# STAT 524
# HW3

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4.2

**4.2.** Consider a bivariate normal population with $\mu_1 = 0$, $\mu_2 = 2$, $\sigma_{11} = 2$, $\sigma_{22} = 1$, and $\rho_{12} = .5$.

(a) Write out the bivariate normal density.

(b) Write out the squared generalized distance expression $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ as a function of $x_1$ and $x_2$.

(c) Determine (and sketch) the constant-density contour that contains 50% of the probability.

(a) $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ $\quad |\Sigma| = \sigma_{11}\sigma_{22} - \sigma_{12}^2$

$$\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} \iff \sigma_{12} = \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} = \frac{\sqrt{2}}{2}$$

$$\therefore |\Sigma| = \sigma_{11}\sigma_{22} - \rho_{12}^2\sigma_{11}\sigma_{22} = \sigma_{11}\sigma_{22}(1-\rho_{12}^2)$$

A Bivariate normal density formula is

$$f(x) = \frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} \exp\left\{ \frac{-1}{2} (\underset{\sim}{x} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \right\}$$

$$= \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)}} \exp\left\{ \frac{-1}{2(1-\rho_{12}^2)} \left[ \left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sigma_{22}}\right) \right] \right\}$$

$$= \frac{1}{2\pi\sqrt{2\times1\times(1-0.5^2)}} \exp\left\{ \frac{-1}{2(1-0.5^2)} \left[ \left(\frac{x_1-0}{\sqrt{2}}\right)^2 + (x_2-2)^2 - 2\cdot0.5\left(\frac{x_1-0}{\sqrt{2}}\right)^2 (x_2-2)^2 \right] \right\}$$

$$= \frac{1}{2\pi\cdot\sqrt{1.5}} \exp\left\{ \frac{-2}{3} \left[ \frac{x_1^2}{2} + (x_2-2)^2 - \frac{\sqrt{2}x_1^2}{2}(x_2-2)^2 \right] \right\}$$

$$= \frac{1}{\sqrt{6}\pi} \exp\left\{ \frac{-x_1^2}{3} - \frac{2(x_2-2)^2}{3} + \frac{\sqrt{2}x_1^2}{3}(x_2-2)^2 \right\} \quad //$$

(d) $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$ $\quad \Sigma^{-1} = \dfrac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{21}} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$

$$\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} \iff \sigma_{12} = \rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} = \sigma_{21}$$

$$\therefore \Sigma^{-1} = \frac{1}{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)} \begin{pmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{21} & \sigma_{11} \end{pmatrix}$$

$$(x-\mu)^T \Sigma^{-1} (x-\mu)$$

$$= [x_1-\mu_1, \; x_2-\mu_2] \frac{1}{\sigma_{11}\sigma_{22}(1-\rho_{12})^2} \begin{bmatrix} \sigma_{22} & -\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} \\ -\rho_{12}\sqrt{\sigma_{11}}\sqrt{\sigma_{22}} & \sigma_{11} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= \frac{1}{1-\rho_{12}^2}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right]$$

$$= \frac{1}{1-0.5^2}\left[\left(\frac{x_1-0}{\sqrt{2}}\right)^2 + \left(\frac{x_2-2}{\sqrt{1}}\right)^2 - 2\cdot 0.5\left(\frac{x_1-0}{\sqrt{2}}\right)\left(\frac{x_2-2}{\sqrt{1}}\right)\right]$$

$$= \frac{4}{3}\left[\frac{x_1^2}{2} + (x_2-2)^2 - \left(\frac{x_1}{\sqrt{2}}\right)(x_2-2)\right]$$

$$= \frac{2}{3}x_1^2 + \frac{4}{3}(x_2-2)^2 - \frac{2\sqrt{2}}{3}x_1(x_2-2) \quad //$$

(e) $P(d^2(x_1, x_2) \le c^2) = 0.5$

$$d^2(x_1, x_2) = (\underset{\sim}{x} - \underset{\sim}{\mu})\Sigma^{-1}(\underset{\sim}{x} - \underset{\sim}{\mu}) \le \chi_2^2(0.5) = 1.386$$

Let $(\lambda, e)$ be an eigenvalue - eigenvector pair for $\Sigma$.

