# Divide and Conquer 2

CS240 Spring 2024

*Rui Fan*

# Counting inversions

- Given a sequence $a_1, \ldots, a_n$, the pair $(a_i, a_j)$ is an inversion if $i < j$ but $a_i > a_j$.
- Ex In 5,2,6,1,4,3, there are 9 inversions.
- Can count number of inversions in $O(n^2)$ time.
- Use divide and conquer to solve in $O(n \log n)$ time.
- Basic idea Divide sequence in half.
  - Count the number of inversions in both halves.
  - Count number of inversions between the halves.

# Counting inversions

- Let L and R be the left and right halves of a sequence.
- Observation No matter how we permute L and R, the number of inversions between L and R stays the same.
  - Ex There are 7 inversions betweem 5,2,6 and 1,3,4. There are also 7 inversions between 2,5,6 and 1,3,4.
- Counting inversions between halves is easy if the halves are sorted in nondecreasing order.
  - Keep a pointer i for L, j for R, initially both 0.
  - If $L_i > R_j$, increment $j$.
    - Also increment number of inversions by $|L| - i$, because $L_k > R_j$ for all $k \geq i$, because $L$ and $R$ are sorted.
  - Otherwise increment i.
  - Just like merging L and R.
  - Takes $O(n)$ time, where $n = |L| + |R|$.

# Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.
- Combine: count inversions where $a_i$ and $a_j$ are in different halves, and return sum of three quantities.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4 | 8 | 10 | 2 | 6 | 9 | 12 | 11 | 3 | 7 |

Divide: O(1).

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 5 | 4 | 8 | 10 | 2 | 6 | 9 | 12 | 11 | 3 | 7 |

5 blue-blue inversions

8 green-green inversions

Conquer: 2T(n / 2)

9 blue-green inversions

5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

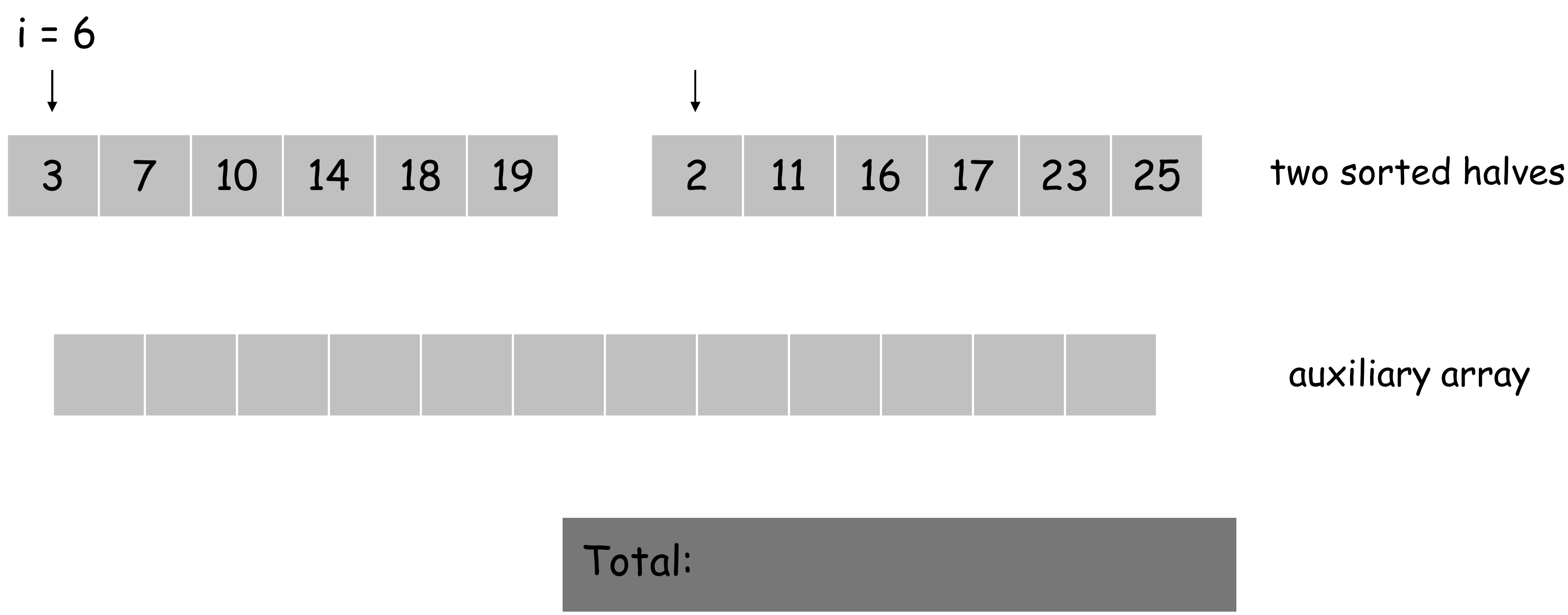
Combine: ???

Total = 5 + 8 + 9 = 22.

# Merge and Count

Merge and count step.

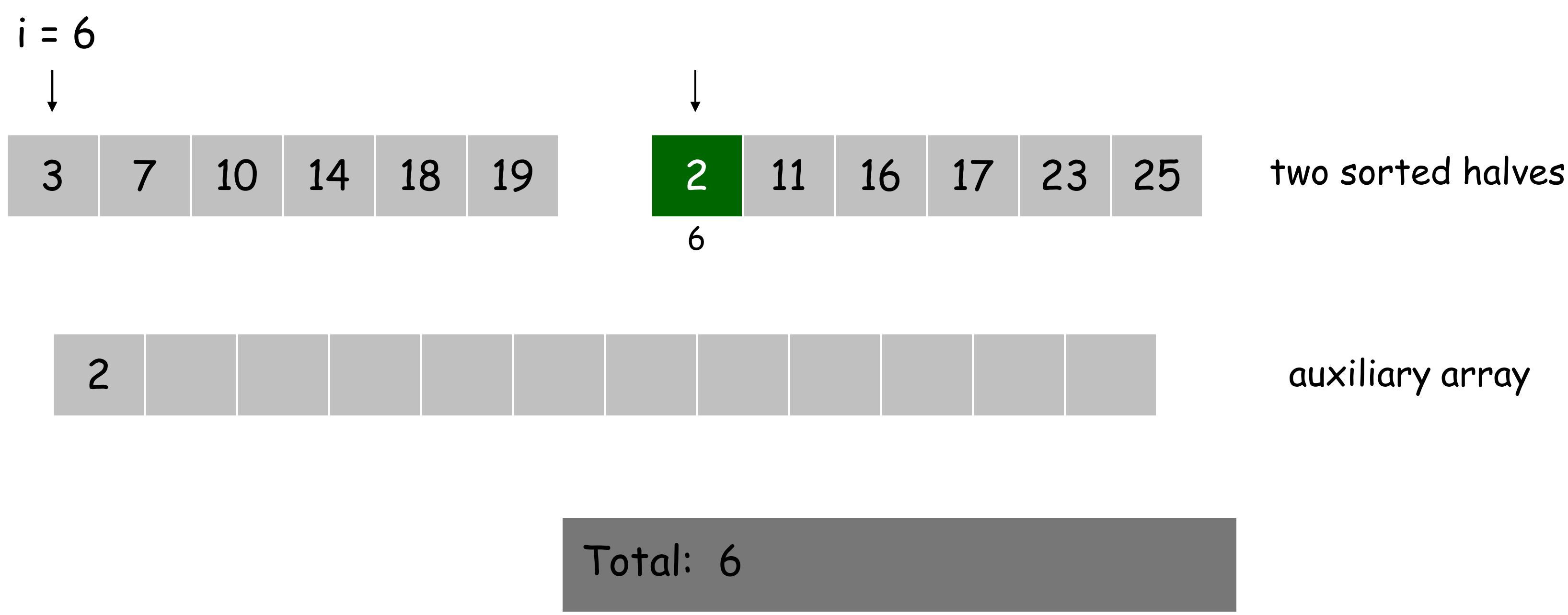
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 6

two sorted halves

| | | | | | |
|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 |

| | | | | | |
|---|---|---|---|---|---|
| 2 | 11 | 16 | 17 | 23 | 25 |
| 6 | | | | | |

auxiliary array

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | |

Total: 6

# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 6

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 |
| | | | | | | | 6 | | | | | |

