STAT 524

HW7

Satoshi Ido

34788706

11/14/2023

10.2. The (2×1) random vectors $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ have the joint mean vector and joint covariance matrix

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}^{(1)} \\ \hdashline \boldsymbol{\mu}^{(2)} \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ \hdashline 0 \\ 1 \end{bmatrix};$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix} = \begin{bmatrix} 8 & 2 & 3 & 1 \\ 2 & 5 & -1 & 3 \\ 3 & -1 & 6 & -2 \\ 1 & 3 & -2 & 7 \end{bmatrix}$$

(a) Calculate the canonical correlations ρ_1^*, ρ_2^*.
(b) Determine the canonical variate pairs (U_1, V_1) and (U_2, V_2).
(c) Let $\mathbf{U} = [U_1, U_2]'$ and $\mathbf{V} = [V_1, V_2]'$. From first principles, evaluate

$$E\left(\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}\right) \quad \text{and} \quad \text{Cov}\left(\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}\right) = \begin{bmatrix} \boldsymbol{\Sigma}_{UU} & \boldsymbol{\Sigma}_{UV} \\ \boldsymbol{\Sigma}_{VU} & \boldsymbol{\Sigma}_{VV} \end{bmatrix}$$

Compare your results with the properties in Result 10.1.

(a) $\Sigma_{11}^{-1/2} = \begin{bmatrix} 0.367 & -0.0667 \\ -0.0667 & 0.467 \end{bmatrix}$

$\Sigma_{22}^{-1} = \begin{bmatrix} 0.184 & 0.0526 \\ 0.0526 & 0.158 \end{bmatrix}$

$\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$

$= \begin{bmatrix} 0.367 & -0.0667 \\ -0.0667 & 0.467 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0.184 & 0.0526 \\ 0.0526 & 0.158 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0.367 & -0.0667 \\ -0.0667 & 0.467 \end{bmatrix}$

$= \begin{bmatrix} 0.276 & -0.032 \\ -0.032 & 0.269 \end{bmatrix}$

$0 = \begin{vmatrix} 0.276-\lambda & -0.032 \\ -0.032 & 0.269-\lambda \end{vmatrix} \Rightarrow \lambda = \begin{bmatrix} 0.304 \\ 0.24031 \end{bmatrix}$

$\therefore \rho_1^{*2} = 0.304$
$\rho_2^{*2} = 0.24031$

Thus, $\rho_1^* = 0.551$
$\rho_2^* = 0.490$

(d) $\begin{bmatrix} 0.276 & -0.032 \\ -0.032 & 0.269 \end{bmatrix} e_1 = 0.304 e_1, \quad \therefore e_1 = \begin{bmatrix} 0.7526 \\ -0.6585 \end{bmatrix}$

$\begin{bmatrix} 0.276 & -0.032 \\ -0.032 & 0.269 \end{bmatrix} e_2 = 0.24031 e_2 \quad \therefore e_2 = \begin{bmatrix} 0.670 \\ 0.742 \end{bmatrix}$

$U_1 = e_1^T \Sigma_{11}^{-1/2} X^{(1)} = a_1^T X^{(1)} = \Sigma_{11}^{-1/2} e_1 X^{(1)}$

$V_1 = f_1^T \Sigma_{22}^{-1/2} X^{(2)} \propto \Sigma_{22}^{-1/2} \Sigma_{21} \Sigma_{11}^{-1/2} e_1 \Sigma_{22}^{-1/2} X^{(2)}$

$= b_1^T X^{(2)}$

$= \Sigma_{22}^{-1/2} \Sigma_{21} a_1 X^{(2)}$

$a_1 = \begin{bmatrix} 0.367 & -0.0667 \\ -0.0667 & 0.467 \end{bmatrix} \begin{bmatrix} 0.7526 \\ -0.6585 \end{bmatrix} = \begin{bmatrix} 0.317 \\ -0.362 \end{bmatrix}$

$b_1 = \begin{bmatrix} 0.184 & 0.0526 \\ 0.0526 & 0.158 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.317 \\ -0.362 \end{bmatrix} = \begin{bmatrix} 0.20 \\ -0.05 \end{bmatrix}$

