



Randomized algorithms 1

Intro, hashing

CS240

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Rui Fan



Outline

- Introduction
- Probability review
- Max-cut and randomized quicksort
- Hashing
 - Closed addressing
 - Universal hashing
 - Perfect hashing

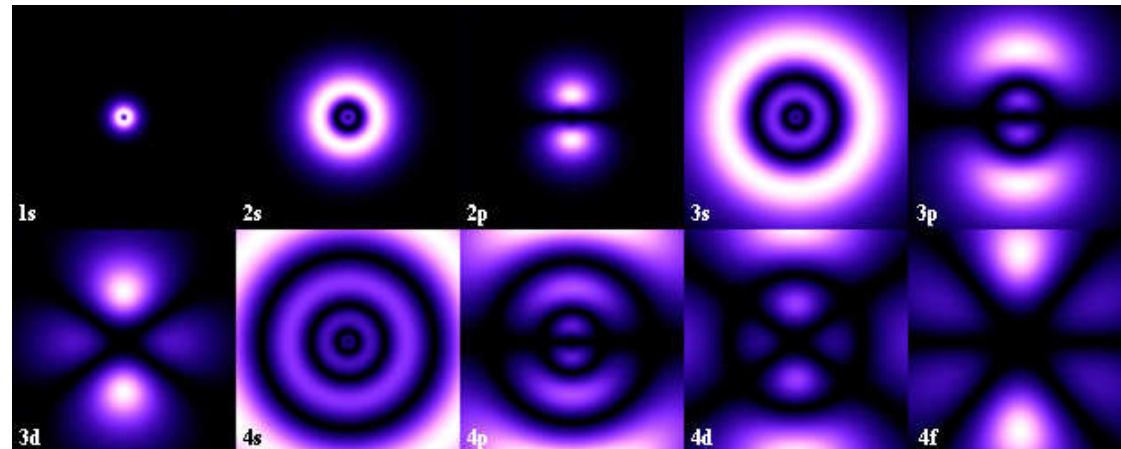
Randomized algorithms

- Till now, all of our algorithms have been deterministic.
 - Given an input, the algorithm always does the same thing.
- It turns out it's very useful to allow algorithms to be nondeterministic.
 - As the algorithm operates, it's allowed to make some random choices.
 - Running the algorithm multiple times on same input can produce different behaviors.



Why randomized algorithms?

- For many problems, randomized algorithms work better than deterministic ones.
 - Faster / uses less memory
 - Simpler, easier to understand.
 - Some problems that provably can't be solved (or solved efficiently) by deterministic algorithms can be solved by randomized ones.
 - According to quantum mechanics, the world is inherently probabilistic, so nature is randomized!



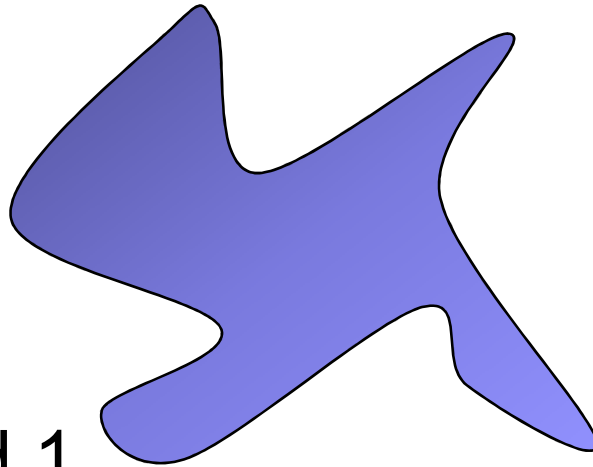


How can randomness help?

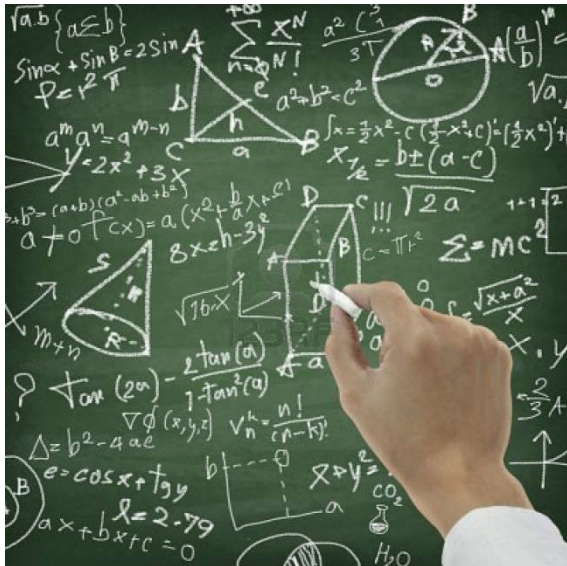
- Say you have a string of length n that's half A's and half B's.
- We want to find a location in the string with an A.
- Any deterministic algorithm takes $n/2+1$ steps in the worst case.
- But by checking random locations, a randomized algorithm finds an A in 2 steps in expectation.

How can randomness help?

