Mathematical Logic Homework 2

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Solution 2.1.

(a)

Let $\mathfrak{A} = (\mathcal{A}, \circ, e^{\mathcal{A}})$, $\mathfrak{B} = (\mathcal{B}, \circ, e^{\mathcal{B}})$ and $\mathfrak{A} \times \mathfrak{B} = (\mathcal{A} \times \mathcal{B}, \circ, e^{\mathcal{A}\times\mathcal{B}})$.

- For all $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathcal{A} \times \mathcal{B}$,

$$\begin{aligned} & ((a_1, b_1) \circ (a_2, b_2)) \circ (a_3, b_3) \\ = \; & (a_1 \circ a_2, b_1 \circ b_2) \circ (a_3, b_3) \\ = \; & ((a_1 \circ a_2) \circ a_3, (b_1 \circ b_2) \circ b_3) \\ = \; & (a_1 \circ (a_2 \circ a_3), b_1 \circ (b_2 \circ b_3)) \\ = \; & (a_1, b_1) \circ (a_2 \circ a_3, b_2 \circ b_3) \\ = \; & (a_1, b_1) \circ ((a_2, b_2) \circ (a_3, b_3)) \end{aligned}$$

- For all $(a, b) \in \mathcal{A} \times \mathcal{B}$,

$$\begin{aligned} & (a, b) \circ (e^{\mathcal{A}}, e^{\mathcal{B}}) \\ = \; & (a \circ e^{\mathcal{A}}, b \circ e^{\mathcal{B}}) \\ = \; & (a, b) \end{aligned}$$

- For all $(a, b) \in \mathcal{A} \times \mathcal{B}$, there exists $c \in \mathcal{A}$ and $d \in \mathcal{B}$ such that

$$a \circ c = e^{\mathcal{A}}, b \circ d = e^{\mathcal{B}}.$$

  Then

$$(a, b) \circ (c, d) = (a \circ c, b \circ d) = (e^{\mathcal{A}}, e^{\mathcal{B}}) = e^{\mathcal{A}\times\mathcal{B}},$$

  where $(c, d) \in \mathcal{A} \times \mathcal{B}$.

Above all, $\mathfrak{A} \times \mathfrak{B}$ is a group.

(b) Let $\mathfrak{A} = (\mathcal{A}, R^{\mathcal{A}})$, $\mathfrak{B} = (\mathcal{B}, R^{\mathcal{B}})$ and $\mathfrak{A} \times \mathfrak{B} = (\mathcal{A} \times \mathcal{B}, R^{\mathcal{A}\times\mathcal{B}})$.

- For all $(a, b) \in \mathcal{A}\times\mathcal{B}$, we have $(a, a) \in R^{\mathcal{A}}$ and $(b, b) \in R^{\mathcal{B}}$ and then$((a, b), (a, b)) \in R^{\mathcal{A}\times\mathcal{B}}$.

- For all $(a_1, b_1), (a_2, b_2) \in \mathcal{A}\times\mathcal{B}$, if $((a_1, b_1), (a_2, b_2) \in R^{\mathcal{A}\times\mathcal{B}}$ then $(a_1, a_2) \in R^{\mathcal{A}}$ and $(b_1, b_2) \in R^{\mathcal{B}}$. Because $\mathfrak{A}$ and $\mathfrak{B}$ are groups, we have $(a_2, a_1) \in R^{\mathcal{A}}$ and $(b_2, b_1) \in R^{\mathcal{B}}$. Then $((a_2, b_2), (a_1, b_1)) \in R^{\mathcal{A}\times\mathcal{B}}$.

- For all $(a_1, b_1), (a_2, b_2), (a_3, b_3)$, if $((a_1, b_1), (a_2, b_2)), ((a_2, b_2), (a_3, b_3)) \in R^{\mathcal{A}\times\mathcal{B}}$, we have $(a_1, a_2), (a_2, a_3) \in R^{\mathcal{A}}$ and $(b_1, b_2), (b_2, b_3) \in R^{\mathcal{B}}$. Then $(a_1, a_3) \in R^{\mathcal{A}}$ and $(b_1, b_3) \in R^{\mathcal{B}}$. Finally, we have $((a_1, b_1), (a_3, b_3)) \in R^{\mathcal{A}\times\mathcal{B}}$.

(c)
Because $(1^{\mathcal{A}}, 0^{\mathcal{B}})$ has no inverse, $\mathfrak{A} \times \mathfrak{B}$ is not a field. $\square$

Solution 2.2. For all $\mathfrak{I} \models \Theta_{Gr}$, we have

$$\mathfrak{I}\frac{a}{v_0}\frac{b}{v_1}\frac{c}{v_2} \models (a \circ b) \circ c = a(b \circ c) \quad \text{(for all } a, b, c \in \mathcal{A})$$

which means for all $a, b, c \in \mathcal{A}$, we have $(a \circ b) \circ c = a \circ (b \circ c)$. Similarly, for all $a \in \mathcal{A}$, we have $a \circ e = a$ and exists a' such that $a \circ a' = e$.

Let $a' \in \mathcal{A}$ such that $a \circ a' = e$ and $a'' \in \mathcal{A}$ such that $a' \circ a'' = e$. Then

$$a' \circ a = a' \circ a \circ e = a' \circ a \circ a' \circ a'' = a' \circ a'' = e$$

and then

$$e \circ a = a \circ a' \circ a = a \circ e = a$$

. which means for all $a \in \mathcal{A}$, we have $\mathfrak{I}\frac{a}{v_0} \models e \circ v_0 \equiv v_0$ and then $\mathfrak{I} \models \forall v_0\; e \circ v_0 = v_0$. Similarly, for all $a \in \mathcal{A}$, exists $b \in \mathcal{A}$ such that $\mathfrak{I}\frac{a}{v_0}\frac{b}{v_1} \models v_1 \circ v_0 = e$, which means $\mathfrak{I} \models \forall v_0 \exists v_1\; v_1 \circ v_0 = e$.

Above all, $\Theta_{Gr} \models \forall v_0\; e \circ v_0 = v_0$ and $\Theta_{Gr} \models \forall v_0 \exists v_1\; v_1 \circ v_0 = e$. $\square$

Solution 2.3.

$\square$