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Домашняя работа №3

№9

x_1	0	1	0	2	2	2	4	3
x_2	-1	0	0	0	1	0	1	2
y	0	0	0	0	0	1	1	1

$N=8,\ K=2$

$N_0=5,\ N_1=3.$

$\hat{P}_N\{Y=0\} = 5/8\ ,\ \hat{P}_N\{Y=1\} = 3/8$

$\hat{\mu}_0 = \begin{pmatrix}1\\0\end{pmatrix},\ \hat{\mu}_1 = \begin{pmatrix}3\\1\end{pmatrix}$

$$\hat{\Sigma}_0 = \frac{1}{N_0-1}\sum_{y^{(i)}=0}(x^{(i)}-\hat{\mu}_0)(x^{(i)}-\hat{\mu}_0)^T = \frac{1}{4}\left(\begin{pmatrix}-1\\-1\end{pmatrix}(-1\ -1) + \begin{pmatrix}0\\0\end{pmatrix}(0\ 0) + \right.$$
$$\left. + \begin{pmatrix}-1\\0\end{pmatrix}(-1\ 0) + \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \begin{pmatrix}1\\1\end{pmatrix}(1\ 1)\right) = \frac{1}{4}\left(\begin{pmatrix}1&1\\1&1\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix} + \right.$$
$$\left. + \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}1&1\\1&1\end{pmatrix}\right) = \frac{1}{4}\begin{pmatrix}4&2\\2&2\end{pmatrix} = \begin{pmatrix}1&1/2\\1/2&1/2\end{pmatrix}$$

$$\hat{\Sigma}_1 = \frac{1}{N_1-1}\sum_{y^{(i)}=1}(x^{(i)}-\hat{\mu}_1)(x^{(i)}-\hat{\mu}_1)^T = \frac{1}{2}\left(\begin{pmatrix}-1\\-1\end{pmatrix}(-1\ -1) + \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \right.$$
$$\left. + \begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\right) = \frac{1}{2}\left(\begin{pmatrix}1&1\\1&1\end{pmatrix} + \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}2&1\\1&2\end{pmatrix} = \begin{pmatrix}1&1/2\\1/2&1\end{pmatrix}$$

$$\hat{\Sigma} = \frac{1}{N-K}\sum_k\sum_{y^{(i)}=k}(x^{(i)}-\hat{\mu}_k)(x^{(i)}-\hat{\mu}_k)^T = \frac{1}{6}\left(\begin{pmatrix}4&2\\2&2\end{pmatrix} + \begin{pmatrix}2&1\\1&2\end{pmatrix}\right) =$$
$$= \frac{1}{6}\begin{pmatrix}6&3\\3&4\end{pmatrix} = \begin{pmatrix}1&1/2\\1/2&2/3\end{pmatrix}$$

$$\hat{\Sigma}_0^{-1} = \begin{pmatrix}2&-2\\-2&4\end{pmatrix},\ \hat{\Sigma}_1^{-1} = \frac{1}{3}\begin{pmatrix}4&-2\\-2&4\end{pmatrix},\ \hat{\Sigma}^{-1} = \frac{1}{5}\begin{pmatrix}8&-6\\-6&12\end{pmatrix}$$

① Линейные дискриминантные функции:

$$\delta_0(x) = x^T\hat{\Sigma}^{-1}\hat{\mu}_0 - \frac{1}{2}\hat{\mu}_0^T\hat{\Sigma}^{-1}\hat{\mu}_0 + \ln\hat{P}_N\{Y=0\} = (x_1\ \ x_2)\frac{1}{5}\begin{pmatrix}8&-6\\-6&12\end{pmatrix}\cdot$$
$$\cdot\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{2}(1\ 0)\frac{1}{5}\begin{pmatrix}8&-6\\-6&12\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \ln 5/8 = \frac{8}{5}x_1 - \frac{6}{5}x_2 - \frac{8}{10} + \ln\frac{5}{8}.$$

$$\delta_1(x) = x^T\hat{\Sigma}^{-1}\hat{\mu}_1 - \frac{1}{2}\hat{\mu}_1^T\hat{\Sigma}^{-1}\hat{\mu}_1 + \ln P_N\{Y=1\} = (x_1\ \ x_2)\frac{1}{5}\begin{pmatrix}8&-6\\-6&12\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} -$$

$$-\frac{1}{2}(3\ \ 1)\frac{1}{5}\begin{pmatrix}8 & -6\\ -6 & 12\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}+\ln{}^{3}/_{8} = \frac{18}{5}x_1-\frac{6}{5}x_2-\frac{48}{10}+\ln{}^{3}/_{8}$$

$$\delta_0 = \delta_1 - \frac{10}{5}x_1+\frac{40}{10}+\ln{}^{5}/_{3}$$

Ур-е разделяющей пов-ти: $-2x_1+4+\ln{}^{5}/_{3}=0$.