- Measure the area of this



- Method 1



- Method 2

- ❖ Print the shape out on a piece of paper.
- ❖ Throw 100 darts at it.
- ❖ See what percent land in the shape.
- ❖ Multiply by area of your paper.



Las Vegas vs Monte Carlo

- A Las Vegas randomized algorithm always produces the right answer. But its running time can vary depending on its random choices.
 - We want to minimize the expected running time of a Las Vegas algorithm.
- A Monte Carlo algorithm always has the same running time. But it sometimes produces the wrong answer, depending on its random choices.
 - We want to minimize the error probability of a Monte Carlo algorithm.





Probability review

- Discrete probability theory is based on events and their probabilities.
 - Events can be composed of more basic events.
 - Ex Event of rolling a 2 on a fair dice, with probability $1/6$.
 - Ex Event of rolling an even number, with probability $1/2$.
Composed of basic events of rolling a 2, 4 or 6.
 - If A is event, write $\Pr[A]=y$. E.g. $\Pr[\text{roll a 2}]=1/6$
- Two events A, B are independent if $\Pr[A \wedge B] = \Pr[A] * \Pr[B]$.
 - Ex Events A="2 on first roll" and B="3 on second roll" are independent, because $\Pr[A \wedge B] = 1/36 = \Pr[A] * \Pr[B] = 1/6 * 1/6$.
 - Ex Events A="2 on first roll" and B="the two rolls sum to 5" are not independent, because $\Pr[A \wedge B] = 1/36 \neq \Pr[A] * \Pr[B] = 1/6 * 4/36$.



Probability review

■ Random variables

- A variable which takes values with certain probabilities. The probabilities sum to 1.
- **Ex** X = value from roll of dice. Values are $\{1,2,3,4,5,6\}$, each with probability $1/6$.
- **Ex** Y = number of heads in 4 flips of fair coin. Values are $\{0,1,2,3,4\}$, with probabilities $\{1/16, 4/16, 6/16, 4/16, 1/16\}$.
- **Ex** Z = number of flips of fair coin till first head. Values are $\{1,2,3,\dots\}$, with probabilities $\{1/2, 1/4, 1/8, \dots\}$.
- We write $\Pr[X=x]=y$, e.g. $\Pr[Z=3]=1/8$.



Probability review

- Expectation of random variable X

- $E[X] = \sum_x x \cdot \Pr[X=x]$.

- The average value of X , over many trials.

- **Ex** X = number of heads in 4 flips.

- $E[X] = 0 \cdot 1/16 + 1 \cdot 4/16 + 2 \cdot 6/16 + 3 \cdot 4/16 + 4 \cdot 1/16 = 2.$

- If you flip a coin 4 times, for 1000 times, on average you see 2 heads per 4 flips.

- An indicator variable X for an event E is a random variable that's 1 if E occurs, and 0 otherwise.

- If event E has probability p of occurring, and X is E 's indicator variable, then $E[X] = p$.

- Because $E[X] = \Pr[E \text{ occurs}] \cdot 1 + \Pr[E \text{ doesn't occur}] \cdot 0 = p$.

- This is a convenient fact we'll frequently use.



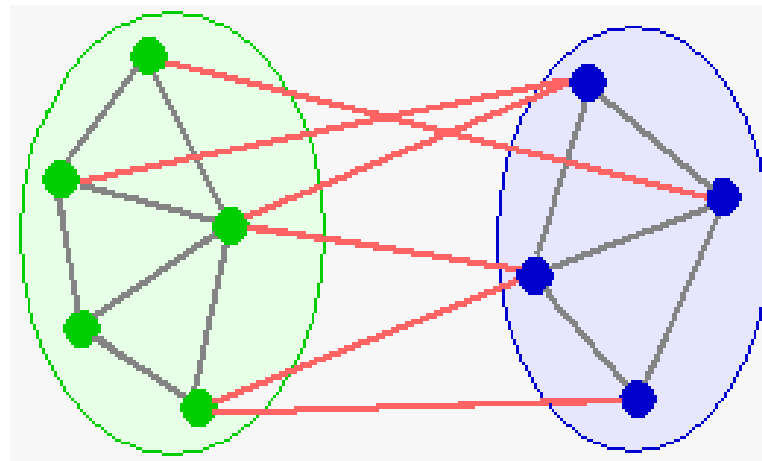
Probability review

■ Linearity of expectations

- Given random variables X, Y , $E[X+Y]=E[X]+E[Y]$.
- Extends to any number of random variables, e.g. $E[X+Y+Z]=E[X]+E[Y]+E[Z]$.
- The random variables do not have to be independent.
- Very useful property!
- **Ex** Let X =number of heads in 100 coin flips. Calculate $E[X]$.
 - **Direct method**: $1*\text{Pr}[1 \text{ head}]+2*\text{Pr}[2 \text{ heads}]+\dots+100*\text{Pr}[100 \text{ heads}]$, a very complicated sum.
 - **Linearity method**: $X=X_1+X_2+\dots+X_{100}$, where X_i =number of heads on i 'th flip.
 - $E[X_i]=0*\text{Pr}[0 \text{ heads}]+1*\text{Pr}[1 \text{ head}]=1/2$.
 - $E[X]=E[X_1]+\dots+E[X_{100}]=100/2=50$.