$$|\Sigma - \lambda I| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} \\ \sigma_{21} & \sigma_{22} - \lambda \end{vmatrix}$$

$$= (\sigma_{11} - \lambda)(\sigma_{22} - \lambda) - \sigma_{12}^2$$

$$= (2-\lambda)(1-\lambda) - \left(\frac{\sqrt{2}}{2}\right)^2$$

$$= 2\lambda^2 - 6\lambda + 3$$

Since $|\Sigma - \lambda I| = 0$

$$\lambda = \frac{3}{2} \pm \frac{\sqrt{3}}{2} \Rightarrow \lambda_1 = \frac{3}{2} + \frac{\sqrt{3}}{2}, \ \lambda_2 = \frac{3}{2} - \frac{\sqrt{3}}{2}$$

For $\lambda_1 = \frac{3}{2} + \frac{\sqrt{3}}{2}$,

$$\begin{bmatrix} 2 & \sqrt{2}/2 \\ \sqrt{2}/2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} 2x + \frac{\sqrt{2}}{2} y = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) x \\ \frac{\sqrt{2}}{2} x + y = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right) y \end{cases}$$

$$\begin{cases} \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) x - \frac{\sqrt{2}}{2} y = 0 \\ \frac{\sqrt{2}}{2} x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) y = 0 \end{cases}$$

$$\begin{cases} \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) x - \frac{2}{4} y = 0 \\ \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) x - \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) y = 0 \end{cases}$$

$$\begin{cases} \frac{(\sqrt{6} - \sqrt{2})}{2} x - y = 0 \\ (\sqrt{6} - \sqrt{2}) x - 2y = 0 \end{cases} \Rightarrow y = \frac{\sqrt{6} - \sqrt{2}}{2} x$$

By using normalization, $x^2 + y^2 = 1$

$$x^2 + \frac{(\sqrt{6}-\sqrt{2})^2}{4} x^2 = 1$$

$$4x^2 + (6 - 2\sqrt{12} + 2)x^2 = 4$$

$$(12 - 2\sqrt{12})\, x^2 = 4$$

$$x^2 = 0.78867$$

$$x = 0.88807 \qquad \therefore e_1 \approx \begin{bmatrix} 0.880 \\ 0.460 \end{bmatrix}$$

$$y = 0.460$$

Since major axis and minor axis are perpendicular to each other, $e_1 \cdot e_2 = 0$ $\therefore e_2 \approx \begin{bmatrix} -0.460 \\ 0.880 \end{bmatrix}$ for $\lambda_2$

To get the lengths of the ellipse,

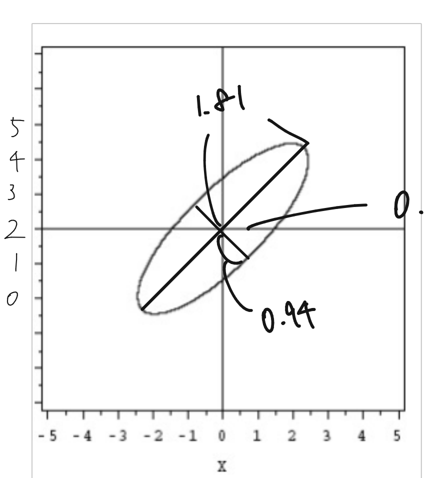
$$c\sqrt{\lambda_1} = \sqrt{1.386} \cdot \sqrt{\frac{3+\sqrt{3}}{2}}$$

$= 1.81$ for the major axis in the direction of $\pm \begin{bmatrix} 0.880 \\ 0.460 \end{bmatrix}$

$$c\sqrt{\lambda_2} = \sqrt{1.386} \cdot \sqrt{\frac{3-\sqrt{3}}{2}}$$

$= 0.94$ for the minor axis in the direction of $\pm \begin{bmatrix} 0.460 \\ 0.880 \end{bmatrix}$

Since the center of eclipse is $(0, 2)$

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4.4

**4.4.** Let $\mathbf{X}$ be $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\mu}' = [2, -3, 1]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

(a) Find the distribution of $3X_1 - 2X_2 + X_3$.

(b) Relabel the variables if necessary, and find a $2 \times 1$ vector $\mathbf{a}$ such that $X_2$ and $X_2 - \mathbf{a}'\begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$ are independent.

