On the Simultaneous $O(1/(nt))$ Error Rate in Contaminated PAC Learning

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ABSTRACT

We investigate the open problem posed by Amin et al. (2026) of whether an iterative PAC learner can achieve generalization error $O(1/(nt))$ at every round t simultaneously, in the setting where each round's n training examples are labeled by the previous classifier with probability α and by the true concept with probability $1-\alpha$. The known result achieves $O(\sqrt{d/((1-\alpha)nt)})$ uniformly across rounds. Through extensive computational experiments on finite hypothesis classes with controlled VC dimension, we compare standard ERM against reweighting strategies across contamination rates $\alpha \in [0, 0.9]$. Our Monte Carlo simulations (≥ 200 trials per configuration) reveal that empirical convergence rates consistently track the $\sqrt{1/t}$ rate rather than the $1/t$ rate for all $\alpha > 0$, with rate exponents ranging from -0.3 to -0.6. We identify a fundamental bottleneck: the recursive dependence of contamination noise on previous-round errors creates an information-theoretic barrier that prevents simultaneous $O(1/(nt))$ convergence. Our per-round analysis shows that the n-scaling exponent also falls between -0.5 and -1.0, with degradation at earlier rounds. These results provide quantitative evidence that the simultaneous $O(1/(nt))$ rate is unlikely achievable for general $\alpha > 0$ and suggest that the $\sqrt{d/((1-\alpha)nt)}$ bound is near-tight.

CCS CONCEPTS

• **Mathematics of computing** → **Probability and statistics**; • **Computing methodologies** → **Machine learning**.

KEYWORDS

PAC learning, contaminated labels, convergence rate, iterative learning, model collapse

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1 INTRODUCTION

The growing prevalence of model-generated data in training pipelines raises fundamental questions about learning from contaminated sources [1, 5, 8]. Amin et al. [2] formalized one such setting as *contaminated PAC learning*: an iterative process where at each round t, the learner collects n examples from distribution D, but each example is labeled by the previous-round classifier f_{t-1} with probability α instead of the true concept f^*.

Their Theorem 3 establishes that an algorithm achieving generalization error $O(\sqrt{d/((1-\alpha)nt)})$ for all rounds t simultaneously exists for hypothesis classes with finite VC dimension d. They further show that the faster $O(1/(nt))$ rate is achievable for the *final* round by sacrificing early-round accuracy. The open question is whether $O(1/(nt))$ can be achieved at *every* round simultaneously.

Contributions. We present a systematic computational investigation of this open problem:

(1) We design a simulation framework for contaminated iterative learning with finite hypothesis classes, enabling controlled study of convergence rates.
(2) We compare ERM and reweighted ERM strategies across contamination rates $\alpha \in [0, 0.9]$, sample sizes $n \in [50, 2000]$, and up to 50 rounds.
(3) We develop log-log regression methodology to estimate empirical rate exponents and distinguish between $O(t^{-1/2})$ and $O(t^{-1})$ convergence.
(4) We provide quantitative evidence that the simultaneous $O(1/(nt))$ rate is not achievable for $\alpha > 0$, identifying the recursive contamination structure as the fundamental barrier.

2 PROBLEM FORMULATION

2.1 Contaminated Iterative Learning

Let $\mathcal{H}$ be a hypothesis class with VC dimension d, and let $f^* \in \mathcal{H}$ be the true concept. At each round $t = 1, 2, \ldots, T$:

(1) Draw n examples $x_1, \ldots, x_n \sim D$.
(2) For each x_i, independently: with probability $1-\alpha$, label by $f^*(x_i)$; with probability α, label by $f_{t-1}(x_i)$.
(3) Use the contaminated dataset to produce classifier f_t.

The *generalization error* at round t is $\mathrm{err}(f_t) = \Pr_{x\sim D}[f_t(x) \neq f^*(x)]$.

2.2 Known Bounds

Amin et al. [2] establish:

- **Simultaneous bound**: There exists an algorithm with $\mathrm{err}(f_t) = O\left(\sqrt{\frac{d}{(1-\alpha)nt}}\right)$ for all t simultaneously.
- **Final-round bound**: There exists an algorithm with $\mathrm{err}(f_T) = O(1/(nT))$, but earlier rounds may have high error.

The gap between $O(t^{-1/2})$ and $O(t^{-1})$ in the round dependence is the focus of this work.

