Pure functional epidemics

An Agent-Based Approach

Jonathan Thaler
Thorsten Altenkirch
Peer-Olaf Siebers
jonathan.thaler@nottingham.ac.uk
thorsten.altenkirch@nottingham.ac.uk
peer-olaf.siebers@nottingham.ac.uk
University of Nottingham
Nottingham, United Kingdom

Abstract

Agent-Based Simulation (ABS) is a methodology in which a system is simulated in a bottom-up approach by modelling the micro interactions of its constituting parts, called agents, out of which the global macro system behaviour emerges.

So far mainly object-oriented techniques and languages have been used in ABS. Using the SIR model of epidemiology, which allows to simulate the spreading of an infectious disease through a population, we show how to use Functional Reactive Programming to implement ABS.

With our approach we can guarantee the reproducibility of the simulation already at compile time, which is not possible with traditional object-oriented languages. Also, we claim that this representation is conceptually cleaner and opens the way to formally reason about ABS.

Keywords Functional Reactive Programming, Monadic Stream Functions, Agent-Based Simulation

ACM Reference Format:
Jonathan Thaler, Thorsten Altenkirch, and Peer-Olaf Siebers. 2018. Pure functional epidemics: An Agent-Based Approach. In *Proceedings of Haskell Symposium (HS18).* ACM, New York, NY, USA, 12 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 Introduction

The traditional approach to Agent-Based Simulation (ABS) has so far always been object-oriented techniques due to the influence of the seminal work of Epstein et al [7] in which the authors claim "[..] object-oriented programming to be a particularly natural development environment for Sugarscape specifically and artificial societies generally [..]" (p. 179). This work established the metaphor in the ABS community, that *agents map naturally to objects* [17] which still holds up today.

In this paper we fundamentally challenge this metaphor and explore ways of approaching ABS in a pure functional way using Haskell. By doing this we expect to leverage the benefits of pure functional programming [9]: higher expressivity through declarative code, being polymorph and explicit about side-effects through monads, more robust and less susceptible for bugs due to explicit data flow and lack of implicit side-effects.

As use-case we introduce the simple SIR model of epidemiology with which one can simulate epidemics, that is the spreading of an infectious disease through a population, in a realistic way.

Over the course of four steps we derive all necessary concepts required for a full agent-based implementation. We start from a very simple solution running in the Random Monad which has all general concepts already there and then refine it in various ways, making the transition to Functional Reactive Programming (FRP) [31] and to Monadic Stream Functions (MSF) [21].

The aim of this paper is to show how ABS can be done in *pure* Haskell and what the benefits and drawbacks are. By doing this we give the reader a good understanding of what ABS is, what the challenges are when implementing it and how we solve these in our approach.

The contributions of this paper are:

- To the best of our knowledge, we are the first to *systematically* introduce the concepts of ABS to the *pure* functional programming paradigm in a step-by-step approach. It is also the first paper to show how to apply arrowized FRP to ABS on a technical level, presenting a new field of application to FRP.
- Our approach shows how robustness can be achieved through purity which guarantees reproducibility at compile time, something not possible with traditional object-oriented approaches.
- The result of using arrowized FRP is a unique, hybrid approach to ABS because FRP allows expressing continuous time-semantics not possible with the traditional imperative object-oriented approaches.

Section 2 discusses related work. In section 3 we introduce functional reactive programming, arrowized programming and monadic stream functions because our approach builds

HS18, 2018, 09
2018. ACM ISBN 978-x-xxxx-xxxx-x/YY/MM...$15.00
https://doi.org/10.1145/nnnnnnn.nnnnnnn

heavily on these concepts. Section 4 defines agent-based simulation. In section 5 we introduce the SIR model of epidemiology as an example model to explain the concepts of ABS. The heart of the paper is section 6 in which we derive the concepts of a pure functional approach to ABS in four steps, using the SIR model. Finally we draw conclusions and discuss issues in section 7 and point to further research 8.

2 Related Work

The amount of research on using pure functional programming with Haskell in the field of ABS has been moderate so far. Most of the papers are more related to the field of Multi Agent Systems (MAS) and look into how agents can be specified using the belief-desire-intention paradigm [6], [27], [12].

A library for Discrete Event Simulation (DES) and System Dynamics (SD) in Haskell called *Aivika 3* is described in the technical report [26]. It is not pure as it uses the IO Monad under the hood and comes only with very basic features for event-driven ABS which allows to specify simple state-based agents with timed transitions.

The authors of [12] discuss using functional programming for DES and explicitly mention the paradigm of FRP to be very suitable to DES.

The authors of [30] present a domain-specific language for developing functional reactive agent-based simulations. This language called FRABJOUS is human readable and easily understandable by domain-experts. It is not directly implemented in FRP/Haskell but is compiled to Yampa code which they claim is also readable. This supports that FRP is a suitable approach to implement ABS in Haskell. Unfortunately the authors do not discuss their mapping of ABS to FRP on a technical level, which would be of most interest to functional programmers.

3 Background

3.1 Functional Reactive Programming

Functional Reactive Programming (FRP) is a way to implement systems with continuous and discrete time-semantics in pure functional languages. There are many different approaches and implementations but in our approach we use *arrowized* [10], [11] FRP as implemented in the library Yampa [8], [5], [16].

The central concept in arrowized FRP is the Signal Function (SF) which can be understood as a *process over time* which maps an input- to an output-signal. A signal can be understood as a value which varies over time. Thus, signal functions have an awareness of the passing of time by having access to Δt which are positive time-steps with which the system is sampled.

$$Signal\ \alpha \approx Time \rightarrow \alpha$$
$$SF\ \alpha\ \beta \approx Signal\ \alpha \rightarrow Signal\ \beta$$

Yampa provides a number of combinators for expressing time-semantics, events and state-changes of the system. They allow to change system behaviour in case of events, run signal functions and generate stochastic events and random-number streams. We shortly discuss the relevant combinators and concepts we use throughout the paper. For a more in-depth discussion we refer to [8], [5], [16].

Event An event in FRP is an occurrence at a specific point in time which has no duration e.g. the the recovery of an infected agent. Yampa represents events through the *Event* type which is programmatically equivalent to the *Maybe* type.

Dynamic behaviour To change the behaviour of a signal function at an occurrence of an event during run-time, the combinator *switch :: SF a (b, Event c) -> (c -> SF a b) -> SF a b* is provided. It takes a signal function which is run until it generates an event. When this event occurs, the function in the second argument is evaluated which receives the data of the event and has to return the new signal function which will then replace the previous one.

