Approximation algorithms 1
Set cover, vertex cover, scheduling

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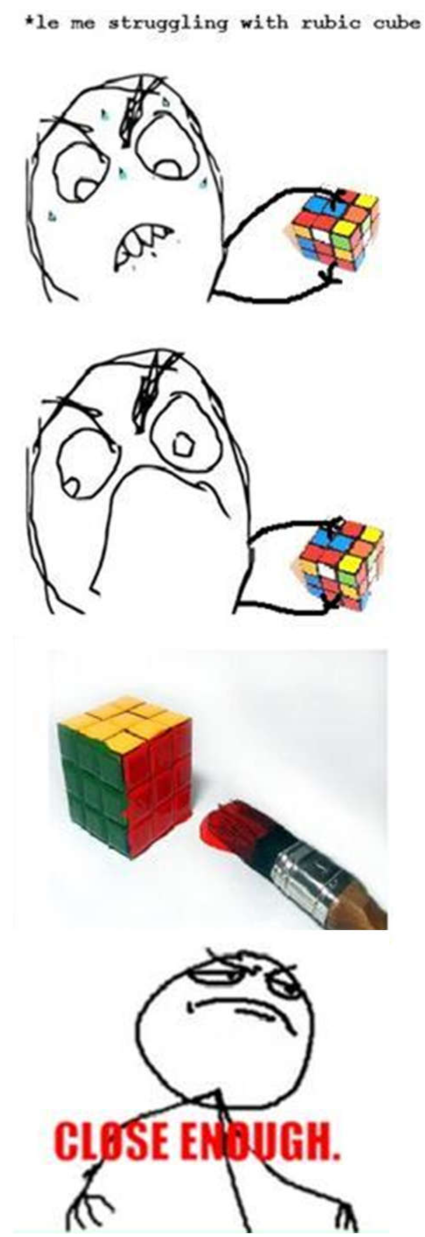
Rui Fan

Approximation algorithms

- Up to now, most of our algorithms have been exact. I.e. they find an optimal solution.
- But there are many problems for which we don't know how to find an optimal solution.
  - A key example is NP-complete problems. We don't know efficient algorithms for any NPC problem.
- Many such problems are important in practice. What do we do?
- If we can't get find the best answer, let's try for good enough.
- Approximation algorithms find an approximately optimal answer.

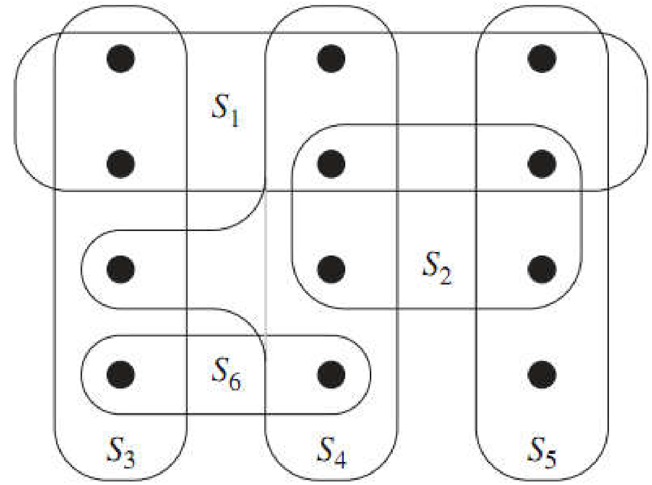


Approximation ratio

- Let X be a maximization problem. Let A be an algorithm for X.
- Let α>1 be a constant.
- A is an α-approximation algorithm for X if A always returns an answer that's at least 1/α times the optimal.
  - Ex If X is max-flow, A is a 2-approx algorithm if it always returns a flow that's at least ½ the optimal.
  - The closer α is to 1, the better the approximation.
- If X is a minimization problem, A is an α-approximation algorithm for X if it always returns an answer that's at most α times larger than the optimal.
  - Ex If X is min-cut, A is a 2-approx algorithm if it always returns a cut that's at most 2 times the size of the optimal.
  - Again, the closer α is to 1, the better the approximation.

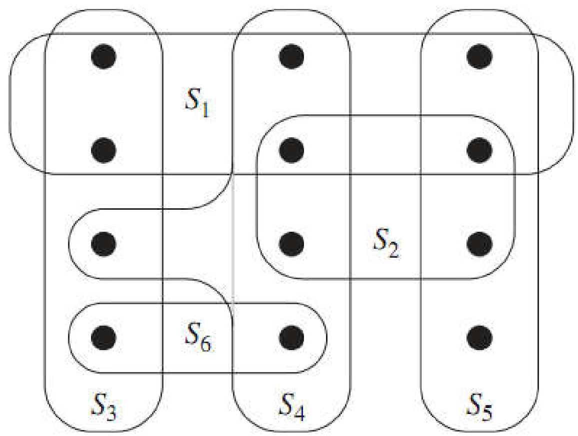
Coverings

- Suppose there's a set of teachers, and each can teach a certain set of classes.
  - Let S_i be the set of classes teach i can teach.
- The entire set of classes is X.
- We want to pick the minimum set of teachers to teach all the classes.
  - Let T be set of teachers we pick.
  - We want $\bigcup_{i \in T} S_i = X$, and T to be the smallest possible.