two sorted halves

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | | | | | | | | | | |

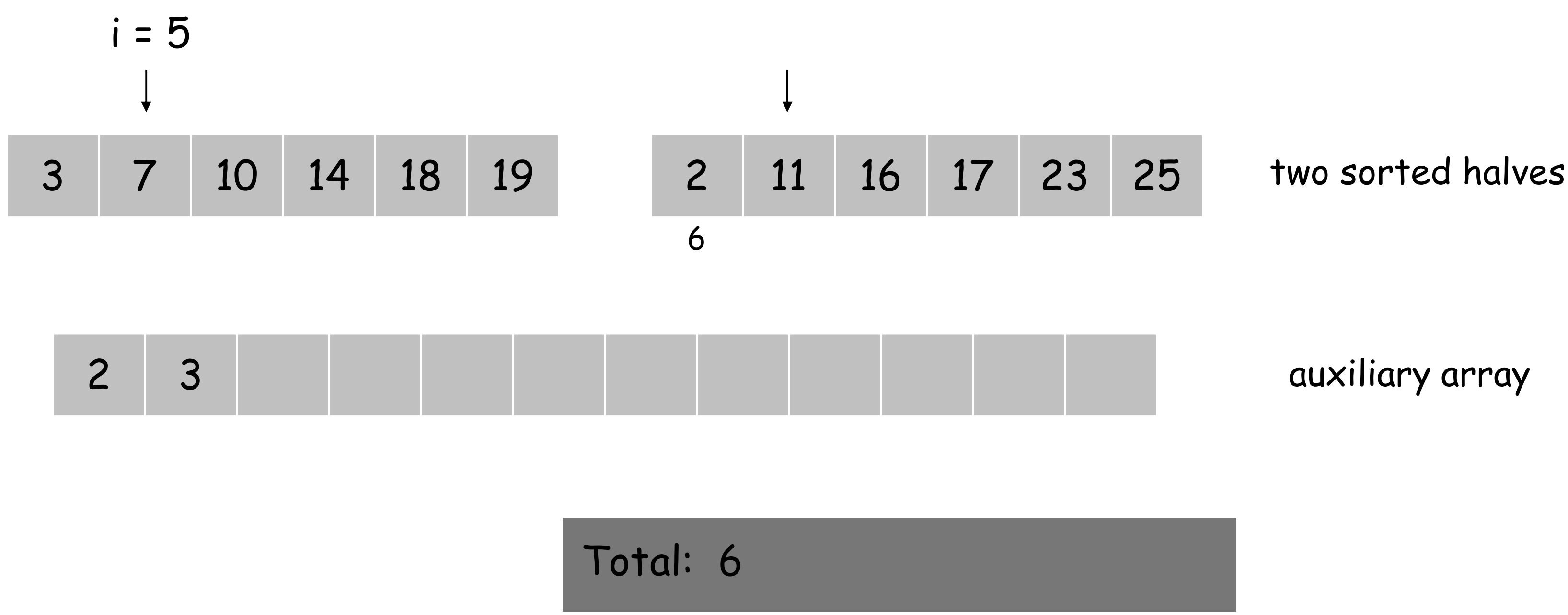
auxiliary array

Total: 6

# Merge and Count

Merge and count step.

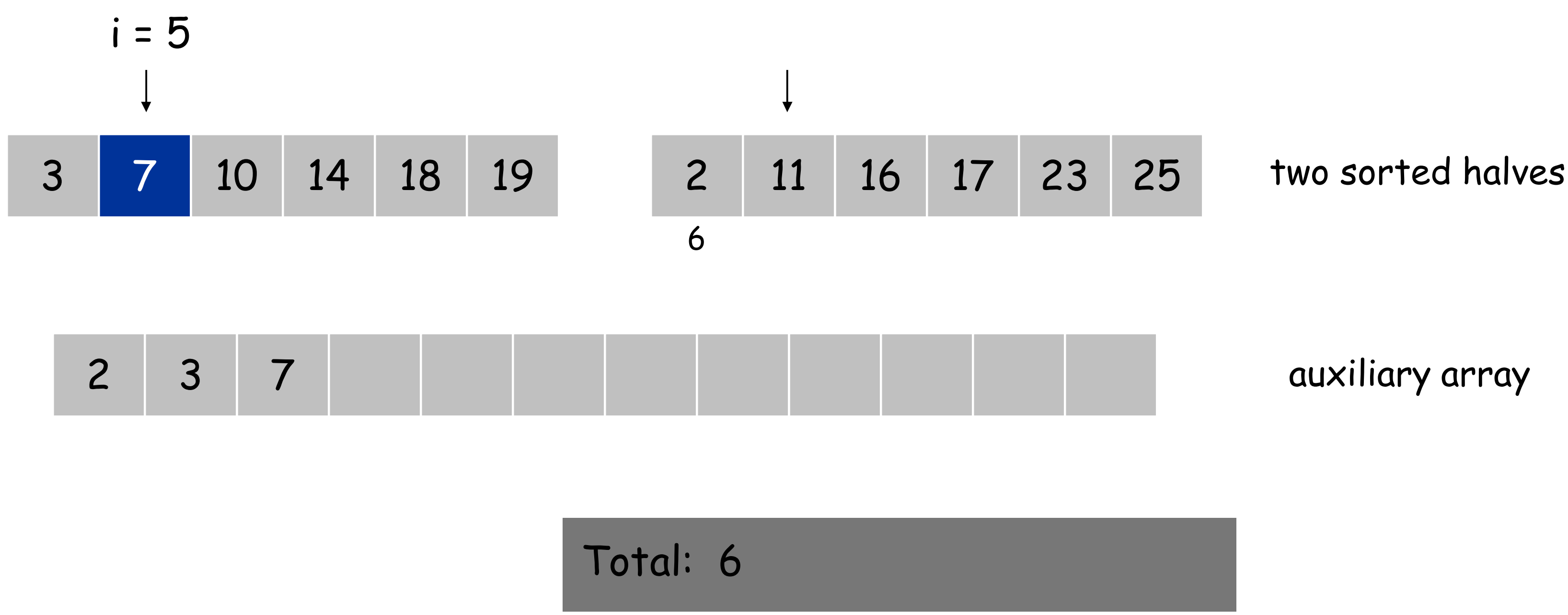
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

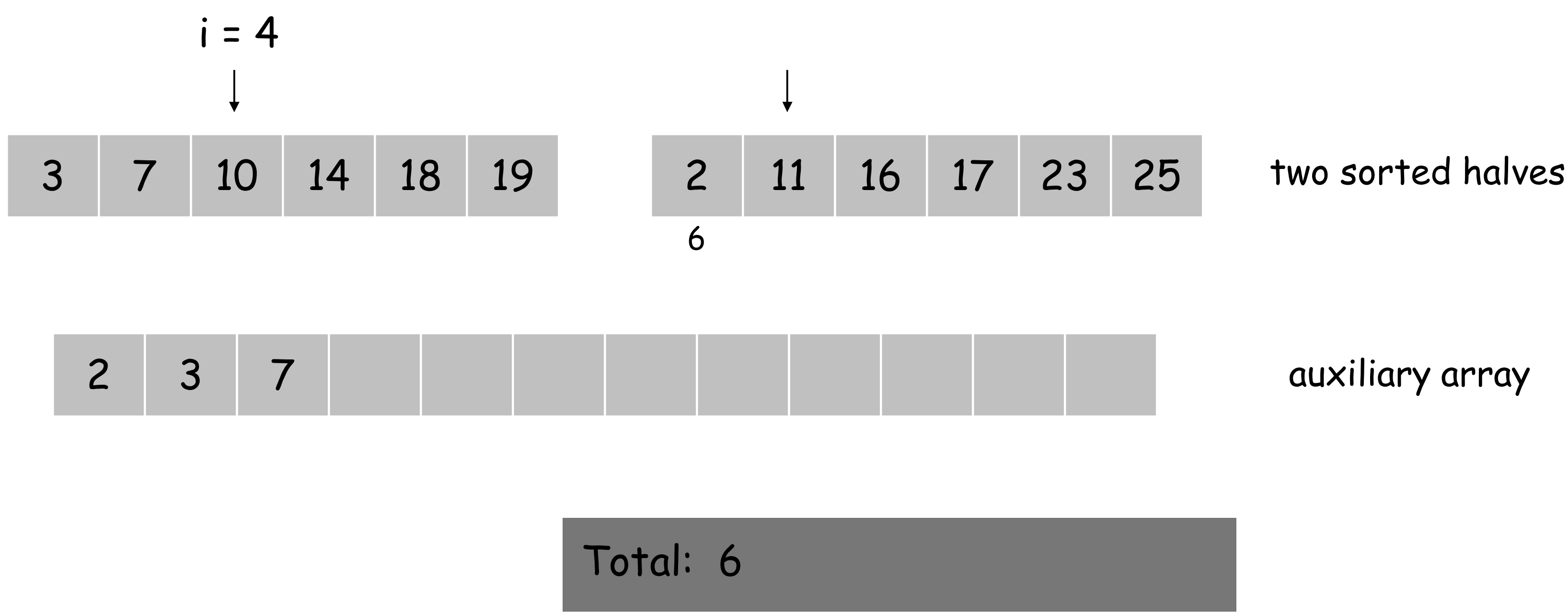
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

