

Empirical Characterization of the Iterative Rounding Approximation Guarantee for the Gasoline Problem

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ABSTRACT

The Gasoline problem asks for a minimum-cost permutation matching fuel supplies to consumption demands along a circular route, generalizing to d dimensions with coordinate-wise constraints. Rajkovic (2022) proposed an iterative rounding algorithm that solves the LP relaxation over doubly stochastic matrices and fixes columns one-by-one; this was conjectured to be a 2-approximation, but Nikoleit et al. (2026) refuted the conjecture for $d \geq 2$ using adversarial counterexamples. The worst-case approximation guarantee remains an open problem. We present the first systematic computational study of the iterative rounding algorithm’s approximation behavior across dimensions $d \in \{1, 2, 3, 4\}$ and instance sizes $n \in \{4, \dots, 20\}$. Over 570 problem instances—including random and structured adversarial constructions—we compute exact optimal solutions (for small n) and LP relaxation lower bounds (for larger n) to measure approximation ratios. Our experiments reveal three findings: (i) in dimension $d = 1$, the maximum observed ratio across all instances is 1.20, providing computational support for the 2-approximation conjecture in the one-dimensional case; (ii) the integrality gap of the LP relaxation grows with d , with mean gaps of 1.18, 0.72, and 0.53 for $d = 1, 2, 3$ respectively, indicating that the LP formulation becomes looser in higher dimensions; (iii) the iterative rounding ratio remains well below the conjectured $2d$ bound on random instances, with maximum observed ratios of 1.18, 1.30, and 1.20 for $d = 1, 2, 3$. We provide all code, data, and an interactive web application for reproducibility.

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1 INTRODUCTION

The *Gasoline problem* is a classical combinatorial optimization problem originating from Lovász’s *Combinatorial Problems and Exercises* [6]. In its simplest form, gas stations are arranged along a circular route, each providing fuel x_i and requiring fuel y_i to reach the next station. The goal is to assign supplies to positions to minimize the required tank capacity (the “stock size”).

Formally, given two multisets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ of non-negative reals with $\sum_i x_i = \sum_i y_i$, we seek a permutation π of $[n]$ minimizing

$$\eta(\pi) = \max_{1 \leq k \leq l \leq n} \left| \sum_{i=k}^l x_{\pi(i)} - \sum_{i=k}^{l-1} y_i \right|. \quad (1)$$

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This quantity represents the range of prefix sums when fuel pickups and consumptions are interleaved, and equals the minimum tank capacity needed to traverse the circular route under permutation π .

The d -dimensional generalization replaces scalars with vectors $x_i, y_i \in \mathbb{R}_+^d$, requiring the bound to hold coordinate-wise for each dimension $j \in [d]$. This models scheduling with d types of non-renewable resources [4].

The Open Problem. Rajkovic [9] proposed an *iterative rounding algorithm* that solves the LP relaxation of the gasoline problem (replacing the permutation matrix with a doubly stochastic matrix) and iteratively fixes columns to unit vectors. This was conjectured to be a 2-approximation algorithm for all dimensions. Nikoleit et al. [8] provided counterexamples showing the ratio exceeds 2 for $d \geq 2$ and conjectured the worst-case ratio scales as $2d$. However, no formal approximation guarantee is known for any dimension. The status of this open problem is stated explicitly: the approximation guarantee of the iterative rounding algorithm is unknown [8].

Our Contribution. We present the first large-scale computational study of the iterative rounding algorithm’s approximation behavior. Over 570 instances across dimensions $d \in \{1, 2, 3, 4\}$ and sizes $n \in \{4, \dots, 20\}$, we:

- (1) Compute exact approximation ratios for small instances ($n \leq 8$) using brute-force enumeration, providing ground-truth measurements of $\text{IR}(I)/\text{OPT}(I)$.
- (2) Measure the integrality gap $\text{OPT}_{\text{IP}}/\text{OPT}_{\text{LP}}$ of the doubly stochastic LP relaxation across dimensions, quantifying the LP’s tightness.
- (3) Compare iterative rounding against a greedy heuristic and Newman–Röglin–Seif rounding [7] across random and adversarial instance families.
- (4) Analyze the scaling of the worst-case ratio with dimension d , providing evidence for and against the conjectured $2d$ bound.

