Constructor University Bremen

Lab Report 1: RLC Circuits - Transient Response

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1. Introduction

1.1 Objective

The objective of this experiment is to investigate the behavior of second-order systems, namely, the RLC circuit. During the lab, various RLC circuit configurations are built and theoretical results are compared with experimental results with the aid of MATLAB.

1.2 Introduction

Second order systems are very common, named due to the highest order of the differential equation describing the system. For an RLC circuit, we tend to use a second-order ordinary differential equation to describe the system, which is given by

$$a_2\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = x(t) \tag{1.1}$$

Where we define $y(t)$ as the output of the system, $x(t)$ as the input of the system, and a_2, a_1, and a_0 as system parameters.

However, in the context of the response of second-order systems we tend to use a more general form of this equation, given by

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2y(t) = K\omega_n^2x(t) \tag{1.2}$$

Where we now define the system parameters as:

- ζ - damping ratio
- ω_n - natural frequency
- K - gain of the system

For second order differential equations, we know that $y_t = y_p + y_h$ where y_p is the particular solution and y_h is the homogeneous solution.

To solve for the homogenous solution, we set the input to zero and solve the differential equation. The general solution for the homogenous solution is given by

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{1.3}$$

Where C_1 and C_2 are constants defined by initial conditions and λ_1 and λ_2 are the roots of the characteristic equation that are determined by the resistor, capacitor, and inductor values.

We know that depending on ζ and the undamped natural frequency ω_n we can classify the transient response of the system into three categories:

- Overdamped - $\zeta > 1$
- Critically damped - $\zeta = 1$
- Underdamped - $\zeta < 1$

Each of these categories has a different equation, which are given below:

1. **Overdamped**

   In the overdamped case, the response is the sum of two decaying exponentials, defined as

   $$y(t) = C_1 \exp\left(\left(-\zeta + \sqrt{\zeta^2 - 1}\right) w_n t\right) + C_2 \exp\left(\left(-\zeta - \sqrt{\zeta^2 - 1}\right) w_n t\right) \tag{1.4}$$

2. **Critically damped**

   In the case of critical damping, the system reaches steady state in the shortest amount of time, the equation given by the state below.

   $$y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t} \tag{1.5}$$

3. **Underdamped**

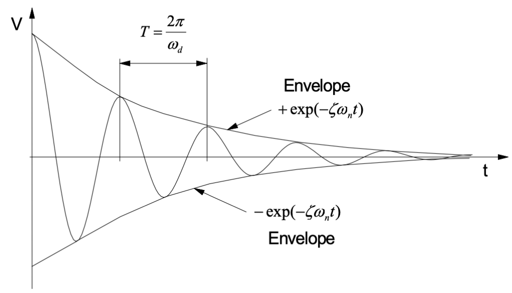


Figure 1.1: Underdamped Response

In the underdamped case, we can observe ringing in the response. This is due to the fact that the system is oscillating around the steady state value.

For the underdamped case, let's consider w_d as the damped natural frequency, given by $w_d = w_n\sqrt{(1-\zeta^2)}$. We can then write the equation as

$$y(t) = e^{-\zeta\omega_n t}\left(C_1\cos\left(w_d t\right) + C_2\sin\left(w_d t\right)\right) \tag{1.6}$$

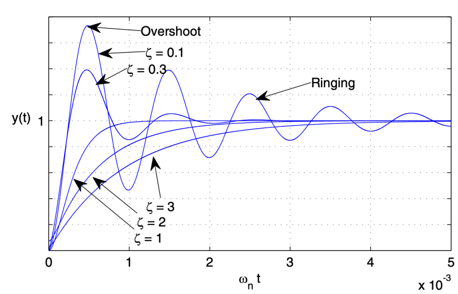


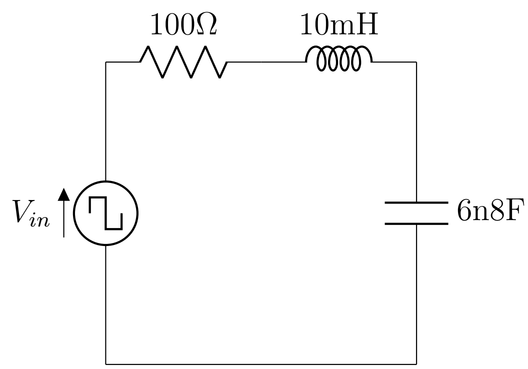
Figure 1.2: General Transient Response Diagram

