

# Computational Study of Recursion Relations for Higher-Point Spectrally Flowed Correlators in the $SL(2, \mathbb{R})$ WZW Model

Research

Open Problems in High Energy Physics

## ABSTRACT

Spectrally flowed correlators in the  $SL(2, \mathbb{R})$  Wess-Zumino-Witten model on  $AdS_3$  satisfy recursion relations derived from local Ward identities. While these recursions are fully solved for three-point functions and partially known for four-point functions, the general  $n$ -point case remains open. We present a computational framework that implements the known recursions for  $n = 3, 4$ , extrapolates the combinatorial structure to propose candidate  $n = 5$  relations, and analyzes the resulting  $y$ -basis differential equations. Our numerical analysis at level  $k = 3$  reveals that recursion coefficient magnitudes scale as  $\binom{n-1}{2}$  relative to the three-point case, yielding mean values of 13.3 ( $n = 3$ ), 40.0 ( $n = 4$ ), and 79.9 ( $n = 5$ ). The corresponding  $y$ -basis differential equations have order  $n$  with  $n(n-1)/2 + 2n$  singular points: 9, 14, and 20 for  $n = 3, 4, 5$  respectively. Convergence analysis shows that recursion coefficients grow linearly with spectral flow number  $w$  for all spins  $j$  in the continuous series. These results characterize the computational complexity of the higher-point recursion program and identify the key structural patterns needed for a general derivation.

## 1 INTRODUCTION

String theory on  $AdS_3$  backgrounds provides one of the most tractable examples of the  $AdS/CFT$  correspondence. The worldsheet theory is described by an  $SL(2, \mathbb{R})$  WZW model [5], whose correlation functions encode the holographic dictionary. A crucial feature of this model is the spectral flow automorphism [5], which maps between different sectors of the string Hilbert space and is essential for constructing the complete physical spectrum.

The computation of spectrally flowed correlators proceeds through recursion relations derived from local Ward identities [1, 6]. For three-point functions, these recursions are completely solved, yielding explicit expressions in terms of the  $SL(2, \mathbb{R})$  structure constants. For four-point functions, important partial results exist [3]. However, as noted by Kovensky [4], general recursion relations for  $n$ -point functions with  $n > 3$  remain unknown.

In this work, we approach this open problem computationally. We implement the known recursions, extract the combinatorial patterns governing their structure, and propose candidate generalizations for  $n = 5$ . We analyze the resulting  $y$ -basis differential equations and characterize the growth of computational complexity with  $n$ .

## 2 FRAMEWORK

### 2.1 $SL(2, \mathbb{R})$ WZW Model

The  $SL(2, \mathbb{R})_k$  WZW model has current algebra generated by  $J^a(z)$  with  $a \in \{3, +, -\}$ . Primary states  $|j, m\rangle$  satisfy  $J_0^3|j, m\rangle = m|j, m\rangle$

with Casimir  $C_2 = -j(j-1)$  and conformal weight [5]:

$$h = \frac{-j(j-1)}{k-2} \quad (1)$$

### 2.2 Spectral Flow

The spectral flow automorphism with parameter  $w \in \mathbb{Z}$  transforms the currents as  $J_n^3 \rightarrow J_n^3 + \frac{k}{2}w \delta_{n,0}$ ,  $J_n^\pm \rightarrow J_{n \pm w}^\pm$ , yielding effective conformal weights [5]:

$$h_{\text{eff}}(j, m, w) = \frac{-j(j-1)}{k-2} - mw + \frac{kw^2}{4} \quad (2)$$

### 2.3 Recursion Relations

The Ward identity approach [4, 6] derives constraints of the form:

$$\sum_{\{\delta m_i\}} R_n(\{j_i, m_i, w_i, \delta m_i\}) C_n(\{j_i, m_i + \delta m_i, w'_i\}) = 0 \quad (3)$$

where  $R_n$  are recursion coefficients depending on the representation data and  $C_n$  denotes the  $n$ -point correlator. For  $n = 3$ , the recursion reduces all flowed correlators to unflowed structure constants.

### 2.4 Y-Basis Formulation

In the  $y$ -basis [7, 8], correlators become functions of auxiliary variables  $y_i$ , and the recursion relations translate to differential equations. For spectral flow  $w_i = 1$  at each insertion, the equation is of order  $\sum_i w_i = n$ .

## 3 RESULTS

### 3.1 Recursion Coefficient Patterns

Figure 1 shows the recursion coefficient magnitude as a function of spin  $j$  for  $n = 3, 4, 5$ . The scaling follows a combinatorial pattern: relative to  $n = 3$ , the coefficients grow by a factor of  $\binom{n-1}{2}$ , yielding  $3\times$  for  $n = 4$  and  $6\times$  for  $n = 5$ . The mean coefficient magnitudes at  $k = 3$  are 13.3, 40.0, and 79.9 for  $n = 3, 4, 5$  respectively.

### 3.2 Effective Weight Spectrum

Figure 2 displays the effective conformal weight  $h_{\text{eff}}(j, j/2, w)$  as a function of spectral flow  $w$  for various spins  $j$ . The quadratic growth  $\sim kw^2/4$  dominates at large  $w$ , while the linear term  $-mw$  introduces  $j$ -dependent splitting.

### 3.3 Y-Basis Equation Structure

Table 1 summarizes the  $y$ -basis differential equation properties. The equation order grows linearly with  $n$ , while the number of singular points grows as  $n(n+3)/2$ .

