Constructor University Bremen

Lab Report 2: The Fourier Series and The Fourier Transform

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1. Prelab

1.1 Problem 1: Decibels

1. Given $x(t) = 5\cos(2\pi 1000t)$

   - What is the amplitude and the V_{pp} of the signal?
     The amplitude of the signal is $5V$ and the V_{pp} is $2 \cdot 5V = 10V$.
   - What is the V_{rms} of the signal?
     The V_{rms} of a sinusoidal signal is given by

     $$V_{rms} = \frac{V_A}{2\sqrt{2}} = \frac{5V}{\sqrt{2}} = 3.535V \tag{1.1}$$

   - What is the amplitude of the spectral peak in dBVrms? The amplitude of the spectral peak in dBVrms is given by the following relation:

     $$A_{dBV_{rms}} = 20\log_{10}(V_{rms}) = 20\log_{10}(3.535V) = 10.9dBV_{rms} \tag{1.2}$$

2. Given a square wave of $1V_{pp}$ the voltage changes between -0.5V to 0.5V

   - What is the signal amplitude in V_{rms}? To find the RMS value of a square wave, the following relation is used:

     $$V_{rms} = \sqrt{\frac{1}{T}\left[\int_0^T f(t)^2 dt\right]} \tag{1.3}$$

     The square wave can be specified as a piecewise function:

     $$f(t) = \begin{cases} -0.5 & \text{if } 0 \leq t \leq \frac{T}{2} \\ 0.5 & \text{if } \frac{T}{2} \leq t \leq T \end{cases} \tag{1.4}$$

Performing the integral over the function,

$$\begin{aligned} V_{rms} &= \sqrt{\frac{1}{T}\left[\int_0^{T/2}(-0.5)^2dt + \int_{T/2}^{T}(0.5)^2dt\right]} \\ &= \sqrt{0.25\cdot\frac{1}{T}\left(\frac{T}{2}+T-\frac{T}{2}\right)} \\ &= \sqrt{0.25}V \\ &= 0.5V \end{aligned} \tag{1.5}$$

- What is the amplitude in dBVrms?

$$V_{dB_{rms}} = 20\log_{10}\left(V_{rms}\right) = 20\log_{10}\left(0.5V\right) = -6.02dBV_{rms} \tag{1.6}$$

1.2 Problem 2: Determination of Fourier Series Coefficients

1. Determine the Fourier series coefficients up to the 5th harmonic of the function $f(t) = 4t^2$

   To determine the fourier series coefficients, first recognize that the function is an even function by the very fact that it is a polynomial of an even degree. Knowing this, the relations used to compute the fourier series coefficients are:

$$\begin{aligned} a_n &= \frac{4}{T}\int_{-T}^{T} t^2\cos(n\omega_0 t)dt \\ a_0 &= \frac{1}{T}\int_0^T t^2dt \\ b_n &= 0 \end{aligned} \tag{1.7}$$

   Where it is known that $T = 1$ due to the interval being given as $[-0.5, 0.5]$. Furthermore, $\omega_0 = \frac{2\pi}{T} = 2\pi$. By using the help of an integration table, the integral for a_n is

$$a_n = 4\cdot\left[\frac{2t\cos(n\omega_0 t)}{4\pi^2n^2} + \frac{n^2 4\pi^2 - 2}{8\pi^3n^3}sin(n\omega_0 t)\right]_{-T}^{T} \tag{1.8}$$

Evaluating this integral:

$$a_n = 4 \cdot \left(\frac{4 \cdot (-1)^n}{4\pi^2 n^2} \right) = \frac{4 \cdot (-1)^n}{\pi^2 n^2} \tag{1.9}$$

For a_0, the integral is straightforward:

$$a_0 = \frac{1}{T} \int_0^T t^2 dt = \left[\frac{t^3}{3} \right]_0^T = \frac{1}{3} \tag{1.10}$$