Problem 1: Max-Cut

- We studied the Min-Cut problem, which is closely related to finding the max flow in a network.
- Max-Cut is the opposite of Min-Cut.
- Given a graph G , split vertices into two sides to maximize the number of edges between the sides.



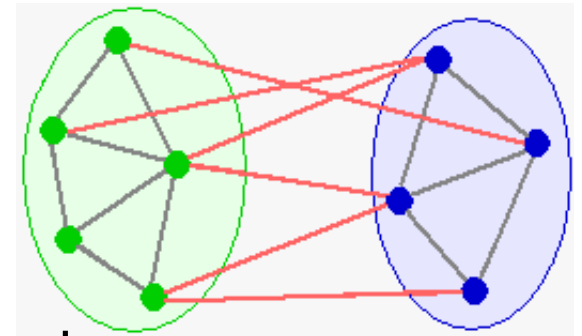


Max-Cut

- Unlike Min-Cut, Max-Cut is NP-complete.
- We'll give a very simple randomized Monte Carlo 2-approximation algorithm.
 - Monte Carlo means the algorithm sometimes returns the wrong answer, i.e. a cut that's not a 2-approximation.
 - Monte Carlo also means the algorithm always runs in a fixed amount of time.
- ❖ Put each node in a random side with probability $\frac{1}{2}$.

Correctness

- **Lemma** In a graph with e edges, the algorithm produces a cut with expected size $e/2$.
- **Proof** Let X be a random variable equal to the size of the cut. We want to bound $E[X]$.
 - For each edge e , let X_e be the indicator variable of whether e is in the cut.
 - I.e. $X_e=1$ if e is in the cut and 0 otherwise.
 - So $X = \sum_e X_e$.
 - Given an edge $e=(i,j)$, e is in the cut if i and j are on different sides.
 - So $\Pr[e \text{ in cut}] = \Pr[(i \text{ in } L) \wedge (j \text{ in } R)] + \Pr[(j \text{ in } L) \wedge (i \text{ in } R)] = 1/4 + 1/4 = 1/2$.
 - So $E[X_e] = 1/2$.
 - So $E[X] = e/2$ by linearity of expectations.



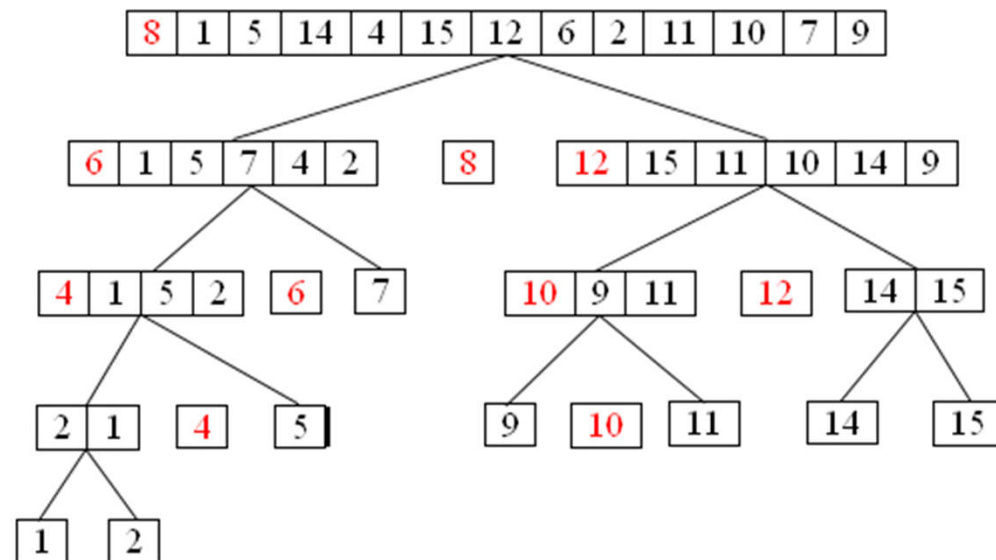


Correctness

- Since a cut can have at most e edges, the $e/2$ edges the algorithm outputs in expectation is a 2 approximation.
- Note that we only bounded expected size of the algorithm's cut.
 - In any particular execution, the algorithm can output a cut that's smaller or larger than $e/2$.
 - On average, the cut has size $e/2$.
 - It's possible to bound the probability the algorithm outputs a cut significantly smaller than $e/2$, but we won't do this.