$\Sigma = \frac{XX^T}{N}$ or $\frac{XX^T}{n-1}$

$A \Sigma A^T =$

(a) $X \sim N(\mu, \Sigma)$

Consider the linear combination $A^T X$,

$$A^T X = 3x_1 - 2x_2 + x_3$$

$$= \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A^T \mu = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 13$$

$$A^T \Sigma A = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 9$$

$\therefore 3x_1 - 2x_2 + x_3 \sim N(13, 9)$ //

(b) If $x_2$ and $x_2 - a^T \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$ are independent, then

$Cov\left(x_2, x_2 - a \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}\right) = 0$ where $a^T = [a_1 \; a_3]$

$$= Cov(x_2, x_2 - a_1 x_1 - a_3 x_3)$$

$$= \begin{bmatrix} Var(x_2) & Cov(x_2, x_2 - a_1 x_1 - a_3 x_3) \\ Cov(x_2, x_2 - a_1 x_1 - a_3 x_3) & Var(x_2 - a_1 x_1 - a_3 x_3) \end{bmatrix}$$

$$= A \Sigma A^T$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -a_1 & 1 & -a_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -a_1 \\ 1 & 1 \\ 0 & -a_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -a_1 + 3 - 2a_3 \\ -a_1 + 3 - a_3 & a_1^2 - 2a_3 + 2a_1a_3 + 3 - 4a_3 + 2a_3^2 \end{bmatrix}$$

$\therefore -a_1 + 3 - 2a_3$ must be equal to $0$

$a_1 = 3 - 2a_3$

$\therefore$ when $a = \begin{bmatrix} 3-2a_3 \\ a_3 \end{bmatrix}$, $X_2$ and $X_2 - a^T \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$ are ind.

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4-6.

**4.6.** Let $\mathbf{X}$ be distributed as $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}' = [1, -1, 2]$ and

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Which of the following random variables are independent? Explain.

(a) $X_1$ and $X_2$
(b) $X_1$ and $X_3$
(c) $X_2$ and $X_3$
(d) $(X_1, X_3)$ and $X_2$
(e) $X_1$ and $X_1 + 3X_2 - 2X_3$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

(a) $Cov(X_1, X_2) = \sigma_{12} = \sigma_{21} = 0$ $\therefore$ Independent

(b) $Cov(x_1, x_3) = \sigma_{13} = \sigma_{31} = -1$ $\therefore$ dependent

(c) $Cov(x_2, x_3) = \sigma_{23} = \sigma_{31} = 0$ $\therefore$ Independent

(d) $Cov((x_1, x_3), x_2)$

$\Rightarrow Cov(x_1, x_3) = -1$

$$X = \begin{bmatrix} x_1 \\ x_3 \\ \hdashline x_2 \end{bmatrix} = \begin{bmatrix} x_* \\ x_2 \end{bmatrix} \quad \text{where } x_* = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

$$\Sigma = \left[\begin{array}{c:c} \Sigma_{**} & \Sigma_{*3} \\ \hdashline \Sigma_{3*} & \Sigma_{33} \end{array}\right] = \left[\begin{array}{cc:c} 4 & -1 & 0 \\ -1 & 2 & 0 \\ \hdashline 0 & 0 & 5 \end{array}\right]$$

$$\Sigma_{*3} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \therefore (x_1, x_3) \text{ and } x_2 \text{ are independent} //$$

(e) $Cov(x_1, (x_1 + 3x_2 - 2x_3))$

Let $a^T = [1 \ 3 \ -2]$

Covariance matrix of $x_1$ and $(x_1 + 3x_2 - 2x_3)$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & -1 \\ 6 & 15 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 6 & 61 \end{bmatrix}$$

$$\therefore Cov(x_1, (x_1 + 3x_2 - 2x_3)) = 6$$

$\Leftrightarrow$ $x_1$ and $(x_1 + 3x_2 - 2x_3)$ are dependent //

4.12

**4.12.** Show that, for **A** symmetric,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1} & \mathbf{0} \\ \mathbf{0}' & \mathbf{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix}$$

Thus, $(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1}$ is the upper left-hand block of $\mathbf{A}^{-1}$.

*Hint:* Premultiply the expression in the hint to Exercise 4.11 by $\begin{bmatrix} \mathbf{I} & -\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ \mathbf{0}' & \mathbf{I} \end{bmatrix}^{-1}$ and postmultiply by $\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21} & \mathbf{I} \end{bmatrix}^{-1}$. Take inverses of the resulting expression.

Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

From Exercise 4.11,

$$\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix}$$

By premultiplying this equation by $\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}^{-1}$

and postmultiplying by $\begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix}^{-1}$

(Left hand side)

$$= \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

(Right hand side)

$$= \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I & 0 \\ -A_{22}^{-1}A_{21} & I \end{bmatrix} \begin{bmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & 0 \\ 0 & A_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \quad //$$

4.16

> **4.16.** Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$, and $\mathbf{X}_4$ be independent $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ random vectors.
> (a) Find the marginal distributions for each of the random vectors
> $$\mathbf{V}_1 = \tfrac{1}{4}\mathbf{X}_1 - \tfrac{1}{4}\mathbf{X}_2 + \tfrac{1}{4}\mathbf{X}_3 - \tfrac{1}{4}\mathbf{X}_4$$
> and
> $$\mathbf{V}_2 = \tfrac{1}{4}\mathbf{X}_1 + \tfrac{1}{4}\mathbf{X}_2 - \tfrac{1}{4}\mathbf{X}_3 - \tfrac{1}{4}\mathbf{X}_4$$
> (b) Find the joint density of the random vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ defined in (a).