So the first five coefficients are:

$$a_n = \left\{ \frac{-4}{\pi^2}, \frac{4}{16\pi^2}, \frac{-4}{36\pi^2}, \frac{4}{64\pi^2}, \frac{-4}{100\pi^2} \right\} \tag{1.11}$$

2. Use MATLAB to plot the original function and the inverse Fourier transform. Put both graphs into the same diagram.

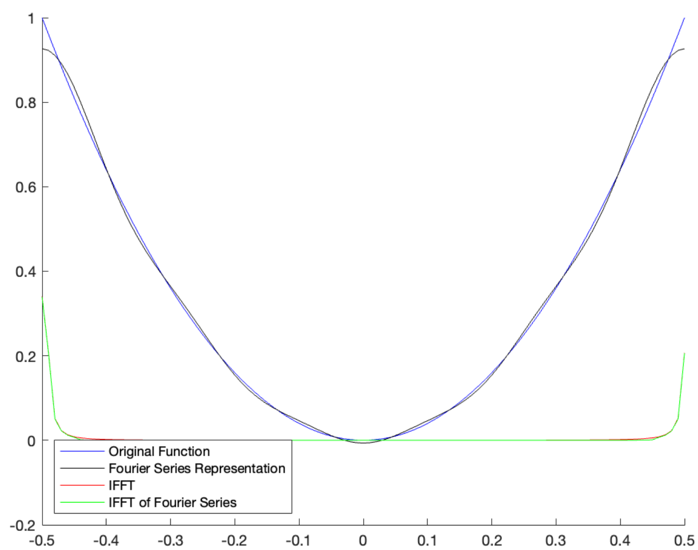

Figure 1.1: The plot of the original function, its inverse fourier transform, the fourier series representation up to the fifth harmonic, and the representation's inverse fourier transform

The code used:

```
t = -0.5:0.01:0.5;
f = 4*t.^2;
f_fs = 1/3;

% Fourier series representation of the function
for i=1:5
    % w_0 = 2pi
    % T = 1
    a_i = (4*(-1)^i)/(pi^2*i^2);
    f_fs = f_fs + (a_i .* cos(i.*2.*pi.*t));
end
hold on

plot(t, f, "blue");
plot(t, f_fs, "black");
xlim([-0.5, 0.5]);

plot(t, abs(ifft(f)), "red");
plot(t, abs(ifft(f_fs)), "green");

legend({"Original Function", ...
    "Fourier Series Representation", ...
    "IFFT", ...
    "IFFT of Fourier Series" ...
    }, "Location", "southwest")
```

1.3 Problem 3: FFT of a Rectangular Wave

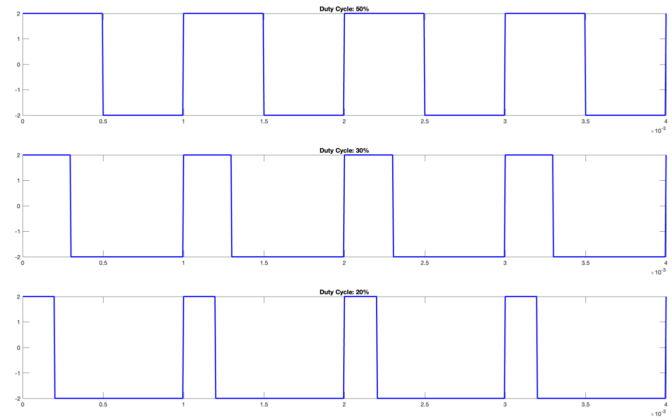

Figure 1.2: A figure showing the different rectangular waves based on their duty cycles. Plotted using MATLAB.