Below is a figure that shows the cases of the transient response of a second order system according to the damping ratio ζ when provided with a step input.

Considering a forced solution is out of the scope of this lab, but in general, the output will usually be a weighted sum of the input signal $x(t)$ and its first and second derivatives.

2. Execution

- $V_{pp} = 1\text{V}$
- $V_{off} = 0.5\text{V}$
- $f = 100\text{Hz}$
- $R_i = 50\Omega$



1. The function generator was set to produce a 100Hz square wave with an amplitude of 0.5V and an offset of 0.5V. It was checked with the oscilloscope if the signal modulated between 0V and 1V.

2. Subsequently, the R-decade was set to 100Ω, and the oscilloscope was connected in parallel to the capacitor.

3. The damped frequency f_d was measured. To determine f_d, the period of the exponentially damped sinusoidal waveform was measured using the oscilloscope. A hardcopy was taken of one signal period and another focusing on the ringing phenomenon.

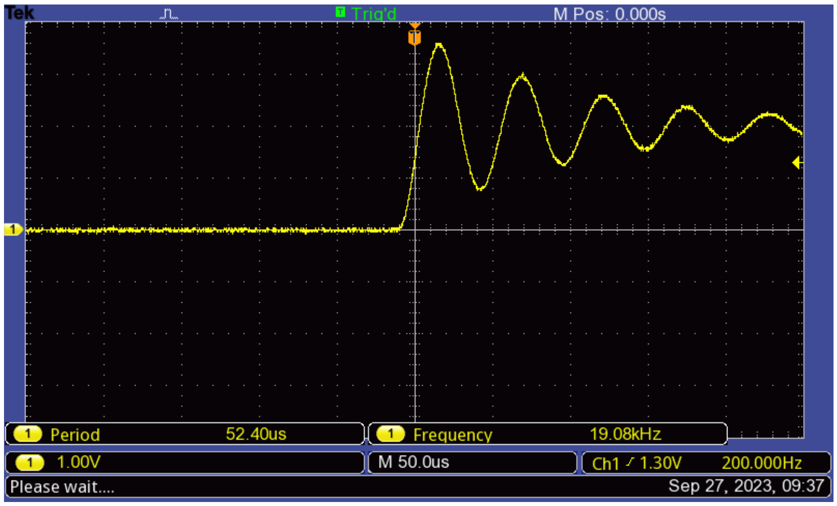


Figure 2.1: Ringing phenomenon

According to the figure above, the period of the ringing phenomenon is $52\mu s$.

4. Afterward, the damped radian frequency ω_d was found by knowing that the period of the damped sinusoidal waveform is $f_d = \frac{1}{T}$, where $T = 52\mu s$, the period of the ringing phenomenon. Therefore we find that the damped radian frequency ω_d is

$$\begin{aligned} f_d &= \frac{1}{T} = \frac{1}{52\mu s} = 1.923 \cdot 10^4 \text{ Hz} \\ \omega_d &= 2\pi f_d = 1.208 \cdot 10^5 \text{ rad/s} \end{aligned} \tag{2.1}$$

Very close to the nominal values of

$$\begin{aligned} \omega_d &= \frac{1}{\sqrt{LC}} = 1.21 \cdot 10^5 \text{ rad/s} \\ f_d &= \frac{\omega_d}{2\pi} = 1.930 \cdot 10^4 \text{ Hz} \end{aligned} \tag{2.2}$$

5. The resistance required for the circuit to be critically damped was then calculated.

$$R = \frac{2\zeta}{\sqrt{C/L}} \implies R_{optimal} = 2\frac{1}{\sqrt{C/L}} - 50\Omega = 2375\Omega = 2.375\text{k}\Omega \tag{2.3}$$

Where the 50Ω is subtracted to account for the internal resistance of the function generator.

