Applications of the Bi-Invariant Mean

by Oliver Brinkmann, and Jan Hakenberg, 2019-06-21

Background

Quote [2012 Pennec, Arsigny; p.20]: "*Although bi-invariant metrics may fail to exist, the group geodesics always exists in a Lie group and one can define a bi-invariant mean implicitly as an exponential barycenter, at least locally. As will be shown in the sequel, this definition has all the desirable invariance properties, even when bi-invariant metrics do not exist. Moreover, we can show the existence and uniqueness of the bi-invariant mean provided the dispersion of the data is small enough.*" ■

Quote [2012 Pennec, Arsigny; p.4]: "*The intuition behind such a bi-invariant mean on matrix Lie groups was present in* [2003 Woods] *along with a practical iterative algorithm to compute it. However, no precise definition nor proof of convergence was provided. The barycentric definition of bi-invariant means on Lie groups based on one-parameter subgroups was developed in the the PhD of Vincent Arsigny* [...] *and in the research report* [2006 Arsigny, Pennec, Ayache]." ■

Introduction

The bi-invariant mean generalizes the notion of affine combinations to Lie-groups: The bi-invariant mean of n points $g_i \in G$ for $i = 1, 2, \ldots, n$ from a Lie-group G, and an affine weight vector with $w \in \mathbb{R}^n$ and $\Sigma_i w_i = 1$, is the group element $m \in G$ that satisfies

$$\Sigma_i w_i \log\left(m^{-1}.g_i\right) = 0.$$

The equation implies the following properties of the bi-invariant mean: 1) m is invariant under permutation of the input sequence of points g_i and weights w_i. 2) the bi-invariant mean of two points $p, q \in G$ coincides with the binary average

$$[p, q]_\lambda = p.\exp\left[\lambda \log\left(p^{-1}.q\right)\right] = m \text{ for } \lambda \in \mathbb{R} \text{ and } p, q \in G$$

with $w_1 = 1 - \lambda$, $w_2 = \lambda$, $g_1 = p$, $g_2 = q$, and $r := p^{-1}.q$

$$(1-\lambda)\log\left[\left(p.\exp\left[\lambda\log\left(p^{-1}.q\right)\right]\right)^{-1}.p\right] + \lambda\log\left[\left(p.\exp\left[\lambda\log\left(p^{-1}.q\right)\right]\right)^{-1}.q\right]$$
$$= (1-\lambda)\log\left[(\exp[\lambda\log r])^{-1}\right] + \lambda\log\left[(\exp[\lambda\log r])^{-1}.r\right] = (1-\lambda)(-\lambda\log r) + \lambda(1-\lambda)\log r = 0.$$

For 3) left-, right- and inverse-invariance of bi-invariant means see [2012 Pennec, Arsigny; p. 21].

[2012 Pennec, Arsigny] state explicit formulas for the bi-invariant mean for the Heisenberg group He(n), ST(n), SE(2), etc.

Contributions

In the open-source repository [2019 IDSC/Frazzoli], we implemented

- the explicit formula of the bi-invariant mean for the covering Lie group $\overline{\mathrm{SE}}(2)$,
- the retrieval of the weight vector w for given g_i and bi-invariant mean m, and
- the applications interpolation, averaging, etc. listed below.

Remark: Further applications are sampling, denoising, and causal filtering. ■

References

[2003 Woods] Characterizing volume and surface deformations in an atlas framework: theory, applications, and implementation.

[2006 Arsigny, Pennec, Ayache] Bi-invariant Means in Lie Groups. Application to Left-invariant Polyaffine Transformations.