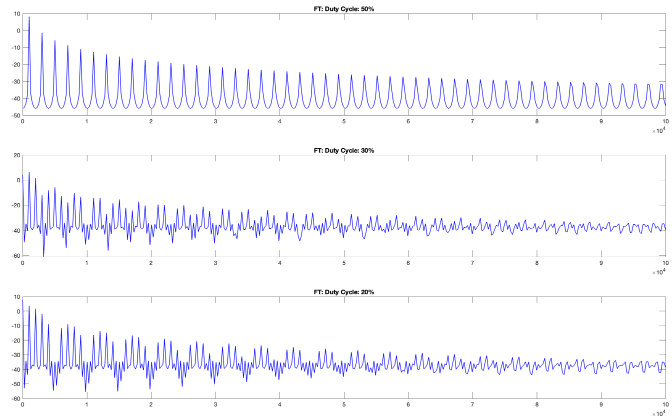

Figure 1.3: A figure showing the different rectangular waves' spectra based on their duty cycles. Plotted using MATLAB.

The code used to plot the figures:

```
%% Time plots

[t_fifty, signal_fifty, f_fifty, y_fifty] = get_square(50);
[t_thirty, signal_thirty, f_thirty, y_thirty] = get_square(30);
[t_twenty, signal_twenty, f_twenty, y_twenty] = get_square(20);

subplot(3, 1, 1);
plot(t_fifty, signal_fifty, "blue", "LineWidth", 2);
title("Duty Cycle: 50%");

subplot(3, 1, 2);
plot(t_thirty, signal_thirty, "blue", "LineWidth", 2);
title("Duty Cycle: 30%");

subplot(3, 1, 3);
plot(t_twenty, signal_twenty, "blue", "LineWidth", 2);
title("Duty Cycle: 20%");

%% Frequency plots
subplot(3, 1, 1);
plot(f_fifty, y_fifty, "blue", "LineWidth", 1);
title("FT: Duty Cycle: 50%");


subplot(3, 1, 2);
plot(f_thirty, y_thirty, "blue", "LineWidth", 1);
title("FT: Duty Cycle: 30%");


subplot(3, 1, 3);
plot(f_twenty, y_twenty, "blue", "LineWidth", 1);
title("FT: Duty Cycle: 20%");
```

Where the function `get_square` is defined as:

```
function [t, signal, f, y_single_db] = get_square(duty_cycle)
    period = 1e-3;
    Fs = 200e3;
    frequency = 1/period;
    Vpp = 2;
    duration = period * 4;

    % The signal
    t = 0:1/Fs:duration;
    signal = Vpp*square((2*pi*frequency)*t, duty_cycle);
    plot(t, signal, "blue", "LineWidth", 2);

    % The fourier transform of the signal
    rms_value = sqrt(mean(signal.^2));
    N = length(signal); % The length of the signal

    y = fft(signal, N);
    y_mag = 2*(abs(y)/N); % Magnitudes of y

    y_single = y_mag(1:floor(N/2)) * 2;
    f_nyquist = Fs / 2;

    y_single_db = 20*log10(y_single / rms_value);
    f = linspace(0, f_nyquist, length(y_single));
end
```

Following by the equation:

$$c_k = \frac{1}{k\pi} \sin(k\omega_0 T_1) \tag{1.12}$$

It is observed that for a lower T_1, for each frequency component the amplitude of the component is lower. This is due to the fact that the pulse width is lower, and thus the signal is more spread out in the time domain. This is also observed in the frequency domain, where the frequency components are more spread out, thereby more frequency components are required to represent the signal.