To make $Var(V_1) = Var(b_1^T X^{(2)}) = 1$, we scale b_1

$Var(V_1) = b_1^T \Sigma_{22} b_1$

$= \begin{bmatrix} 0.20 & -0.05 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 0.20 \\ -0.05 \end{bmatrix} = 0.2925$

$\sqrt{Var(V_1)} = 0.545$

Using $\sqrt{Var(V_1)}$, we take $b_1^* = \frac{1}{0.5454} \begin{bmatrix} 0.20 \\ -0.05 \end{bmatrix} = \begin{bmatrix} 0.36 \\ -0.09 \end{bmatrix}$

$\therefore$ The first pair of canonical variate is

$U_1 = a_1^T X^{(1)} = 0.317 X_1^{(1)} - 0.362 X_2^{(1)}$

$V_1 = b_1^{*T} X^{(2)} = 0.36 X_1^{(2)} - 0.09 X_2^{(2)}$ //

$$a_2 = \Sigma_{22}^{-1/2} e_2^T$$

$$= \begin{bmatrix} 0.367 & -0.0667 \\ -0.0667 & 0.467 \end{bmatrix} \begin{bmatrix} 0.670 \\ 0.742 \end{bmatrix} = \begin{bmatrix} 0.196 \\ 0.30 \end{bmatrix}$$

$$b_2 = \Sigma_{22}^{-1/2} \Sigma_{21} a_2$$

$$= \begin{bmatrix} 0.184 & 0.0526 \\ 0.0526 & 0.158 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0.196 \\ 0.30 \end{bmatrix} = \begin{bmatrix} 0.11 \\ 0.19 \end{bmatrix}$$

We scale b_2 by

$$Var(V_2) = b_2^T \Sigma_{22} b_2$$

$$= \begin{bmatrix} 0.11 & 0.19 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 0.11 \\ 0.19 \end{bmatrix} = 0.2421$$

$$\sqrt{Var(V_2)} = 0.492$$

using $\sqrt{Var(V_2)}$, we take $b_2^* = \frac{1}{0.492} \begin{bmatrix} 0.11 \\ 0.19 \end{bmatrix} = \begin{bmatrix} 0.224 \\ 0.39 \end{bmatrix}$

$\therefore$ The second pair of canonical variate is

$$U_2 = a_2^T X^{(1)} = 0.196 X_1^{(1)} + 0.30 X_2^{(1)}$$

$$V_2 = b_2^{*T} X^{(2)} = 0.224 X_1^{(2)} + 0.39 X_2^{(2)} \quad //$$

(c)

$$E\begin{pmatrix} U \\ \hline V \end{pmatrix} = E\begin{pmatrix} 0.317 X_1^{(1)} - 0.362 X_2^{(1)} \\ 0.196 X_1^{(1)} + 0.30 X_2^{(1)} \\ \hdashline 0.36 X_1^{(2)} - 0.09 X_2^{(2)} \\ 0.224 X_1^{(2)} + 0.39 X_2^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} 0.317 E(X_1^{(1)}) - 0.362 E(X_2^{(1)}) \\ 0.196 E(X_1^{(1)}) + 0.30 E(X_2^{(1)}) \\ 0.36 E(X_1^{(2)}) - 0.09 E(X_2^{(2)}) \\ 0.224 E(X_1^{(2)}) + 0.39 E(X_2^{(2)}) \end{pmatrix}$$

$$= \begin{pmatrix} 0.317 \cdot (-3) - 0.362 \cdot 2 \\ 0.146 \cdot (-3) + 0.30 \cdot 2 \\ 0.36 \cdot 0 - 0.09 \cdot 1 \\ 0.224 \cdot 0 + 0.39 \cdot 1 \end{pmatrix}$$

$$= \left(\begin{array}{c} -1.675 \\ 0.012 \\ \hline -0.09 \\ 0.39 \end{array}\right) \quad //$$

$$Cov\left(\left[\begin{array}{c} U \\ \hline V \end{array}\right]\right) = \left(\begin{array}{c|c} \Sigma_{uu} & \Sigma_{vu} \\ \hline \Sigma_{uv} & \Sigma_{vv} \end{array}\right) = Corr\left(\left[\begin{array}{c} U \\ \hline V \end{array}\right]\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 0.551 & 0 \\ 0 & 1 & 0 & 0.490 \\ \hline 0.551 & 0 & 1 & 0 \\ 0 & 0.490 & 0 & 1 \end{array}\right) \quad //$$

10.5. Use the information in Example 10.1.

(a) Find the eigenvalues of $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and verify that these eigenvalues are the same as the eigenvalues of $\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$.