[2012 Pennec, Arsigny] Exponential Barycenters of the Canonical Cartan Connection and Invariant Means on Lie Groups

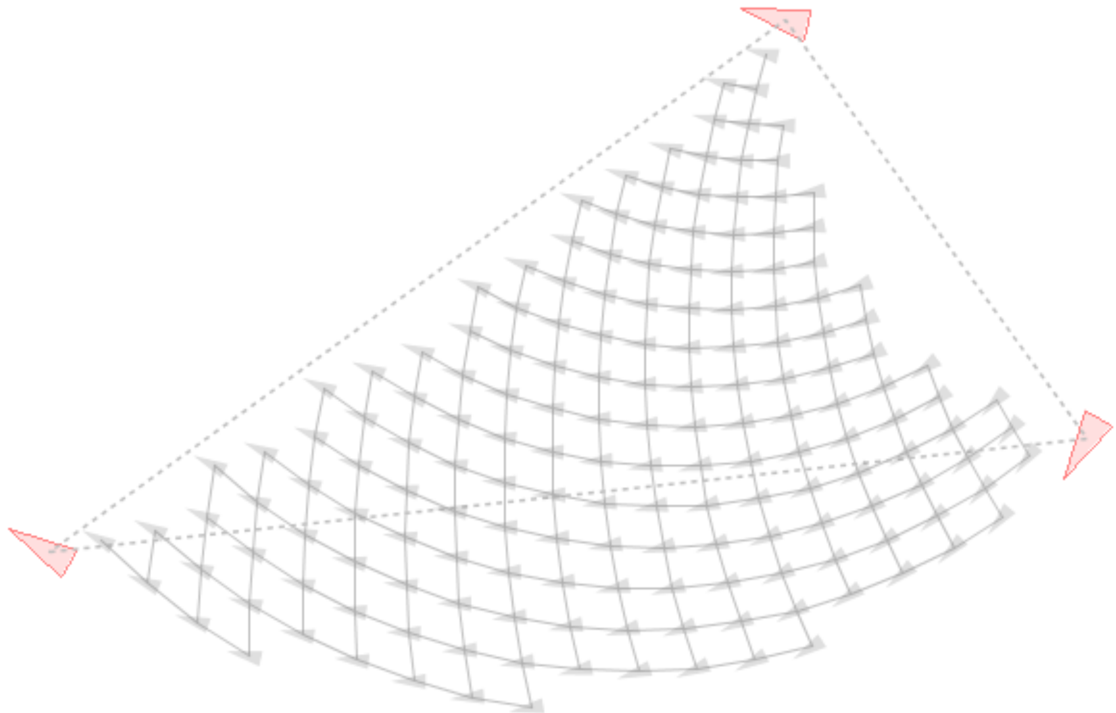
[2016 Budninskiy, Liu, Tong, Desbrun] Power Coordinates: A Geometric Construction of Barycentric Coordinates on Convex Polytopes

