Mathematical Logic Homework 3

Ding Yaoyao

October 12, 2018

Solution 2.1.

(1) Let π be the identify function from A to A. So $\pi(a) = a$ for all $a \in A$. Then

- (i) Identity function π is a bijection.
- (ii) For any n-ary relation symbol $R \in S$ and $a_0, \dots, a_{n-1} \in A$.
$$(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \Leftrightarrow (\pi(a_0), \dots, \pi(a_{n-1}) \in R^{\mathfrak{A}}.$$
- (iii) For any n-ary function symbol $f \in S$ and $a_0, \dots, a_{n-1} \in A$
$$\pi(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = f^{\mathfrak{A}}(\pi(a_0), \dots, \pi(a_{n-1}).$$
- (iv) For any constant $c \in S$
$$\pi(c^{\mathfrak{A}}) = c^{\mathfrak{A}}.$$

So $\mathfrak{A} \cong \mathfrak{B}$

(2) Let π_1 be the isomorphism from A to B. Then define π_2 as the inverse function of π_1.

- (i) The inverse of a bijection is also a bijection.
- (ii) For any n-ary relation symbol $R \in S$ and $b_0, \dots, b_{n-1} \in B$. There exist $a_0, \dots, a_{n-1} \in A$ such that $\pi_1(a_i) = b_i$ and $\pi_2(b_i) = a_i$ for $i \in \{0, \dots, n-1\}$ such that
$$(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \Leftrightarrow (\pi_1(a_0), \dots, \pi_1(a_{n-1})) \in R^{\mathfrak{B}},$$
which is equivalent to
$$(b_0, \dots b_{n-1}) \in R^{\mathfrak{B}} \Leftrightarrow (\pi_2(b_0), \dots, \pi_2(b_{n-1}) \in R^{\mathfrak{A}}.$$
- (iii) For any n-ary function symbol $f \in S$ and $b_0, \dots, b_{n-1} \in B$. There exist $a_0, \dots, a_{n-1} \in A$ such that $\pi_1(a_i) = b_i$ and $\pi_2(b_i) = a_i$ for $i \in \{0, \dots, n-1\}$. Because
$$\pi_1(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{B}}(b_0, \dots, b_{n-1}),$$
then
$$\begin{aligned}\pi_2(f^{\mathfrak{B}}(b_0, \dots, b_{n-1})) &= \pi_2(\pi_1(f^{\mathfrak{A}}(a_0, \dots, a_{n-1}))) \\ &= f^{\mathfrak{A}}(\pi_2(b_0), \dots, \pi_2(b_{n-1})).\end{aligned}$$

- (iv) For any constant $c \in S$

  $$\pi_1(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$$

  and then

  $$\pi_2(c^{\mathfrak{B}}) = \pi_2(\pi_1(c^{\mathfrak{A}})) = c^{\mathfrak{A}}$$

Above all, $\mathfrak{B} \cong \mathfrak{A}$.

(3) Let π_1 be the bijection from A to B and π_2 be the bijection from B to C. Then define $\pi_3 : A \to C$ such that $\pi_3(a) = \pi_2(\pi_1(a))$ for all $a \in A$.

- (i) The composition of two bijections is also a bijection.
- (ii) For any n-ary relation symbol $R \in S$ and $a_0, \dots, a_{n-1} \in A$, we define $b_i = \pi_1(a_i)$ and $c_i = \pi_2(b_i)$ for all $0 \le i < n$. Because π_1 and π_2 are isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{B}$ to $\mathfrak{C}$ respectively, we have

  $$(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \Leftrightarrow (b_0, \dots, b_{n-1}) \in R^{\mathfrak{B}}$$
  $$(b_0, \dots, b_{n-1}) \in R^{\mathfrak{B}} \Leftrightarrow (c_0, \dots, c_{n-1}) \in R^{\mathfrak{C}}$$

  And then

  $$(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \Leftrightarrow (\pi_3(a_0), \dots, \pi_3(a_{n-1}) \in R^{\mathfrak{C}}$$

