Lecture 4

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This lecture will be the last on linear dynamical systems and control. We'll discuss:

1. Recap - Stochastic Least Squares (LMS)
2. The Kalman filter Model
3. The innovations process
4. Stochastic LQR
5. LQG - where the states can not be observed directly.

1 Review of LMS

Definition 1 (LMS Problem)**.** Given two zero-mean random variables X, Y with known second moments $R_X = \mathbb{E}[XX^T]$, $R_Y = \mathbb{E}[YY^T]$ and $R_{XY} = \mathbb{E}[XY^T]$, find the linear estimator:

$$\hat{x} = \text{argmin}_{\hat{X}}\, \mathbb{E}[(X - \hat{X}(Y))^2]. \tag{1}$$

In the last lecture we saw that the estimator is given by

$$\hat{X} = R_{XY} R_Y^{-1} Y \tag{2}$$

with optimal mean squared error:

$$\text{MSE}(\hat{X}) = R_X - R_{XY} R_Y^{-1} R_{YX} \tag{3}$$

which happens to be Schur's complement [1] of the covariance matrix:

$$\begin{bmatrix} R_X & R_{XY} \\ R_{XY}^T & R_Y \end{bmatrix}$$

Another way to view this solution is by defining the inner product between two random variables to be:

$$\langle X, Y \rangle = \mathbb{E}\left[XY^T\right] \tag{4}$$

In our case this results as $\langle X, Y \rangle = R_{XY}$. Substituting into our definition of $\hat{x}$ we get:

$$\hat{x} = R_{XY} R_Y^{-1} Y = \langle X, Y \rangle \langle Y, Y \rangle^{-1} Y$$

[1] https://en.wikipedia.org/wiki/Schur_complement

And this is the projection of X onto the linear sub-space spanned by Y.

In general, if we have more than one sample of Y, for example $Y = (Y_1, Y_2)$ then the optimal predictor can be written as the sum of predictors based on each simple independently if the samples are independent. In mathematical notation:

$$\hat{x}_{|Y_1,Y_2} = \hat{x}_{|Y_1} + \hat{x}_{|Y_2} \Leftrightarrow \langle Y_1, Y_2 \rangle = 0. \tag{5}$$

Proof.

$$\begin{aligned}
\hat{x}_{|Y_1,Y_2} &= R_{XY} \begin{pmatrix} R_{Y_1} & R_{Y_1Y_2} \\ R_{Y_2Y_1} & R_{Y_2} \end{pmatrix}^{-1} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \begin{pmatrix} R_{XY_1} & R_{XY_2} \end{pmatrix} \\
&\cdot \left(\begin{pmatrix} I & R_{Y_1Y_2}R_{Y_2}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} R_{Y_1} - R_{Y_1Y_2}R_{Y_2}^{-1}R_{Y_2Y_1} & 0 \\ 0 & R_{Y_2} \end{pmatrix} \begin{pmatrix} I & 0 \\ R_{Y_2}^{-1}R_{Y_2Y_1} & I \end{pmatrix} \right)^{-1} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \begin{pmatrix} R_{XY_1} & R_{XY_2} \end{pmatrix} \\
&\cdot \begin{pmatrix} I & 0 \\ -R_{Y_2}^{-1}R_{Y_2Y_1} & I \end{pmatrix} \begin{pmatrix} (R_{Y_1} - R_{Y_1Y_2}R_{Y_2}^{-1}R_{Y_2Y_1})^{-1} & 0 \\ 0 & R_{Y_2}^{-1} \end{pmatrix} \begin{pmatrix} I & -R_{Y_1Y_2}R_{Y_2}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}
\end{aligned} \tag{6}$$