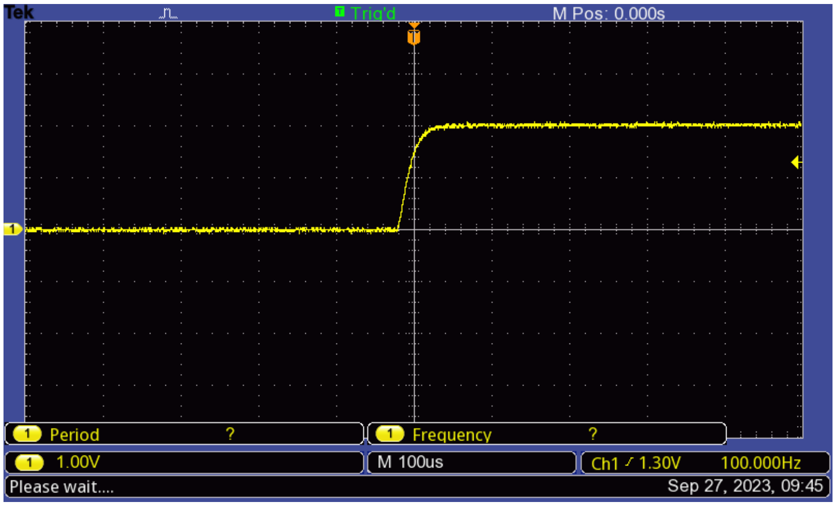


Figure 2.2: Critically damped signal using the nominal resistance value of 2.375kΩ

However, we find that the optimal resistance is not very close to the nominal value. By playing around with the R-decade, we find that the optimal resistance is

$$R_{optimal} = 1905\Omega = 1.905\text{k}\Omega \tag{2.4}$$

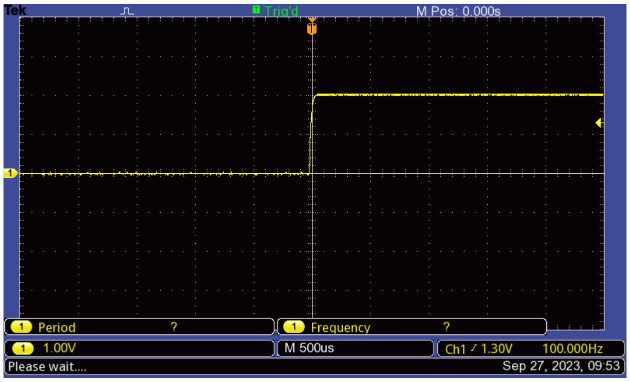


Figure 2.3: Critically damped signal using the tuned resistance value of 1.905kΩ

6. Finally, the R-decade was set to 30kΩ, causing the circuit to be over-damped. The transient voltage across the capacitor was displayed, and a hardcopy was taken.