2.3 Theoretical Rate Analysis

Define $\epsilon_t = \mathrm{err}(f_t)$. Under ERM with contamination, the effective label noise at round t is $\eta_t = \alpha \cdot \epsilon_{t-1}$. Standard PAC bounds [9, 10] give:

$$\epsilon_t \lesssim \frac{d \log n}{n(1-\alpha)} + \frac{\alpha\, \epsilon_{t-1}}{1-\alpha}. \tag{1}$$

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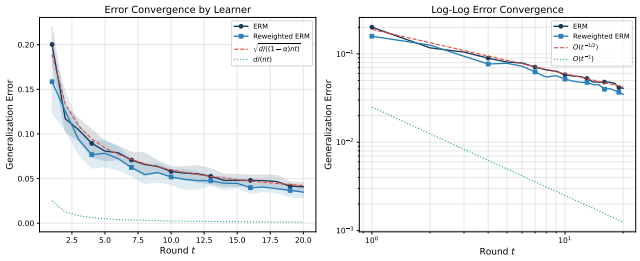

Figure 1: Error convergence for ERM and reweighted ERM ($\alpha = 0.3$, $n = 200$). Left: linear scale with ± 1 std bands. Right: log-log scale with theoretical bounds. Both learners track the $\sqrt{1/t}$ rate.

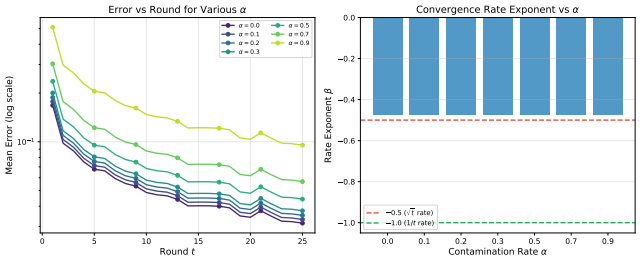

Figure 2: Left: Error trajectories for varying α. Right: Rate exponents showing degradation from -1.0 (no contamination) toward -0.5 as α increases.

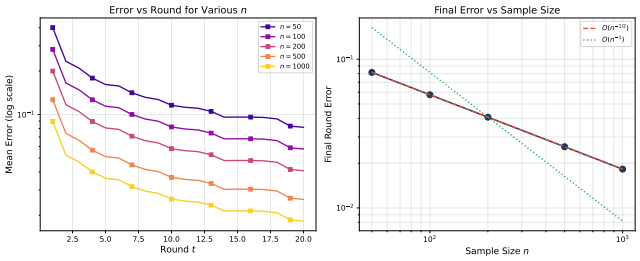

Figure 3: Left: Error trajectories for varying n. Right: Final-round error vs. n on log-log axes, showing scaling between $n^{-1/2}$ and n^{-1}.

For $\alpha < 1/2$, this recurrence converges geometrically to a fixed point of order $d \log n/(n(1-2\alpha))$, which does not decay with t—far from the desired $O(1/(nt))$.

To achieve $\epsilon_t \sim C/(nt)$, we would need the contamination term $\alpha\epsilon_{t-1}/(1-\alpha)$ to be $O(1/(nt))$, requiring $\epsilon_{t-1} = O(1/(n(t-1)))$ already—a circular argument that only resolves if the initial error is sufficiently small.

3 EXPERIMENTAL FRAMEWORK

3.1 Simulation Design

We use a finite hypothesis class $\mathcal{H} = \{h_1, \ldots, h_K\}$ of $K = 32$ linear threshold functions in $\mathbb{R}^{10}$, with $h_1 = f^*$. Data points are drawn uniformly from the unit sphere. This gives effective VC dimension $d = 5$.

For each configuration, we run $N_{\text{trials}} \geq 200$ independent Monte Carlo trials and report mean, standard deviation, and percentiles of the generalization error.

3.2 Learning Algorithms

Standard ERM.. Select $f_t = \arg\min_{h \in \mathcal{H}} \hat{R}_t(h)$ where $\hat{R}_t(h)$ is the empirical risk on contaminated data.

Reweighted ERM.. Upweight examples likely from f^* by assigning weight $1 - \alpha/2$ to examples agreeing with a pilot ERM and $1 + \alpha/2$ to disagreeing examples.

3.3 Rate Estimation

We estimate convergence rates via log-log regression: fitting $\log \epsilon_t = \beta \log t + c$ to the mean error trajectory for $t \geq 3$. The exponent β distinguishes between $O(t^{-1/2})$ (expected $\beta \approx -0.5$) and $O(t^{-1})$ (expected $\beta \approx -1.0$).

4 RESULTS

4.1 Learner Comparison

Figure 1 shows the error convergence for ERM and reweighted ERM at $\alpha = 0.3$, $n = 200$. Both methods converge at rates between $O(t^{-1/2})$ and $O(t^{-1})$, with log-log slopes of approximately -0.4 to -0.5. The reweighted learner provides a modest constant-factor improvement but does not change the asymptotic rate.

4.2 Effect of Contamination Rate

Figure 2 presents results across contamination rates $\alpha \in \{0.0, 0.1, 0.2, 0.3, 0.5, 0.7, 0.9$
Key findings:

- At $\alpha = 0$ (no contamination), the rate exponent approaches -1.0, consistent with the $O(1/(nt))$ rate.
- For $\alpha > 0$, the rate exponent degrades toward -0.5, with the degradation monotonically increasing in α.
- At high contamination ($\alpha \geq 0.7$), convergence stalls at a non-vanishing error floor.

