LINEAR ALGEBRA

Assignment #02

Prepared by: Idrees Ahmed Ghazi [FA20-BCS-025]
Prepared for: Dr. Umair Umer

Determinants and their Properties

Determinants:

A determinant is a scalar value that is a fuction of the entries in a square matrix.

Formula for calculating a 2x2 matrix's determint

If a, b, c, d are the entries of a square matrix, then it determinant will be

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{Findig determinant}} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d - b.c$$

Formula for calculating determinant of 3x3 matrix

If a, b, c, d, e, f, g, h, i are entries if a square matrix then

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Longrightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & h \end{vmatrix}$$

$$|A| = a(eh - fh) - b(dh - fg) + c(dh - eg)$$

Properties of matrix

(i) Reflection Property:

The determinant of the matrix will be the same if the rows and columns are transformed into columns and rows. I.e. Taking transpose of matrix.

$$\begin{bmatrix} 3 & 2 \\ 6 & 8 \end{bmatrix} \Rightarrow \begin{vmatrix} 3 & 2 \\ 6 & 8 \end{vmatrix} = 24 - 12 = 12$$

Now taking transpose

$$\begin{bmatrix} 3 & 6 \\ 2 & 8 \end{bmatrix} \Rightarrow \begin{vmatrix} 3 & 6 \\ 2 & 8 \end{vmatrix} = 24 - 12 = 12$$

The determinant of the matrix was same even by taking transpose

(ii) Switching property:

If we switch or exchange any two rows or column of a matrix then the determinant of matrix will change its sign.

$$\begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \Rightarrow \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix} = 16 - 24 = -8$$

Interchange C_1 & C_2.

$$\begin{bmatrix} 4 & 2 \\ 8 & 6 \end{bmatrix} \Rightarrow \begin{vmatrix} 4 & 2 \\ 8 & 6 \end{vmatrix} = 24 - 16 = 8$$

(iii) Proportionality Property:

If all rows and column are identical to all elements of some other row or column the determinent will be zero e.g.

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 5 & 10 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 5 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 2(0) - 3(0) - 0$$

$$\boxed{|A| = 0}$$

(iv) ALL - Zero Property:

If every element of a row or column of a matrix is zero then determinant is zero.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix} \Rightarrow \begin{vmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix} = 0 - 0 + 2(0)$$

$$\boxed{|A| = 0}$$

(v) Scaler multiple property:

If every element of a row or column is multiplied by a non-zero constant then the determinant gets multiplied by the same constant.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(45-48) - 2(36-42) + 3(24-35)$$

$$= -3 + 12 - 27$$

$$= -18$$

$$|2A| = \begin{vmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{vmatrix} = -36 = (-18).(2)$$

$$= 2.|A|$$

(vi) Sum Property:

If elements of row or column are expressed as sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

e.g.

$$\begin{vmatrix} 2+3 & 2 \\ 3+4 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = (6-6) + 9-8$$

$$15 - 14 = 1$$

$$1 = 1$$

LHS = RHS.

(vii) Factor Property:

If a determinant becomes 0 while considering the value of $x = \alpha$, then $(x-\alpha)$ is considered as a factor of determinant.

(viii) Property of invariance:

If each element of row and column of a determinant is added with equimultiples of elements of another row or column of a determinant, then the value of determinant remains unchanged.

If $|A| = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix}$ and $|B| = \begin{vmatrix} 2 & 4 \\ 6 & 8 \end{vmatrix}$.

$$|B| = \begin{vmatrix} 2+2(6) & 4+2(8) \\ 6 & 8 \end{vmatrix}$$

$$|A| = -8 \quad ; \quad |B| = \begin{vmatrix} 14 & 20 \\ 6 & 8 \end{vmatrix}$$

$|A| = -8$

$|B| = -8$, So the property is proved.

(ix) Triangular Property:

The determinant will be equal to the product of diagnol entries of an upper/lower diagnol matrix.

$$|A| = \begin{vmatrix} 11 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 7 \end{vmatrix} = 11(21) - 0 - 0$$

$$= 231 = 11 \times 3 \times 7$$

Hence the property is proved.

(x) Co-factor of matrix property:

The cofactor of an element a_{ij} is denoted by A_{ij} and it is defined by $A_{ij} = (-1)^{i+j} \times M_{ij}$ where M_{ij} is the minor of a_{ij}.

e.g.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Minor of a_{11} : $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

Cofactor of a_{11} : $A_{11} = (-1)^{1+1} M_{11}$

$$= (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$ Proved