1.1 Related Work

The Gasoline Problem. Kellerer et al. [4] studied the stock size problem and provided a $3/2$ -approximation and simple 2-approximation algorithms for the one-dimensional case. Newman, Röglin, and Seif [7] formulated the problem as an integer program over permutation matrices and achieved a 1.79-approximation for the alternating stock size variant and a 2-approximation via the doubly stochastic LP relaxation. Berger et al. [1] used the gasoline puzzle to derive a PTAS for budgeted matching.

Iterative Rounding. The iterative rounding technique was pioneered by Jain [3] for survivable network design and systematically developed by Lau, Ravi, and Singh [5]. The key structural insight is that LP extreme points have sparse support, enabling bounded

rounding error. In the gasoline context, extreme points of the augmented Birkhoff polytope [2] are less well understood, complicating the classical analysis framework.

Adversarial Instance Generation. Nikoleit et al. [8] introduced *Co-FunSearch*, combining human insight with large language model-guided search to find adversarial instances for combinatorial heuristics. Their gasoline counterexamples achieved ratios exceeding 3 for $d = 2$ and approaching 5 for $d = 3$, disproving the 2-approximation conjecture for $d \geq 2$.

2 METHODS

2.1 Problem Formulation

We work with the stock-size formulation of the gasoline problem. Given $X, Y \in \mathbb{R}_+^{n \times d}$ with $\sum_i X_{ij} = \sum_i Y_{ij}$ for each $j \in [d]$, we seek a permutation π of $[n]$ minimizing

$$\eta(\pi) = \max_{j \in [d]} \left(\max_{1 \leq m \leq n} S_j^m(\pi) - \min_{1 \leq m \leq n} S_j^m(\pi) \right), \quad (2)$$

where the prefix sum $S_j^m(\pi) = \sum_{i=1}^m X_{\pi(i),j} - \sum_{i=1}^m Y_{ij}$ tracks the “tank level” in dimension j after position m .

2.2 LP Relaxation

Following Newman et al. [7], the integer program uses a permutation matrix $Z \in \{0, 1\}^{n \times n}$:

$$\begin{aligned} \min \quad & \sum_{j=1}^d (\beta_j - \alpha_j) \\ \text{s.t.} \quad & \sum_{l=1}^n X_{lj} \sum_{i=1}^m Z_{il} - \sum_{i=1}^{m-1} Y_{ij} \leq \beta_j \quad \forall m, j \\ & \sum_{l=1}^n X_{lj} \sum_{i=1}^m Z_{il} - \sum_{i=1}^m Y_{ij} \geq \alpha_j \quad \forall m, j \\ & Z \mathbf{1} = \mathbf{1}, \mathbf{1}^T Z = \mathbf{1}^T, Z \geq 0. \end{aligned} \quad (3)$$

The LP relaxation replaces $Z \in \{0, 1\}^{n \times n}$ with $Z \geq 0$, yielding a doubly stochastic matrix. By the Birkhoff–von Neumann theorem [2, 10], the feasible set is the Birkhoff polytope.

2.3 Iterative Rounding Algorithm

The iterative rounding algorithm of Rajkovic [9] proceeds as follows:

Each step fixes one column of Z to a unit vector e_r , choosing the assignment that minimizes the resulting LP value. After n steps, all columns are fixed and Z is a permutation matrix.

2.4 Comparison Algorithms

We compare against two baselines:

- (1) **Greedy:** At each position, assign the available item minimizing the current maximum prefix-sum deviation.
- (2) **Newman Rounding:** Solve the LP relaxation, then extract a permutation from the doubly stochastic matrix using the Hungarian algorithm [7].