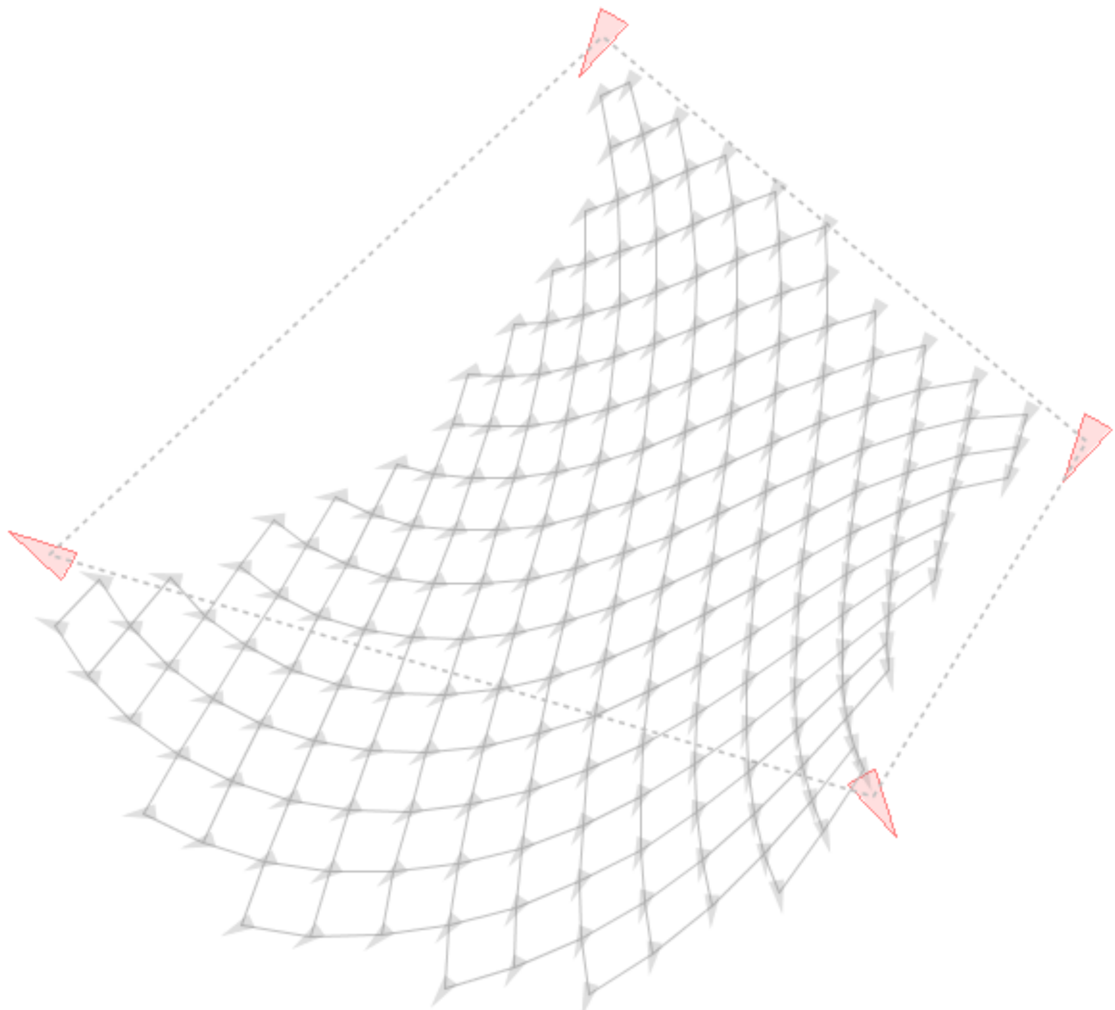
[2019 IDSC/Frazzoli] https://github.com/idsc-frazzoli/owl

