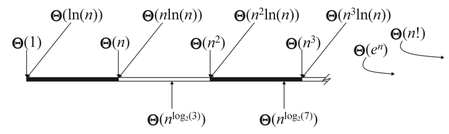
Complexity Analyze

Landau Symbol	limit	rate of growth
$f(n)=o(g(n))$	$\lim_{n\to\infty}\frac{f(n)}{g(n)}=0$	$f<g$
$f(n)=O(g(n))$	$\lim_{n\to\infty}\frac{f(n)}{g(n)}<\infty$	$f\leq g$
$f(n)=\Theta(g(n))$	$\lim_{n\to\infty}\frac{f(n)}{g(n)}=c, 0<c<\infty$	$f\sim g$
$f(n)=\omega(g(n))$	$\lim_{n\to\infty}\frac{f(n)}{g(n)}=\infty$	$f>g$
$f(n)=\Omega(g(n))$	$\lim_{n\to\infty}\frac{f(n)}{g(n)}>0$	$f\geq g$



- when $p<q$, $np=o(\ln(n)n^p)$, but $\ln(n)n^p=o(n^q)$
- $(\log n)^k=o(n^\varepsilon), \forall k\in Z^+, \varepsilon\in R^+$
- $n!\approx\sqrt{2\pi n}(\frac{n}{e})^n$
  - 或者表达为：$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n}=1$
  - 解释了为什么 $n!<n^n$, but $\log(n!)\sim\log(n^n)$

if-else: assume the longest branch runs (worse case complexity)

Analyze Recursive algorithms

4 ways to solve a recurrence relation.

Direction solution

Guess a solution, then prove it by induction.

Substitution method

一直展开式子，直到$S(n)=S(1)+\dots$

Recursion tree method

计算树的深度和每层上的开销

Master theorem

If $T(n)=aT\left(\frac{n}{b}\right)+\Theta(n^d)$ for constants $a>0, b>1, d\geq 0$, then:

$$T(n)=\begin{cases}\Theta(n^d) & if\ d>log_b a\\ \Theta(n^d log n) & if\ d=log_b a\\ \Theta(n^{log_b a}) & if\ d<log_b a\end{cases}$$

- - 可以处理$T(n)=3T(n/4)+n\log n$, 但无法处理$T(n)=2T(n/2)+n\log n$
  - $T(n)=2T(\sqrt{n})+\Theta(\log n)$: Let $n=2^m, S(m)=T(2^m)$.

> $T(n)=a\cdot T\left(\frac{n}{b}\right)+f(n)$
>
> 其中 $a\geq 1$，$b>1$，$f(n)$ 是一个正函数，则 $T(n)$ 的渐进上界可以根据以下三种情况中的一种确定：
>
> 1. 如果 $f(n)=O(n^{\log_b a-\epsilon})$ 对某个常数 $\epsilon>0$ 成立，则 $T(n)=\Theta(n^{\log_b a})$。
> 2. 如果 $f(n)=\Theta(n^{\log_b a})$，则 $T(n)=\Theta(n^{\log_b a}\log n)$。
> 3. 如果 $f(n)=\Omega(n^{\log_b a+\epsilon})$ 对某个常数 $\epsilon>0$ 成立，且对某个常数 $c<1$ 和足够大的 n，有 $a\cdot f\left(\frac{n}{b}\right)\leq c\cdot f(n)$，则 $T(n)=\Theta(f(n))$。**

Divide and Conquer

Selection

find the i-th smallest number in a given array.

---

an algorithm that is similar to quick sort:

randomly choose a piviot. on average, each piviot divide the array equally.

$$T(n)=T(n/2)+O(n)$$
$$T(n)=O(n)$$

But if not random, worse case each pivot only divide out one element.

$$T(n)=T(n-1)+O(n)$$
$$T(n)=O(n^2)$$

---

an algorithm introduced in calss (I think it is just a method introduced to avoid the worse case in the former QS without using random)

`select(A, i)`

Algorithm

Analysis

$$S(n)=S(n/5)+S(u)+O(n)$$

- u is the size of whichever partition we recurse on.

show $u\leq 7n/10$

- n/10 medians less than x
- each group with median less than x, 3 values in this group less than x
- totally at least 3n/10 values less than x
- similarly, at least 3n/10 values larger than x

Therefore, $u\leq 7n/10$ proved, and

$$S(n)\leq S(n/5)+S(7n/10)+O(n)$$

guess and prove $S(n)=O(n)$

suppose $S(n)\leq cn$

$$cn=c(n/5)+c(7n/10)+bn$$
$$c=10b$$

Multiplying complex number – Gauss' s method

$$(a+bi)(c+di)=x+yi$$
$$x=ac-bd$$
$$y=(a+b)(c+d)-ac-bd$$

- less multiplication, more addition -> much faster when used **recursively**

Long multiplication of two integers – Karatsuba' s algorithm

$$\boldsymbol{a}=\underbrace{1000}_{a_1}\underbrace{1101}_{a_0}\qquad \boldsymbol{b}=\underbrace{1110}_{b_1}\underbrace{0001}_{b_0}$$

$$\begin{aligned}a &= 2^{n/2}\cdot a_1+a_0\\ b &= 2^{n/2}\cdot b_1+b_0\\ ab &= 2^n\cdot a_1b_1+2^{n/2}\cdot(a_1b_0+a_0b_1)+a_0b_0\end{aligned}$$

