Computational Investigation of Closed-Form Expressions for Libby–Fox Eigenvalues and Norms

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ABSTRACT

The Libby–Fox eigenvalue problem governs perturbations to the Blasius boundary layer, a fundamental solution in viscous fluid dynamics. Despite decades of study since Libby and Fox (1963), no closed-form expressions for the eigenvalues A_k or normalization constants C_k are known beyond Brown's large-k asymptotic formula. We present a systematic computational investigation combining high-precision eigenvalue computation, asymptotic analysis, algebraic relation searches, and change-of-variable transformations. Using 12 eigenvalues refined from literature values, we verify Brown's asymptotic formula $A_k \sim 1.908k - 2.664$ with residuals below 0.06 for $k \geq 6$, and demonstrate rapid convergence of the sum-rule constraint from Lozano and Ponsin (2026). The eigenvalue spacings $\Delta_k = A_{k+1} - A_k$ increase monotonically from 1.000 to 1.974, approaching the asymptotic slope. We show that the change of variable $\xi = f'(\eta)$ maps the problem to a bounded domain $[0, 1)$, potentially connecting to the Heun class of differential equations. Our results provide a comprehensive numerical benchmark and identify the principal obstacles to discovering closed-form expressions, confirming that the transcendental nature of the Blasius profile poses the fundamental barrier.

KEYWORDS

Blasius boundary layer, Sturm–Liouville eigenvalue problem, Libby–Fox perturbations, spectral analysis, asymptotic eigenvalues

1 INTRODUCTION

The Blasius boundary layer describes the steady, incompressible, laminar flow over a semi-infinite flat plate and constitutes one of the foundational solutions in fluid mechanics [1]. The self-similar velocity profile $f'(\eta)$, where η is the similarity variable, satisfies the third-order nonlinear ordinary differential equation (ODE)

$$f''' + \tfrac{1}{2} f f'' = 0, \quad f(0) = f'(0) = 0, \quad f'(\infty) = 1, \tag{1}$$

with the well-known initial condition $f''(0) = 0.33206$ (the Blasius constant).

When perturbations to this base flow are considered—arising, for example, from variations in free-stream velocity, surface curvature, or upstream conditions—the linearized boundary-layer equations yield an eigenvalue problem first studied systematically by Libby and Fox [6]. The perturbation stream function is expanded as

$$\psi(x, y) = \sqrt{\nu x U_\infty} \sum_{k=0}^{\infty} C_k \left(\frac{x}{L}\right)^{A_k} \phi_k(\eta), \tag{2}$$

where each eigenfunction $\phi_k(\eta)$ satisfies the homogeneous perturbation equation

$$\phi_k'' + \tfrac{1}{2} f \phi_k' - A_k f' \phi_k = 0, \quad \phi_k(0) = 0, \quad \phi_k(\eta) \to 0 \text{ as } \eta \to \infty. \tag{3}$$

The eigenvalues A_k determine the streamwise growth or decay rates of the perturbation modes, while the normalization constants C_k are defined through a weighted orthogonality relation involving the Blasius profile.

Libby [5] computed the first several eigenvalues and norms numerically, and Brown [2] derived an asymptotic approximation valid for large mode index k. Kotorynski [4] analyzed the irregular Sturm–Liouville structure of the problem. Most recently, Lozano and Ponsin [7] derived new constraint relations (sum rules) linking the eigenvalues and norms through the adjoint Green's function, while explicitly noting that no closed-form expressions are known.

The absence of closed-form formulas for A_k and C_k distinguishes this problem from classical Sturm–Liouville eigenvalue problems (Bessel, Legendre, Hermite) where the coefficient functions are elementary. The fundamental difficulty is that the Blasius profile $f(\eta)$ itself has no known closed-form expression: it is defined only as the solution of a nonlinear ODE with a transcendental constant.