Set covering

- Input A collection F of sets. Each set has a cost. The union of all the sets is X.
- Output A subset G of F, whose union is X.
- Goal Minimize the total cost of the sets in G.



If all sets have same cost, S_3, S_4 and S_5 is a min cost set cover of X.

- Minimum cost set cover is NP-complete.
- We'll see a ln(n)-approximation algorithm, where n=|X|.

A greedy approximation alg

- A natural greedy heuristic is to choose sets which cover points most cheaply.
  - For each set, let c be its cost, and m be the number of points it covers.
  - We want to use the set with the smallest c/m value, because this is the cheapest way to cover some new points.
- After we pick this set, remove all the points it covers. Then we consider the per unit cost of the remaining sets and again choose the cheapest.

A greedy approximation alg

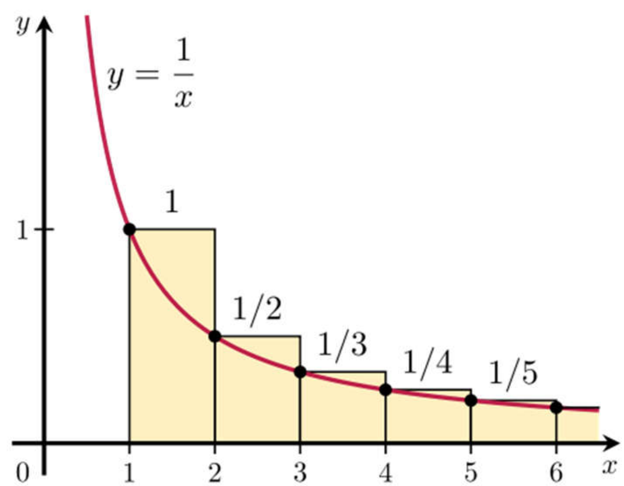
- F is the entire collection of sets. The union of these sets is X.
- Each set S in F has a cost cost(S).
- U is the set of elements of X we haven't covered yet.
- C is the set cover we eventually output.

Per unit cost to cover new elements.

- $U = X$
- $C = \emptyset$
- while $U \neq \emptyset$
  - choose $S \in F - C$ with min $|cost(S)|/|S \cap U|$
  - $C = C \cup \{S\}$
  - $U = U - S$
- output C

Proof of correctness

- We always output a set cover, because the while loop continues till X is covered.
- We'll prove the approximation ratio is at most $1+1/2+1/3+...+1/n \approx \ln(n)$.
  - If the min cost of a set cover is V, our set cover costs at most ln(n)*V.
- The basic plan is to bound the cost of the set cover the algorithm outputs using the "average cost" per element.



Proof of correctness

- Order the sets in C by when they're added to C, earliest set first.
  - Let the order be $S_1, S_2, \ldots, S_m$.
- Cost of the set cover is $L = \Sigma_i \, cost(S_i)$.
- Order the elements in X by when they're added, earliest element first.
  - Let the order be $e_1, e_2, \ldots, e_n$.
  - So, the first few e's are added by S_1, the next few added by S_2, etc.
  - Every element in X is in the list, because C covers X.

Proof of correctness

- Let n_i be the number of new elements S_i covers.
  - So, n_i is the number of elements in S_i, but not in $S_1,\ldots,S_{i-1}$.
- Divide the cost of S_i evenly among the new elements it covers.
  - If e is newly covered by S_i, then cost(e)= $\text{cost}(S_i)/n_i$.
- $\Sigma_k \text{cost}(e_k) = \Sigma_i n_i * \text{cost}(S_i)/n_i = \Sigma_i \text{cost}(S_i) = L$.
  - Every element is covered by some S_i, and S_i covers n_i new elements.
- We'll prove $\text{cost}(e_k) \le OPT/(n-k+1)$, for any k.
- Suppose this is true, then

$$L = \Sigma_k \text{cost}(e_k) \le \Sigma_k OPT/(n-k+1) \approx \ln(n)*OPT$$

The per element cost

- Let's focus on some element e_k, and let S_j be the set which covers e_k for the first time.
- Let $C_1,\ldots,C_r$ be the sets in an optimal cover, each of which covers some elements of $U=\{e_k,e_{k+1},e_{k+2},\ldots,e_n\}$.
  - Let $n'_1,\ldots,n'_r$ be the number of elements of U which $C_1,\ldots,C_r$ cover.
- Obs 1 $\sum_i n'_i \geq n-k+1$.
  - Because $C_1,\ldots,C_r$ cover U.
- Obs 2 $\sum_i \mathrm{cost}(C_i) \leq \mathrm{OPT}$.
  - Because $C_1,\ldots,C_r$ are a subset of an optimal cover, which has cost OPT.

