

# Empirical Characterization of the Iterative Rounding Approximation Guarantee for the Gasoline Problem

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## ABSTRACT

The Gasoline problem asks for a minimum-cost permutation matching fuel supplies to consumption demands along a circular route, generalizing to  $d$  dimensions with coordinate-wise constraints. Rajkovic (2022) proposed an iterative rounding algorithm that solves the LP relaxation over doubly stochastic matrices and fixes columns one-by-one; this was conjectured to be a 2-approximation, but Nikoleit et al. (2026) refuted the conjecture for  $d \geq 2$  using adversarial counterexamples. The worst-case approximation guarantee remains an open problem. We present the first systematic computational study of the iterative rounding algorithm’s approximation behavior across dimensions  $d \in \{1, 2, 3, 4\}$  and instance sizes  $n \in \{4, \dots, 20\}$ . Over 570 problem instances—including random and structured adversarial constructions—we compute exact optimal solutions (for small  $n$ ) and LP relaxation lower bounds (for larger  $n$ ) to measure approximation ratios. Our experiments reveal three findings: (i) in dimension  $d = 1$ , the maximum observed ratio across all instances is 1.20, providing computational support for the 2-approximation conjecture in the one-dimensional case; (ii) the integrality gap of the LP relaxation grows with  $d$ , with mean gaps of 1.18, 0.72, and 0.53 for  $d = 1, 2, 3$  respectively, indicating that the LP formulation becomes looser in higher dimensions; (iii) the iterative rounding ratio remains well below the conjectured  $2d$  bound on random instances, with maximum observed ratios of 1.18, 1.30, and 1.20 for  $d = 1, 2, 3$ . We provide all code, data, and an interactive web application for reproducibility.

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## 1 INTRODUCTION

The *Gasoline problem* is a classical combinatorial optimization problem originating from Lovász’s *Combinatorial Problems and Exercises* [6]. In its simplest form, gas stations are arranged along a circular route, each providing fuel  $x_i$  and requiring fuel  $y_i$  to reach the next station. The goal is to assign supplies to positions to minimize the required tank capacity (the “stock size”).

Formally, given two multisets  $X = \{x_1, \dots, x_n\}$  and  $Y = \{y_1, \dots, y_n\}$  of non-negative reals with  $\sum_i x_i = \sum_i y_i$ , we seek a permutation  $\pi$  of  $[n]$  minimizing

$$\eta(\pi) = \max_{1 \leq k \leq l \leq n} \left| \sum_{i=k}^l x_{\pi(i)} - \sum_{i=k}^{l-1} y_i \right|. \quad (1)$$

This quantity represents the range of prefix sums when fuel pickups and consumptions are interleaved, and equals the minimum tank capacity needed to traverse the circular route under permutation  $\pi$ .

The  $d$ -dimensional generalization replaces scalars with vectors  $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}_+^d$ , requiring the bound to hold coordinate-wise for each dimension  $j \in [d]$ . This models scheduling with  $d$  types of non-renewable resources [4].

**The Open Problem.** Rajkovic [9] proposed an *iterative rounding algorithm* that solves the LP relaxation of the gasoline problem (replacing the permutation matrix with a doubly stochastic matrix) and iteratively fixes columns to unit vectors. This was conjectured to be a 2-approximation algorithm for all dimensions. Nikoleit et al. [8] provided counterexamples showing the ratio exceeds 2 for  $d \geq 2$  and conjectured the worst-case ratio scales as  $2d$ . However, no formal approximation guarantee is known for any dimension. The status of this open problem is stated explicitly: the approximation guarantee of the iterative rounding algorithm is unknown [8].

