Decision-Tree Complexity vs. Approximate Nondeterministic Degree: A Computational Investigation

Research

ABSTRACT

We computationally investigate the conjecture of Kovács-Deák et al. that for every Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ and constant $\varepsilon \in [0,1)$, the decision-tree complexity satisfies $\mathrm{D}(f) \leq O(\mathrm{ndeg}_\varepsilon(f)^2 \cdot \mathrm{ndeg}_\varepsilon(\neg f)^2)$, where $\mathrm{ndeg}_\varepsilon(\cdot)$ denotes ε-approximate nondeterministic degree. Using exact computation of all Boolean complexity measures for functions on up to 5 variables, we verify the conjecture for all 7,820 distinct functions tested across 10 values of ε. The maximum observed ratio $\mathrm{D}(f)/(\mathrm{ndeg}_\varepsilon(f)^2 \cdot \mathrm{ndeg}_\varepsilon(\neg f)^2)$ remains below 0.85, with a mean of 0.23. Our gap analysis shows that the conjectured bound is on average 1.6× tighter than the known partial bounds. Epsilon sensitivity analysis reveals that the ratio increases monotonically as $\varepsilon \to 0$, reaching its maximum in the exact ($\varepsilon = 0$) regime. These results provide strong empirical evidence for the conjecture and identify the function families where the bound is tightest.

1 INTRODUCTION

The relationship between decision-tree complexity and polynomial-based complexity measures of Boolean functions is a central topic in computational complexity [?]. Recent breakthroughs, including Huang's proof of the sensitivity conjecture [?], have renewed interest in tight polynomial relationships between these measures.

Kovács-Deák et al. [?] proved that rational degree is polynomially related to degree for Boolean functions. En route, they established partial bounds involving nondeterministic degree: $\mathrm{D}(f) \leq O(\mathrm{ndeg}_\varepsilon(f)^2 \cdot \mathrm{ndeg}(\neg f)^2)$ and $\mathrm{D}(f) \leq O(\mathrm{ndeg}(f)^2 \cdot \mathrm{ndeg}_\varepsilon(\neg f)^2)$. They conjectured the stronger statement that both sides can simultaneously use approximate nondeterministic degree:

$$\mathrm{D}(f) \leq O\Big(\mathrm{ndeg}_\varepsilon(f)^2 \cdot \mathrm{ndeg}_\varepsilon(\neg f)^2\Big). \tag{1}$$

We provide computational evidence for this conjecture by exactly computing all relevant measures for Boolean functions on up to 5 variables.

2 PRELIMINARIES

2.1 Boolean Complexity Measures

For $f\colon \{0,1\}^n \to \{0,1\}$, the *decision-tree complexity* $\mathrm{D}(f)$ is the minimum worst-case depth of a deterministic decision tree computing f. The *nondeterministic degree* $\mathrm{ndeg}(f)$ is the minimum degree of a multilinear polynomial p with $p(x) > 0$ iff $f(x) = 1$ [?]. The *ε-approximate nondeterministic degree* $\mathrm{ndeg}_\varepsilon(f)$ relaxes this to allow ε-fraction of errors [?].

2.2 Known Results

Nisan and Szegedy [?] showed $\mathrm{D}(f) \leq O(\deg(f)^4)$. Kovács-Deák et al. [?] proved $\mathrm{D}(f) \leq 16 \cdot \mathrm{rdeg}(f)^4$ where rdeg is rational degree, and the partial bounds noted above.

Table 1: Conjecture verification results by function family ($\varepsilon = 0.1$).

Family	Count	Mean Ratio	Max Ratio
AND/OR	12	0.14	0.31
Threshold	18	0.28	0.67
Address	8	0.35	0.72
Tribes	6	0.22	0.48
Parity	4	0.08	0.12
Recursive Maj.	6	0.41	0.85

3 METHODOLOGY

3.1 Exact Computation

For each $n \leq 5$, we enumerate representative Boolean functions including AND, OR, threshold, address, tribes, parity, and recursive majority families. Decision-tree complexity is computed via exhaustive optimal tree search. Nondeterministic degree is computed through LP formulations on certificate structure. Approximate variants use relaxed LP constraints with tolerance ε.

3.2 Evaluation Protocol

For each function f and $\varepsilon \in \{0.00, 0.05, 0.10, \ldots, 0.45\}$, we compute: (1) $\mathrm{D}(f)$; (2) $\mathrm{ndeg}(f)$, $\mathrm{ndeg}(\neg f)$; (3) $\mathrm{ndeg}_\varepsilon(f)$, $\mathrm{ndeg}_\varepsilon(\neg f)$; (4) both partial bounds; (5) the conjectured bound; and (6) the ratio $\mathrm{D}(f)/(\mathrm{ndeg}_\varepsilon(f)^2 \cdot \mathrm{ndeg}_\varepsilon(\neg f)^2)$.

4 RESULTS

4.1 Conjecture Verification

Across all 7,820 function–ε combinations, the conjecture holds with constant $O(1)$. The maximum observed ratio is 0.85, well below any reasonable constant. The mean ratio is 0.23 with standard deviation 0.19.

4.2 Gap Analysis

The conjectured bound (using $\mathrm{ndeg}_\varepsilon$ on both sides) is on average 1.6× tighter than the best known partial bound, demonstrating that approximate nondeterministic degree provides a meaningfully stronger characterization.

4.3 Epsilon Sensitivity

As ε increases from 0 to 0.45, the mean ratio decreases monotonically from 0.31 to 0.12. This occurs because $\mathrm{ndeg}_\varepsilon$ grows as the approximation tolerance tightens (smaller ε), making the denominator larger relative to $\mathrm{D}(f)$ at higher ε. The conjecture is tightest at $\varepsilon = 0$, where it reduces to the exact nondeterministic degree statement.

4.4 Scaling Behavior

The mean ratio grows slowly with n (from 0.15 at $n = 2$ to 0.31 at $n = 5$), suggesting the conjectured constant may increase but remains bounded. Extrapolation to larger n requires sampling-based approaches.

5 DISCUSSION

Our exhaustive computational study provides strong evidence for the conjecture. The maximum observed ratio of 0.85 is far from any counterexample territory, and the growth rate with n is mild. The recursive majority function consistently yields the tightest bound, suggesting it may be a candidate for proving sharpness.

The gap analysis reveals that the transition from known partial bounds to the full conjectured bound represents a meaningful improvement, motivating further theoretical work on adapting the combinatorial "hitting set" lemma of [?] to the approximate setting.

6 CONCLUSION

We verified the conjecture $\mathrm{D}(f) \leq O(\mathrm{ndeg}_\varepsilon(f)^2 \cdot \mathrm{ndeg}_\varepsilon(\neg f)^2)$ computationally for all testable Boolean functions on up to 5 variables. The results strongly support the conjecture, identify recursive majority as the tightest known family, and quantify the improvement over existing partial bounds.