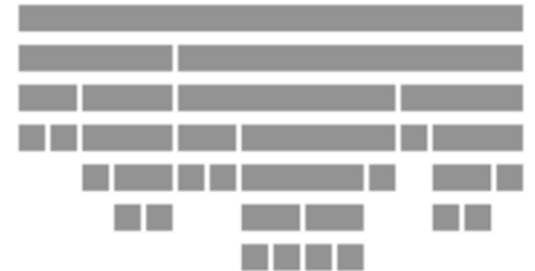
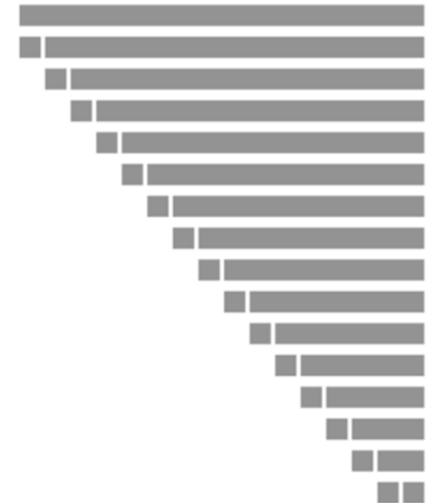
Problem 2: Quicksort

- Recall the Quicksort algorithm.
 - ❖ Pick a pivot element s .
 - ❖ Partition the elements into two sets, those less than s and those more than s .
 - ❖ Recursively Quicksort the two sets.



Complexity of Quicksort

- Let $T(n)$ be the time to Quicksort n numbers.
- $T(n)$ is small in practice.
- But in the worst case, $T(n)=O(n^2)$.
 - Occurs with very uneven splits. I.e. the rank of the pivot is very small or large.
 - **Ex** If pivot is smallest element, then $T(n)=T(1)+T(n-1)+n-1$. This solves to $T(n)=O(n^2)$.
 - $T(1)$ and $T(n-1)$ to recursively sort each side, $n-1$ to partition the elements wrt the pivot.
- As long as the pivot is near the middle, Quicksort takes $O(n \log n)$ time.
 - **Ex** If the pivot is always in the middle half, $[n/4, 3n/4]$, then $T(n) \leq T(n/4)+T(3n/4)+n-1$, which solves to $O(n \log n)$.





Randomized Quicksort

- Quicksort is only slow if we keep picking very small or large pivots.
- Let's pick the pivot at random. Intuitively, we shouldn't be unlucky and always pick small or large pivots.
- ❖ Pick a random pivot element s .
- ❖ Partition the elements into two sets, those less than s and those more than s .
- ❖ Recursively RQuicksort the two sets.



Complexity of RQuicksort

- Let $R(n)$ be the expected time to RQuicksort n numbers.
- With probability $1/n$, the pivot has rank 1 (is smallest element), in which case $R(n)=R(1)+R(n-1)+n-1$.
- With probability $1/n$, the pivot has rank 2, and $R(n)=R(2)+R(n-2)+n-1$.
- ...
- With probability $1/n$, the pivot has rank k , and $R(n)=R(k)+R(n-k)+n-1$.
- Putting these together, we have
$$R(n) = 1/n \cdot (R(1)+R(n-1)+R(2)+R(n-2)+\dots+R(n-1)+R(1) + n \cdot 1/n \cdot (n-1) =$$
$$2/n \cdot \sum_k R(k) + n-1.$$

Complexity of RQuicksort

- We solve the recurrence for $R(n)$ using the substitution method. We guess $R(n) \leq an \log n + b$ for some constants $a, b > 0$ to be determined.
- We first need the following lemma.

Lemma 1.1 $\sum_{k=1}^{n-1} k \log k \leq \frac{1}{2}n^2 \log n - \frac{1}{8}n^2$.

Proof.

$$\begin{aligned} \sum_{k=1}^{n-1} k \log k &= \sum_{k=1}^{\lceil n/2 \rceil - 1} k \log k + \sum_{k=\lceil n/2 \rceil}^{n-1} k \log k \\ &\leq (\log n - 1) \sum_{k=1}^{\lceil n/2 \rceil - 1} k + \log n \sum_{k=\lceil n/2 \rceil}^{n-1} k \\ &= \log n \sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k \\ &\leq \frac{1}{2}n(n-1) \log n - \frac{1}{2}\left(\frac{n}{2} - 1\right)\frac{n}{2} \\ &\leq \frac{1}{2}n^2 \log n - \frac{1}{8}n^2 \end{aligned}$$

Complexity of RQuicksort

- Now we can solve for $R(n)$.

$$\begin{aligned} R(n) &= \frac{2}{n} \sum_{k=1}^{n-1} R(k) + \Theta(n) \\ &\leq \frac{2}{n} \sum_{k=1}^{n-1} (ak \log k + b) + \Theta(n) \\ &= \frac{2a}{n} \sum_{k=1}^{n-1} k \log k + \frac{2b(n-1)}{n} + \Theta(n) \\ &\leq \frac{2a}{n} \left(\frac{1}{2} n^2 \log n - \frac{1}{8} n^2 \right) + \frac{2b}{n} (n-1) + \Theta(n) \\ &\leq an \log n - \frac{a}{4} n + 2b + \Theta(n) \\ &= an \log n + b + (\Theta(n) + b - \frac{a}{4} n) \\ &\leq an \log n + b \end{aligned}$$

by choosing a so that $\frac{a}{4}n > \Theta(n) + b$.

