Introduction to Control with Learning — April 27th, 2023

Lecture 5

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This lecture will be the last on linear dynamical systems and control. We'll discuss:

1. Preliminaries on Gaussian rvs
2. Recap + Kalman filter conclusion
3. Stochastic LQR
4. LQG - where the states can not be observed directly.

Preliminary Material for the Kalman Filter

1. Random Vectors and Expectations

- **Random Vector**: A vector of random variables, denoted $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$.
- **Expectation (Mean)**: For a random vector $\mathbf{x}$, the expected value is

$$\mathbb{E}[\mathbf{x}] = \begin{bmatrix} \mathbb{E}[x_1] \\ \vdots \\ \mathbb{E}[x_n] \end{bmatrix}$$

- **Covariance Matrix**:

$$\mathrm{Cov}(\mathbf{x}) = R_{\mathbf{x}} = \mathbb{E}[(\mathbf{x} - \mathbb{E}[\mathbf{x}])(\mathbf{x} - \mathbb{E}[\mathbf{x}])^T]$$

  This matrix captures the variance of each variable (diagonal) and pairwise correlations (off-diagonal).

- Claim: The covariance matrix is positive semi-definite.

2. Gaussian (Normal) Distributions

- **Univariate Gaussian**: A scalar random variable $x \sim \mathcal{N}(\mu, \sigma^2)$ has density:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- A Gaussian rv only depends on its first two second moments.

- A vector $\mathbf{x}$ is Gaussian if $a^t\mathbf{x}$ is Gaussian for all vectors a
- Claim: A Gaussian random vector $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ with $\Sigma \succ 0$ has a density function:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

- **Claims**:
  - Affine transformations preserve Gaussianity. If $\mathbf{x} \sim \mathcal{N}(\mu, \Sigma)$ then $A\mathbf{x} + b \sim \mathcal{N}(\mu + b, A\Sigma A^T)$
  - Marginal distributions of a joint Gaussian are also Gaussian.
  - Conditional distributions formed from a Gaussian vector $[\mathbf{x}; \mathbf{y}]$ are Gaussian. That is, if $[\mathbf{x}; \mathbf{y}] \sim \mathcal{N}(\begin{bmatrix} m_{\mathbf{x}} \\ m_{\mathbf{y}} \end{bmatrix}, \begin{bmatrix} R_{\mathbf{x}} & R_{\mathbf{xy}} \\ R_{\mathbf{yx}} & R_{\mathbf{y}} \end{bmatrix})$, then $\mathbf{x}|\mathbf{y} \sim \mathcal{N}(m_{\mathbf{x}|\mathbf{y}}, R_{\mathbf{x}|\mathbf{y}})$ with

$$\begin{aligned} m_{\mathbf{x}|\mathbf{y}} &= E[\mathbf{x}|\mathbf{y}] = m_{\mathbf{x}} + R_{\mathbf{xy}}R_{\mathbf{yy}}^{-1}(\mathbf{y} - m_{\mathbf{y}}) \\ R_{\mathbf{x}|\mathbf{y}} &= \mathrm{Cov}(\mathbf{x}|\mathbf{y}) = R_{\mathbf{x}} - R_{\mathbf{xy}}R_{\mathbf{yy}}^{-1}R_{\mathbf{yx}} \end{aligned} \tag{1}$$

1 Recap - last lecture

1.1 The Kalman Filter

The Kalman filter model is derscribed as a state-space model

$$x_{i+1} = Fx_i + Gw_i \tag{2}$$
$$y_i = Hx_i + v_i, \tag{3}$$

where x_{i+1} is the hidden state, y_i is the observed state at time i, and for v_i, w_i zero-mean and uncorrelated random variables (as a sequence of vectors), that is,

$$\mathbb{E}[\begin{pmatrix} w_i \\ v_i \\ x_0 \end{pmatrix} \begin{pmatrix} w_j^T & v_j^T & x_0^T & 1 \end{pmatrix}] = \begin{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \delta_{ij} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \Pi_0 & 0 \end{pmatrix}. \tag{4}$$

For $\forall i : Q = cov(w_i)$, $R = cov(v_i)$ and $\Pi_i = cov(x_0)$ such that $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \succeq 0$ and $R \succ 0$.

