Mathematical Modelling, Project 1

Group F

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1 Problem Statement

"A road must be constructed between two cities. One of the cities B is 40km north and 70km east of the other A. The two cities are separated by a strip of forest running from east to west, with a north-south extent of about 7.5km. You are asked to design the most cost-efficient route for the road, taking into consideration that the construction cost per km of road depends in the terrain and is twice as much in the forest area. Is a route that runs through the forest in the north-south direction the most cost efficient?"

2 Introduction

This optimisation problem seemed from the beginning to be based on angles and triangles. The route can be broken down into the distance covered inside the forest and outside of the forest.

3 Methods and Analysis

Note: All answers given to 2 d.p unless stated otherwise.

3.1 Assumptions

We begin this section by defining the assumptions used in the model.

- All terrain is flat and hence price is uniform
- The forest is perfectly rectangular. North and south boundary are parallel.
- The cities are point locations.
- The road is a a straight line with no bends and detours (the only time angle changes is when road is entering and leaving a change in environment, terrain or forest).
- (for route 1 and 2 only) The forest is exactly half way between the cities in a northern direction.
- (for route 3 only) The forest is immediately north of city A

3.2 Route 1

We begin this section by defining the variables used in Route 1 and 2:

- a = northern distance between the two cities
- b = eastern distance between the two cities
- c = total distance between the two cities
- d = new distance from forest to city B (route 2)

We first model the shortest and most direct route. Considering the known relative location of the second city to the first, we use the Pythagorean theorem to find the total distance of the modelled route c where the northern distance a is 40km and the eastern distance b is 70km:

$$a^2 + b^2 = c^2 \tag{3.1}$$

$$40^2 + 70^2 = c^2 = 6500$$

$$c = 10\sqrt{65}$$

Using the total northerly and easterly distances a and b, as well as the assumption that the forest is exactly half way between the two cities, we obviously find the northerly distance either side of the forest to be 16.25km. We can also find the angle of the road to the horizontal:

$$\arctan(\frac{a}{b}) \tag{3.2}$$

$$\arctan(\frac{40}{70}) = 0.5191...$$

$$= 0.52rads$$

This angle is the same for each point that the road meets the horizontal. We now find the distance either side of the forest using:

$$\frac{16.25}{sin(0.52...)} = 32.75 \tag{3.3}$$

and the distance inside the forest using:

$$\frac{7.5}{sin(0.52...)} = 15.12 \tag{3.4}$$

These calculations are summarised in figure 1 and can be checked by finding the sum of the distances to be $10\sqrt{65}$ as seen in equation 3.1.

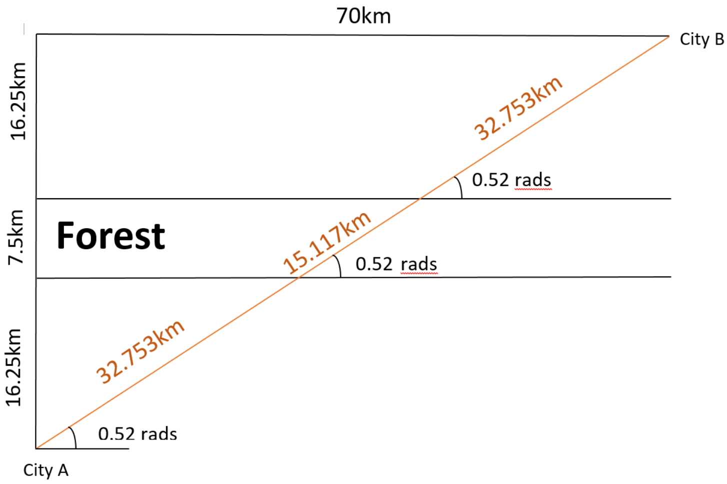

Figure 1: Route of model 1

NOTE: In this case and in route 2 it can be seen that the position of the forest does not affect the cost of the journey, as the distance travelled both inside and outside of the forest will always remain the same, regardless of where the forest section is positioned.