- not faster, still $\Theta(n^2)$

use Gauss' s method

$$\boldsymbol{a}=\underbrace{1000}_{a_1}\underbrace{1101}_{a_0}\qquad \boldsymbol{b}=\underbrace{1110}_{b_1}\underbrace{0001}_{b_0}$$

$$\begin{aligned}a &= 2^{n/2}\cdot a_1+a_0\\ b &= 2^{n/2}\cdot b_1+b_0\\ ab &= 2^n\cdot a_1b_1+2^{n/2}\cdot(a_1b_0+a_0b_1)+a_0b_0\\ &= 2^n\cdot a_1b_1+2^{n/2}\cdot((a_1+a_0)(b_1+b_0)-a_1b_1-a_0b_0)+a_0b_0\end{aligned}$$

$$\boxed{\begin{aligned}(a+bi)(c+di)&=x+yi\\ x&=ac-bd\\ y&=(a+b)(c+d)-ac-bd\end{aligned}}$$

- It does 3 multiplications of digit numbers instead of 4

$$S(n)=3S(n/2)+O(n)$$
$$S(n)=\Theta(n^{\log_2 3})=O(n^{1.59})<O(n^2)$$

Block matrix multiplication

naive, $\Theta(n^3)$

$$\begin{bmatrix} 152 & 158 & 164 & 170 \\ 504 & 526 & 548 & 570 \\ 856 & 894 & 932 & 970 \\ 1208 & 1262 & 1316 & 1370 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \\ 12 & 13 & 14 & 15 \end{bmatrix} \times \begin{bmatrix} 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 \\ 24 & 25 & 26 & 27 \\ 28 & 29 & 30 & 31 \end{bmatrix}$$

$$C_{11} = A_{11} \times B_{11} + A_{12} \times B_{21} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 16 & 17 \\ 20 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix} \times \begin{bmatrix} 24 & 25 \\ 28 & 29 \end{bmatrix} = \begin{bmatrix} 152 & 158 \\ 504 & 526 \end{bmatrix}$$

$$S(n) = 8S(n/2) + O(n^2)$$
$$S(n) = \Theta(n^3)$$

- not faster

use Strassen' s algorithm

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$\begin{aligned} C_{11} &= P_5 + P_4 - P_2 + P_6 \\ C_{12} &= P_1 + P_2 \\ C_{21} &= P_3 + P_4 \\ C_{22} &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} P_1 &= A_{11} \times (B_{12} - B_{22}) \\ P_2 &= (A_{11} + A_{12}) \times B_{22} \\ P_3 &= (A_{21} + A_{22}) \times B_{11} \\ P_4 &= A_{22} \times (B_{21} - B_{11}) \\ P_5 &= (A_{11} + A_{22}) \times (B_{11} + B_{22}) \\ P_6 &= (A_{12} - A_{22}) \times (B_{21} + B_{22}) \\ P_7 &= (A_{11} - A_{21}) \times (B_{11} + B_{12}) \end{aligned}$$

$$S(n) = 7S(n/2) + O(n^2)$$
$$S(n) = \Theta(n^{\log_2 7}) = O(n^{2.81})$$

Counting Inversions

> CS101 notes/Sort/Merge Sort

- divide the array into 2 halves

total number of inversions = # in the left half + # in the right half + # between the 2 halves

count # between the 2 halves:

naively, 每两个都比， $\frac{n^2}{4}$

$$S(n) = 2S(n/2) + O(n^2)$$
$$S(n) = O(n^2)$$

- not faster

Sort (which doesn't change the number of inversions)

merge and count

$$S(n) = 2S(n/2) + O(n)$$
$$S(n) = O(n \log n)$$

> 不要哦从各自的头开始比较，从一个的头和另一个的尾。如果从尾开始的某个数已经比从头开始的某个小，那剩下的就全是Inversion。这应该能做到一个常数级的优化?
>
> 不行。一头一尾 没法把sorted 作为结果

Maximum Subarray

> 在动态规划里也有

Goal. Given array A, find with max $\sum_{k=i+1}^{j} A[k]$

Solution with D&C.

divide the array from the middle, the wanted array is either:

- in the right half
- in the left half
- across the 2 halves

Find max subarray crossing the midpoint:

在左右各自找，和mid连续的最大subarray

$O(n)$

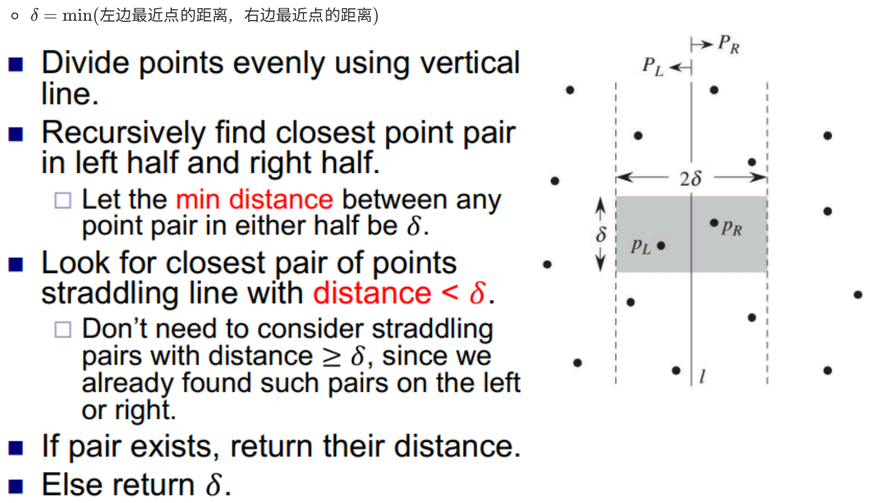
$$S(n) = 2S(n/2) + O(n)$$
$$S(n) = O(n \log n)$$