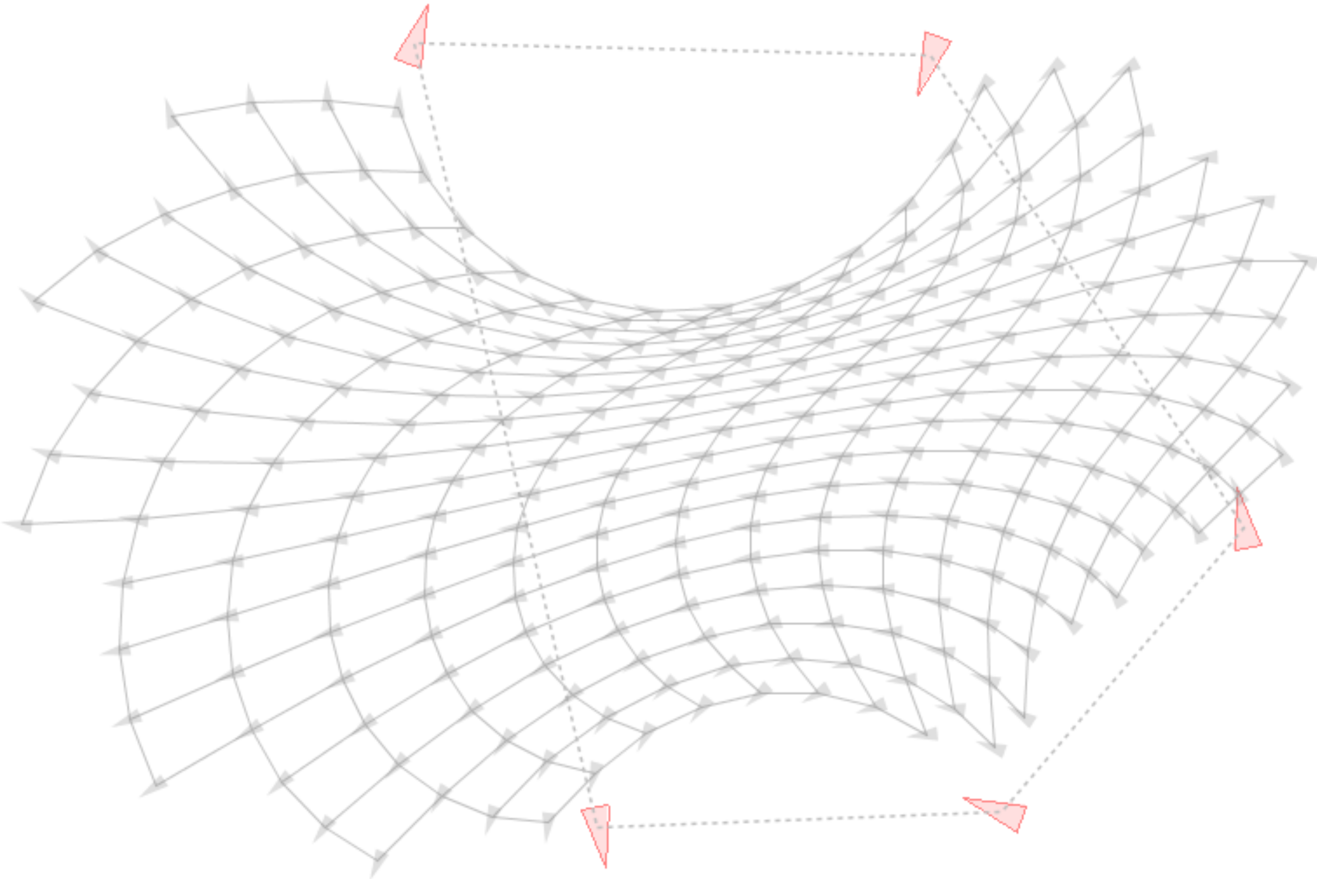
Applications

Interpolation using Barycentric Coordinates

Control points $g_i \in \overline{\mathrm{SE}}(2)$ are plotted as red arrowheads. The interpolation function is C^∞ inside the convex domain.