- (iii) For any n-ary function symbol $f \in S$ and $a_0, \dots, a_{n-1} \in A$. We define $b_i = \pi_1(a_i)$ and $c_i = \pi_2(b_i)$ for all $0 \le i < n$. Because π_1 and π_2 are isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ and from $\mathfrak{B}$ to $\mathfrak{C}$ respectively, we have

  $$\pi_1(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{B}}(b_0, \dots, b_{n-1})$$
  $$\pi_2(f^{\mathfrak{B}}(b_0, \dots, b_{n-1})) = f^{\mathfrak{C}}(c_0, \dots, c_{n-1}),$$

  which means

  $$\pi_3(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{C}}(\pi_3(a_0), \dots, \pi_3(a_{n-1}))$$

- (iv) For any constant $c \in S$,

  $$\pi_3(c^{\mathfrak{A}}) = \pi_2(\pi_1(c^{\mathfrak{A}})) = \pi_2(c^{\mathfrak{B}}) = c^{\mathfrak{C}}.$$

Above all, $\mathfrak{A} \cong \mathfrak{C}$. □

Solution 2.2.

(a)

(b)

$$\models \varphi \vee \psi \iff \text{ for all } \mathfrak{I},\, \mathfrak{I} \models \varphi \text{ implies } \mathfrak{I} \models \psi \iff \varphi \models \psi$$

□

Solution 2.3.

⇒: By Isomorphism Theorem, it's true.

$\Leftarrow$: A is finite and assume that $A = \{a_0, a_1, \dots, a_{n-1}\}$. Then we can construct the φ as

$$\exists_{v_0}\exists_{v_1}\cdots\exists_{v_{n-1}}\left(\bigwedge_{0\le i<j<n}\neg(v_i \equiv v_j)\right) \tag{1}$$

$$\wedge\left(\neg\exists_{v_n}\bigwedge_{0\le i<n}\neg(v_n = v_i)\right) \tag{2}$$

$$\wedge\left(\bigwedge_{1\le k\le n_1}\ \bigwedge_{\text{k-ary Rela. R}}\ \bigwedge_{(a_{j_1},\dots,a_{j_k})\in R} R(v_{j_1},\dots,v_{j_k}) \bigwedge_{(a_{j_1},\dots,a_{j_k})\notin R}\neg R(v_{j_1},\dots,v_{j_k})\right) \tag{3}$$

$$\wedge\left(\bigwedge_{1\le k\le n_2}\ \bigwedge_{\text{k-ary function } f}\ \bigwedge_{f(a_{j_1},\dots,a_{j_k})=a_{j_{k+1}}} f(v_{j_1},\dots,v_{j_k}) \equiv v_{j_{k+1}}\right) \tag{4}$$

$$\wedge\left(\bigwedge_{c\in S \text{ and } c^{\mathfrak{A}}=a_j} c \equiv v_j\right) \tag{5}$$

where n_1 is the max number of arguments of a relation and n_2 is the max number of arguments of a function. Intuitively, Condition (1) makes S-structure has at least n elements in A. Condition (2) makes it at most n. Condition (3), (4), (5) makes the relations, functions and constants isomorphic respectively.

For simplicity, We use

$$\exists_{v_0}\exists_{v_1}\cdots\exists_{v_{n-1}}P(v_0,\dots,v_{n-1})$$

to represent φ.

Of course $\mathfrak{A} \models \varphi$, then there exists $a_0, \dots, a_{n-1} \in A$ such that

$$\mathfrak{A} \models P(a_0,\dots,a_{n-1}).$$

Because $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$, then $\mathfrak{B} \models \varphi$. There exists $b_0, \dots, b_{n-1} \in B$ such that

$$\mathfrak{B} \models P(b_0,\dots,b_{n-1})$$

By (1)(2), we can know that a_i are distinct and b_i are distinct and $|A| = |B|$. Then we can construct a bijection π from A to B such that

$$\pi(a_i) = b_i \text{ for all } 0 \le i < n$$

For any n-ary relation symbol $R \in S$ and $x_0, \dots, x_{n-1} \in A$, by (3),

$$(x_0,\dots,x_{n-1}) \in R^{\mathfrak{A}} \Leftrightarrow (\pi(x_0),\dots,\pi(x_{n-1}) \in R^{\mathfrak{B}}$$

For any n-ary function symbol $R \in S$ and $x_0, \ldots, x_{n-1} \in A$, by (4),

$$\pi(f^{\mathfrak{A}}(x_0, \ldots, x_{n-1})) = f^{\mathfrak{B}}(\pi(x_0), \ldots, \pi(x_{n-1})$$

For any $c \in S$, by (5),

$$\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$$

$\square$