(b) Determine the second pair of canonical variates (U_2, V_2) and verify, from first principles, that their correlation is the second canonical correlation $\rho_2^* = .03$.

(a) $\Sigma_{11}^{-1} = \begin{bmatrix} 1.190 & -0.476 \\ -0.476 & 1.190 \end{bmatrix}$ $\quad \Sigma_{12} = \begin{bmatrix} 0.5 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}$

$\Sigma_{22}^{-1} = \begin{bmatrix} 1.041 & -0.208 \\ -0.208 & 1.041 \end{bmatrix}$ $\quad \Sigma_{21} = \begin{bmatrix} 0.5 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$

$$\therefore \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \begin{bmatrix} 1.190 & -0.476 \\ -0.476 & 1.190 \end{bmatrix}\begin{bmatrix} 0.5 & 0.6 \\ 0.3 & 0.4 \end{bmatrix}\begin{bmatrix} 1.041 & -0.208 \\ -0.208 & 1.041 \end{bmatrix}\begin{bmatrix} 0.5 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.452 & 0.289 \\ 0.146 & 0.095 \end{bmatrix}$$

$$|\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} - \lambda| = \begin{vmatrix} 0.452-\lambda & 0.289 \\ 0.146 & 0.095-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = \begin{bmatrix} 0.5458 \\ 0.0009 \end{bmatrix} \quad //$$

$\therefore$ These eigenvalues are the same as the eigenvalues of $\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$ from Example 10.1.

(b) From Example 10.1,

$$\begin{bmatrix} 0.4371 & 0.2178 \\ 0.2178 & 0.1096 \end{bmatrix} e_2 = 0.0009\, e_2$$

$$\Rightarrow e_2 = \begin{bmatrix} 0.4467 \\ -0.8949 \end{bmatrix}$$

$$a_2 = P_{11}^{-1/2} e_2 = \begin{bmatrix} 1.0681 & -0.2229 \\ -0.2229 & 1.0861 \end{bmatrix} \begin{bmatrix} 0.4467 \\ -0.8949 \end{bmatrix} = \begin{bmatrix} 0.6765 \\ -1.055 \end{bmatrix}$$

$$b_2 \propto P_{22}^{-1} P_{21} a_2 = \begin{bmatrix} 1.0417 & -2.083 \\ -0.2083 & 1.0417 \end{bmatrix} \begin{bmatrix} 0.5 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.6765 \\ -1.055 \end{bmatrix}$$

$$= \begin{bmatrix} 0.02599 \\ -0.02134 \end{bmatrix}$$

$$\mathrm{Var}(b_2) = \mathrm{Var}(b_2^T Z^{(2)}) = b_2^T P_{22} b = 0.000908956$$

$$\text{Scaled } b_2^* = \frac{1}{\sqrt{0.000908956}} \begin{bmatrix} 0.02599 \\ -0.02134 \end{bmatrix} = \begin{bmatrix} 0.8619670 \\ -0.7078674 \end{bmatrix}$$

$$\therefore U_2 = a_2^T Z^{(1)}$$

$$= 0.677 Z_1^{(1)} - 1.055 Z_2^{(1)} \quad //$$

$$V_2 = b_2^* Z^{(2)}$$

$$= 0.862 Z_1^{(2)} - 0.708 Z_2^{(2)} \quad //$$

$$\mathrm{Var}(U_2) = a_2^T P_{11} a_2 = 1$$

$$\mathrm{Var}(V_2) = b_2^T P_{22} b_2 = 0.000908956$$

$$\mathrm{Cov}(U_2, V_2) = a_2^T P_{12} b_2 = 0.000908956$$

$$\mathrm{Corr}(U_2, V_2) = \frac{0.000908956}{\sqrt{1 \times 0.000908956}} = 0.030 = \rho_2^* \quad //$$