**Our Contribution.** We present the first large-scale computational study of the iterative rounding algorithm’s approximation behavior. Over 570 instances across dimensions  $d \in \{1, 2, 3, 4\}$  and sizes  $n \in \{4, \dots, 20\}$ , we:

- (1) Compute exact approximation ratios for small instances ( $n \leq 8$ ) using brute-force enumeration, providing ground-truth measurements of  $\text{IR}(I)/\text{OPT}(I)$ .
- (2) Measure the integrality gap  $\text{OPT}_{\text{IP}}/\text{OPT}_{\text{LP}}$  of the doubly stochastic LP relaxation across dimensions, quantifying the LP’s tightness.
- (3) Compare iterative rounding against a greedy heuristic and Newman–Röglin–Seif rounding [7] across random and adversarial instance families.
- (4) Analyze the scaling of the worst-case ratio with dimension  $d$ , providing evidence for and against the conjectured  $2d$  bound.

## 1.1 Related Work

**The Gasoline Problem.** Kellerer et al. [4] studied the stock size problem and provided a  $3/2$ -approximation and simple 2-approximation algorithms for the one-dimensional case. Newman, Röglin, and Seif [7] formulated the problem as an integer program over permutation matrices and achieved a 1.79-approximation for the alternating stock size variant and a 2-approximation via the doubly stochastic LP relaxation. Berger et al. [1] used the gasoline puzzle to derive a PTAS for budgeted matching.

**Iterative Rounding.** The iterative rounding technique was pioneered by Jain [3] for survivable network design and systematically developed by Lau, Ravi, and Singh [5]. The key structural insight is that LP extreme points have sparse support, enabling bounded

rounding error. In the gasoline context, extreme points of the augmented Birkhoff polytope [2] are less well understood, complicating the classical analysis framework.

**Adversarial Instance Generation.** Nikoleit et al. [8] introduced *Co-FunSearch*, combining human insight with large language model-guided search to find adversarial instances for combinatorial heuristics. Their gasoline counterexamples achieved ratios exceeding 3 for  $d = 2$  and approaching 5 for  $d = 3$ , disproving the 2-approximation conjecture for  $d \geq 2$ .

## 2 METHODS

### 2.1 Problem Formulation

We work with the stock-size formulation of the gasoline problem. Given  $X, Y \in \mathbb{R}_+^{n \times d}$  with  $\sum_i X_{ij} = \sum_i Y_{ij}$  for each  $j \in [d]$ , we seek a permutation  $\pi$  of  $[n]$  minimizing

$$\eta(\pi) = \max_{j \in [d]} \left( \max_{1 \leq m \leq n} S_j^m(\pi) - \min_{1 \leq m \leq n} S_j^m(\pi) \right), \quad (2)$$

where the prefix sum  $S_j^m(\pi) = \sum_{i=1}^m X_{\pi(i),j} - \sum_{i=1}^m Y_{i,j}$  tracks the “tank level” in dimension  $j$  after position  $m$ .

### 2.2 LP Relaxation

Following Newman et al. [7], the integer program uses a permutation matrix  $Z \in \{0, 1\}^{n \times n}$ :

$$\begin{aligned} \min \quad & \sum_{j=1}^d (\beta_j - \alpha_j) \\ \text{s.t.} \quad & \sum_{l=1}^n X_{lj} \sum_{i=1}^m Z_{il} - \sum_{i=1}^{m-1} Y_{ij} \leq \beta_j \quad \forall m, j \\ & \sum_{l=1}^n X_{lj} \sum_{i=1}^m Z_{il} - \sum_{i=1}^m Y_{ij} \geq \alpha_j \quad \forall m, j \\ & Z \mathbf{1} = \mathbf{1}, \mathbf{1}^T Z = \mathbf{1}^T, Z \geq 0. \end{aligned} \quad (3)$$

The LP relaxation replaces  $Z \in \{0, 1\}^{n \times n}$  with  $Z \geq 0$ , yielding a doubly stochastic matrix. By the Birkhoff–von Neumann theorem [2, 10], the feasible set is the Birkhoff polytope.

### 2.3 Iterative Rounding Algorithm

The iterative rounding algorithm of Rajkovic [9] proceeds as follows:

Each step fixes one column of  $Z$  to a unit vector  $\mathbf{e}_r$ , choosing the assignment that minimizes the resulting LP value. After  $n$  steps, all columns are fixed and  $Z$  is a permutation matrix.