② Квадратичные дискриминантные ф-ии:

$$\delta_0(x) = -\frac{1}{2}\ln\det\hat{\Sigma}_0-\frac{1}{2}(x-\hat{\mu}_0)^T\hat{\Sigma}_0^{-1}(x-\hat{\mu}_0)+\ln P_r\{Y=0\} \ ③$$

$$\det\hat{\Sigma}_0 = \det\begin{vmatrix}1 & 1/2\\ 1/2 & 1/2\end{vmatrix} = 1/2-1/4=1/4$$

$$③ -\frac{1}{2}\ln\frac{1}{4}-\frac{1}{2}\begin{pmatrix}x_1-1\\x_2\end{pmatrix}^T\begin{pmatrix}2 & -2\\-2 & 4\end{pmatrix}\begin{pmatrix}x_1-1\\x_2\end{pmatrix}+\ln{}^{5}/_{8} =$$

$$= -\frac{1}{2}\ln\frac{1}{4}+\ln\frac{5}{8}-\frac{1}{2}(2x_1^2-4x_1+4x_2-4x_1x_2+4x_2^2+2) =$$

$$= \ln\frac{5}{4}-x_1^2+2x_1-2x_2+2x_1x_2-2x_2^2-1$$

$$\delta_1(x) = -\frac{1}{2}\ln\det\hat{\Sigma}_1-\frac{1}{2}(x-\hat{\mu}_1)^T\hat{\Sigma}_1^{-1}(x-\hat{\mu}_1)+\ln\hat{P}_r\{Y=1\} \ ③$$

$$\det\hat{\Sigma}_1 = \det\begin{vmatrix}1 & 1/2\\ 1/2 & 1\end{vmatrix} = 1-1/4=3/4$$

$$③ -\frac{1}{2}\ln{}^{3}/_{4}-\frac{1}{6}\begin{pmatrix}x_1-3\\x_2-1\end{pmatrix}^T\begin{pmatrix}4 & -2\\-2 & 4\end{pmatrix}\begin{pmatrix}x_1-3\\x_2-1\end{pmatrix}+\ln{}^{3}/_{8} =$$

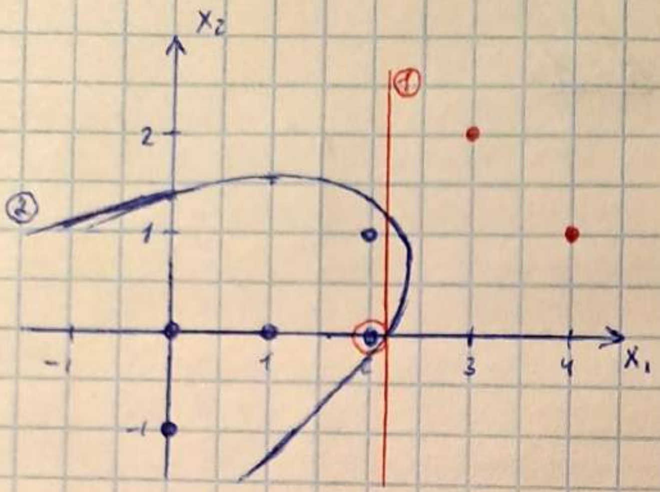
$$= -\frac{1}{2}\ln\frac{3}{4}+\ln\frac{3}{8}-\frac{1}{6}(4x_1^2+4x_2^2-4x_1x_2+28+4x_2-20x_1) =$$

$$= \ln\frac{\sqrt{3}}{4}+(-2x_1^2-2x_2^2+2x_1x_2-14-2x_2+10x_1)\frac{1}{3} =$$

$$= \ln\frac{\sqrt{3}}{4}-\frac{2}{3}x_1^2-\frac{2}{3}x_2^2+\frac{2}{3}x_1x_2-\frac{14}{3}-\frac{2}{3}x_2+\frac{10}{3}x_1$$

$$\delta_0 = \delta_1-\ln{}^{\sqrt{3}}/_{5}-\frac{1}{3}x_1^2-\frac{4}{3}x_2^2+\frac{4}{3}x_1x_2+\frac{11}{3}-\frac{4}{3}x_2-\frac{4}{3}x_1$$

Уравнение разд-й пов-ти: $\ln\frac{\sqrt{3}}{5}+\frac{1}{3}x_1^2+\frac{4}{3}x_2^2-\frac{4}{3}x_1x_2-\frac{11}{3}+$
$+\frac{4}{3}x_2+\frac{4}{3}x_1=0$