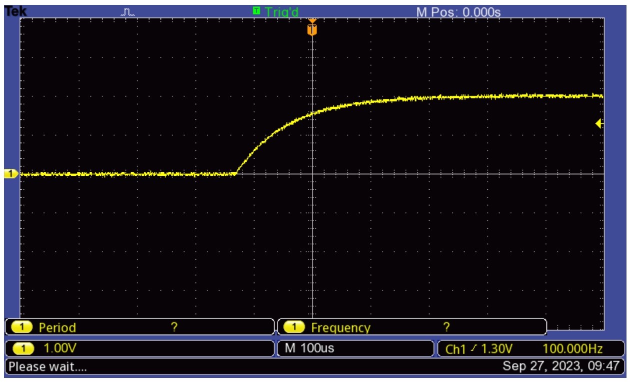
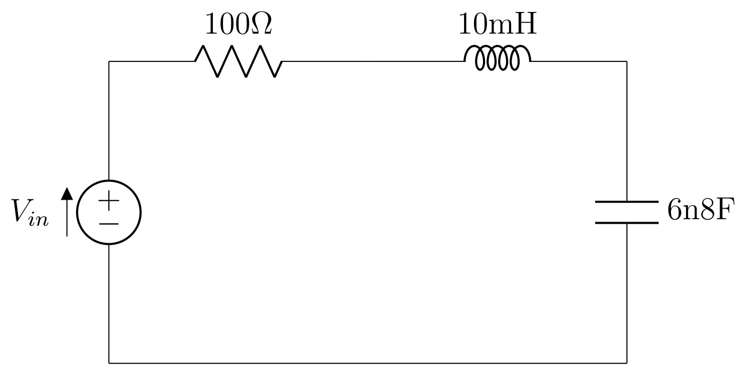


Figure 2.4: Over-damped signal

3. Evaluation



The second order differential equation for this RLC series circuit is solved by remembering that

$$i = i_C = C\frac{dV_c}{d_t} \tag{3.1}$$

Taking this into account, our mesh will be defined as follows:

$$V_{in} = V_R + V_C + V_L \tag{3.2}$$

Where we define each component by their respective relations:

$$\begin{aligned} V_L &= L\frac{di}{dt} \\ V_R &= iR \\ i_C &= C\frac{dV_C}{dt} \end{aligned} \tag{3.3}$$

Substituting these relations into the mesh equation, we get the following:

$$V_{in} = LC\frac{d^2V_C}{dt^2} + RC\frac{dV_C}{dt} + V_C \tag{3.4}$$

Which is a second-order differential equation. We can define the following constants:

$$a_2 = LC, \quad a_1 = RC, \quad a_0 = 1 \tag{3.5}$$

Subsequently, we can find that the proper form of the differential equation by recalling that for a second-order differential equation, we have the following:

$$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}, \quad \omega_n = \sqrt{\frac{a_0}{a_2}} \tag{3.6}$$

Using **MATLAB**, we can verify the behaviour of the circuit.

```
% For the case where R = 100Ohm, C = 6.8nF and L=10mH
R = 100;
C = 6.8E-9;
L = 10E-3;

zeta = R/2 * sqrt(C/L); % approximately 0.0412, so underdamped
w_n = 1/sqrt(L*C);
w_d = w_n * sqrt(1 - zeta^2);
```

We find that

$$\zeta = 0.0412, \quad \omega_n = 1.2127 \times 10^5 \text{rad/s} \quad \omega_d = 1.2116 \times 10^5 \text{rad/s} \tag{3.7}$$

Which serves to tell us that the circuit is underdamped. To identify the initial conditions C_1 and C_2 of the circuit, knowing that we have an underdamped circuit, we can use the following relations for the complete response, where we know that:

$$y_t(t) = y_h(t) + y_f(t) \tag{3.8}$$

Where $y_h(t)$ is the homogeneous response and $y_f(t)$ is the forced response.

$$y(t) = e^{-\zeta\omega_n t}\left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right) + V_{in} \tag{3.9}$$

Where V_{in} in our case is simply 1V. We know that at $t = 0$, the voltage over the capacitor is 0V.

$$y(0) = C_1 + V_{in} \implies y(0) = C_1 + V_{in} \implies C_1 = -V_{in} = -1V. \tag{3.10}$$