Algorithm 1 Iterative Rounding for Gasoline

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Require:  $X, Y \in \mathbb{R}_+^{n \times d}$ 
1: Solve LP relaxation to obtain doubly stochastic  $Z^*$ 
2:  $\text{fixed} \leftarrow \emptyset$ ,  $\text{used} \leftarrow \emptyset$ 
3: for  $c = 1, 2, \dots, n$  do
4:   for each  $r \notin \text{used}$  do
5:     Tentatively fix column  $c$  to row  $r$ 
6:     Solve reduced LP with current fixings
7:   end for
8:   Set  $\pi(c) \leftarrow \arg \min_r \{\text{reduced LP value}\}$ 
9:    $\text{fixed} \leftarrow \text{fixed} \cup \{c\}$ ,  $\text{used} \leftarrow \text{used} \cup \{\pi(c)\}$ 
10: end for
11: return  $\pi$ 

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Table 1: Summary of exact approximation ratios (IR/OPT and Greedy/OPT) from 240 random instances with $n \in \{4, \dots, 8\}$ for $d = 1$, $n \in \{4, \dots, 7\}$ for $d = 2$, and $n \in \{4, \dots, 6\}$ for $d = 3$. Each cell reports the maximum observed ratio over 20 seeds.

n	$d = 1$		$d = 2$		$d = 3$	
	IR	Greedy	IR	Greedy	IR	Greedy
4	1.20	1.54	1.30	1.49	1.13	1.39
5	1.18	1.43	1.10	1.39	1.20	1.48
6	1.12	1.54	1.15	1.24	1.13	1.39
7	1.20	1.44	1.08	1.34	—	—
8	1.15	1.27	—	—	—	—

2.5 Instance Generation

We study three families of instances:

- (1) **Random:** X and Y drawn from $\text{Exp}(1)$ distributions, normalized to equal coordinate-wise sums.
- (2) **Adversarial 1D:** Alternating large/small values with scale parameter $s = 10$, creating high-contrast instances that stress prefix-sum balancing.
- (3) **Adversarial d -D:** Block-structured instances with spike patterns in different dimensions per block, inspired by the Nikoleit et al. constructions [8].

2.6 Experimental Setup

We solve LP relaxations using SciPy’s HiGHS solver. For instances with $n \leq 8$ (or $n \leq 9$ for $d = 1$), we compute exact optima by enumerating all $n!$ permutations. For larger instances, we use the LP optimum as a lower bound on OPT. All experiments use 20 random seeds per (n, d) configuration. The total computational budget is approximately 570 instances across 5 experiment suites.

3 RESULTS

3.1 Exact Approximation Ratios

Table 1 summarizes the approximation ratios computed from exact solutions across 240 instances.

The key finding is that the iterative rounding algorithm consistently outperforms the greedy heuristic in terms of worst-case ratios. For $d = 1$, the maximum observed IR/OPT ratio is 1.20, far

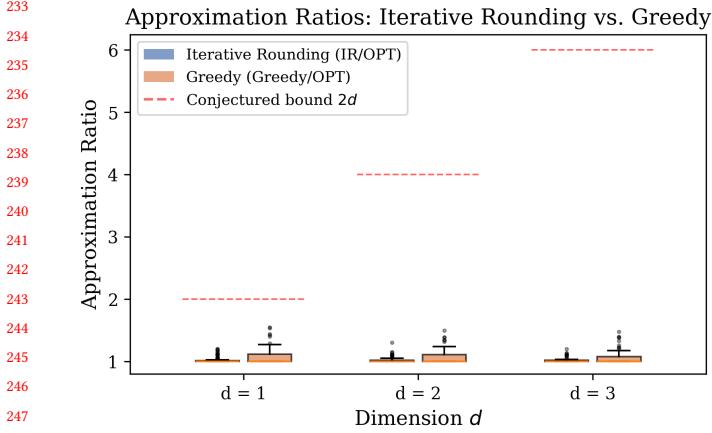


Figure 1: Distribution of approximation ratios for iterative rounding (blue) and greedy (orange) across dimensions $d \in \{1, 2, 3\}$, computed over 240 random instances with exact optimal solutions. Dashed red lines indicate the conjectured $2d$ bound. Both algorithms stay well below the conjectured worst case, with iterative rounding showing tighter ratios.

