Mathematical Logic Homework 6

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Solution 6.1. (I think the exercise wants to prove consistent Ψ that contains witness does not exist)

Let's construct a S-interpretation $\mathfrak{I}$ that models Φ to prove Φ is consistent.

Let the universe set $A = \{p, q\}$, where p, q are two different elements.

For any n-ary relation $R \in S$ and $x_1, \dots, a_n \in A$,

$$(a_1, \dots, a_n) \in R$$

For any n-ary relation $f \in S$ and $a_1, \dots, a_n \in A$,

$$f(a_1, \dots, a_n) = p$$

For any variable v,

$$\beta(v) = p$$

Then for any term $t \in T^S$, $\mathfrak{I}(t) = p$, in particular $\mathfrak{I}(v_0) = p$. So $\mathfrak{I} \models v_0 \equiv t$ for any $t \in T^S$. Because A has two different elements, $\mathfrak{I} \models \exists v_0 \exists v_1 \neg v_0 \equiv v_1$. Eventually, $\mathfrak{I} \models \Phi$, thus Φ is consistent.

Let's prove that such Ψ does not exist by contradiction. Assume that there exists Ψ such that $\Phi \subseteq \Psi \subseteq L^S$ and Ψ contains witness. Then there exists $t_0, t_1 \in T^S$ such that

$$\Phi \models \exists v_0 \exists v_1 \neg v_0 \equiv v_1 \to \exists v_1 \neg t_0 \equiv v_1$$

$$\Phi \models \exists v_1 \neg t_0 \equiv v_1 \to \neg t_0 \equiv t_1$$

Because $\Phi \models \exists v_0 \exists v_1 \neg v_0 \equiv v_1$, we can derive $\Phi \models \neg t_0 \equiv t_1$. Because $\Phi \models v_0 \equiv t$ for all $t \in T^S$, $\Phi \models t_0 \equiv t_1$. Then Φ is inconsistent and contradiction occurs. So such Ψ does not exist.

□

Solution 6.2. Let β be any assignment of $\mathfrak{A}$ and let $\mathfrak{I} = (\mathfrak{A}, \beta)$. For any $\bar{t} \in T^\Phi$, define $h : T^\Phi \to A$ as

$$h(\bar{t}) = \mathfrak{I}(t)$$

The function h is well-defined because for any $t_1, t_2 \in T^S$ such that $t_1 \sim t_2$, we have $\Phi \models t_1 \equiv t_2$ and we can derive $\mathfrak{A} \models t_1 \equiv t_2$ from $\mathfrak{A} \models \Phi$ and thus $\mathfrak{I}(t_1) = \mathfrak{I}(t_2)$.

Then let's check the required three properties.

(1) For every n-ary relation symbol $R \in S$ and $\overline{t_1}, \dots, \overline{t_n} \in T^\Phi$, we have

$$(\overline{t_1}, \dots, \overline{t_n}) \in R^{\mathfrak{T}^\Phi} \text{ implies } \Phi \models R(t_1, \dots, t_n) \text{ implies } \mathfrak{A} \models R(t_1, \dots, t_n)$$

And then $(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n)) \in R^{\mathfrak{A}}$

(2) For every n-ary function symbol $f \in S$ and $\overline{t_1}, \ldots, \overline{t_n} \in T^{\Phi}$, we have

$$h(f^{\mathfrak{T}^{\Phi}}(\overline{t_1}, \ldots, \overline{t_n})) = h(\overline{f(t_1, \ldots, t_n)}) = \mathfrak{I}(f(t_1, \ldots, t_n)) \tag{1}$$
$$= f^{\mathfrak{A}}(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n)) = f^{\mathfrak{A}}(h(\overline{t_1}), \ldots, h(\overline{t_n})) \tag{2}$$

(3) For every constant $c \in S$,

$$h(c^{\mathfrak{T}^{\Phi}}) = \mathfrak{I}(c) = c^{\mathfrak{A}}$$

Above all, h is a homomorphism from $\mathfrak{T}^{\Phi}$ to $\mathfrak{A}$. $\square$