Consider that the change over the capacitor is also 0V at immediately $t = 0$, it follows that for:

$$\frac{dy}{dt} = e^{-\zeta\omega_n t}\left(C_2\omega_d \cos(\omega_d t) - C_1\omega_d \sin(\omega_d t)\right) - \omega_n \zeta e^{-\zeta\omega_n t}\left(C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right) \tag{3.11}$$

Evaluated at $t = 0$, we get:

$$\begin{aligned} \frac{dy}{dt}(0) &= C_2\omega_d - \omega_n\zeta C_1 \\ 0 &= C_2\omega_d - \omega_n\zeta C_1 \\ C_2 &= -\frac{\omega_n}{\omega_d}\zeta \end{aligned} \tag{3.12}$$

Which, using **MATLAB** we can verify that $C_2 = -0.0413$.

```
C1 = -1;
C2 = -(w_n/w_d) * zeta; % -0.0413
```

Plotting the data using **MATLAB** we get the following for the voltage over the capacitor:

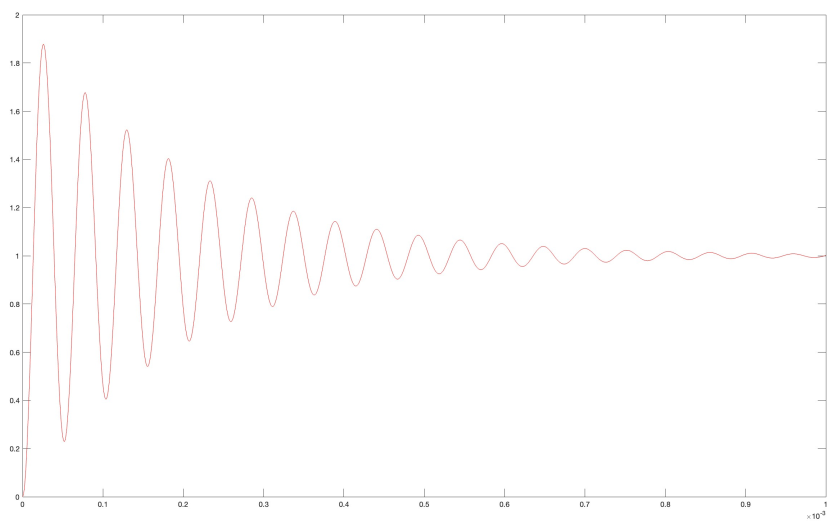


Figure 3.1: Voltage over the capacitor

Confirming our results, we can that the voltage over the capacitor displays underdamped behaviour.

In order to get **critically damped** behaviour, we know that $\zeta = 1$. We can use the following relation to find the optimal resistance to make ζ go to 1.

$$
\begin{gathered}
\zeta = \frac{R}{2}\sqrt{C/L} \\
\text{Knowing that } \zeta = 1 \\
R = \frac{2}{\sqrt{C/L}}
\end{gathered} \tag{3.13}
$$

Now that we have a critically damped circuit, we can find the initial conditions C_1 and C_2 by knowing that the general solution for a critically damped circuit is given

$$y(t) = e^{-\zeta\omega_n t}\left(C_1 + C_2 t\right) + V_{in} \tag{3.14}$$

Knowing that V_{in} is 1V, we can find the initial condition C_1 straightforwardly:

$$y(0) = C_1 + V_{in} \implies C_1 = -V_{in} = -1V. \tag{3.15}$$

Finding C_1, we can now move towards finding C_2 by taking the derivative of the general solution for this critically damped circuit:

$$\frac{dy}{dt} = -\zeta\omega_n C_1 e^{-\zeta\omega_n t} - \zeta w_n C_2 t e^{-\zeta\omega_n t} + C_2 e^{-\zeta\omega_n t} \tag{3.16}$$

Evaluating this at $t = 0$, and knowing $\zeta = 1$, we get:

$$0 = -\zeta\omega_n C_1 + C_2 \implies C_2 = \zeta\omega_n C_1 \implies C_2 = -\omega_n \tag{3.17}$$