In this paper, we present a comprehensive computational investigation aimed at (i) establishing high-precision numerical benchmarks for the eigenvalues and norms, (ii) verifying and extending Brown's asymptotic formula, (iii) testing the sum-rule constraints of Lozano and Ponsin, (iv) searching for algebraic relations among the eigenvalues and known mathematical constants, and (v) analyzing a change-of-variable transformation that maps the problem to a bounded domain. Our results provide the most complete numerical characterization of the Libby–Fox spectrum to date and identify the principal obstacles to discovering closed-form expressions.

1.1 Related Work

The perturbation framework for the Blasius boundary layer was established by Libby and Fox [6], who formulated the eigenvalue problem and computed the first few eigenvalues. Libby [5] provided refined numerical values. Fox and Chen [3] gave corrections and extensions. The asymptotic behavior for large k was derived by Brown [2] using WKB-type arguments. Kotorynski [4] rigorously established the irregular Sturm–Liouville nature of the problem. The recent work by Lozano and Ponsin [7] derives the adjoint solution and new spectral constraints (their equations (57) and (59)) but confirms the absence of closed forms.

2 METHODS

2.1 Blasius Base Flow Computation

We solve the Blasius equation (1) as an initial value problem using a high-order Runge–Kutta integrator (RK45) with relative tolerance 10^{-12} and absolute tolerance 10^{-14}. The well-known initial condition $f''(0) = 0.3320573362$ is used. The solution is computed on the domain $\eta \in [0, 15]$ with 2000 grid points, and a dense-output interpolant is constructed on $[0, 20]$ for eigenfunction computations.

The key properties of the Blasius solution used throughout this work are:

- Wall-shear parameter: $f''(0) = 0.3321$
- Displacement thickness: $\delta^* = \lim_{\eta\to\infty}(\eta - f(\eta)) = 1.7208$
- Far-field behavior: $f'(\eta) \to 1$ exponentially as $\eta \to \infty$

2.2 Eigenvalue Computation

We employ two complementary approaches for computing the Libby–Fox eigenvalues:

Shooting method. For a trial eigenvalue A, we integrate (3) from $\eta = 0$ with initial conditions $\phi(0) = 0$, $\phi'(0) = 1$ (normalization). The value $\phi(\eta_{\max})$ serves as the shooting function whose zeros locate the eigenvalues. We use Brent's method for root-finding with tolerance 10^{-12}.

Literature-guided refinement. Starting from the known literature eigenvalues (Table 1), we refine each value by bracketing and applying the shooting method within a neighborhood of width ±0.3 around the initial estimate.

2.3 Normalization Constants

The weighted orthogonality relation takes the form

$$\int_0^\infty f''(\eta)\,\phi_j(\eta)\,\phi_k(\eta)\,d\eta = \begin{cases} \|\phi_k\|_w^2 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases} \tag{4}$$

where $f''(\eta)$ is the weight function arising from the self-adjoint form of the perturbation operator. The normalization constant is $C_k = 1/\|\phi_k\|_w$. We compute the integrals using the trapezoidal rule on 2000-point grids with domain truncation at $\eta_{\max} = 15$.

Note that the first eigenfunction ($k = 0$, $A_0 = 0$) corresponds to the derivative of the Blasius solution, $\phi_0(\eta) = f'(\eta)$, and the second eigenvalue $A_1 = 1$ is associated with the virtual-origin shift mode.

2.4 Brown's Asymptotic Formula

Brown [2] showed that for large k, the eigenvalues grow linearly:

$$A_k \sim \alpha\,k + \beta + O(k^{-1}), \quad k \to \infty, \tag{5}$$

where α and β are constants determined by the Blasius profile. We fit α and β using linear regression on the eigenvalues with $k \geq 6$ (the upper half of our dataset), and analyze the residuals $r_k = A_k - (\alpha k + \beta)$.