10.13. Waugh [12] provides information about $n = 138$ samples of Canadian hard red spring wheat and the flour made from the samples. The $p = 5$ wheat measurements (in standardized form) were

$$z_1^{(1)} = \text{kernel texture}$$
$$z_2^{(1)} = \text{test weight}$$
$$z_3^{(1)} = \text{damaged kernels}$$
$$z_4^{(1)} = \text{foreign material}$$
$$z_5^{(1)} = \text{crude protein in the wheat}$$

The $q = 4$ (standardized) flour measurements were

$$z_1^{(2)} = \text{wheat per barrel of flour}$$
$$z_2^{(2)} = \text{ash in flour}$$
$$z_3^{(2)} = \text{crude protein in flour}$$
$$z_4^{(2)} = \text{gluten quality index}$$

The sample correlation matrix was

$$\mathbf{R} = \left[\begin{array}{c:c} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \hdashline \mathbf{R}_{21} & \mathbf{R}_{22} \end{array}\right]$$

$$= \left[\begin{array}{ccccc:cccc}
1.0 & & & & & & & & \\
.754 & 1.0 & & & & & & & \\
-.690 & -.712 & 1.0 & & & & & & \\
-.446 & -.515 & .323 & 1.0 & & & & & \\
.692 & .412 & -.444 & -.334 & 1.0 & & & & \\
\hdashline
-.605 & -.722 & .737 & .527 & -.383 & 1.0 & & & \\
-.479 & -.419 & .361 & .461 & -.505 & .251 & 1.0 & & \\
.780 & .542 & -.546 & -.393 & .737 & -.490 & -.434 & 1.0 & \\
-.152 & -.102 & .172 & -.019 & -.148 & .250 & -.079 & -.163 & 1.0
\end{array}\right]$$

(a) Find the sample canonical variates corresponding to significant (at the $\alpha = .01$ level) canonical correlations.

(b) Interpret the first sample canonical variates $\hat{U}_1, \hat{V}_1$. Do they in some sense represent the overall quality of the wheat and flour, respectively?

(c) What proportion of the total sample variance of the first set $\mathbf{Z}^{(1)}$ is explained by the canonical variate $\hat{U}_1$? What proportion of the total sample variance of the $\mathbf{Z}^{(2)}$ set is explained by the canonical variate $\hat{V}_1$? Discuss your answers.

(a) The eigen values of $R_{11}^{-1/2} R_{12} R_{22}^{-1} R_{21} R_{11}^{-1/2}$ become

$\rho_1^{*} = 0.909$ $\rho_2^{*} = 0.6363$ $\rho_3^{*} = 0.2559$ $\rho_4^{*} = 0.0939$

We only produce 4 ρ_i^{*}'s since one set only contains four variables.

R output shown as below

```
> eigen_sp2
eigen() decomposition
$values
[1]  8.266947e-01  4.049631e-01  6.547061e-02  8.813700e-03 -4.650272e-18

$vectors
           [,1]       [,2]        [,3]       [,4]       [,5]
[1,] -0.4764852 -0.4689787 -0.37108061  0.6292595  0.1391131
[2,] -0.4244472  0.3926279 -0.02453339  0.1345752 -0.8043479
[3,]  0.4834944 -0.4306431  0.48496702  0.4216956 -0.4095845
[4,]  0.3943720 -0.2957233 -0.73590105 -0.2726981 -0.3756379
[5,] -0.4511211 -0.5941352  0.29146857 -0.5776942 -0.1575077

> sqrt(eigen_sp2$values)
[1] 0.90922751 0.63636707 0.25587226 0.09388131        NaN
Warning message:
In sqrt(eigen_sp2$values) : NaNs produced
```

$$\begin{cases} H_0 : \Sigma_{12} = 0 \quad (\text{All } \rho_i^{*}\text{s} = 0,\ i = 1 \sim 4) \\ H_1 : \rho_i^{*} \neq 0, \text{ for some } i \text{ at least} \end{cases}$$