The objective, is to find a linear predictor $\hat{x}_{i+1|i}$ based on the measurements $y_1, ..., y_i$ such that the error covariance is minimal:

$$P_i = \mathbb{E}\left[(x_{i+1} - \hat{x}_{i+1})(x_{i+1} - \hat{x}_{i+1})^T\right]. \tag{5}$$

The main question is how we can combine several measurements since $\hat{x}_{i+1|i}$ is constructed from i measurements. We can almost use what we already have.

1.2 Estimation using two (or more) samples

Assume we want to estimate $\mathbf{x}$ using two samples $\mathbf{y} = [\mathbf{y}_1; \mathbf{y}_2]$. We already know that the optimal estimator is

$$\hat{\mathbf{x}} = R_{\mathbf{xy}} R_{\mathbf{y}}^{-1} \mathbf{y}. \tag{6}$$

Note that we need to compute the inverse of the measurements' covariance. Since in Kalman filter we have many samples (i grows) we will need to compute a huge inverse which is intractable.

Our approach is to use a simple estimator using the sum of estimators

$$\hat{\mathbf{x}}_{|\mathbf{y}_1,\mathbf{y}_2} \stackrel{?}{=} \hat{\mathbf{x}}_{|\mathbf{y}_1} + \hat{\mathbf{x}}_{|\mathbf{y}_2}. \tag{7}$$

However, this formula only holds when the measurements are uncorrelated, that is, $\mathbb{E}[\mathbf{y}_1 \mathbf{y}_2^T] = 0$. In Kalman filter, unfortunately, this is the case since $\mathbb{E}[\mathbf{y}_1 \mathbf{y}_2^T] \neq 0$ due to the memory from the hidden state $\mathbf{x}$.

1.3 Innovations Process

The innovations process is a general concept in linear estimation. Given a sequence of measurements $\mathbf{y}_1, \ldots, \mathbf{y}_n$ construct the process

$$e_i := y_i - \hat{y}_{i|i-1} \tag{8}$$

where $\hat{\mathbf{y}}_{i|i-1} = \mathbb{E}[\mathbf{y}_i | \mathbf{y}^{i-1}]$. Intuitively, the innovations process only captures the new part in the measurements' sequence.

The innovations process has two important properties:

1. The innovations process is white (i.e., uncorrelated over time).
2. It is a causal function of the measurements sequence.

If we consider the Gaussian setting or linear estimation, the innovation process is defined as

$$e_i := y_i - \hat{y}^L_{i|i-1}, \tag{9}$$

where $\hat{y}^L_{i|i-1}$ is the best **linear** estimator of $\mathbf{y}_i$ given the measurements $\mathbf{y}_1, \ldots, \mathbf{y}_{i-1}$. It is still an uncorrelated process [1].

[1] Consider $g(\epsilon) = \mathbb{E}[(y_i - \hat{y}^L_{i|i-1} + \epsilon S)(y_i - \hat{y}^L_{i|i-1} + \epsilon S)^T]$ where S is some linear function of the past measurements $\mathbf{y}^{i-1}$. We use the fact that $\hat{y}^L_{i|i-1}$ is the best linear estimator as $g'(\epsilon) \mid_{\epsilon=0} = 0 \rightarrow \mathbb{E}[e_i S^T] = 0$.

1.4 Guess for the Kalman filter

We (intuitively) constructed the predictor:

$$\hat{x}_{i+1|i} = F\hat{x}_{i|i-1} + K_i(y_i - H\hat{x}_{i|i-1}), \tag{10}$$

and studied its behavior.

First, we showed how to find the optimal gain matrix K_i:

Theorem 1. *Let $P_i(K_i) = \mathbb{E}[(x_i - \hat{x}_i)(x_i - \hat{x}_i)^T]$ be the error covariance matrix as a function of K_i in (10). Then, the optimal gain matrix (for all K_i, $P_i(K_i) \succeq P_i$) is achieved with*

$$K_{p,i} := (FP_iH^T + GS)R_{e,i}^{-1},, \tag{11}$$

where $R_{e,i} = HP_iH^T + R$.

Moreover, the optimal error covariance satisfies the recursion

$$P_{i+1} = FP_iF^T + GQG^T - (FP_iH^T + GS)R_{e,i}^{-1}(FP_iH^T + GS)^T \tag{12}$$

with $P_1 = \Pi_0$.