Sometimes one needs to run a collection of signal functions in parallel and collect all of their outputs in a list. Yampa provides the combinator *dpSwitch* for it. It is quite involved and has the following type-signature:

```haskell
dpSwitch :: Functor col
         -- routing function
         => (forall sf. a -> col sf -> col (b, sf))
         -- SF collection
         -> col (SF b c)
         -- SF generating switching event
         -> SF (a, col c) (Event d)
         -- continuation to invoke upon event
         -> (col (SF b c) -> d -> SF a (col c))
         -> SF a (col c)
```

Its first argument is the pairing-function which pairs up the input to the signal functions - it has to preserve the structure of the signal function collection. The second argument is the collection of signal functions to run. The third argument is a signal function generating the switching event. The last argument is a function which generates the continuation after the switching event has occurred. *dpSwitch* returns a new signal function which runs all the signal functions in parallel (thus the p) and switches into the continuation when the switching event occurs. The d in *dpSwitch* stands for delayed which guarantees that it delays the switching until the next time-step: the function into which we switch is only applied in the next step which prevents an infinite loop if we switch into a recursive continuation.

Randomness In ABS one often needs to generate stochastic events which occur based on an exponential distribution. Yampa provides the combinator *occasionally :: RandomGen g => g -> Time -> b -> SF a (Event b)* for this. It takes a random-number generator, a rate and a value the stochastic event will carry. It generates events on average with the given rate. Note that at most one event will be generated and no 'backlog' is kept. This means that when this function is not sampled with a sufficiently high frequency, depending on the rate, it will loose events.

Yampa also provides the combinator *noise :: (RandomGen g, Random b) => g -> SF a b* which generates a stream of noise by returning a random number in the default range for the type *b*.

Running signal functions To *purely* run a signal function Yampa provides the function *embed :: SF a b -> (a, [(DTime, Maybe a)]) -> [b]* which allows to run a SF for a given number of steps where in each step one provides the Δt and an input *a*. The function then returns the output of the signal function for each step. Note that the input is optional, indicated by *Maybe*. In the first step at $t = 0$, the initial a is applied and whenever the input is *Nothing* in subsequent steps, the last a which was not *Nothing* is re-used.

3.2 Arrowized programming

Yampas signal functions are arrows, requiring us to program with arrows. Arrows are a generalisation of monads which, in addition to the already familiar parameterisation over the output type, allow parameterisation over their input type as well [10], [11].

In general, arrows can be understood to be computations that represent processes, which have an input of a specific type, process it and output a new type. This is the reason why Yampa is using arrows to represent their signal functions: the concept of processes, which signal functions are, maps naturally to arrows. There exists a number of arrow combinators which allow arrowized programing in a point-free style but due to lack of space we will not discuss them here. Instead we make use of Patersons do-notation for arrows [18] which makes the code much more readable as it allows us to program with points instead of using only the point-free arrow combinators.

To show how arrowized programming works we implement a simple signal function which calculates the acceleration of a falling mass on its vertical axis as an example [22].

```haskell
fallingMass :: Double -> Double -> SF () Double
fallingMass p0 v0 = proc _ -> do
  v <- arr (+v0) <<< integral -< (-9.8)
  p <- arr (+p0) <<< integral -< v
  returnA -< p
```

To create an arrow, we use the *proc* keyword, which binds a variable after which then the *do* of Patersons do-notation [18] follows. Using the signal function *integral :: SF v v* of Yampa which integrates the input value over time using the rectangle rule, we calculate the current velocity and the position based on the initial position *p0* and velocity *v0*. The <<< is one of the arrow combinators which composes two arrow computations and *arr* simply lifts a pure function into an arrow. To pass an input to an arrow, -< is used and <- to bind the result of an arrow computation to a variable. Finally to return a value from an arrow, the function *returnA* is used which is itself an arrow - the identity function lifted.

3.3 Monadic Stream Functions

Monadic Stream Functions (MSF) are a generalisation of Yampas signal functions with additional combinators to control and stack side effects. An MSF is a polymorphic type and an evaluation function which applies an MSF to an input and returns an output and a continuation, both in a monadic context [21], [20]:

```haskell
newtype MSF m a b =
  MSF { unMSF :: MSF m a b -> a -> m (b, MSF m a b) }
```

MSFs are also arrows which means we can apply arrowized programming with Patersons do-notation as well. The authors [21] implement the library Dunai, which is available on Hackage. It allows us to run monadic computations within a signal function by providing the combinators *arrM :: Monad m => (a -> m b) -> MSF m a b* and *arrM_ :: Monad m => m b -> MSF m a b*.

4 Defining Agent-Based Simulation

Agent-Based Simulation (ABS) is a methodology to model and simulate a system where the global behaviour may be unknown but the behaviour and interactions of the parts making up the system is of knowledge. Those parts, called agents, are modelled and simulated out of which then the aggregate global behaviour of the whole system emerges.

So the central aspect of ABS is the concept of an agent which can be understood as a metaphor for a pro-active unit, situated in an environment, able to spawn new agents and interacting with other agents in some neighbourhood by exchange of messages.

We informally assume the following about our agents [25], [32], [15]:

- They are uniquely addressable entities with some internal state over which they have full, exclusive control.
- They are pro-active which means they can initiate actions on their own e.g. change their internal state, send messages, create new agents, terminate themselves.
- They are situated in an environment and can interact with it.
- They can interact with other agents which are situated in the same environment by means of messaging.

Figure 1. States and transitions in the SIR compartment model.

5 The SIR Model

To explain the concepts of ABS and of our pure functional approach to it, we introduce the SIR model as a motivating example and use-case for our implementation. It is a very well studied and understood compartment model from epidemiology [13] which allows to simulate the dynamics of an infectious disease like influenza, tuberculosis, chicken pox, rubella and measles spreading through a population.

In this model, people in a population of size N can be in either one of three states *Susceptible*, *Infected* or *Recovered* at a particular time, where it is assumed that initially there is at least one infected person in the population. People interact *on average* with a given rate of β other people per time-unit and become infected with a given probability γ when interacting with an infected person. When infected, a person recovers *on average* after δ time-units and is then immune to further infections. An interaction between infected persons does not lead to re-infection, thus these interactions are ignored in this model. This definition gives rise to three compartments with the transitions as seen in Figure 1.

Before looking into how one can simulate this model in an agent-based approach we first explain how to formalize it using System Dynamics (SD) [23]. In SD one models a system through differential equations, allowing to conveniently express continuous systems which change over time. The advantage of a SD solution is that one has an analytically tractable solution against which e.g. agent-based solutions can be validated. The problem is that the more complex a system, the more difficult it is to derive differential equations describing the global system, to a point where it simply becomes impossible. This is the strength of an agent-based approach over SD, which allows to model a system when only the constituting parts and their interactions are known but not the macro behaviour of the whole system. As will be shown later, the agent-based approach exhibits further benefits over SD.

