Suggested due date: 2018/03/21

Combinatorics: Homework 3

1 Combinatorial identities

Problem 1. Prove that for any $n \in \mathbf{N}$,

$$\sum_{k\geq 0} \binom{n}{2k}\binom{2k}{k} 2^{n-2k} = \binom{2n}{n}.$$

There exists a combinatorial proof for this, but I don't suggest you try to find it.

Solution. Firstly, we can observe the left to get that:

$$\binom{n}{2k}\binom{2k}{k} = \frac{n!}{(n-2k)!(2k)!}\frac{(2k)!}{k!k!} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{k!(n-2k)!} = \binom{n}{k}\binom{n-k}{n-2k}$$

Then we only need to prove the following formula:

$$\sum_{k\geq 0} \binom{n}{k}\binom{n-k}{n-2k} 2^{n-2k} = \binom{2n}{n}$$

□

Problem 2. Prove that for any $m, n \in \mathbf{N}$,

$$\sum_{r\geq 0} \binom{2n}{2r-1}\binom{r-1}{m-1} = \binom{2n-m}{m-1} 2^{2n-2m+1}.$$

Try to find a combinatorial proof for this one.

2 Practice on PIE

In each problem, clearly specify what is the universe, what are the bad sets, how to calculate the size of the bad sets, etc.

Problem 3. Complete the combinatorial proof of the following: For any positive integer n,

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k} = 2n+1.$$

Problem 4. Count the number of permutations π of $[2n]$ such that $\pi(i)+\pi(i+1) \neq 2n+1$ for all $1 \leq i \leq 2n-1$.

3 More for the pie day

Problem 5. The Euler function $\phi(n)$ is defined to be the number of elements in $[n]$ that are relatively prime to n. Define $f(n) = \sum_{i=1}^{n}\phi(i)$. Approximately how big is $f(n)$?

4 Some optional hard problems

Problem 6. A *tournament* is a complete graph with exactly one direction on each edge. A *Hamilton path* in a graph is a path in the graph that visits every vertex exactly once.
Prove that, in any tournament,
(a) there exists at least one Hamilton path;
(b) the number of Hamilton paths is odd.

Problem 7. Suppose G is a connected graph. A *spanning subgraph* of G is a subgraph that connects all the vertices of G. Prove that, G has an odd number of spanning subgraphs if and only if G is bipartite.