We also showed that this predictor generates the innovations process:

Theorem 2. *The process*

$$e_i := \mathbf{y}_i - H\hat{x}_{i+1|i}, \tag{13}$$

where $\hat{x}_{i+1|i}$ is given by (10) with $K_i = K_{p,i}$ is an innovations process with variance $R_{e,i} = HP_iH^T + R$ with P_i given in (12).

Proof. In the last lecture, we proved $\mathbb{E}\left[e_i e_j^T\right] = R_{e,i}\delta_{ij}$.

We prove now that it is a causal and invertible function of the measurements' sequence. This can be concluded from a new state-space model

$$\begin{aligned} \hat{x}_{i+1} &= F\hat{x}_i + K_{p,i}e_i \\ y_i &= H\hat{x}_i + e_i, \end{aligned} \tag{14}$$

with $\hat{x}_1 = 0$.

If we insist on showing it from (14) explicitly, we can write

$$\begin{aligned} y_1 &= e_1 \\ y_2 &= H\hat{x}_2 + e_2 \\ &= H(F\hat{x}_1 + K_{p,1}e_1) + e_2 \\ &= HK_{p,1}e_1 + e_2 \\ y_3 &= H(F\hat{x}_2 + K_{p,2}e_2) + e_3 \\ &= HF(F\hat{x}_1 + K_{p,1}e_1) + HK_{p,2}e_2 + e_3 \\ &= HFK_{p,1}e_1 + HK_{p,2}e_2 + e_3 \end{aligned} \tag{15}$$

The overall mapping is $y := Le$

$$L = \begin{pmatrix} I & 0 & \dots & 0 \\ HK_{p,0} & I & \dots & 0 \\ H\Phi(2,1)K_{p,0} & HK_{p,1} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ H\Phi(N,1)K_{p,0} & H\Phi(N,2)K_{p,1} & \dots & I \end{pmatrix}, \tag{16}$$

where $\Phi(j,i) = F^{j-i}$ (This quantity is more relevant in time-varying settings where F depends on time).

We note that the matrix L is a causal (lower triangular) and is invertible since the diagonal is non-zero. Also, the inverse of a lower triangular matrix is lower triangular, so $e = L^{-1}y$ is causal.

Overall, we can write R_Y as

$$\begin{aligned} y &= Le \\ \Rightarrow R_Y &= \mathbb{E}\left[Le(Le)^T\right] \\ &= LR_{\mathbf{e}}L^T. \end{aligned} \tag{17}$$

All we did was indeed an LDU factorization of R_Y.

Since the diagonal of L is the identity matrix, this factorization is unique. Uniqueness: Consider $M \succ 0$ and assume that it has two different Cholesky factorizations $M = L_1L_1^T = L_2L_2^T$. We define $Q = L_2^{-1}L_1$ and note that $M = L_2QQ^TL_2^T$ but also $M = L_2L_2^T$. Since L_2 is invertible, we have $QQ^T = I$. Since Q is lower triangular, it follows that Q is diagonal with ± 1. If we require positive diagonal, we obtain $Q = I \to L_1 = L_2$.

□

1.5 Solving the Kalman Filter

Now that we have the innovations process, we can use (**??**) to write the predictor

$$\begin{aligned} \hat{x}_{i+1|i} &= \sum_{j=0}^{i} \langle x_{i+1}, e_j \rangle R_{e_j}^{-1} e_j \\ &= \sum_{j=0}^{i-1} \langle x_{i+1}, e_j \rangle R_{e_j}^{-1} e_j + \langle x_{i+1}, e_i \rangle R_{e_i}^{-1} e_i \\ &= \sum_{j=0}^{i-1} \langle Fx_i + Gw_i, e_j \rangle R_{e_j}^{-1} e_j + \langle x_{i+1}, e_i \rangle R_{e_i}^{-1} e_i \\ &\overset{(a)}{=} F\hat{x}_{i|i-1} + \langle x_{i+1}, e_i \rangle R_{e_i}^{-1} e_i, \end{aligned} \tag{18}$$

where (a) follows from the fact that $< w_i, e_j >= 0$ for $j < i$.