The dynamics of the SIR model can be formalized in SD with the following equations:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -infectionRate \tag{1}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = infectionRate - recoveryRate \tag{2}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = recoveryRate \tag{3}$$

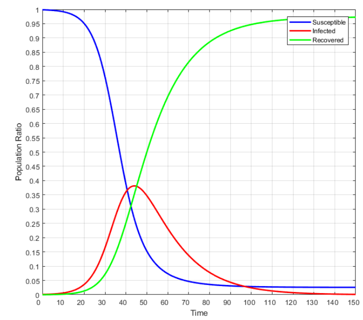


Figure 2. Dynamics of the SIR compartment model using the System Dynamics approach. Population Size $N = 1{,}000$, contact rate $\beta = \frac{1}{5}$, infection probability $\gamma = 0.05$, illness duration $\delta = 15$ with initially 1 infected agent. Simulation run for 150 time-steps.

$$infectionRate = \frac{I \beta S \gamma}{N} \tag{4}$$

$$recoveryRate = \frac{I}{\delta} \tag{5}$$

Solving these equations is done by numerically integrating over time which results in the dynamics as shown in Figure 2 with the given variables.

An Agent-Based approach

The SD approach is inherently top-down because the behaviour of the system is formalized in differential equations. This requires that the macro behaviour of the system is known a priori which may not always be the case. In the case of the SIR model we have already a top-down description of the system in the form of the differential equations from SD. We want now to derive an agent-based approach which exhibits the same dynamics as shown in 2.

The question is whether such top-down dynamics can be achieved using ABS as well and whether there are fundamental drawbacks or benefits when doing so. Such questions were asked before and modelling the SIR model using an agent-based approach is indeed possible [14].

The fundamental difference is that SD is operating on averages, treating the population completely continuous which results in non-discrete values of stocks e.g. 3.1415 infected persons. The approach of mapping the SIR model to an ABS is to discretize the population and model each person in the population as an individual agent. The transitions between the states are no longer happening according to continuous differential equations but due to discrete events caused both by interactions amongst the agents and time-outs. Besides the already mentioned differences, the true advantage of ABS becomes now apparent: with it we can incorporate spatiality as shown in section 6.4 and simulate heterogenity of population e.g. different sex, age,... Note that the latter is theoretically possible in SD as well but with increasing number of population properties, it quickly becomes intractable.

According to the model, every agent makes *on average* contact with β random other agents per time unit. In ABS we can only contact discrete agents thus we model this by generating a random event on average every $\frac{1}{\beta}$ time units. We need to sample from an exponential distribution because the rate is proportional to the size of the population [3]. Note that an agent does not know the other agents' state when making contact with it, thus we need a mechanism in which agents reveal their state in which they are in *at the moment of making contact*. This mechanism is an implementation detail which we will derive in our implementation steps. For now we only assume that agents can make contact with each other somehow.

This results in the following agent behaviour:

- *Susceptible*: A susceptible agent makes contact *on average* with β other random agents. For every *infected* agent it gets into contact with, it becomes infected with a probability of γ. If an infection happens, it makes the transition to the *Infected* state.
- *Infected*: An infected agent recovers *on average* after δ time units. This is implemented by drawing the duration from an exponential distribution [3] with $\lambda = \frac{1}{\delta}$ and making the transition to the *Recovered* state after this duration.
- *Recovered*: These agents do nothing because this state is a terminating state from which there is no escape: recovered agents stay immune and can not get infected again in this model.

6 Deriving a pure functional approach

We presented a high-level agent-based approach to the SIR model in the previous section, which focused only on the states and the transitions, but we haven't talked about technical implementation details on how to actually implement such a state-machine.

The authors of [28] discuss two fundamental problems of implementing an agent-based simulation from a programming language agnostic point of view. The first problem is how agents can be pro-active and the second how interactions and communication between agents can happen. For agents to be pro-active they must be able to perceive the passing of time, which means there must be a concept of an agent-process which executes over time. Interactions between agents can be reduced to the problem of how an agent can expose information about its internal state which can be perceived by other agents.

In this section we will derive a pure functional approach for an agent-based simulation of the SIR model in which we will pose solutions to the previously mentioned problems. We will start out with a very naive approach and show its limitations which we overcome by adding FRP. Then in further steps we will add more concepts and generalisations, ending up at the final approach which utilises monadic stream functions (MSF), a generalisation of FRP [1].

6.1 Naive beginnings

We start by modelling the states of the agents:

```
data SIRState = Susceptible | Infected | Recovered
```

Agents are ill for some duration, meaning we need to keep track when a potentially infected agent recovers. Also a simulation is stepped in discrete or continuous time-steps thus we introduce a notion of *time* and Δt by defining:

```
type Time      = Double
type TimeDelta = Double
```

Now we can represent every agent simply as a tuple of its SIR state and its potential recovery time. We hold all our agents in a list:

```
type SIRAgent = (SIRState, Time)
type Agents   = [SIRAgent]
```

Next we need to think about how to actually step our simulation. For this we define a function which advances our simulation with a fixed Δt until a given time t where in each step the agents are processed and the output is fed back into the next step. This is the source of pro-activity as agents are executed in every time step and can thus initiate actions based on the passing of time. As already mentioned, the agent-based implementation of the SIR model is inherently stochastic which means we need access to a random-number generator. We decided to use the Random Monad at this point as threading a generator through the simulation and the agents could become very cumbersome. Thus our simulation stepping runs in the Random Monad:

```
runSimulation :: RandomGen g
  => Time -> TimeDelta -> Agents -> Rand g [Agents]
runSimulation tEnd dt as = runSimulationAux 0 as []
  where
    runSimulationAux :: RandomGen g
      => Time -> Agents -> [Agents] -> Rand g [Agents]
    runSimulationAux t as acc
      | t >= tEnd = return (reverse (as : acc))
      | otherwise = do
        as' <- stepSimulation dt as
        runSimulationAux (t + dt) as' (as : acc)

stepSimulation :: RandomGen g
  => TimeDelta -> Agents -> Rand g Agents
stepSimulation dt as = mapM (runAgent dt as) as
```

Now we can implement the behaviour of an individual agent. First we need to distinguish between the agents SIR states:

[1]The code of all steps can be accessed freely through the following URL: https://github.com/thalerjonathan/phd/tree/master/public/purefunctionalepidemics/code

```haskell
runAgent :: RandomGen g
  => TimeDelta -> Agents -> SIRAgent -> Rand g SIRAgent
runAgent _  as   (Susceptible, _) = susceptibleAgent as
runAgent dt _  a@(Infected   , _) = return (infectedAgent dt a)
runAgent _  _  a@(Recovered  , _) = return a
```

An agent gets fed the states of all agents in the system from the previous time-step so it can draw random contacts - this is one, very naive way of implementing the interactions between agents.

From our implementation it becomes apparent that only the behaviour of a susceptible agent involves randomness and that a recovered agent is simply a sink - it does nothing and stays constant.