The per element cost

- Obs 3 None of $C_1,\ldots,C_r$ are among $S_1,\ldots,S_{j-1}$.
  - If some C_i is among $S_1,\ldots,S_{j-1}$, then since C_i covers some e in U, e would be covered by $\{S_1,\ldots,S_{j-1}\}$. So, e would be among the first k-1 elements covered. Contradiction.
- Obs 4 There exists some C_i among $C_1,\ldots,C_r$ with $\mathrm{cost}(C_i)/n'_i \leq OPT/(n-k+1)$.
  - If every C_i in $C_1,\ldots,C_r$ has $\mathrm{cost}(C_i)/n'_i > OPT/(n-k+1)$, then

$OPT \geq \Sigma_i \mathrm{cost}(C_i) = \Sigma_i n'_i * \mathrm{cost}(C_i)/n'_i >$

$\Sigma_i n'_i * OPT/(n-k+1) \geq OPT/(n-k+1)\ \Sigma_i n'_i \geq$

$OPT/(n-k+1)*(n-k+1) = OPT.$

Contradiction.

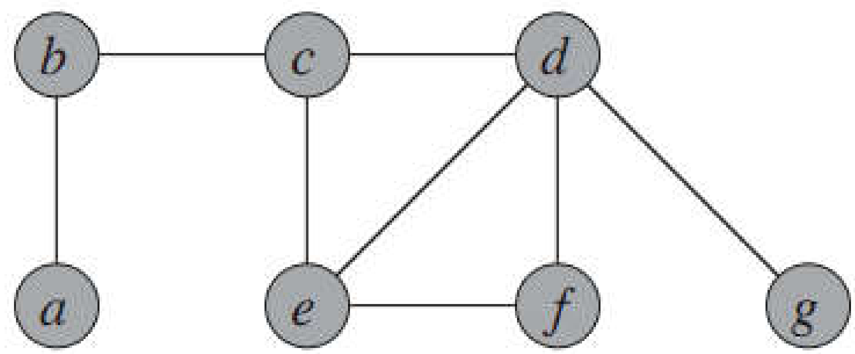
Proof of approximation ratio

- Lemma $cost(S_j)/n_j \leq OPT/(n-k+1)$.
- Proof When choosing S_j, the only sets the algorithm is not allowed to choose are $S_1,\ldots,S_{j-1}$.
  - By obs 3, $C_1,\ldots,C_r$ aren't in $S_1,\ldots,S_{j-1}$.
  - By obs 4, there's some C_i in $C_1,\ldots,C_r$, with $cost(C_i)/n'_i \leq OPT/(n-k+1)$.
  - S_j was chosen so that $cost(S_j)/n_j$ is min among all sets not in $S_1,\ldots,S_{j-1}$.
  - So $cost(S_j)/n_j \leq cost(C_i)/n'_i \leq OPT/(n-k+1)$.
- Since $cost(S_j)/n_j = cost(e_k)$, we have $cost(e_k) \leq OPT/(n-k+1)$.
- The approx ratio follows because

$$L = \Sigma_k\, cost(e_k) = \Sigma_k\, OPT/(n-k+1) \approx \ln(n)*OPT$$

Vertex cover

- Input A graph with vertices V and edges E.
- Output A subset V’ of the vertices, so that every edge in E touches some vertex in V’.
- Goal Make |V’| as small as possible.



source: *Introduction to Algorithms*, Cormen et al.