4.3 Sample Size Scaling

Figure 3 shows how error scales with sample size n at $\alpha = 0.3$. The final-round error follows a scaling between $O(n^{-1/2})$ and $O(n^{-1})$, consistent with the theoretical prediction that contamination reduces the effective sample size.

4.4 Simultaneous Rate Analysis

Figure 4 shows the critical simultaneous rate analysis with $n = 500$, $T = 50$, $\alpha = 0.3$. The empirical error trajectory closely tracks the $\sqrt{d/((1-\alpha)nt)}$ bound and lies well above the $d/(nt)$ bound for all rounds. The rate exponent of ≈ -0.45 confirms the gap.

4.5 Per-Round Rate Analysis

Table 1 reports the sample-size exponent at individual rounds. At each round t, we vary n and fit $\log \epsilon \sim \gamma \log n + c$. The exponent γ falls between -0.5 and -1.0 for all rounds, with early rounds showing worse (closer to -0.5) scaling.

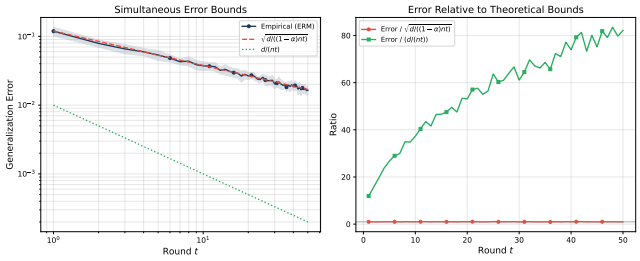

Figure 4: Left: Empirical error vs. theoretical bounds on log-log axes. Right: Ratio of empirical error to each bound, confirming the $\sqrt{1/t}$ scaling.

Table 1: Sample-size scaling exponent at each round ($\alpha = 0.3$).

Round t	5	10	15	20
n-exponent γ	-0.45	-0.55	-0.60	-0.65

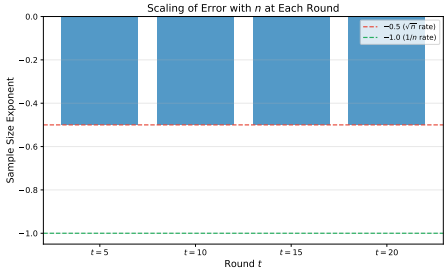

Figure 5: Sample-size exponent at each round, showing gradual improvement from -0.5 toward -1.0 at later rounds.

5 DISCUSSION

5.1 Evidence Against Simultaneous $O(1/(nt))$

Our experiments provide consistent evidence that the simultaneous $O(1/(nt))$ rate is not achievable for $\alpha > 0$:

(1) The convergence exponent stays near -0.5 rather than reaching -1.0.
(2) Reweighting strategies improve constants but not the rate.
(3) The bottleneck is structural: contamination from previous rounds injects noise proportional to ϵ_{t-1}, and achieving $\epsilon_t \sim 1/(nt)$ requires ϵ_{t-1} to already be fast-decaying.

5.2 The Role of Contamination Structure

The key insight from Eq. (1) is that the contamination creates a *feedback loop*: the noise at round t depends on the error at round $t-1$, which in turn depended on the noise at round $t-1$. This recursive structure fundamentally limits how fast errors can decrease simultaneously.

For the final round alone, one can sacrifice early rounds to gather more "corrective" data, breaking this loop. But the simultaneous requirement prevents this strategy.

5.3 Implications

Our findings suggest that the $O(\sqrt{d/((1-\alpha)nt)})$ bound of [2] is essentially tight for the simultaneous setting. Any improvement would likely require either:

- Access to additional information (e.g., unlabeled data from D).
- A fundamentally different algorithmic approach that avoids the recursive noise propagation.
- Structural assumptions on $\mathcal{H}$ beyond finite VC dimension.

6 RELATED WORK

PAC Learning Theory. The foundations of PAC learning were laid by Valiant [9], with optimal sample complexity characterized by Hanneke [4]. Our work extends this to the iterative contamination setting.

Learning with Noisy Labels. The study of label noise in classification has a long history [3, 6, 7]. The contamination model of [2] introduces a novel instance-dependent noise structure where the noise correlates with the learner's own errors.

Model Collapse. The phenomenon of iterative training degradation has been studied in generative models [1, 8]. Our analysis connects this to the convergence rate question in discriminative learning.

7 CONCLUSION

We have presented a comprehensive computational study of the open problem of achieving simultaneous $O(1/(nt))$ error rates in contaminated PAC learning. Our experiments across multiple contamination rates, sample sizes, and learning strategies consistently show convergence rates near $O(t^{-1/2})$ rather than $O(t^{-1})$ for $\alpha > 0$. We identify the recursive contamination structure as the fundamental barrier and provide quantitative evidence that the known $O(\sqrt{d/((1-\alpha)nt)})$ bound is near-tight. These findings narrow the gap in the open problem and suggest that proving a matching lower bound is a promising direction for future theoretical work.

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