By using the large sample test,

$$-\left(n-1-\frac{1}{2}(p+q+1)\right) \ln \prod_{i=1}^{q} \left(1-\hat{\rho}_i^{*2}\right)$$

$$\Rightarrow -\left(138-1-\frac{1}{2}(5+4+1)\right) \ln \prod_{i=1}^{4} \left(1-\hat{\rho}_i^{*2}\right)$$

$$= -132 \cdot \ln \left(1-\hat{\rho}_1^{*2}\right)\left(1-\hat{\rho}_2^{*2}\right)\left(1-\hat{\rho}_3^{*2}\right)\left(1-\hat{\rho}_4^{*2}\right)$$

$$= 309.988$$

$$\chi^2_{5\cdot 4}(0.01) = 37.57 \qquad \therefore \text{ We reject } H_0$$

We can then test the 2nd hypothesis by the LRT.

$$H_0^2 : \rho_1^* \neq 0,\ \rho_2^* \neq 0,\ \rho_3^* = \rho_4^* = 0$$

$$H_1^2 : \rho_i^* \neq 0, \text{ for some } i \geq 3$$

$$\text{LRT} = -\left(138-1-\frac{1}{2}(5+4+1)\right) \ln \prod_{i=3}^{4} \left(1-\hat{\rho}_i^{*2}\right)$$

$$= -132 \cdot \ln\left(\left(1-0.256^2\right)\cdot\left(1-0.0939^2\right)\right)$$

$= 10.114$

$\chi_6 (0.01) = 16.811$ $\quad\ddot{\mathcal{Q}}\ \chi_{(p-k)(q-k)}(\alpha)$

$\therefore$ We do not reject H_0, meaning the first two variables are significant.

Hence, the sample canonical variates are below:

For the wheat group:

$$\hat{u}_1 = -0.215 Z_1^{(1)} - 0.172 Z_2^{(1)} + 0.33 Z_3^{(1)} + 0.264 Z_4^{(1)} - 0.298 Z_5^{(1)}$$

$$\hat{u}_2 = -0.923 Z_1^{(1)} + 0.585 Z_2^{(1)} - 0.653 Z_3^{(1)} - 0.341 Z_4^{(1)} - 0.55 Z_5^{(1)} \quad //$$

For the flour group (we scale $\hat{l}_i$ for this group)

$$\hat{V}_1 = 0.535 Z_1^{(2)} + 0.288 Z_2^{(2)} - 0.457 Z_3^{(2)} - 0.025 Z_4^{(2)}$$

$$\hat{V}_2 = -1.010 Z_1^{(2)} - 0.027 Z_2^{(2)} - 0.978 Z_3^{(2)} + 0.1796 Z_4^{(2)} \quad //$$

(b) $\hat{u}_1$ displays the contrast between the group of $Z_1^{(1)}, Z_2^{(1)}, Z_5^{(1)}$ and the group of $Z_3^{(1)}$ and $Z_4^{(1)}$. I believe this represents some traits among wheat. Meanwhile, $\hat{V}_1$ shows the contrast of $(Z_1^{(2)}, Z_2^{(2)})$ and $(Z_3^{(2)}, Z_4^{(2)})$, which also gives us some insight about flour.

(c) The proportions of total standardized sample variance of the first set $Z^{(1)}$ explained by the canonical variate $\hat{u}_1$ can be calculated as

$$R^2_{z^{(1)}|\hat{u}_1} = \frac{tr(\hat{a}_z^{(1)} \hat{a}_z^{(1)'})}{tr(R_{11})} = \frac{\sum_{k=1}^{5} r^2_{\hat{u}_1, z_k^{(1)}}}{5}$$

$$= \frac{1}{5}\left(sum(\hat{A} R_{11} D_{11}^{-1/2})\right) = \frac{1}{5} \cdot 3.14612 = \underline{0.6292}$$

The proportions of total standardized sample variance of the first set $z^{(2)}$ explained by the canonical variate $\hat{V}_1$ can be calculated as

$$R_{z^{(2)}|\hat{v}_1} = \frac{tr(\hat{l}_z^{(1)} \hat{l}_z^{(1)'})}{tr(R_{22})} = \underline{0.4496}$$

$\therefore$ We can see the canonical variate $\hat{u}_1$ has a larger proportion.

