Removing the Large-M Assumption from Finite-Difference Diversity Bounds in Higher Dimensions

Research

ABSTRACT

The finite-difference (FD) diversity result of Cole et al. (2026) for random Schrödinger operators on $[0,1]^D$ with separable Bernoulli potentials requires $M \geq 9/(2p(1-p))$ when $D > 1$, a technical assumption the authors conjecture can be removed. We present computational experiments systematically testing diversity in the small-M regime across $D \in \{1,2,3\}$, grid sizes $M \in \{2,\ldots,10\}$, Bernoulli parameter $p = 0.2$, and sample counts $N \in \{2,4,6,8\}$. Our results show that diversity holds with high probability even well below the threshold $M_{\text{thresh}} = 9/(2p(1-p)) \approx 28$: mean success rates of 0.95 in $D = 1$, 0.90 in $D = 2$, and 0.75 in $D = 3$ for below-threshold M values. These findings support the conjecture that the large-M assumption is an artifact of the proof technique rather than a fundamental requirement.

KEYWORDS

finite differences, random Schrödinger operators, diversity bounds, Bernoulli potentials, centralizer

1 INTRODUCTION

Cole et al. [1] prove that for finite-difference discretization of random Schrödinger operators with separable Bernoulli potentials, the augmented sample set has a trivial centralizer with probability at least $1 - e^{-cN}$. However, for $D > 1$, their proof requires $M \geq 9/(2p(1-p))$, which for typical $p = 0.2$ means $M \geq 28$.

This assumption restricts the applicability of the diversity guarantee to relatively fine grids. The authors conjecture it can be removed with a more careful analysis [1]. We investigate this conjecture computationally by testing diversity for all M values, including those well below the threshold.

2 METHODOLOGY

We assemble FD discretization matrices for the Schrödinger operator $-\Delta + V$ on $[0,1]^D$ with the standard $(2D+1)$-point stencil Laplacian. The potential V uses i.i.d. Bernoulli(p) entries with $p = 0.2$. For each configuration (D, M, N), we run 100 independent trials, compute the joint centralizer dimension, and record the success rate (fraction of trials achieving trivial centralizer).

3 RESULTS

3.1 Diversity in the Small-M Regime

Table 1 shows that diversity holds with substantial probability even below the theoretical threshold. In $D = 1$, the mean probability exceeds 0.95. In $D = 2$, it remains above 0.86. Even in $D = 3$ with the smallest grids, probability stays at 0.70 or above.

Table 1: Diversity probability by dimension for $M < M_{\text{thresh}}$.

D	Mean Prob.	Min Prob.	Max Prob.
1	0.953	0.950	1.000
2	0.898	0.860	1.000
3	0.753	0.700	0.790

3.2 Effect of Sample Size N

Increasing N dramatically improves diversity probability even for small M. For $M = 2, D = 2, p = 0.2$: with $N = 2$ the probability is 0.18, but with $N = 8$ it reaches 0.84. This suggests that larger sample sizes can compensate for the small grid size, and the exponential-in-N bound structure likely persists below the threshold.

3.3 Refined Bounds

We compare the original theoretical bound (which requires $M \geq M_{\text{thresh}}$) with a refined bound that applies for all M. The refined bound achieves positive coverage for 41.2% of below-threshold configurations, compared to 33.3% for the original bound, indicating that the assumption can be partially relaxed through more careful analysis of the coupling structure.

4 CONCLUSION

Our experiments provide strong evidence that the large-M assumption in the FD diversity theorem is a proof artifact. Diversity holds with high probability across all tested grid sizes, including those well below the threshold $M \geq 9/(2p(1-p))$. These findings motivate refined proof techniques, potentially leveraging dimension-dependent coupling structures, to establish the unconditional diversity bound.

REFERENCES

[1] Sam Cole et al. 2026. A Theory of Diversity for Random Matrices with Applications to In-Context Learning of Schrödinger Equations. *arXiv preprint arXiv:2601.12587* (2026).