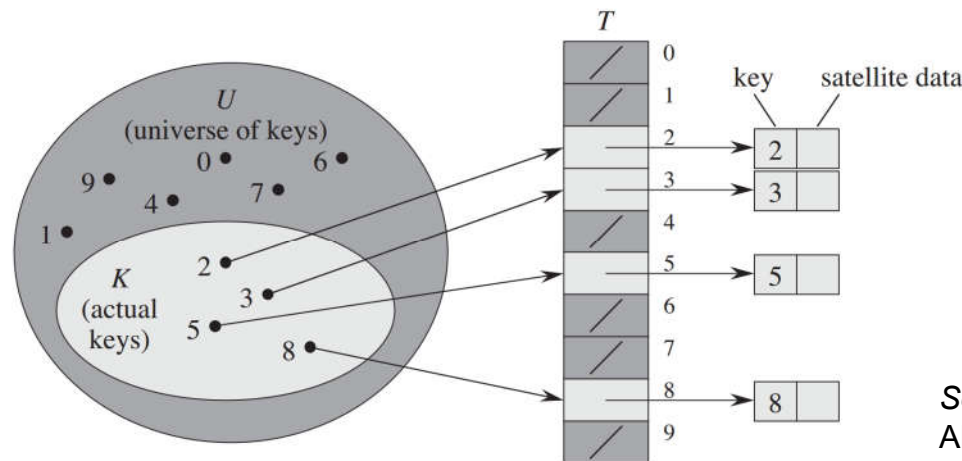


Hash tables

- A hash table is a randomized data structure to efficiently implement a dictionary.
- Supports find, insert, and delete operations all in expected $O(1)$ time.
 - But in the worst case, all operations are $O(n)$.
 - The worst case is provably very unlikely to occur.
- A hash table does not support efficient min / max or predecessor / successor functions.
 - All these take $O(n)$ time on average.
- A practical, efficient alternative to binary search trees if only find, insert and delete needed.

Direct addressing

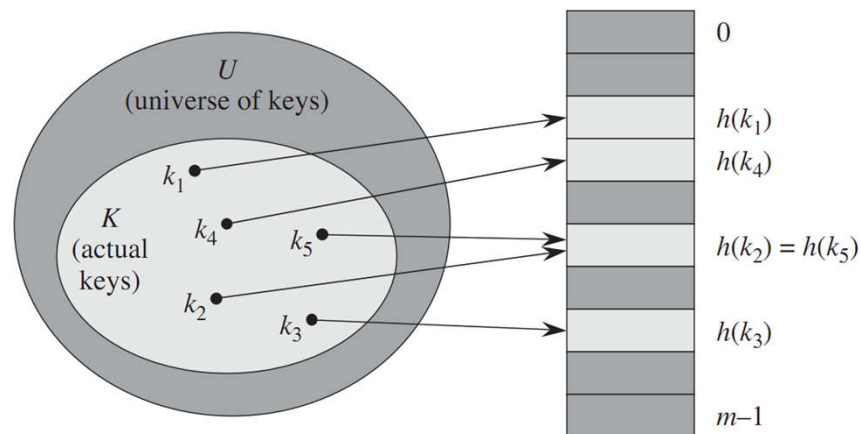
- Suppose we want to store (key, value) pairs, where keys come from a finite universe $U = \{0, 1, \dots, m-1\}$.
- Use an array of size m .
 - `insert(k, v)` Store v in array position k .
 - `find(k)` Return the value in array position k .
 - `delete(k)` Clear the value in array position k .
- All operations take $O(1)$ time.
- The problem is, if m is large, then we need to use a lot of memory.
 - Uses $O(|U|)$ space.
 - **Ex** For 32 bit keys, need 16 GB memory. For 64 bit keys, more memory than in world.
 - **Ex** What about string based keys?
- Also, if only need to store few values, lots of space wasted.



