Mathematical Logic Homework 10

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Solution 10.1.

We call the sentences in S_{ar} sentence P_1 to P_7 in order.

$P_1 : \forall x \neg x + 1 \equiv 0$

$P_2 : \forall x\ x + 0 = x$

$P_3 : \forall x\ x \cdot 0 = 0$

$P_4 : \forall x \forall y (x + 1 \equiv y + 1 \to x \equiv y)$

$P_5 : \forall x \forall y\ x + (y + 1) \equiv (x + y) + 1$

$P_6 : \forall x \forall y\ x \cdot (y + 1) = x \cdot y + x$

$P_7 : \forall x_1 \cdots \forall x_n \left(\left(\varphi \frac{0}{y} \wedge \forall y (\varphi \to \varphi \frac{y+1}{y}) \right) \to \forall y \varphi \right)$

By P_2, for all x, y, if $x + 0 \equiv y + 0$, we have $x \equiv x + 0 \equiv y + 0 \equiv y$, then

$$\forall x \forall y\ x + 0 \equiv y + 0 \to x \equiv y \tag{1}$$

By P_5, $x + (z + 1) \equiv y + (z + 1)$ is equivalent to $(x + z) + 1 \equiv (y + z) + 1$. By P_4, $x + z \equiv y + z$. Then $x + z \equiv y + z \to x \equiv y$ implies $x \equiv y$. Then

$$\forall x \forall y (x + z \equiv y + z \to x \equiv y) \to (x + (z + 1) \equiv y + (z + 1) \to x \equiv y) \tag{2}$$

By $(1), (2), P_7$, we have

$$\forall x \forall y \forall z\ x + z \equiv y + z \to x \equiv y \tag{3}$$

Let $y = 0$ in P_5, we have $\forall x\ x + (0 + 1) \equiv (x + 0) + 1 \equiv x + 1$. By (3), we have $0 + 1 \equiv 1$. By P_2, we have

$$0 + 1 \equiv 1 + 0 \tag{4}$$

Because $x + 1 \equiv 1 + x$ implies $(x + 1) + 1 \equiv (1 + x) + 1 \equiv 1 + (x + 1)$, we have

$$\forall x\ x + 1 \equiv 1 + x \to (x + 1) + 1 \equiv 1 + (x + 1) \tag{5}$$

By $(4), (5), P_7$, we have

$$\forall x\ x + 1 \equiv 1 + x \tag{6}$$

Because $0 + x \equiv x + 0$ implies $0 + (x + 1) \equiv (0 + x) + 1 \equiv (x + 0) + 1 \equiv x + 1 \equiv (x + 1) + 0$, we have

$$\forall x\ 0 + x \equiv x + 0 \tag{7}$$

Assume $\forall x y + x \equiv x + y$, then $(y+1)+x \equiv (1+y)+x \equiv 1+(y+x) \equiv 1+(x+y) \equiv (1+x)+y \equiv (x+1)+y \equiv x+(1+y) \equiv x+(y+1)$. Then we have

$$\forall x\; y + x \equiv x + y \to (y+1)+x \equiv x+(y+1) \tag{8}$$

By $(7), (8), P_7$, we have

$$\forall x \forall y\; x + y \equiv y + x$$

□

Solution 10.2. Because T is R-enumerable, we can construct a program P enumerate T. Assume $\varphi_1, \varphi_2, \ldots$ is an enumeration. Then we can define

$$\Theta = \{\varphi_1, \varphi_1 \wedge \varphi_2, \varphi_1 \wedge \varphi_2 \wedge \varphi_3, \ldots\},$$

which is strictly R-enumerable(Because the length of sentences increases). Then Θ is R-decidable. By the definition of modeling, we have $T \models \Theta$(thus, $\Theta \subseteq T$) and $\Theta \models T$(also T is a theory, then $T = \Theta^{\models}$).Then T is R-axiomatizable.

□

Solution 10.3.

$$\begin{aligned}\varphi_{exp}(x,y,z) = &\exists u \exists v(\varphi_\beta(u,v,0,1) \wedge \varphi_\beta(u,v,y,z) \wedge &(9)\\ &\forall i\; (i < y \to (\forall w \varphi_\beta(u,v,i,w) \to \varphi_\beta(u,v,i+1,w \cdot x)))) &(10)\end{aligned}$$

□