Lets look how we can implement the behaviour of a susceptible agent. It simply makes contact on average with a number of other agents and gets infected with a given probability if an agent it has contact with is infected. When the agent gets infected it calculates also its time of recovery by drawing a random number from the exponential distribution meaning it is ill on average for *illnessDuration*.

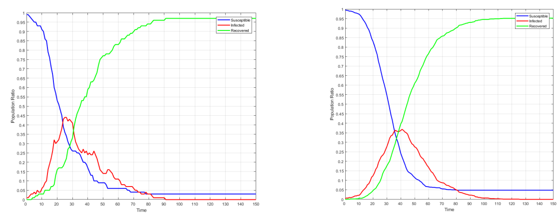
```haskell
susceptibleAgent :: RandomGen g => Agents -> Rand g SIRAgent
susceptibleAgent as = do
    -- draws from exponential distribution
    rc <- randomExpM (1 / contactRate)
    cs <- forM [1..floor rc] (const (makeContact as))
    if elem True cs
      then infect
      else return (Susceptible, 0)
  where
    makeContact :: RandomGen g => Agents -> Rand g Bool
    makeContact as = do
      randContact <- randomElem as
      case fst randContact of
        -- returns True with given probability
        Infected -> randomBoolM infectivity
        _        -> return False

    infect :: RandomGen g => Rand g SIRAgent
    infect = randomExpM (1 / illnessDuration)
               >>= \rd -> return (Infected, rd)
```

The infected agent is trivial. It simply recovers after the given illness duration which is implemented as follows:

```haskell
infectedAgent :: TimeDelta -> SIRAgent -> SIRAgent
infectedAgent dt (_, t)
    | t' <= 0   = (Recovered, 0)
    | otherwise = (Infected, t')
  where
    t' = t - dt
```

6.1.1 Results

When running our naive implementation with increasing population sizes we get the dynamics as seen in Figure 3. With increasing number of agents [14] our solution becomes increasingly smoother and approaches the SD dynamics from Figure 2 but doesn't quite match them because we are undersampling the contact-rate. We will will address this problem in the next section.



(a) 100 Agents **(b)** 1,000 Agents

Figure 3. Naive simulation of SIR using the agent-based approach. Varying population size, contact rate $\beta = \frac{1}{5}$, infection probability $\gamma = 0.05$, illness duration $\delta = 15$ with initially 1 infected agent. Simulation run for 150 time-steps with fixed $\Delta t = 1.0$.

6.1.2 Discussion

Reflecting on our first naive approach we can conclude that it already introduced most of the fundamental concepts of ABS

- Time - the simulation occurs over virtual time which is modelled explicitly divided into *fixed* Δt where at each step all agents are executed.
- Agents - we implement each agent as an individual, with the behaviour depending on its state.
- Feedback - the output state of the agent in the current time-step t is the input state for the next time-step $t + \Delta t$.
- Environment - as environment we implicitly assume a fully-connected network (complete graph) where every agent 'knows' every other agents, including itself and thus can make contact with all of them.
- Stochasticity - it is an inherently stochastic simulation, which is indicated by the Random Monad type and the usage of *randomBoolM* and *randomExpM*.
- Deterministic - repeated runs with the same initial random-number generator result in same dynamics. This may not come as a surprise but in Haskell we can guarantee that property statically already at compile time because our simulation runs in the Random Monad and *not* in the IO Monad. This guarantees that no external, uncontrollable sources of randomness can interfere with the simulation.
- Dynamics - with increasing number of agents the dynamics smooth out [14].

Nonetheless our approach has also weaknesses and dangers:

1. Δt is passed explicitly as argument to the agent and needs to be dealt with explicitly. This is not very elegant and a potential source of errors - can we do better and find a more elegant solution?

2. The way our agents are represented is not very modular. The state of the agent is explicitly encoded in an ADT and when processing the agent, the function needs always first distinguish between the states. Can we express it in a more modular way e.g. continuations?

We now move on to the next section in which we will address these points and the under-sampling issue.

6.2 Adding Functional Reactive Programming

As shown in the first step, the need to handle Δt explicitly can be quite messy, is inelegant and a potential source of errors, also the explicit handling of the state of an agent and its behavioural function is not very modular. We can solve both these weaknesses by switching to the functional reactive programming paradigm (FRP), because it allows to express systems with discrete and continuous time-semantics.

In this step we are focusing on arrowized [10] FRP using the library Yampa [8]. In it, time is handled implicit, meaning it cannot be messed with, which is achieved by building the whole system on the concept of signal functions (SF). An SF can be understood as a process over time and is technically a continuation which allows to capture state using closures. Both these fundamental features allow us to tackle the weaknesses of our first step and push our approach further towards a truly functional approach.

6.2.1 Implementation

We start by defining an agent now as an SF which receives the states of all agents as input and outputs the state of the agent:

```
type SIRAgent = SF [SIRState] SIRState
```

Now we can define the behaviour of an agent to be the following:

```
sirAgent :: RandomGen g => g -> SIRState -> SIRAgent
sirAgent g Susceptible = susceptibleAgent g
sirAgent g Infected    = infectedAgent g
sirAgent _ Recovered   = recoveredAgent
```

Depending on the initial state we return the corresponding behaviour. Most notably is the difference that we are now passing a random-number generator instead of running in the Random Monad because signal functions as implemented in Yampa are not capable of being monadic. We see that the recovered agent ignores the random-number generator which is in accordance with the implementation in the previous step where it acts as a sink which returns constantly the same state:

```
recoveredAgent :: SIRAgent
recoveredAgent = arr (const Recovered)
```

When an event occurs we can change the behaviour of an agent using the Yampa combinator *switch*, which is much more elegant and expressive than the initial approach as it makes the change of behaviour at the occurrence of an event explicit. Thus a susceptible agent behaves as susceptible until it becomes infected. Upon infection an *Event* is returned which results in switching into the *infectedAgent* SF, which causes the agent to behave as an infected agent from that moment on. Instead of randomly drawing the number of contacts to make, we now follow a fundamentally different approach by using Yampas *occasionally* function. This requires a fundamental different approach in selecting the right Δt and sampling the system as will be shown in results.

```
susceptibleAgent :: RandomGen g => g -> SIRAgent
susceptibleAgent g =
    switch (susceptible g) (const (infectedAgent g))
  where
    susceptible :: RandomGen g
      => g -> SF [SIRState] (SIRState, Event ())
    susceptible g = proc as -> do
      makeContact <- occasionally g (1 / contactRate) () -< ()
      if isEvent makeContact
        then (do
          a <- drawRandomElemSF g -< as
          case a of
            Infected -> do
              i <- randomBoolSF g infectivity -< ()
              if i
                then returnA -< (Infected, Event ())
                else returnA -< (Susceptible, NoEvent)
            _       -> returnA -< (Susceptible, NoEvent))
        else returnA -< (Susceptible, NoEvent)
```