Source: Introduction to Algorithms, Cormen et al

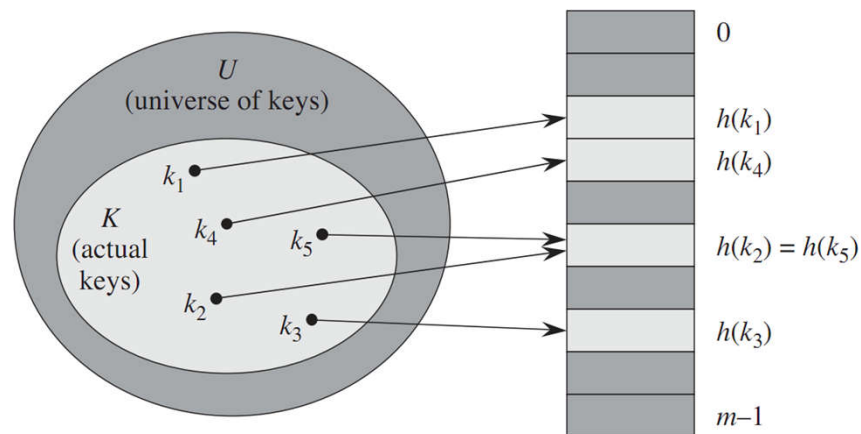
Hash table

- Similar to direct addressing, but uses much less space.
- **Idea** Instead of storing directly at key's location, convert key to much smaller value, and store at this location.
- A hash table consists of the following.
 - A universe U of keys.
 - An array of T of size m .
 - A hashing function $h:U \rightarrow \{0,1,\dots,m-1\}$.
- We'll talk later about how to pick good hash functions.
- **insert(k, v)** Hash key to $h(k)$. Store v in $T[h(k)]$.
- **find(k)** Return the value in $T[h(k)]$
- **delete(k)** Delete the value in $T[h(k)]$
- Assuming $h(k)$ takes $O(1)$ time to compute, all ops still take $O(1)$ time. Uses $O(m)$ space.
- If $m \ll |U|$, then hashing uses much less space than direct addressing.
- However, our current scheme doesn't quite work, due to collisions.



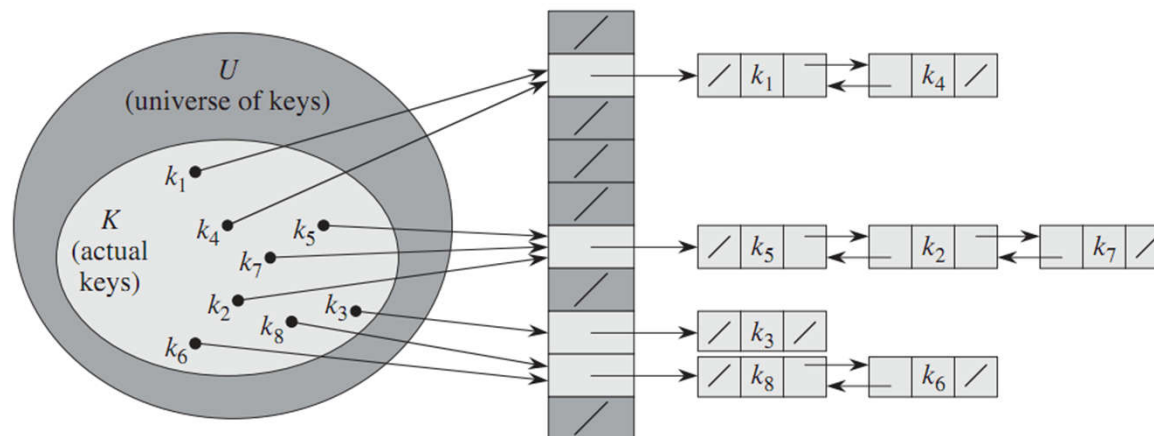
Collisions

- We store a key at array position $h(k)$.
- But what if two keys hash to the same location, i.e. $k_1 \neq k_2$, but $h(k_1) = h(k_2)$?
 - This is called a collision.
- Collisions are unavoidable when $|U| > m$.
 - By Pigeonhole Principle, must exist at least two different keys in U that hash to same value.
- Two basic ways to deal with collisions, closed and open addressing.



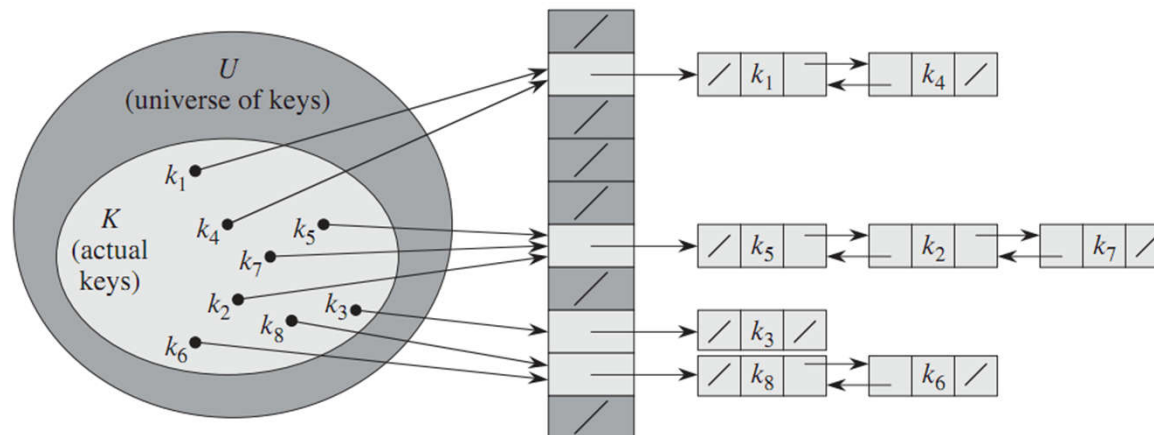
Closed addressing

- In closed addressing, every entry in hash table points to a linked list.
 - Keys that hash to the same location get added to the linked list.
 - For simplicity, we'll ignore values from now on and only focus on keys.
- **insert(k)** Add k to the linked list in $T[h(k)]$.
- **find(k)** Search the linked list in $T[h(k)]$ for k .
- **delete(k)** Delete k from the linked list in $T[h(k)]$.
- Suppose the longest list has length \hat{n} , and average length list is \bar{n} .
 - Each operation takes worst case $O(\hat{n})$ time.
 - An operation on a random key takes $O(\bar{n})$ time.



