Mathematical Logic Homework 9

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Solution 9.1.

(1)

$\mathfrak{A} \models \varphi_0$: When $P : \Box \to \infty$, we have $A_{\mathbb{P}} = \mathbb{N}$, $<^{\mathfrak{A}}$ is the less order of $\mathbb{N}$ and $f^{\mathfrak{A}}(x) = x + 1$. Of course, $<^{\mathfrak{A}}$ is an order. $x < x + 1$ for all $x \in \mathbb{N}$ implies $\mathfrak{A} \models \forall x(x < fx)$. No $y \in \mathbb{N}$ such that $x < y < x + 1$ implies $\mathfrak{A} \models \forall x(\forall z(x < z \to (fx < z \vee fx \equiv z)))$. So $\mathfrak{A} \models \varphi_0$. When $P : \Box \to halt$, we have $A_{\mathbb{P}} = \{0, \dots, e\}$, $<^{\mathfrak{A}} = \{(i, j) \mid 0 \leq i < j \leq e\}$ and $f^{\mathfrak{A}}(x) = min(x + 1, e)$. $c^{\mathfrak{A}} = 0$ implies $\mathfrak{A} \models \forall x(c < x \vee c \equiv x)$. For all $x \in A$, $x < min(x + 1, e)$ or $x = min(x + 1, e)$, which implies $\mathfrak{A} \models \forall x(x < fx \vee x \equiv fx)$. The last part of φ_0 is the same as above when $x < e$. When $x = e$, no $y \in A$ such that $x < y$, so it's true directly. So $\mathfrak{A} \models \varphi_0$.

$\mathfrak{A} \models R\overline{0}\overline{0}\dots\overline{0}$: When $\mathbb{P}$ starts with $\Box$, the initial configuration is $(0, 0, \dots, 0)$. The definition of R implies $(\overline{0}, \dots, \overline{0}) \in R$, which means $\mathfrak{A} \models R\overline{0}\overline{0}\dots\overline{0}$.

$\mathfrak{A} \models \varphi_{\alpha_i}$: Assume c_t is the configuration of the machine at step t and the next instruction is i and c_{t+1} is the configuration that the i-th instruction has been executed. Because φ_{α_i} describe that 'if c_t is the configuration before executing instruction i, then c_{t+1} is also a configuration'. By the definition of configuration, c_{t+1} must also be a configuration of the program $\mathbb{P}$, then $\mathfrak{A} \models \varphi_{\alpha_i}$.

Above all, $\mathfrak{A} \models \varphi_{\mathbb{P}}$.

(2)

Because $(L, m_0, \dots, m_n)$ is a configuration $\mathbb{P}$ after s steps, there exists s configurations:

$$c_0, c_1, \dots, c_{s-1}$$

such that the configuration of $\mathbb{P}$ transforms from c_0 initially to c_{s-1} and then $c_s = (L, m_0, \dots, m_n)$.

Every configuration corresponds a state $S_i = (\overline{s_i}, \overline{L_i}, \overline{m_0^{(i)}}, \dots, \overline{m_n^{(i)}})$, specially $S_0 = (\overline{0}, \dots, \overline{0})$. $\mathfrak{A} \models R(\overline{0}, \dots, \overline{0})$ implies $\mathfrak{A} \models R(S_0)$. Assume $\mathfrak{A} \models R(S_i)$, and the next instruction of S_i is instruction j, then by definition of φ_{α_j}, we have $\mathfrak{A} \models R(S_{i+1})$. Finally we have $\mathfrak{A} \models R(S_s)$, which means $\mathfrak{A} \models R(\overline{(s)}, \overline{L}, \overline{m_0}, \dots, \overline{m_n})$.

$\Box$

Solution 9.2.

$A = \mathbb{N}$

$f(x) = x + 1$,

$R = \{(s, L, m_0, \dots, m_n) \mid$ $\mathbb{P}$ reachs configuration $(L, m_0, \dots, m_n)$ after s steps $\}$.

$c^{\mathfrak{A}} = 0$. $\Box$

Solution 9.3.[1] Assume it's R-enumerable and the program is P_1. Because $\{\varphi \in L_0^{S_\infty} \mid \models \varphi\}$ is enumerable, there exists a program P_2 that can enumerable all such sentences. For a given formula $\varphi \in L_0^{S_\infty}$, run the two programs simultaneously. Once P_2 prints φ or P_1 prints $\neg\varphi$, we can decide whether $\models \varphi$ holds. This process will stops in finite steps for any φ because one of the two conditions must hold. As a result, we construct a program P_3 that can decide $\{\varphi \in L_0^{S_\infty} \mid \models \varphi\}$, which is a contradiction.

Above all, $\{\varphi \in L_0^{S_\infty} \mid \varphi \text{ is satisfiable } \}$ is not R-enumerable. $\square$

[1]refers: https://github.com/blargoner/math-logic-ebbinghaus/blob/master/exercises.pdf