We deal with randomness different now and implement signal functions built on the *noiseR* function provided by Yampa. This is an example for the stream character and statefulness of a signal function as it needs to keep track of the changed random-number generator internally through the use of continuations and closures. Here we provide the implementation of *randomBoolSF*, *drawRandomElemSF* works similar but takes a list as input and returns a randomly chosen element from it:

```
randomBoolSF :: RandomGen g => g -> Double -> SF () Bool
randomBoolSF g p = proc _ -> do
  r <- noiseR ((0, 1) :: (Double, Double)) g -< ()
  returnA -< (r <= p)
```

The infected agent behaves as infected until it recovers on average after the illness duration after which it behaves as a recovered agent by switching into *recoveredAgent*. As in the case of the susceptible agent, we use the *occasionally* function to generate the event when the agent recovers. Note that the infected agent ignores the states of the other agents as its behaviour is completely independent of them.

```
infectedAgent :: RandomGen g => g -> SIRAgent
infectedAgent g = switch infected (const recoveredAgent)
  where
    infected :: SF [SIRState] (SIRState, Event ())
    infected = proc _ -> do
      recEvt <- occasionally g illnessDuration () -< ()
      let a = event Infected (const Recovered) recEvt
      returnA -< (a, recEvt)
```

Running and stepping the simulation works now a bit differently, using Yampas function *embed*:

```
runSimulation :: RandomGen g
  => g -> Time -> DTime -> [SIRState] -> [[SIRState]]
runSimulation g t dt as
    = embed (stepSimulation sfs as) ((), dts)
  where
    steps     = floor (t / dt)
    dts       = replicate steps (dt, Nothing)
    n         = length as
    (rngs, _) = rngSplits g n [] -- unique rngs for each agent
    sfs       = map (\ (g', a) -> sirAgent g' a) (zip rngs as)
```

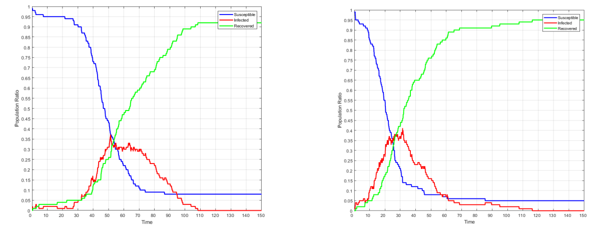
What we need to implement next is a closed feedback-loop. Fortunately, [16], [5] discusses implementing this in Yampa. The function *stepSimulation* is an implementation of such a closed feedback-loop. It takes the current signal functions and states of all agents, runs them all in parallel and returns the new agent states of this step. Yampa provides the *dpSwitch* combinator for running signal functions in parallel, which is quite involved and discussed more in-depth in section 3. It allows us to recursively switch back into the *stepSimulation* with the continuations and new states of all the agents after they were run in parallel. Note the use of *notYet* which is required because in Yampa switching occurs immediately at $t = 0$.

```
stepSimulation :: [SIRAgent] -> [SIRState] -> SF () [SIRState]
stepSimulation sfs as =
    dpSwitch
      -- feeding the agent states to each SF
      (\_ sfs' -> (map (\sf -> (as, sf)) sfs'))
      -- the signal functions
      sfs
      -- switching event, ignored at t = 0
      (switchingEvt >>> notYet)
      -- recursively switch back into stepSimulation
      stepSimulation
  where
    switchingEvt :: SF ((), [SIRState]) (Event [SIRState])
    switchingEvt = arr (\ (_, newAs) -> Event newAs)
```

6.2.2 Results

The function which drives the dynamics of our simulation is *occasionally*, which randomly generates an event on average with a given rate following the exponential distribution. To arrive at the correct dynamics, this requires us to sample *occasionally*, and thus the whole system, with small enough Δt which matches the rate. If we choose a too large Δt, we loose events which will result in dynamics which do not approach the SD dynamics sufficiently enough, see Figure 4.

Clearly by keeping the population size constant and just increasing the Δt results in a closer approximation to the SD dynamics. To increasingly approximate the SD dynamics with ABS we still need a bigger population size and even smaller Δt. Unfortunately increasing both the number of agents and the sample rate results in severe performance and memory problems. A possible solution would be to implement super-sampling which would allow us to run the whole simulation with $\Delta t = 1.0$ and only sample the *occasionally* function with a much higher frequency.



(a) $\Delta t = 0.1$ **(b)** $\Delta t = 0.01$

Figure 4. FRP simulation of agent-based SIR showing the influence of different Δt. Population size of 100 with contact rate $\beta = \frac{1}{5}$, infection probability $\gamma = 0.05$, illness duration $\delta = 15$ with initially 1 infected agent. Simulation run for 150 time-steps with respective Δt.

6.2.3 Discussion

By moving on to FRP using Yampa we made a huge improvement in clarity, expressivity and robustness of our implementation. State is now implicitly encoded, depending on which signal function is active. Also by using explicit time-semantics with *occasionally* we can achieve extremely fine grained stochastics. Compared to drawing a random number of events we create only a single event or none at all. This requires to sample the system with a much smaller Δt: we are treating it as a continuous agent-based system, resulting in a hybrid SD/ABS approach.

So far we have an acceptable implementation of an agent-based SIR approach. The next steps focus on introducing more concepts and generalising our implementation so far. What we are lacking at the moment is a general treatment of an environment. To conveniently introduce it we want to make use of monads which is not possible using Yampa. In the next step we make the transition to Monadic Stream Functions (MSF) as introduced in Dunai [21] which allows to do FRP within a monadic context.

6.3 Generalising to Monadic Stream Functions

A part of the library Dunai is BearRiver, a wrapper which re-implements Yampa on top of Dunai, which should allow us to easily replace Yampa with MSFs. This will enable us to run arbitrary monadic computations in a signal function, which we will need in the next step when adding an environment.

6.3.1 Identity Monad

We start by making the transition to BearRiver by simply replacing Yampas signal function by BearRivers which is the same but takes an additional type parameter *m* indicating the monadic context. If we replace this type-parameter with the Identity Monad we should be able to keep the code exactly the same, except from a few type-declarations, because BearRiver re-implements all necessary functions we are using from Yampa. We start by re-defining our general agent signal function, introducing the monad stack our SIR implementation runs in and the agents signal function:

```
type SIRMonad    = Identity
type SIRAgent    = SF SIRMonad [SIRState] SIRState
```

6.3.2 Random Monad

Using the Identity Monad does not gain us anything but it is a first step towards a more general solution. Our next step is to replace the Identity Monad by the Random Monad which will allow us to get rid of the RandomGen arguments to our functions and run the whole simulation within the Random Monad *again* just as we started but now with the full features functional reactive programming. We start by re-defining the SIRMonad and SIRAgent:

```
type SIRMonad g = Rand g
type SIRAgent g = SF (SIRMonad g) [SIRState] SIRState
```

The question is now how to access this Random Monad functionality within the MSF context. For the function *occasionally*, there exists a monadic pendant *occasionallyM* which requires a MonadRandom type-class. Because we are now running within a MonadRandom instance we simply replace *occasionally* with *occasionallyM*.