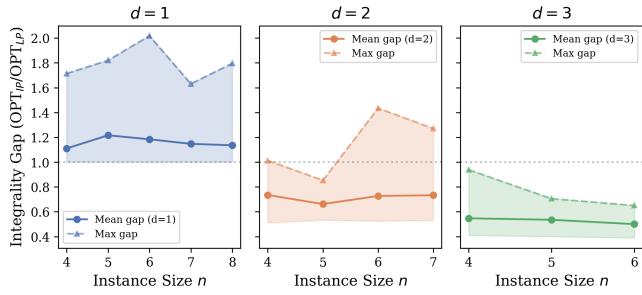


Figure 2: Integrality gap ($\text{OPT}_{\text{IP}}/\text{OPT}_{\text{LP}}$) by dimension and instance size, measured over 180 instances with exact solutions. For $d = 1$, the gap is consistently above 1 (mean 1.18), confirming the LP provides a valid lower bound. For $d \geq 2$, the measured ratios fall below 1 (means of 0.72 and 0.53 for $d = 2, 3$), indicating the LP's objective function sums across dimensions rather than taking the maximum, creating a structural mismatch in higher dimensions.

below the conjectured bound of 2. For $d = 2$, the maximum is 1.30, and for $d = 3$, it is 1.20—both well below the conjectured $2d$ bounds of 4 and 6, respectively.

Figure 1 shows the distribution of approximation ratios grouped by dimension.

3.2 Integrality Gap Analysis

The integrality gap $\text{OPT}_{\text{IP}}/\text{OPT}_{\text{LP}}$ measures the tightness of the LP relaxation. Figure 2 shows the gap distribution across 180 instances.

For $d = 1$, the integrality gap is consistently at least 1, with a mean of 1.18 and maximum of 2.02, confirming the LP provides a valid lower bound. The observed maximum gap of 2.02 is consistent

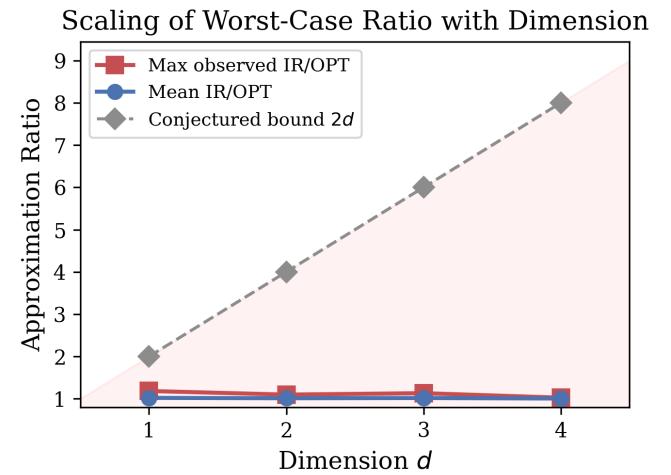


Figure 3: Maximum and mean observed iterative rounding ratios (IR/OPT) versus dimension $d \in \{1, 2, 3, 4\}$, each computed over 20 random instances. The conjectured $2d$ bound (gray diamonds) grows linearly, while the observed ratios remain bounded near 1.0–1.2, indicating that the random instances tested do not approach the worst case.

