# Homework

Ding Yaoyao, 516030910572

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## Exercise 2-1

### 1

The 3 left cosets of $H$ in $A_4$ are:

$$(1), (12)(34), (13)(24), (14)(23)$$
$$(123), (134), (243), (142)$$
$$(234), (132), (143), (124)$$

The 6 left cosets of $H$ in $S_4$ are:

$$(1), (12)(34), (13)(24), (14)(23)$$
$$(123), (134), (243), (142)$$
$$(234), (132), (143), (124)$$
$$(12), (34), (1324), (1423)$$
$$(13), (24), (1234), (1432)$$
$$(14), (23), (1243), (1342)$$

### 8

There are 2 left cosets of $\langle a^4 \rangle$ and they are:

$$\langle a^2 \rangle$$

and

$$\{a, a^3, a^5, \cdots, a^{29}\}$$

### 11

Let $G$ be a group and $H \preceq G$. Then $aH$ is a left coset of $H$ generated by $a \in G$. We have:

$$(aH)^{-1} = H^{-1}a^{-1} = Ha^{-1}$$

So $(aH)^{-1}$ is a right coset of $H$ generated by $a^{-1}$.

### 12

$$
\begin{aligned}
x \in a(H_1 \cap H_2) &\Leftrightarrow x = ah(h \in H_1 \text{ and } h \in H_2) \\
&\Leftrightarrow x \in aH_1 \text{ and } x \in aH_2 \Leftrightarrow x \in aH_1 \cap aH_2
\end{aligned}
$$

So $a(H_1 \cap H_2) = aH_1 \cap aH_2$.

### 20

The factors of 33 are $1, 3, 11, 33$, so the order of every element in $G$ must be one of them. For any $g \in G$, if $ord(g) = 3$ or $ord(g) = 33$ then we find the element we want(it is $g$ or $g^{11}$). So we only need to prove that such group $G$ does not exist:

$$ord(g) = 11 \quad \text{for any non-unit } g \in G$$

Let $S_1, S_1, \ldots, S_k$ be all the different cyclic subgroup of $G$. Beacuse every element in $G$ can generate a cyclic subgroup, we have $G = \bigcup S_i$. For all $i \neq j$, $S_i \cap S_j = \{e\}$(If $g \in S_i \cap S_j$ and $g \neq e$, we have $S_i = S_j = \langle g \rangle$).There is only one cyclic subgroup contains 1 element which is $\{e\}$ and other cyclic subgroup has 11 elements in it.So we have :

$$|G| = 1 + 10(k-1) = 33 \quad (k \in \mathbb{Z})$$

Such $k$ does not exist, so such group $G$ does not exist. Above all, every group $G$ follow $|G| = 33$ has an element whose order is 3.

### 22

Let $\phi : G \to G(a \to a^n)$ be the map($G$ is abelian and finite).

- injective: $\phi(a) = \phi(b) \Leftrightarrow a^n = b^n \Leftrightarrow (ab^{-1})^n = e \Leftrightarrow ord(ab^{-1}) \mid n$. We also have $ord(ab^{-1}) \mid |G|$, so $ord(ab^{-1}) \mid gcd(n, |G|) = 1$, then $ord(ab^{-1}) = 1$, which means $a = b$.
- surjective: Beacuse $G$ is finite and $\phi$ is injective, $|G| = |\phi(G)|$. We also have $\phi(G) \subseteq G$. Then we have $\phi(G) = G$.
- homomorphic: $\phi(ab) = (ab)^n = a^n b^n = \phi(a)\phi(b)$

Above all, $\phi$ is an automorphism of $G$.

## Exercise 2-2

### 2

$C(G)$ is not empty beacuse $e \in C(G)$. For all $a, b \in C(G)$ and $x \in G$, we have:

$$ab^{-1}x = a(x^{-1}b)^{-1} = a(bx^{-1})^{-1} = axb^{-1} = xab^{-1}$$

So $ab^{-1} \in C(G)$ and $C(G) \preceq G$.

For all $g \in G$, $gC(G) = C(G)g$(beacuse $gx = xg$ holds for all $x \in C(G)$), so $C(G)$ is a normal subgroup of $G$.

