STAT 656 HW 1

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Question 1

AR(1) model:

$$y_i = p \cdot y_{i-1} + e_i$$

Assume that the error terms (e_i) are independently and identically distributed (i.i.d.) with a gaussian distribution $N(0, \sigma^2)$.

The PDF of a gaussian distribution is given by:

$$f(e_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{e_i^2}{2\sigma^2}\right)$$

The likelihood function, $L(p, \sigma^2 | y_0, y_1, y_2, ..., y_n)$, is the joint pdf of the observed data $(y_0, y_1, y_2, ..., y_n)$ given the parameters (p, σ^2). Since the observations are assumed to be independent, the joint probability can be expressed as the product of individual probabilities:

$$L(p, \sigma^2 | y_0, y_1, y_2, ..., y_n) = \prod f(y_i | p, \sigma^2)$$

Substituting the AR(1) model equation into the likelihood function, we have:

$$L(p, \sigma^2 | y_0, y_1, y_2, ..., y_n) = \prod f(y_i | y_{i-1}, p, \sigma^2)$$

Using the PDF of the gaussian distribution, we can express the conditional probability $f(y_i | y_{i-1}, p, \sigma^2)$ as:

$$f(y_i | y_{i-1}, p, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(y_i - p \cdot y_{i-1})^2}{2\sigma^2}\right)$$

Taking the log of the likelihood function to obtain the log-likelihood function:

$$\begin{aligned} \log L(p, \sigma^2 | y_0, y_1, y_2, ..., y_n) &= \log\left(\prod f(y_i | y_{i-1}, p, \sigma^2)\right) \\ &= \sum \log\left(\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{(y_i - p \cdot y_{i-1})^2}{2\sigma^2}\right)\right) \\ &= -\frac{n}{2} \cdot \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \cdot \sum (y_i - p \cdot y_{i-1})^2 \end{aligned}$$

\end{align*}

The first term represents a constant that does not depend on the parameters and can be ignored during optimization. The second term quantifies the sum of squared residuals, which measures the discrepancy between the observed values and the predictions made by the AR(1) model.

Question 2

Instruction

STAT 656: Bayesian Data Analysis
Fall 2023
Homework 1

Note: For each question, no credit will be given unless work is shown.

Synthetic data

40 points

The autoregressive model is frequently used to analyze time series data. The simplest autoregressive model has order 1, and is abbreviated as AR(1). This model assumes that an observation y_i at time point i ($i = 1, \ldots, n$) is generated according to

$$y_i = \rho y_{i-1} + \epsilon_i,$$

where $\epsilon_i \sim \mathrm{N}\left(0, \sigma^2\right)$ independently, and ρ and σ are unknown parameters. For simplicity, we shall assume that y_0 is a fixed constant. We will also assume $|\rho| < 1$.

1. (5 points) Write the log-likelihood function $\log L(\rho, \sigma^2 | y_0, y_1, \ldots, y_n)$ for $(\rho, \sigma^2)^\mathsf{T}$ for the AR(1) model.

For rest of this problem, we shall take ρ to be our estimand of interest, and consider data in the file `computation_data_hw_1.csv`, generated from this type of process with $y_0 = 0$.

2. (10 points) Write an R function that computes the log of the likelihood functions for $(\rho, \log(\sigma))^\mathsf{T}$ for this data. Specifically, this function should take the form:

```
ar_loglik <- function(rho,sig) {
# Do stuff with data loaded from computation_data_hw_1.csv
# You can make the data also an input to the function,
#   or treat it as a global variable

    ....

}
```

Provide a visualizations of both as a contour plot. Hint: The `outer` and `contour` function in `R` can be useful for creating the visualization.

3. (10 points) For the purposes of this problem, suppose we specify $\rho \sim \mathrm{Uniform}(-1, 1)$, $\log(\sigma) \sim \mathrm{N}(0, 10^2)$ independently *a priori* (note that this may not be an appropriate prior for the parameters of an AR(1) model in general). Write an R function that computes the log of the posterior density (upto a constant) for $(\rho, \log(\sigma))^\mathsf{T}$ under this prior. Provide a visualizations of this function as above. How does this function correspond to the log-likelihood function? Would you say that this prior specification is overly informative? Why or why not?

5. (10 points) Draw 1000 values of $(\rho, \log(\sigma))^\mathsf{T}$ from a discrete grid approximation to the posterior. Be sure to describe your choice of discrete grid. Hint: The previous step can be helpful in this regard, together with the R function `sample`.

7. (5 points) Use these draws to calculate the following summaries for each of ρ and $\log(\sigma)$: 0.025, 0.25, 0.5, 0.75, 0.975 quantiles, mean, standard deviation, skewness, and kurtosis. You can use the library `moments` if you want.

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Figure 1: PDF File