1.4 Problem 4: FFT of an audio file

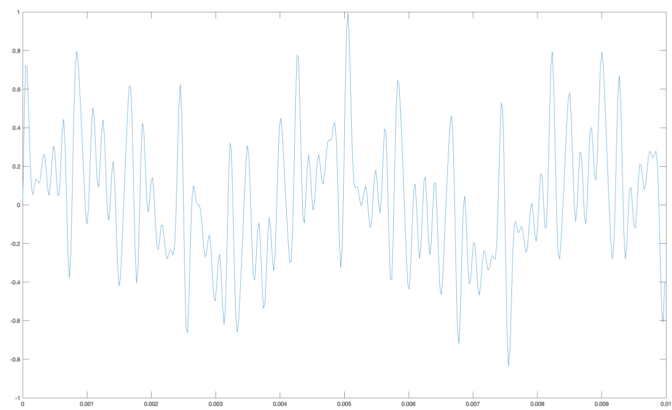

Figure 1.4: A figure showing the plot of the audio file

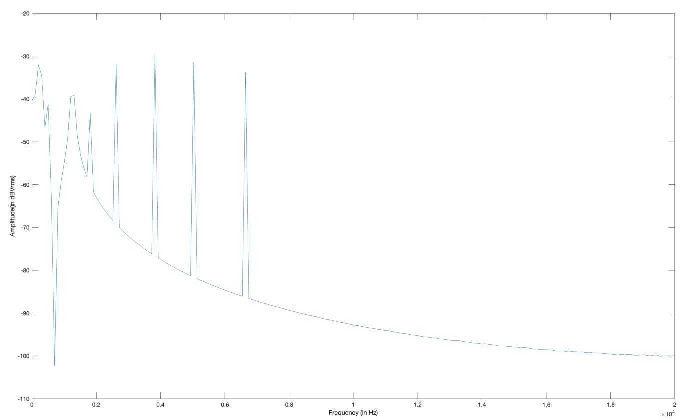

Figure 1.5: A figure showing the frequency spectrum of the audio file

The following code was used for the plots:

```
%% Problem 4:
[y, Fs] = audioread("s_samp.wav");

N_samples = Fs * 10E-3;
y = y(1:N_samples);
t = 0:1/Fs:((10E-3)-1/Fs);
plot(t, y);
%%
Y = fft(y);
rms = sqrt(mean(abs(Y.^2)));

Fs_nyquist = Fs / 2;

Y_single = 2 * abs(Y) / N_samples;
Y_single = Y_single(1:floor(N_samples/2));
f = linspace(0,Fs_nyquist,length(Y_single));
Y_db = 20*log10(Y_single ./ rms);

plot(f, Y_db);
ylabel("Amplitude(in dBVrms)")
xlabel("Frequency (in Hz)");
xlim([0, 20000]);
```

The following tones are observed:

- G3 - 201Hz
- B4 - 503Hz
- E6 - 1308Hz
- A6 - 1812Hz
- E7 - 2617Hz
- A#7 - 3826Hz
- D#8 - 5034Hz
- G#8 - 6645Hz

2. Introduction

In this lab, the Fourier Transform was explored by means of the use of the FFT button on the oscilloscope. The Fourier Transform is a mathematical tool that allows us to convert a signal from the time domain to the frequency domain, thereby allowing the extraction of the frequency components that make up a signal.

A continuous time signal can be described by a sum of sinusoids of different frequencies, amplitudes, and phases. The Fourier Series is a mathematical tool that allows us to decompose a periodic signal into a sum of sinusoids, to use it, however, it must first be established what it means for a signal to be periodic. We say that a signal is periodic if for some positive period T the following holds:

$$x(t) = x(t + nT) \tag{2.1}$$

This must hold for all t. The fundamental period is the smallest positive period T for which the above holds. The fundamental frequency is defined as:

$$\omega_0 = \frac{2\pi}{T} \tag{2.2}$$

To determine the complex Fourier Series coefficients, the following equation is used,

$$c_\nu = \frac{1}{T} \int_T f(t) e^{-j\nu\omega_0 t} dt \tag{2.3}$$

Where $f(t)$ can then be expressed as,

$$f(t) = \sum_{\nu=-\infty}^{+\infty} c_\nu e^{j\nu\omega_0 t} \tag{2.4}$$

Where the DC component is obtained by substituting $\nu = 0$.

$$c_0 = \frac{1}{T} \int_T f(t) dt \tag{2.5}$$

The signal does not have to be represented by complex exponentials, indeed, it can also be represented by sines and cosines, where the fourier coefficients a_0, a_ν, and b_ν are used.

$$f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{+\infty} a_\nu \cos(\nu\omega_0 t) + b_\nu \sin(\nu\omega_0 t) \tag{2.6}$$