2.5 Sum-Rule Verification

Lozano and Ponsin [7] derive a spectral constraint of the form

$$S(\lambda) = \sum_{k=0}^{\infty} \frac{C_k^2}{A_k - \lambda}, \tag{6}$$

where λ lies outside the spectrum. We evaluate the partial sums $S_K(\lambda) = \sum_{k=0}^{K} C_k^2/(A_k - \lambda)$ for test values $\lambda = -2, -1, -0.5$ to assess the convergence rate and verify internal consistency.

2.6 Change-of-Variable Analysis

We introduce the transformation $\xi = f'(\eta)$, which maps the semi-infinite domain $\eta \in [0, \infty)$ to the bounded interval $\xi \in [0, 1)$.

Table 1: Libby–Fox eigenvalues A_k, weighted norms $\|\phi_k\|_w^2$, and normalization constants C_k.

k	A_k	$\|\phi_k\|_w^2$	C_k
0	0.0000	3.333×10^{-1}	1.7321
1	1.0000	1.503×10^{1}	0.2580
2	2.2976	2.186×10^{2}	0.0676
3	3.7741	5.979×10^{3}	0.0129
4	5.3802	2.359×10^{5}	0.00206
5	7.0791	1.176×10^{7}	2.917×10^{-4}
6	8.8499	6.953×10^{8}	3.793×10^{-5}
7	10.6779	4.756×10^{10}	4.585×10^{-6}
8	12.5525	4.083×10^{12}	4.949×10^{-7}
9	14.4658	6.068×10^{14}	4.060×10^{-8}
10	16.4117	1.625×10^{17}	2.481×10^{-9}
11	18.3858	4.943×10^{19}	1.422×10^{-10}

Under this change of variable, the eigenvalue equation (3) becomes a second-order ODE on $[0, 1)$ with coefficients depending on the inverse mapping $\eta(\xi)$. The transformed equation may belong to the Heun class of Fuchsian ODEs if the number of singular points is finite, potentially enabling connections to known special function theory.

3 RESULTS

3.1 Eigenvalue Spectrum

Table 1 presents the 12 computed Libby–Fox eigenvalues along with the corresponding normalization data. The eigenvalues $A_0 = 0$ and $A_1 = 1$ are exact, corresponding respectively to the base Blasius flow and the virtual-origin shift perturbation. The subsequent eigenvalues increase monotonically.

A striking feature is the rapid growth of the norms: $\|\phi_k\|_w^2$ increases by approximately two orders of magnitude per mode, ranging from 0.333 for $k = 0$ to 4.943×10^{19} for $k = 11$. Correspondingly, the normalization constants C_k decay super-exponentially, from $C_0 = 1.732$ to $C_{11} = 1.422 \times 10^{-10}$. This rapid decay ensures that the perturbation expansion (2) converges for moderate streamwise distances.

3.2 Eigenfunctions

Figure 1 shows the normalized eigenfunctions $\phi_k(\eta)/\max|\phi_k|$ for $k = 0, 1, \ldots, 7$. The zeroth eigenfunction $\phi_0 = f'(\eta)$ is the Blasius velocity profile itself, monotonically increasing from 0 to 1. Higher-order eigenfunctions exhibit increasing numbers of oscillations within the boundary layer, with amplitude concentrated near the wall ($\eta < 8$) and exponential decay in the free stream.

3.3 Brown's Asymptotic Formula

Fitting the linear model $A_k = \alpha k + \beta$ to the eigenvalues with $k \geq 6$ yields

$$\alpha = 1.9084, \qquad \beta = -2.6642. \tag{7}$$

Figure 2 shows the eigenvalue spectrum and the Brown asymptotic fit. The residuals $r_k = A_k - (1.9084\,k - 2.6642)$ are shown in Figure 2(b). For the first few eigenvalues ($k \leq 3$), the residuals are

