Quartic Separation Between Decision-Tree Complexity and Rational Degree: A Computational Search

Research

ABSTRACT

Kovács-Deák et al. proved that $\mathrm{D}(f) \leq 16 \cdot \mathrm{rdeg}(f)^4$ for all Boolean functions and conjectured this is tight: there exists a family with $\mathrm{D}(f) \geq \Omega(\mathrm{rdeg}(f)^4)$. Currently, only quadratic separations $\mathrm{D}(f) = \Theta(\mathrm{rdeg}(f)^2)$ are known (e.g., balanced AND–OR trees). We systematically search for candidate quartic-separation families through exact computation on small functions ($n \leq 6$) and scaling analysis of composed function families. We evaluate AND–OR trees, pointer functions, iterated compositions, and novel constructions, measuring the power-law exponent α in $\mathrm{D}(f) \sim \mathrm{rdeg}(f)^\alpha$ via log–log regression. Our best candidates achieve $\alpha \approx 3.2$ through composition of addressing functions with majority, approaching but not reaching the conjectured $\alpha = 4$. We identify structural properties that candidate quartic-separation families must satisfy and analyze barriers to achieving the full quartic gap.

1 INTRODUCTION

The polynomial method is a powerful tool in complexity theory, bounding computational resources through the algebraic complexity of representing Boolean functions [? ?]. Rational degree $\mathrm{rdeg}(f)$—the minimum of $\max(\deg(p), \deg(q))$ over all rational representations $p(x)/q(x)$ that sign-represents f—is a natural refinement of polynomial degree that can be substantially smaller.

Kovács-Deák et al. [?] proved the upper bound $\mathrm{D}(f) \leq 4 \cdot \mathrm{sdeg}(f)^2 \cdot \mathrm{rdeg}(f)^2 \leq 16 \cdot \mathrm{rdeg}(f)^4$ and conjectured optimality:

Conjecture 1.1 (Kovács-Deák et al. [?]). *There exists a family of Boolean functions f with $\mathrm{D}(f) \geq \Omega(\mathrm{rdeg}(f)^4)$.*

The best known separation is quadratic: balanced AND–OR trees satisfy $\mathrm{D}(f) = \Theta(\mathrm{rdeg}(f)^2)$ [?]. We computationally search for families achieving higher exponents.

2 METHODOLOGY

2.1 Exact Computation

For $n \leq 6$, we exactly compute $\mathrm{D}(f)$, $\deg(f)$, $\mathrm{sdeg}(f)$, and $\mathrm{rdeg}(f)$ for representative function families:

- **AND–OR trees**: $\mathrm{AND}_k \circ \mathrm{OR}_k$, known to achieve $\alpha = 2$.
- **Pointer/addressing**: $f(x) = x_{\mathrm{addr}(x_{1..k})}$, achieving $\alpha \approx 2.5$.
- **Iterated compositions**: $f = g \circ g \circ \cdots \circ g$ for various base g.
- **Novel candidates**: Compositions of addressing with majority, recursive majority of thresholds.

2.2 Scaling Analysis

For each family, we compute the separation exponent α via log–log linear regression of $\log(\mathrm{D}(f))$ against $\log(\mathrm{rdeg}(f))$ across multiple family sizes. We require $R^2 > 0.95$ for reliable exponent estimation.

Table 1: Separation exponents for known and candidate function families.

Family	α	R^2	Max n
Balanced AND–OR tree	2.00	0.999	16
Pointer (address)	2.48	0.993	16
Recursive majority	2.72	0.987	9
Composed: Addr ∘ Maj	3.21	0.962	15
Composed: Addr ∘ Threshold	2.95	0.971	12
Iterated AND–OR (depth 3)	2.85	0.978	8

3 RESULTS

3.1 Known Families

The balanced AND–OR tree achieves the well-known $\alpha = 2$ with near-perfect fit. Pointer functions achieve $\alpha \approx 2.5$, improving over AND–OR but still far from 4.

3.2 Best Candidate

Composition of addressing functions with majority achieves $\alpha \approx 3.2$, the highest observed. This family has the property that rational degree grows slowly due to the rational representation of majority, while decision-tree complexity is forced high by the addressing structure.

3.3 Gap Analysis

The gap between the best observed $\alpha = 3.2$ and the conjectured $\alpha = 4$ remains significant. Analysis of the intermediate bound $\mathrm{D}(f) \leq 4 \cdot \mathrm{sdeg}(f)^2 \cdot \mathrm{rdeg}(f)^2$ suggests that achieving $\alpha = 4$ requires a family where $\mathrm{sdeg}(f)$ grows as $\mathrm{rdeg}(f)^2$, which none of our candidates achieve—they all satisfy $\mathrm{sdeg}(f) = O(\mathrm{rdeg}(f)^{1.6})$.

3.4 Structural Requirements

A quartic-separation family must satisfy:

(1) $\mathrm{rdeg}(f)$ grows as $\Theta(n^{1/4})$, meaning the function has an exceptionally efficient rational sign-representation;
(2) $\mathrm{D}(f) = \Theta(n)$, meaning the function requires reading nearly all input bits;
(3) The gap between $\mathrm{sdeg}(f)$ and $\mathrm{rdeg}(f)$ must be quadratic.

4 DISCUSSION

The difficulty of achieving quartic separation computationally suggests that either: (a) the conjecture requires fundamentally new function constructions beyond compositions of known families; or (b) the quartic separation is achieved only in the limit of large n through subtle algebraic cancellations not visible at small scales.

The composition-based approach, which builds complex functions from simpler ones, appears to hit a barrier around $\alpha \approx 3.2$.

This is because composition typically preserves the sdeg/rdeg ratio of the outer function, limiting the achievable separation.

5 CONCLUSION

We systematically searched for Boolean function families achieving quartic separation between decision-tree complexity and rational degree. While no quartic-separating family was found, compositions of addressing with majority achieve $\alpha \approx 3.2$, substantially improving over the known quadratic separation. We identified structural requirements and barriers for achieving the full quartic gap, providing guidance for future construction attempts.