№15

X_1	0	0	1	1	0	0	1	1	1	0
X_2	0	1	0	1	1	1	1	1	1	1
y	0	0	0	0	0	1	1	1	1	1

$\hat{P}_n\{Y=0\} = 1/2$, $\hat{P}_n\{Y=1\} = 1/2$

$\hat{P}_n\{X_1=0/Y=0\} = 3/5$, $\hat{P}_n\{X_1=1/Y=0\} = 2/5$, $\hat{P}_n\{X_2=0/Y=0\} = 2/5$,
$\hat{P}_n\{X_2=1/Y=1\} = 3/5$

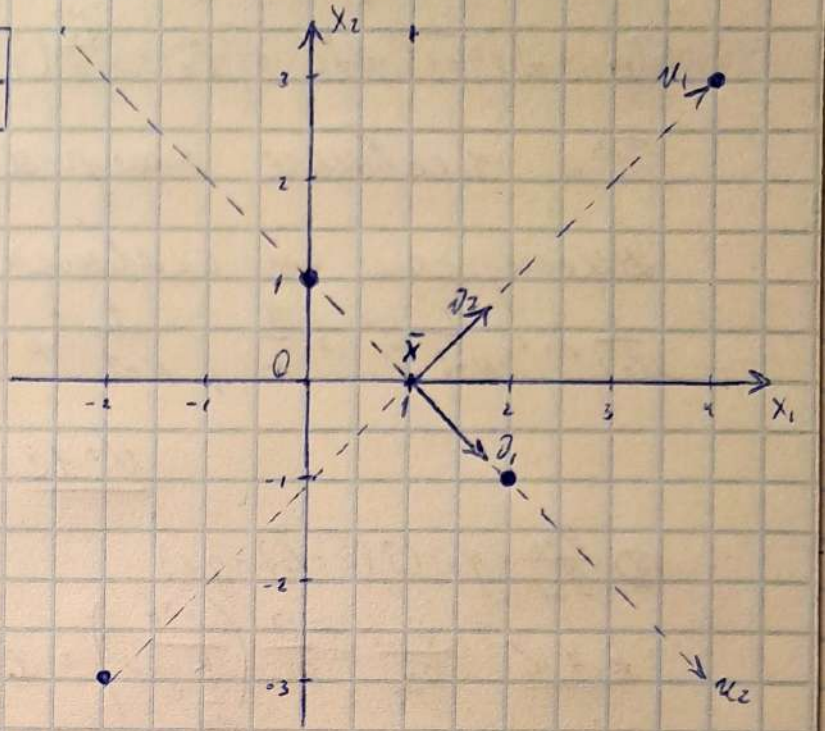
$\hat{P}_n\{X_1=0/Y=1\} = 2/5$, $\hat{P}_n\{X_1=1/Y=1\} = 3/5$, $\hat{P}_n\{X_2=0/Y=1\} = 0$,
$P_n\{X_2=1/Y=1\} = 1$

$$P_n\{Y=0/X_1=1, X_2=1\} = \frac{P_n\{X_1=1/Y=0\}\,P_n\{X_2=1/Y=0\}\,P_n\{Y=0\}}{P_n\{X_1=1, X_2=1\}} \approx$$
$$\approx \frac{2/5 \cdot 3/5 \cdot 1/2}{3/25 + 3/10} = 2/7$$

$$P_n\{Y=1/X_1=1, X_2=1\} = \frac{P_n\{X_1=1/Y=1\}\,P_n\{X_2=1/Y=1\}\,P_n\{Y=1\}}{P_n\{X_1=1, X_2=1\}} \approx$$
$$\approx \frac{3/5 \cdot 1 \cdot 1/2}{3/25 + 3/10} = 5/7.$$

№16

x_1	4	0	-2	2
x_2	3	1	-3	-1



$$X = \begin{pmatrix} 4 & 3 \\ 0 & 1 \\ -2 & -3 \\ 2 & -1 \end{pmatrix}$$

$\bar{x} = (1, 0)$

$$X_c = \begin{pmatrix} 3 & 3 \\ -1 & 1 \\ -3 & -3 \\ 1 & -1 \end{pmatrix}$$

$$C = X_c^T X_c = \begin{pmatrix} 3 & -1 & -3 & 1 \\ 3 & 1 & -3 & -1 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ -1 & 1 \\ -3 & -3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 20 & 16 \\ 16 & 20 \end{pmatrix}$$

$$\frac{1}{N-1} C = \begin{pmatrix} 20/3 & 16/3 \\ 16/3 & 20/3 \end{pmatrix}$$

$$\det(C - \lambda I) = \begin{vmatrix} 20-\lambda & 16 \\ 16 & 20-\lambda \end{vmatrix} = 400 - 40\lambda + \lambda^2 - 256 =$$
$$= \lambda^2 - 40\lambda + 144 = (\lambda - 4)(\lambda - 36) = 0$$

