STAT 524
HW2

Satoshi Ido
34788706
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2.1

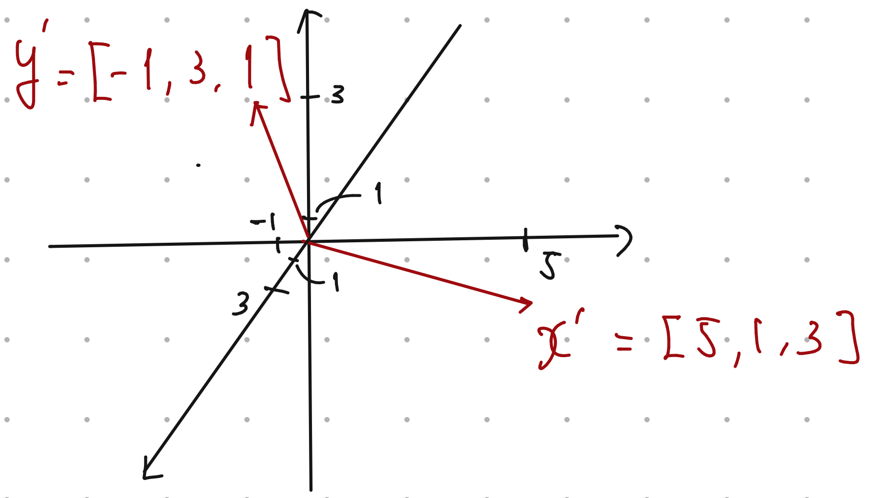
2.1. Let $\mathbf{x}' = [5, \ 1, \ 3]$ and $\mathbf{y}' = [-1, \ 3, \ 1]$.

(a) Graph the two vectors.

(b) Find (i) the length of $\mathbf{x}$, (ii) the angle between $\mathbf{x}$ and $\mathbf{y}$, and (iii) the projection of $\mathbf{y}$ on $\mathbf{x}$.

(c) Since $\bar{x} = 3$ and $\bar{y} = 1$, graph $[5-3, 1-3, 3-3] = [2, -2, 0]$ and $[-1-1, 3-1, 1-1] = [-2, 2, 0]$.

a)



b) i) $\|x\| = \sqrt{5^2+1^2+3^2} = \sqrt{35}$

$\|y\| = \sqrt{(-1)^2+3^2+1^2} = \sqrt{11}$

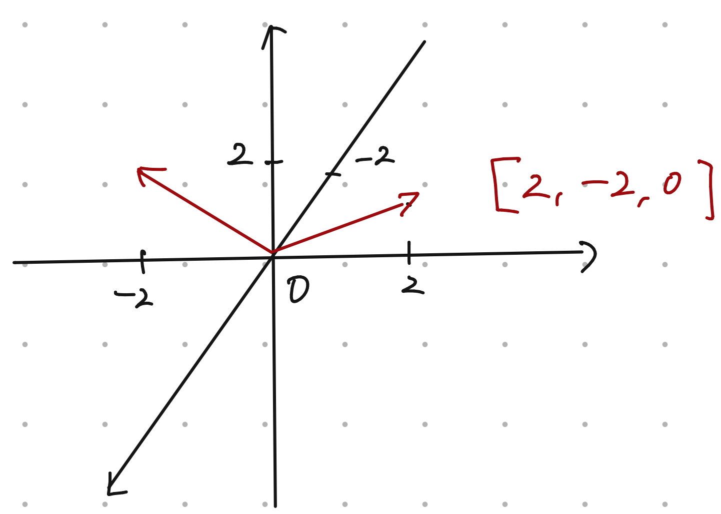
ii) $\cos\theta = \dfrac{-5+3+3}{\sqrt{35}\sqrt{11}} = \dfrac{1}{\sqrt{385}} = 0.05285$