Therefore we can calculate the total unit cost of the road for this route to be:

$$\begin{aligned} & 32.753 + 2(15.117) + 32.753 \\ &= 65.506 + 30.234 \\ &= 95.74 \end{aligned} \tag{3.5}$$

3.3 Route 2

Expanding the model above we now model the second extremity, a route that travels the shortest possible distance through the forest; directly through it. Route 2 is identical to route 1 from City A to the point of entering the forest. The route inside the forest is now directly north for 7.5km. This changes the distance between the point of exit from the forest and City B. This new distance d can be found by first finding the easterly distance travelled at the point of entering the forest,using the Pythagorean theorem, and subtracting this value from the total easterly distance.

$$\sqrt{(32.753^2 - 16.25^2)} = 28.44 \tag{3.6}$$

$$70 - 28.44 = 41.56$$

Using the northerly and easterly distance between the point of leaving the forest and City B, the Pythagorean theorem 3.1 can be used to find distance d.

$$\sqrt{(41.56^2 + 16.25^2)} = 44.63 \tag{3.7}$$

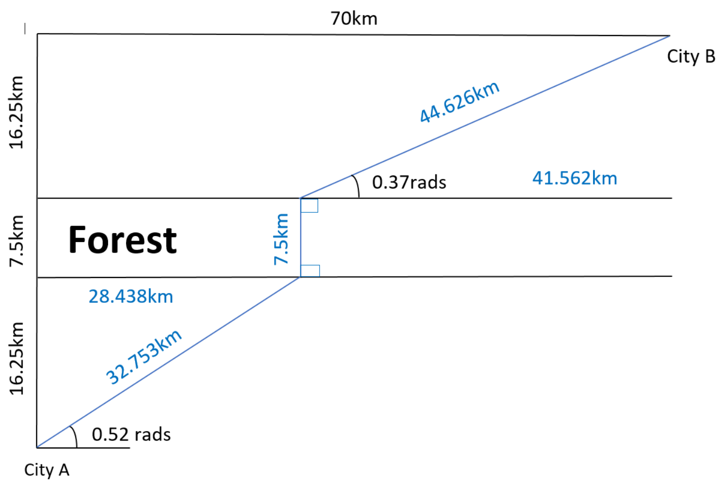

Figure 2: route 2

We now find the total unit cost of route 2:

$$\begin{aligned} &32.753 + 2(7.5) + 44.53 \\ &= 92.28 \end{aligned} \tag{3.8}$$

3.4 Optimal Route Based on Angle

After considering both extremities, outlined in 3.2 and 3.3, it is clear that neither the shortest nor longest route through the forest provide the most cost efficient journey from A to B. Thus we now expand the model to consider an optimum angle of the route through the forest, that lies between the two paths outlined in the previous sections: See figures 1 and 2

It is clear from the diagrams that the angle from the horizontal at which we must leave the forest in order to reach city B, call this angle θ_2, depends on the angle (to the horizontal) at which we leave city A, θ_1. Hence it is possible to form a set of equations in θ_1, θ_2, and x, where x is the horizontal distance between the point of entry and point of exit of the forest. The equations are outlined below:

$$\tan(\theta_1) = \frac{7.5}{x} \tag{3.9}$$

and

$$\tan(\theta_2) = \frac{32.5}{70 - x} \tag{3.10}$$

Combining these equations we can form an equation for the cost, C, to complete the journey which depends solely on angle θ_1:

$$C = \sqrt{(32.5)^2 + \left(70 - \frac{7.5}{\tan(\theta_1)}\right)^2} + 2\sqrt{(7.5)^2 + \left(\frac{7.5}{\tan(\theta_1)}\right)^2} \tag{3.11}$$

The minimum cost of the journey and corresponding value of θ_1 can subsequently be found by plotting the graph of the equation, as illustrated in the below diagrams:

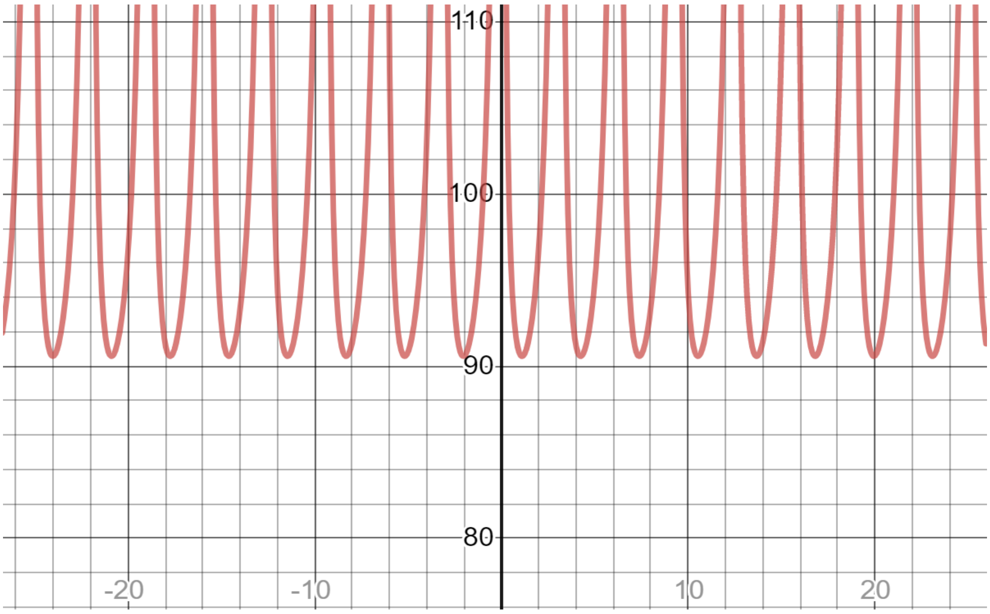

Figure 3: Graph
[1]