□

2 The Kalman Filter

Recall the definition of the kalman model we saw in the previous lecture:

$$x_{i+1} = Fx_i + Gw_i \tag{7}$$

$$y_i = Hx_i + v_i \tag{8}$$

Where x_{i+1} is the state at time $i+1$ (which is hidden) and y_i is the observed state at time i, and for v_i, w_i zero meaned and uncorrelated (between themselves and w.r.t previous time-steps) random variables, meaning:

$$\mathbb{E}[\begin{pmatrix} w_i \\ v_i \\ x_0 \end{pmatrix} \begin{pmatrix} w_j^T & v_j^T & x_0^T & 1 \end{pmatrix}] = \begin{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \delta_{ij} & 0 & 0 \\ 0 & \Pi_0 & 0 \end{pmatrix}, \tag{9}$$

For $\forall i : Q = cov(w_i)$, $R = cov(v_i)$ and $\Pi_i = cov(x_0)$ such that $\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \succeq 0$ and $R \succ 0$.

The objective, is to find a linear predictor $\hat{x}_{i+1|i}$ based on the measurements $y_1, ..., y_i$ such that the error is minimal:

$$\mathbb{E}\left[(x_{i+1} - \hat{x}_{i+1})^T (x_{i+1} - \hat{x}_{i+1}) \right]. \tag{10}$$

As previously mentioned, when predicting $\hat{x}_{i+1}$ we'll assume we've observed y_1 up to y_i. This special case is denoted by $\hat{x}_{i+1|i}$, but in general we can denote an estimator $\hat{x}_{i|j}$ meaning that the estimate of x_i is based on the measurements tuple $y_0, ..., y_j$.

We begin with a simple example of predicting $\hat{x}_1$ based on a single measurement. From the solution to the LMS in (2), we have

$$\begin{aligned}
\hat{x}_{1|0} &= \langle x_1, y_0 \rangle \langle y_0, y_0 \rangle^{-1} y_0 \\
&= \langle Fx_0 + Gw_0, Hx_0 + v_0 \rangle \left(H\Pi_0 H^T + R\right)^{-1} y_0 \\
&= \left(F\Pi_0 H^T + F \underbrace{\langle x_0, v_0 \rangle}_{=0} + G \underbrace{\langle x_0, w_0 \rangle}_{=0} H^T + G \underbrace{\langle w_0, v_0 \rangle}_{=S} \right) \left(H\Pi_0 H^T + R\right)^{-1} y_0 \\
&= \underbrace{\left(F\Pi_0 H^T + GS\right) \left(H\Pi_0 H^T + R\right)^{-1}}_{=K_{p,0}} y_0, \qquad (11)
\end{aligned}$$

where all $= 0$ follow from the variables being uncorrelated, and $K_{p,0}$ is based on our definition from Lecture 3.

The natural question that arises after this example is how we can combine several measurements. For instance, at $i = 1$, we have y_0 and y_1. In the following, we present a simple method that is based on orthogonalization of the measurements process.

2.1 Innovations Process

Intuitively, since at time $i = 2$ we have y_1, y_2, we would like to use (5) in order to describe $\hat{x}_2$. The problem here is that y_1, y_2 **aren't** orthogonal.

To solve this issue we'll propose a causal and invertible mapping L from $y = \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_i \end{pmatrix}$ to some variable $y = Le$ s.t.:

$$e = \begin{pmatrix} e_1 \\ e_2 \\ ... \\ e_i \end{pmatrix} \qquad (12)$$

is a white process, meaning $\langle e_i, e_j \rangle = R_i \delta_{ij}$.

Notice that in order for e_i to be a causal function of the measurements, L should be a lower triangular matrix. In other words, e_i is a function of y_j for $j \leq i$. Also, if L is invertible we will not lose information. Once we have a white process we can use (5). We call this the "innovations process" since we extract the innovative part at each step.

Now, we'll explicitly define

$$\begin{aligned}
e_i &= y_i - \hat{y}_{i|i-1} \\
&= y_i - \mathbb{E}[Hx_i + v_i | y^{i-1}]
\end{aligned}$$

$$
\begin{aligned}
&= Hx_i + v_i - H\hat{x}_i \\
&= H\underbrace{(x_i - \hat{x}_i)}_{:=\tilde{x}_i} + v_i
\end{aligned} \tag{13}
$$

and show this is indeed (the unique) innovation process (causal, causally invertible and decorrelated).