Solution with DP

> CS101/General Resources/DP
>
> CS101/My Notes/DynamicProgramming
>
> 两个里面都有讲

Closest Point Pair

- 用一条与x轴或y轴垂直的直线，将平面分成两个区域。每个区域点数接近 - $O(n)$
  - see **Dividing points evenly**
- 最近点可能在：
  - 左
  - 右
  - 横跨左右
- 找横跨左右的最近点 - $O(n)$
  - **see Sorting R points by y coordinate**
  - $\delta = \min$(左边最近点的距离，右边最近点的距离)



$$S(n) = 2S(n/2) + O(n)$$
$$S(n) = O(n \log n)$$

Sparsity lemma

Dividing points evenly

在算法最初，先把所有的点以某一个维度sort $O(n \log n)$

然后每次从这个sort好的array，中分数量

- $O(n)$

Sorting R points by y coordinate

算法初sort $O(n \log n)$

Polynomial multiplication – FFT

naively, multiplying 2 polynomials each of degree n takes $O(n^2)$ time.

can also be solve using ***Karatsuba's method***, but higher time complexity $\Theta(n^{\log_2 3})$, while ***FFT*** has $\Theta(n \log n)$

> see CS101/PA/PA2/prob1

多项式的系数表示

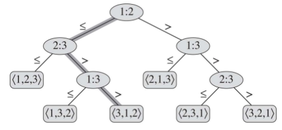
point-value representation：$n-1$次的多项式可以通过n个不同的点取值来确定

那么求$C(x)$相当于：求C在 $x = x_k$ 时的取值，那么如果已知A和B 在 $x = x_k$ 的取值，直接 $C(x_k) = A(x_k)B(x_k)$ for all k

Therefore, $C(x) = A(x)B(x)$.

Linear time sorting

基于比较的排序，理论上时间复杂度的下限就是 $O(n \log n)$

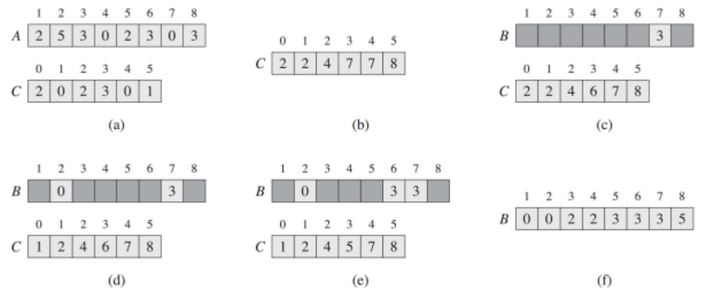


而非基于比较的排序理论下限 $O(n)$，因为至少要把输入读一遍并且输出

Counting sort

Requirement: integers in the range 0 to k

- In (a), contains number of occurrences of each input value in A
- 计算prefix sum也就是图(b)中的C
- iterate through in reverse order.



$\Theta(n+k)$

if $k=O(n)$, then complexity is $\Theta(n)$

stable -> can be used for radix sort

Radix sort

Sort digit by digit, from least to most significant digit.(从最低位起sort，必须stable算法)

- Suppose we sort n d-digit numbers, where each digit is between 0 to k-1. Then radix sort takes $O(d(n+k))$ time.
- Given n b-bit numbers and $r \leq b$. Radix sort takes $O(\frac{b}{r}(n+2^r))$ time.
    - Break the b bits into blocks of r digits, having values between 0 and 2^r-1
- Setting $r=\min(\lfloor \log n \rfloor, b)$ minimizes the running time $\Theta(\frac{b}{r}(n+2^r))$
- radix sort is efficient when there are many short numbers, but not when there are a few long numbers.

Greedy

Selecting Breakpoints

gas refueling problem

save refuel time

proof.

the greedy algorithm always choose to go as far as possible, so OPT can't go further than Greedy.

If $g_{r+1} > f_{r+1}$, than OPT must require one more step than Greedy, then it's not optimal. Contradicts.

Coin Changing

Theorem. Greedy is optimal for U.S. coinage: 1, 5, 10, 25, 100.

proof.

find max vlaue of coins in any OPT
prove that without the least max coin, can't achieve opt

Such greedy algorithm **doesn't work for all** coin combinations.

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500 .

Interval scheduling

Goal. Given a set of intervals, pick the largest number of non-overlapping ones.

- sort by finishing time
- choose **earliest finishing** one that doesn't overlap previous selected interval

Overall $O(n \log n)$ time.

proof.

greedy is not worse than the optimal solution. prove by **induction**

Greedy solution S, optimal solution T.
Let s_k and t_k be k'th interval in S and T, resp.
Let $fin(i)$ be finishing time of an interval i.
Claim. $fin(s_k) \leq fin(t_k)$ for all k

Interval coloring problem

Goal. schedule all the intervals on some number of resources. Intervals on the same resource cannot overlap. Minimize the number of resources used.

Observation. Suppose k intervals intersect at some time point. Then the optimal schedule needs at least k resources.
Def. ***Depth*** of a set of intervals is the max number of intervals that intersect at any time.

Corollary. Let d be the depth of a set of intervals, and suppose we find a schedule using d resources. Then the schedule is optimal.

- Sweep through intervals in order of **increasing start time**.
- For each interval, assign it to smallest resource not already assigned to an intersecting interval.

