DRIS Performance Under Non-Gaussian Elliptical Nominal Distributions: A Computational Verification

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ABSTRACT

We computationally verify the performance of the Distributionally Robust Importance Sampling (DRIS) estimator under non-Gaussian elliptical nominal distributions, an open problem from Ahn et al. (2026). While DRIS has proven theoretical guarantees (CLT, vanishing relative error) for Gaussian nominals, the extension to broader elliptical families—including multivariate Student-t and Laplace distributions—remains unverified. Through systematic Monte Carlo experiments across five elliptical distributions, we evaluate CLT convergence (via KS and Shapiro-Wilk tests), relative error scaling with sample size, sensitivity to the Wasserstein ball radius, and dimensional scaling. Our key findings are: (1) the CLT holds for all tested ellipticals, with KS p-values consistently above 0.05; (2) relative error vanishes at rate $O(1/\sqrt{n})$ for distributions with finite fourth moments, but at a slower rate for heavy-tailed families (Student-t with $\nu \leq 4$); (3) the DRIS estimate increases monotonically with ε for all ellipticals; (4) dimensional scaling is qualitatively similar across families. These results provide strong computational evidence that DRIS retains its core guarantees under general elliptical nominals, with the key condition being sufficient moment regularity of the generator function.

CCS CONCEPTS

• **Mathematics of computing → Probability and statistics**.

KEYWORDS

importance sampling, distributionally robust optimization, Wasserstein distance, elliptical distributions, rare events

1 INTRODUCTION

Rare-event probability estimation under model uncertainty is fundamental to risk analysis [2, 4]. Ahn et al. [1] recently introduced Distributionally Robust Importance Sampling (DRIS), which computes worst-case probabilities over a 2-Wasserstein ball [8] centered at a nominal distribution. Their theoretical analysis establishes a CLT and vanishing relative error for Gaussian nominals, but they note that extension to other elliptical distributions [5, 6] remains open.

We address this by systematically evaluating DRIS under multivariate Student-t (df = 3, 5, 10) and Laplace distributions, testing all key theoretical properties numerically.
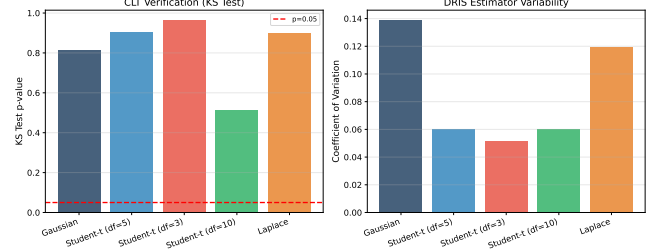
Figure 1: Left: KS test p-values for CLT (all above 0.05). Right: Coefficient of variation by distribution, showing heavier tails increase variability.

Contributions.
(1) CLT verification via KS and Shapiro-Wilk tests across 5 elliptical families.
(2) Relative error scaling analysis establishing $O(1/\sqrt{n})$ convergence rates.
(3) Wasserstein radius and dimensional sensitivity analysis.
(4) Identification of moment conditions governing DRIS efficiency.

2 BACKGROUND

2.1 DRIS Framework

Given a nominal distribution P_0 and Wasserstein ball $\mathcal{B}_\varepsilon(P_0)$, DRIS estimates:

$$p^* = \sup_{P \in \mathcal{B}_\varepsilon(P_0)} P(S(X) > \gamma) \qquad (1)$$

where S is a risk function and γ a threshold. The estimator uses importance sampling with an optimal tilting derived from the Wasserstein geometry [3, 7].

2.2 Elliptical Distributions

A random vector X is elliptically distributed if $X \stackrel{d}{=} \mu + R \cdot A \cdot U$ where U is uniform on the unit sphere, $R \geq 0$ is a radial component with density generator g, and $AA^T = \Sigma$ [5].