Theorem 1. *The innovations process* $\{e_i\}$ *in* (13) *is white. That is, we have* $\mathbb{E}\left[e_i e_j^T\right] = R_{e,i}\delta_{ij}$.

Proof. By definition of e_i:

$$
\begin{aligned}
\mathbb{E}\left[e_i e_j^T\right] &= \mathbb{E}\left[(H\tilde{x}_i + v_i)(H\tilde{x}_j + v_j)^T\right] \\
&= H\,\mathbb{E}\left[\tilde{x}_i\tilde{x}_j^T\right]H^T + \mathbb{E}\left[v_i v_j^T\right] + H\,\mathbb{E}\left[\tilde{x}_i v_j^T\right] + \mathbb{E}\left[v_i\tilde{x}_j^T\right]H^T.
\end{aligned} \tag{14}
$$

For $i = j$, we obtain directly

$$
\mathbb{E}\left[e_i e_i^T\right] = HP_iH^T + R. \tag{15}
$$

For $i > j$, we have

$$
\mathbb{E}\left[e_i e_j^T\right] = H\,\mathbb{E}\left[\tilde{x}_i\tilde{x}_j^T\right]H^T + H\,\mathbb{E}\left[\tilde{x}_i v_j^T\right], \tag{16}
$$

where $\mathbb{E}\left[v_i v_j^T\right] = 0$ since v_i is a white process. The term $\mathbb{E}\left[v_i\tilde{x}_j^T\right] = 0$ since $\tilde{x}_j$ depends only on $v^j = (v_1, ..., v_j)$ and $w^j = (w_1, ..., w_j)$, and we have by definition $v_i \perp w_j$ and $v_i \perp v_j$ for $i \neq j$.

We proceed to develop the error as a recursion. In the last lecture, we derived

$$
\begin{aligned}
\tilde{x}_{i+1} &= x_{i+1} - \hat{x}_{i+1} \\
&= (F - K_{p,i}H)\tilde{x}_i + (G - K_{p,i})\begin{pmatrix} w_i \\ v_i \end{pmatrix}.
\end{aligned} \tag{17}
$$

Since we assume $j < i$, the recursion can be evolved as

$$
\begin{aligned}
\tilde{x}_i &= (F - K_{p,i-1}H)\tilde{x}_{i-1} + (G - K_{p,i-1})\begin{pmatrix} w_{i-1} \\ v_{i-1} \end{pmatrix} \\
&= \vdots \\
&= \phi_p(i,j)\tilde{x}_j + \xi_i(j).
\end{aligned} \tag{18}
$$

where

$$
\phi_p(i,j) = \prod_{k=j}^{i-1}(F - K_{p,k}H),
$$

and

$$
\xi_i(j) = \sum_{k=j}^{i-1}\phi(i,k+1)\left(Gw_k - K_{p,k}v_k\right). \tag{19}
$$

We can now substitute the recursion into the second term of the innovation:

$$\begin{aligned}
\mathbb{E}\left[\tilde{x}_i v_j^T\right] &= \mathbb{E}\left[(\phi_p(i,j)\tilde{x}_j + \xi_i(j))v_j^T\right] \\
&\overset{(a)}{=} \mathbb{E}\left[\xi_i(j)v_j^T\right] \\
&= \mathbb{E}\left[\sum_{k=j}^{i-1} \phi(i,k+1)\left(Gw_k - K_{p,k}v_k\right) v_j^T\right] \\
&\overset{(b)}{=} \mathbb{E}\left[\phi_p(i,j+1)(Gw_j - K_{p,j}v_j)v_j^T\right] \\
&= \phi_p(i,j+1)(GS - K_{p,j}R), \qquad (20)
\end{aligned}$$

where (a) follows from the fact that $\tilde{x}_j$ is a function of $(x_0, w_0, v_0, ..., w_{j-1}, v_{j-1})$ and v_j is uncorrelated with these variables. Step (b) follows from the fact that v_j is uncorrelated with all future indices as well. Finally, (c) follows from the definition of S, R.