Claim. Let the set of intervals have depth d. Then the algorithm uses d resources

Scheduling to Minimizing Lateness

- **Earliest deadline first** -> optimal

Observation. There exists an optimal schedule with no idle time.

Observation. The greedy schedule has no idle time.

Def. An ***inversion*** in schedule S is a pair of jobs i and j such that: i < j but j scheduled before i. *(I suppose the i and j here means deadline time)*

Observation. Greedy schedule has no inversions.

an existing inversion always make the lateness larger.

Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Pf. Greedy schedule S is optimal.

Define $S*$ to be the optimal solution. Then $S*$ must has the fewest number of inversions.

- Can assume $S*$ has no idle time.
- If $S*$ has no inversions, then $S = S*$.
- If S* has an inversion, let i-j be an adjacent inversion.
    - swapping i and j does not increase the maximum lateness and strictly decreases the number of inversions
    - this contradicts definition of $S*$

Caching

cache: fast, closer to CPU core, not quite large

- store only the most important data in cache
- caching algorithm to decide which data to store

Optimal Offline Caching

offline: at the beginning, **know** which data are going to be accessed

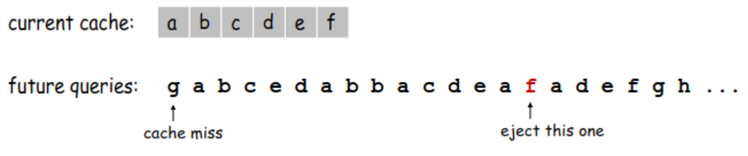
assume only one layer of cache
Intuition. Can transform an unreduced schedule into a reduced one with no more evictions.

cache hit: in cache
cache miss: if not find in cache, go to memory, load in into cache, choose to evict some existing item.

Goal. Eviction schedule that minimizes number of evictions.

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future

- FF is optimal offline eviction algorithm



Proof.

Def. A ***reduced*** schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more evictions. *(There is no need to prepare ahead of time)*

> see proof on ppt

prove by induction

Lemma. If the optimal reduced schedule S share the same first j requests as S_{FF}, then they also share the same $j+1$ request.

pf: Consider $(j+1)$ st request $d = d_{j+1}$

- **Case 1**: (d is already in the cache). $S' = S$
- **Case 2**: (d is not in the cache and S and S_{FF} evict the same element). $S' = S$
- **Case 3**: (d is not in the cache; S_{FF} evicts e; S evicts $f \neq e$).

  From request j+2 onward, we make S' the same as S, but this becomes impossible when e or f is involved

  - **Case 3a**: $g = e$
  - **Case 3b**: $g \neq e, g \neq f$
  - **Case 3c**: $g = f$

Online Caching

Online (reality): requests are not known in advance.

- **LIFO**. Evict page brought in most recently.
  - LIFO is arbitrarily bad: For n requests, it may do $O(n)$ times more loads than optimal.
- **LRU**. Evict page whose most recent access was earliest
  - LRU is k-competitive: it does times more loads than the optimal eviction algorithm

Huffman Coding

prefix-free codes -> unique decoding
Huffman encoding is an optimal prefix-free code

n different chars to encode -> $O(n^2)$ time

Pf. optimal

Dynamic Programming

Optimal substructure

After making a decision, the rest of the solution should be optimal for the rest of the problem.

Not all problems have optimal substructure.

Weighted Interval Scheduling

Goal. Pick a set of non-overlapping intervals with the largest combined weight.

- Given interval I_j, let $p(j)$ be the maximum index k s.t. I_k finishes before I_j starts.
  finishes before starts.
- Let $OPT(j)$ be the weight of a max weight non-overlapping subset of $I_1, \ldots, I_j$.

> see CS101/LectureNotes/dp
>
> or CS101/mynotes

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max\{OPT(j-1),\ v_j + OPT(p(j))\} & \text{if } j > 0 \end{cases}$$

Segmented Least Squares

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with $x_1 < x_2 < \ldots < x_n$, find a sequence of lines that minimizes $E + cL$
  - E: the sum of the sums of the squared errors in each segment
  - L: the number of lines
  - c: some positive constant

$OPT(j)$ = minimum cost for points $p_1, p_{i+1}, \ldots, p_j$.

$e(i, j)$ = minimum sum of squares for points $p_i, p_{i+1}, \ldots, p_j$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max_{1 \le i \le j}\{e(i,j) + c + OPT(i-1)\} & \text{otherwise} \end{cases}$$

Running time. $O(n^3)$

Subset Sum

背包问题的变种

Goal. Each job i takes w_i resource. We have W resources. Find a set of jobs $S \subseteq \{1, \ldots n\}$ to maximize $\sum_{i \in S} w_i$, subject to $\sum_{i \in S} w_i \le W$.

Greedy algorithm. Sort items from largest to smallest. Insert items sequentially, as many as possible. -> Always achieves at least 1/2 the max possible sum.

DP. Let $OPT(i, W')$ be max weight of a subset $S \subseteq \{w_1, \ldots, w_i\}$, subject to $\sum_{i \in S} w_i \le W'$.

$$OPT(i, W') = \max(OPT(i-1, W'), w_i + OPT(i-1, W' - w_i))$$

Time complexity and memory complexity are both $O(nW)$.

Matrix-chain multiplication

choose which point to break on

Goal. Given a sequence $A_1, A_2, \ldots, A_n$ of matrices, where A_i has dimensions $p_{i-1} \times p_i$, for $i = 1, \ldots n$, compute the product $A_1 \times A_2 \times \cdots \times A_n$ in a way that minimizes the cost.