Load factor

- The key to making closed addressing hashing fast is to make sure list lengths aren't too long.
- For this, we want the hash function to appear random.
 - Assume that any key is uniformly likely to be hashed to any table location.
- Suppose the hash table contains n items, and has size m .
- Then under the uniform hashing assumption, each table location has on average n/m keys.
 - Call $\alpha = n/m$ the load factor.
- So the average time for each operation is $O(\alpha)$.
- However, even with uniform hashing, in the worst case, all keys can hash to the same location. So the worst case performance is $O(n)$.





Picking a hash function

- We saw that we want hash functions to hash keys to “random” locations.
 - However, note that each hash function is itself a deterministic function, i.e. $h(k)$ always has the same value.
 - If $h(k)$ can produce different values, we can’t find key k in the hash table anymore.
- It’s hard to find such random hash functions, since we don’t assume anything about the distribution of input keys.
 - **Ex** For any hash function, there are always $\geq |U|/m$ keys from the universe hashing to the same location. So if the input is exactly this set, and $|U|/m \geq n$, then all ops take $O(n)$ time.
- In practice, we use a number of heuristic functions.



Heuristic hash functions

- Assume the keys are natural numbers.
 - Convert other data types to numbers.
 - **Ex** To convert ASCII string to natural number, treat the string as a radix 128 number. E.g. “pt”
 $\rightarrow (112 \cdot 128) + 116 = 14452$.
- **Division method** $h(k) = k \bmod m$
 - Often choose m a prime number not too close to a power of 2.
- **Multiplication method** $h(k) = \lfloor m (k A \bmod 1) \rfloor$, where A is some constant.
 - Knuth's suggestion is $A = \frac{\sqrt{5}-1}{2} \approx 0.618034 \dots$



Universal hashing

- As we said, regardless of the hash function, an adversary can choose a set of n inputs to make all operations $O(n)$ time.
- Universal hashing overcomes this using randomization.
 - No matter what the n input keys are, every operation takes $O(n/m)$ time in expectation, for a size m hash table.
 - Note $O(n/m)$ time is optimal.
- Instead of using a fixed hash function, universal hashing uses a random hash function, chosen from some set of functions H .
- Say H is a universal hash family if for any keys $x \neq y$

$$\Pr_{h \in H} [h(x) = h(y)] = 1/m$$

- So if we randomly choose a hash function from H and use it to hash any keys x, y , they have $1/m$ probability of colliding.
- Note the hash functions in H are not random. However, we choose which function to use from H randomly.

Universal hashing

- **Thm** Let H be a universal hash family. Let S be a set of n keys, and let $x \in S$. If $h \in H$ is chosen at random, then the expected number of $y \in S$ s.t. $h(x) = h(y)$ is n/m .
- **Proof** Say $S = \{x_1, \dots, x_n\}$.
 - Let X be a random variable equal to the number of $y \in S$ s.t. $h(x) = h(y)$.
 - Let $X_i = 1$ if $h(x_i) = h(x)$ and 0 otherwise.
 - $E[X_i] = \Pr_{h \in H} [h(x_i) = h(x)] \times 1 + \Pr_{h \in H} [h(x_i) \neq h(x)] \times 0 = 1/m$.
 - First equality follows by universal hashing property.
 - $E[X] = E[X_1] + \dots + E[X_n] = n/m$.

Constructing universal hash family 1

- Choose a prime number p such that $p > m$, and $p >$ all keys.
- Let $h_{ab}(k) = ((ak + b) \bmod p) \bmod m$.
- Let $H_{pm} = \{h_{ab} \mid a \in \{1, 2, \dots, p-1\}, b \in \{0, 1, \dots, p-1\}\}$.
- **Thm** H_{pm} is a universal hash family.
- **Proof** Let $x, y < p$ be two different keys. For a given h_{ab} let
$$r = (ax + b) \bmod p, \quad s = (ay + b) \bmod p$$
- We have $r \neq s$, because $r - s \equiv a(x - y) \bmod p \neq 0$, since neither a nor $x - y$ divide p .
- Also, each pair (a, b) leads to a different pair (r, s) , since
$$a = ((r - s)(x - y)^{-1} \bmod p), \quad b = (r - ax) \bmod p$$
 - Here, $(x - y)^{-1} \bmod p$ is the unique multiplicative inverse of $x - y$ in \mathbb{Z}_p^* .



