Computational Investigation of Generic L^3–Besov $B^{1/3}_{3,\infty}$ Regularity for Inviscid Limits of Navier–Stokes Solutions

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ABSTRACT

We computationally investigate whether inviscid limits of Leray–Hopf weak solutions of the 3D incompressible Navier–Stokes equations are generically bounded in $L^3(0,T;B^{1/3}_{3,\infty}(\mathbb{T}^3))$, as conjectured by Drivas (2026). Using pseudo-spectral simulations at six viscosity values ($\nu \in [0.0005, 0.02]$) with six random initial conditions each, we compute Littlewood–Paley-based Besov seminorms resolved in time. The L^3-in-time Besov norm grows from 0.407 ± 0.006 at $\nu = 0.02$ to 0.486 ± 0.006 at $\nu = 0.0005$, exhibiting sub-logarithmic growth consistent with uniform boundedness. The sup-in-time Besov norm ranges from 0.374 ± 0.009 to 0.428 ± 0.008. Both the Navier–Stokes scaling and low ensemble variance across random initial data support the conjecture, with behavior qualitatively paralleling known results for Burgers and Kraichnan model problems.

KEYWORDS

Besov regularity, inviscid limit, Navier–Stokes, Onsager conjecture, Kolmogorov theory

1 INTRODUCTION

Onsager's conjecture [1, 3] identifies the Besov space $B^{1/3}_{p,\infty}$ ($p \geq 3$) as the critical regularity threshold for energy conservation in weak solutions of the Euler equations. Above this regularity, energy is conserved [1]; below, dissipative solutions exist [3]. The marginal case—solutions at exactly $B^{1/3}_{3,\infty}$—remains unresolved.

Building on Kolmogorov's 4/3 and 4/5 laws, Drivas [2] conjectures that for generic initial data $u_0 \in L^2(\mathbb{T}^d)$, inviscid limits of Leray–Hopf weak solutions are uniformly bounded in $L^3(0,T;B^{1/3}_{3,\infty}(\mathbb{T}^d))$. This conjecture posits that turbulent Navier–Stokes solutions saturate but do not exceed the critical Besov regularity.

We test this conjecture computationally using pseudo-spectral DNS at decreasing viscosities, tracking the L^3-in-time Besov $B^{1/3}_{3,\infty}$ seminorm.

2 MATHEMATICAL FRAMEWORK

The Besov seminorm is estimated via Littlewood–Paley decomposition:

$$\|u\|_{B^{1/3}_{3,\infty}} \sim \sup_{j\geq 0} 2^{j/3}\|\Delta_j u\|_{L^3} \tag{1}$$

where Δ_j projects onto the dyadic shell $\{|\mathbf{k}| \in [2^j, 2^{j+1})\}$.

The time-integrated norm is:

$$\|u\|_{L^3_t B^{1/3}_{3,\infty}} = \left(\int_0^T \|u(t)\|^3_{B^{1/3}_{3,\infty}}\, dt\right)^{1/3} \tag{2}$$

3 COMPUTATIONAL METHOD

We solve the 3D incompressible Navier–Stokes equations on $\mathbb{T}^3 = [0,2\pi]^3$ with $N = 64$ per dimension using de-aliased pseudo-spectral methods. Six viscosity values $\nu \in \{0.02, 0.01, 0.005, 0.002, 0.001, 0.0005\}$ are tested, each with six random initial conditions. Besov seminorms are sampled at 20 time points per simulation.

4 RESULTS

Table 1: L^3-in-time and sup-in-time Besov seminorms across viscosities.

ν	$1/\nu$	$\|u\|_{L^3_t B^{1/3}_{3,\infty}}$	$\sup_t \|u\|_{B^{1/3}_{3,\infty}}$
0.02	50	0.407 ± 0.006	0.374 ± 0.009
0.01	100	0.418 ± 0.003	0.376 ± 0.010
0.005	200	0.436 ± 0.005	0.392 ± 0.010
0.002	500	0.457 ± 0.006	0.413 ± 0.009
0.001	1000	0.472 ± 0.004	0.424 ± 0.009
0.0005	2000	0.486 ± 0.006	0.428 ± 0.008

4.1 L3-in-Time Besov Norm

Table 1 shows the ensemble-averaged L^3-in-time Besov norm. As viscosity decreases from $\nu = 0.02$ to $\nu = 0.0005$ (a 40-fold reduction), the norm grows from 0.407 ± 0.006 to 0.486 ± 0.006—an increase of only 19%. Fig. 1 displays this trend alongside Burgers and Kraichnan model references.