Let $M(i, j)$ be the smallest time to multiply matrices of $A_i \times A_{i+1} \times \cdots \times A_j$

$$M(1,n) = \min_{1 \le j \le n-1}(M(1,j) + M(j+1,n) + p_0 p_j p_n)$$

Cost of multiplying all the matrices $A_1, \ldots, A_n$. | Choose the best break point j | Cost of the first part | Cost of the second part | Cost of multiplying the matrices produced by the first and second parts

Table method to solve the DP space

关键在于先找出DP的前后依赖关系

Total time cost: $O(n^3)$

Space complexity: $O(n^2)$

Problems with(out) optimal substructure

Shortest paths has optimal substructure

$d(x, y) = \min_{(z,y) \in E}(d(x, z) + w(z, y))$

shortest path from x to y is to take shortest path from x to one of y's neighbors, then go to y

Longest simple path does not have optimal substructure.

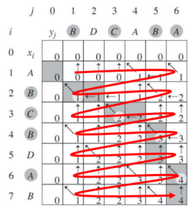
When sub-solutions aren't combinable, we say the subproblems aren't independent

Longest common subsequence

Remove the last letter of the long string

- Let $S[1, i]$ be the first i letters of a string S, and $S[i]$ be the i'th letter of S.
  - Let $S[i, 0]$ be the empty string, for any i.
- Let $LCS(X[1, i], Y[1, j])$ be the LCS of $X[1, i]$ and $Y[1, j]$.
- Let $c(i, j)$ be the length of $LCS(X[1, i], Y[1, j])$.

$$c[i, 0] = 0 \text{ for all i}$$
$$c[0, j] = 0 \text{ for all j}$$
$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } X[i] = Y[j] \\ \max(c[i-1.j, c[i, j-1]]) & \text{if } X[i] \neq Y[j] \end{cases}$$



Time complexity and memory complexity are both $O(mn)$.

Network Flow

duality between ***max flow*** and ***min cut***

The weight of the edge stands for its capacity.

Flows

s-t flow

- Each edge can have an amount of flow less than its capacity.
  - $0 \leq f(e) \leq c(e)$ for each $e \in E$
- The sum of in-flow equals the sum of out-flow (except the source and sink)
  - $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ for each $v \in C - \{s, t\}$

***value* of a flow f**

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

Greedy

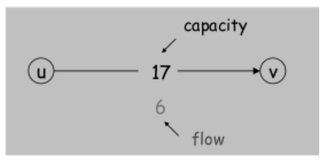
- 单走一条source 到 t 的路径，取上面所有edge的剩余的capacity的最小值作为f(e)
- 重复此步骤直到stuck

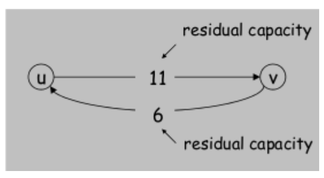
not always optimal

- 局部最优无法达到全局最优

undo to go out of the trapped decision

Residual Graph





Augmenting Path

a simple $s - t$ path P in the residual graph G_f.

Ford-Fulkerson Algorithm

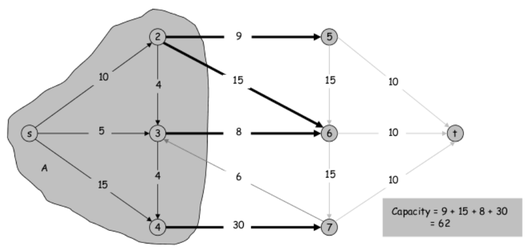
可以反悔的greedy

如果没有e^R这条反向edge每次循环后的添加，就等于纯粹的greedy

Cuts

An ***s-t cut*** is a partition (A, B) of V with $s \in A$ and $t \in B$.

The ***capacity*** of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of A}} c(e)$



Flow value lemma

Let v be any vlow, and let (A, B) be any s-t cut.
Then, the net vlow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of A}} f(e) - \sum_{e \text{ in to A}} f(e) = v(f)$$

Weak Duality

Let v be any flow, and let (A, B) be any s-t cut. Then thenvalue of the flow is at most the capacity of the cut.

Corollary. Let v be any vlow, and let (A, B) be any cut.
Iv v(v) = cap(A, B), then v is a max vlow and (A, B) is a min cut.

the equivalence of the following three conditions for any flow f:

- There exists a cut (A, B) such that v(v) = cap(A, B).
- Flow v is a max vlow.
- There is no augmenting path relative to v.

Choosing Good Augmenting Paths

Assumption. All capacities are integers between 1 and C.

n 是所有edge capacity之和?

Running time of Ford-Fulkerson is $O(mnC)$.

NP, reductions

Def. P is the set of all problems that can be solved by a polytime algorithm.

NP = Nondeterministic polynomial time.

Def. NP is the class of problems for which the solvability of an instance can be verified in polynomial time. (未知是否有多项式时间的解，但是解的可行性可以在多项式时间内得到验证)

- To show a problem is in P, give an algorithm solving the problem that runs in polynomial time.
- To show a decision problem is in NP, give a polynomial time verifier for the problem satisfying the properties on the previous slide.