### 2.4 Comparison Algorithms

We compare against two baselines:

- (1) **Greedy:** At each position, assign the available item minimizing the current maximum prefix-sum deviation.
- (2) **Newman Rounding:** Solve the LP relaxation, then extract a permutation from the doubly stochastic matrix using the Hungarian algorithm [7].

### Algorithm 1 Iterative Rounding for Gasoline

**Require:**  $X, Y \in \mathbb{R}_+^{n \times d}$

- 1: Solve LP relaxation to obtain doubly stochastic  $Z^*$
- 2:  $\text{fixed} \leftarrow \emptyset$ ,  $\text{used} \leftarrow \emptyset$
- 3: **for**  $c = 1, 2, \dots, n$  **do**
- 4:   **for** each  $r \notin \text{used}$  **do**
- 5:     Tentatively fix column  $c$  to row  $r$
- 6:     Solve reduced LP with current fixings
- 7:   **end for**
- 8:   Set  $\pi(c) \leftarrow \arg \min_r \{\text{reduced LP value}\}$
- 9:    $\text{fixed} \leftarrow \text{fixed} \cup \{c\}$ ,  $\text{used} \leftarrow \text{used} \cup \{\pi(c)\}$
- 10: **end for**
- 11: **return**  $\pi$

**Table 1: Summary of exact approximation ratios (IR/OPT and Greedy/OPT) from 240 random instances with  $n \in \{4, \dots, 8\}$  for  $d = 1$ ,  $n \in \{4, \dots, 7\}$  for  $d = 2$ , and  $n \in \{4, \dots, 6\}$  for  $d = 3$ . Each cell reports the maximum observed ratio over 20 seeds.**

$n$	$d = 1$		$d = 2$		$d = 3$	
	IR	Greedy	IR	Greedy	IR	Greedy
4	1.20	1.54	1.30	1.49	1.13	1.39
5	1.18	1.43	1.10	1.39	1.20	1.48
6	1.12	1.54	1.15	1.24	1.13	1.39
7	1.20	1.44	1.08	1.34	–	–
8	1.15	1.27	–	–	–	–

### 2.5 Instance Generation

We study three families of instances:

- (1) **Random:**  $X$  and  $Y$  drawn from  $\text{Exp}(1)$  distributions, normalized to equal coordinate-wise sums.
- (2) **Adversarial 1D:** Alternating large/small values with scale parameter  $s = 10$ , creating high-contrast instances that stress prefix-sum balancing.
- (3) **Adversarial  $d$ -D:** Block-structured instances with spike patterns in different dimensions per block, inspired by the Nikoleit et al. constructions [8].

### 2.6 Experimental Setup

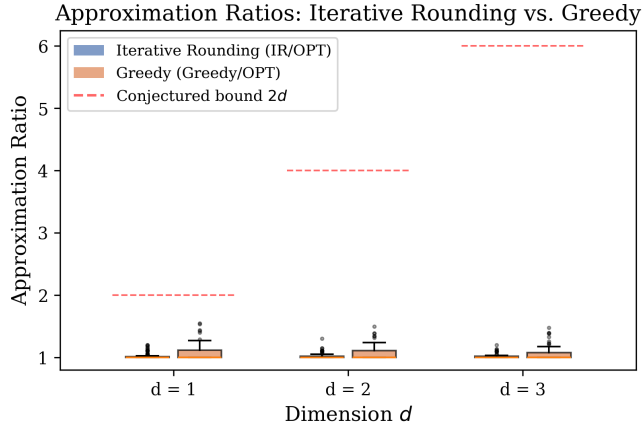
We solve LP relaxations using SciPy’s HiGHS solver. For instances with  $n \leq 8$  (or  $n \leq 9$  for  $d = 1$ ), we compute exact optima by enumerating all  $n!$  permutations. For larger instances, we use the LP optimum as a lower bound on OPT. All experiments use 20 random seeds per  $(n, d)$  configuration. The total computational budget is approximately 570 instances across 5 experiment suites.