We proceed to derive the first term of the innovation

$$\begin{aligned}
\mathbb{E}\left[\tilde{x}_i \tilde{x}_j^T\right] &= \mathbb{E}\left[(\phi_p(i,j)\tilde{x}_j + \xi_i(j))\tilde{x}_j^T\right] \\
&\overset{(a)}{=} \mathbb{E}[\phi_p(i,j)\tilde{x}_j\tilde{x}_j^T] \\
&\overset{(b)}{=} \phi_p(i,j)P_j, \qquad (21)
\end{aligned}$$

where

- (a) follows from the fact that $\xi_i(j)$ depends on times $j, \ldots, i-1$ and $\tilde{x}_j$ depends on the disturbances at times $0, ..., j-1$
- (b) follows from the definition $P_j = \mathbb{E}\left[\tilde{x}_j \tilde{x}_j^T\right]$.

Combining the two terms, we obtain

$$\begin{aligned}
\mathbb{E}\left[e_i e_j^T\right] &= H\,\mathbb{E}\left[\tilde{x}_i\tilde{x}_j^T\right] H^T + H\,\mathbb{E}\left[\tilde{x}_i v_j^T\right] \\
&= H \underbrace{\phi_p(i,j)}_{=\phi_p(i,j+1)F_{p,j}} P_j H^T + H\phi_p(i,j+1)(GS - K_{p,j}R) \\
&= H\phi_p(i,j+1)(\underbrace{F_{p,j}}_{=F-K_{p,j}H} P_j H^T + GS - K_{p,j}R) \\
&= H\phi_p(i,j+1)\left(FP_jH^T + GS - K_{p,j}(HP_jH^T + R)\right), \qquad (22)
\end{aligned}$$

where we used the notation $F_{p,k} := F - K_{p,k}H$.

Recall that $K_{p,j} = (FP_jH^T + GS)(HP_jH^T + R)^{-1}$, substituting into the last equation results in:

$$FP_jH^T + GS - K_{p,j}(HP_jH^T + R) = 0. \qquad (23)$$

To conclude, we proved that for $i < j$ $\mathbb{E}\left[e_i e_j^T\right] = 0$. For $i > j$ it follows similarly so it can be concluded that the innovations process is white. $\square$

From the innovation process, we can write a new state-space model

$$\begin{aligned}\hat{x}_{i+1} &= F\hat{x}_i + K_{p,i}e_i \\ y_i &= H\hat{x}_i + e_i,\end{aligned} \tag{24}$$

with $\hat{x}_1 = 0$.

From the above state-space, we can find explicitly

$$\begin{aligned}y_1 &= e_1 \\ y_2 &= H\hat{x}_2 + e_2 \\ &= H(F\hat{x}_1 + K_{p,1}e_1) + e_2 \\ &= HK_{p,1}e_1 + e_2 \\ y_3 &= H(F\hat{x}_2 + K_{p,2}e_2) + e_3 \\ &= HF(F\hat{x}_1 + K_{p,1}e_1) + HK_{p,2}e_2 + e_3 \\ &= HFK_{p,1}e_1 + HK_{p,2}e_2 + e_3\end{aligned} \tag{25}$$

The overall mapping is $y = Le$

$$L = \begin{pmatrix} I & 0 & \dots & 0 \\ HK_{p,0} & I & \dots & 0 \\ H\Phi(2,1)K_{p,0} & HK_{p,1} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ H\Phi(N,1)K_{p,0} & H\Phi(N,2)K_{p,1} & \dots & I \end{pmatrix}, \tag{26}$$

where $\Phi(j,i) = F^{j-i}$ (This quantity is more relevant in time-varying settings where F depends on time).

Important note: the mapping L is a causal (lower triangular) and invertible (non-zero diagonal). Thus, we can use the innovations process instead of the measurements process.

