Mathematical Logic Homework 1

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September 17, 2018

Solution 2.1. $(a) \Rightarrow (b)$: If M is a limited set, then there exists a bijection between M and $[|M|]$. If M is a countable set, then there is a bijection between M and $\mathbb{N}$. Because M is at most countable, there exists a bijection g from a subset of $\mathbb{N}$ to M. Then we define f from $\mathbb{N}$ to M such that

$$f(n) = \begin{cases} g(n) & n \in Dom(f) \\ \text{an arbitrary element of } M & n \notin Dom(f) \end{cases} \tag{1}$$

Then $Range(f) = M$, which means f is a surjection from $\mathbb{N}$ to M.

$(b) \Rightarrow (c)$: For each $x \in M$, let $S_x = \{n \mid f(n) = x\}$. Then we define g from M to $\mathbb{N}$ such that

$$g(x) = \text{an arbitrary element of } S_x \tag{2}$$

Because f is a function from $\mathbb{N}$ to M, $S_x \cap S_y = \emptyset$ for different x and y. Then g is a injection from M to $\mathbb{N}$.

$(c) \Rightarrow (a)$: There is a bijection between M and $Range(f)$. When the number of elements of M is limited, M is at most countable. When it's infinite, we only need to show that there is a bijection between $Range(f)$ and $\mathbb{N}$. Because $Range(f)$ is a subset of $\mathbb{N}$, we can list the elements of $Range(f)$ in a line in ascending order. Then let $g(i)$ be the i-th element in the line. Then g is a bijection between $\mathbb{N}$ and $Range(f)$. At the same time, there is a bijection between $Range(f)$ and M. So there is a bijection between $\mathbb{N}$ and M, which means M is countable. Above all, M is at most countable. □

Solution 2.2. Firstly, it's obvious that for any $n \in \mathbb{N}$, A^n is at most countable. Let f_n be the bijection from $\mathbb{N}$ to A^n. We can place the elements in A^* in such an order:

$$f_0(0), f_0(1), f_1(0), f_0(2), f_1(1), f_2(0), \ldots$$

(Similar method are used to prove $\mathbb{Q}$ is countable). □

Solution 2.3. Prove by contradiction.

Assume f is a surjective function from M to $Pow(M)$.

We construct a set S such that $x \in S$ if and only if $x \notin f(x)$.

There does not exist an element x in M such that $f(x) = S$. So f is not surjective.

Thus, there is not a surjective function from M to $Pow(M)$. □