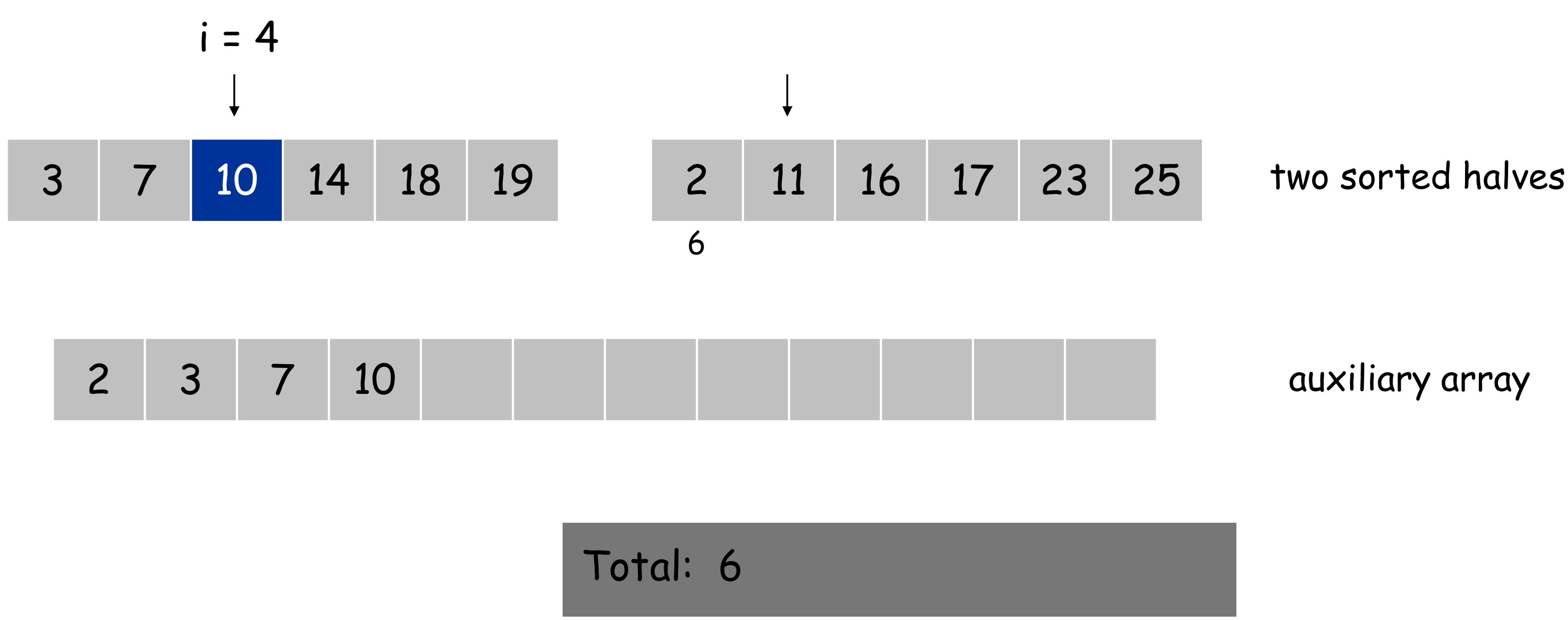
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

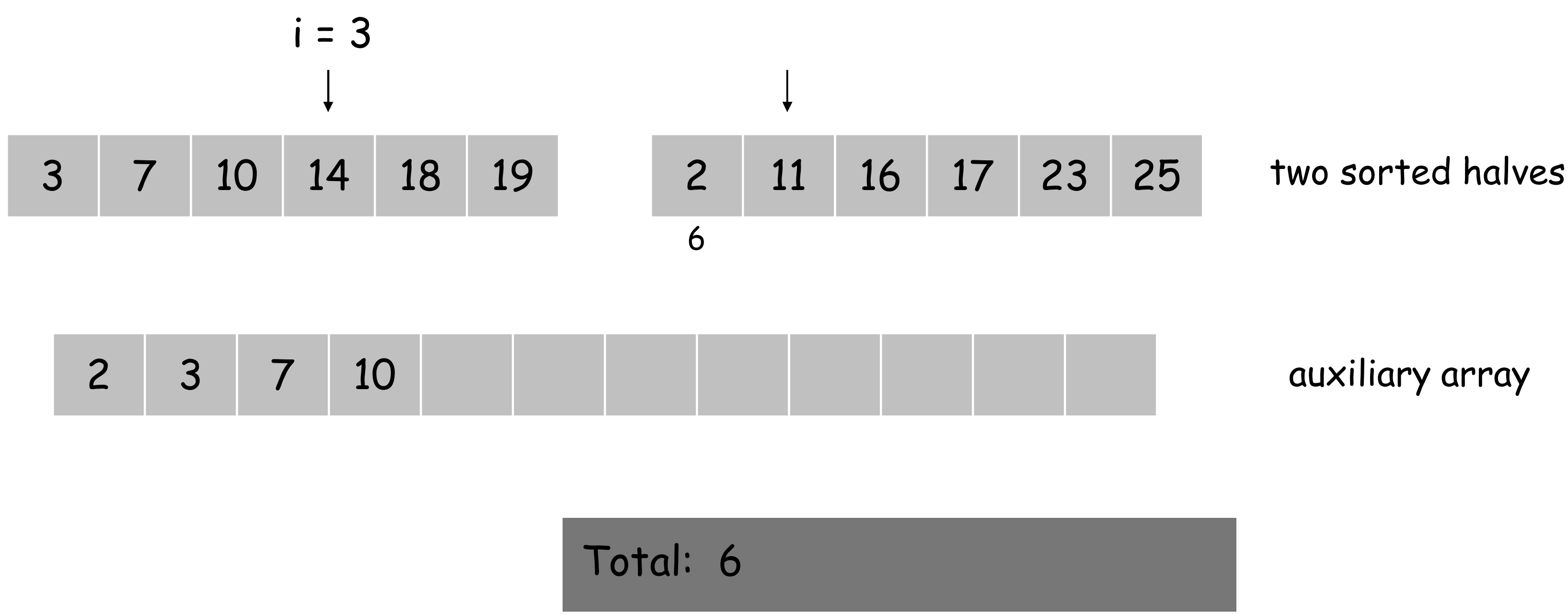
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

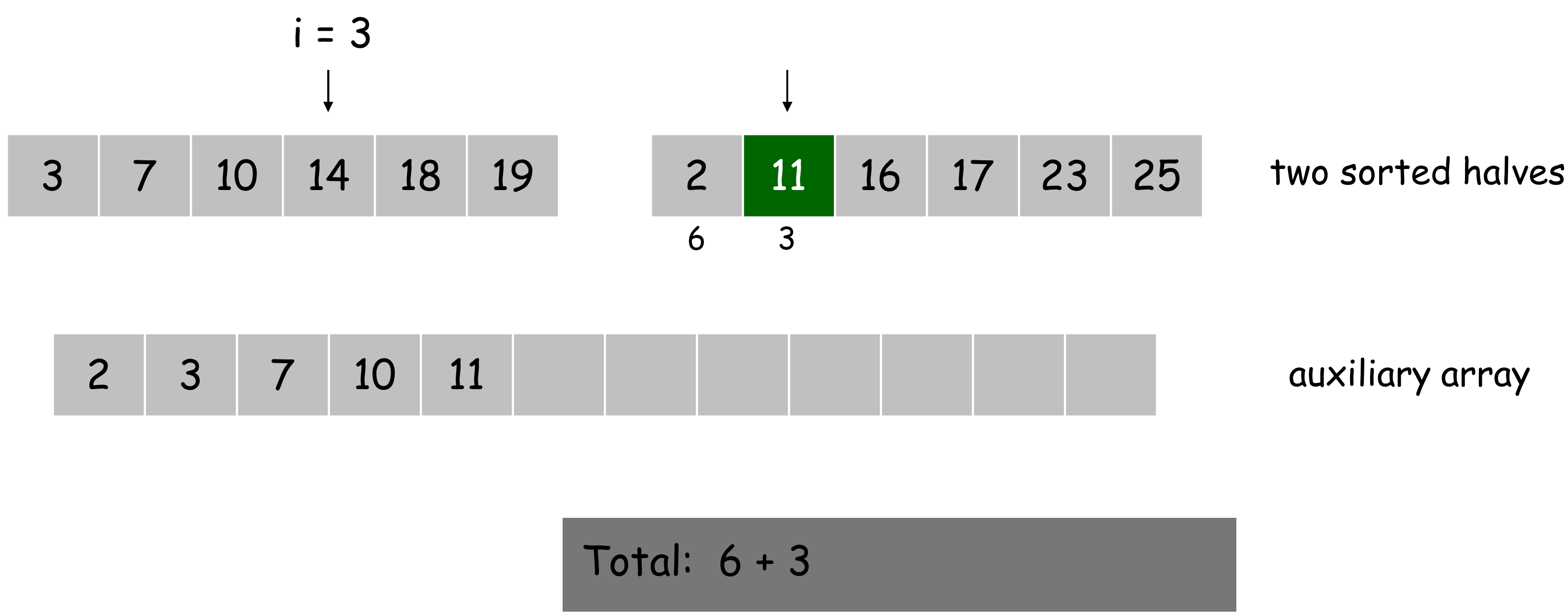
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 3

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 |
| | | | | | | | 6 | 3 | | | | |

two sorted halves

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 7 | 10 | 11 | | | | | | | |

