Suggested due date: 2018/04/11

Combinatorics: Homework 6

1 New problems

Problem 1. Draw the Hasse diagram for the posets $(X, |)$, where $|$ is the divisible relation, and (a) X is the set of positive divisors of 16. (b) X is the set of positive divisors of 12. (c) X is the positive divisors of 30.

Solution. $\square$

Figure 1: 16

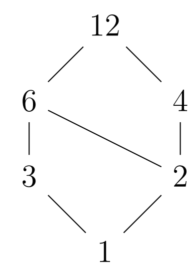

Figure 2: 12

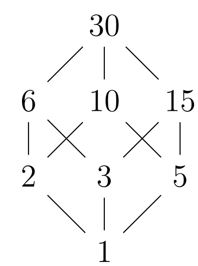

Figure 3: 30

Problem 2. Determine the number of non-isomorphic partial orders on $[1]$, $[2]$, $[3]$, $[4]$, (and extra, $[5]$,) respectively.

Solution. Let $f([n])$ be the answer for $[n]$. Then

$$f([1]) = 1, f([2]) = 2, f([3]) = 5, f([4]) = 15$$

$\square$

Problem 3. In class we proved both Dilworth theorem and a dual theorem: In a poset the maximum size of an anti-chain equals the minimum number of chains that can cover the set; the size of the longest chain equals the minimum number of anti-chains that can cover the set.
Deduce the following Erdős-Szekeres theorem from each of the theorems above:
In a sequence of $mn + 1$ distinct real numbers, we can always find a subsequence of length $m + 1$ that is increasing, or a subsequence of length $n + 1$ that is decreasing.

Solution.

$\square$

Problem 4. 101 distinct (closed) segments on a line, prove that either there are 11 pairwise disjoint segments, or one can find a point that lies in at least 11 segments. How can you deduce this from Dilworth's theorem?

Problem 5. Let $G = (V, E)$ be a graph whose chromatic number is k, and let $\phi : V \to [k]$ be a proper vertex colouring with exactly k colours. Prove that we can always find in G a path $(p_1, p_2, \ldots, p_k)$ so that $\phi(p_i) = i$ for all $1 \le i \le k$.

2 Still haunting

Problem 6. Consider all the permutations on $[100]$ and their cycle representations. Let N be the number of those permutations with exactly 50 cycles. What is $N \mod 3$? Prove your answer.

Problem 7. Prove that, for any positive integer k, there exists a graph whose chromatic number is k, yet it does not contain a triangle.

3 Still open

This is the correct version of the problem I mentioned in class:

Problem 8. (Open) Given two bipartite graphs $G_1 = (A, B)$ and $G_2 = (A, B)$ such that

$$\forall X \subseteq A, |N_{G_1}(X)| \geq |N_{G_2}(X)|.$$

Prove or disprove that the number of A-perfect matchings in G_1 is no less than that in G_2.