2.1 The Square Wave

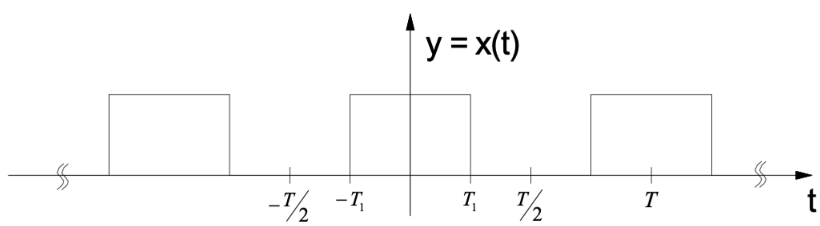

Figure 2.1: Square Wave

The square wave shown is a periodic signal that is defined as follows:

$$x(t) = \begin{cases} 1 & \text{if } |t| \leq T/2 \\ 0 & \text{if } T_1 < |t| < T/2 \end{cases} \tag{2.7}$$

Using the equations for the Fourier Series coefficients, and knowing the signal is 1 between $-T/4$ and $T/4$, and 0 elsewhere,

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T/4}^{T/4} 1dt = \frac{1}{T}\left(\frac{T}{4} - \left(-\frac{T}{4}\right)\right) = \frac{1}{2} \tag{2.8}$$

For the c_ν coefficients,

$$\begin{aligned} c_\nu &= \frac{1}{T}\int_{-T_1}^{T_1} x(t)e^{-j\nu\omega_0 t}dt \\ &= -\frac{1}{j\nu\omega_0 T}e^{-j\nu\omega_0 t}\Big|_{-T_1}^{T_1} \\ &= \frac{2}{\nu\omega_0 T}\left(\frac{e^{j\nu\omega_0 T_1} - e^{-j\nu\omega_0 T_1}}{2j}\right) \end{aligned} \tag{2.9}$$

Where from Euler's identities, it can be extracted:

$$c_\nu = \frac{1}{\nu\pi}\sin(\nu\omega_0 T_1) \tag{2.10}$$

Where T is the period, and T_1 is the width of the pulses. Similarly, for the a_ν and b_ν coefficients extracted from the complex exponential form of the Fourier Series,

$$a_\nu = 2\Re(c_\nu) = \frac{2}{\nu\pi}\sin\left(\nu\frac{\pi}{2}\right) = \begin{cases} 0 & \nu \text{ even} \\ \frac{2}{\nu\pi} & \nu \text{ odd} \end{cases} \tag{2.11}$$

$$b_\nu = -2\Im(c_\nu) = 0 \tag{2.12}$$

Which leads to,

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{\nu=1,3,5,\ldots}^{\infty} sin(\nu\frac{\pi}{2}) \cdot \frac{cos(\nu\omega_0 t)}{\nu} \tag{2.13}$$

The Gibbs Phenomenon is the overshoot of the Fourier Series approximation near the discontinuities of a periodic signal, which most surely can be observed in the case of the square wave. The discontinuities in the pulses makes it necessary to have more and more sinusoids to approximate it, and even then, the approximation is not perfect, leading to "ringing" near the discontinuities.