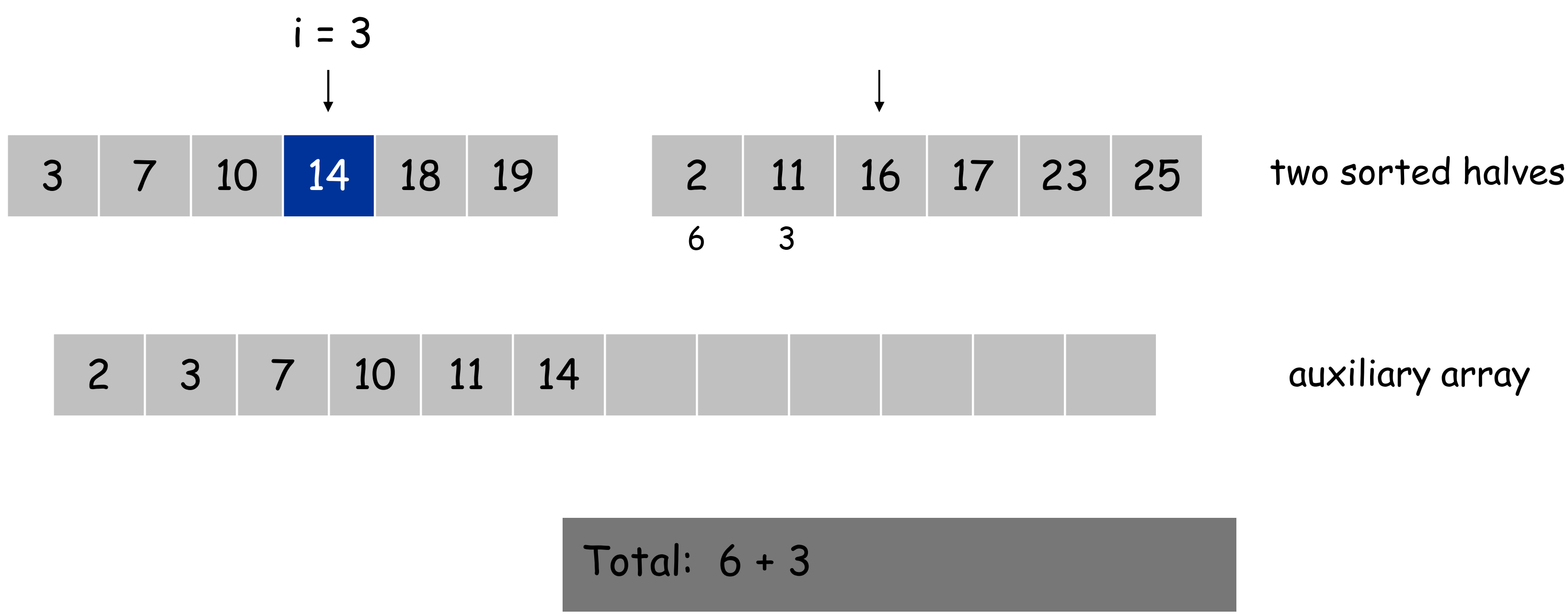
auxiliary array

Total: 6 + 3

# Merge and Count

Merge and count step.

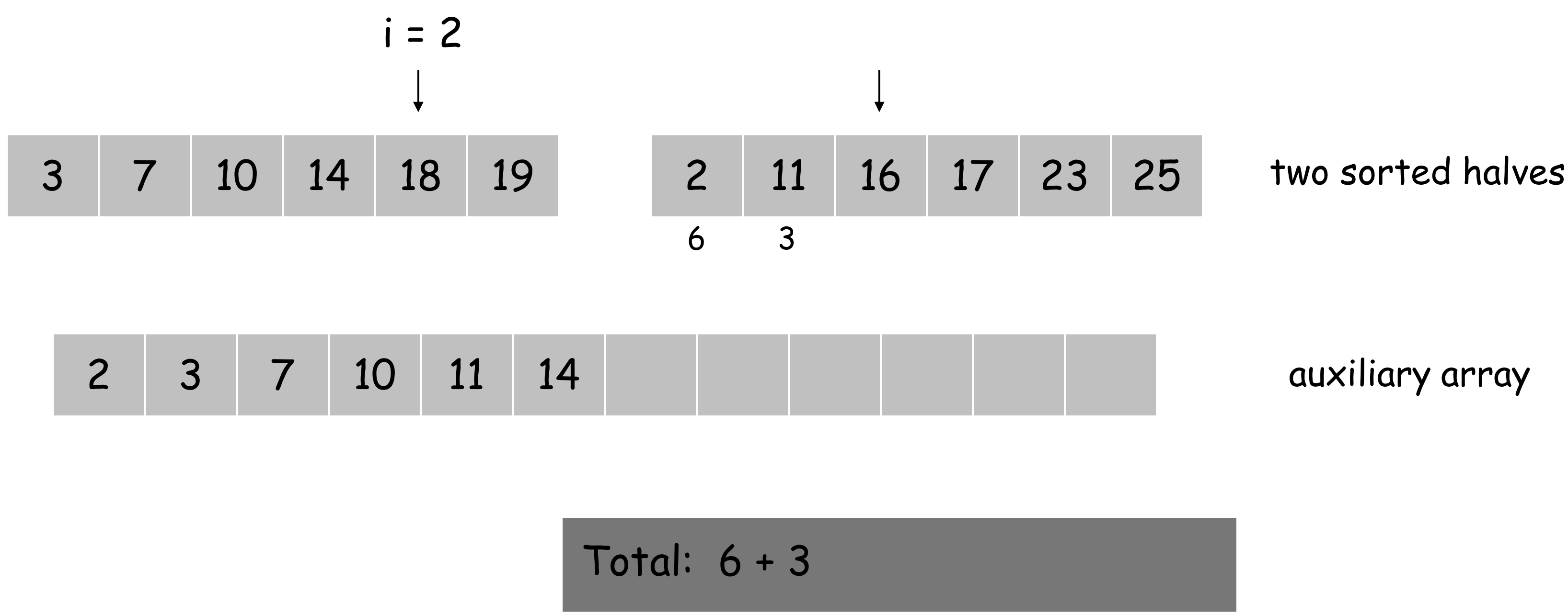
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

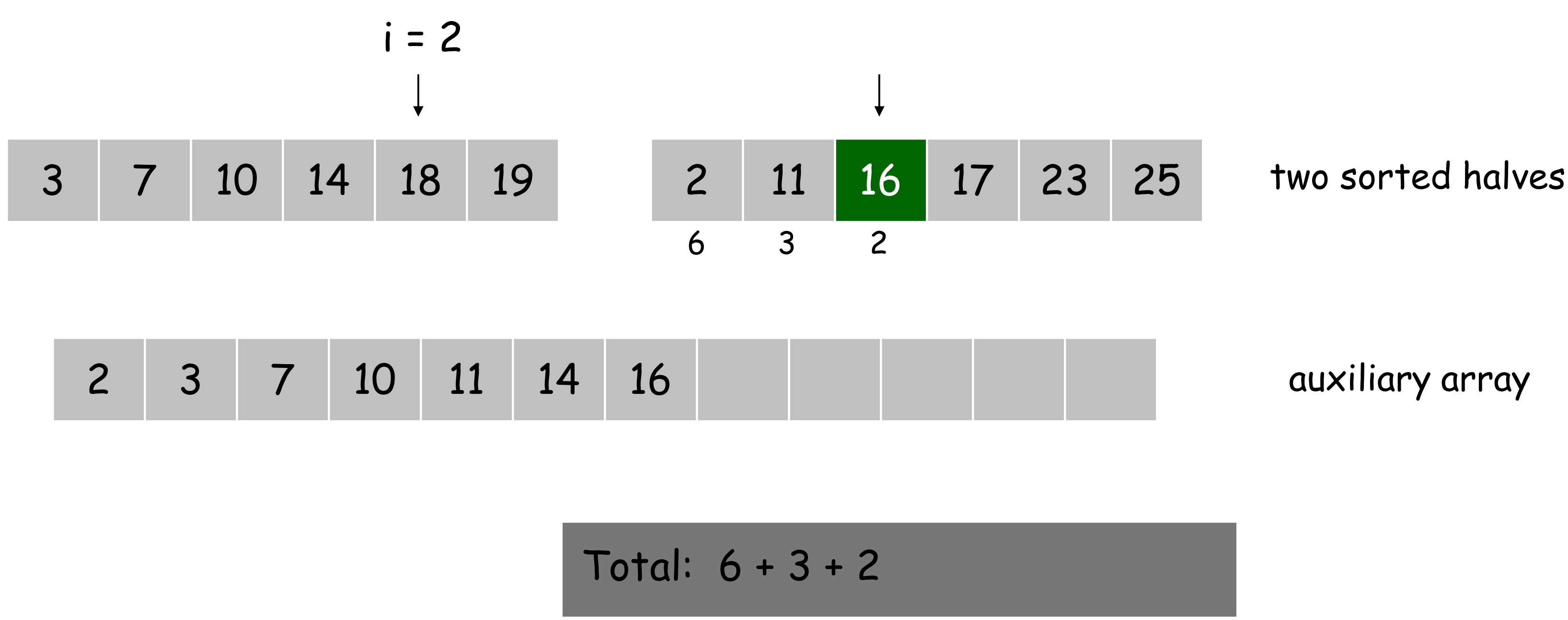
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