$\theta = \cos^{-1}(0.05285) = \underline{86.97}$

iii) $P_x y = \dfrac{(x^T y)}{L_x^2} x$

$= \dfrac{1}{35} \times (5, 1, 3) = \underline{\left(\dfrac{1}{7}, \dfrac{1}{35}, \dfrac{3}{35}\right)}$

c)



2-3

2.3. Verify the following properties of the transpose when

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

(a) $(\mathbf{A}')' = \mathbf{A}$
(b) $(\mathbf{C}')^{-1} = (\mathbf{C}^{-1})'$
(c) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
(d) For general $\underset{(m\times k)}{\mathbf{A}}$ and $\underset{(k\times \ell)}{\mathbf{B}}$, $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

(a) $A^T = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ $\quad (A^T)^T = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A$

$\therefore (A^T)^T = A$ //

(b) $C^T = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ $\quad (C^T)^{-1} = \frac{-1}{10}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1/5 & 3/10 \\ 2/5 & -1/10 \end{bmatrix}$

$(C^{-1}) = \begin{bmatrix} -1/5 & 2/5 \\ 3/10 & -1/10 \end{bmatrix}$ $\quad (C^{-1})^T = \begin{bmatrix} -1/5 & 3/10 \\ 2/5 & -1/10 \end{bmatrix}$

$\therefore (C^T)^{-1} = (C^{-1})^T$ //

(c) $(AB)^T = \left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}\right)^T = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$

$\therefore (AB)^T = (B^T A^T)$ //

(d) We know that $(AB)_{ji} = (AB)^T_{ij}$

$$(AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}$$

Also, $(B^T A^T)_{ij} = \sum_{k=1}^{n} B^T_{ik} A^T_{kj}$

$$= \sum_{k=1}^{n} B_{ki} A_{jk}$$

$$= \sum_{k=1}^{n} A_{jk} B_{ki}$$

$\therefore (AB)^T_{ij} = (B^T A^T)_{ij}$

, for all $i = 1, \dots, m$ $j = 1, \dots, l$

$(AB)^T = B^T A^T$

2-6

2.6. Let

$$\mathbf{A} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

(a) Is $\mathbf{A}$ symmetric?
(b) Show that $\mathbf{A}$ is positive definite.

(a) $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

Since $A = A^T$, A is symmetric.

(b) Assume λ as eigen value of matrix A.

$$\underset{\sim}{A} - \lambda \underset{\sim}{I} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}$$

$$\det | \underset{\sim}{A} - \lambda \underset{\sim}{I} | = (9-\lambda)(6-\lambda) - 4$$

$$= (\lambda - 5)(\lambda - 10)$$

Since $\det | \underset{\sim}{A} - \lambda \underset{\sim}{I} | = 0$, $\lambda = 5, 10$

$\lambda = 5, 10$. Both of them are positive $\therefore$ $\underline{A \text{ is P.D}}$ //

$\lambda = 5$,

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I}) \underset{\sim}{x} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= 2x_1 - x_2 = 0$$

$$\therefore \quad \underset{\sim}{x} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$\lambda = 10$,

$$(\underset{\sim}{A} - \lambda \underset{\sim}{I}) \underset{\sim}{x} = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= -x_1 - 2x_2 = 0$$

$$\therefore \quad \underset{\sim}{x} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

2.14

2.14. Show that $\underset{(p\times p)}{\mathbf{Q}'}\underset{(p\times p)}{\mathbf{A}}\underset{(p\times p)}{\mathbf{Q}}$ and $\underset{(p\times p)}{\mathbf{A}}$ have the same eigenvalues if $\mathbf{Q}$ is orthogonal.

Hint: Let λ be an eigenvalue of $\mathbf{A}$. Then $0 = |\mathbf{A} - \lambda\mathbf{I}|$. By Exercise 2.13 and Result 2A.11(e), we can write $0 = |\mathbf{Q}'||\mathbf{A} - \lambda\mathbf{I}||\mathbf{Q}| = |\mathbf{Q}'\mathbf{A}\mathbf{Q} - \lambda\mathbf{I}|$, since $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$.