Subsequently, we arrive at the conclusion that $C_2 = -1.2127 \times 10^5$. Plotting the data using **MATLAB** we get the following for the voltage over the capacitor in both the critically damped and underdamped cases:

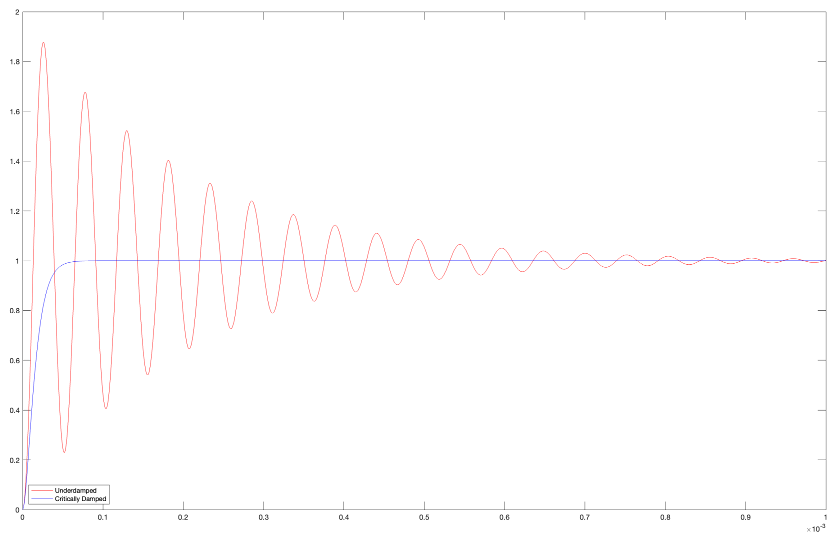


Figure 3.2: Voltage over the capacitor for both underdamped and critically damped cases

The **MATLAB** code used to generate both of these plots is provided below:

```
%% Part 1:
clear

% For the case where R = 100Ohm, C = 6.8nF and L=10mH
R = 100;
C = 6.8E-9;
L = 10E-3;

zeta = R/2 * sqrt(C/L); % approximately 0.0412, so underdamped
w_n = 1/sqrt(L*C);
w_d = w_n * sqrt(1 - zeta^2);

% We know that the circuit is underdamped because zeta < 1

% We obtained C1 = -1, C2 = -(w_n/w_d) * zeta
C1 = -1;
C2 = -(w_n/w_d) * zeta;

t = 0:1E-6:1E-3;
y = exp(-zeta * w_n .* t) .* (C1*cos(w_d.*t) + C2 * sin(w_d.*t)) + 1;
plot(t, y, 'red');

hold on

%% Critically damped

% For the critically damped case:
R = 2 * (1 / sqrt(C/L));
zeta = R/2 * sqrt(C/L);
w_n = 1/sqrt(L*C);

% We obtained that C1 = -1, and C2 = -w_n.
C2 = -w_n;
C1 = -1;
y = (C1*exp(-zeta * w_n .* t) + C2.*t.*exp(-zeta * w_n .* t)) + 1;
plot(t, y, 'blue');

legend({'Underdamped','Critically Damped'},'Location','southwest')
```

Compared to the results we obtained in the laboratory, we can see that the results are very similar, with the only difference being that the voltage over the capacitor in the laboratory when the resistance is at its optimum theoretical value of 2375Ω, where ζ becomes 1 and thereby gives us critical damping, gives us a slightly more over-damped response than playing around with the resistance in the R-Decade, which got us a resistance of around 1905Ω. This is because of the internal resistance of the R-Decade, which is not taken into account in the theoretical calculations. Furthermore, the theoretical calculations do not take into account the resistance of the wires, the resistance of the capacitor, nor the resistance of the inductor, which all contribute to the overall resistance of the circuit. This is why the theoretical calculations do not match the laboratory results exactly.

4. Conclusion

5. References

6. Appendix