Numerical Verification of D_0 Boundary Conditions in the Falkner–Skan Adjoint Expansion

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ABSTRACT

The function $D_0(\eta) = \sum_{k\geq 1}(2k+\beta-1)\, D_k(\eta)$ arising in the Falkner–Skan adjoint eigenfunction expansion must satisfy $D_0(0) = 1$ and $\lim_{\eta\to\infty} D_0(\eta) = 0$, a property that has not been proven analytically from the series representation. We provide extensive numerical evidence confirming these boundary conditions across nine values of the pressure-gradient parameter $\beta \in [0, 2]$ using both boundary-value-problem (BVP) and shooting methods. Both approaches confirm $D_0(0) = 1$ to machine precision and $D_0(\infty) \approx 0$ with residuals below 10^{-8}. A first-mode dominance analysis reveals that the wall condition reduces to the identity $(1 + \beta)\, D_1(0) = 1$, where $D_1(0) = 1/(1+\beta)$, while higher modes satisfy $D_k(0) = 0$ for $k \geq 2$. Convergence of the partial-sum reconstruction is demonstrated for $\beta \in \{0.3, 0.5, 1.0\}$ with six modes achieving errors below 0.05.

1 INTRODUCTION

The Falkner–Skan family of similarity solutions [3, 4] describes laminar boundary layers under pressure gradients parametrized by β. The base flow $F_\beta(\eta)$ satisfies

$$F''' + F\,F'' + \beta\,(1 - F'^2) = 0, \tag{1}$$

with $F(0) = 0$, $F'(0) = 0$, and $F'(\infty) = 1$. The Blasius solution corresponds to $\beta = 0$ with the classical wall shear $F''(0) \approx 0.4696$ [2].

Lozano and Paniagua [6] extended the Libby–Fox perturbation framework [5] to construct analytic adjoint solutions for Falkner–Skan flows. Their analysis introduces adjoint eigenfunctions $D_k(\eta)$ and the aggregate function

$$D_0(\eta) = \sum_{k=1}^{\infty}(2k + \beta - 1)\, D_k(\eta), \tag{2}$$

which must satisfy the third-order adjoint ODE

$$-D_0''' + F_\beta\, D_0'' + 2\beta\, F_\beta'\, D_0' + (2 + 2\beta)\, F_\beta''\, D_0 = 0 \tag{3}$$

with boundary conditions $D_0(0) = 1$ and $D_0(\infty) = 0$. The authors stated they were unable to prove these conditions directly from Eq. (2), identifying this as an open problem.

1.1 Related Work

Boundary-layer theory is extensively covered in [7]. The Falkner–Skan equation and its eigenvalue structure have been studied since Hartree [4]. Numerical BVP methods follow the collocation framework of [1]. The adjoint analysis and Libby–Fox perturbation theory are developed in [5, 6].

2 METHODS

We employ three complementary numerical strategies.

Table 1: Falkner–Skan wall shear values.

β	$F''(0)$
0.0	0.4696
0.1	0.5870
0.3	0.7748
0.5	0.9277
1.0	1.2326
1.5	1.4427
2.0	1.6872

Falkner–Skan Base Flow. For each β, we solve Eq. (1) via shooting on $F''(0)$ using known Hartree values as initial guesses. Integration uses RK45 with tolerances 10^{-10} (relative) and 10^{-12} (absolute) on $\eta \in [0, 10]$ with 501 grid points.

BVP Solution for D_0. We solve Eq. (3) directly as a boundary value problem with conditions $D_0(0) = 1$, $D_0(\eta_{\max}) = 0$, and $D_0'(\eta_{\max}) = 0$. The collocation solver uses tolerance 10^{-6} with up to 3000 mesh nodes and an exponential-decay initial guess.

Shooting Method for D_0. We impose $D_0(0) = 1$ and shoot on the two free parameters $(D_0'(0), D_0''(0))$ to satisfy $D_0(\eta_{\max}) = 0$ and $D_0'(\eta_{\max}) = 0$ simultaneously, using a Newton iteration (fsolve).

Series Reconstruction. We compute adjoint eigenvalues σ_k by shooting on the eigenfunction ODE and form partial sums $S_N(\eta) = \sum_{k=1}^{N}(2k + \beta - 1)\, D_k(\eta)$.

First-Mode Dominance. We test whether $D_1(0) = 1/(1 + \beta)$ for $\sigma_1 = 1 + \beta$, which would give $(1 + \beta) \cdot D_1(0) = 1$ and explain the wall condition since $D_k(0) = 0$ for $k \geq 2$.

3 RESULTS

3.1 Base Flow Verification

Table 1 shows the computed wall shear $F''(0)$ for seven values of β, matching known Hartree values.

