Suggested due date: 2018/03/21

Combinatorics: Homework 3

1 Combinatorial identities

Problem 1. Prove that for any $n \in \mathbf{N}$,

$$\sum_{k\geq 0} \binom{n}{2k}\binom{2k}{k} 2^{n-2k} = \binom{2n}{n}.$$

There exists a combinatorial proof for this, but I don't suggest you try to find it.

Solution. Firstly, we can observe the left to get that:

$$\binom{n}{2k}\binom{2k}{k} = \frac{n!}{(n-2k)!(2k)!}\frac{(2k)!}{k!k!} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{k!(n-2k)!} = \binom{n}{k}\binom{n-k}{n-2k}$$

Then we only need to prove the following formula:

$$\sum_{k\geq 0} \binom{n}{k}\binom{n-k}{n-2k} 2^{n-2k} = \binom{2n}{n}$$

There exists a set $A = [2n]$ and we want to count the number of subset S of A, whose size is n. It's obvious that the number is $\binom{2n}{n}$.
Assuming there exists a 01 string whose length is $2n$. We call the first half s_1 and the second half s_2. The right side of the formula count the number of such string that contains n 1.
Let's consider what the left side of the formula count. Firstly, it enumerate the intersection of s_1 and s_2 that contains k 1. Then there are totally $2k$ 1 in the whole string and we need to choose another $n-2k$ 1 to be 1. We can select $n-2k$ places in the rest $n-k$ places. For any chosen place, it can be in either s_1 or s_2. There are totally 2^{n-2k} chooses. Finally, we get the left side of the formula. □

Problem 2. Prove that for any $m, n \in \mathbf{N}$,

$$\sum_{r\geq 0}\binom{2n}{2r-1}\binom{r-1}{m-1} = \binom{2n-m}{m-1}2^{2n-2m+1}.$$

Try to find a combinatorial proof for this one.

2 Practice on PIE

In each problem, clearly specify what is the universe, what are the bad sets, how to calculate the size of the bad sets, etc.

Problem 3. Complete the combinatorial proof of the following: For any positive integer n,

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k} = 2n+1.$$

Solution. Let $\{a_n\}$ be a sequence of R and B and its length is $2n$. A sequence is good if and only if it does not contain RB. Let B_i $(1 \leq i \leq 2n-1)$ be the set that contains all the bad sequences that $a_i = R$ and $a_{i+1} = B$.
By enumerating the position of the first B, we can know that there are totally $2n+1$ good sequences.
By PIE, we know that the number of all the good sequences is:

$$\sum_{S\subset[2n-1]}(-1)^{|S|}B_S = \sum_{k=0}^{2n-1}\sum_{\substack{S\subset[2n-1]\\|S|=k}}(-1)^{|S|}B_S$$
$$= \sum_{k=0}^{2n-1}(-1)^k\sum_{\substack{S\subset[2n-1]\\|S|=k}}B_S = \sum_{k=0}^{2n-1}(-1)^k\binom{2n-k}{k}2^{2n-2k}$$

The last equation may need some explanation. If the exist i such that $i \in S$ and $i+1 \in S$, B_S is empty. Because we can not assign two different values to the same place. If there does not exist such i, we can use $\binom{2n-2k+k}{k}$ to find the number of such S. For each S, we have determined the values of $2k$ places and we can assign R or B to other $2n-2k$ places and there are totally 2^{2n-2k} ways.

□

Problem 4. Count the number of permutations π of $[2n]$ such that $\pi(i)+\pi(i+1) \neq 2n+1$ for all $1 \leq i \leq 2n-1$.

Solution. Let $B_i (1 \leq i \leq 2n-1)$ be the set of permutation π satisfying $\pi(i)+\pi(i+1) = 2n+1$. Then the answer is

$$\sum_{S \subset [2n-1]} (-1)^{|S|} B_S = \sum_{k=0}^{2n-1} (-1)^k \binom{2n-k}{k} \frac{n!}{(n-k)!} (n-2k)!$$

For a B_S, if $i, i+1 \in S$, the B_S is an empty set. So there are $\binom{2n-k}{k}$ nonempty B_S. For each S that has the size k, we can enumerate the minor number in each of the k couple places. Then we have $\frac{n!}{(n-k)!}$ schemes. For the rest $n-2k$ numbers, we can random places them and there are $(n-2k)!$ schemes. □

3 More for the pie day

Problem 5. The Euler function $\phi(n)$ is defined to be the number of elements in $[n]$ that are relatively prime to n. Define $f(n) = \sum_{i=1}^{n} \phi(i)$. Approximately how big is $f(n)$?

4 Some optional hard problems

Problem 6. A *tournament* is a complete graph with exactly one direction on each edge. A *Hamilton path* in a graph is a path in the graph that visits every vertex exactly once.
Prove that, in any tournament,
(a) there exists at least one Hamilton path;
(b) the number of Hamilton paths is odd.

Problem 7. Suppose G is a connected graph. A *spanning subgraph* of G is a subgraph that connects all the vertices of G. Prove that, G has an odd number of spanning subgraphs if and only if G is bipartite.