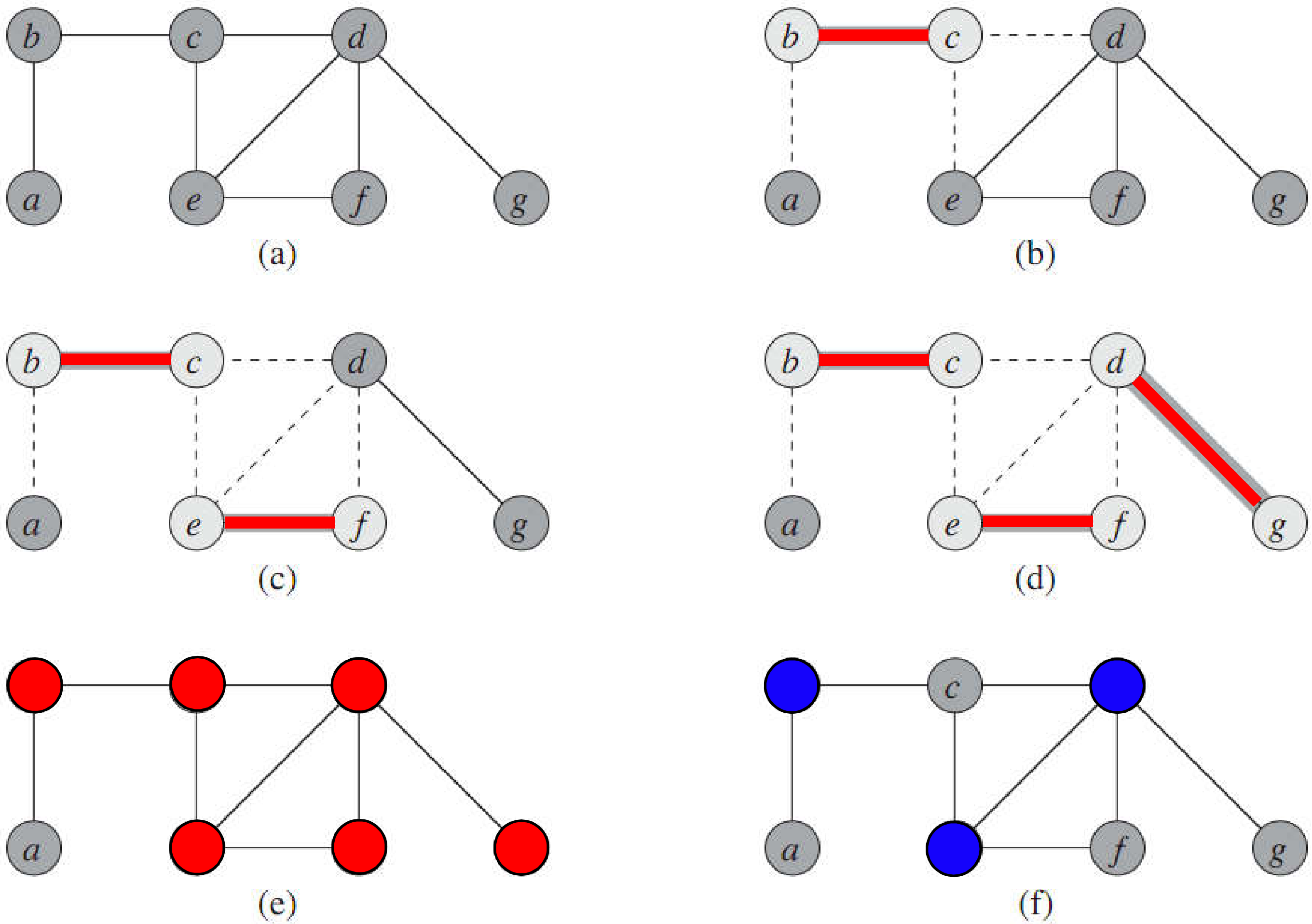
- Finding the minimum vertex cover is NP-complete.
- Vertex cover is a special case of (unweighted) set cover, where each element (edge) can be covered by at most two sets (vertices).
- We’ll see a simple 2 approximation for this problem.

A vertex cover algorithm

- Initially, let D be all the edges in the graph, and C be the empty set.
  - C is our eventual vertex cover.
- Repeat as long as there are edge left in D.
  - Take any edge (u,v) in D.
  - Add {u,v} to C.
  - Remove all the edges adjacent to u or v from D.
- Output C as the vertex cover.

Example

source: CLRS



Algorithm's vertex cover

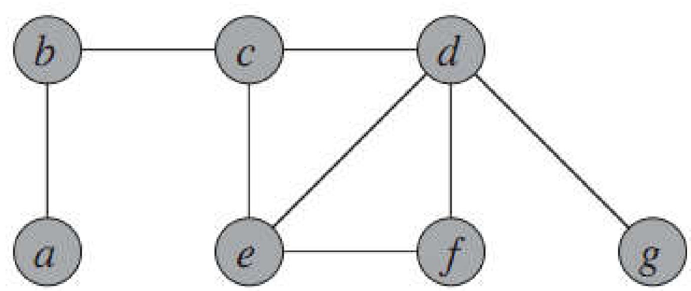
Optimal vertex cover

Proof of correctness

- The output is certainly a vertex cover.
  - In each iteration, we only take out edges that get covered.
  - We keep adding vertices till all edges are covered.
- Now, we show it’s a 2 approximation.
- Let C* be an optimal vertex cover.
- Let A be the set of edges the algorithm picked.

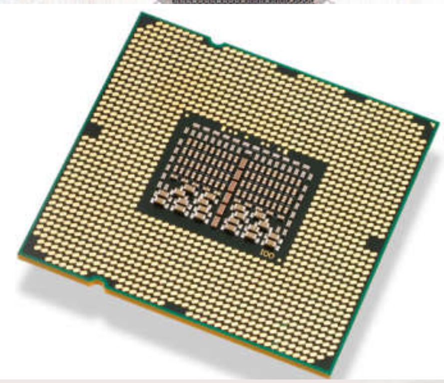
Proof of Correctness

- None of the edges in A touch each other.
  - Each time we pick an edge, we remove all adjacent edges.
- So each vertex in C* covers at most one edge in A.
  - The edges covered by a vertex all touch each other.
- Every edge in A is covered by a vertex in C*.
  - Because C* is a vertex cover.
- So $|C^*| \geq |A|$.
- The number of vertices the algorithm uses is $2|A|$.
  - If alg picks edge (u,v), it uses {u,v} in the cover.
- So (# vertices alg uses) / (# vertices in opt cover) = $2|A| / |C^*| \leq 2|A| / |A| = 2$.