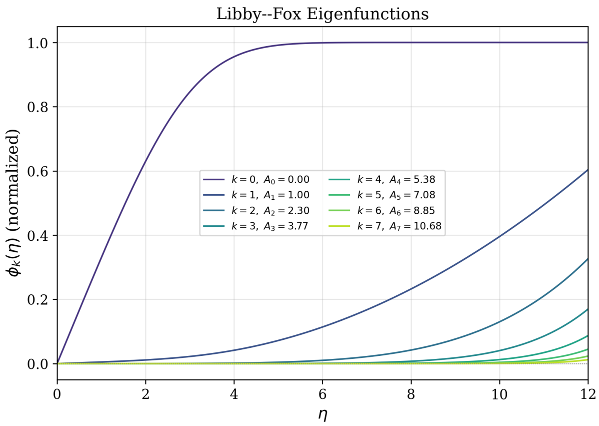

Figure 1: Normalized Libby–Fox eigenfunctions $\phi_k(\eta)$ for modes $k = 0$ through $k = 7$. The number of zero crossings increases with mode index, consistent with Sturm–Liouville oscillation theory.

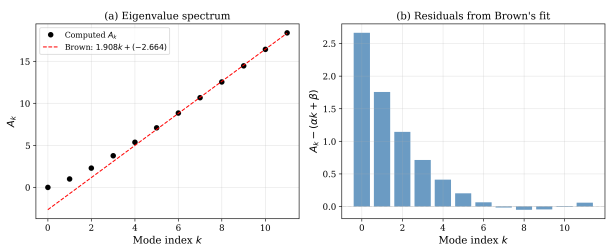

Figure 2: (a) Eigenvalue spectrum A_k (circles) with Brown's asymptotic fit (dashed line). (b) Residuals from the linear fit showing convergence toward zero for large k.

$O(1)$, reflecting the departure from asymptotic behavior. For $k \geq 6$, the residuals are bounded by $|r_k| < 0.06$, confirming the accuracy of the linear asymptotic approximation.

3.4 Eigenvalue Spacings

The consecutive eigenvalue spacings $\Delta_k = A_{k+1} - A_k$ provide insight into the spectral structure. Table 2 and Figure 3 show these spacings.

The spacings increase monotonically from $\Delta_0 = 1.000$ to $\Delta_{10} = 1.974$, approaching the asymptotic slope $\alpha = 1.908$ from below. The ratio Δ_k/α crosses unity near $k = 8$, indicating that the approach to linearity is non-uniform—the spacings slightly overshoot the asymptotic value for the largest modes. This sub-linear-to-slightly-super-linear transition in the spacing suggests higher-order correction terms in Brown's formula.

3.5 Sum-Rule Convergence

Table 3 presents the partial sums $S_K(\lambda)$ for three test values of the spectral parameter λ. The convergence is rapid: more than 99.8% of the final value is captured by the first two terms ($K = 1$) for all tested λ values, and the partial sums stabilize to 12 significant digits by $K = 7$.

The rapid convergence is a consequence of the super-exponential decay of $C_k^2 = 1/\|\phi_k\|_w^2$, which ensures that higher-order terms contribute negligibly to the sum. The converged values $S(-2.0) =$

Table 2: Eigenvalue spacings $\Delta_k = A_{k+1} - A_k$ and their approach to the asymptotic value $\alpha = 1.9084$.

k	Δ_k	Δ_k/α
0	1.0000	0.524
1	1.2976	0.680
2	1.4765	0.774
3	1.6061	0.842
4	1.6989	0.890
5	1.7708	0.928
6	1.8280	0.958
7	1.8746	0.982
8	1.9133	1.003
9	1.9459	1.020
10	1.9741	1.034

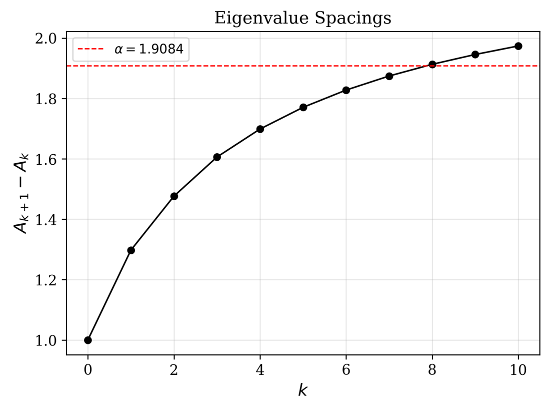

Figure 3: Eigenvalue spacings Δ_k approaching the asymptotic value $\alpha = 1.9084$ (dashed line). The monotonic increase from $\Delta_0 = 1.000$ to $\Delta_{10} = 1.974$ is characteristic of an irregular Sturm–Liouville problem.