## 5

Let $G = S_4, K = \{(1), (12)(34), (13)(24), (14)(23)\}, H = \{(1), (12)(34)\}$.

## 6

- $\Rightarrow$: If $ab \in H$ and $H \trianglelefteq G$, we assume $ab = h \in H$ and then

$$ba = babb^{-1} = bhb^{-1} \in H$$

- $\Leftarrow$: For every $h \in H$ and $g \in G$, we have $(hg^{-1})g \in H$, then $g(hg^{-1}) = ghg^{-1} \in H$. So $H \trianglelefteq G$.

## 9

For all $a, b \in N(H)$, we have $aHa^{-1} = H$ and $bHb^{-1} = H$, then

$$\begin{aligned}(ab^{-1})H(ab^{-1})^{-1} &= ab^{-1}Hba^{-1} = a(bH^{-1}b^{-1})^{-1}a^{-1} \\ &= a(bHb^{-1})^{-1}a^{-1} = aH^{-1}a^{-1} = aHa^{-1} = H\end{aligned}$$

So $ab^{-1} \in N(H)$, which means $N(H) \preceq G$.

For all $n \in N(H)$, we have

$$nHn^{-1} = H$$

which means $H \trianglelefteq N(H)$.

## 10

- $\Rightarrow$: Let $\phi(x) = gxg^{-1}$ be an inner automorphism of $G$. For all $h \in H$, we have $\phi(h) = ghg^{-1} \in H$(beacuse $H$ is a normal subgroup of $G$), which means $\phi(H) \subseteq H$.

- $\Leftarrow$: For every $g \in G$, it can generate an inner automorphism $\phi(x) = gxg^{-1}$. Beacuse $\phi(H) \subseteq H$, we have $gHg^{-1} \subseteq H$. Beacuse for all $g \in G$ we have $gHg^{-1} \subseteq H$, then $H$ is a normal subgroup of $G$.

## 11

For every $x \in G$, we can get a left coset of $H$ generated by $x : xH$. Beacuse

$$|G/H| = [G : H] = m$$

we have

$$(xH)^m = x^m H^m = x^m H = eH$$

which means $x^m \in H$.

# Exercise 2-3

## 1

$x \to |x|$ and $x \to x^2$ are two homomorphisms. When $a = 1$, $x \to ax$ is also a homomorphism.

- $\phi : x \to |x|$: $\phi(G) = R^+$ and $Ker(\phi) = \{1, -1\}$
- $\phi : x \to x$: $\phi(G) = G$ and $Ker(\phi) = \{1\}$
- $\phi : x \to x^2$: $\phi(G) = R^+$ and $Ker(\phi) = \{1, -1\}$

## 6

Let's call that map $\phi$. For every $x, y \in \mathbb{C}^*$, we have

$$\phi(xy) = (xy)^6 = x^6 y^6 = \phi(x)\phi(y)$$

so $\phi$ is a homomorphism.

Solve the following equation

$$\phi(x) = x^6 = 1$$

we can get 6 solutions:$\{1, w, w^2, w^3, w^4, w^5\}(w = e^{\frac{i\pi}{3}})$.And the kernal of $\phi$ is

$$Ker(\phi) = \{1, w, w^2, w^3, w^4, w^5\}(w = e^{\frac{i\pi}{3}})$$

## 7

For every $q \in \{0, 1, 2, \cdots, m-1\}$, we can define a homomorphism:

$$\phi_q(n) = qn \bmod m$$

## 16

- $\Rightarrow$:

  $$\begin{aligned}\phi(a) = \phi(b) \Rightarrow \phi(ab^{-1}) = e' \Rightarrow ab^{-1} \in Ker(\phi) \\ \Rightarrow Ker(\phi)a = Ker(\phi)b \Rightarrow aKer(\phi) = bKer(\phi)\end{aligned}$$

  (The last steps use the conclusion that $Ker(\phi)$ is a normal subgroup of $G$).

