Homework

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Exercise 2-4

1

If $(a, b) \in Z_9 \oplus Z_6$ and $|(a, b)| = 9$, we have

$$lcm(|a|, |b|) = 9.$$

Also $|a|$ may be one of $\{1, 3, 9\}$ and $|b|$ may be one of $\{1, 2, 3, 6\}$. Thus $|a| = 9$ and $|b| = 1$ or 3. Thus there are totally $6 \times 3 = 18$ elements whose order is 9.

3

$Z \oplus Z$ is not a cyclic group.

Proof by contradiction:

If $Z \oplus Z$ is a cyclic group, we let (a, b) be a generator. Then we can get $|a| = 1$ and $|b| = 1$(otherwise $(1, 1)$ can not be generated by (a, b),here $|a|$ means the absolute value of a).

But it's obvious that any one of $(1, 1), (1, -1), (-1, 1), (-1, -1)$ can not generate $Z \oplus Z$(for example, none can generate $(1, 0)$).

So $Z \oplus Z$ is not a cyclic group.

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Let $a = (1, 0) \in Z_8 \oplus Z_2$, we have $|a| = 8$.

If $(b, c) \in Z_4 \oplus Z_4$, we have $|(b, c)| = lcm(|b|, |c|) = max(|b|, |c|) \leq 4$.

So $Z_8 \oplus Z_2$ is not isomorpic to $Z_4 \oplus Z_4$.

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Let $S = \{1, -1\}$.

- $S \trianglelefteq R^*$ and $R^+ \trianglelefteq R^*$: Beacuse the group R^* is an abelian group, all of its subgroup are normal.

- $R^* = R^+S$: By $R^+ \preceq R^*$ and $S \preceq R^*$, we can get $R^+S \subseteq R^*$.

  For any $x \in R^*$, we have
  $$x = |x|\phi(x)$$
  $\phi(x) = 1$ if $x > 0$ and $\phi(x) = -1$ if $x < 0$. Beacuse $\phi(x) \in \{-1, 1\}$ and $|x| \in Z^+$, we have $x \in R^+S$. Then we can get $R^* \subseteq R^+S$.

  Thus $R^* = R^+S$.

- $R^+ \cap S = \{1\}$: $-1 \notin R^+$ and $1 \in R^+$, so $R^+ \cap S = \{1\}$.

Thus R^* is the internal direct product of R^+ and $\{1, -1\}$.

8

Beacuse $|(1,0) + H| = 2$, $|(0,1) + H| = 2$ and $(1,0) + H \neq (0,1) + H$, we know that $G/H \cong Z_2 \oplus Z_2$.

Beacuse $|(1,1) + K| = 4$, we know that $G/K \cong Z_4$.

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Let $a = ord(a_1, a_2, \ldots, a_n)$ and $b = lcm(ord(a_1), ord(a_2), \cdots, ord(a_n))$.

- $a \mid b$:
  $$(a_1, a_2, \ldots, a_n)^b = (a_1^b, a_2^b, \ldots, a_n^b) = (1, 1, \cdots, 1)$$
  Then $a \mid b$.

- $b \mid a$:
  $$(a_1, a_2, \ldots, a_n)^a = (a_1^a, a_2^a, \ldots, a_n^a) = (1, 1, \cdots, 1)$$
  Then
  $$a_i^a = 1 \Rightarrow ord(a_i) \mid a (\text{for all} i = 1, 2, \ldots, n)$$
  So
  $$b = lcm(ord(a_1), ord(a_2), \ldots, ord(a_n)) \mid a$$
  Thus $b \mid a$.

So $ord(a_1, a_2, \ldots, a_n) = lcm(ord(a_1), ord(a_2), \ldots, ord(a_n))$.

Addition

Problem:Show that Z_{145} is a cyclic group.

By Sylow Theorem, there are Sylow 29-subgroup and Sylow 5-subgroup. Beacuse
$$n_{29} \equiv 1 \pmod{29} \text{ and } n_{29} \mid 5$$
We have $n_{29} = 1$. Similarly, we can get $n_5 = 1$. So there are only one Sylow 29-subgroup and one Sylow 5-subgroup, we call them H and K, respectively. Then H and K are both normal subgroups.

If $a \in H \cap K$, $|a| \mid 29$ and $|a| \mid 5$, then $|a| = 1$, thus $H \cap K = \{1\}$.

$$|HK| = \frac{|H||K|}{|H \cap K|} = 145.$$

So $HK = Z_{145}$.

So $G = HK$ is an internal direct product of H and K, Thus $G \cong H \times K$.

The order of $(1, 1) \in H \times K$ is $lcm(29, 5) = 145$, so $H \times K$ is a cylic group, thus G is a cylic group.