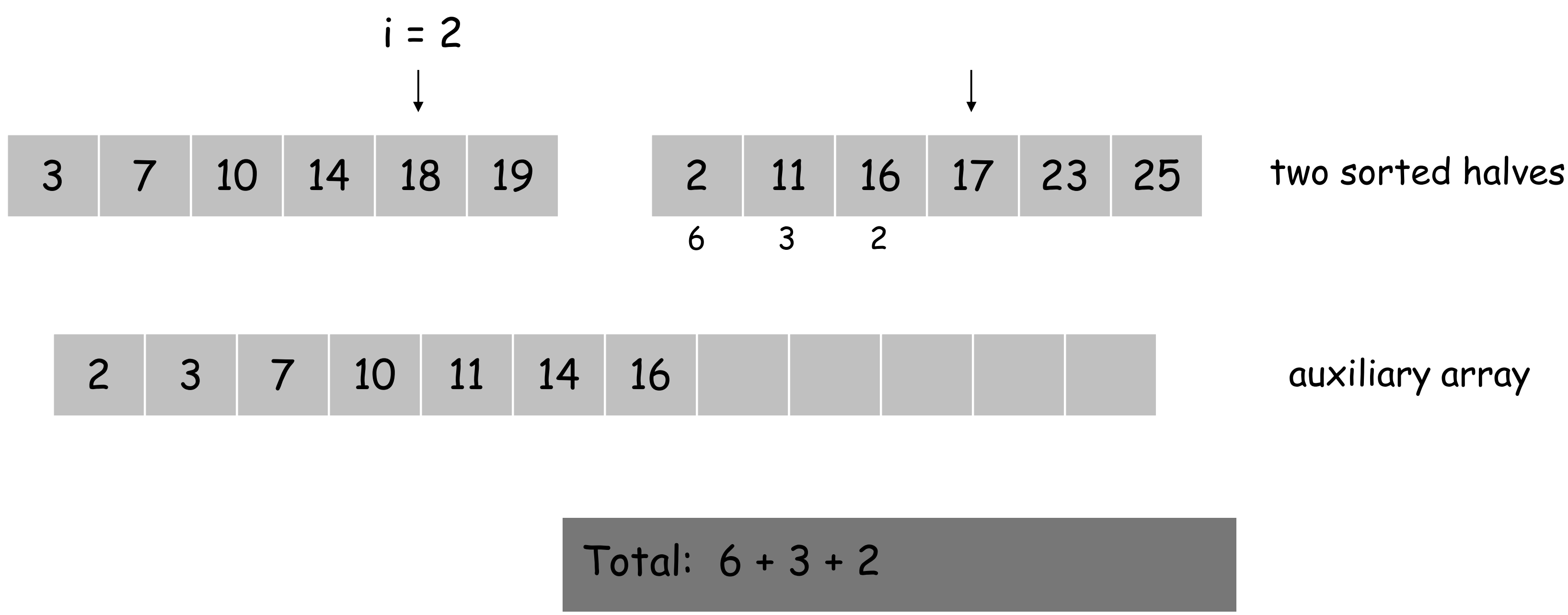
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

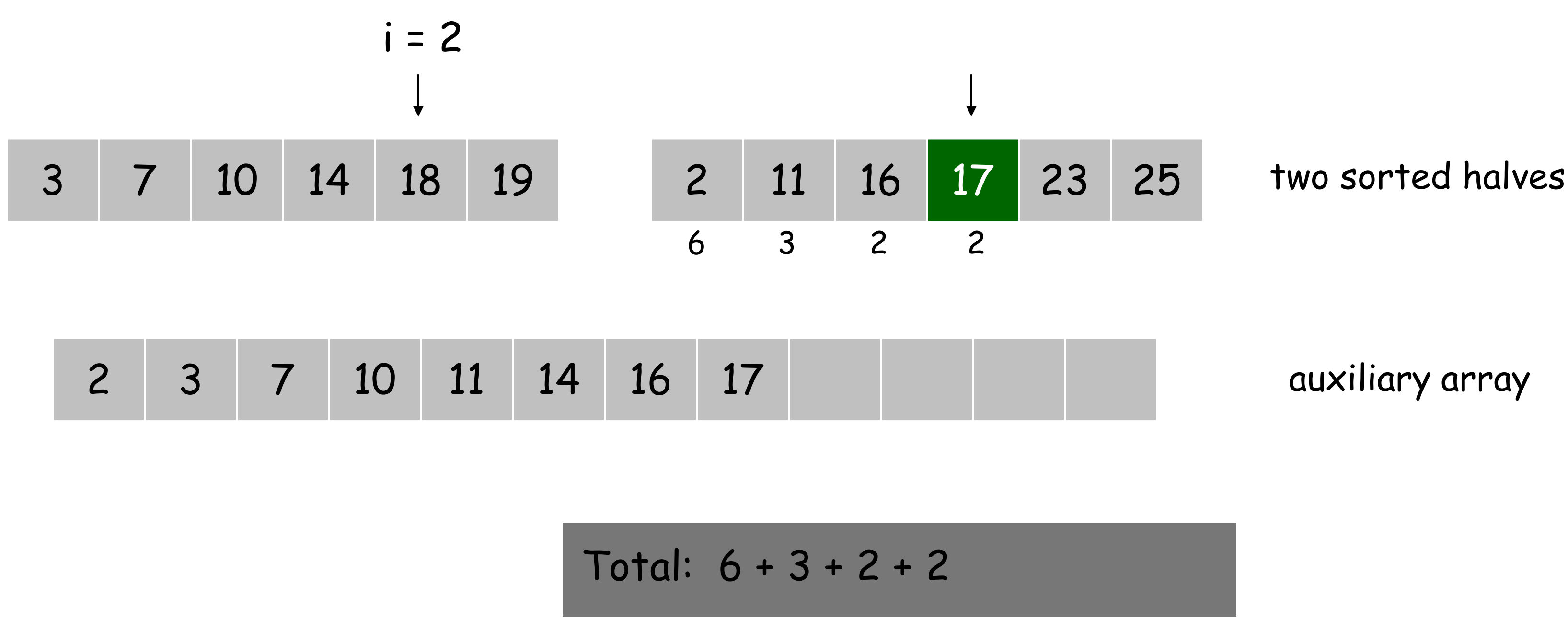
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 2

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 |
| | | | | | | | 6 | 3 | 2 | 2 | | |

two sorted halves

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 7 | 10 | 11 | 14 | 16 | 17 | | | | |

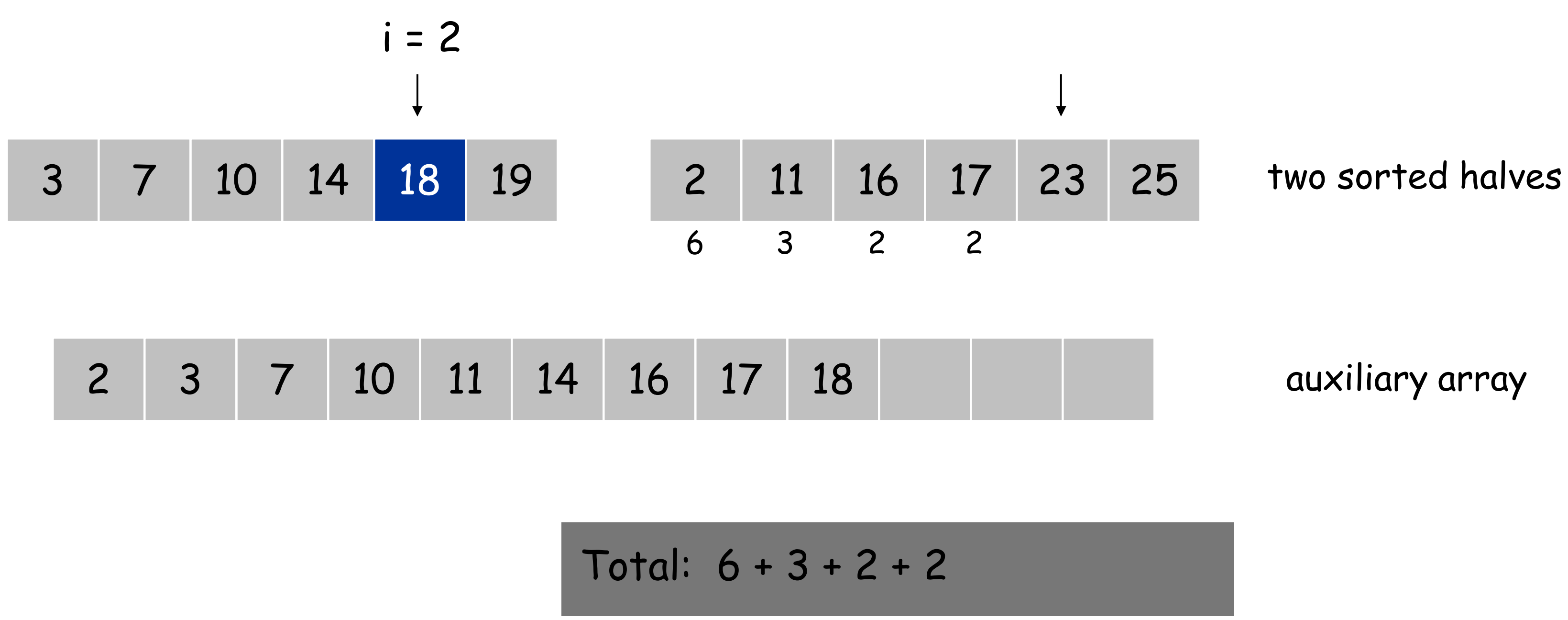
auxiliary array

Total: 6 + 3 + 2 + 2

# Merge and Count

Merge and count step.

