Mathematical Logic Homework 5

Ding Yaoyao

October 26, 2018

Solution 5.1. Because Φ is inconsistent, for any φ, $\Phi \vdash \varphi$. Then by the definition of $\sim$ relation, there is only one element of the universe T^Φ and the only element is the whole set of the terms T^S. Let e denote the lonely element. For any relation R, we have $(e, \dots, e) \in R$ because $\Phi \vdash R(t_1, \dots, t_n)$. For any function f, because we have only one element in universe, $f(e, \dots, e) = e$. For any constant c, $c^\Phi = e$. $\square$

Solution 5.2. (1) Let's construct a S-interpretation $\mathfrak{I}$ that is satisfied by the Φ. Let the universe $A = \{a, b\}$ and $a \in R, b \notin R$. Let $\beta(x) = b$ for all $x \in A$. Then $\mathfrak{I} \models \Phi$. Because Φ is satisfiable, then Φ is consistent.

(2) Because no function and constant symbol exists, T^S only contains variables(i.e. $T^S = \{v_0, v_1, \dots\}$). Then $\neg Rt \in \Phi$. By the lemma 2.6.a in Logic5.pdf, $\Phi \vdash Rt$ is equivalent to that $\Phi \cup \{\neg Rt\}$ is inconsistent. Because $\neg Rt \in \Phi$, then $\Phi \cup \{\neg Rt\} = \Phi$. By (1), Φ is consistent, which is a contradiction. So such term $t \in T^S$ does not exist. $\square$

Solution 5.3. (1) Let's construct a S-interpretation $\mathfrak{I}$ that is satisfied by Φ. Let the universe $A = \{a\}$ and $a \in R$. Then $\mathfrak{I} \models \Phi$.

(2) It's equivalent to show that $\Phi \cup \{\neg Rx\}$ and $\Phi \cup \{\neg Ry\}$ are consistent by lemma 2.6.a. Let's prove $\Phi \cup \{\neg Rx\}$ is consistent and the proof of $\Phi \cup \{\neq Ry\}$ is completely the same. Let the universe $A = \{a, b\}$, $a \in R, b \notin R$ and $\beta(x) = a$ and $\beta(y) = b$. Then $\mathfrak{I} \models Rx \vee Ry$ and $\mathfrak{I} \models \neg Rx$. Then $\mathfrak{I} \models \Phi \cup \{\neg Rx\}$ is consistent.

(3) Because $\Phi \nvdash Rx$, then $\mathfrak{T}^\Phi \not\models Rx$. Similarly, $\mathfrak{T}^\Phi \not\models Ry$. Thus $\mathfrak{T}^\Phi \not\models Rx \vee Ry$, which means $\mathfrak{T}^\Phi \not\models \Phi$. $\square$