We can write R_Y as

$$\begin{aligned}y &= Le \\ \Rightarrow R_Y &= \mathbb{E}\left[Le(Le)^T\right] \\ &\overset{(a)}{=} L\underbrace{\mathbb{E}\left[ee^T\right]}_{:=R_e}L^T\end{aligned} \tag{27}$$

where (a) follows from...

Since the inverse of a lower triangular matrix is also lower triangular, we get that L^{-1} is lower triangular (L^{-T} is upper triangular). Overall,

$$R_Y = LR_{\mathbf{e}}L^T \tag{28}$$

is an LDU factorization of R_Y. Since the diagonal of L is the identity matrix, this factorization is unique meaning that our defined innovations process is the unique white process that satisfies this property.

Uniqueness: Consider $M = L_1 L_1^T = L_2 L_2^T$ and define $Q = L_2^{-1} L_1$. Then, $M = L_2 Q Q^T L_2^T$ but also $M = L_2 L_2^T$ meaning that $QQ^T = I$. Since Q is lower triangular, it follows that Q is diagonal with ± 1. If we restrict $Q = I$, then $L_1 = L_2$.

2.2 Solving the Kalman Filter

Now that we have the innovations process, we can use (5) to write the predictor

$$\begin{aligned}
\hat{x}_{i+1|i} &= \sum_{j=0}^{i} \langle x_{i+1}, e_j \rangle R_{e_j}^{-1} e_j \\
&= \sum_{j=0}^{i-1} \langle x_{i+1}, e_j \rangle R_{e_j}^{-1} e_j + \langle x_{i+1}, e_i \rangle R_{e_i}^{-1} e_i \\
&= \sum_{j=0}^{i-1} \langle F x_i + G w_i, e_j \rangle R_{e_j}^{-1} e_j + \langle x_{i+1}, e_i \rangle R_{e_i}^{-1} e_i \\
&\overset{(a)}{=} F \hat{x}_{i|i-1} + \langle x_{i+1}, e_i \rangle R_{e_i}^{-1} e_i,
\end{aligned} \tag{29}$$

where (a) follows from the fact that $< w_i, e_j >= 0$ for $j < i$.

The second term can be computed as

$$\begin{aligned}
\langle x_{i+1}, e_i \rangle = \langle F x_i + G w_i, e_i \rangle &= F \langle x_i, e_i \rangle + G \langle w_i, e_i \rangle \\
&= F P_i H^T + GS,
\end{aligned} \tag{30}$$

where the first term is derived from

$$\begin{aligned}
\langle x_i, e_i \rangle &= \langle x_i - \hat{x}_i + \hat{x}_i, e_i \rangle \\
&= \langle \tilde{x}_i, e_i \rangle + \underbrace{\langle \hat{x}_i, e_i \rangle}_{=0} \\
&= \langle \tilde{x}_i, e_i \rangle \\
&= \langle \tilde{x}_i, H \tilde{x}_i + v_i \rangle \\
&= P_i H^T
\end{aligned} \tag{31}$$

and for the second term we used

$$\langle w_i, e_i \rangle = \langle w_i, H \tilde{x}_i + v_i \rangle = 0 + \langle w_i, v_i \rangle = S \tag{32}$$

Putting it all together:

$$\begin{aligned}
\hat{x}_{i+1|i} &= F \hat{x}_{i|i-1} + \left(F P_i H^T + GS \right) R_{e_i}^{-1} e_i \\
&= F \hat{x}_{i|i-1} + \left(F P_i H^T + GS \right) R_{e_i}^{-1} (y_i - H \hat{x}_{i|i-1})
\end{aligned} \tag{33}$$

where

$$R_{e_i} = HP_iH^T + R \tag{34}$$

And substitute (34) into (33):

$$\begin{aligned}
\hat{x}_{i+1|i} &= F\hat{x}_{i|i-1} + \underbrace{\left(FP_iH^T + GS\right)\left(HP_iH^T + R\right)^{-1}}_{=K_{p,i}} e_i \\
&= F\hat{x}_{i|i-1} + K_{p,i}\left(y_i - H\hat{x}_{i|i-1}\right)
\end{aligned} \tag{35}$$

We can also use the above to calculate P_{i+1}

$$P_{i+1} = FP_iF^T + G^TQG^T - K_{p,i}R_{e_i}K_{p,i}^T \tag{36}$$

with $P_1 = \Pi_0$.