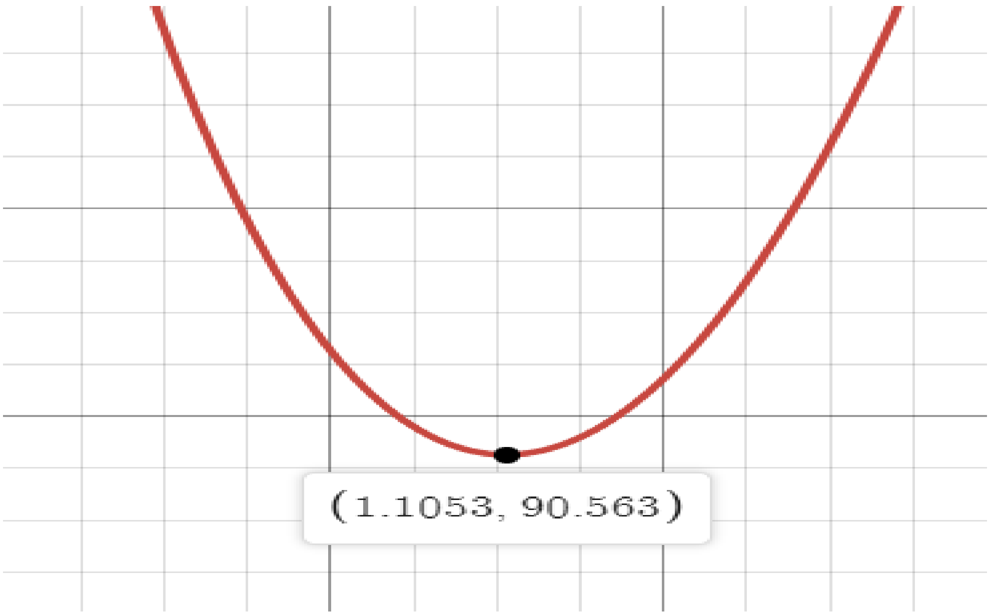

Figure 4: Minimum Point
[1]

Thus the lowest possible unit cost of the journey is 90.56 which occurs when $\theta_1 = 1.1053$ rads and it follows that θ_2= 0.4562 rads (to 4 d.p).

3.5 Optimal Route Based on Horizontal Distance

The optimal route is dependent on the distance travelled in the forest. The optimal route can also be calculated without incorporating angles.

Let ϕ be the horizontal distance travelled in the forest. 7.5km in the vertical distance in the forest. The resultant distance travelled in the forest is then given by,

$$\sqrt{7.5^2 + \phi^2}. \tag{3.12}$$

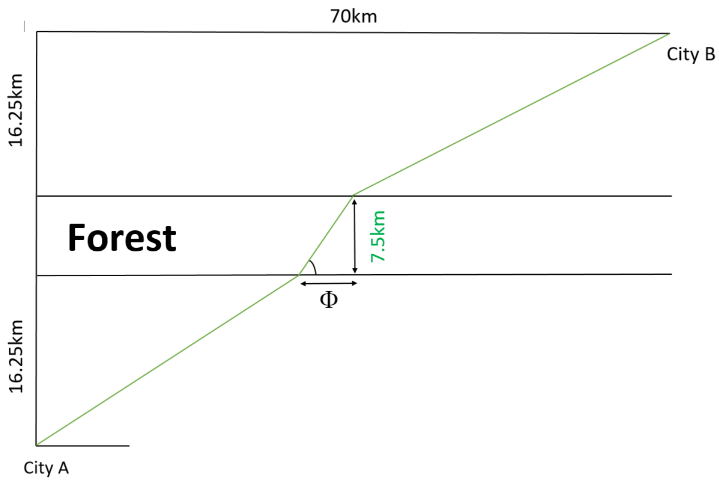

Figure 5: Optimal route showing horizontal distance ϕ

Similarly, outside of the forest, the total distance travelled horizontally is $(70-\phi)$ and the total distance vertically is $40 - 7.5 = 32.5$. Then the resultant distance travelled outside of the forest is,

$$\sqrt{32.5^2 + (70 - \phi)^2}. \tag{3.13}$$

The two expressions above then allow a function for the cost to be created:

$$Cost = 2\sqrt{7.5^2 + \phi^2} + \sqrt{32.5^2 + (70 - \phi)^2}. \tag{3.14}$$

As with the previous method, this function can be plotted and from this the minimum cost found, along with the corresponding value for ϕ. Shown below is the graph of the cost function.