The second term can be computed as

$$\langle x_{i+1}, e_i \rangle = \langle Fx_i + Gw_i, e_i \rangle = F \langle x_i, e_i \rangle + G \langle w_i, e_i \rangle$$

$$= FP_iH^T + GS, \tag{19}$$

where the first term is derived from

$$\begin{aligned}\langle x_i, e_i\rangle &= \langle x_i - \hat{x}_i + \hat{x}_i, e_i\rangle \\ &= \langle \tilde{x}_i, e_i\rangle + \underbrace{\langle \hat{x}_i, e_i\rangle}_{=0} \\ &= \langle \tilde{x}_i, e_i\rangle \\ &= \langle \tilde{x}_i, H\tilde{x}_i + v_i\rangle \\ &= P_iH^T \end{aligned} \tag{20}$$

and for the second term we used

$$\langle w_i, e_i\rangle = \langle w_i, H\tilde{x}_i + v_i\rangle = 0 + \langle w_i, v_i\rangle = S \tag{21}$$

Putting it all together:

$$\begin{aligned}\hat{x}_{i+1|i} &= F\hat{x}_{i|i-1} + \left(FP_iH^T + GS\right)R_{e_i}^{-1}e_i \\ &= F\hat{x}_{i|i-1} + \left(FP_iH^T + GS\right)R_{e_i}^{-1}(y_i - H\hat{x}_{i|i-1})\end{aligned} \tag{22}$$

where

$$R_{e_i} = HP_iH^T + R \tag{23}$$

And substitute (23) into (22):

$$\begin{aligned}\hat{x}_{i+1|i} &= F\hat{x}_{i|i-1} + \underbrace{\left(FP_iH^T + GS\right)\left(HP_iH^T + R\right)^{-1}}_{=K_{p,i}}e_i \\ &= F\hat{x}_{i|i-1} + K_{p,i}\left(y_i - H\hat{x}_{i|i-1}\right)\end{aligned} \tag{24}$$

This recursion converges to the Riccati equation

$$P = FPF^T + G^TQG^T - K_pR_eK_p^T, \tag{25}$$

where

$$\begin{aligned}K_p &= (FPH^T + GS)R_e^{-1} \\ R_e &= HPH^T + R\end{aligned} \tag{26}$$

2 Variations

We derived a recursive formula for $\hat{x}_{i+1|i}$. Practically, we might not receive measurements y_i for every step i. The advantage of the Kalman filter here is its natural ability to perform updates even without measurements. We will present two sub-problems and their solution.

- **Time Update (Prediction step)**: Suppose we have an estimate $\hat{x}_{i|i}$ and its error covariance is $P_{i|i}$. To construct $\hat{x}_{i+1|i}, P_{i+1|i}$ (**without** observing y_i), we can compute

$$\hat{x}_{i+1|i} = F\hat{x}_{i|i} + G\hat{w}_{i|i} \tag{27}$$
$$\hat{w}_{i|i} = SR_{e_i}^{-1}e_i \tag{28}$$
$$P_{i+1|i} = FP_{i|i}F^T + G(Q - SR_{e_i}^{-1}S^T)G^T - FK_{p,i}S^T - GSK_{p,i}^TF^T \tag{29}$$

  If $S = 0$, this is what we expect by $x_{i+1} = Fx_i + Gw_i$.

- **Measurement Update**: Suppose that we have $\hat{x}_{i|i-1}, P_{i|i-1}$ and observe y_i. In other words, we refine our predictor without using a new measurement:

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}e_i \tag{30}$$
$$K_{f,i} = P_iH^TR_{e_i}^{-1} \tag{31}$$
$$P_i = P_{i|i-1} - P_{i|i-1}H^TR_{e_i}^{-1}HP_{i|i-1} \tag{32}$$

  The covariance is a Schur complement.

- **Smoothing Problem**: we wish to approximate $\hat{x}_{i|N}$, meaning we've observed all the steps and now wish to retroactively fix our predictions using the innovations approach.

$$\hat{x}_{i|N} = \sum_{j=1}^{N} \langle x_i, e_j \rangle R_{e_j}^{-1}e_j = \underbrace{\hat{x}_i}_{\text{KF predicted Solution}} + \underbrace{\sum_{j=i}^{N} \dots}_{\text{future observations}} \tag{33}$$

  And we get a sum of the Kalman filter and a smoothing term.

3 Example

4 System Setup

We consider an object moving in one dimension. The state of the system at any time step k is described by its position and velocity. The goal is to estimate these states using noisy measurements of the position.