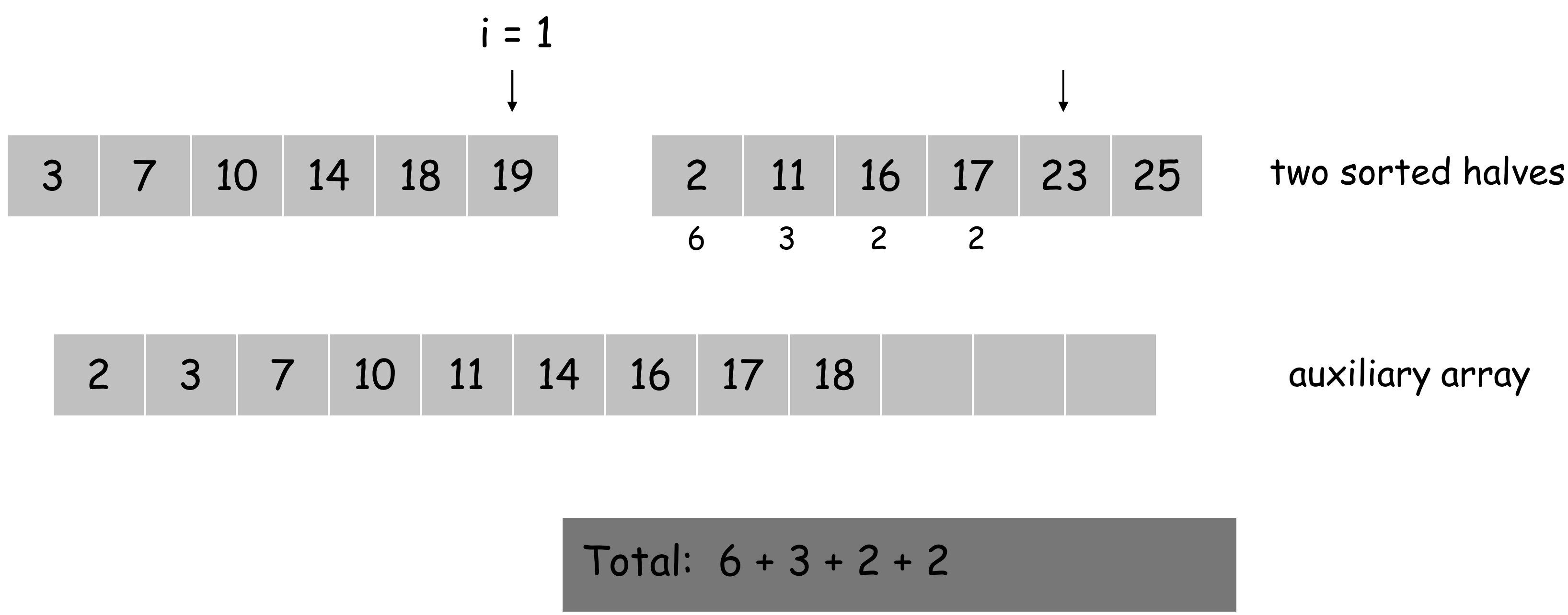
- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.



# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 1

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 | two sorted halves |
| | | | | | | | 6 | 3 | 2 | 2 | | | |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 7 | 10 | 11 | 14 | 16 | 17 | 18 | 19 | | | auxiliary array |

Total: 6 + 3 + 2 + 2

# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

first half exhausted

i = 0

two sorted halves

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 |
| | | | | | | | 6 | 3 | 2 | 2 | | |

auxiliary array

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 7 | 10 | 11 | 14 | 16 | 17 | 18 | 19 | | |

Total: 6 + 3 + 2 + 2

# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 0

two sorted halves

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 |
| | | | | | | | 6 | 3 | 2 | 2 | 0 | |

auxiliary array

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 7 | 10 | 11 | 14 | 16 | 17 | 18 | 19 | 23 | |

Total: 6 + 3 + 2 + 2 + 0

# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 0

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 |
| | | | | | | | 6 | 3 | 2 | 2 | 0 | |

two sorted halves

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 7 | 10 | 11 | 14 | 16 | 17 | 18 | 19 | 23 | |

auxiliary array

Total: 6 + 3 + 2 + 2 + 0

# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

i = 0

two sorted halves

| 3 | 7 | 10 | 14 | 18 | 19 | | 2 | 11 | 16 | 17 | 23 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | 6 | 3 | 2 | 2 | 0 | 0 |

auxiliary array

| 2 | 3 | 7 | 10 | 11 | 14 | 16 | 17 | 18 | 19 | 23 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|---|

Total: 6 + 3 + 2 + 2 + 0 + 0

# Merge and Count

Merge and count step.

- Given two sorted halves, count number of inversions where $a_i$ and $a_j$ are in different halves.
- Combine two sorted halves into sorted whole.

$i = 0$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 7 | 10 | 14 | 18 | 19 | 2 | 11 | 16 | 17 | 23 | 25 | two sorted halves |
| | | | | | | 6 | 3 | 2 | 2 | 0 | 0 | |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 7 | 10 | 11 | 14 | 16 | 17 | 18 | 19 | 23 | 25 | auxiliary array |

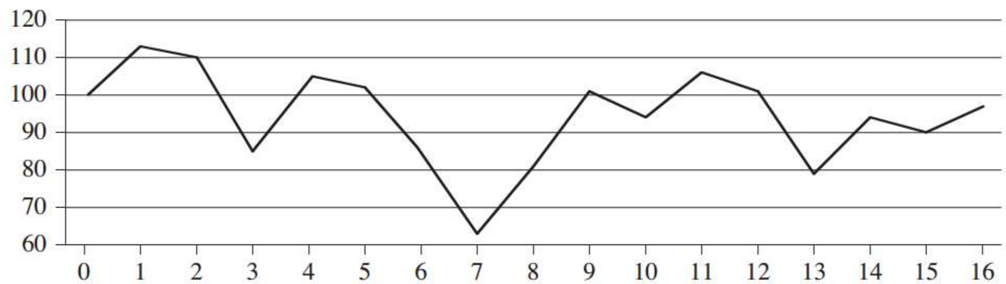
Total: $6 + 3 + 2 + 2 + 0 + 0 = 13$

# Counting inversions

- Let A be a sequence with left / right halves L / R.
- Function C counts the number of inversions in A, and returns A in sorted order.
  - Compute x = C(L), y = C(R).
    - After this, L and R are sorted.
  - Merge L and R, while counting number of inversions z.
  - Return x+y+z, and the merged sequence.
- Let T be the time complexity for C.
  - $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$.
  - Thus, $T(n) = O(n \log n)$.

# Maximum subarray

- Motivation Make money on stocks by buying and selling on days with largest price difference.

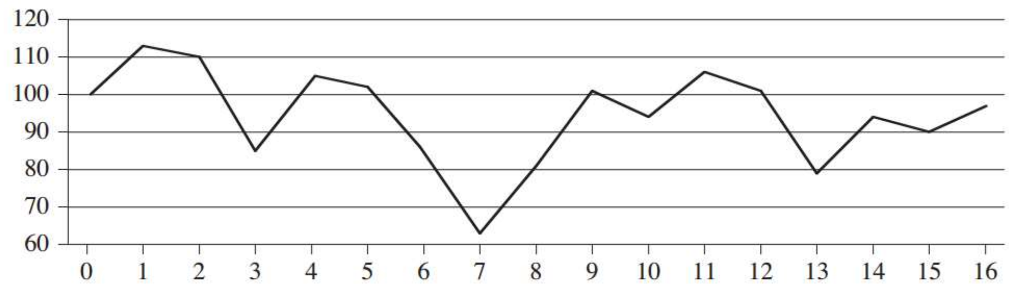



| Day | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Price | 100 | 113 | 110 | 85 | 105 | 102 | 86 | 63 | 81 | 101 | 94 | 106 | 101 | 79 | 94 | 90 | 97 |
| Change | | 13 | −3 | −25 | 20 | −3 | −16 | −23 | 18 | 20 | −7 | 12 | −5 | −22 | 15 | −4 | 7 |

Source: *Introduction to Algorithms* Cormen et al

- Ex Buy on day 7, sell on day 11, make \$106 - \$63 = \$43.
- If there are n days, can compute price difference of all $O(n^2)$ pairs of days and take the max.
- Is there a faster way?

# Maximum subarray




| Day | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Price | 100 | 113 | 110 | 85 | 105 | 102 | 86 | 63 | 81 | 101 | 94 | 106 | 101 | 79 | 94 | 90 | 97 |
| Change | | 13 | −3 | −25 | 20 | −3 | −16 | −23 | 18 | 20 | −7 | 12 | −5 | −22 | 15 | −4 | 7 |

- Let P be the array of stock prices.
- Goal Find $i < j$ such that $P[j] - P[i]$ is maximum.
- We first compute the price change on consecutive days.
  - Ex On day 4, the price change is \$105-\$85=\$20.
  - Call the array of changes A.
  - So $A[i] = P[i] - P[i-1]$, for $i = 1, \ldots, n$.

# Maximum subarray

| Day | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Price | 100 | 113 | 110 | 85 | 105 | 102 | 86 | 63 | 81 | 101 | 94 | 106 | 101 | 79 | 94 | 90 | 97 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | 13 | –3 | –25 | 20 | –3 | –16 | –23 | 18 | 20 | –7 | 12 | –5 | –22 | 15 | –4 | 7 |

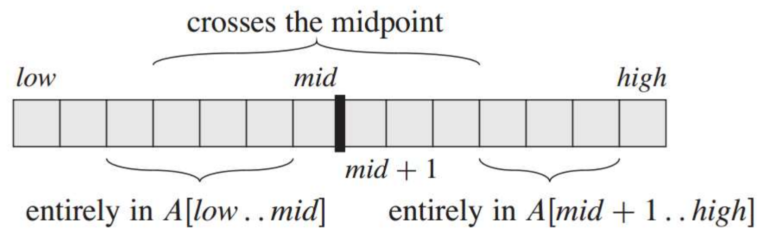
maximum subarray

- Observation Finding $i < j$ with max $P[j] - P[i]$ is the same as finding $i < j$ with max $\sum_{k=i+1}^{j} A[k]$ .
  - Ex $P[11] - P[7] = 43 = A[8] + A[9] + A[10] + A[11]$.
- Thus, we want to find a subarray of A with the maximum sum.
  - I.e. want to find a continuous set of elements of A with the largest sum.
  - Ex For A above, it's the 8th to 11th elements.