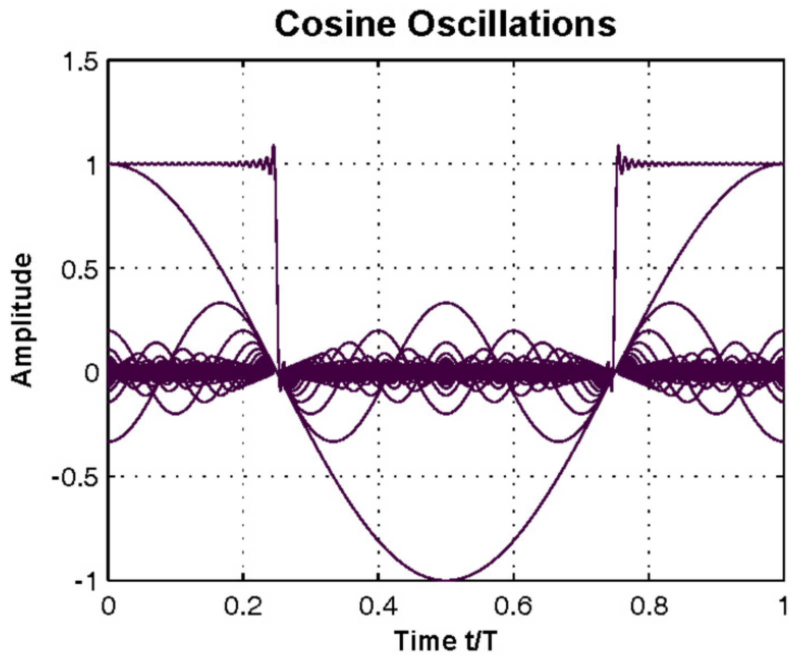

Figure 2.2: Gibbs Phenomenon, the overshoot near the discontinuities of the square wave approximated by 50 sinusoids.

2.2 The Fourier Transform

The Continuous Time Fourier Transform is what is obtained when the period of the Fourier Series goes to infinity. It is defined by,

$$X(j\omega) = \int_{\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{2.14}$$

And the inverse is given by,

$$x(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} x(t)e^{j\omega t}d\omega \tag{2.15}$$

For a square pulse, which is essentially a square wave of infinite period, the Fourier Transform is given by,

$$X(j\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}dt \tag{2.16}$$

By using Euler's trigonometric identities, it is obtained that the Fourier Transform of the square pulse is given by,

$$X(j\omega) = \tau \cdot \left[\frac{\sin(\frac{\omega\tau}{2})}{\frac{\omega\tau}{2}} \right] = \tau \cdot sinc\left(\frac{\omega\tau}{2}\right) \tag{2.17}$$

2.3 The Discrete Fourier Series

The Discrete Fourier Series is the discrete time equivalent of the Fourier Series, and the coefficients are defined as follows:

$$a_\nu = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-j\frac{2\pi\nu n}{N}} \tag{2.18}$$

Furthermore, the fourier series summation can be expressed as,

$$x[n] = \frac{1}{N} \sum_{\nu=(N)} a_\nu e^{j\frac{2\pi\nu n}{N}} \tag{2.19}$$

2.4 The Discrete Fourier Transform

Because computers can not handle continuous time signals, an analogous transform to the Fourier Transform is necessary, one that can handle discrete time signals. This is the Discrete Fourier Transform, which is defined as follows:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \tag{2.20}$$

And the inverse is given by,

$$x[k] = \frac{1}{N} \sum_{n=0}^{N-1} X[n] e^{j\frac{2\pi kn}{N}} \tag{2.21}$$

The DFT, and the more commonly used algorithm to perform the DFT, the FFT, are widely used for signal filtering, for correlation analysis and for spectral analysis. The FFT simply uses the properties of periodicity, symmetry, and uses a divide-and-conquer approach to reduce the number of computations needed to perform the DFT. It is not an approximation of the DFT, it is simply a more efficient way to compute it.

During the experiment, the Hanning Window is used on the oscilloscope as it is the best window to use when the signal is periodic. A window function simply cuts out the signal so that the FFT can be performed on it. The Hanning Window is some sort of rectangular pulse multiplied by the time signal, however, which leads to a sinc in the frequency domain. It will affect the transform, but it is the only viable option to perform the FFT on the signal.

3. Execution

In Problem 1 of section 6.3.1, a sinusoidal wave with a frequency of 500 Hz and an amplitude of 2 Vpp was generated using the function generator, with no offset. All properties were verified using the measure function, and a hard copy was taken in the time domain. The FFT spectrum was obtained using the oscilloscope FFT function, and the properties were measured using the cursor. Hard copies were taken of the complete spectra and the zoomed spectra peak. A sinusoidal wave with a frequency of 2 KHz and an amplitude value of 0 dB spectrum peak, without a dc offset, was generated using the measure function and the cursors. Hard copies were taken of the time and frequency domain.

4. Evaluation

5. Conclusion

6. References

7. Appendix