Suggested due date: 2018/03/07

Combinatorics: Homework 1

Problem 1. According to Lucas's theorem, we have

$$\binom{n}{i} \equiv \binom{(n_1n_2\dots n_k)_2}{(i_1i_2\dots i_k)_2} \equiv \binom{n_1}{i_1}\binom{n_2}{i_2}\cdots\binom{n_k}{i_k} \pmod 2$$

where $(n_1n_2\dots n_k)_2$ and $(i_1i_2\dots i_k)_2$ are the binary expressions of n and i.
$\binom{n}{i}$ is odd if and only if $i_k \le n_k$ for all k, so the number of such i is 2^{cnt} where cnt is the number of 1 in the binary expression of n. $2018 = (0111\ 1110\ 0010)_2$ so the answer is $2^7 = 128$.

Problem 2. (a) If we use a bit string to express a set, we have:
$A_0 = 0000000000000001$
$A_1 = 0000000000000010$
$A_2 = 0000000000000101$
$A_3 = 0000000000001000$
$A_4 = 0000000000010101$
$A_5 = 0000000000100010$
$A_6 = 0000000001010001$
$A_7 = 0000000010000000$
$A_8 = 0000000101010001$
$A_9 = 0000001000100010$
$A_{10} = 0000010100010101$
$A_{11} = 0000100000001000$
$A_{12} = 0001010100000101$
$A_{13} = 0010001000000010$
$A_{14} = 0101000100000001$
$A_{15} = 1000000000000000$

(b) For simplification, let's define $B_0 = \emptyset$ and $B_{i+1} = \{a+1 | a \in A_i\}$ for all $i \ge 0$, $S^{(n)} = S + \{1\} + \{1\} + \cdots + \{1\}$ (add n times) and $AB = A\Delta B$.
So $B_1 = \{1\}$, $B_2 = \{2\}$, $B_3 = \{1, 3\}$ and so on.

Let's define the proposition $P(N)$ be :

$$B_{2^N+i} = B_{2^N-i} B_i^{(2^N)} \text{ for all } 1 \leq i \leq 2^N - 1$$

and

$$B_{2^N} = \{2^N\}$$

We can check and get that $P(N)$ holds for $N = 0, 1$.
Assuming $P(N)$ is true, let's prove $P(N+1)$ is also true:

- $B_{2^{N+1}} = \{2^{N+1}\}$:

$$\begin{aligned}
B_{2^{N+1}} &= B_{2^{N+1}-1}^{(1)} B_{2^{N+1}-2} \\
&= (B_1 B_{2^N-1}^{(2^N)})^{(1)} B_2 B_{2^N-2}^{(2^N)} \\
&= B_1^{(1)} B_2 B_{2^N-1}^{(2^N+1)} B_{2^N-2}^{2^N} \\
&= B_{2^N-1}^{(2^N+1)} B_{2^N-2}^{2^N} \\
&= (B_{2^N})^{(2^N)} \\
&= (\{2^N\})^{(2^N)} \\
&= \{2^{N+1}\}
\end{aligned}$$

- $i = 1$:

$$\begin{aligned}
B_{2^{N+1}+1} &= B_{2^{N+1}}^{(1)} B_{2^{N+1}-1} \\
&= \{2^{N+1} + 1\} B_{2^{N+1}-1} \\
&= B_1^{(2^{N+1})} B_{2^{N+1}-1}
\end{aligned}$$

- $2 \leq i \leq 2^N - 1$:(This part itself is also an induction proof)

$$\begin{aligned}
B_{2^{N+1}+i} &= (B_{2^{N+1}-i+1} B_{i-1}^{(2^{N+1})})^{(1)} B_{2^{N+1}-i+2} B_{i-2}^{2^{N+1}} \\
&= (B_{2^{N+1}-i+1})^{(1)} B_{2^{N+1}-i+2} (B_{i-1} B_{i-2})^{(2^{N+1})} \\
&= B_{2^{N+1}-i} B_i^{2^{N+1}}
\end{aligned}$$

So $|A_{2^n-1}| = |B_{2^n}| = 1$ for all $n >= 1$.

Problem 3. (b) Let's define a state function $g(s)$ equal to the sum of the distances between all the n points for the state s. Explicitly,

$$g(s) = \sum_{i=0}^{n} \sum_{j=i+1}^{n} s[i]s[j]min(j-i, n-j+i)$$

where $s[0], s[1], \ldots, s[n-1]$ are the number of points at the places $0, 1, \cdots, n-1$.
It's easy to observe the fact that any movement will cause $g(s)$ increasing and g will stop increasing when no movement can be done.(Only when n is odd, g has this property)

I've known the answers of the rest problems from Wang Tianzhe.