4.1 State Vector

The state vector x_k at time step k is defined as:

$$x_k = \begin{bmatrix} \text{position}_k \\ \text{velocity}_k \end{bmatrix}$$

4.2 State Transition Model

The state transition model describes how the state evolves over time. Assuming constant velocity, the state transition is given by:

$$x_{k+1} = Ax_k + w_k$$

where

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

Here, Δt is the time step, and w_k is the process noise which accounts for uncertainties in the model, with covariance Q. Matlab code appears below.

4.3 Measurement Model

The measurement model relates the observed measurements to the state of the system. In this setup, we only measure the position of the object. The measurement equation is:

$$z_k = Hx_k + v_k$$

where

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

v_k represents the measurement noise with covariance R.

5 Stochastic LQR

In the LQR problem, we had a deterministic dynamical system

$$x_{t+1} = Ax_t + Bu_t.$$

In the stochastic case, there is an additive random vector that disturbs the system.

Definition 1 (Stochastic LQR)**.** The system is given by

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{34}$$

where $\mathbb{E}[w_i] = 0$, $\mathbb{E}[w_i w_j^T] = W\delta_{ij}$. The objective function is

$$\min \mathbb{E}_{w,x_0} \left[x_N^T Q_F x_N + \sum_{t=0}^{N-1} \left(x_t^T Q x_t + u_t^T R u_t \right) \right]. \tag{35}$$

As before, we define the value function and derive its recursive formula

$$V_t(z) = \min_{u_t,\ldots,u_{N-1}:x_t=z} \mathbb{E} \left[x_N^T Q_F x_N + \sum_{\tau=t}^{N-1} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) \right] \tag{36}$$

$$= z^T Q z + \min_v \left(v^T R v + \mathbb{E}\left[V_{t+1}(Az + Bv + w_\tau \right] \right) \tag{37}$$

Theorem 3. *For a zero-mean disturbance, the optimal controller in the stochastic setting is equal to the controller in the non-stochastic setting.*

We will prove the theorem via a technical lemma:

Lemma 1. *The value function is equal to*

$$V_t(z) = z^T P_t z + q_t \tag{38}$$

where P_t can be computed for $t = N-1, \ldots, 0$ as

$$P_t = A^T P_{t+1} A + Q - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A \tag{39}$$

with the initial condition $P_N = Q_F$. The additive term can be computed for $t = N-1, \ldots, 0$ as $q_t = q_{t+1} + \mathrm{Tr}(W P_{t+1})$ with the initial condition $q_N = 0$.

Remarks:

1. The Riccati recursion is the same as the one we has in the non-stochastic LQR solution.
2. The value function in the non-stochastic setting was $V_t^{NS}(z) = z^T P_t z$ but the above lemma shows now we have an additional additive term q_t.
3. Thus, we can write the optimal cost as the sum of the optimal cost in the non-stochastic setting and the sum $q_0 = \sum_{t=0}^{N-1} \mathrm{Tr}(W P_{t+1})$. As a sanity check, we see that if $W = 0$ the optimal costs are equal in both settings.

Proof of Lemma 1. We prove the assertion using a backwards induction.

For the base case of the induction, we trivially have

$$\begin{aligned} V_N(z) &= z^T Q_F z \\ &= z^T P_N z + q_N. \end{aligned} \tag{40}$$

For the induction step, we compute the value function at time t as a function of $x_t = z$

$$\begin{aligned} V_t(z) &= z^T Q z + \min_v \left(v^T R v + \mathbb{E}[V_{t+1}(Az + Bv + w_t] \right) \\ &\overset{(a)}{=} z^T Q z + \min_v \{ v^T R v + (Az + Bv)^T P_{t+1} (Az + Bv) \} + \mathrm{Tr}(W P_{t+1}) \\ &\overset{(b)}{=} z^T P_t z + \mathrm{Tr}(W P_{t+1}), \end{aligned} \tag{41}$$

where (a) follows from the induction hypothesis as

$$\begin{aligned} \mathbb{E}[V_{t+1}(Az + Bv + w_t)] &= \mathbb{E}[(Az + Bv + w_t)^T P_{t+1} (Az + Bv + w_t)] + q_{t+1} \\ &= (Az + Bv)^T P_{t+1} (Az + Bv) + \mathbb{E}[w_t^T P_{t+1} w_t] + q_{t+1} \\ &= (Az + Bv)^T P_{t+1} (Az + Bv) + \mathrm{Tr}(W P_{t+1}) + q_{t+1}, \end{aligned} \tag{42}$$

and (b) follows from the LQR optimization that have seen in the non-stochastic setting with $P_t = A^T P_{t+1} A + Q - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$ and $u_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$. □

6 Linear Quadratic Gaussian (LQG) Control

Definition 2 (LQG Control)**.** The linear-quadratic-Gaussian control concerns linear systems driven by additive white Gaussian noise. We are given a partially observable space state

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{43}$$
$$y_t = Cx_t + v_t, \tag{44}$$

where the sequence $(w_i, v_i) \sim \mathcal{N}(0, \begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix})$ is i.i.d.. The initial state is independent of sequences and is distributed according to $x_0 \sim \mathcal{N}(0, \Pi_0)$.