6.3.3 Discussion

So far making the transition to MSFs does not seem as compelling as making the move from the Random Monad to FRP in the beginning. Running in the Random Monad within FRP is convenient but we could achieve the same with passing RandomGen around as we already demonstrated. In the next step we introduce the concept of a read/write environment which we realise using a StateT monad. This will show the real benefit of the transition to MSFs as without it, implementing a general environment access would be quite cumbersome.

6.4 Adding an environment

In this step we will add an environment in which the agents exist and through which they interact with each other. This is a fundamental different approach to agent interaction but is as valid as the approach in the previous steps.

In ABS agents are often situated within a discrete 2D environment [7] which is simply a finite NxM grid with either a Moore or von Neumann neighbourhood (Figure 5). Agents are either static or can move freely around with cells allowing either single or multiple occupants.

We can directly map the SIR model to a discrete 2D environment by placing the agents on a corresponding 2D grid with an unrestricted neighbourhood. The behaviour of the agents is the same but they select their interactions directly from the environment. Also instead of feeding back the states of all agents as inputs, agents now communicate through the environment by revealing their current state to their neighbours by placing it on their cell. Agents can read the

(a) von Neumann **(b)** Moore

Figure 5. Common neighbourhoods in discrete 2D environments of Agent-Based Simulation.

states of all their neighbours which tells them if a neighbour is infected or not. This allows us to implement the infection mechanism as in the beginning. For purposes of a more interesting approach, we restrict the neighbourhood to Moore (Figure 5b).

6.4.1 Implementation

We start by defining our discrete 2D environment for which we use an indexed two dimensional array. In each cell the agents will store their current state, thus we use the *SIRState* as type for our array data:

```
type Disc2dCoord = (Int, Int)
type SIREnv      = Array Disc2dCoord SIRState
```

Next we redefine our monad stack and agent signal function. We use a StateT transformer on top of our Random Monad from the previous step with *SIREnv* as type for the state. Our agent signal function now has unit input and output type, which indicates by the types that the actions of the agents are only visible through side-effects in the monad stack they are running in.

```
type SIRMonad g = StateT SIREnv (Rand g)
type SIRAgent g = SF (SIRMonad g) () ()
```

The implementation of a susceptible agent is now a bit different and a mix between previous steps. The agent directly queries the environment for its neighbours and randomly selects one of them. The remaining behaviour is similar:

```
susceptibleAgent :: RandomGen g => Disc2dCoord -> SIRAgent g
susceptibleAgent coord
    = switch susceptible (const (infectedAgent coord))
  where
    susceptible :: RandomGen g
      => SF (SIRMonad g) () ((), Event ())
    susceptible = proc _ -> do
      makeContact <- occasionallyM (1 / contactRate) () -< ()
      if not (isEvent makeContact)
        then returnA -< ((), NoEvent)
        else (do
          env <- arrM_ (lift get) -< ()
          let ns = neighbours env coord agentGridSize moore
          s <- drawRandomElemS -< ns
          case s of
            Infected -> do
              infected <- arrM_
                (lift $ lift $ randomBoolM infectivity) -< ()
              if infected
                then (do
```

```
                  arrM (put . changeCell coord Infected) -< env
                  returnA -< ((), Event ()))
                else returnA -< ((), NoEvent)
           _          -> returnA -< ((), NoEvent))
```

Querying the neighbourhood is done using the *neighbours :: SIREnv -> Disc2dCoord -> Disc2dCoord -> [Disc2dCoord] -> [SIRState]* function. It takes the environment, the coordinate for which to query the neighbours for, the dimensions of the 2D grid and the neighbourhood information and returns the data of all neighbours it could find. Note that on the edge of the environment, it could be the case that fewer neighbours than provided in the neighbourhood information will be found due to clipping.

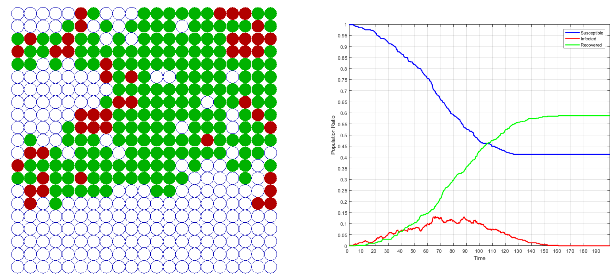
The behaviour of an infected agent is nearly the same as in the previous step, with the difference that upon recovery the infected agent updates its state in the environment from Infected to Recovered.

Running the simulation with MSFs works slightly different. The function *embed* we used before is not provided by BearRiver but by Dunai which has important implications. Dunai does not know about time in MSFs, which is exactly what BearRiver builds on top of MSFs. It does so by adding a ReaderT Double which carries the Δt. This means that *embed* returns a computation in the ReaderT Double Monad which we need to peel away using *runReaderT*. This then results in a StateT computation which we evaluate by using *evalStateT* and an initial environment as initial state. This then results in another monadic computation of the Random Monad type which we evaluate using *evalRand* which delivers the final result. This is also the reason why we need lifts e.g. in case of getting the environment. Note that instead of returning agent states we simply return a list of environments, one for each step. The agent states can then be extracted from each environment.

```
runSimulation :: RandomGen g => g -> Time -> DTime
  -> SIREnv -> [(Disc2dCoord, SIRState)] -> [SIREnv]
runSimulation g t dt env as = evalRand esRand g
  where
    steps    = floor (t / dt)
    dts      = replicate steps ()
    -- initial SFs of all agents
    sfs      = map (uncurry sirAgent) as
    -- running the simulation
    esReader = embed (stepSimulation sfs) dts
    esState  = runReaderT esReader dt
    esRand   = evalStateT esState env
```

Due to the different approach of returning the SIREnv in every step, we implemented our own MSF:

```
stepSimulation :: RandomGen g
  => [SIRAgent g] -> SF (SIRMonad g) () SIREnv
stepSimulation sfs = MSF (\_ -> do
  -- running all SFs with unit input
  res <- mapM (`unMSF` ()) sfs
  -- extracting continuations, ignore output
  let sfs' = fmap snd res
  -- getting environment of current step
  env <- get
```



(a) $t = 100$ **(b)** Dynamics over time

Figure 6. Simulating the agent-based SIR model on a 21x21 2D grid with Moore neighbourhood (Figure 5b), a single infected agent at the center and same SIR parameters as in Figure 2. Simulation run until $t = 200$ with fixed $\Delta t = 0.1$. Last infected agent recovers shortly after $t = 160$. The susceptible agents are rendered as blue hollow circles for better contrast.

```
  -- recursive continuation
  let ct = stepSimulation sfs'
  return (env, ct))
```

6.4.2 Results

We implemented rendering of the environments using the gloss library which allows us to cycle arbitrarily through the steps and inspect the spreading of the disease over time visually as in Figure 6.