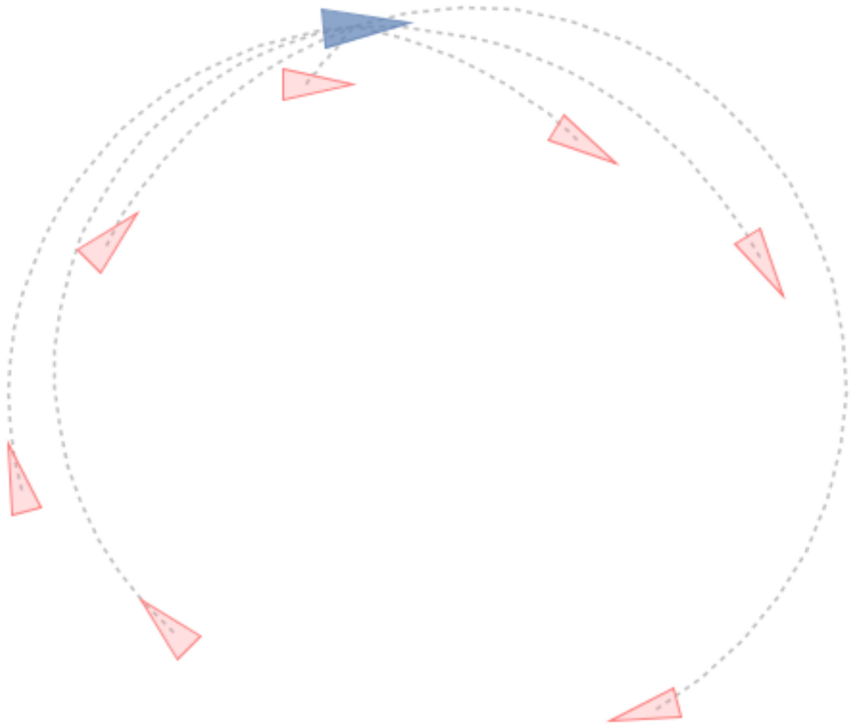


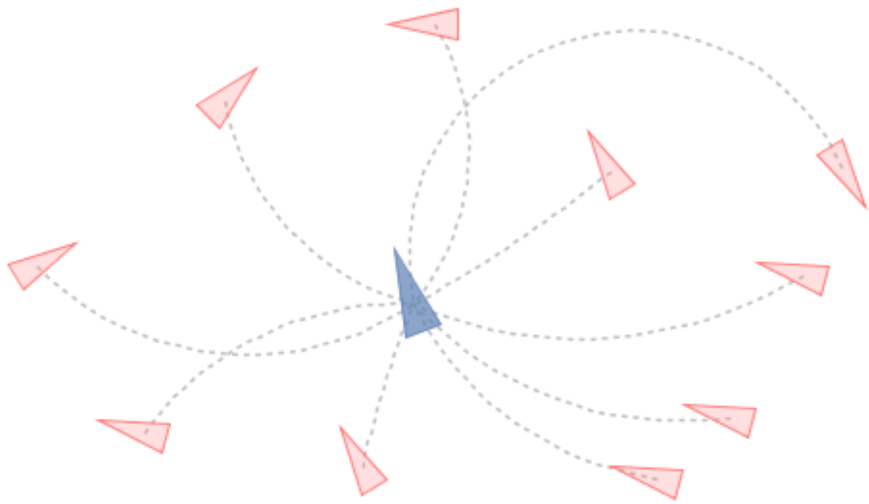


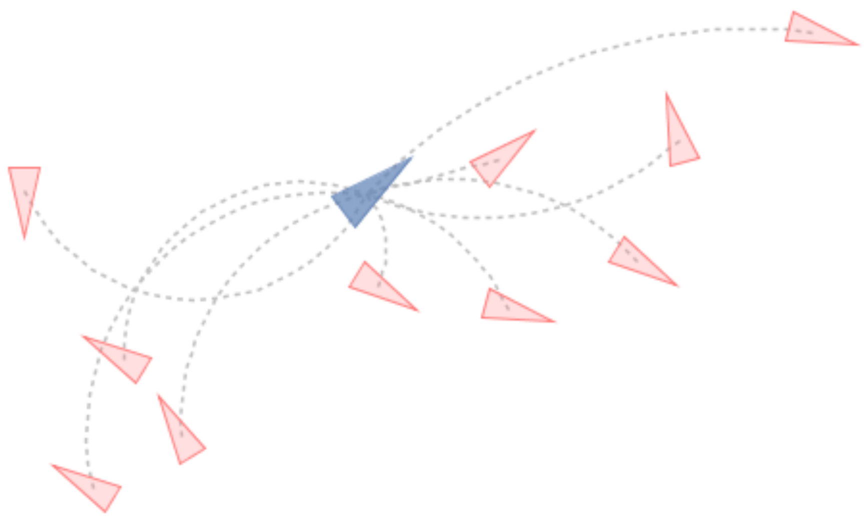


Mean

Control points $g_i \in \overline{\mathrm{SE}}(2)$ in red, mean in blue. All weights are $w_i = 1/n$, where n is the number of control points.