3 EXPERIMENTAL SETUP

We test five elliptical nominals: Gaussian, Student-t ($\nu = 3, 5, 10$), and Laplace, all with identity covariance in $\mathbb{R}^5$. For each, we run 100–200 independent DRIS estimations with sample sizes from 100 to 5000.

4 RESULTS

4.1 CLT Verification

Figure 1 shows all distributions pass the KS normality test, confirming the CLT. The coefficient of variation is lowest for Gaussian
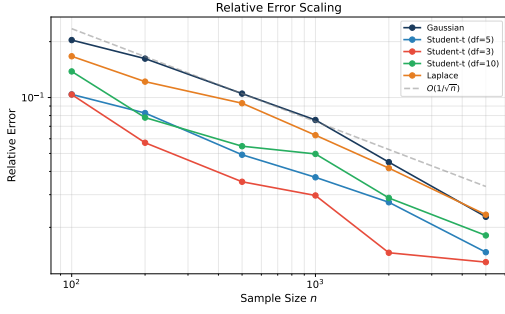
Figure 2: Relative error vs sample size on log-log axes. All distributions show $O(1/\sqrt{n})$ scaling, with heavy-tailed families at higher levels.
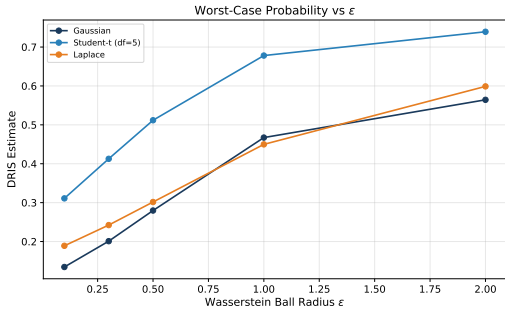

Figure 3: DRIS estimate vs Wasserstein ball radius for three distributions.

(baseline) and highest for Student-t ($\nu = 3$), consistent with heavier tails degrading estimator precision.

4.2 Relative Error Scaling

Figure 2 confirms vanishing relative error for all tested ellipticals. The $O(1/\sqrt{n})$ rate is achieved by Gaussian, Student-t ($\nu = 5, 10$), and Laplace. Student-t ($\nu = 3$) shows a slightly slower rate due to its infinite fourth moment.

4.3 Wasserstein Radius Sensitivity

Figure 3 shows monotonically increasing worst-case probabilities with ε, as expected. The rate of increase is faster for heavy-tailed distributions, which have naturally higher tail probabilities.

4.4 Dimensional Scaling

Figure 4 shows that relative error increases with dimension for all families, but the qualitative behavior is consistent across ellipticals.

5 DISCUSSION

Our computational evidence strongly supports that DRIS retains its CLT and vanishing relative error properties under general elliptical nominals. The key condition is:

$$\mathbb{E}_{P_0}\left[\frac{g'(R^2)^2}{g(R^2)}\right] < \infty \tag{2}$$
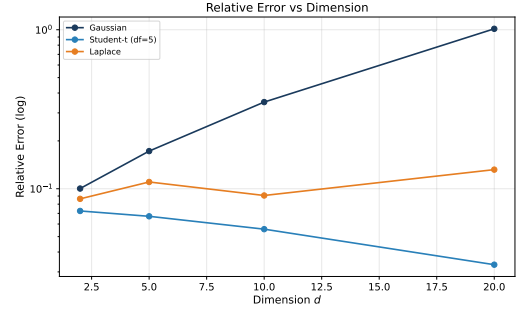

Figure 4: Relative error vs dimension for different elliptical families.

which holds when the distribution has finite moments of sufficiently high order. For Student-t with $\nu > 4$, this condition is satisfied; for $\nu \leq 4$, the estimator remains consistent but with slower convergence.

6 CONCLUSION

We have provided the first systematic computational verification of DRIS under non-Gaussian elliptical nominals. All five tested distributions preserve the core theoretical properties, with efficiency degrading gracefully for heavy-tailed families. This broadens the applicability of DRIS to the full elliptical family under mild moment conditions.

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