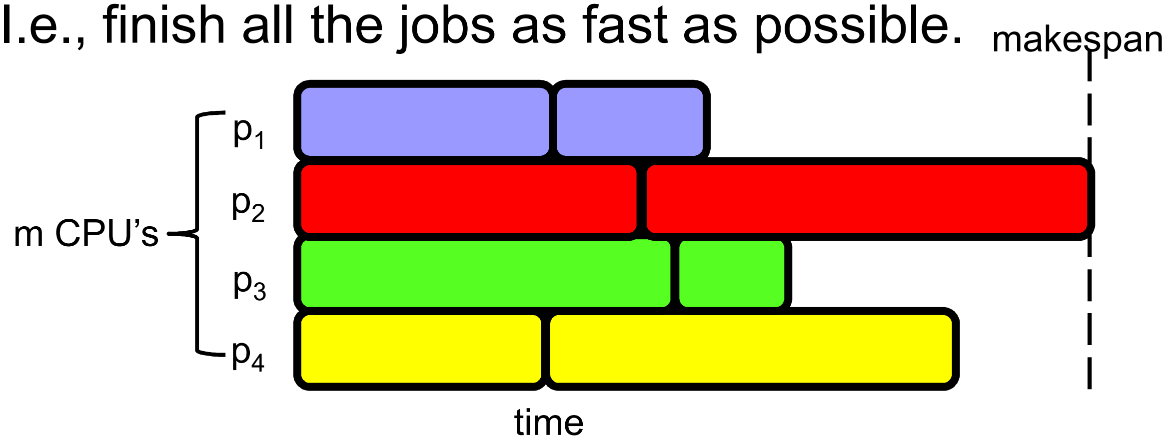
Parallel computing and scheduling

- Computers today are parallel.
  - Multiple processors in a system.
  - Multiple tasks for the processors to run.
- Multiprocessor scheduling is the problem of deciding which tasks to run on which processors at what time.
- Many possible objectives.
  - Throughput, fairness, energy usage.
  - Latency, i.e. finishing all jobs as fast as possible.



Makespan scheduling

- n independent jobs.
  - Jobs have different sizes, i.e. time needed to perform job.
  - Jobs can be done in any order.
  - Any job can be done on any machine.
- m processors.
  - All have the same speed.
  - Each processors can do one job at a time.
- Assign the jobs to the processors.
- Makespan is when the last processor finishes all its jobs.
- Minimize the makespan.
  - I.e., finish all the jobs as fast as possible.



Minimizing makespan is NPC

- Show that minimizing makespan on even two processors is NP-complete.
- Decision version of problem is in NP.
- SUBSET-SUM problem: Given a set of numbers S and target t, is there a subset of S summing to t?
  - Ex S={1,3,8,9}. For t=9, yes. For t=14, no.
  - SUBSET-SUM is NP-complete. Will reduce it to 2 processor makespan scheduling.
- Let (S,t) be an instance of SUBSET-SUM, and let s be sum of all elements in S.
- Make a set of tasks J = $S \cup \{s-2t\}$, and schedule them on 2 processors.
- Show that SUBSET-SUM reduces to min makespan, i.e. SUBSET-SUM has a solution iff min makespan has a certain solution.

Minimizing makespan is NPC

- Claim If some subset of S sums to t, then min makespan is s-t.
- Proof Say S'⊆S sums to t. Schedule the tasks in S' and task s-2t on processor 1. So processor 1 finishes at time t+s-2t=s-t. Processor 2 does the tasks in S-S', so it finishes at time s-t as well. Since processors finish at same time, the makespan is minimal.
- Claim If the min makespan is s-t, there exists a subset of S that sums to t.
- Proof Suppose WLOG processor 1 does the s-2t task. Since makespan is s-t, the other tasks processor 1 does must have total size s-t-(s-2t)=t.
- So (S,t) is yes instance of SUBSET-SUM iff minimum makespan = s-t, so minimizing makespan is NPC.

