Problem 1

Define indicator random variables I_i for $i = 1, 2, \ldots, 2n$ such that:

$$I_i = \begin{cases} 1 & \text{if ball } i \text{ is thrown into bin 1} \\ 0 & \text{if ball } i \text{ is thrown into bin 2} \end{cases}$$

$$X_1 = \sum_{i=1}^{2n} I_i, \quad X_2 = 2n - X_1$$

$$\mathbb{E}[X_1] = \sum_{i=1}^{2n} \mathbb{E}[I_i] = \sum_{i=1}^{2n} \frac{1}{2} = n$$

$$\mathrm{Var}(X_1) = \sum_{i=1}^{2n} \mathrm{Var}(I_i) = \sum_{i=1}^{2n} \left(\frac{1}{2} \cdot \frac{1}{2}\right) = \frac{2n}{4} = \frac{n}{2}$$

$$X_1 - X_2 = X_1 - (2n - X_1) = 2X_1 - 2n$$

$$\Pr[X_1 - X_2 > c\sqrt{n}] = \Pr[2X_1 - 2n > c\sqrt{n}] = \Pr[X_1 - n > \frac{c\sqrt{n}}{2}]$$

According to Chebychev's Inequality,

$$\Pr(X_1 - n \geq a) \leq \frac{n}{2a^2}$$

Let $a = \frac{c\sqrt{n}}{2}$, then

$$\Pr[X_1 - n > a] \leq \frac{2}{c^2}$$

$$\frac{2}{c^2} \leq \epsilon$$

$$c \geq \sqrt{\frac{2}{\epsilon}}$$

Problem 2

Let X_i be the indicator random variable for the i-th run, where $X_i = 1$ if the algorithm produces the correct result on the i-th run and $X_i = 0$ otherwise.

$$\Pr(X_i = 1) = p \geq \frac{2}{3}$$

Define X as the total number of correct results out of n runs:

$$X = \sum_{i=1}^{n} X_i$$

$$\mu = \mathbb{E}[X] = np$$

Suppose we need the majority of the runs to be correct for the final answer to be correct. Then the probability that this modified algorithm produces an incorrect result is:

$$\Pr\left(X \le \frac{n}{2}\right)$$

According to Chernoff Bounds,

$$\Pr(X \le (1-\delta)\mu) \le e^{-\frac{\delta^2\mu}{2}}$$
$$\frac{n}{2} = (1-\delta)\mu$$
$$\delta = 1 - \frac{\frac{n}{2}}{np} = 1 - \frac{1}{2p} \le \frac{1}{4}$$
$$\Pr\left(X \le \frac{n}{2}\right) \le e^{-\frac{\left(\frac{1}{4}\right)^2 np}{2}} = e^{-\frac{np}{32}} \le e^{-\frac{n\cdot\frac{2}{3}}{32}} = e^{-\frac{n}{48}}$$

Problem 3

Suppose the greedy algorithm uses A piggy banks.

Let the optimal schedule uses A^* piggy banks.

The total size of all coins is $C = \sum_{i=1}^{n} c_i$.

$$A^* \ge \frac{S}{V}$$
$$S \le A^*V$$

According to our greedy algorithm, there is at most 1 piggy bank that is less than half full.

$$S \ge \frac{V}{2}(A-1)$$
$$A^*V \ge \frac{V}{2}(A-1)$$
$$A^* \ge \frac{1}{2}(A-1)$$
$$A^* > \frac{A}{2}$$

Therefore, this algorithm is a 2-approximation.

Problem 4

1

Consider any node $v \in T$.

- If $v \in S$, the condition is trivially satisfied.
- If $v \notin S$, it must have been removed from V at some step in the algorithm when one of its neighbors, say v', was added to S.
  - By the algorithm, v' was chosen because $w(v')$ was at least as large as any other weight in the remaining V at that step, which means $w(v) \le w(v')$

2

Let S be the independent set given by the greedy algorithm with weight $W(S)$.

Let S^* be an optimal independent set with maximum total weight $W(S^*)$.

Each node $v \in S^*$ is either in S or is adjacent to a node in S with weight at least as large as v.

Each node v can have up to 4 neighbors in this grid graph G.

Since each node in S can be adjacent to up to 4 nodes in S^*, the total weight of nodes in S^* covered by one node in S is at most 4 times the weight of that node in S.

Therefore, $W(S) \geq \frac{1}{4}W(S^*)$, the greedy algorithm is a 4-approximation for the problem.