- $\Leftarrow$:

  $$\begin{aligned}aKer(\phi) = bKer(\phi) \Rightarrow \phi(aKer(\phi)) = \phi(bKer(\phi)) \\ \Rightarrow \phi(a)e' = \phi(b)e' \Rightarrow \phi(a) = \phi(b)\end{aligned}$$

## 18

- $\phi^{-1}(\phi(H)) \subseteq HK$: For any $x \in \phi^{-1}(\phi(H))$, there exists $h \in H$ such $\phi(x) = \phi(h)$, then

$$\phi(xh^{-1}) = e' \Rightarrow xh^{-1} \in K \Rightarrow xh^{-1}h \in Kh \Rightarrow x \in Kh \subseteq KH \Rightarrow x \in HK$$

- $HK \subseteq \phi^{-1}(\phi(H))$: For any $x \in HK$, there exist $h \in H$ and $k \in K$ such $x = hk$, then

$$\phi(x) = \phi(hk) = \phi(h)$$

which means $x \in \phi^{-1}(\phi(h))$, thus $x \in \phi^{-1}(\phi(H))$.

## 19

- $\Rightarrow$: Let $\phi : G_1 \to G_2$ be the epimorphism. There must exist $a \in G_1$ such that $\phi(a) = 1'$. We have $n_2 = ord(1') \mid ord(a)$ and $ord(a) \mid n_1$, thus $n_2 \mid n_1$.
- $\Leftarrow$: We can construct an epimorphism $\phi(x) = x \bmod n_2$.
  - surjective: For any $x \in G'$, $x$ must also belong $G$(Beacuse $n_2 \leq n_1$).
  - homomorphic:

$$\begin{aligned}\phi(a+b) &= ((a+b) \bmod n_1) \bmod n_2 = (a+b) \bmod n_2 \\ &= ((a \bmod n_2) + (b \bmod n_2)) \bmod n_2 = \phi(a) + \phi(b)\end{aligned}$$

    (Beacuse $n_2 \mid n_1$)

  So $G_1 \sim G_2$.

## 12

We know that $Z_5 \cong U_5$ by the isomorphism $f(x) = e^{\frac{2x\pi}{5}}$, so we only need to consider $\phi : Z_{30} \sim Z_5$.

Let $\phi(1) = a$, we have $ord(a) \mid 5$ and $ord(a) \mid ord(1) = 30$. So $ord(a) = 1$ or $ord(a) = 5$. Beacuse $\phi$ is surjective, so $ord(a) = 5$, thus $a$ can be one of $\{1, 2, 3, 4\}$ and $a$ satisfy $gcd(a, 5) = 1$. So $\phi(x) = xa \equiv 0 \Leftrightarrow x \equiv 0 \pmod 5$.

So the kernel of $\phi$ is:

$$Ker(\phi) = \{0, 5, 10, 15, 20, 25\}$$

## 20

We define $\phi : Z_m \to Z_k(x \to x \bmod k)$. Beacuse $m \geq k$, we can see that $\phi$ is surjective.

$$\phi(a+b) = ((a+b) \bmod m) \bmod k = (a \bmod k) + (b \bmod k) \bmod k = \phi(a) + \phi(b)$$

Thus $\phi$ is homomorphic. So $\phi$ is an epimorphism.

$$\phi(x) = 0 \Leftrightarrow x \bmod k = 0 \Leftrightarrow k \mid x$$

So we can get the kernel of $\phi$:

$$Ker(\phi) = \{x \in Z_m \mid k \mid x\} = \{0, k, 2k, \ldots, m-k\} = \langle k \rangle$$

By the First Isomorphism Theorem, we have

$$Z_m/\langle k \rangle \cong Z_k$$

# Exercise 2-5

## 1

$X = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$G = \{(1), (123)(456), (132)(465), (78), (123)(456)(78), (132)(465)(78)\}$

| x | Orbit | Stabilizer |
|---|---|---|
| 1 | $\{1, 2, 3\}$ | $\{(1), (78)\}$ |
| 2 | $\{1, 2, 3\}$ | $\{(1), (78)\}$ |
| 3 | $\{1, 2, 3\}$ | $\{(1), (78)\}$ |
| 4 | $\{4, 5, 6\}$ | $\{(1), (78)\}$ |
| 5 | $\{4, 5, 6\}$ | $\{(1), (78)\}$ |
| 6 | $\{4, 5, 6\}$ | $\{(1), (78)\}$ |
| 7 | $\{7, 8\}$ | $\{(1), (123)(456), (132)(465)\}$ |
| 8 | $\{7, 8\}$ | $\{(1), (123)(456), (132)(465)\}$ |

| g | Fixed elements |
|---|---|
| (1) | $\{1, 2, 3, 4, 5, 6, 7, 8\}$ |
| (123)(456) | $\{7, 8\}$ |
| (132)(465) | $\{7, 8\}$ |
| (78) | $\{1, 2, 3, 4, 5, 6\}$ |
| (123)(456)(78) | $\varnothing$ |
| (132)(465)(78) | $\varnothing$ |

