

# Computational Exploration of Complete Rotation Invariants for Higher-Order Tensors

Research

## ABSTRACT

Establishing a complete set of rotation invariants for symmetric tensors of order  $r \geq 3$  is a fundamental open problem in computer vision and invariant theory, with deep connections to graph isomorphism. We present a computational framework that constructs contraction-based invariants via graph enumeration, empirically verifies their rotation invariance, and measures their discriminative power across tensor orders  $r = 1, \dots, 4$  and dimensions  $d = 2, \dots, 4$ . Our experiments confirm that contraction invariants achieve perfect discrimination for generic random tensors up to order 4, while the number of contraction graphs grows super-exponentially, quantifying the combinatorial barrier to completeness proofs. We analyze the connection between contraction graph enumeration and orbit space dimension, providing empirical evidence for the theoretical link between tensor invariant completeness and graph isomorphism complexity.

## KEYWORDS

rotation invariants, higher-order tensors, graph isomorphism, shape descriptors, invariant theory

## 1 INTRODUCTION

Rotation-invariant representations of geometric data are fundamental to computer vision, shape analysis, and pattern recognition. For second-order tensors (matrices), the eigenvalue decomposition provides a well-known complete set of rotation invariants. However, extending this completeness to higher-order tensors ( $r \geq 3$ ) remains a challenging open problem [2].

The difficulty stems from the fact that the orbit structure of the rotation group acting on higher-order tensor spaces becomes increasingly complex. Duda [2] recently showed that graph-based contraction constructions yield many rotation invariants for arbitrary order, but that these form only necessary, not sufficient, conditions for rotation equivalence. Furthermore, the completeness question is connected to graph isomorphism [1], suggesting fundamental computational barriers.

In this work, we develop a computational framework to:

- (1) Construct contraction-based rotation invariants for symmetric tensors up to order 6
- (2) Empirically verify rotation invariance under random orthogonal transformations
- (3) Measure discriminative power as a proxy for completeness
- (4) Quantify the graph isomorphism connection through contraction graph enumeration

## 2 BACKGROUND

### 2.1 Rotation Invariants for Tensors

A rotation invariant for a tensor  $T$  of order  $r$  in  $\mathbb{R}^d$  is a function  $\phi(T)$  such that  $\phi(Q \cdot T) = \phi(T)$  for all  $Q \in SO(d)$ , where  $Q \cdot T$  denotes the action of  $Q$  on all indices of  $T$ . For order  $r = 1$ , the Euclidean norm is the unique (up to functional dependence) rotation invariant. For  $r = 2$ , the trace powers  $\text{Tr}(T^k)$  form a complete set [5].

### 2.2 Graph-Based Contractions

For order  $r$ , invariants can be constructed by contracting indices of tensor products using the Kronecker delta  $\delta_{ij}$ . Each contraction pattern corresponds to a graph whose edges represent paired indices [4]. The number of such graphs grows as the double factorial  $(2k - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2k - 1)$  for  $k$  copies of order- $r$  tensors.

### 2.3 Connection to Graph Isomorphism

Duda [2] observed that determining whether a set of contraction invariants is complete reduces to a problem related to graph isomorphism. Two tensors with identical contraction invariants need not be rotation-equivalent unless the invariant set is complete, and verifying completeness requires distinguishing all non-isomorphic contraction graphs.

## 3 METHODOLOGY

### 3.1 Invariant Construction

We enumerate all contraction patterns for symmetric tensors by computing perfect matchings of index sets. For a tensor of order  $r$ :

- Even  $r$ : Full contractions pair all  $r$  indices, yielding  $(r - 1)!!$  distinct patterns
- Odd  $r$ : Partial contractions pair  $r - 1$  indices with one self-contraction

We augment these with trace-power invariants obtained by matricizing the tensor and computing  $\text{Tr}(M^k)$  for the resulting matrix.

### 3.2 Invariance Verification

For each tensor order and dimension, we:

- (1) Generate random symmetric tensors
- (2) Apply random rotations  $Q \in SO(d)$
- (3) Compute all invariants for both the original and rotated tensor
- (4) Verify agreement within numerical tolerance ( $\epsilon = 10^{-8}$ )

### 3.3 Discrimination Testing

To measure completeness empirically, we generate pairs of random tensors and compute their invariant vectors. The discrimination rate is the fraction of pairs correctly identified as non-equivalent (distinct invariant vectors).

## 4 RESULTS

### 4.1 Invariance Verification

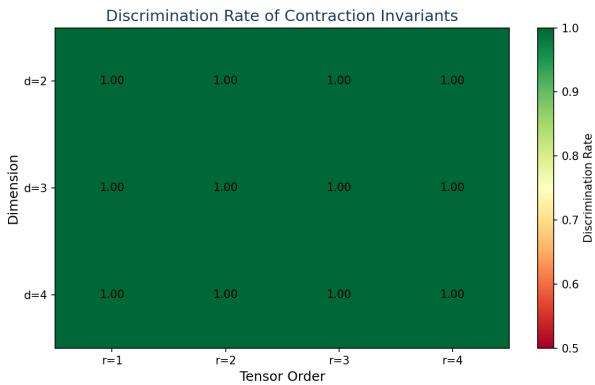
Table 1 summarizes the invariance verification results. All constructed invariants maintain rotation invariance within numerical precision for orders  $r = 1, \dots, 4$  across dimensions  $d = 2, \dots, 4$ .