1. 4-coloring is in NP
2. Factoring is in NP
3. Traveling salesman is in NP
4. k-Clique is in NP
5. **All problems in P are in NP**
   - Let A be a problem in P. I.e. there's a polytime algorithm S s.t. on every instance x of A
     - If x has a solution, S returns a solution.
     - If x has no solution, S returns fail.
   - Verifier
     - V runs S. If S finds a solution, V outputs 1. Otherwise V outputs 0.
   - If x is yes instance
     - S finds a solution, so V outputs 1.
   - If x is no instance
     - S returns fail, so V outputs 0.
   - V runs in polytime.
     - Because V just runs S, which runs in polytime.
6. Primes is in NP (Pratt certificates https://en.wikipedia.org/wiki/Primality_certificate)

Incorrect verifiers

- Let's see some incorrect verifiers.
  - None of these verifiers can be used to prove k-Clique is in NP.
- Verifier 1 Always outputs 1, regardless of y.
  - Wrong, because when graph doesn't contain a k-clique, V is supposed to output 0.
- Verifier 2 Always output 0, regardless of y.
  - Wrong, because when the graph does contain a k-clique, V is supposed to output 1, for some y.
- Verifier 3 Check all subsets of k nodes. If any form a clique, output 1, else output 0.
  - Seems OK. When x has a k-clique, V outputs 1, and when x doesn't, it outputs 0.
  - But V is still wrong, because it doesn't run in polytime.
    - There are $O(n^k)$ subsets of k nodes, and V checks all of them.

P vs NP

$P \subseteq NP$

But we don't know whether $NP \subseteq P$.

That is, we don't know whether $P = NP$

Reduction

- Let A and B be two decision problems.
- Let X and Y be the set of yes instances for A and B, resp.
- Ex Say A = PRIME and B = k-CLIQUE.
  - X is the set of prime numbers.
  - Y is the set of graphs containing a k-clique.
- Let f be a function that maps instances of A to instances of B.
- Def A reduces to B if there exists $f: A \to B$ s.t. for all instances x of A, $x \in X \Leftrightarrow f(x) \in Y$.
  - We write $A \leq_R B$.
- To show $A \leq_R B$, just give the mapping f.
- If $A \leq_R B$, then we can use an algorithm for B to solve A.
  - To solve an instance of A, first map it to an instance of B using f.
  - Then run the B algorithm.
  - Return the same answer for A as the B algorithm gives.
  - By definition, A is true ⇔ f(A) is true.

- If the mapping function from A to B runs in polynomial time, then it's a polynomial time reduction, and we write $A \leq_P B$.

Thm 1. Let A, B and C be three problems, and suppose $A \leq_P B$ and $B \leq_P C$. Then $A \leq_P C$

NP-completeness

Def. A problem A is NP-complete (NPC) if the following are true

- $A \in NP$
- Given any other problem $B \in NP, B \leq_P A$

SAT = satisfiable Boolean formulas

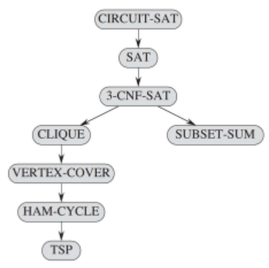
Given a Boolean formula, is there any setting for the variables which makes the formula true?

Steve Cook and Leonid Levin proved around 1970 that **SAT is NP-complete**.

Cook-Levin theorem says 2 things:

1. SAT $\in NP$
2. Every NP problem reduces to SAT. I.e. every problem A in NP can be mapped to an SAT formula in polytime, such that
   - If A is true, then ϕ is satisfiable.
   - If A is false, then ϕ is not satisfiable.

For every problem in the picture, if A points to B, it means $A \leq_P B$.



Thm. Given two NP problems A and B, suppose A is NP-complete, and $A \leq_P B$. Then B is also NP-complete.

To prove a problem B is NP-complete: Take a problem A you know is NPC, and prove $A \leq_P B$.

Thm 2. Suppose a problem is NP-complete, and $A \in P$. Then $P = NP$.

Cor. Suppose a problem A is NP-complete, and $A \notin P$. Then for any NP-complete problem B, $B \notin P$

Beyond NP

co-NP. All problems whose "complement" is in NP

- E.g. GRAPH-ISO∈NP, so GRAPH-NONISO∈co-NP

Reduction

Reducing 3-CNF-SAT to CLIQUE

- Let ϕ be a 3-CNF formula with m clauses.
- Let C be a clause in ϕ. Then C has 3 literals.
  - Make 3 vertices in G corresponding to the literals.
    - So G has 3m vertices total.
    - Let n be the number of nodes in G. Then m = n/3.
- Now, add in an edge between two vertices u, v if both conditions below hold.
  - u, v correspond to literals from different clauses of ϕ.
  - The literals corresponding to u and v are not negations of each other.
    - We say u and v are consistent.

Reducing 3-CNF-SAT to SUBSET-SUM

- For each variable x_i, S contains two numbers v_i and v_i'.
  - v_i and v_i' both have a 1 in x_i's digit, and 0's in all the other variable digits.
  - If x_i appears in clause C_j, then the j'th clause digit in v_i is 1.
  - If $\neg x_i$ appears in clause C_j, then the j'th clause digit in v_i' is 1.
  - All other clause digits in v_i and v_i' are 0.
- For each clause C_j, S contains two numbers s_j and s_j'.
  - s_j has a 1 in the C_j digit, and s_j' has a 2 in this digit.
  - s_j and s_j' are 0's elsewhere.
- Target t is 1 in all the variable digits and 4 in all the clause digits.

		x_1	x_2	x_3	C_1	C_2	C_3	C_4
v_1	=	1	0	0	1	0	0	1
v_1'	=	1	0	0	0	1	1	0
v_2	=	0	1	0	0	0	0	1
v_2'	=	0	1	0	1	1	1	0
v_3	=	0	0	1	0	0	1	1
v_3'	=	0	0	1	1	1	0	0
s_1	=	0	0	0	1	0	0	0
s_1'	=	0	0	0	2	0	0	0
s_2	=	0	0	0	0	1	0	0
s_2'	=	0	0	0	0	2	0	0
s_3	=	0	0	0	0	0	1	0
s_3'	=	0	0	0	0	0	2	0
s_4	=	0	0	0	0	0	0	1
s_4'	=	0	0	0	0	0	0	2
t	=	1	1	1	4	4	4	4

- SUBSET-SUM instance for $\phi = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$.
- Each row except last represents a base-10 number in S. Last row is target t.

