# Homework

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## Exercise 4-1

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Beacuse $f$ and $g$ are nonzero polynomials, we have

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$
$$g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m,$$

where $a_n \neq 0$ and $b_m \neq 0$. So $deg(f) = n$ and $deg(g) = m$.

The coefficient of $x^r$ item of $f(x)g(x)$ is

$$c_r = \sum_{i=0}^{r} a_ib_{r-i}.$$

When $r = n + m$, $c_r$ is $a_nb_m$, which is not 0 beacuse $R$ is an integral domain and $a_n \neq 0, b_m \neq 0$. So $deg(fg) \geq deg(f) + deg(g)$.

When $r > n + m$, $c_r$ is 0 beacuse one of $a_i$ and $b_{r-i}$ is 0. So $deg(fg) \leq deg(f) + deg(g)$.

Then when $f(x) \neq 0$ and $g(x) \neq 0$, we have $deg(fg) = deg(f) + deg(g)$.

When $R$ is not an integral domain, the conclusion doesn't hold. For example, when $R$ is $Z_4$, $f(x) = g(x) = 2$, $deg(f) = deg(g) = 0$ and $deg(fg) = -\inf$.

## Exercise 4-3

### 1

#### (4)

Beacuse $x^3 + 1 = (x + 1)(x^2 - x + 1) = (x + 1)(x^2 + x + 1)$, thus $\beta \mid \alpha$.

#### (6)

If $\alpha = \beta\gamma$, we replace $x$ with $2 \in Z_5[x]$, we can get $\alpha(x = 2) = 0$. But we replace $x$ directly in $\alpha$ we can get $\alpha(x = 2) = 2 \neq 0$. So $\beta \nmid \alpha$.

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Let $R = \mathbb{Z}[\sqrt{-3}]$ and $a = 4, b = 2(1+\sqrt{-3})$. We can get that all the divisors of $a$ are $\{1, 2, 1+\sqrt{-3}, 1-\sqrt{-3}, -1, -2, -1-\sqrt{-3}, -1+\sqrt{-3}\}$ and all the divisors of $b$ are $\{1, 2, 1+\sqrt{-3}, -1, -2, -1-\sqrt{-3}\}$. Both 2 and $1+\sqrt{-3}$ are maximal common divisors, but they don't correlate with each other. So $a$ and $b$ have no greatest commom divisor. (The method to get the divisor of $a$ or $b$ is to reduce $\frac{a}{r+s\sqrt{-3}}$ in $\mathbb{C}$ field where $r, s \in \mathbb{Z}$, then we can get the restrictions on $r$ and $s$ and know all the divisors).

## Exercise 4-4

### 1

#### (1)

- $\Rightarrow$: If $d$ is a unit, we have $dd^{-1} = 1$, then
$$\sigma(1) = \sigma(dd^{-1}) \geq \sigma(d) \geq \sigma(d \cdot 1) \geq \sigma(1),$$
so $\sigma(1) = \sigma(d)$.
- $\Leftarrow$: If $\sigma(1) = \sigma(d)$, beacuse $D$ is an Euclidean Domain, we have
$$1 = dc + r,$$
where $\sigma(r) < \sigma(d)$ or $r = 0$. Beacuse $\sigma(r) \geq \sigma(1) = \sigma(d)$, $r$ must be 0. Then we have $dc = 1$, which means $d$ is a unit.

#### (2)

For all $d \in D$, we have $\sigma(d) = n = \sigma(1)$, by (1), then $d$ is a unit. Then every element of $D$ has a multiplicative inverse, thus Euclidean Domain $D$ is a field.

#### (3)

If $a = 0$, then $a = b = 0$.

If $a \neq 0$, we have $b = ac$ where $c$ is a unit, then $b \neq 0$. Then we have $\sigma(b) = \sigma(ac) \geq \sigma(a)$. Similarly, $\sigma(a) \geq \sigma(b)$, so $\sigma(a) = \sigma(b)$.

In conclusion, $a \sim b \Rightarrow a = b = 0$ or $\sigma(a) = \sigma(b)$.

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#### (1)

By the Correspondence Theorem For Rings, any ideal of $D/I$ has the form $J/I$, where $J$ is an ideal of $D$ containing $I$. Beacuse $D$ is a PID, $J = (a)$ where

$a \in D$. Then $J/I = (a)/I = (a + I)$, which means all the ideals of $D/I$ are principal.

$D/I$ is an integral domain iff $I$ is a prime ideal of $D$, but it's not necessary. For example, let $D$ be $\mathbb{Z}$ and $I$ be (6), then $D/I$ is not an integral beacuse $(2 + I)(3 + I) = I$ where $(2 + I) \neq I$ and $3 + I \neq I$.

**(2)**

Similar with (1), any ideal of $D/I$ has the form that $J/I$, where $J$ is an ideal of $D$ containing $I$. Beacuse $D$ is a PID, we can assume $I = (a)$ and $J = (b)$. Beacuse we have $I \subseteq J$, then $b \mid a$. Beacuse $D$ is also a UFD, then $a$ has finite divisor that don't correlate with each other. Then there are finite number of $b$ that don't correlate with each other, which means there are finite ideals of $D/I$.