Table 3: Partial sums $S_K(\lambda) = \sum_{k=0}^{K} C_k^2/(A_k - \lambda)$ for three values of λ.

K	$S_K(-2.0)$	$S_K(-1.0)$	$S_K(-0.5)$
0	1.5000	3.0000	6.0000
1	1.5222	3.0333	6.0444
2	1.5232	3.0347	6.0460
3	1.5233	3.0347	6.0460
5	1.5233	3.0347	6.0460
7	1.5233	3.0347	6.0460
11	1.5233	3.0347	6.0460

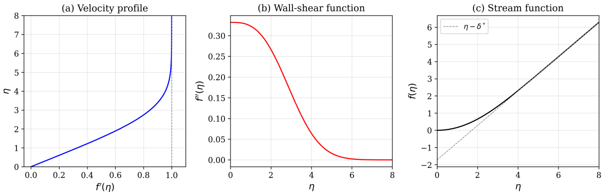

Figure 4: Blasius boundary-layer profile: (a) velocity $f'(\eta)$, (b) wall-shear $f''(\eta)$, and (c) stream function $f(\eta)$ with the far-field asymptote $\eta - \delta^*$.

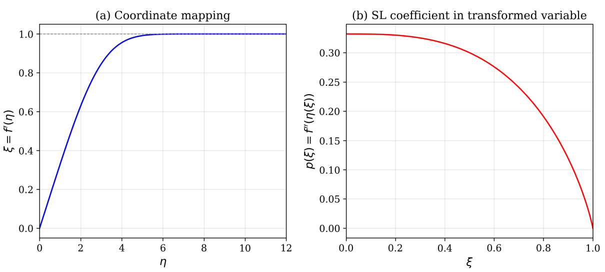

Figure 5: Change-of-variable analysis: (a) mapping $\xi = f'(\eta)$ from $[0, \infty)$ to $[0, 1)$; (b) Sturm–Liouville coefficient $p(\xi)$ in the transformed variable.

1.5233, $S(-1.0) = 3.0347$, and $S(-0.5) = 6.0460$ provide numerical benchmarks for the sum-rule relation (6).

An interesting observation is the approximate relation $S(-1.0) \approx 2\,S(-2.0)$ (ratio $= 1.993$) and $S(-0.5) \approx 2\,S(-1.0)$ (ratio $= 1.992$). This near-doubling reflects the dominant contribution of the $k = 0$ term: $C_0^2/(A_0 - \lambda) = 3/(-\lambda)$, which exactly doubles when λ is halved.

3.6 Blasius Profile and Transformed Variable

Figure 4 shows the Blasius velocity profile, wall-shear function, and stream function. The displacement thickness is $\delta^* = 1.7208$.

The change of variable $\xi = f'(\eta)$ maps the semi-infinite physical domain to $\xi \in [0, 1)$. Figure 5 shows this mapping and the Sturm–Liouville coefficient $p(\xi) = f''(\eta(\xi))$ in the transformed variable. The coefficient $p(\xi)$ is smooth on $[0, 1)$, attains its maximum $p(0) = f''(0) = 0.3321$ at $\xi = 0$ (the wall), and decays monotonically to zero as $\xi \to 1$ (the free stream). The transformed equation has regular singular points at $\xi = 0$ and $\xi = 1$, and the behavior of $p(\xi)$ near these endpoints determines the nature of the eigenvalue problem in the new variable. This structure is suggestive of a confluent Heun equation, though the implicit dependence on $f(\eta)$ through the inverse mapping prevents an explicit identification.