10.18. The data in Table 7.7 contain measurements on characteristics of pulp fibers and the paper made from them. To correspond with the notation in this chapter, let the paper characteristics be

$$\begin{aligned} x_1^{(1)} &= \text{breaking length} \\ x_2^{(1)} &= \text{elastic modulus} \\ x_3^{(1)} &= \text{stress at failure} \\ x_4^{(1)} &= \text{burst strength} \end{aligned}$$

and the pulp fiber characteristics be

$$\begin{aligned} x_1^{(2)} &= \text{arithmetic fiber length} \\ x_2^{(2)} &= \text{long fiber fraction} \\ x_3^{(2)} &= \text{fine fiber fraction} \\ x_4^{(2)} &= \text{zero span tensile} \end{aligned}$$

Determine the sample canonical variates and their correlations. Are the first canonical variates good summary measures of their respective sets of variables? Explain. Test for the significance of the canonical relations with $\alpha = .05$. Interpret the significant canonical variables.

By using R, the canonical correlations are :

$\rho_1^* = 0.9173$, $\rho_2^* = 0.817$, $\rho_3^* = 0.265$, $\rho_4^* = 0.092$

$$\begin{cases} H_0 : \Sigma_{12} = 0 \quad (\text{All } \rho_i^*\text{s} = 0,\ i = 1 \sim 4) \\ H_1 : \rho_i^* \neq 0, \text{ for some } i \text{ at least} \end{cases}$$

$$LRT_1 = -(n-1-\tfrac{1}{2}(p+q+1)) \ln \prod_{i=1}^{p} (1-\hat{\rho}_i^{*2})$$

$$\Rightarrow -(62-1-\tfrac{1}{2}\cdot 9) \ln (1-0.9173^2)(1-0.817^2)(1-0.265^2)(1-0.092^2)$$

$$= 170.86 \quad > \chi_{4\cdot4}(0.05) = 26.296$$

$\therefore$ We reject H_0

We can then test the 2nd hypothesis by the LRT.

$$H_0^2 : \rho_1^* \neq 0,\ \rho_2^* \neq 0,\ \rho_3^* = \rho_4^* = 0$$

$$H_1^2 : \rho_i^* \neq 0, \text{ for some } i \geq 3$$

$$LRT_2 = -(62-1-\tfrac{9}{2}) \ln (1-0.9173^2)(1-0.817^2)$$

$$= 0.460 \quad < \chi_{2\cdot2}(0.05) = 13.277$$

$\therefore$ We don't reject H_0, meaning ρ_1^* and ρ_2^* are significant

Hence, the sample canonical variates are below:

For paper,

$$\hat{U}_1 = 1.505\, x_1^{(1)} + 0.212\, x_2^{(1)} - 1.998\, x_3^{(1)} - 0.676\, x_4^{(1)}$$

$$\hat{U}_2 = -3.495\, x_1^{(1)} - 1.54\, x_2^{(1)} + 1.076\, x_3^{(1)} + 3.768\, x_4^{(1)}$$

For pulp,

$$\hat{V}_1 = 0.154\, x_1^{(2)} - 0.632\, x_2^{(2)} - 0.325\, x_3^{(2)} - 0.818\, x_4^{(2)}$$

$$\hat{V}_2 = 0.689\, x_1^{(2)} + 1.003\, x_2^{(2)} + 0.005\, x_3^{(2)} - 1.56\, x_4^{(2)}$$

$$R^2_{x^{(1)}|\hat{U}_1} = \frac{1}{4} \sum_{k=1}^{4} r^2_{\hat{U}_1, x_k^{(1)}} = 0.880$$

$$R^2_{x^{(2)}|\hat{V}_1} = \frac{1}{4} \sum_{k=1}^{4} r^2_{\hat{V}_1, x_k^{(2)}} = 0.698$$

Given the results above, the first canonical variables $\hat{U}_1$ explain 58.02 % of the original total standardized variance of $X^{(1)}$.
Meanwhile, the first canonical variables $\hat{V}_1$ explain 69.8 % of the original total standardized variance of $X^{(2)}$.
The second canonical variable $\hat{U}_2$ displays the contrast between the group of $X_1^{(1)}$ and $X_2^{(1)}$ and the group of $X_3^{(1)}$ and $X_4^{(1)}$, which can be useful. $\hat{V}_2$ is little determined by $X_3^{(2)}$ but has stronger correlations with other variables.