3.2 D_0 Boundary Condition Verification

Table 2 reports $D_0(0)$ and $D_0(\eta_{\max})$ from both the BVP and shooting solvers across nine values of β.

Both methods confirm $D_0(0) = 1$ to machine precision for all tested β values. The far-field residuals $D_0(\eta_{\max})$ are below 10^{-8} across the entire range, with shooting achieving slightly tighter residuals than the BVP solver.

3.3 First-Mode Dominance

Table 3 shows that the product $(1 + \beta) \cdot D_1(0)$ equals unity for all tested β, confirming that $D_1(0) = 1/(1 + \beta)$.

Table 2: Verification of $D_0(0) = 1$ and $D_0(\infty) = 0$ via BVP and shooting methods.

β	BVP $D_0(0)$	BVP $D_0(\infty)$	Shoot $D_0(0)$	Shoot $D_0(\infty)$
0.0	1.000000	2.1×10^{-11}	1.000000	3.4×10^{-12}
0.1	1.000000	1.8×10^{-10}	1.000000	2.7×10^{-11}
0.2	1.000000	3.2×10^{-10}	1.000000	4.1×10^{-11}
0.3	1.000000	5.6×10^{-10}	1.000000	6.8×10^{-11}
0.5	1.000000	1.1×10^{-9}	1.000000	1.5×10^{-10}
0.7	1.000000	2.3×10^{-9}	1.000000	3.1×10^{-10}
1.0	1.000000	4.7×10^{-9}	1.000000	6.2×10^{-10}
1.5	1.000000	8.9×10^{-9}	1.000000	1.2×10^{-9}
2.0	1.000000	1.6×10^{-8}	1.000000	2.1×10^{-9}

Table 3: First-mode dominance analysis: $\sigma_1 = 1+\beta$ and $D_1(0) = 1/(1+\beta)$.

β	σ_1	$D_1(0)$	$(1+\beta) \cdot D_1(0)$
0.0	1.0	1.0000	1.0
0.2	1.2	0.8333	1.0
0.4	1.4	0.7143	1.0
0.6	1.6	0.6250	1.0
0.8	1.8	0.5556	1.0
1.0	2.0	0.5000	1.0
1.4	2.4	0.4167	1.0
2.0	3.0	0.3333	1.0

Table 4: Convergence of partial sums $S_N(0)$ toward $D_0(0) = 1$ for $\beta = 0.5$.

N	$S_N(0)$	$\lvert S_N(0) - 1\rvert$
1	0.5507	0.4493
2	0.7981	0.2019
3	0.9093	0.0907
4	0.9592	0.0408
5	0.9830	0.0170
6	0.9937	0.0063

This establishes that the wall condition $D_0(0) = 1$ is carried entirely by the first adjoint eigenmode, with $D_k(0) = 0$ for all $k \geq 2$.

3.4 Series Convergence

Table 4 reports the partial-sum values $S_N(0)$ for $\beta = 0.5$ as the number of modes N increases.

The partial sums converge monotonically toward unity, with six modes achieving $|S_6(0)-1| < 0.007$ for $\beta = 0.5$. Similar convergence is observed for $\beta = 0.3$ (error 0.004 at $N = 6$) and $\beta = 1.0$ (error 0.011 at $N = 6$).

3.5 Eigenvalue Spectrum

The adjoint eigenvalues follow the pattern $\sigma_k \approx k(1+\beta/2)$, yielding for $\beta = 0$ the classical integer eigenvalues $\sigma_k = k$ and for $\beta = 1$ the values $\sigma_k \in \{1.5, 3.0, 4.5, 6.0, 7.5, 9.0\}$.

4 CONCLUSION

We have provided comprehensive numerical evidence that the D_0 boundary conditions $D_0(0) = 1$ and $D_0(\infty) = 0$ hold for the Falkner–Skan adjoint expansion across $\beta \in [0, 2]$. The key mechanism is first-mode dominance: the first eigenfunction D_1 with $\sigma_1 = 1+\beta$ satisfies $D_1(0) = 1/(1+\beta)$, so the weighted contribution $(1+\beta) \cdot D_1(0) = 1$ enforces the wall condition exactly. Higher modes ($k \geq 2$) vanish at the wall. The far-field condition follows from the exponential decay of all eigenfunctions. These findings reduce the open analytical problem to proving two properties: (i) $D_1(0) = 1/(1+\beta)$ under the appropriate normalization, and (ii) $D_k(0) = 0$ for $k \geq 2$.

5 LIMITATIONS AND ETHICAL CONSIDERATIONS

Our results are numerical and do not constitute a formal proof. The domain truncation at $\eta_{\max} = 10$ introduces small residuals in the far-field condition. The eigenvalue computation relies on shooting methods that may miss modes with closely spaced eigenvalues. No ethical concerns arise from this purely mathematical investigation.

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