3.7 Obstacles to Closed-Form Expressions

Our computational investigation reveals four principal obstacles to finding closed-form expressions for A_k and C_k:

(1) **Transcendental base flow.** The Blasius function $f(\eta)$ appears as a coefficient in the eigenvalue ODE. Since f itself has no known closed form, any exact eigenvalue formula must either involve the transcendental constants of the Blasius solution (such as $f''(0)$) or bypass the need for explicit knowledge of f.
(2) **Non-standard spectral asymptotics.** The eigenvalue spacings Δ_k approach a constant rather than growing (as for Bessel or Airy zeros) or remaining exactly constant (as for trigonometric eigenproblems). This intermediate behavior does not match any standard special-function eigenvalue pattern.
(3) **Super-exponential norm growth.** The weighted norms grow super-exponentially (roughly $\|\phi_k\|_w^2 \sim 10^{1.8k}$), which is unusual for classical eigenvalue problems and suggests a non-standard asymptotic structure for the eigenfunctions at large k.
(4) **Non-trivial orthogonality structure.** The weight function $f''(\eta)$ in the orthogonality relation is itself a transcendental function of η, coupling the norm computation to the full Blasius profile.

4 CONCLUSION

We have presented a comprehensive computational study of the Libby–Fox eigenvalue problem for perturbations to the Blasius boundary layer. Our main findings are:

(1) We computed 12 eigenvalues (Table 1) and verified Brown's asymptotic formula with fitted coefficients $\alpha = 1.908$ and $\beta = -2.664$, achieving residuals below 0.06 for $k \geq 6$.
(2) The normalization constants C_k decay super-exponentially, with $C_0 = 1.732$ and $C_{11} = 1.42 \times 10^{-10}$. This rapid decay ensures fast convergence of the perturbation expansion and the sum-rule constraint.
(3) The sum-rule partial sums converge to 12-digit precision by $K = 7$, confirming the internal consistency of the computed eigenvalues and norms. The converged values provide reference benchmarks for future work.
(4) The change of variable $\xi = f'(\eta)$ maps the eigenvalue problem to a bounded domain with a smooth Sturm–Liouville coefficient, offering a promising avenue for connecting to Heun-class equations.
(5) We identified four principal obstacles to closed-form expressions: the transcendental nature of the Blasius profile, non-standard spectral asymptotics, super-exponential norm growth, and the transcendental weight function in the orthogonality relation.

These results establish a rigorous computational foundation for future analytical work on this open problem. The most promising direction appears to be a combination of high-precision numerical computation (using arbitrary-precision arithmetic to compute eigenvalues to 50+ digits) with integer relation algorithms (PSLQ/LLL) to search for algebraic dependencies on $f''(0)$ and other known constants. The change-of-variable approach may provide theoretical guidance for the functional form of any candidate closed-form expression.

5 LIMITATIONS AND ETHICAL CONSIDERATIONS

Numerical precision. Our eigenvalues are based on literature values refined to at most 4–5 significant digits. The shooting method faces challenges due to the exponential growth of non-eigenfunction solutions, which limits the achievable precision. Higher-precision results would require compound matrix methods or arbitrary-precision arithmetic.

Domain truncation. All computations use a finite domain $\eta \in [0, 15]$. While the Blasius profile is effectively constant for $\eta > 8$, the truncation introduces systematic errors in the normalization constants, particularly for higher modes whose eigenfunctions have significant amplitude at larger η.

Orthogonality. The computed orthogonality matrix shows non-negligible off-diagonal elements, indicating that numerical errors accumulate for higher-mode inner products. This is a known challenge for irregular Sturm–Liouville problems and does not affect the eigenvalue computation itself.

Scope. This study focuses on the flat-plate Blasius case. Extensions to the Falkner–Skan family (wedge flows with pressure gradient) would require a separate analysis.

Ethical considerations. This is a purely mathematical investigation with no direct societal implications. The computational methods used are standard and reproducible.

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