Constructing universal hash family 2

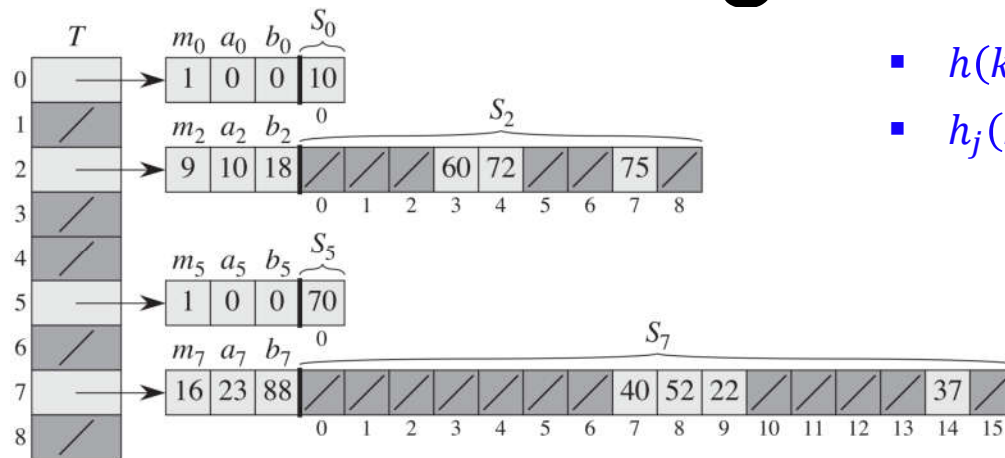
- Since there are $p(p - 1)$ pairs (a, b) and $p(p - 1)$ pairs (r, s) with $r \neq s$, then a random (a, b) produces a random (r, s) .
- The probability x and y collide equals the probability $r \equiv s \pmod{m}$.
- For fixed r , number of $s \neq r$ s.t. $r \equiv s \pmod{m}$ is $(p - 1)/m$.
- So for each r and random $s \neq r$, probability that $r \equiv s \pmod{m}$ is $((p - 1)/m)/(p - 1) = 1/m$.
- So $\Pr_{h_{ab} \in H_{pm}} [h_{ab}(x) = h_{ab}(y)] = 1/m$ and H_{pm} is universal.



Perfect hashing

- The hashing methods we've seen can ensure $O(1)$ expected performance, but are $O(n)$ in the worst case due to collisions.
- However, if we have a fixed set of keys, perfect hashing can ensure no collisions at all.
 - Perfect hashing maintains a static set, and allows $\text{find}(k)$ and $\text{delete}(k)$ in $O(1)$ time.
 - It doesn't support $\text{insert}(k)$.
- **Ex** The fixed set of keys may represent the file names on a non-writable DVD.

Perfect hashing



- $h(k) = ((3k + 42) \bmod 101) \bmod 9$
- $h_j(k) = ((a_j k + b_j) \bmod 101) \bmod m_j$

- Suppose we want to store n items with no collisions.
- Perfect hashing uses two levels of universal hashing.
 - The first layer hash table has size $m = n$.
 - Use first layer hash function h to hash key to a location in T .
 - Each location j in T points to a hash table S_j with hash function h_j .
 - If n_j keys hash to location j , the size of S_j is $m_j = n_j^2$.
- We'll ensure there are no collisions in the secondary hash tables S_1, \dots, S_m .
 - So all operations take worst case $O(1)$ time.
- Overall the space use is $O(m + \sum_{j=1}^m n_j^2)$.
 - We'll show this is $O(n) = O(m)$.
 - So perfect hashing uses same amount of space as normal hashing.

Avoiding collisions

- **Lemma** Suppose we store n keys in a hash table of size $m = n^2$ using universal hashing. Then with probability $\geq 1/2$ there are no collision.
- **Proof** There are $\binom{n}{2}$ pairs of keys that can collide.
 - Each collision occurs with probability $1/m = 1/n^2$, by universal hashing.
 - So the expected number of collisions is $\frac{\binom{n}{2}}{n^2} \leq \frac{1}{2}$.
 - By Markov's inequality the $\Pr[\# \text{ collisions} \geq 1] \leq E[\# \text{ collisions}] \leq 1/2$.
- When building each hash table S_j , there's $< 1/2$ probability of having any collisions.
 - If collisions occur, pick another random hash function from the universal family and try again.
 - In expectation, we try twice before finding a hash function causing no collisions.

Space complexity

- **Lemma** Suppose we store n keys in a hash table of size $m=n$. Then the secondary hash tables use space $E\left[\sum_{j=0}^{m-1} n_j^2\right] \leq 2n$, where n_j is the number of keys hashing to location j .
- **Proof** $E\left[\sum_{j=0}^{m-1} n_j^2\right] = E\left[\sum_{j=0}^{m-1} (n_j + 2 \binom{n_j}{2})\right] = E\left[\sum_{j=0}^{m-1} n_j\right] + 2 E\left[\sum_{j=0}^{m-1} \binom{n_j}{2}\right]$
- $\sum_{j=0}^{m-1} \binom{n_j}{2}$ is the total number of pairs of hash keys which collide in the first level hash table.
 - By universal hashing, this equals $\binom{n}{2} \frac{1}{m} = \frac{n-1}{2}$.
- $E\left[\sum_{j=0}^{m-1} n_j\right] = n$.
- So $E\left[\sum_{j=0}^{m-1} n_j^2\right] \leq n + \frac{2(n-1)}{2} < 2n$.