Let λ be an eigenvalue of $\underset{p\times p}{A}$ and Q be an orthogonal matrix. Using the fact that $\underset{p\times p}{Q^TQ} = \underset{p\times p}{I}$,

$$
\begin{aligned}
|Q^T||A-\lambda I||Q| &= |Q^TAQ - Q^T\lambda I Q| \\
&= |Q^TAQ - \lambda I| \\
&= |A - \lambda I| \\
&= 0
\end{aligned}
$$

$\therefore$ λ is also an eigen value of Q^TAQ.

2.19

2.19. Let $\underset{(m\times m)}{\mathbf{A}^{1/2}} = \sum_{i=1}^{m} \sqrt{\lambda_i}\,\mathbf{e}_i\mathbf{e}_i' = \mathbf{P}\Lambda^{1/2}\mathbf{P}'$, where $\mathbf{PP}' = \mathbf{P}'\mathbf{P} = \mathbf{I}$. (The λ_i's and the $\mathbf{e}_i$'s are the eigenvalues and associated normalized eigenvectors of the matrix $\mathbf{A}$.) Show Properties (1)–(4) of the square-root matrix in (2-22).

Let $A^{1/2} = \sum_{i=1}^{m} \sqrt{\lambda_i}\, e_i e_i^T = P\Lambda^{1/2}P^T$ where $PP^T = P^TP = I$.

λ_i's eigenvalues

e_i's eigenvectors.

Λ diagonal matrix

$$
\begin{aligned}
(1)\quad (A^{1/2})^T &= (P\Lambda^{1/2}P^T)^T \\
&= P^T(\Lambda^{1/2})^TP \\
&= P\Lambda^{1/2}P^T \\
&= A^{1/2}
\end{aligned}
$$

$\therefore (A^{1/2})^T = A^{1/2}$

(2) $A^{1/2}A^{1/2} = (P\Lambda^{1/2}P^T)(P\Lambda^{1/2}P^T)$

$= P\Lambda^{1/2} I \Lambda^{1/2} P^T \qquad \because PP^T = P^TP = I$

$= P\Lambda P^T \qquad = A$

$\therefore \underline{A^{1/2}A^{1/2} = A}$ //

(3) $(A^{1/2})^{-1} = (P\Lambda^{1/2}P^T)^{-1}$

$= (P^T)^{-1}\Lambda^{-1/2}P^{-1}$

In a context of the spectral decomposition of a symmetric matrix A, the eigenvectors can be chosen to to be orthonormal as long as a matrix is real and symmetric.

$\therefore$ P is an orthogonal matrix $\Leftrightarrow P^{-1} = P^T$

$\therefore (A^{1/2})^{-1} = (P^{-1})^{-1}\Lambda^{-1/2}P^T$

$= \underline{P\Lambda^{-1/2}P^T}$ //

Since, $\Lambda^{-1/2} = \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{1}{\sqrt{\lambda_n}} \end{bmatrix}$,

$(A^{1/2})^{-1} = \underline{\sum_{i=1}^{k} \frac{1}{\sqrt{\lambda_i}} e_i e_i^T}$ //