Graham's list scheduling

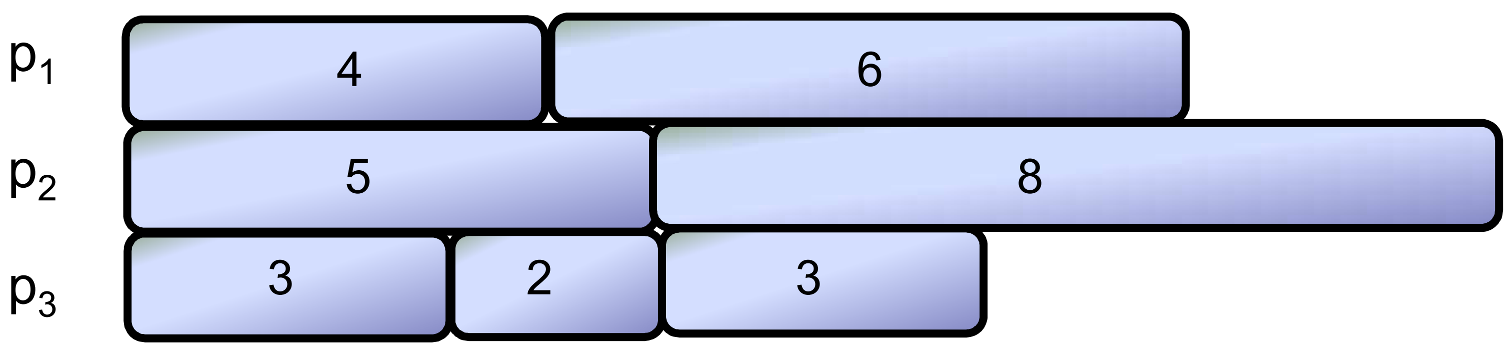
- Since scheduling is NPC, it's unlikely we can find the min makespan in polytime.
- List scheduling is a simple greedy algorithm.
  - Finds a schedule with makespan at most twice the minimum.
  - A 2-approximation.
- If there are n tasks and m processors, list scheduling only takes O(n log m) time.
  - Compare this to n! C(n+m-1, m-1) time to try all possible schedules and pick the best.

Graham's list scheduling

- List the jobs in any order.
- As long as there are unfinished jobs.
  - If any processor doesn't have a job now, give it the next job in the list.

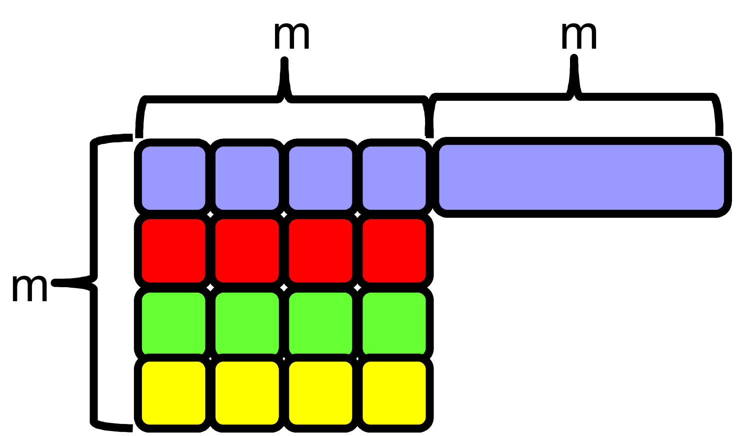
Example

- 3 processors. The jobs have length 2, 3, 3, 4, 5, 6, 8.
- List them in any order. Say 4, 5, 3, 2, 6, 8, 3.
- Initially, no proc has a job. Give first 3 jobs to the 3 procs.
- At time 3, proc 3 is done. Give it next job in list, 2.
- At time 4, proc 2 is done. Give it next job in list, 6.
- At time 5, both 1, 3 are done. Give them next jobs in list, 8,3.
- Everybody finishes by time 13.
  - The makespan of this schedule is 13.

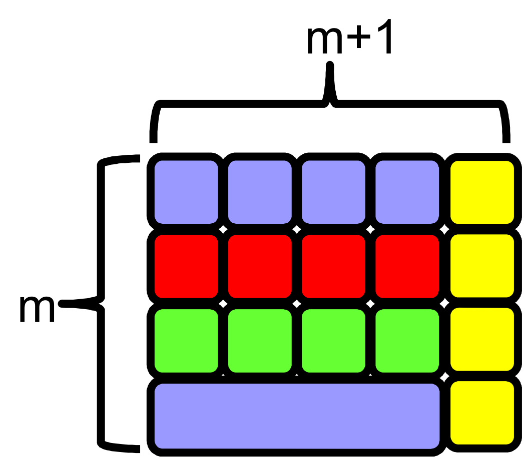


The worst case for LS

- How badly can list scheduling do compared to optimal?
- Say there are m^2 jobs with length 1, and one job with length m.
  - Suppose they're listed in the order 1,1,1,...,1,m.
  - LS has makespan 2m. Optimal makespan is m+1.
  - makespan(LS) / makespan(opt) = 2m/(m+1) $\approx$ 2.
- This is worst possible case for list scheduling.