Extending Tractability

Suppose I need to solve an NP-complete problem in poly-time:

Must sacrifice one of three desired features.

- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

This lecture. **Solve some special cases of NP-complete problems.**

finding small vertex covers

Claim. Let (u, v) be an edge of G. G has a vertex cover of size ≤ k iff at least one of G − { u } and G − { v } has a vertex cover of size ≤ k − 1.

Claim. If G has a vertex cover of size k, it has ≤ k (n − 1) edges

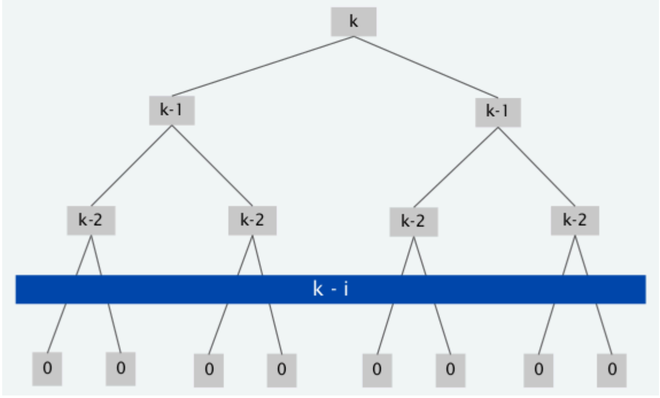
Claim. The following algorithm determines **if G has a vertex cover of** $size \leq k$ **in** $O(2^k kn)$ **time.**

```
Vertex-Cover(G, k) {
    if (G contains no edges)    return true
    if (G contains ≥ kn edges)  return false

    let (u, v) be any edge of G
    a = Vertex-Cover(G - {u}, k-1)
    b = Vertex-Cover(G - {v}, k-1)
    return a or b
}
```

Running time:

$$T(n,k) \le \begin{cases} c & \text{if } k = 0 \\ cn & \text{if } k = 1 \\ 2T(n,k-1) + ckn & \text{if } k > 1 \end{cases} \quad \Rightarrow \quad T(n,k) \le 2^k ckn$$



Solving NP-hard problems on trees

independent set on general graph is NP-problem.

independent set on trees

Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two share an edge.

Key observation. If v is a leaf, there exists a maximum size independent set containing v.

Theorem. The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

```
Independent-Set-In-A-Forest(F) {
    S ← φ
    while (F has at least one edge) {
        Let e = (u, v) be an edge such that v is a leaf
        Add v to S
        Delete from F nodes u and v, and all edges
        incident to them.
    }
    return S
}
```

Weighted independent set on trees

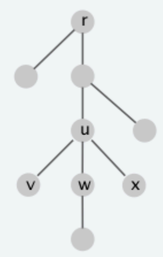
Weighted independent set on trees. Given a tree and node weights $w_v > 0$, find an independent set S that maximizes $\Sigma_{v \in S} w_v$.

Dynamic programming solution. Root tree at some node, say r.

- $OPT_{in}(u)$ = max weight independent set of subtree rooted at u, containing u.
- $OPT_{out}(u)$ = max weight independent set of subtree rooted at u, not containing u.

$$OPT_{in}(u) = w_u + \sum_{v \in \text{children}(u)} OPT_{out}(v)$$

$$OPT_{out}(u) = \sum_{v \in \text{children}(u)} \max \{ OPT_{in}(v), OPT_{out}(v) \}$$



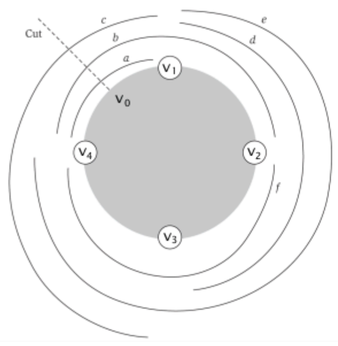
children(u) = { v, w, x }

circular arc coverings

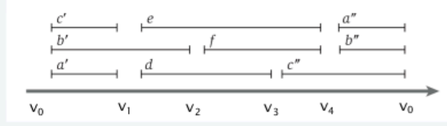
Weak duality: number of colors ≥ depth.

(Almost) transforming circular arc coloring to interval coloring

Equivalent problem. Cut the network between nodes v_1 and v_n . The arcs can be colored with k colors iff the intervals can be colored with k colors in such a way that **"sliced" arcs have the same color**.

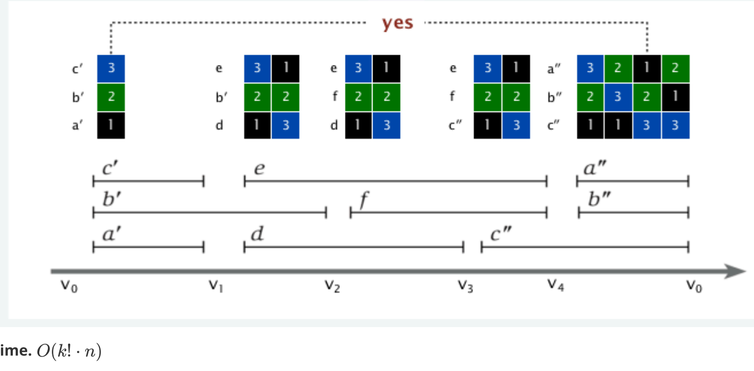


colors of a′, b′, and c′ must correspond to colors of a″, b″, and c″



Dynamic programming algorithm.