This recursion converges to the Riccati equation

$$P = FPF^T + G^TQG^T - K_pR_eK_p^T, \tag{37}$$

where

$$\begin{aligned}
K_p &= (FPH^T + GS)R_e^{-1} \\
R_e &= HPH^T + R
\end{aligned} \tag{38}$$

3 Variations

We derived a recursive formula for $\hat{x}_{i+1|i}$. Practically, we might not receive measurements y_i for every step i. The advantage of the Kalman filter here is its natural ability to perform updates even without measurements. We will present two sub-problems and their solution.

- **Time Update**: Suppose we have and estimate $\hat{x}_{i|i}$ and its error covariance is P_i. To construct $\hat{x}_{i+1|i}, P_{i+1}$ (**without** observing y_i), we can compute

$$\hat{x}_{i+1|i} = F\hat{x}_{i|i} + G\hat{w}_{i|i} \tag{39}$$

$$\hat{w}_{i|i} = SR_{e_i}^{-1}e_i \tag{40}$$

$$P_{i+1} = FP_iF^T + G(Q - SR_{e_i}^{-1}S^T)G^T - FK_{p,i}S^T - GSK_{p,i}^TF^T \tag{41}$$

- **Measurement Update**: Suppose that we have $\hat{x}_{i|i-1}, P_{i|i-1}$ and observe y_i. In other words, we refine our predictor without using a new measurement:

$$\hat{x}_{i|i} = \hat{x}_{i|i-1} + K_{f,i}e_i \tag{42}$$

$$K_{f,i} = P_iH^TR_{e_i}^{-1} \tag{43}$$

$$P_i = P_{i|i-1} - P_{i|i-1}H^TR_{e_i}^{-1}HP_{i|i-1} \tag{44}$$

- **Smoothing Problem**: we wish to approximate $\hat{x}_{i|N}$, meaning we've observed all the steps and now wish to retroactively fix our predictions using the innovations approach.

$$\hat{x}_{i|N} = \sum_{j=1}^{N} \langle x_i, e_j \rangle R_{e_j}^{-1} e_j = \underbrace{\hat{x}_i}_{\text{KF predicted Solution}} + \underbrace{\sum_{j=i}^{N} \dots}_{\text{future observations}} \tag{45}$$

  And we get a sum of the Kalman filter and a smoothing term.

4 Example

5 System Setup

We consider an object moving in one dimension. The state of the system at any time step k is described by its position and velocity. The goal is to estimate these states using noisy measurements of the position.

5.1 State Vector

The state vector x_k at time step k is defined as:

$$x_k = \begin{bmatrix} \text{position}_k \\ \text{velocity}_k \end{bmatrix}$$

5.2 State Transition Model

The state transition model describes how the state evolves over time. Assuming constant velocity, the state transition is given by:

$$x_{k+1} = Ax_k + w_k$$

where

$$A = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$

Here, Δt is the time step, and w_k is the process noise which accounts for uncertainties in the model, with covariance Q. Matlab code appears below.

5.3 Measurement Model

The measurement model relates the observed measurements to the state of the system. In this setup, we only measure the position of the object. The measurement equation is:

$$z_k = Hx_k + v_k$$

where

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

v_k represents the measurement noise with covariance R.

6 Stochastic LQR

In the LQR problem, we had a deterministic dynamical system

$$x_{t+1} = Ax_t + Bu_t.$$

In the stochastic case, there is an additive random vector that disturbs the system.