Note that the dynamics of the spatial SIR simulation which are seen in Figure 6b look quite different from the SD dynamics of Figure 2. This is due to a much more restricted neighbourhood which results in far fewer infected agents at a time and a lower number of recovered agents at the end of the epidemic, meaning that fewer agents got infected overall.

6.4.3 Discussion

At first the environment approach might seem a bit overcomplicated and one might ask what we have gained by using an unrestricted neighbourhood where all agents can contact all others. The real win is that we can introduce arbitrary restrictions on the neighbourhood as shown with the Moore neighbourhood.

Of course an environment is not restricted to be a discrete 2D grid and can be anything from a continuous N-dimensional space to a complex network - one only needs to change the type of the StateT monad and provide corresponding neighbourhood querying functions. The ability to place the heterogeneous agents in a generic environment is also the fundamental advantage of an agent-based over the SD approach and allows to simulate much more realistic scenarios.

7 Conclusions

Our approach is radically different from traditional approaches in the ABS community. First it builds on the already quite powerful FRP paradigm. Second, due to our hybrid approach, it forces one to think properly of time-semantics of the model, how small Δt should be, whether one needs super-sampling or not and how many samples one should take. Third it requires to think about agent-interactions in a new way instead of being just method-calls.

Because no part of the simulation runs in the IO Monad and we do not use unsafePerformIO we can rule out a serious class of bugs caused by implicit data-dependencies and side-effects which can occur in traditional imperative implementations.

Also we can statically guarantee the reproducibility of the simulation. Within the agents there are no side effects possible which could result in differences between same runs. Every agent has access to its own random-number generator or the Random Monad, allowing randomness to occur in the simulation but the random-generator seed is fixed in the beginning and can never be changed within an agent. This means that after initialising the agents, which *could* run in the IO Monad, the simulation itself runs completely deterministic.

Determinism is also ensured by fixing the Δt and not making it dependent on the performance of e.g. a rendering-loop or other system-dependent sources of non-determinism as described by [22]. Also by using FRP we gain all the benefits from it and can use research on testing, debugging and exploring FRP systems [22], [19].

Issues

Unfortunately, the hybrid approach of SD/ABS amplifies the performance issues of agent-based approaches, which requires much more processing power compared to SD, because each agent is modelled individually in contrast to aggregates in SD [14]. With the need to sample the system with high frequency, this issue gets worse. We haven't investigated how to optimize the performance by using efficient functional data structures, hence in the moment our program performs much worse than an imperative implementation that exploits in-place updates.

Despite the strengths and benefits we get by leveraging on FRP, there are errors that are not raised at compile-time, e.g. we can still have infinite loops and run-time errors. This was for example investigated by [24] who use dependent types to avoid some runt-time errors in FRP. We suggest that one could go further and develop a domain specific type system for FRP that makes the FRP based ABS more predictable and that would support further mathematical analysis of its properties.

We can conclude that the main difficulty of a pure functional approach evolves around the communication and interaction between agents. This is straight-forward in object-oriented programming where it is achieved using method-calls mutating the internal state of the agent but which comes at the cost of a new class of bugs due to implicit data flow. In pure functional programming these data flows are explicit but our current approach of feeding back the states of all agents as inputs is not very general and we have added further mechanisms of agent interaction which we had to omit due to lack of space. We hypothesise that MSFs allow us to conveniently express agent communication but but leave this for further research.

We started with high hopes for the pure functional approach and hypothesized that it will be truly superior to existing traditional object-oriented approaches but we come to the conclusion that this is not so. The single real benefit is the lack of implicit side-effects and reproducibility guaranteed at compile time. Still, our research was not in vain as we see it as an intermediary step towards using dependent types. Moving to dependent types would pose a unique benefit over the object-oriented approach and should allow us to express and guarantee properties at compile time which is not possible with imperative approaches. We leave this for further research.

8 Further Research

We see this paper as an intermediary and necessary step towards dependent types for which we first needed to understand the potentials and limitations of a non-dependently typed pure functional approach in Haskell. Dependent types are extremely promising in functional programming as they allow us to express stronger guarantees about the correctness of programs and go as far as allowing to formulate programs and types as constructive proofs which must be total by definition [29], [2], [1].

So far no research using dependent types in agent-based simulation exists at all and it is not clear whether dependent types make sense in this context. In our next paper we want to explore this for the first time and ask more specifically how we can add dependent types to our pure functional approach, which conceptual implications this has for ABS and what we gain from doing so. We plan on using Idris [4] as the language of choice as it is very close to Haskell with focus on real-world application and running programs as opposed to other languages with dependent types e.g. Agda and Coq which serve primarily as proof assistants.

It would be of immense interest whether we could apply dependent types to the model meta-level or not - this boils down to the question if we can encode our model specification in a dependently typed way. This would allow the ABS community for the first time to reason about a proper formalisation of a model in code.

Acknowledgments

The authors would like to thank I. Perez, H. Nilsson, J. Greensmith, M. Baerenz, H. Vollbrecht, S. Venkatesan and J. Hey for constructive comments and valuable discussions.

References

[1] Thorsten Altenkirch, Nils Anders Danielsson, Andres Loeh, and Nicolas Oury. 2010. Pi Sigma: Dependent Types Without the Sugar. In *Proceedings of the 10th International Conference on Functional and Logic Programming (FLOPS'10)*. Springer-Verlag, Berlin, Heidelberg, 40–55. https://doi.org/10.1007/978-3-642-12251-4_5

[2] Thorsten Altenkirch, Conor Mcbride, and James Mckinna. 2005. Why dependent types matter. In *In preparation, http://www.e-pig.org/downloads/ydtm.pdf*.

[3] Andrei Borshchev and Alexei Filippov. 2004. From System Dynamics and Discrete Event to Practical Agent Based Modeling: Reasons, Techniques, Tools. Oxford.

[4] Edwin Brady. 2013. Idris, a general-purpose dependently typed programming language: Design and implementation. *Journal of Functional Programming* 23, 05 (2013), 552–593. https://doi.org/10.1017/S095679681300018X

[5] Antony Courtney, Henrik Nilsson, and John Peterson. 2003. The Yampa Arcade. In *Proceedings of the 2003 ACM SIGPLAN Workshop on Haskell (Haskell '03)*. ACM, New York, NY, USA, 7–18. https://doi.org/10.1145/871895.871897

[6] Tanja De Jong. 2014. *Suitability of Haskell for Multi-Agent Systems*. Technical Report. University of Twente.