makespan(LS) = 2m

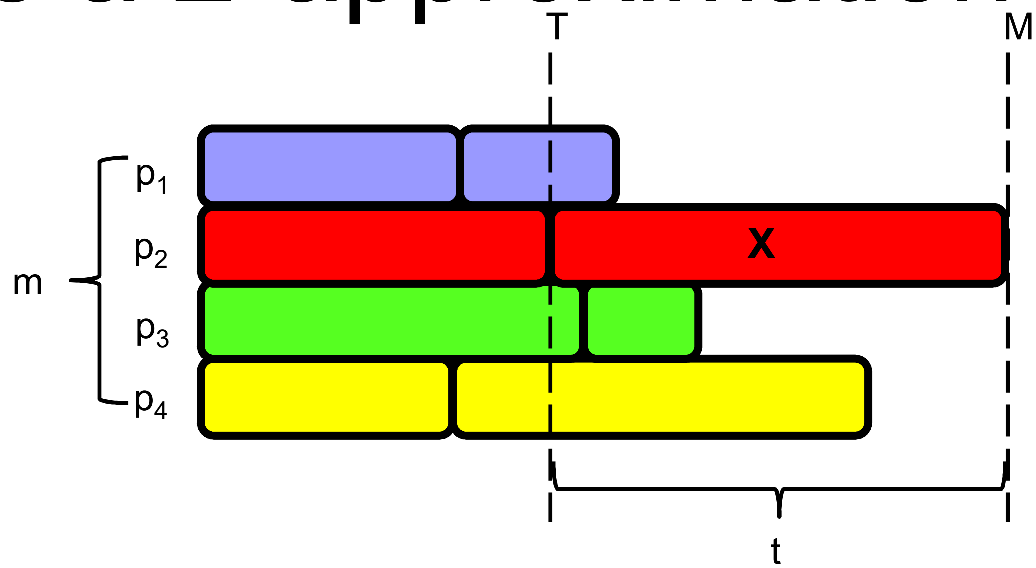


makespan(opt) = m+1

LS is a 2-approximation

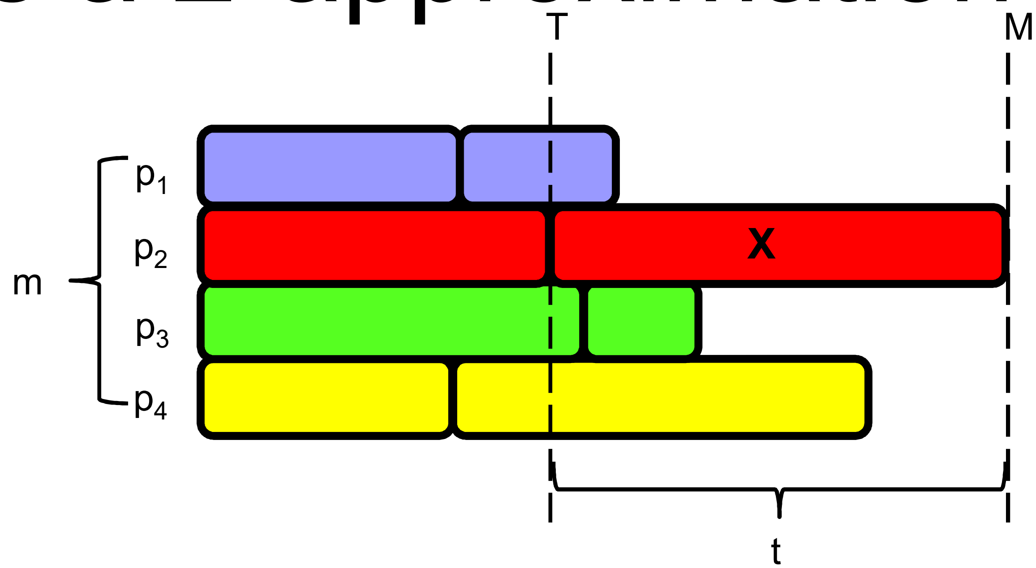
- Next, we prove LS always gives a schedule at most twice the optimal.
- Suppose LS gives makespan of M.
- Let the optimal schedule have makespan M*.
- We prove that $M \leq 2M^*$.