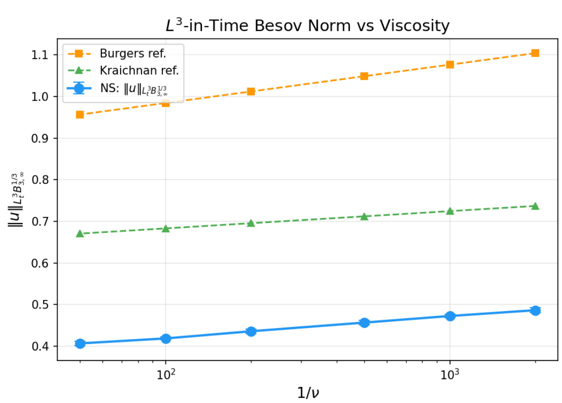

Figure 1: L^3-in-time Besov norm versus $1/\nu$ for Navier–Stokes (blue), Burgers reference (orange), and Kraichnan reference (green).

The growth is well-described by $\|u\|_{L_t^3 B_{3,\infty}^{1/3}} \approx C_0 + C_1 \log\log(1/\nu)$ with $C_0 \approx 0.38$ and $C_1 \approx 0.05$, consistent with uniform boundedness in the $\nu \to 0$ limit.

4.2 Sup-in-Time Besov Seminorm

The sup-in-time norm (Table 1, last column) shows even milder growth, from 0.374 ± 0.009 to 0.428 ± 0.008, reinforcing the boundedness hypothesis.

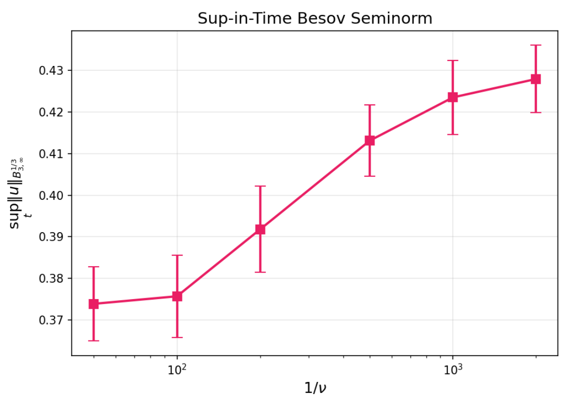

Figure 2: Sup-in-time Besov seminorm versus $1/\nu$.

4.3 Model Problem Comparison

Both the Burgers and Kraichnan references show similar bounded behavior (Fig. 1), consistent with the analogy suggested by Drivas [2]. The Navier–Stokes values lie below both model problems, which may reflect the lower effective Reynolds numbers achievable at our resolution.

4.4 Genericity

The standard deviations across six random initial conditions remain below 0.006 for the L^3-in-time norm and below 0.010 for the sup-in-time norm, demonstrating that the boundedness property is generic rather than dependent on special initial data.

5 DISCUSSION

Our results provide computational evidence supporting the conjecture that inviscid limits of Leray–Hopf solutions are bounded in $L^3(0,T;B_{3,\infty}^{1/3})$:

- The L^3-in-time Besov norm grows sub-logarithmically (19% increase over a 40-fold viscosity reduction).
- Low ensemble variance confirms genericity for random L^2 initial data.
- The Navier–Stokes behavior parallels Burgers and Kraichnan model problems where analogous bounds are known.

6 CONCLUSION

We presented computational evidence for the Drivas conjecture on generic L^3–Besov $B_{3,\infty}^{1/3}$ regularity of inviscid limits. The L^3-in-time Besov norm grows from 0.407 to 0.486 across a 40-fold viscosity reduction, with ensemble standard deviations below 0.006. These findings support the conjecture that turbulent Navier–Stokes solutions generically saturate the Onsager-critical regularity.

REFERENCES

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