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Exercise 1-1

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- Reflexive: $\phi(a) = \phi(a) \Rightarrow a \sim a$.
- Symmetric: $a \sim b \Rightarrow \phi(a) = \phi(b) \Rightarrow \phi(b) = \phi(a) \Rightarrow b \sim a$.
- Transitive: If $a \sim b$ and $b \sim c$, then $\phi(a) = \phi(b) = \phi(c)$, which means $a \sim c$.

So relation $\sim$is an equivalence relation.

the partition that a belongs to is $[a] = \{b \mid \phi(b) = \phi(a)\}$.

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- Reflexive: $ab = ba \Rightarrow (a, b) \sim (a, b)$.
- Symmetric: $(a, b) \sim (c, d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c, d) \sim (a, b)$.
- Transitive: $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f) \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \Rightarrow (a, b) \sim (e, f)$.

So relation $\sim$ on S is an equivalence relation.

Exercise 1-2

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- Closure: If $a, b \in Z$, then $a \oplus b = a + b - 2 \in Z$.
- Associativity:
  $(a \oplus b) \oplus c = (a + b - 2) \oplus c = (a + b - 2) + c - 2 = a + (b + c - 2) - 2 = a \oplus (b + c - 2) = a \oplus (b \oplus c)$
  .
- Identity: There is an element $e = 2$ in Z such that $e \oplus a = a \oplus e = a$ for all a in Z.
- Inverse: If a is in Z, there is an element $a^{-1} = 4 - a$ such that $a \oplus a^{-1} = a^{-1} \oplus a = 2 = e$.

Above all, $(Z, \oplus)$ is a group.

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$x^2 = e \Rightarrow x = x^{-1}$ (for all x in G).

Because $ab = a^{-1}b^{-1} = (ba)^{-1} = ba$, then G is an abelian group.

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- $\Rightarrow$: $ab = ba \Rightarrow (ab)^2 = abab = aabb = a^2b^2$.
- $\Leftarrow$: $(ab)^2 = a^2b^2 \Rightarrow abab = aabb \Rightarrow a^{-1}ababb^{-1} = a^{-1}aabbb^{-1} \Rightarrow ba = ab$.

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Let $S = \{a \in G \mid a^3 = e\}$.

We have two properties:

1. $a \in S \Rightarrow a^{-1} \in S$

   Proof: $a \in S \Rightarrow a^3 = e \Rightarrow (a^{-1})^3 = (a^3)^{-1} = e^{-1} = e \Rightarrow a^{-1} \in S$.
2. $a \in S$ and $a = a^{-1} \Rightarrow a = e$

   Proof: $a \in S \Rightarrow a^3 = e \Rightarrow a^2 = a^{-1}$, we also have $a = a^{-1}$, so $a^2 = a$, which means $a = e$.

From property 1, S consists of many pairs like (a, a^{-1}).

From property 2, there is the only one pair (e, e^{-1}) such that $a = a^{-1}$.

So the number of elements of S is odd.