+m,10+m,5+m,0),
           c(0,5+m,10+m,5+m),
           c(0,0,5+m,10+m)))
```

```
##      [,1] [,2] [,3] [,4]
## [1,]   11    6    0    0
## [2,]    6   11    6    0
## [3,]    0    6   11    6
## [4,]    0    0    6   11
```

We have to find $v(A)$, the value of the above matrix game A in mixed extensions.

Let $A = ((a_{ij}))$, where a_{ij} represents the payoff to player 1 if player 1 chooses row i and player 2 chooses column j. The strategies of player 1 are denoted by x_i, which in turn represents the probability of player 1 choosing row i. Similarly, the strategies of player 2 are denoted by y_j.

The mixed extension of the matrix game involves finding the **optimal strategies** for both players that maximize the payoff of player 1 and minimizes the payoff of player 2 simultaneously. Introducing a dummy variable z, the problem of obtaining the value of the mixed extension of the matrix game can be formulated as the following optimization problem:

$$\textbf{Maximize } z$$

subject to:

- $\sum_{i=1}^{4} x_i = 1$ (sum of probabilities of strategies of player 1 is 1)
- $\sum_{j=1}^{4} y_j = 1$ (sum of probabilities of strategies of player 2 is 1)
- $x_i \geq 0 \quad i = 1, 2, 3, 4$ (probabilities are non-negative)
- $y_j \geq 0 \quad j = 1, 2, 3, 4$ (probabilities are non-negative)
- $z \leq \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij} x_i y_j$ (maximize payoff of player 1)
- $z \geq \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij} x_i y_j$ (minimize payoff of player 2)

This can be equivalently formulated as (by removing z):

$$\textbf{Maximize } \sum_{i=1}^{4} \sum_{j=1}^{4} a_{ij} x_i y_j$$

subject to:

- $\sum_{i=1}^{4} x_i = 1$ (sum of probabilities of strategies of player 1 is 1)
- $\sum_{j=1}^{4} y_j = 1$ (sum of probabilities of strategies of player 2 is 1)
- $x_i \geq 0 \quad i = 1, 2, 3, 4$ (probabilities are non-negative)
- $y_j \geq 0 \quad j = 1, 2, 3, 4$ (probabilities are non-negative)

The **R** code for performing the following task is given below:

```
## matrix game A : same as printed earlier


## objective function ....
# formed as minimization problem
objective_function <- function(x){
  return(-(x[1:4]%*%A%*%x[5:8]))
}


## constraints ....
# Constraint: Probability vectors sum is 1
constraints <- function(x){
  c(sum(x[1:4])-1, # constraint 1
    sum(x[5:8])-1) # constraint 2
}
## bounds ....
# probabilities are in [0,1]
l = rep(0,8)
u = rep(1,8)

## initial guess ....
x0 = c(rep(1/4,4),rep(1/4,4))


# Perform constrained optimization ....
result=nloptr::slsqp(x0,objective_function,
             lower=l,upper=u,heq=constraints)
value = -result$value
```

Implementing the above code, we get $v(A)$ as follows