# Maximum subarray

- Goal Given array A, find $i < j$ with max $\sum_{k=i+1}^{j} A[k]$.
- Seems no easier than initial problem... Still $O(n^2)$ pairs i, j to consider.
  - In fact, computing $\sum_{k=i+1}^{j} A[k]$ takes O(n) time, so finding max subarray seems to take $O(n^3)$ time!
  - Actually, can find $\sum_{k=i+1}^{j} A[k]$ for all pairs i,j in $O(n^2)$ time. How? DP思想优化（存起来已经算过的，DP空间为n^2）
- But with divide and conquer, can find max subarray in $O(n \log n)$ time.

# A divide and conquer algorithm




### Observation

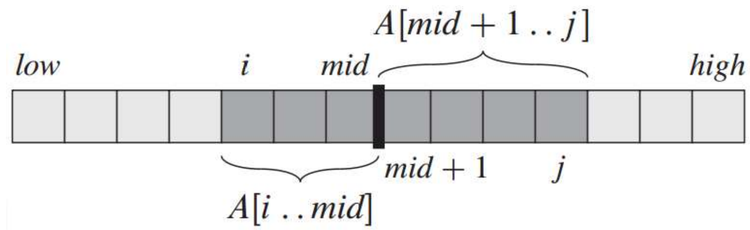
- Divide A down the middle. Then a max subarray of A either
  - Lies entirely in the left half.
  - Lies entirely in the right half.
  - Crosses the midpoint.

### Algorithm

- Break A into left and right halves.
- Compute the max subarrays in each half.
- Compute the max subarray crossing the midpoint.
- Return max of these three subarrays.

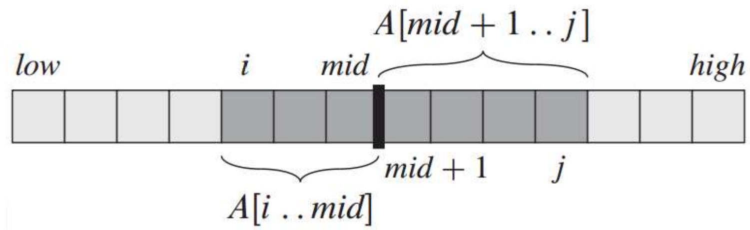
- Analysis $S(n) = 2S(n/2) + T(n) + O(1)$.
  - Finding max subarray in each half takes $S(n/2)$ time.
  - $T(n)$ = time to find max subarray crossing midpoint.
- We will show $T(n) = O(n)$.
- So $S(n) = O(n \log n)$.

# Max crossing subarray




- Goal Find max subarray crossing the midpoint.
- Solution Find the max leftwards subarray from the midpoint.
  - I.e. find a subarray containing the midpoint and lying to the left, that has the max sum.
  - Also find the max rightwards subarray from the midpoint.
  - Combine them and return this.
- Ex A = [3,2,-8,1,6,7,-4,2,8,2,-4,1,-2, 3,1].
  - Max leftwards subarray from 2 is [1,6,7,-4,2].
  - Max rightwards subarray from 2 is [2,8,2].
  - Max crossing subarray is [1,6,7,-4,2,8,2].

# Max crossing subarray



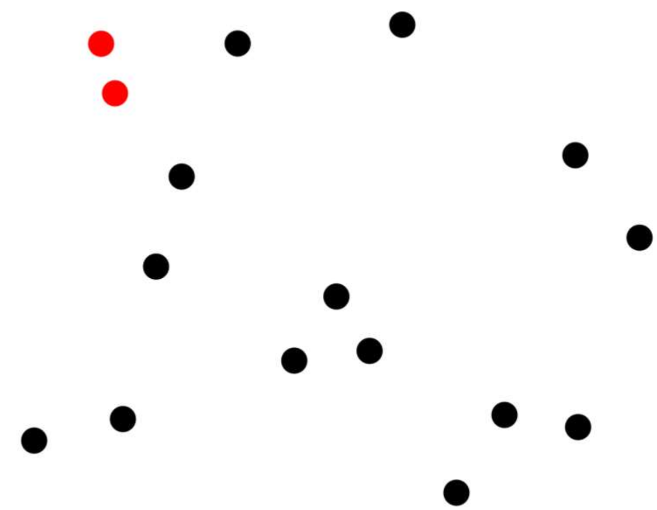

- Algorithm To find max leftwards subarray, sum array elements leftwards starting from midpoint.
  - Whenever sum exceeds current max, remember the index as the current max.
  - Similar for rightwards subarray.
- Analysis Scan through once to left and right. $O(n)$ time.

FIND-MAX-CROSSING-SUBARRAY$(A, low, mid, high)$

```
left-sum = −∞
sum = 0
for i = mid downto low
    sum = sum + A[i]
    if sum > left-sum
        left-sum = sum
        max-left = i
right-sum = −∞
sum = 0
for j = mid + 1 to high
    sum = sum + A[j]
    if sum > right-sum
        right-sum = sum
        max-right = j
return (max-left, max-right, left-sum + right-sum)
```

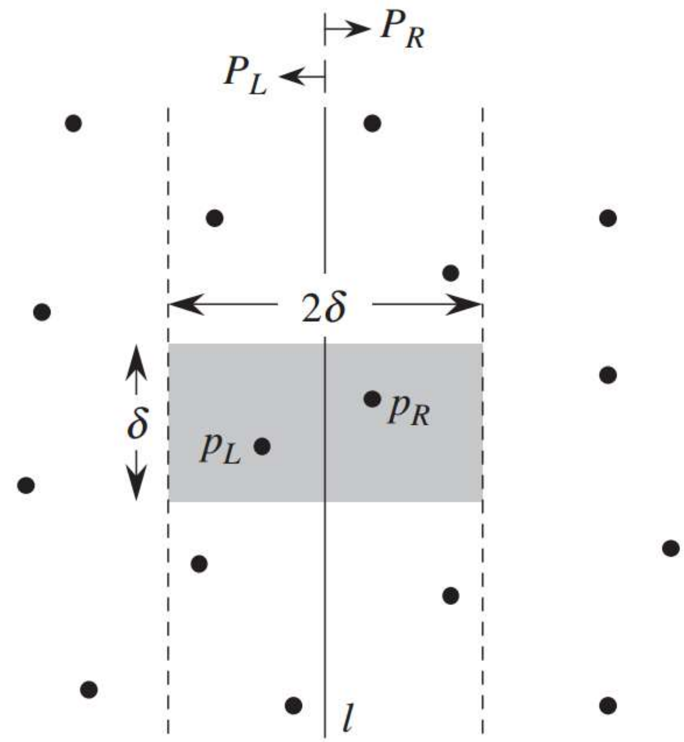
# Closest point pair

- Given a set of n points in the plane, find the pair that's closest.
- Naive algorithm computes distances between all $O(n^2)$ pairs of points and chooses min.
- Use divide and conquer to improve complexity to $O(n \log n)$.



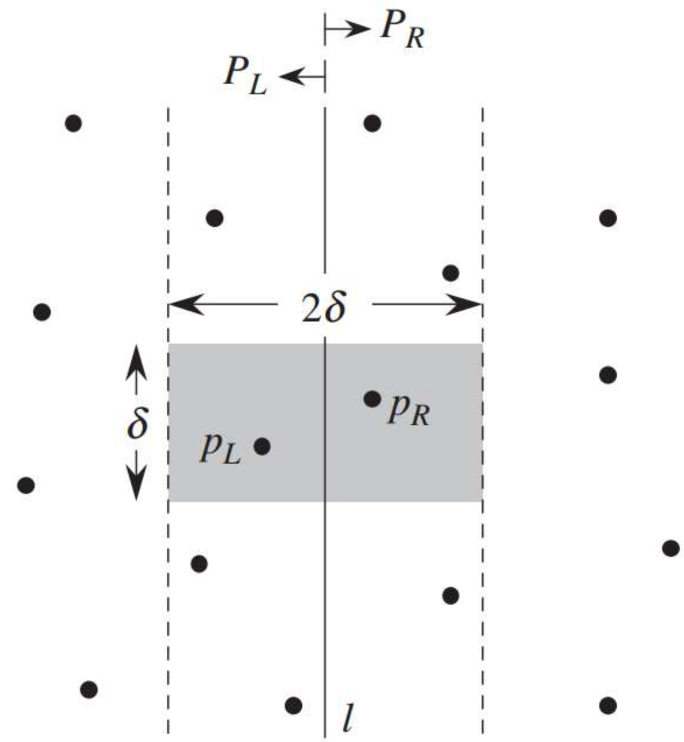
# Closest point pair

- Split the points evenly using a vertical line, i.e. half the points lie on the left and half on the right.
- Observation The closest pair of points either
  - Both lie in the left half
  - Both lie in the right half, or
  - Straddles the line, i.e. one point on each side.
- This suggests the following algorithm.



