STAT 656: Bayesian Data Analysis
Fall 2023
Homework 3

Note: For each question, no credit will be given unless work is shown.

Gibbs sampling for normal hierarchical models (100 points)

In Lecture 7, we modeled the NBA data with a normal hierarchical model with known variances as follows:

$$p\left(\mu, \tau^2\right) = p\left(\mu \mid \tau^2\right) p\left(\tau^2\right) \propto p\left(\tau^2\right)$$

$$\theta_j \mid \mu, \tau^2 \overset{\text{iid}}{\sim} \mathrm{N}\left(\mu, \tau^2\right), \quad j = 1, \ldots, J$$

$$y_{ij} \mid \theta, \sigma^2, \mu, \tau^2 \overset{\perp\!\!\!\perp}{\sim} \mathrm{N}\left(\theta_j, \sigma_j^2\right), \quad i = 1, \ldots, n_j.$$

. Now we will extend this to allow the variances σ_j^2 to be unknown as well. We will consider two models:

1. **Unknown but identical variances**: $\sigma_1^2 = \ldots, \sigma_J^2 = \sigma^2$, with $\sigma^2 \sim p_v$
2. **Unknown and independent variances**: $\sigma_1^2, \ldots, \sigma_J^2 \overset{\text{iid}}{\sim} p_v$

For both models set $p_v(\sigma^2) \propto 1/\sigma$. Your task is to implement a Gibbs sampler to simulate all latent variables given the data from `nba_data.csv`. The Gibbs sampler involves the following steps each iteration:

1. *(For both models)* Sample from $\mu, \tau^2 \mid \theta_1, \ldots, \theta_J, \sigma_1^2, \ldots, \sigma_J^2, Y$, and then

2a. *(For model 1)* Sample from $\theta_1, \ldots, \theta_J \mid \sigma^2, \mu, \tau^2, Y$ and then $\sigma^2 \mid \theta_1, \ldots, \theta_J, \mu, \tau^2, Y$

2b. *(For model 2)* Sample from $(\theta_1, \sigma_1^2), \ldots, (\theta_J, \sigma_J^2) \mid \mu, \tau^2, Y$

For full credit, for each model, you need to:

2. (15 points) Write down the form of the conditional distributions above. Here, the material in Lecture 3, 7 and 13 will be useful.

3. (25 points) Write down R code to implement the Gibbs samplers.

4. (30 points) Run the two Gibbs samplers on the NBA dataset and summarize the results. Specifically, for μ, τ^2 and a few θ_j and σ_j^2, plot the MCMC traceplots, and diagnose mixing. Also plot the corresponding posterior distributions.

5. (30 points) Comment on how the posterior distributions differ from each other. Use each model to make predictions on the NBA games from the rest of the season (following code from Lecture 7), and comment on which model (or the model with known variances from Lecture 7) performs best.

Hint: To debug/verify your code, note that if you keep the σ_j^2's fixed each Gibbs step, you will get the same results as the code from Lecture 7 (with known variances).

The Stroop effect (100 points)

Consider a psychological task where subjects are presented with a word at the center of a computer screen (`'red'`, `'blue'`, or `'green'`). Further, the word is colored either red, blue, or green. In some trials, the word matches the color of the text ('congruent' condition); otherwise they do not match ('incongruent' condition, e.g. the word is `'red'` but it is colored blue). Subjects are told to focus only on the color that the word is written in, and press 1 if the color is red, 2 if it is blue, and 3 if it is green. In each case, the experimenter measures the reaction time (i.e. how long it takes them to press the correct button). The Stroop effect (Stroop, 1935) is a robust effect in psychology where the reaction time in the incongruent condition is on average larger than in the congruent condition.

Your task is to use the data in `stroop_data.csv` to verify if this is the case. The data measures multiple reactions times of different subjects in congruent and incongruent settings. You will model this with a hierarchical Bayesian model, with the goal of determining:

- how much longer reaction times are for each color in incongruent vs congruent cases, and whether this difference is significant.
- how different the effect is for each color, and whether these differences are significant.
- how different individuals in the study are from each other.

Your model should account for the fact that

- each response of each individual involves random fluctuations
- reaction times and the magnitude of the Stroop effect can be different for different individuals
- reaction times and the magnitude of the Stroop effect can be different for different colors (e.g. it might be smaller for red where you have to press 1 vs others)

Your hierarchical model should allow statistical sharing among individuals and possibly among different colors. Justify your model and prior choices, implement it in `Stan` and discuss your findings, being sure to include visualizations and predictive checks of model fit. You must present your final results in a form that can be readily understood by a general audience.

Feedback (optional)

Brief comments on each of these points would be greatly appreciated.

(a) Do the instructors present material at an adequate pace during lecture (too slow/too fast)?

(b) What general material would you like the instructors to spend more time on?

(c) Which topics/ideas/concepts in lecture were not well-explained? Brief comments are appreciated.

(d) Any further comments/questions/feedback?

References

J Ridley Stroop. Studies of interference in serial verbal reactions. *Journal of experimental psychology*, 18(6): 643, 1935.