(4) $A^{1/2}A^{-1/2} = (P\Lambda^{1/2}P^T)(P\Lambda^{1/2}P^T)^{-1}$

$= P\Lambda^{1/2}P^T(P^T)^{-1}\Lambda^{-1/2}P^{-1}$

$= I$

$P^T\cdot(P^T)^{-1} = I$

$\Lambda^{1/2}\Lambda^{-1/2} = I$

$PP^{-1} = I$

$A^{-1/2}A^{1/2} = (P\Lambda^{1/2}P^T)^{-1}(P\Lambda^{1/2}P^T)$

$= (P^T)^{-1}\Lambda^{-1/2}P^{-1}P\Lambda^{1/2}P^T$

$= I$

$\therefore A^{1/2}A^{-1/2} = A^{-1/2}A^{1/2} = I$ //

$A^{-1/2}A^{-1/2} = (P\Lambda^{-1/2}P^T)(P\Lambda^{-1/2}P^T)$

$= P\Lambda^{-1/2}\Lambda^{-1/2}P^T$

$= P\Lambda^{-1}P^T = A^{-1}$ //

$\therefore A^{-1/2} = (A^{1/2})^{-1}$

2.28

2.28. Show that

$$\mathrm{Cov}(c_{11}X_1 + c_{12}X_2 + \cdots + c_{1p}X_p, c_{21}X_1 + c_{22}X_2 + \cdots + c_{2p}X_p) = \mathbf{c}_1'\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{c}_2$$

where $\mathbf{c}_1' = [c_{11}, c_{12}, \ldots, c_{1p}]$ and $\mathbf{c}_2' = [c_{21}, c_{22}, \ldots, c_{2p}]$. This verifies the off-diagonal elements $\mathbf{C}\boldsymbol{\Sigma}_{\mathbf{X}}\mathbf{C}'$ in (2–45) or diagonal elements if $\mathbf{c}_1 = \mathbf{c}_2$.

Hint: By (2–43), $Z_1 - E(Z_1) = c_{11}(X_1 - \mu_1) + \cdots + c_{1p}(X_p - \mu_p)$ and $Z_2 - E(Z_2) = c_{21}(X_1 - \mu_1) + \cdots + c_{2p}(X_p - \mu_p)$. So $\mathrm{Cov}(Z_1, Z_2) = E[(Z_1 - E(Z_1))(Z_2 - E(Z_2))] = E[(c_{11}(X_1 - \mu_1) + \cdots + c_{1p}(X_p - \mu_p))(c_{21}(X_1 - \mu_1) + c_{22}(X_2 - \mu_2) + \cdots + c_{2p}(X_p - \mu_p))]$.

The product

$$(c_{11}(X_1 - \mu_1) + c_{12}(X_2 - \mu_2) + \cdots + c_{1p}(X_p - \mu_p))(c_{21}(X_1 - \mu_1) + c_{22}(X_2 - \mu_2) + \cdots + c_{2p}(X_p - \mu_p))$$

$$= \left(\sum_{\ell=1}^{p} c_{1\ell}(X_\ell - \mu_\ell)\right)\left(\sum_{m=1}^{p} c_{2m}(X_m - \mu_m)\right)$$

$$= \sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}(X_\ell - \mu_\ell)(X_m - \mu_m)$$

has expected value

$$\sum_{\ell=1}^{p}\sum_{m=1}^{p} c_{1\ell}c_{2m}\sigma_{\ell m} = [c_{11}, \ldots, c_{1p}]\boldsymbol{\Sigma}[c_{21}, \ldots, c_{2p}]'.$$

Verify the last step by the definition of matrix multiplication. The same steps hold for all elements.

Let $Z_1 = C_{11}X_1 + C_{12}X_2 + \cdots + C_{1p}X_p$

$Z_2 = C_{21}X_1 + C_{22}X_2 + \cdots + C_{2p}X_p$

$$E(Z_1) = C_{11}E(X_1) + C_{12}E(X_2) + \cdots + C_{1p}E(X_p)$$

$$= C_{11}\mu_1 + C_{12}\mu_2 + \cdots + C_{1p}\mu_p$$

$$E(Z_2) = C_{21}\mu_1 + C_{22}\mu_2 + \cdots + C_{2p}\mu_p$$

$$Z_1 - E(Z_1) = C_{11}(X_1 - \mu_1) + C_{12}(X_2 - \mu_2) + \cdots + C_{1p}(X_p - \mu_p)$$

$$Z_2 - E(Z_2) = C_{21}(X_2 - \mu_2) + C_{22}(X_2 - \mu_2) + \cdots + C_{2p}(X_p - \mu_p)$$