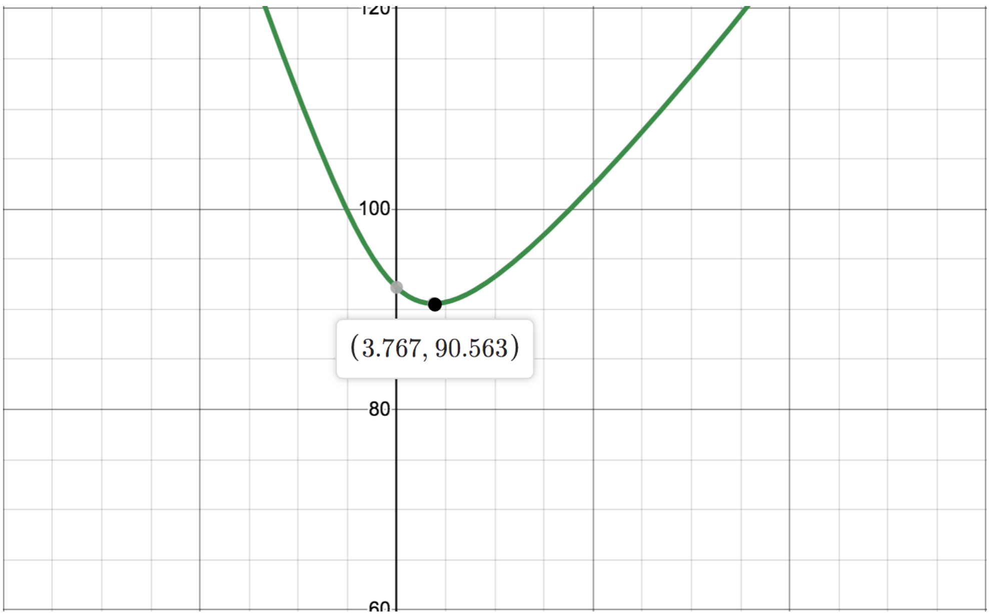

Figure 6: Minimum Point of Cost Function
[1]

Thus, the minimum unit cost is 90.56 which is when the horizontal distance ϕ travelled in the forest is $3.77km$. This gives and angle of $1.11rads$ from the horizontal for the section of road in the forest.

4 Results

The two optimal route solutions show that the cost of the journey is dependent only on the route taken in the forest. Both solutions give that the optimal route is when the angle of the path to the horizontal is 1.05 rads and the horizontal component of the distance travelled in $3.77km$.

This gives a minimum cost for the route as 90.56 units based on our assumption that the cost per km is 1 unit outside of the forest region and 2 units within the forest region.

5 Discussion

To extend the model, now consider features that will complicate the model and the construction cost of the road. An example of this is the likelihood that there will be more cities and/or other obstacles in between the two cities A and B. Therefore there may be certain routes that are not possible or may cost more. The model could be extended to incorporate this by saying our angles θ_1 and θ_2 are not equal to certain angles that would pass through these obstructions. Moreover, if these obstacles can not be avoided, the model can be further refined by also modelling these areas with a higher unit cost.

Also consider the assumption that each city is a point. This is inaccurate as cities are often huge areas of land, with specific entry/exit points that may force the road to leave the city at a different angle to what has been modeled. This can have a significant effect on the construction, and therefore price, of the road. A more accurate model would consider these exact locations of the point of entry/exit to the cities.

It is also highly unlikely and unrealistic that all terrain between the two cities is flat. This could greatly affect out calculations as the presence of any significant hills could extend the length of the route and thus the cost of the journey would be greater.

6 Conclusions

Modelling this problem first in two extreme cases (Route 1 and 2) allowed us to see that the optimal route through the forest lay between these two extreme cases.

From this we developed two models based on Pythagoras' Theorem which calculated the optimal route based on the angle in the forest and the horizontal distance. Both models gave the optimal route at an angle of 1.11 rads to the horizontal in the forest. The second model is also not dependent on the location of the forest which proves that the optimal route is not dependent on the location of the forest.

This clearly shows that the optimal route is not directly though the forest, but when a path of 1.11 rads from the horizontal is taken in the forest.

References

[1] Online Graphing software. *https://www.desmos.com/calculator*