- Assign distinct color to each interval which begins at cut node v_0.
- At each node v_i, some intervals may finish, and others may begin.
- Enumerate all k-colorings of the intervals through v_i that are consistent with the colorings of the intervals through v_{i-1}.
- The arcs are k-colorable iff some coloring of intervals ending at cut node v_0 is consistent with original coloring of the same intervals.



Running time. $O(k! \cdot n)$

vertex cover in bipartite graphs

Weak duality. Let M be a matching, and let S be a vertex cover. Then, $|M| \le |S|$.

Theorem. [König-Egerváry] In a **bipartite** graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

$$\min \text{ cut} \overset{\text{strong duality}}{=} \max \text{ flow} \overset{\text{bipartite}}{=} \max \text{ matching} \overset{\text{weak duality and bipartite}}{=} \min \text{ vertex cover}$$

没懂 有点晕晕了

Local Search

Solving NP-hard problems

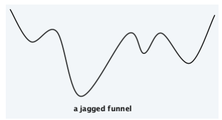
Gradient descent 梯度下降

Vertex cover. Given a graph G = (V, E), find a subset of nodes S of minimal cardinality such that for each (u, v) ∈ E, either u or v (or both) are in S.
Neighbor relation. $S \sim S\prime$ (S is a neighbor of S') if S ' can be obtained from S by adding or deleting a single node. Each vertex cover S has at most n neighbors.

Local optimum 局部最优无法达到全局最优

Gradient descent. Let S denote current solution. If there is a neighbor S ' of S with strictly lower cost, replace S with the neighbor whose cost is as small as possible. Otherwise, terminate the algorithm.

可以做到局部最优，但不一定是全局最优



Metropolis algorithm

大多数时候朝"downhil"方向走，但偶尔 "go uphills" to break out of local minima

Metropolis algorithm.

- Given a fixed temperature T, maintain current state S.
- Randomly perturb current state S to new state $S\prime \in N(S)$.
- If $E(S\prime) \leq E(S)$, update current state to $S\prime$.
  Otherwise, update current state to $S\prime$ with probability $e^{-\Delta E/(kT)}$, where $\Delta E = E(S\prime) – E(S) > 0$.

这个概率函数是基于一个物理原理？***Gibbs-Boltzmann function***

但还是有小概率离开global optimal 反而跑到 local optimal去了

Simulated annealing.

- T large ⇒ probability of accepting an uphill move is large.
- T small ⇒ uphill moves are almost never accepted.
- Idea: turn knob to control T.
- Cooling schedule: T = T(i) at iteration i.

一开始的时候 T 较大，allows to explore the whole graph.

gradually, lower T -> smaller probability of leaving a good solution

---

Hopfield neural networks

Intuition. If $w_{uv} < 0$, then u and v want to have the same state; if $w_{uv} > 0$ then u and v want different states.

Def. With respect to a configuration S, edge e = (u, v) is ***good*** if $w_e \times s_u \times s_v < 0$. That is, if $w_e < 0$ then $s_u = s_v$; if w e > 0, then s u ≠ s v .

Def. With respect to a configuration S, a node u is ***satisfied*** if the weight of incident good edges ≥ weight of incident bad edges.

$$\sum_{e=(u,v)\in E} w_e s_u s_v \leq 0$$

Def. A configuration is ***stable*** if all nodes are satisfied.

Goal. Find a stable configuration, if such a configuration exists.

State-flipping algorithm. Repeated flip state of an unsatisfied node.

```
HOPFIELD-FLIP (G, w)
S ← arbitrary configuration.
WHILE (current configuration is not stable)
  u ← unsatisfied node.
  s_u ← -s_u.
RETURN S.
```

每次 flip 会导致 从$\sum_{e=(u,v)\in E} w_e s_u s_v \leq 0$ 变为 $\sum_{e=(u,v)\in E} w_e s_u s_v \geq 0$ (因为逆转会导致所有原来的bad edge 变成good edge 而good 变成bad)

Theorem. The state-flipping algorithm terminates with a stable configuration after at most $W = \Sigma_e |w_e|$ iterations. – **but not ploy-time**

Maximum cut

Single-flip neighborhood. Given a cut (A, B), move one node from A to B, or one from B to A if it improves the solution.

-> gives a locally optimal solution

Theorem. Let (A, B) be a locally optimal cut and let $(A*, B*)$ be an optimal cut. Then $w(A, B) \geq$ ½ $\Sigma_e w_e \geq$ ½ $w(A*, B*)$.

Local search. Within a factor of 2 for MAX-CUT , but not poly-time!

Big-improvement-flip algorithm. Only choose a node which, when flipped, increases the cut value by at least $\frac{2\varepsilon}{n} w(A, B)$

Claim. Upon termination, big-improvement-flip algorithm returns a cut (A, B) such that $(2 + \epsilon) w(A, B) \geq w(A*, b*)$.

Claim. Big-improvement-flip algorithm terminates after O(ε -1 n log W) flips, where W = Σ e w e .

- 可以减少循环数量，但是使结果离最优更远