$\lambda_1 = 4, \lambda_2 = 36.$

Собственные вектора:

$\underline{\lambda = 4}$: $\begin{pmatrix} 16 & 16 \\ 16 & 16 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \alpha + \beta = 0 \Rightarrow \alpha = -\beta.$

$\vartheta_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$; нормализ-я:

$|\vartheta_1| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$, $\vartheta_1 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\underline{\lambda = 36}$: $\begin{pmatrix} -16 & 16 \\ 16 & -16 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \alpha - \beta = 0 \Rightarrow \alpha = \beta$

$\vartheta_2 = \begin{pmatrix}1\\1\end{pmatrix}$; нормализ-я:

$|\vartheta_2| = \sqrt{1^2+1^2} = \sqrt{2}$, $\vartheta_2 = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}$

Т.о. главные напр-я: $\vartheta_1 = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\-1\end{pmatrix}$, $\vartheta_2 = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}$.

Дисперсии по главным компонентам:

$\frac{1}{N-1}\lambda_1 = \frac{1}{N-1}\sigma_1^2 = \frac{4}{3}$; $\frac{1}{N-1}\lambda_2 = \frac{1}{N-1}\sigma_2^2 = \frac{36}{3} = 12$

№ 19

① $\frac{\partial g_k}{\partial s_l} = g_k \cdot (I(k=l) - g_l)$

$k \neq l$: $\frac{\partial g_k}{\partial s_l} = \frac{\partial}{\partial s_l}\left(\frac{e^{s_k}}{\sum\limits_{l^*=1}^{k} e^{s_{l^*}}}\right) = e^{s_k}\left(\frac{-e^{s_l}}{(\sum\limits_{l^*=1}^{k} e^{s_{l^*}})^2}\right) = \underbrace{\frac{e^{s_k}}{\sum\limits_{l^*=1}^{k} e^{s_{l^*}}}}_{=g_k} \cdot \underbrace{\left(\frac{-e^{s_l}}{\sum\limits_{l^*=1}^{k} e^{s_{l^*}}}\right)}_{=-g_l} = g_k(-g_l)$

$k \neq l$: $\frac{\partial g_k}{\partial s_k} = \frac{\partial}{\partial s_k}\left(\frac{e^{s_k}}{\sum\limits_{l^*=1}^{k} e^{s_{l^*}}}\right) = \frac{e^{s_k}\left(\sum\limits_{l^*=1}^{k} e^{s_{l^*}}\right) - e^{s_k} \cdot e^{s_k}}{\left(\sum\limits_{l^*=1}^{k} e^{s_{l^*}}\right)^2} =$

$= \underbrace{\frac{e^{s_k}}{\sum\limits_{l^*=1}^{k} e^{s_{l^*}}}}_{=g_k} \cdot \left(1 - \underbrace{\frac{e^{s_k}}{\sum\limits_{l^*=1}^{k} e^{s_{l^*}}}}_{=g_k}\right) = g_k(1-g_k)$

$\frac{\partial g_k}{\partial s_l} = g_k(I(k=l) - g_l)$.

② $\frac{\partial R^{(i)}}{\partial g_k} = -\frac{I(y^{(i)})}{g_k}$

$\frac{\partial R^{(i)}}{\partial g_k} = \frac{\partial}{\partial g_k}\left(-\sum\limits_{m=1}^{k} I(y^{(i)}=m)\ln g_m\right) = -\frac{I(y^{(i)}=k)}{g_k}$

③ $\frac{\partial R^{(i)}}{\partial s_l} = g_l - I(l=y^{(i)})$

$\frac{\partial R^{(i)}}{\partial s_l} = \frac{\partial}{\partial s_l}\left(-\sum\limits_{m=1}^{k} I(y^{(i)}=m)\ln g_m(s_1, s_2, \dots, s_k)\right) =$

$$= -\sum_{m=1}^{k} \frac{I(y(i)=m)}{g_m} \cdot \frac{\partial g_m}{\partial s_l} = -\sum_{m=1}^{k} \frac{I(y(i)=m)}{g_m} \cdot g_m \left(I(m=l) - g_l\right) =$$

$$= g_l \underbrace{\sum_{m=1}^{k} I(y(i)=m)}_{=1} - \sum_{m=1}^{k} I(m=l)\, I(y(i)=m) = g_l - I(y(i)=l)$$