Problem 1

First prove Set-Packing is NP.

Verifier

- Certificate y is k subsets from D.
- Check whether these subsets are mutually disjoint.
- If so, output 1, else output 0.

If x is yes instance

- Then there exist k subsets from D which are mutually disjoint.
- Give y to V, and V outputs 1.

If x is no instance

- Then every k subsets from D which are not mutually disjoint.
- So V outputs 0, no matter what k subsets it gets.

V runs in polynomial time

- Checking whether these k subsets are mutually disjoint takes $O(C_k^2) = O(k^2)$ time.

Then reduce from Independent Set.

Given a graph $G = (V, E)$ and an integer, is there a subset $V' \subseteq V$ such that $|V'| \geq k$ and no two vertices in V' are adjacent?

Reduction

- $C = E$
- $S_v = \{e | e \in E, e \text{ incident with } v\}$
- Use the same k from the Independent Set instance.

The reduction takes polynomial time $O(E + V^2)$.

Proof of correctness

- if V' is a yes instance for Independent Set, then $|V'| \geq k$ and no two vertices in V' are adjacent. For any two vertices $v', u' \in V'$, $S_{v'}$ and $S_{u'}$ are disjoint, since $S_{v'}$ are the set of all edges incident with v', $S_{u'}$ are the set of all edges incident with u', and v', u' are not adjacent. Therefore, there are k subsets from the collection which are pairwise disjoint.
- If there are k subsets from the collection which are pairwise disjoint, then for any two subsets $S_{v'}$ and $S_{u'}$, v', u' are not adjacent. Therefore there is a subset $V' \subseteq V$ such that $|V'| = k$ and no two vertices in V' are adjacent.

Problem 2

First prove Stingy-SAT is NP.

Verifier

- Certificate y is a Boolean CNF with at most k variables are set to true.
- Check whether the formula has a satisfying assignment.
- If so, output 1, else output 0.

If x is yes instance

- Then the formula has a satisfying assignment.
- Give y to V, and V outputs 1.

If x is no instance

- Then the formula doesn't have a satisfying assignment.
- So V outputs 0.

V runs in polynomial time

- Checking whether the formula has a satisfying assignment takes $O(n)$ time if n is the number of variables in the formula.

Then reduce from SAT.

Reduction

Given an instance of SAT I, let (I, k) be an instance of stingy SAT where k = the number of variables in SAT instance I.

The reduction takes polynomial time $O(k)$.

Proof of correctness

- If the SAT with k variables has a satisfying assignment, the corresponding STINGY SAT have a satisfiying assignment with true variables number less than k.
- If the SAT with k variables has no satisfying assignment, the STINGY SAT has no satisfying assignment.

Problem 3

First prove Set-Cover is NP.

Verifier

- Certificate y is at most k subsets from the collection of subsets of C.
- Check whether the subsets cover C.
- If so, output 1, else output 0.

If x is yes instance

- Then the subsets cover C.
- Give y to V, and V outputs 1.

If x is no instance

- Then the subsets don't cover C.
- So V outputs 0.

V runs in polynomial time

- Checking whether the subsets cover C by iterating over each element in the subsets of collection and mark the elements in C which are covered. It takes polynomial time.

Then reduce from Vertex Cover.

Given a graph $G = (V, E)$ and an integer k, whether the graph has a vertex cover less or equal to k.

Reduction

- $C = E$
- $S_v = \{e | e \in E, e \text{ incident with } v\}$
- Use the same k from the Vertex Cover instance.

The reduction takes polynomial time $O(E + V^2)$.

Proof of correctness

- If G has a vertex cover V' less or equal to k, then the union of S_v for any vertex $v \in V'$ can cover all the edges, that is C since $C = E$. Therefore, there are at most k subsets from the collection which cover C.
- If there are at most k subsets from the collection which cover C, then the union of these subsets includes all of E, therefore G has a vertex cover V' less or equal to k.

Problem 4

Denote the list of unique positive integers as $\{a_1, a_2, \ldots, a_n\}$.

For $1 \leq k \leq n$, the probability that a_k is the largest among the first k numbers is $\frac{1}{k}$.

$$E = \sum_{k=1}^{n} \frac{1}{k}$$

When $n \to \infty$, $E \to \ln(n)$