# Closest point pair

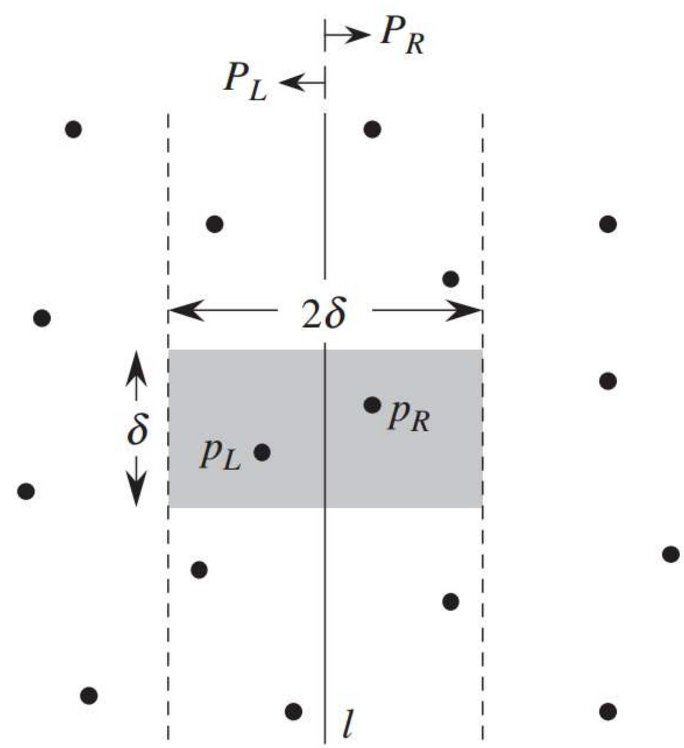
- Divide points evenly using vertical line.
- Recursively find closest point pair in left half and right half.
  - Let the min distance between any point pair in either half be $\delta$.
- Look for closest pair of points straddling line with distance < $\delta$.
  - Don't need to consider straddling pairs with distance $\geq \delta$, since we already found such pairs on the left or right.
- If pair exists, return their distance.
- Else return $\delta$.



# Algorithm analysis

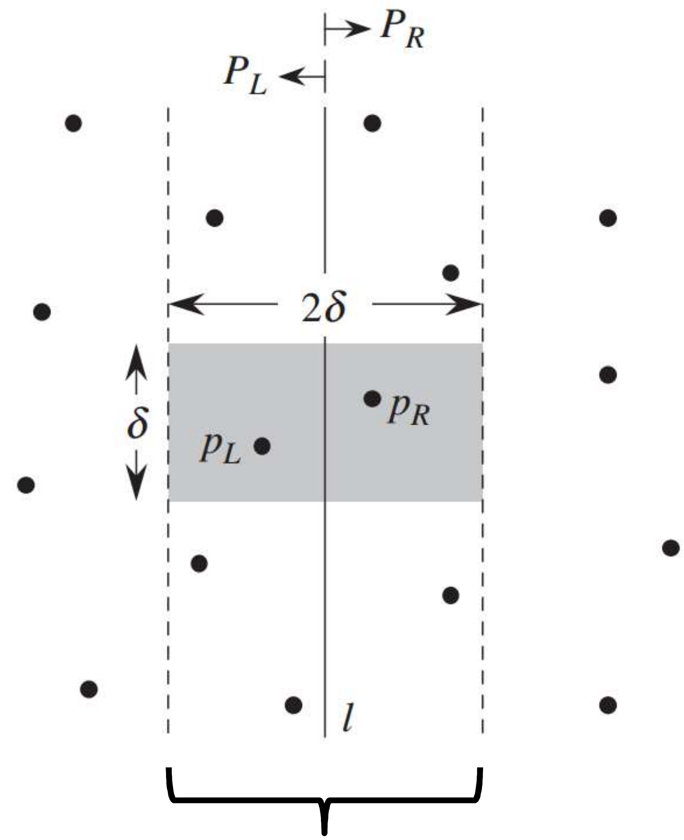
- Let $S(n)$ be time to find closest point pair of n points.
- $S(n) = 2S(n/2) + O(n)$
  - Can divide the points in $O(n)$ time.
    - Details on slide 46.
  - $2S(n/2)$ time to recursively find closest point pair in both halfs.
  - Can find closest straddling pair in $O(n)$ time.
    - Details next slide.
- $S(2) = O(1)$.
  - If only two points, they're the closest pair.
- So $S(n) = O(n \log n)$.

- Divide the points evenly.
- Recursively find closest pair on left and right.
- Find closest straddling pair.
- Return the min of the three.



# Closest straddling point pair

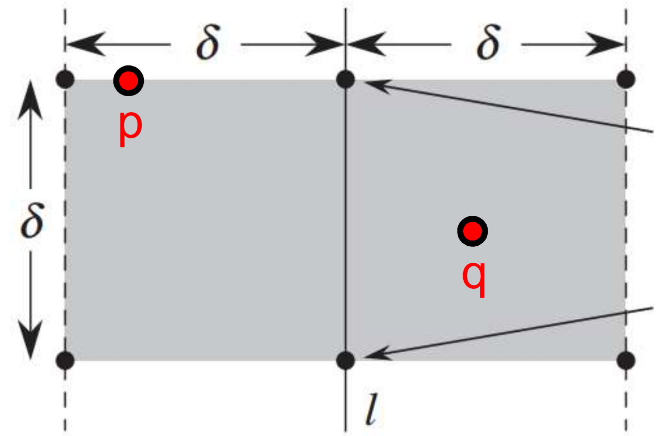
- Goal Find closest straddling pair, assuming their distance is $< \delta$.
- Only need to consider points within a band of width $2\delta$ centered on dividing line.
  - Pairs outside band can't be closer than $\delta$.
- Let B be set of points in band.
  - To form B, iterate through all points in any order, pick ones within distance $\delta$ from line.
  - Takes $O(n)$ time.
- Assume points in B sorted by y coordinate, i.e. from top to bottom.
  - By iterating in the right order when forming B, can get this property "for free", without actually sorting B.
  - Details later.
- Now, use following lemma to find closest straddling pairs.




2δ band around divding line

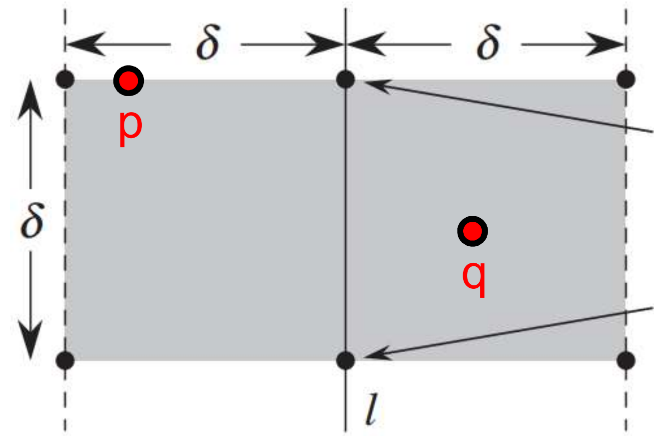
# Sparsity lemma

- Lemma Let $p, q \in B$. Suppose q is below p, and has distance $< \delta$ from p. Then
  1. q lies in a $\delta \times 2\delta$ rectangle centered on the dividing line, and with p on the top edge.
  2. The rectangle contains at most 6 points from B (including p and q).
  3. If we list the points in order B from top to bottom, the points in the rectangle immediately follow p in the ordering.



# Sparsity lemma proof

1. Any point below the rectangle is $> \delta$ distance from p.
2. Any two points in rectangle on same side of the line are distance $\geq \delta$ apart.
   - Because $\delta$ is the min distance between any pair of points on either side.
   - So, at most 6 points in B fit in the rectangle.
   - Ex The 6 points can fit in the corners and the middle, as shown.
3. Points in the rectangle precede any points below it in y ordering.



# Closest straddling point pair

- Algorithm Sweep through points in B from top to bottom.
  - For each point p, check next 5 points in R below it.
  - Let $\delta_p$ be distance to nearest one.
  - After sweeping through all points in B, return the minimum $\delta_p$ value or $\delta$, whichever is smaller.
- Correctness By sparsity lemma, only next 5 points in B below p can be distance $< \delta$ from p.
  - Since we return the closest pair among these 5 points, we find overall closest straddling pair.
  - If no straddling pairs have distance $< \delta$, we return $\delta$.
- Analysis Algorithm takes O(n) time.
  - B contains O(n) points.
  - For each point in B, check its distance to 5 other points.

# Dividing points evenly

- At the beginning of the algorithm, sort all points horizontally and store in an array H.
  - Takes $O(n \log n)$ time.
- Assume at some level of recursion, input array is sorted horizontally.
- Then points to the left / right of dividing line are points in the first / second half of array.
  - Outputting either half takes $O(n)$ time.
- These points are sorted horizontally, for the next level of recursion.
  - So at every level of recursion, can get points in sorted order in $O(n)$ time.
- Add this $O(n \log n)$ preprocessing time to algorithm's running time.
  - Algorithm still $O(n \log n)$.

# Sorting R points by y coordinate

- At the beginning of the algorithm, also sort all the points vertically. Store them in a separate array V.
  - Takes $O(n \log n)$ time.
- Points in H and V have pointers to each other.
  - I.e. given p in H, its pointer gives p's index in V. Similarly given p in V, we can get p's index in H.
- When picking out points left (or right) of dividing line using H, mark them in V by following the pointers.
- Next, iterate through V (in vertical order) and pick out marked points.
  - These points are again sorted vertically.
  - Takes $O(n)$ time.
- Add this $O(n \log n)$ preprocessing time to algorithm's running time. Algorithm still $O(n \log n)$.