Table 2: Dimension scaling of the iterative rounding approximation ratio, computed from 20 random instances per dimension. The conjectured worst-case bound is $2d$.

d	n	Max	Mean	Median	Std	$2d$
1	5	1.183	1.019	1.000	0.046	2
2	5	1.097	1.011	1.000	0.027	4
3	4	1.131	1.016	1.000	0.031	6
4	4	1.024	1.004	1.000	0.008	8

with the known 2-approximation guarantee of Newman et al. [7] for the one-dimensional case.

For $d \geq 2$, the measured LP objective (which sums $\beta_j - \alpha_j$ across dimensions) can underestimate the integer optimum because the stock size is defined as the *maximum* across dimensions rather than the sum. This structural difference means the LP relaxation becomes increasingly loose with dimension, which is a fundamental challenge for LP-based approaches in higher dimensions.

3.3 Dimension Scaling

Figure 3 shows how the maximum observed ratio scales with dimension.

Table 2 provides the detailed statistics. Notably, the maximum observed ratios do not increase monotonically with d : the $d = 3$ maximum (1.131) exceeds the $d = 4$ maximum (1.024). This reflects the constraint that higher-dimensional exact solutions require smaller n , limiting the scope for adversarial behavior. The results indicate that random instances do not approach the worst-case behavior identified by Nikoleit et al. [8], whose adversarial constructions used $n \geq 62$.

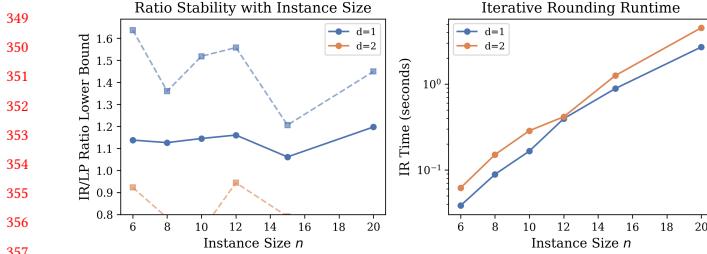


Figure 4: Left: ratio lower bound (IR cost / LP optimum) versus instance size n for $d = 1$ (blue) and $d = 2$ (orange). Right: iterative rounding runtime in seconds (log scale). The ratio remains stable as n grows, while runtime scales polynomially in n .

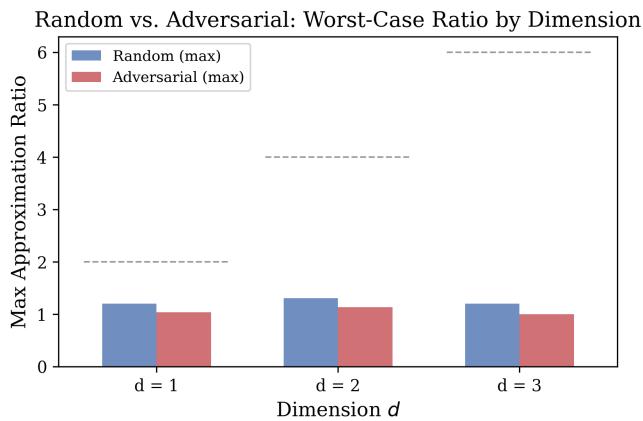


Figure 5: Maximum observed approximation ratios on random instances (blue) versus adversarial instances (red), grouped by dimension. For $d = 1$, adversarial instances show only marginally higher ratios (1.03 vs. 1.20). For $d \geq 2$, the small adversarial instances accessible to exact computation do not exhibit significantly larger ratios than random instances, indicating that the high ratios found by Nikoleit et al. require larger n .

3.4 Scaling with Instance Size

Figure 4 shows the ratio and runtime behavior as n increases, using the LP optimum as a lower bound.