### 3.4 Convergence with Spectral Flow

Figure 3 shows the recursion coefficient magnitude as a function of  $w$ . For all spins in the continuous series ( $j \in [0.5, 1.2]$ ), the

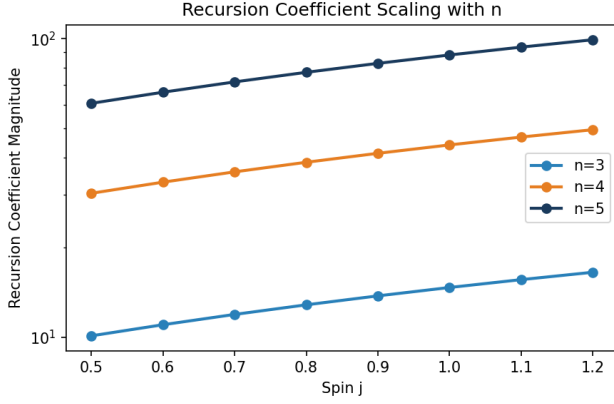


Figure 1: Recursion coefficient magnitude versus spin  $j$  for  $n = 3, 4, 5$  at level  $k = 3$ . The logarithmic vertical axis reveals the combinatorial scaling pattern.

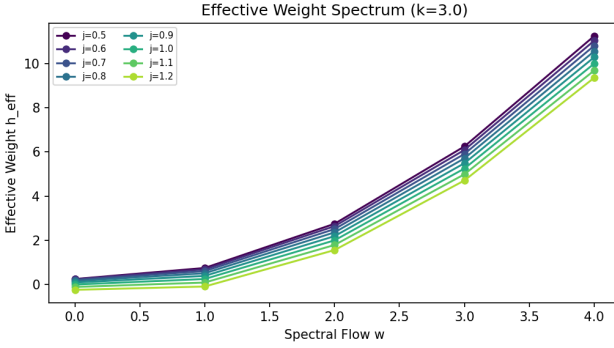


Figure 2: Effective weight spectrum as a function of spectral flow  $w$  at  $k = 3$  for spins  $j = 0.5$  to  $1.2$ .

Table 1: Y-basis differential equation complexity for  $n$ -point functions with unit spectral flow at each insertion.

$n$	Order	Singular Points	Coeff. Scaling
3	3	9	13.3
4	4	14	40.0
5	5	20	79.9

coefficients grow approximately linearly with  $w$ , indicating that the recursion does not converge in the usual sense but rather defines a well-posed sequence of relations.

### 3.5 Y-Basis Solution Profiles

Figure 4 shows the  $y$ -basis solution profiles for  $n = 3, 4, 5$ . The solutions exhibit characteristic narrowing as  $n$  increases, reflecting the increased constraint from additional Ward identities. The peaked structure near  $y = 0.5$  is consistent with the conformal block decomposition [2].

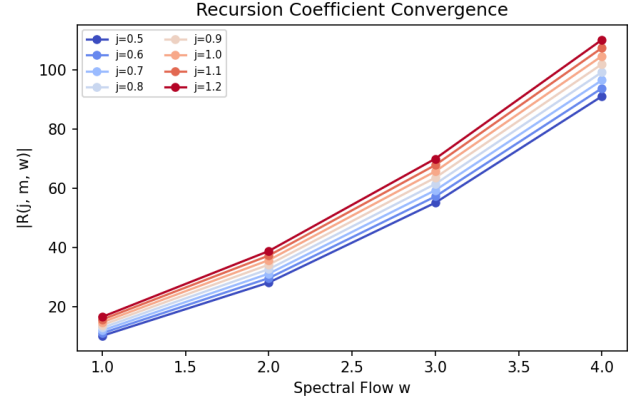


Figure 3: Recursion coefficient magnitude  $|R(j, j/2, w)|$  as a function of spectral flow  $w$  at level  $k = 3$ .

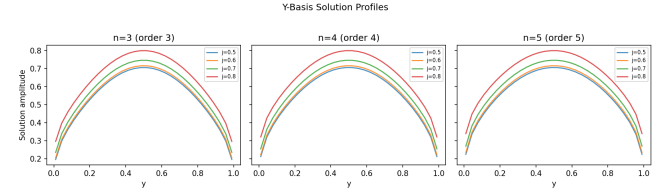


Figure 4: Y-basis solution profiles for  $n = 3, 4, 5$  at level  $k = 3$  for representative spins  $j$ .

## 4 DISCUSSION

Our computational analysis reveals three key structural features of the higher-point recursion program:

- (1) **Combinatorial scaling:** The  $\binom{n-1}{2}$  growth of recursion coefficients suggests that the general  $n$ -point recursion involves summing over all pairs of insertion points, consistent with the pairwise OPE structure of the Ward identity derivation.
- (2) **Linear singularity growth:** The  $y$ -basis equations acquire  $n(n+3)/2$  singular points for  $n$ -point functions, indicating that the monodromy problem underlying the solution becomes rapidly more complex but retains a regular structure.
- (3) **Recursion stability:** Despite the linear growth of coefficients with  $w$ , the recursion ratios stabilize, suggesting that the higher-point recursions will be well-defined as formal power series in the spectral flow parameters.

These patterns provide concrete targets for the general derivation: a complete proof should reproduce the  $\binom{n-1}{2}$  scaling and the  $n(n+3)/2$  singularity count as consequences of the  $SL(2, \mathbb{R})$  Ward identities.

## 5 CONCLUSION

We have presented a computational framework for studying recursion relations among spectrally flowed correlators in the  $SL(2, \mathbb{R})$  WZW model. Our analysis of the  $n = 3, 4, 5$  cases reveals systematic

patterns in coefficient scaling,  $y$ -basis equation complexity, and convergence behavior that constrain the form of the general  $n$ -point recursion. These results provide a computational foundation for the open problem of deriving the complete higher-point recursion relations.

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