LS is a 2-approximation



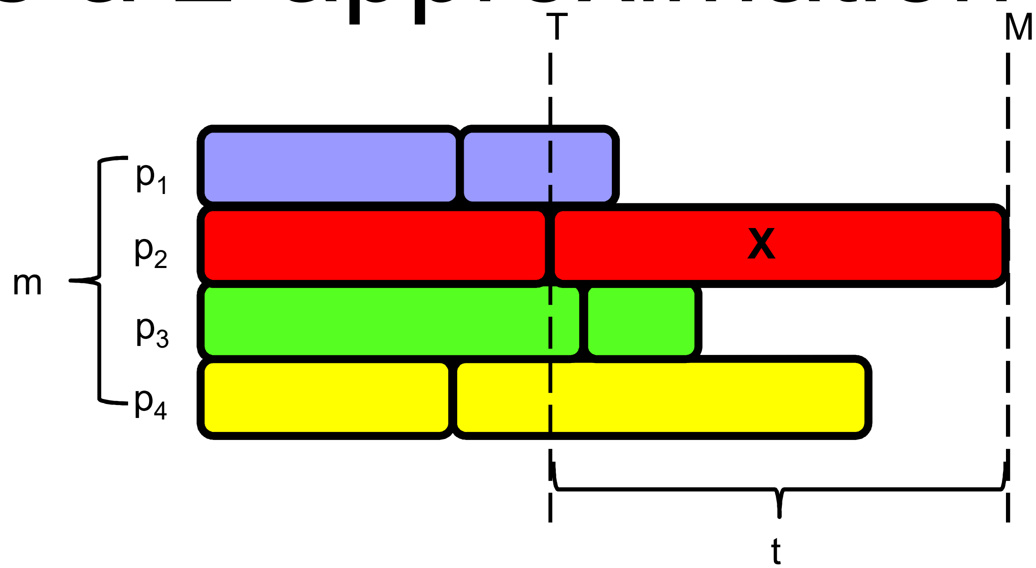
- The picture above is the schedule produced by list scheduling.
- Consider task X that finishes last.
  - Say X starts at time T, and has length t.
- Claim 1 M* ≥ t.
  - In any schedule, X has to run on some process.
  - Since X takes t time, every schedule, including the opt, takes ≥ t time.

LS is a 2-approximation



- Claim 2 $M^* \geq T$.
  - Up to time T, no processor is ever idle.
    - Up to T, there's always some unfinished job.
    - As soon as a processor finishes one job, it's assigned another one.
  - So at time T, each processor completed T units of work.
  - So total amount of work in all the jobs is $\geq mT$.
  - In the opt schedule, m processors complete at most m units of work per time unit.
  - So length of opt schedule is $\geq$ (total work)/m $\geq mT/m = T$.

LS is a 2-approximation



- From Claims 1 and 2, we have $M^* \geq t$ and $M^* \geq T$.
- So $M^* \geq \max(T,t)$.
- $M = T + t$, because X is last job to finish.
- So $M/M^* \leq (T+t)/\max(T,t) \leq 2$.

LPT scheduling

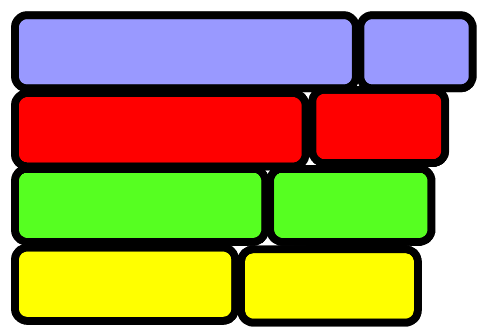
- Worst case for LS occurred when longest job was scheduled last.
  - Large jobs are “dangerous” at end.
- Let’s try to schedule longest jobs first.
- Longest processing time (LPT) schedule is just like list scheduling, except it first sorts tasks by nonincreasing order of size.
- Ex For three processors and tasks with sizes 2, 3, 3, 4, 5, 6, 8, LPT first sorts the jobs as 8,6,5,4,3,3,2. Then it assigns p_1 tasks 8,3, p_2 tasks 6,3, p_3 tasks 5,4,2, for a makespan of 11.
- LPT has an approximation ratio of 4/3.

LPT is a 4/3-approximation

- Thm Suppose the optimal makespan is M^*, and LPT produces a schedule with makespan M. Then $M \leq 4/3\ M^*$.
- Let X be the last job to finish. Assume it starts at time T and has size t.
- Assume WLOG that X is the last job to start.
  - If not, then say Y starts after T.
  - Y finishes before T+t. So we can remove Y without increasing the makespan.
- Cor 1 X is the smallest job.
  - X is the last job to start, so due to LPT scheduling it's the smallest.

LPT is a 4/3-approximation

- Claim 1 LPT's makespan = $T+t \leq M^*+t$.
  - As in LS, no processor is idle up to time T, so $M^* \geq T$.
- Case 1 $t \leq M^*/3$.
  - Then LPT's makespan $\leq M^* + t \leq M^* + M^*/3 = 4/3\ M^*$.
- Case 2 $t > M^*/3$.
  - Since X is the smallest task, all tasks have size $> M^*/3$.
  - So the optimal schedule has at most 2 tasks per processor. So $n \leq 2m$.
  - If $1 \leq n \leq m$, then LPT and optimal schedule both put one task per processor.
  - If $m < n \leq 2m$, then optimal schedule is to put tasks in nonincreasing order on processors 1,...,m, then on m,...,1.
    - LPT also schedules tasks this way, so it's optimal.



LS vs LPT

- LPT gives better approx ratio, has same running time. Why bother with LS?
- LS is online.
  - Imagine the jobs are coming one by one.
    - LS just puts them on any idle computer.
- LPT is offline
  - It needs to know all the jobs that will ever arrive, in order to sort them.
- In a realistic parallel computation, you get jobs on the fly.
  - Online is more realistic.
  - LS is usually more useful.