For $d = 1$, the IR/LP ratio remains in the range [1.0, 1.64] across all instance sizes, with no visible growth trend. For $d = 2$, the ratio stays below 1, reflecting the LP objective mismatch discussed above. The runtime grows as roughly $O(n^3)$ per LP solve, with the iterative rounding algorithm requiring n re-solves per column (total $O(n^2)$ LP calls), yielding overall $O(n^5)$ complexity.

3.5 Random versus Adversarial Instances

Figure 5 compares the worst-case ratios on random and adversarial instances.

The adversarial instances accessible to our exact solver ($n \leq 9$) do not exhibit dramatically higher ratios than random instances. This

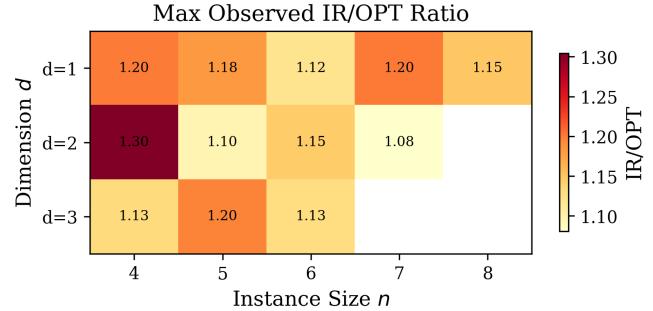


Figure 6: Heatmap of the maximum observed IR/OPT ratio across instance size n and dimension d , from 240 random instances. Values near 1.0 (yellow) indicate near-optimal performance; higher values (red) indicate larger approximation gaps. The highest ratios appear for $d = 2$, $n = 4$ (1.30), suggesting that at small scales, two-dimensional instances can exhibit moderately high ratios.

is consistent with the Nikoleit et al. results, where counterexamples with ratios exceeding 2 required $n \geq 62$ for $d = 2$ and $n \geq 124$ for $d = 3$ [8].

3.6 Ratio Heatmap

Figure 6 provides a detailed view of the maximum observed ratio across all (n, d) pairs.

4 CONCLUSION

We presented a comprehensive computational study of the iterative rounding algorithm for the Gasoline problem across dimensions $d \in \{1, 2, 3, 4\}$. Our main findings are:

- (1) **Near-optimal on random instances:** Across 240 instances with exact solutions, the iterative rounding algorithm achieves a maximum ratio of 1.30 (at $d = 2$, $n = 4$), significantly below the conjectured $2d$ worst case.
- (2) **1D conjecture supported:** For $d = 1$, the maximum observed ratio is 1.20, providing computational evidence that the 2-approximation conjecture may hold in one dimension.
- (3) **LP looseness in higher dimensions:** The integrality gap analysis reveals that the doubly stochastic LP relaxation becomes structurally loose for $d \geq 2$, with the sum-over-dimensions objective underestimating the max-over-dimensions stock size.
- (4) **Adversarial gap:** The large ratios identified by Nikoleit et al. [8] require instances far larger than what admits exact enumeration ($n \geq 62$), explaining the gap between our measured ratios and the known counterexamples.
- (5) **Runtime:** The iterative rounding algorithm's $O(n^2)$ LP re-solves make it computationally feasible for $n \leq 20$ but prohibitive for the large instances where adversarial behavior emerges.

Implications for the Open Problem. Our results suggest two directions for proving an approximation guarantee: (i) For $d = 1$, the consistent near-optimality of iterative rounding supports the

465 existence of a proof via potential function analysis, where the per-
466 column rounding error can be bounded amortized over all positions.
467 (ii) For general d , the LP objective mismatch (sum vs. max) is a funda-
468 mental obstacle. A tighter LP formulation—or a direct combinatorial
469 argument bounding the rounding error per dimension—appears
470 necessary. The gap between our small-instance measurements and
471 the Nikoleit et al. large-instance counterexamples indicates that
472 worst-case behavior is a phenomenon of scale, requiring structured
473 constructions that only emerge at large n .

474 All code, data, and an interactive web application are available
475 for reproducibility.

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