```
## [1] 8.5
```

> *Note* :: **nloptr::slsqp()** uses Sequential Least Squares Programming (SLSQP) for solving minimization problems. SLSQP is a gradient-based method which iteratively approximates the objective and constraint functions using quadratic models and solves a sequence of constrained quadratic subproblems to find the optimal solution.

Problem 2.

There is a set of players $N = \{1, 2, 3, ..., 10\}$ and it is divided into two subsets L and R, such that $L \neq \phi$, $R \neq \phi$, $L \cap R = \phi$ and $L \cup R = N$. Each player of L has p left hand gloves and no right hand glove. Similarly, each player of R has q right hand gloves and no left hand glove. Here, $p, q \in \mathbb{N}$. It is given that a single glove is worth nothing and a right-left pair of gloves is worth Rs. 50.

Let $|L|$ and $|R|$ denote the cardinalities of the subsets L and R respectively. We note that $|L|, |R| \in \mathbb{N}$

(i): Characteristic Function

For any coalition of players S, the characteristic function will be:

$$\boxed{\boldsymbol{v(S) = 50.\min\{p|L \cap S|, q|R \cap S|\}}}$$

Justification: Assuming that each left glove has its own specific right glove, the total number of *(complete)* pairs which can be formed by each member of S would be the smaller number of common members between S and L or that with R. Now, since each member of L has p left gloves and correspondingly each member of R has q right gloves, the above characteristic function is formulated.

(ii): Superadditivity of Characteristic Function

To show: $\boldsymbol{v(S \cup T) \geq v(S) + v(T)}$ for $S \cap T = \phi$

$$\begin{aligned}
v(S \cup T) &= 50.\min\{p|L \cap (S \cup T)|, q|R \cap (S \cup T)|\} \\
&= 50.\min\{p|(L \cap S) \cup (L \cap T)|, q|(R \cap S) \cup (R \cap T)|\} \\
&\geq 50.\min\{p|L \cap S| + p|L \cap T|, q|R \cap S| + q|R \cap T|\} \\
&\geq 50.\min\{p|L \cap S|, q|R \cap S|\} + 50.\min\{p|L \cap T|, q|R \cap T|\} \\
&= v(S) + v(T)
\end{aligned}$$

This proves that $v(S)$ is indeed **superadditive**.

(iii)

We assume that $|L| = |R| = 5$. Also,

$$p + q = 2[\lfloor \frac{1}{2} \rfloor + 4] = 2[0 + 4] = 8$$

$$\implies q = 8 - p$$

Our characteristic function thus becomes,

$$v(S) = 50 \times \min\{p|L \cap S|, (8-p)|R \cap S|\}$$

Now, we have to compute $\boldsymbol{f(p) := \frac{1}{2^{10}-1} \sum_{S \subseteq N, S \neq \phi} \frac{v_p(S)}{|S|}}$. Let us first try to compute $f(p)$ analytically.

Let us consider a specific non-empty subset $S = S_1$ such that $|S_1| = k \quad 1 \leq k \leq 10$. Since the two coalitions L and R are of fixed size 5, the following situations may arise:

- $|L \cap S| = k$ and $|R \cap S| = 0$ (i.e., all members of S are common with L). The total number of possible subsets in which this is possible is $\binom{5}{k}\binom{5}{0}$ and the total payoff would be $50 \times \min\{kp, 0.q\} = 50 \times 0 = 0$
- $|L \cap S| = k-1$ and $|R \cap S| = 1$. The total number of possible subsets in which this is possible is $\binom{5}{k-1}\binom{5}{1}$ and the total payoff would be $50 \times \min\{(k-1)p, 1.q\}$.

...

- $|L \cap S| = 0$ and $|R \cap S| = k$. The total number of possible subsets in which this is possible is $\binom{5}{0}\binom{5}{k}$ and the total payoff would be $50 \times \min\{0.p, kq\} = 0$.

Generalizing this idea for all subsets $S \subseteq N, S \neq \phi$ (i.e. k=1,2,...,10), we get the following expression for $f(p)$:

$$\boxed{\boldsymbol{f(p) = \frac{50}{2^{10}-1} \sum_{k=1}^{10} \frac{1}{k} \sum_{j=1}^{k} \binom{5}{j}\binom{5}{|k-j|} \min\{jp, |k-j|(8-p)\}}, \quad 1 \leq p \leq 7}$$

Note: $\binom{n}{k} = 0$ when $k > n$.

We then use **R** to find the maximum value of $f(p)$. The following code carries out the necessary computation.

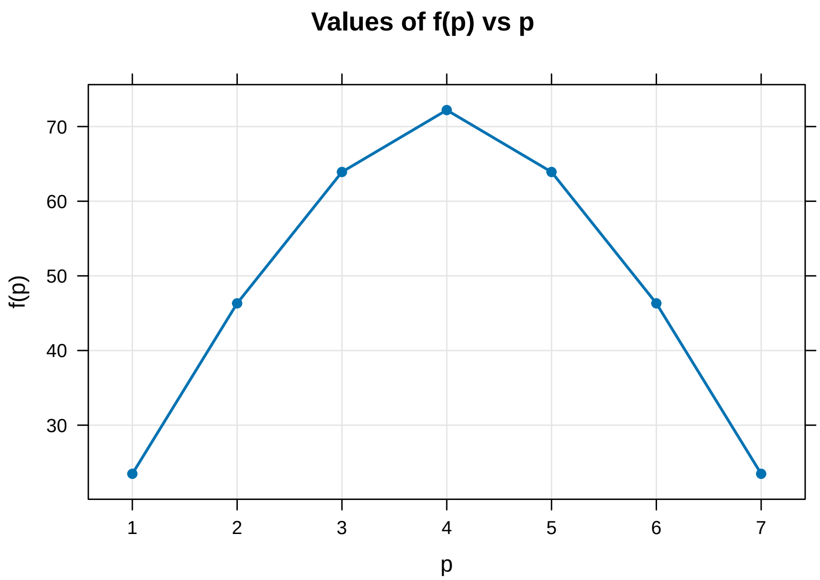
```
L = 1:5
R = 6:10
p_plus_q = 2*(floor(m/2)+4)

## Characteristic function v_p(S) ....
v_p = function(p,S){
  return(50 * min(p*length(intersect(S,L)), (p_plus_q-p)*length(intersect(S,R))))
}

## the function to be computed ....
```

```r
library(foreach)
f = function(p){
  # sum over possible values of k
  sum_v = foreach(k = 1:10,.combine = sum) %do% {
    subsets = combn(10,k,simplify = F)
    # v_p(p,S)/k over all possible subsets of size k
    (sapply(subsets, function(S) v_p(p,S)))/k
  }
  # return value
  return(sum_v/(2^10 - 1))
}
```

We want to check for which value of p if $f(p)$ maximum. It reveals that $f(p)$ is maximum when $\boldsymbol{p = \frac{p+q}{2} = 4}$, the middlemost value in the range of variation of p.



The following table lists the values of $f(p)$ against p.

p	f(p)
1	23.48485
2	46.31802
3	63.91449
4	72.20903
5	63.91449
6	46.31802
7	23.48485