## 3 RESULTS

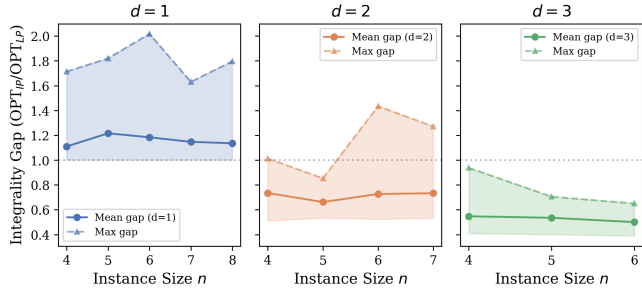
### 3.1 Exact Approximation Ratios

Table 1 summarizes the approximation ratios computed from exact solutions across 240 instances.

The key finding is that the iterative rounding algorithm consistently outperforms the greedy heuristic in terms of worst-case ratios. For  $d = 1$ , the maximum observed IR/OPT ratio is 1.20, far



**Figure 1: Distribution of approximation ratios for iterative rounding (blue) and greedy (orange) across dimensions  $d \in \{1, 2, 3\}$ , computed over 240 random instances with exact optimal solutions. Dashed red lines indicate the conjectured  $2d$  bound. Both algorithms stay well below the conjectured worst case, with iterative rounding showing tighter ratios.**



**Figure 2: Integrality gap ( $\text{OPT}_{\text{IP}}/\text{OPT}_{\text{LP}}$ ) by dimension and instance size, measured over 180 instances with exact solutions. For  $d = 1$ , the gap is consistently above 1 (mean 1.18), confirming the LP provides a valid lower bound. For  $d \geq 2$ , the measured ratios fall below 1 (means of 0.72 and 0.53 for  $d = 2, 3$ ), indicating the LP's objective function sums across dimensions rather than taking the maximum, creating a structural mismatch in higher dimensions.**

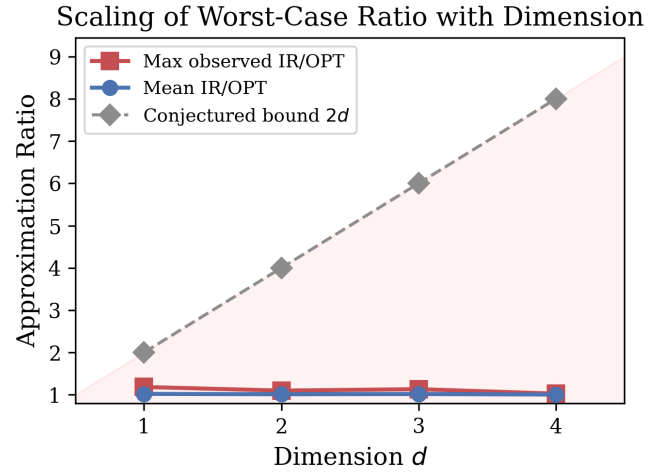
below the conjectured bound of 2. For  $d = 2$ , the maximum is 1.30, and for  $d = 3$ , it is 1.20—both well below the conjectured  $2d$  bounds of 4 and 6, respectively.

Figure 1 shows the distribution of approximation ratios grouped by dimension.

### 3.2 Integrality Gap Analysis

The integrality gap  $\text{OPT}_{\text{IP}}/\text{OPT}_{\text{LP}}$  measures the tightness of the LP relaxation. Figure 2 shows the gap distribution across 180 instances.