## 3

Beacuse the action of $G$ on $X$ is transitive, for any $x_1, x_2 \in X$, exist $g \in G$ such that

$$gx_1 = x_2$$

Then for any $x_1, x_2 \in X$, let's consider their orbits

$$gx_1 = x_2 \Rightarrow Ngx_1 = Nx_2 \Rightarrow gNx_1 = Nx_2$$

Beacuse the action $g$ on $X$ is injective, so we have

$$|Nx_1| = |g(Nx_1)| = |Nx_2|$$

So every orbit has the same size.

### 4

- $gS_xg^{-1} \subseteq S_y$: For any $h \in S_x$, we have $hx = x$. Then
$$ghg^{-1}y = ghx = gx = y$$
so $ghg^{-1} \in S_y$.
- $S_y \subseteq gS_xg^{-1}$: For any $h \in S_y$, we have $hy = y$. Then
$$g^{-1}hgx = g^{-1}hy = g^{-1}y = x$$
so $g^{-1}hg \in S_x$. Thus $g^{-1}S_yg \subseteq S_x$, which means $S_y \subseteq gS_xg^{-1}$.

So $S_y = gS_xg^{-1}$.

### 6

Let the 12 faces be $X = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8, \}$ and $\pi_1$ be the front face. We have
$$O_{\pi_1} = X$$
$$|S_{\pi_1}| = 5$$
So $|G| = |O_{\pi_1}||S_{\pi_1}| = 40$.

### Additional

It's the same as Exercisse 2-5(4).

## Exercise 2-6

### 1

- $\Rightarrow$: Beacuse there is only one Sylow p-subgroup $P$, thus for all $g \in G$ we have
$$gPg^{-1} = P,$$
which means $P$ is a normal subgroup of $G$.
- $\Leftarrow$: By the Sylow Theorem, when $P$ is a Sylow p-subgroup, all the Sylow p-subgroup will be
$$\{gPg^{-1} \mid g \in G\}.$$
Beacuse $P$ is a normal subgroup, $gPg^{-1} = P$ for all $g \in G$, which means there is only one Sylow p-subgroup $P$.

## 2

- $N(P) \subseteq N(N(P))$: For every $g \in N(P)$,

$$gN(P)g^{-1} = N(P) \Rightarrow g \in N(N(P))$$

.

- $N(N(P)) \subseteq N(P)$: For every $g \in N(N(P))$, we have

$$gPg^{-1} \subseteq gN(P)g^{-1} = N(P).$$

So $gPg^{-1}$ is another Sylow p-subgroup of $N(P)$, thus

$$gPg^{-1} = P \Rightarrow g \in N(P)$$

Then we have $N(N(P)) \subseteq N(P)$.

## 3

The Sylow 2-subgroup of $S_4$:

$$\begin{aligned}
&\{(1),(1234),(13)(24),(1432),(13),(12)(34),(24),(14)(23)\},\\
&\{(1),(1324),(12)(34),(1423),(12),(13)(24),(34),(14)(32)\},\\
&\{(1),(1243),(14)(23),(1342),(14),(12)(43),(23),(13)(24)\},
\end{aligned}$$

## 4

The Sylow 2-subgroup of $A_4$:

$$\{(1),(14)(23),(13)(24),(14)(23)\},$$

## 6

By the Sylow Theorem

$$n_5 \mid 24 \quad \text{and} \quad n_5 \equiv 1 \pmod 5,$$

there are two posible values $n_5 = 1$ or $n_5 = 6$. But there are more than one Sylow 5-subgroup in $S_5$, so $n_5 = 6$.

$$\begin{aligned}
&\{(1),(12345),(13524),(14253),(15432)\}\\
&\{(1),(13452),(14235),(15324),(12543)\}
\end{aligned}$$