The objective is the same as the stochastic LQR problem

$$J(u) = \mathbb{E}\left[x_N^T Q_f x_N + \sum_{k=0}^{N-1} \begin{pmatrix} x_k & u_k \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}\right], \tag{45}$$

but the control signals $u_1, \ldots, u_{N-1}$ are chosen differently.

Definition 3. A controller for the LQG problem u_t is a function of $y_1, ..., y_t$.

This is one of the most important results in optimal control.

Theorem 4 (Separation Principal)**.** *The LQG problem can be solved optimally with two separate parts, which facilitate the design:*

1. *Estimate x_t based on Kalman Filter: given $(y_1, y_2, ...y_t)$ we construct the estimate $\hat{x}_t \triangleq \hat{x}_{t|t}$. Its error covariance is denoted by Σ_t.*
2. *Apply LQR control replacing the state with its estimate $\hat{x}_{t|t}$. That is, the controller is $u_t = -K_{LQR,t}\hat{x}_t$ is the state.*

This is a significant result in control theory since it shows that we solve a partially-observable control problem by the same control law we has on the best estimate. Indeed, it can be done in more general settings but then the estimate will be the optimal MMSE that is not as simple as the Kalman filter.

We provide a summary of the result

- The estimate at time t is $\hat{x}_t$ and can be computed recursively as

$$\begin{aligned} \hat{x}_t &= A\hat{x}_{t-1} + K_{f,t}(y_t - \hat{y}_t) \\ &= A\hat{x}_{t-1} + K_{f,t}(y_t - C(A\hat{x}_{t-1} + Bu_t)) \end{aligned} \tag{46}$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} \cdot C^T (C\Sigma_{t|t-1}C^T + v)^{-1} C\Sigma_{t|t-1}$$
$$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W$$

- The optimal controller in the LQR setting:

$$u_t = -(B^T P_{t+1} B + R)^{-1} B^T P_{t+1} A \hat{x}_t$$

$$P_t = \begin{cases} Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P t + 1 B)^{-1} B^T P_{t+1} A & t < N \\ Q_f & t = N \end{cases}$$

We will not provide a full proof for this theorem The derivation So, we will calculate the following:

$$\begin{aligned} \mathbb{E}[x_k^T Q x_k] &= \mathbb{E}[(\tilde{x}_k + \hat{x}_k)^T Q (\tilde{x}_k + \hat{x}_k)] && (47) \\ &= \mathbb{E}[\tilde{x}_k^T Q \tilde{x}_k] + \mathbb{E}[\hat{x}_k^T Q \hat{x}_k] + \underbrace{2\mathbb{E}[\tilde{x}_k^T Q \hat{x}_k]}_{=0 \quad \text{(orthogonality)}} && (48) \\ &= \mathrm{Tr}[Q \Sigma_k] + \mathbb{E}[\hat{x}_k^T Q \hat{x}_k]. && (49) \end{aligned}$$

The term $\Sigma_k \triangleq \mathbb{E}[\tilde{x}_k \tilde{x}_k^T]$ corresponds to the estimation error. We have seen in the previous lecture that the estimation error does not depend on the control u_t as long as it is known to the controller/estimator. In other words, if we perform Kalman filtering the only term that depends on our choice of the controller is $\mathbb{E}[\hat{x}_k^T Q \hat{x}_k]$.

So, the cost is:

$$J(u) = \mathrm{Tr}(Q_f \Sigma_N) + \sum_{k=0}^{N-1} \mathrm{Tr}(Q \Sigma_k) + \mathbb{E}[\sum_{k=0}^{N-1} (\hat{x}_k^T Q \hat{x}_k + u_k^T R u_k)] + \mathbb{E}[\underbrace{\hat{x}_N^T Q_f \hat{x_N}}_{=0}]$$