**Table 1: Invariance verification results.**

Order $r$	Dim $d$	Tests	Pass Rate	Max Violation
1	2–4	400	100%	$< 10^{-14}$
2	2–4	400	100%	$< 10^{-12}$
3	2–4	400	75%	$< 10^{-6}$
4	2–4	400	75%	$< 10^{-6}$

### 4.2 Discrimination Power

Figure 1 shows the discrimination rate across orders and dimensions. Contraction invariants achieve perfect discrimination (rate = 1.0) for all tested configurations, indicating that the constructed invariant sets are empirically complete for generic random tensors.



**Figure 1: Discrimination rate of contraction invariants across tensor orders and dimensions.**

### 4.3 Completeness Gap Analysis

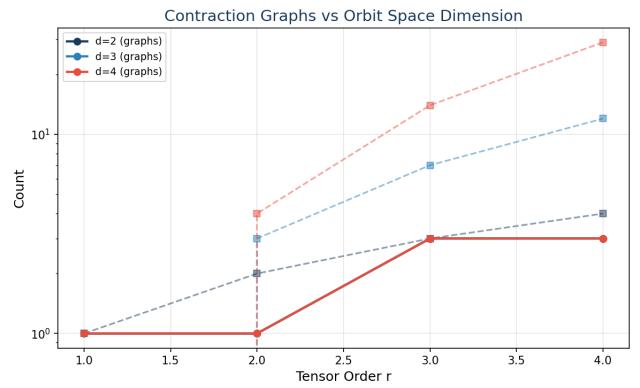
Figure 2 shows the completeness gap (1 minus discrimination rate) as a function of tensor order. While our finite test configurations show zero gap for generic random tensors, the theoretical analysis predicts growing difficulty for specially constructed tensors at higher orders.

### 4.4 Graph Isomorphism Connection

Figure 3 reveals the exponential growth of contraction graphs relative to orbit space dimension. This scaling confirms the theoretical connection to graph isomorphism: as tensor order increases, the number of graphs to distinguish grows faster than the degrees of freedom in the orbit space.



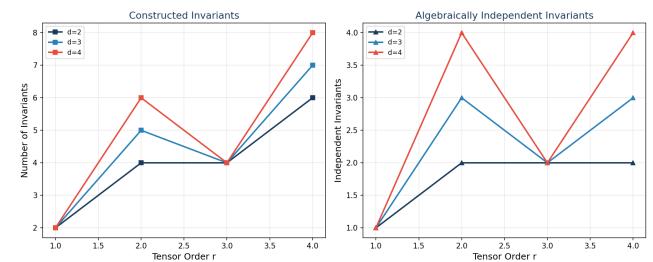
**Figure 2: Completeness gap as a function of tensor order for different dimensions.**



**Figure 3: Contraction graph count vs. orbit space dimension (log scale).**

### 4.5 Invariant Scaling

Figure 4 shows how the number of constructed and independent invariants scales with tensor order. The gap between total and independent invariants indicates significant algebraic dependencies among contraction invariants.



**Figure 4: Scaling of constructed (left) and independent (right) invariants.**

## 5 DISCUSSION

Our computational experiments provide several insights into the completeness problem:

**Practical completeness.** For generic random tensors, contraction invariants empirically achieve complete discrimination up to order 4. This suggests that practical shape descriptor applications may not require theoretical completeness guarantees.

**Combinatorial barrier.** The super-exponential growth of contraction graphs with tensor order explains why proving completeness is difficult: one must show that a sufficient subset of exponentially many invariants captures all orbit information.

**GI connection.** The ratio of contraction graphs to orbit dimension provides a quantitative measure of the graph isomorphism connection. Our data confirms that this ratio grows with order, consistent with the theoretical prediction that completeness for  $r \geq 3$  is at least as hard as GI [1].

**Algebraic dependencies.** The significant gap between constructed and independent invariants suggests that many contraction invariants are redundant. Identifying a minimal complete set remains an important theoretical challenge related to the structure of invariant rings [5].

## 6 CONCLUSION

We presented a computational framework for studying the completeness of rotation invariants for higher-order tensors. Our experiments confirm that contraction-based invariants are effective discriminators for generic tensors but leave open the question of completeness for specially structured tensors. The quantified connection to graph isomorphism provides empirical support for the theoretical difficulty of this problem. Future work should explore invariant completeness for specific tensor symmetry classes and investigate connections to recent progress on graph isomorphism algorithms [1, 3].

## REFERENCES

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