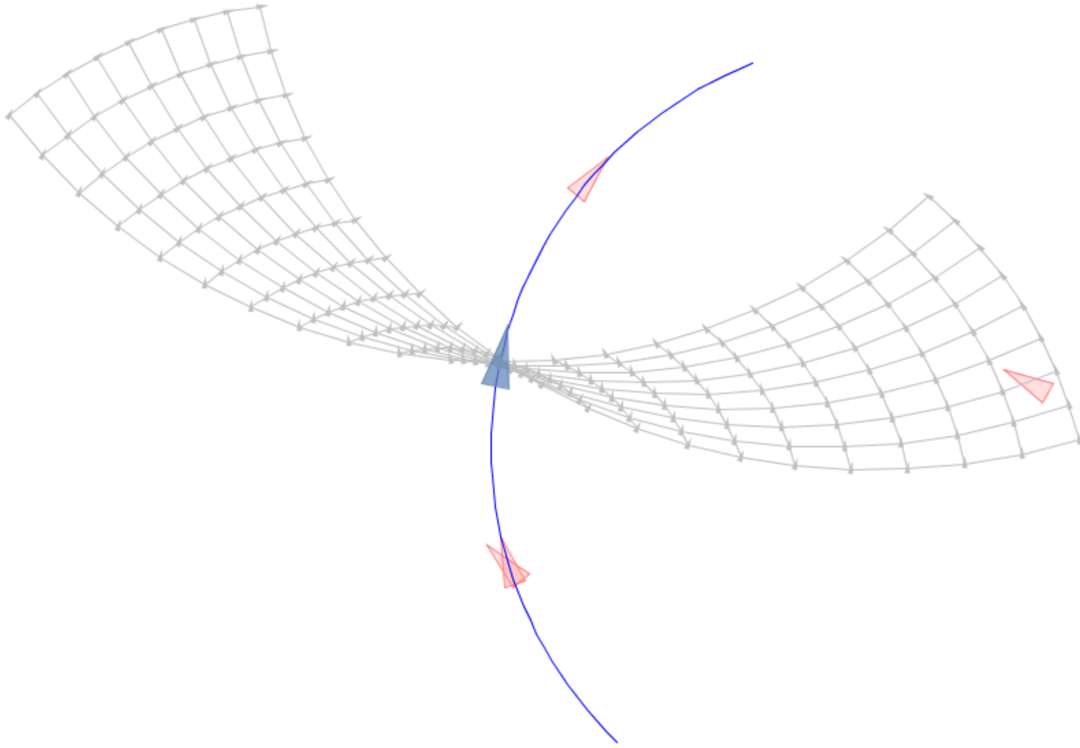


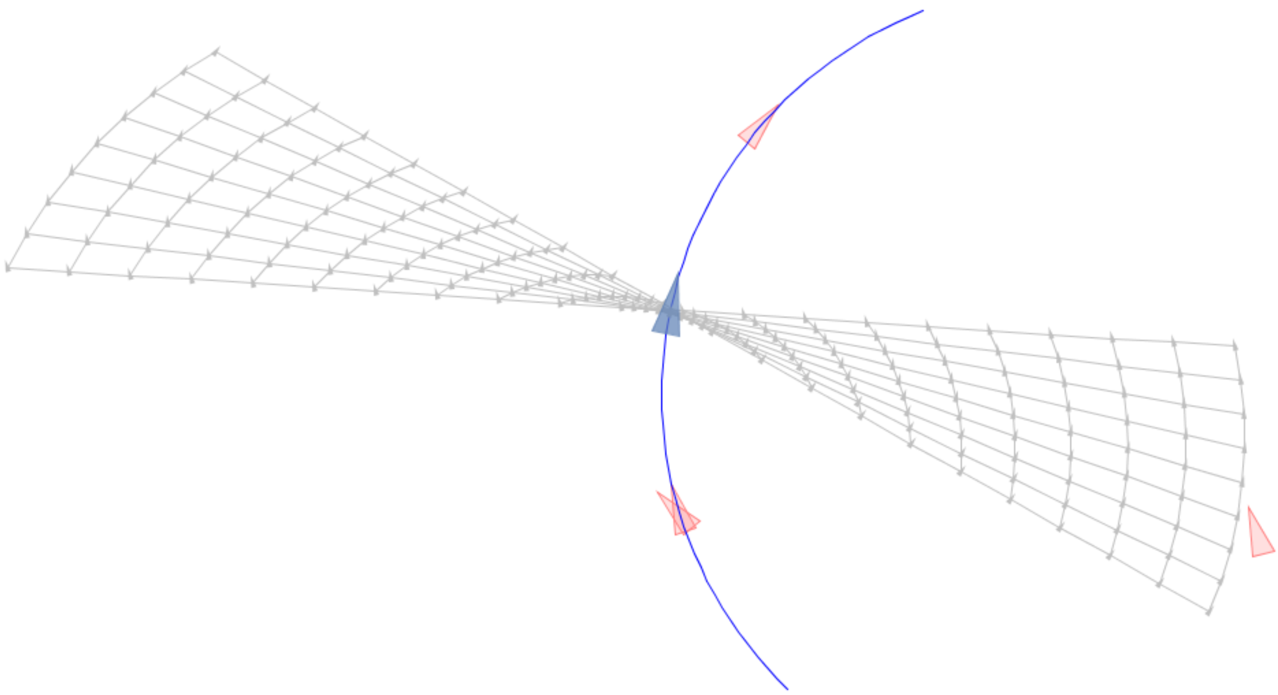


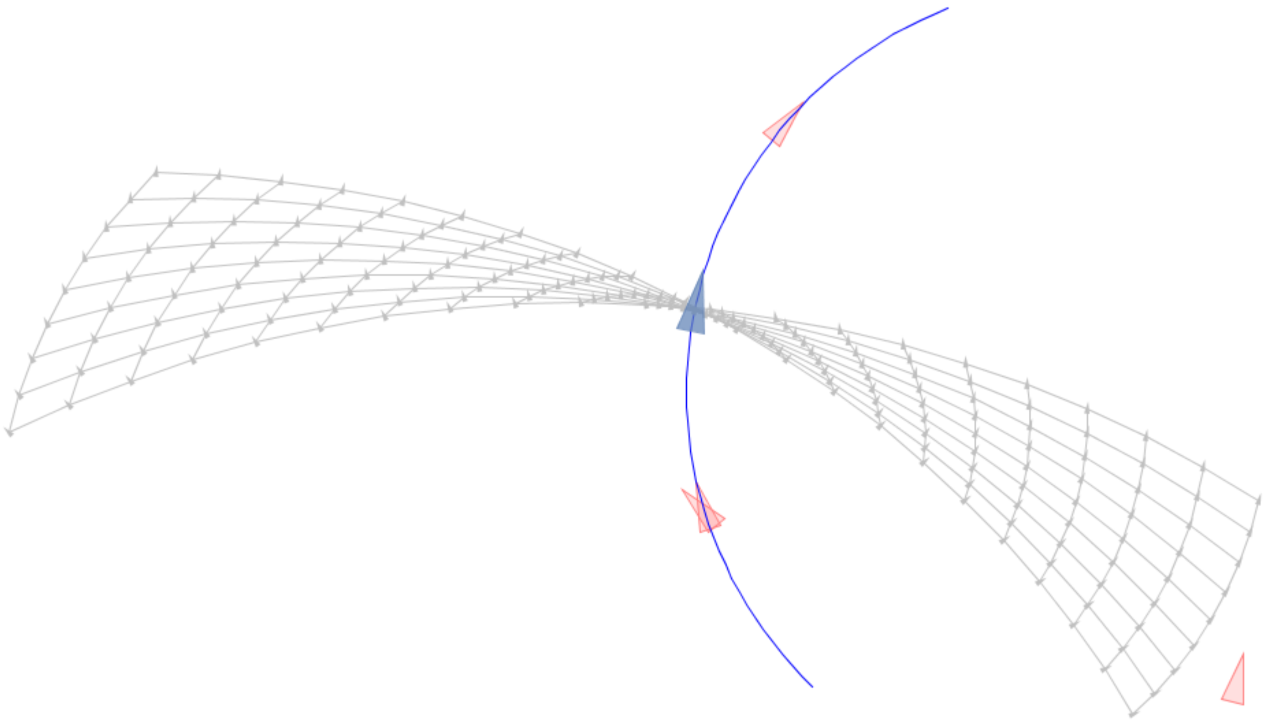


Projection to Geodesic Subspaces

The 4 control points $g_i \in \overline{\mathrm{SE}}(2)$ in red define a parametrization of a subset of $\overline{\mathrm{SE}}(2)$. The gray points are projected to the same point (in blue) along the geodesic between g_1 and g_2.







Mesh Subdivision

Catmull-Clark