Mathematical Logic Homework 7

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Solution 7.1. By Completeness Theorem, we can derive $\Theta \models \varphi$ from $\Theta \vdash \varphi$. Then for any S-interpretation $\mathfrak{I}$

$$\mathfrak{I} \models \Theta \text{ implies } \mathfrak{I} \models \varphi$$

We can construct a S_0-interpretation $\mathfrak{I}'$ by retaining the symbols occurring in Θ and φ and keep their interpretation unchanged. By Coincidence Lemma,

$$\mathfrak{I}' \models \Theta \text{ implies } \mathfrak{I}' \models \varphi$$

It's obvious that any S_0-interpretation can be expanded to a S-interpretation without changing the interpretation of symbols in S_0. So for any S_0-interpretation $\mathfrak{I}'$,

$$\mathfrak{I}' \models \Theta \text{ implies } \mathfrak{I}' \models \varphi$$

which means $\Theta \models \varphi$ (Now φ is a S_0-formula and so does the formulas in Θ). By Completeness Theorem, $\Theta \vdash \varphi$ and every formula occurs in the proof is a S_0-formula.

□

Let's prove the general version of Zorn' Lemma, which can be described as:

Zorn's Lemma. Assume A is an nonempty set and $\preceq$ is a partial order of A. For any chain $C \subseteq A$, there is an upper bound s of C such that $s \in A$. Then there exists a maximal element c in A.

(The Zorn's Lemma discussed in class is a special case of this theorem, where A is the power set of M and $\preceq$ is the $\subseteq$(subset) relation)

Solution 7.2.[1] Let $\leq$ be a well order of A and $\preceq$ be a partial order of A. Let's construct a function f by

$$f(x) = \begin{cases} 1, & \text{for any } y \leq x, y \neq x \text{ and } f(y) = 1,\ x \preceq y \\ 0, & \text{otherwise} \end{cases}$$

Let

$$C = \{x \mid f(x) = 1\}$$

Then C is a chain for $\preceq$ because for any $x, y \in C$, $x \preceq y$ or $y \preceq x$ by the definition of f.

[1]Reference: https://www.drmaciver.com/2015/12/direct-proofs-from-the-well-ordering-theorem

By the assumption of Zorn's Lemma, there is some upper bound for C, call it s.

Firstly, because $\{y \mid y \leq s, f(y) = 1\} \subseteq C$, $y \preceq s$ for all $y \in C$, we must have $s \in C$.

Then s must be a maximal element of A for relation $\preceq$. We can prove this by contradiction. If there exist $t \in A$ such that $s \neq t$ and $s \preceq t$, then t must be in C, which contradicts that s is an upper bound of C. $\square$