[7] Joshua M. Epstein and Robert Axtell. 1996. *Growing Artificial Societies: Social Science from the Bottom Up*. The Brookings Institution, Washington, DC, USA.

[8] Paul Hudak, Antony Courtney, Henrik Nilsson, and John Peterson. 2003. Arrows, Robots, and Functional Reactive Programming. In *Advanced Functional Programming*, Johan Jeuring and Simon L. Peyton Jones (Eds.). Number 2638 in Lecture Notes in Computer Science. Springer Berlin Heidelberg, 159–187. http://link.springer.com/chapter/10.1007/978-3-540-44833-4_6 DOI: 10.1007/978-3-540-44833-4_6.

[9] Paul Hudak, John Hughes, Simon Peyton Jones, and Philip Wadler. 2007. A History of Haskell: Being Lazy with Class. In *Proceedings of the Third ACM SIGPLAN Conference on History of Programming Languages (HOPL III)*. ACM, New York, NY, USA, 12–1–12–55. https://doi.org/10.1145/1238844.1238856

[10] John Hughes. 2000. Generalising Monads to Arrows. *Sci. Comput. Program.* 37, 1-3 (May 2000), 67–111. https://doi.org/10.1016/S0167-6423(99)00023-4

[11] John Hughes. 2005. Programming with Arrows. In *Proceedings of the 5th International Conference on Advanced Functional Programming (AFP'04)*. Springer-Verlag, Berlin, Heidelberg, 73–129. https://doi.org/10.1007/11546382_2

[12] Peter Jankovic and Ondrej Such. 2007. *Functional Programming and Discrete Simulation*. Technical Report.

[13] W. O. Kermack and A. G. McKendrick. 1927. A Contribution to the Mathematical Theory of Epidemics. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 115, 772 (Aug. 1927), 700–721. https://doi.org/10.1098/rspa.1927.0118

[14] Charles M. Macal. 2010. To Agent-based Simulation from System Dynamics. In *Proceedings of the Winter Simulation Conference (WSC '10)*. Winter Simulation Conference, Baltimore, Maryland, 371–382. http://dl.acm.org/citation.cfm?id=2433508.2433551

[15] C. M. Macal. 2016. Everything you need to know about agent-based modelling and simulation. *Journal of Simulation* 10, 2 (May 2016), 144–156. https://doi.org/10.1057/jos.2016.7

[16] Henrik Nilsson, Antony Courtney, and John Peterson. 2002. Functional Reactive Programming, Continued. In *Proceedings of the 2002 ACM SIGPLAN Workshop on Haskell (Haskell '02)*. ACM, New York, NY, USA, 51–64. https://doi.org/10.1145/581690.581695

[17] Michael J. North and Charles M. Macal. 2007. *Managing Business Complexity: Discovering Strategic Solutions with Agent-Based Modeling and Simulation*. Oxford University Press, USA. Google-Books-ID: gRATDAAAQBAJ.

[18] Ross Paterson. 2001. A New Notation for Arrows. In *Proceedings of the Sixth ACM SIGPLAN International Conference on Functional Programming (ICFP '01)*. ACM, New York, NY, USA, 229–240. https://doi.org/10.1145/507635.507664

[19] Ivan Perez. 2017. Back to the Future: Time Travel in FRP. In *Proceedings of the 10th ACM SIGPLAN International Symposium on Haskell (Haskell 2017)*. ACM, New York, NY, USA, 105–116. https://doi.org/10.1145/3122955.3122957

[20] Ivan Perez. 2017. *Extensible and Robust Functional Reactive Programming*. Doctoral Thesis. University Of Nottingham, Nottingham.

[21] Ivan Perez, Manuel Baerenz, and Henrik Nilsson. 2016. Functional Reactive Programming, Refactored. In *Proceedings of the 9th International Symposium on Haskell (Haskell 2016)*. ACM, New York, NY, USA, 33–44. https://doi.org/10.1145/2976002.2976010

[22] Ivan Perez and Henrik Nilsson. 2017. Testing and Debugging Functional Reactive Programming. *Proc. ACM Program. Lang.* 1, ICFP (Aug. 2017), 2:1–2:27. https://doi.org/10.1145/3110246

[23] Donald E. Porter. 1962. Industrial Dynamics. Jay Forrester. M.I.T. Press, Cambridge, Mass.; Wiley, New York, 1961. xv + 464 pp. Illus. $18. *Science* 135, 3502 (Feb. 1962), 426–427. https://doi.org/10.1126/science.135.3502.426-a

[24] Neil Sculthorpe and Henrik Nilsson. 2009. Safe Functional Reactive Programming Through Dependent Types. In *Proceedings of the 14th ACM SIGPLAN International Conference on Functional Programming (ICFP '09)*. ACM, New York, NY, USA, 23–34. https://doi.org/10.1145/1596550.1596558

[25] Peer-Olaf Siebers and Uwe Aickelin. 2008. Introduction to Multi-Agent Simulation. *arXiv:0803.3905 [cs]* (March 2008). http://arxiv.org/abs/0803.3905 arXiv: 0803.3905.

[26] David Sorokin. 2015. *Aivika 3: Creating a Simulation Library based on Functional Programming*.

[27] Martin Sulzmann and Edmund Lam. 2007. *Specifying and Controlling Agents in Haskell*. Technical Report.

[28] Jonathan Thaler and Peer-Olaf Siebers. 2017. The Art Of Iterating: Update-Strategies in Agent-Based Simulation. Dublin.

[29] Simon Thompson. 1991. *Type Theory and Functional Programming*. Addison Wesley Longman Publishing Co., Inc., Redwood City, CA, USA.

[30] Ivan Vendrov, Christopher Dutchyn, and Nathaniel D. Osgood. 2014. Frabjous: A Declarative Domain-Specific Language for Agent-Based Modeling. In *Social Computing, Behavioral-Cultural Modeling and Prediction*, William G. Kennedy, Nitin Agarwal, and Shanchieh Jay Yang (Eds.). Number 8393 in Lecture Notes in Computer Science. Springer International Publishing, 385–392. http://link.springer.com/chapter/10.1007/978-3-319-05579-4_47 DOI: 10.1007/978-3-319-05579-4_47.

[31] Zhanyong Wan and Paul Hudak. 2000. Functional Reactive Programming from First Principles. In *Proceedings of the ACM SIGPLAN 2000 Conference on Programming Language Design and Implementation (PLDI '00)*. ACM, New York, NY, USA, 242–252. https://doi.org/10.1145/349299.349331

[32] Michael Wooldridge. 2009. *An Introduction to MultiAgent Systems* (2nd ed.). Wiley Publishing.

Received March 2018