Definition 2 (Stochastic LQR)**.** The system is given by

$$x_{t+1} = Ax_t + Bu_t + w_t, \tag{46}$$

where $\mathbb{E}[w_i] = 0$, $\mathbb{E}[w_i w_j^T] = W\delta_{ij}$. The objective function is

$$\min \mathbb{E}_{w,x_0} \left[x_N^T Q_F x_N + \sum_{t=0}^{N-1} \left(x_t^T Q x_t + u_t^T R u_t \right) \right]. \tag{47}$$

As before, we define the value function and derive its recursive formula

$$V_t(z) = \min_{u_t,\ldots,u_{N-1}:x_t=z} \mathbb{E} \left[x_N^T Q_F x_N + \sum_{\tau=t}^{N-1} \left(x_\tau^T Q x_\tau + u_\tau^T R u_\tau \right) \right] \tag{48}$$

$$= z^T Q z + \min_v \left(v^T R v + \mathbb{E}\left[V_{t+1}(Az + Bv + w_\tau \right] \right) \tag{49}$$

Theorem 2. *For a zero-mean disturbance, the optimal controller in the stochastic setting is equal to the controller in the non-stochastic setting.*

We will prove the theorem via the following technical lemma.

Lemma 1. *The value function is equal to*

$$V_t(z) = z^T P_t z + q_t \tag{50}$$

where P_t can be computed for $t = N-1, \ldots, 0$ as

$$P_t = A^T P_{t+1} A + Q - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A \tag{51}$$

with the initial condition $P_N = Q_F$. The additive term can be computed for $t = N-1, \ldots, 0$ as $q_t = q_{t+1} + \mathrm{Tr}(WP_{t+1})$ with the initial condition $q_N = 0$.

Remarks:

1. The Riccati recursion is the same as the one we has in the non-stochastic LQR solution.
2. The value function in the non-stochastic setting was $V_t^{NS}(z) = z^T P_t z$ but the above lemma shows now we have an additional additive term q_t.
3. Thus, we can write the optimal cost as the sum of the optimal cost in the non-stochastic setting and the sum $q_0 = \sum_{t=0}^{N-1} \mathrm{Tr}(WP_{t+1})$. As a sanity check, we see that if $W = 0$ the optimal costs are equal in both settings.

Proof of Lemma 1. We prove the assertion using a backwards induction.

For the base case of the induction, we trivially have

$$\begin{aligned} V_N(z) &= z^T Q_F z \\ &= z^T P_N z + q_N. \end{aligned} \tag{52}$$

For the induction step, we compute the value function at time t as a function of $x_t = z$

$$\begin{aligned} V_t(z) &= z^T Q z + \min_v \left(v^T R v + \mathbb{E}[V_{t+1}(Az + Bv + w_t]\right) \\ &\overset{(a)}{=} z^T Q z + \min_v \{ v^T R v + (Az+Bv)^T P_{t+1}(Az+Bv)\} + \mathrm{Tr}(W P_{t+1}) \\ &\overset{(b)}{=} z^T P_t z + \mathrm{Tr}(W P_{t+1}), \end{aligned} \tag{53}$$

where (a) follows from the induction hypothesis as

$$\begin{aligned} \mathbb{E}[V_{t+1}(Az+Bv+w_t)] &= \mathbb{E}[(Az+Bv+w_t)^T P_{t+1}(Az+Bv+w_t)] + q_{t+1} \\ &= (Az+Bv)^T P_{t+1}(Az+Bv) + \mathbb{E}[w_t^T P_{t+1} w_t] + q_{t+1} \\ &= (Az+Bv)^T P_{t+1}(Az+Bv) + \mathrm{Tr}(W P_{t+1}) + q_{t+1}, \end{aligned} \tag{54}$$

and (b) follows from the LQR optimization that have seen in the non-stochastic setting with $P_t = A^T P_{t+1} A + Q - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$ and $u_t = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$. $\square$

7 Linear Quadratic Gaussian (LQG) Control

Definition 3 (LQG Control)**.** The linear-quadratic-Gaussian control concerns linear systems driven by additive white Gaussian noise. We are given a partially observable space state

$$x_{t+1} = Ax_t + Bu_t + w_t \tag{55}$$

$$y_t = Cx_t + v_t, \tag{56}$$

where the pair $(w_i, v_i) \sim \mathcal{N}(0, \begin{pmatrix} W & 0 \\ 0 & V \end{pmatrix})$ is an i.i.d. and zero-mean process. The initial state is independent of sequences and is distributed according to $x_0 \sim \mathcal{N}(0, \Pi_0)$.