$$\mathrm{Cov}(Z_1, Z_2) = E[(Z_1 - E(Z_1))(Z_2 - E(Z_2))]$$

$$= E[(C_{11}(X_1 - \mu_1) + \cdots + C_{1p}(X_p - \mu_p)) \times (C_{21}(X_2 - \mu_2) + \cdots + C_{2p}(X_p - \mu_p))]$$

$$= \left(\sum_{l=1}^{p} C_{1l}(X_l - \mu_l)\right)\left(\sum_{m=1}^{p} C_{2m}(X_m - \mu_m)\right)$$

$$= \sum_{l=1}^{p}\sum_{m=1}^{p} C_{1l}C_{2m}(X_l - \mu_l)(X_m - \mu_m)$$

$$= \sum_{l=1}^{p}\sum_{m=1}^{p} C_{1l}C_{2m}\sigma_{lm}$$

where σ_{lm} = covariance matrix

C_{1l}, C_{2m} are vectors.

Σ is variance - covariance matrix

If we let $\Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1p} \\ \vdots & \ddots & \vdots \\ \sigma_{pp} & \cdots & \sigma_{pp} \end{bmatrix}$,

$$\mathrm{Cov}(Z_1, Z_2) = [C_{11}\ C_{12} \cdots C_{1p}]\ \Sigma\ [C_{21} \cdots C_{2p}]^T$$

//

2-30

2.30. You are given the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = [4, 3, 2, 1]$ and variance–covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = [1 \quad 2] \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

and consider the linear combinations $\mathbf{AX}^{(1)}$ and $\mathbf{BX}^{(2)}$. Find

(a) $E(\mathbf{X}^{(1)})$
(b) $E(\mathbf{AX}^{(1)})$
(c) $\text{Cov}(\mathbf{X}^{(1)})$
(d) $\text{Cov}(\mathbf{AX}^{(1)})$
(e) $E(\mathbf{X}^{(2)})$
(f) $E(\mathbf{BX}^{(2)})$
(g) $\text{Cov}(\mathbf{X}^{(2)})$
(h) $\text{Cov}(\mathbf{BX}^{(2)})$
(i) $\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$
(j) $\text{Cov}(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)})$

a) $E(x^{(1)}) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ //

$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$

b) $E(Ax^{(1)}) = [1 \quad 2] \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 10$ //

c) $Cov(x^{(1)}) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ //

√ d) $Cov(Ax^{(1)}) = A\, Cov(x_1, x_2)\, A^T$

$= [1 \quad 2] \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 7$ //

e) $E(x^{(2)}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ //

(f) $E(Bx^{(2)}) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \end{bmatrix}$ //

(g) $Cov(x^{(2)}) = \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix}$ //

(h) $Cov(Bx^{(2)}) = B\,Cov(x_3, x_4)\,B^T$

$= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 33 & 36 \\ 36 & 48 \end{bmatrix}$ //

(i) $Cov(x^{(1)}, x^{(2)}) = \Sigma_{12} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix}$ //

(j) $Cov(Ax^{(1)}, Bx^{(2)}) = A\,\Sigma_{12}\,B^T$

$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 6 \end{bmatrix}$ //

2.41

2.41. You are given the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = [3, 2, -2, 0]$ and variance–covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

(a) Find $E(\mathbf{AX})$, the mean of $\mathbf{AX}$.
(b) Find Cov $(\mathbf{AX})$, the variances and covariances of $\mathbf{AX}$.
(c) Which pairs of linear combinations have zero covariances?

a) $E(Ax) = A\mu^T x$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix} \quad //$$

b) $Cov(Ax) = A\, Cov(x)\, A^T$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & & & 0 \\ & 3 & & \\ & & 3 & \\ 0 & & & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 36 \end{bmatrix} \quad //$$

c) Since every covariance $= 0$,
all pairs of linear combination have zero covariance.

//