(a) $x_1, x_2, x_3, x_4 \overset{iid}{\sim} N_p(\mu, \Sigma)$

$$V_1 = \frac{1}{4}x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 - \frac{1}{4}x_4$$

$$V_1^T \mu = \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = 0$$

$$V_1^T \Sigma = \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} & \frac{-1}{4} \end{bmatrix} \Sigma \begin{bmatrix} 1/4 \\ -1/4 \\ 1/4 \\ -1/4 \end{bmatrix} = \frac{1}{4}\Sigma$$

$\therefore$ The marginal distribution of $V_1 = (0, \frac{1}{4}\Sigma)$

$$V_2 = \frac{1}{4}x_1 + \frac{1}{4}x_2 - \frac{1}{4}x_3 - \frac{1}{4}x_4$$

$$V_2^T \mu = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = 0$$

$$V_2^T \mu = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix} \Sigma \begin{bmatrix} 1/4 \\ 1/4 \\ -1/4 \\ -1/4 \end{bmatrix} = \frac{1}{4}\Sigma$$

$\therefore$ The marginal distribution of $V_2 = (0, \frac{1}{4}\Sigma)$

(e) $\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} & \frac{-1}{4} \end{bmatrix} \begin{bmatrix} 1/4 \\ 1/4 \\ -1/4 \\ -1/4 \end{bmatrix} = 0$

$\therefore$ Covariance matrix of $V_1$ and $V_2$

$$= \begin{bmatrix} \frac{1}{4}\Sigma & 0\Sigma \\ 0\Sigma & \frac{1}{4}\Sigma \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\Sigma & 0 \\ 0 & \frac{1}{4}\Sigma \end{bmatrix}$$

$\therefore$ The joint distribution of $V_1, V_2 \sim N_{2p}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/4\Sigma & 0 \\ 0 & 1/4\Sigma \end{bmatrix}\right)$

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4.39 P181

**4.39.** The data in Table 4.6 (see the psychological profile data: www.prenhall.com/statistics) consist of 130 observations generated by scores on a psychological test administered to Peruvian teenagers (ages 15, 16, and 17). For each of these teenagers the gender (male = 1, female = 2) and socioeconomic status (low = 1, medium = 2) were also recorded. The scores were accumulated into five subscale scores labeled *independence* (indep), *support* (supp), *benevolence* (benev), *conformity* (conform), and *leadership* (leader).

**Table 4.6** Psychological Profile Data

| Indep | Supp | Benev | Conform | Leader | Gender | Socio |
|---|---|---|---|---|---|---|
| 27 | 13 | 14 | 20 | 11 | 2 | 1 |
| 12 | 13 | 24 | 25 | 6 | 2 | 1 |
| 14 | 20 | 15 | 16 | 7 | 2 | 1 |
| 18 | 20 | 17 | 12 | 6 | 2 | 1 |
| 9 | 22 | 22 | 21 | 6 | 2 | 1 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | 11 | 26 | 17 | 10 | 1 | 2 |
| 14 | 12 | 14 | 11 | 29 | 1 | 2 |
| 19 | 11 | 23 | 18 | 13 | 2 | 2 |
| 27 | 19 | 22 | 7 | 9 | 2 | 2 |
| 10 | 17 | 22 | 22 | 8 | 2 | 2 |

Source: Data courtesy of C. Soto.

(a) Examine each of the variables independence, support, benevolence, conformity and leadership for marginal normality.

(b) Using all five variables, check for multivariate normality.

(c) Refer to part (a). For those variables that are nonnormal, determine the transformation that makes them more nearly normal.

(a) Critical points for the Q-Q plot Correlation Coeff test for Normality when $n = 130$.

$$(\alpha = 0.01,\ \alpha = 0.05) = (0.9854,\ 0.9897)$$

From the R-output,

$r_Q$ of Independence, Support, and Leadership $< 0.9897$

$\therefore$ We reject the hypothesis of marginal normality for these three variables.