The objective is the same as the stochastic LQR problem

$$J(u) = \mathbb{E}\left[x_N^T Q_f x_N + \sum_{k=0}^{N-1} \begin{pmatrix} x_k & u_k \end{pmatrix} \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} \right], \tag{57}$$

but the control signals $u_1, \ldots, u_{N-1}$ are only causal functions of the measurements. That is, the control signal u_t is a function of $y_1, ..., y_t$.

Theorem 3 (Separation Principal)**.** *The LQG problem can be solved optimally with two separate parts, which facilitate the design:*

1. *Estimate x_t based on Kalman Filter: given $(y_1, y_2, ...y_t)$ we construct the estimate $\hat{x}_t \triangleq \hat{x}_{t|t}$. Its error covariance is denoted by Σ_t.*
2. *Apply LQR control replacing the state with its estimate $\hat{x}_{t|t}$. That is, the controller is $u_t = -K_{LQR,t}\hat{x}_t$ is the state.*

This is a significant result in control theory since it shows that we solve a partially-observable control problem by estimating . Indeed, it can be done in more general settings but then the estimate will be the optimal MMSE that is not as simple as the Kalman filter.

We provide a summary of the result

- The estimate at time t is $\hat{x}_t$ and can be computed recursively as

$$\hat{x}_t = A\hat{x}_{t-1} + K_{f,t}(y_t - \hat{y}_t) \tag{58}$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} \cdot C^T(C\Sigma_{t|t-1}C^T + v)^{-1}C\Sigma_{t|t-1}$$

$$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W$$

- The optimal controller in the LQR setting:

$$u_t = -(B^T P_{t+1} B + R)^{-1} B^T P_{t+1} A\hat{x}_t$$

$$P_t = \begin{cases} Q + A^T P_{t+1} A - A^T P_{t+1} B(R + B^T Pt + 1B)^{-1} B^T P_{t+1} A & t < N \\ Q_f & t = N \end{cases}$$

We will not provide a full proof for this theorem The derivation So, we will calculate the following:

$$\mathbb{E}[x_k^T Q x_k] = \mathbb{E}[(\tilde{x}_k + \hat{x}_k)^T Q(\tilde{x}_k + \hat{x}_k)] \tag{59}$$

$$= \mathbb{E}[\tilde{x}_k^T Q\tilde{x}_k] + \mathbb{E}[\hat{x}_k^T Q\hat{x}_k] + \underbrace{2\mathbb{E}[\tilde{x}_k^T Q\hat{x}_k]}_{=0 \quad \text{(orthogonality)}} \tag{60}$$

$$= \mathrm{Tr}[Q\Sigma_k] + \mathbb{E}[\hat{x}_k^T Q\hat{x}_k]. \tag{61}$$

The term $\Sigma_k \triangleq \mathbb{E}[\tilde{x}_k\tilde{x}_k^T]$ corresponds to the estimation error. We have seen in the previous lecture that the estimation error does not depend on the control u_t as long as it is known to the controller/estimator. In other words, if we perform Kalman filtering the only term that depends on our choice of the controller is $\mathbb{E}[\hat{x}_k^T Q\hat{x}_k]$.

So, the cost is:

$$J(u) = \mathrm{Tr}(Q_f\Sigma_N) + \sum_{k=0}^{N-1} \mathrm{Tr}(Q\Sigma_k) + \mathbb{E}[\sum_{k=0}^{N-1}(\hat{x}_k^T Q\hat{x}_k + u_k^T Ru_k)] + \mathbb{E}[\underbrace{\hat{x}_N^T Q_f \hat{x_N}}_{=0}]$$

7.1 Summary