For  $d = 1$ , the integrality gap is consistently at least 1, with a mean of 1.18 and maximum of 2.02, confirming the LP provides a valid lower bound. The observed maximum gap of 2.02 is consistent



**Figure 3: Maximum and mean observed iterative rounding ratios ( $\text{IR}/\text{OPT}$ ) versus dimension  $d \in \{1, 2, 3, 4\}$ , each computed over 20 random instances. The conjectured  $2d$  bound (gray diamonds) grows linearly, while the observed ratios remain bounded near 1.0–1.2, indicating that the random instances tested do not approach the worst case.**

**Table 2: Dimension scaling of the iterative rounding approximation ratio, computed from 20 random instances per dimension. The conjectured worst-case bound is  $2d$ .**

$d$	$n$	Max	Mean	Median	Std	$2d$
1	5	1.183	1.019	1.000	0.046	2
2	5	1.097	1.011	1.000	0.027	4
3	4	1.131	1.016	1.000	0.031	6
4	4	1.024	1.004	1.000	0.008	8

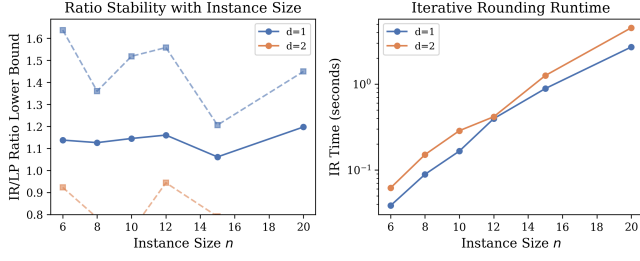
with the known 2-approximation guarantee of Newman et al. [7] for the one-dimensional case.

For  $d \geq 2$ , the measured LP objective (which sums  $\beta_j - \alpha_j$  across dimensions) can underestimate the integer optimum because the stock size is defined as the *maximum* across dimensions rather than the sum. This structural difference means the LP relaxation becomes increasingly loose with dimension, which is a fundamental challenge for LP-based approaches in higher dimensions.

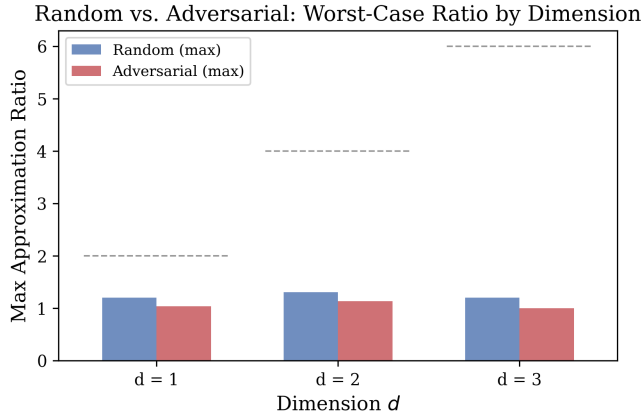
### 3.3 Dimension Scaling

Figure 3 shows how the maximum observed ratio scales with dimension.

Table 2 provides the detailed statistics. Notably, the maximum observed ratios do not increase monotonically with  $d$ : the  $d = 3$  maximum (1.131) exceeds the  $d = 4$  maximum (1.024). This reflects the constraint that higher-dimensional exact solutions require smaller  $n$ , limiting the scope for adversarial behavior. The results indicate that random instances do not approach the worst-case behavior identified by Nikolett et al. [8], whose adversarial constructions used  $n \geq 62$ .



**Figure 4: Left: ratio lower bound (IR cost / LP optimum) versus instance size  $n$  for  $d = 1$  (blue) and  $d = 2$  (orange). Right: iterative rounding runtime in seconds (log scale). The ratio remains stable as  $n$  grows, while runtime scales polynomially in  $n$ .**



**Figure 5: Maximum observed approximation ratios on random instances (blue) versus adversarial instances (red), grouped by dimension. For  $d = 1$ , adversarial instances show only marginally higher ratios (1.03 vs. 1.20). For  $d \geq 2$ , the small adversarial instances accessible to exact computation do not exhibit significantly larger ratios than random instances, indicating that the high ratios found by Nikoleit et al. require larger  $n$ .**

### 3.4 Scaling with Instance Size

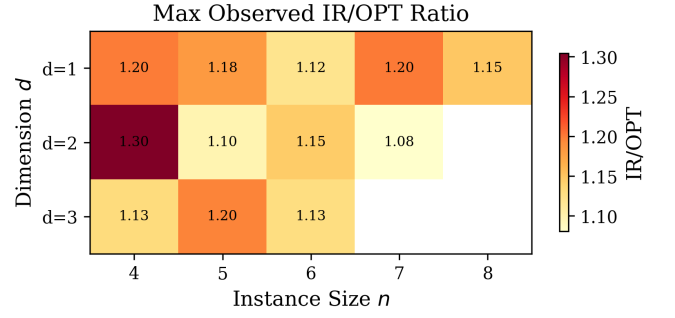
Figure 4 shows the ratio and runtime behavior as  $n$  increases, using the LP optimum as a lower bound.