$r_Q$ of benevolence and conformity $> 0.9854$

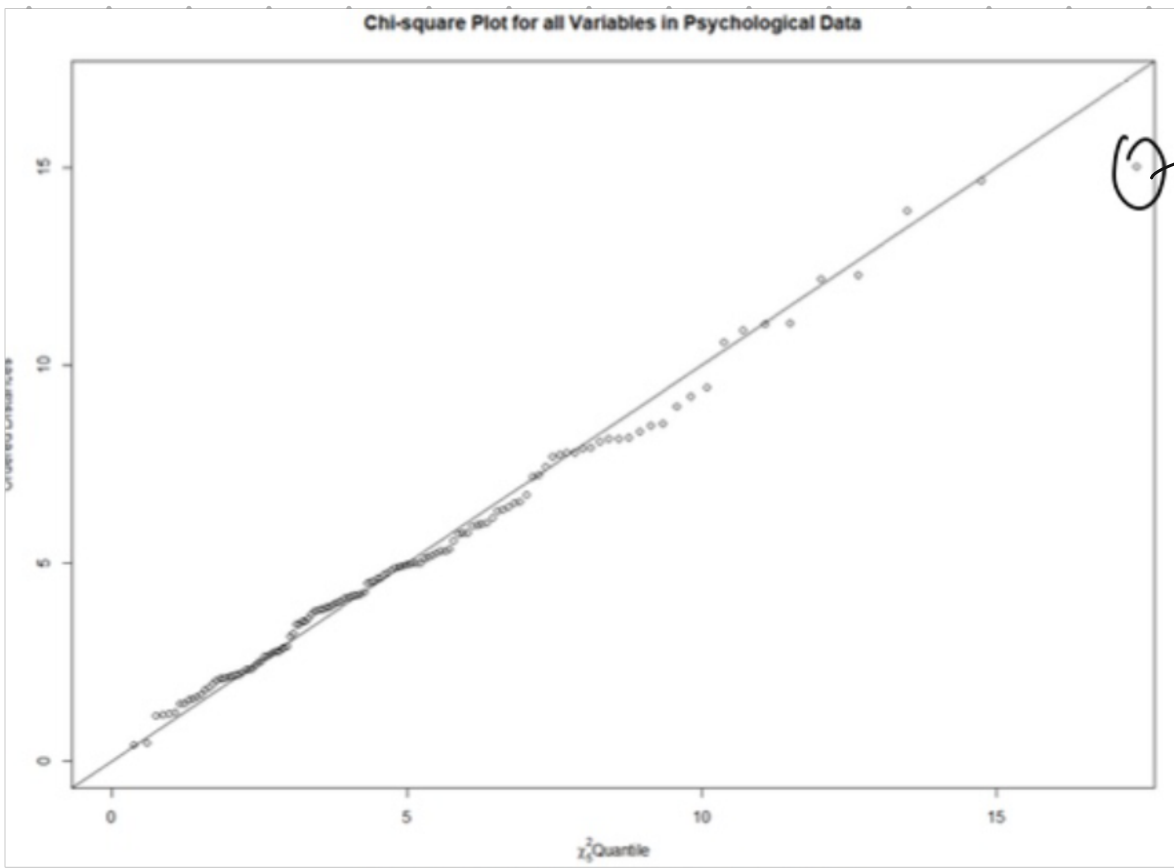
$\therefore$ We cannot reject the hypothesis of marginal normality for these two Variables.

```
> cor(v2$x, v2$y)
[1] 0.989288
>
> v3 <- qqnorm(df$Benev, main="Normal Q-Q Plot", xlab="Theoretical Quantiles", ylab="Sample Quantiles")
cor(v3$x, v3$y)

v4 <- qqnorm(df$Conform, main="Normal Q-Q Plot", x> cor(v3$x, v3$y)
[1] 0.9925086
>
> v4 <- qqnorm(df$Conform, main="Normal Q-Q Plot", xlab="Theoretical Quantiles", ylab="Sample Quantiles")
cor(v4$x, v4$y)

> cor(v4$x, v4$y)
[1] 0.99338
>
> v5 <- qqnorm(df$Leader, main="Normal Q-Q Plot", xlab="Theoretical Quantiles", ylab="Sample Quantiles")
cor(v5$x, v5$y)
> cor(v5$x, v5$y)
```

(b) By the plot below, almost all the obs follow the 45° line. Hence, it's hard to argue against multivariate normality.




outlier

(c) $r_Q$ of transformed Indep, Ledership, and support $> 0.9599$.

$\therefore$ We cannot reject normality