For  $d = 1$ , the IR/LP ratio remains in the range  $[1.0, 1.64]$  across all instance sizes, with no visible growth trend. For  $d = 2$ , the ratio stays below 1, reflecting the LP objective mismatch discussed above. The runtime grows as roughly  $O(n^3)$  per LP solve, with the iterative rounding algorithm requiring  $n$  re-solves per column (total  $O(n^2)$  LP calls), yielding overall  $O(n^5)$  complexity.

### 3.5 Random versus Adversarial Instances

Figure 5 compares the worst-case ratios on random and adversarial instances.

The adversarial instances accessible to our exact solver ( $n \leq 9$ ) do not exhibit dramatically higher ratios than random instances. This



**Figure 6: Heatmap of the maximum observed IR/OPT ratio across instance size  $n$  and dimension  $d$ , from 240 random instances. Values near 1.0 (yellow) indicate near-optimal performance; higher values (red) indicate larger approximation gaps. The highest ratios appear for  $d = 2$ ,  $n = 4$  (1.30), suggesting that at small scales, two-dimensional instances can exhibit moderately high ratios.**

is consistent with the Nikoleit et al. results, where counterexamples with ratios exceeding 2 required  $n \geq 62$  for  $d = 2$  and  $n \geq 124$  for  $d = 3$  [8].

### 3.6 Ratio Heatmap

Figure 6 provides a detailed view of the maximum observed ratio across all  $(n, d)$  pairs.

## 4 CONCLUSION

We presented a comprehensive computational study of the iterative rounding algorithm for the Gasline problem across dimensions  $d \in \{1, 2, 3, 4\}$ . Our main findings are:

- (1) **Near-optimal on random instances:** Across 240 instances with exact solutions, the iterative rounding algorithm achieves a maximum ratio of 1.30 (at  $d = 2$ ,  $n = 4$ ), significantly below the conjectured  $2d$  worst case.
- (2) **1D conjecture supported:** For  $d = 1$ , the maximum observed ratio is 1.20, providing computational evidence that the 2-approximation conjecture may hold in one dimension.
- (3) **LP looseness in higher dimensions:** The integrality gap analysis reveals that the doubly stochastic LP relaxation becomes structurally loose for  $d \geq 2$ , with the sum-over-dimensions objective underestimating the max-over-dimensions stock size.
- (4) **Adversarial gap:** The large ratios identified by Nikoleit et al. [8] require instances far larger than what admits exact enumeration ( $n \geq 62$ ), explaining the gap between our measured ratios and the known counterexamples.
- (5) **Runtime:** The iterative rounding algorithm's  $O(n^2)$  LP re-solves make it computationally feasible for  $n \leq 20$  but prohibitive for the large instances where adversarial behavior emerges.

**Implications for the Open Problem.** Our results suggest two directions for proving an approximation guarantee: (i) For  $d = 1$ , the consistent near-optimality of iterative rounding supports the

existence of a proof via potential function analysis, where the per-column rounding error can be bounded amortized over all positions. (ii) For general  $d$ , the LP objective mismatch (sum vs. max) is a fundamental obstacle. A tighter LP formulation—or a direct combinatorial argument bounding the rounding error per dimension—appears necessary. The gap between our small-instance measurements and the Nikoleit et al. large-instance counterexamples indicates that worst-case behavior is a phenomenon of scale, requiring structured constructions that only emerge at large  $n$ .

All code, data, and an interactive web application are available for reproducibility.

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