Sampling and Bootstrapping

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<review>

IID Random Variables

- Consider n random variables X_1, X_2, ... X_n
  - X_i are all independently and identically distributed (I.I.D.)
  - All have the same PMF (if discrete) or PDF (if continuous)
  - All have the same expectation
  - All have the same variance

IID

iid

The sum of independent, identically distributed variables:

$$Y = \sum_{i=0}^{n} X_i$$

Is normally distributed:

$$Y \sim N(n\mu, n\sigma^2)$$

where $\quad \mu = E[X_i]$

$$\sigma^2 = \mathrm{Var}(X_i)$$

By the Central Limit Theorem, the sample mean of IID variables are distributed normally.

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

</review>

Motivating Example

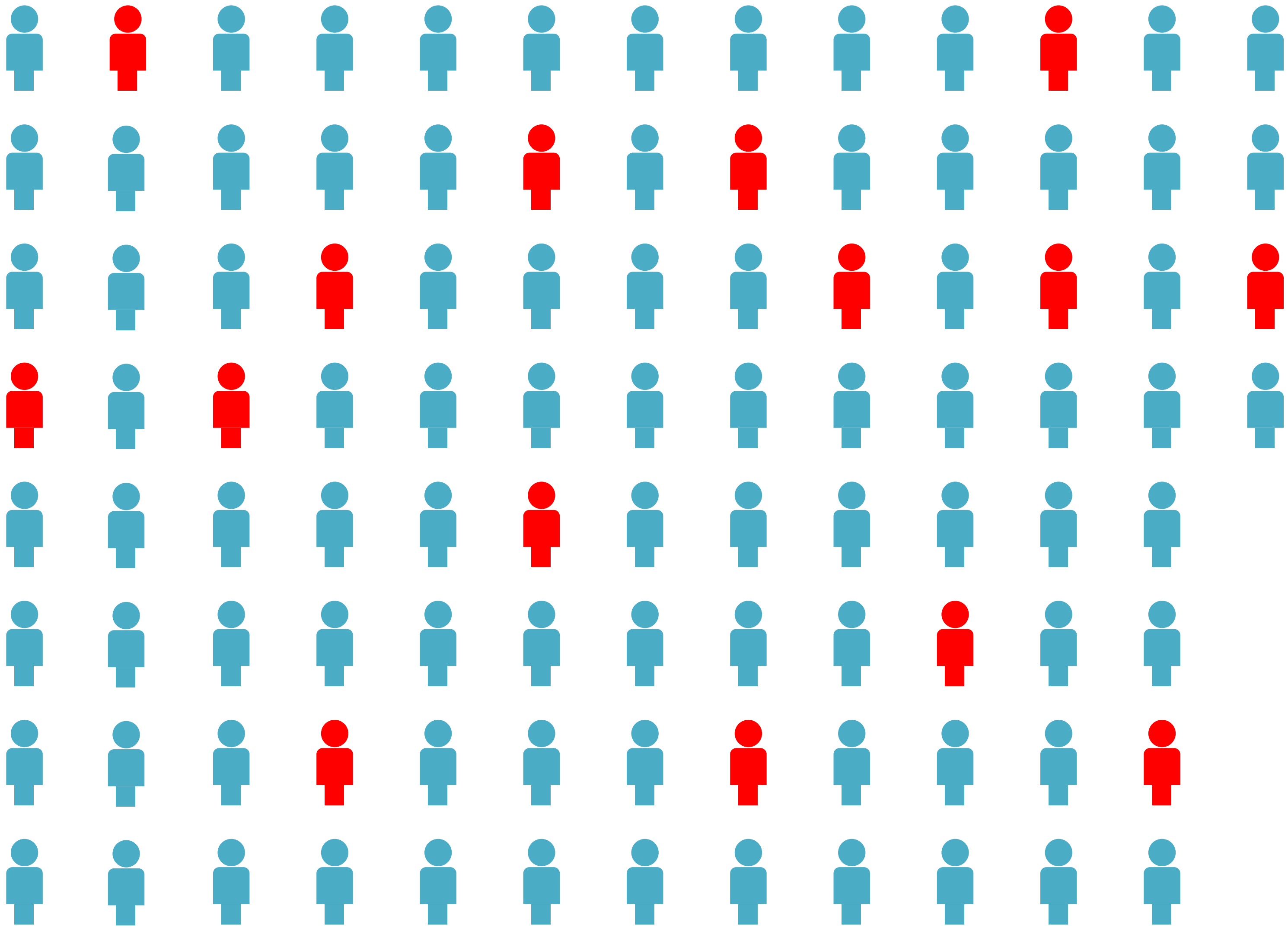
- You want to know the true mean and variance of happiness in Buthan
  - But you can’t ask everyone.
  - Randomly sample 200 people.
  - Your data looks like this:

Happiness = {72, 85, 79, 91, 68, … , 71}

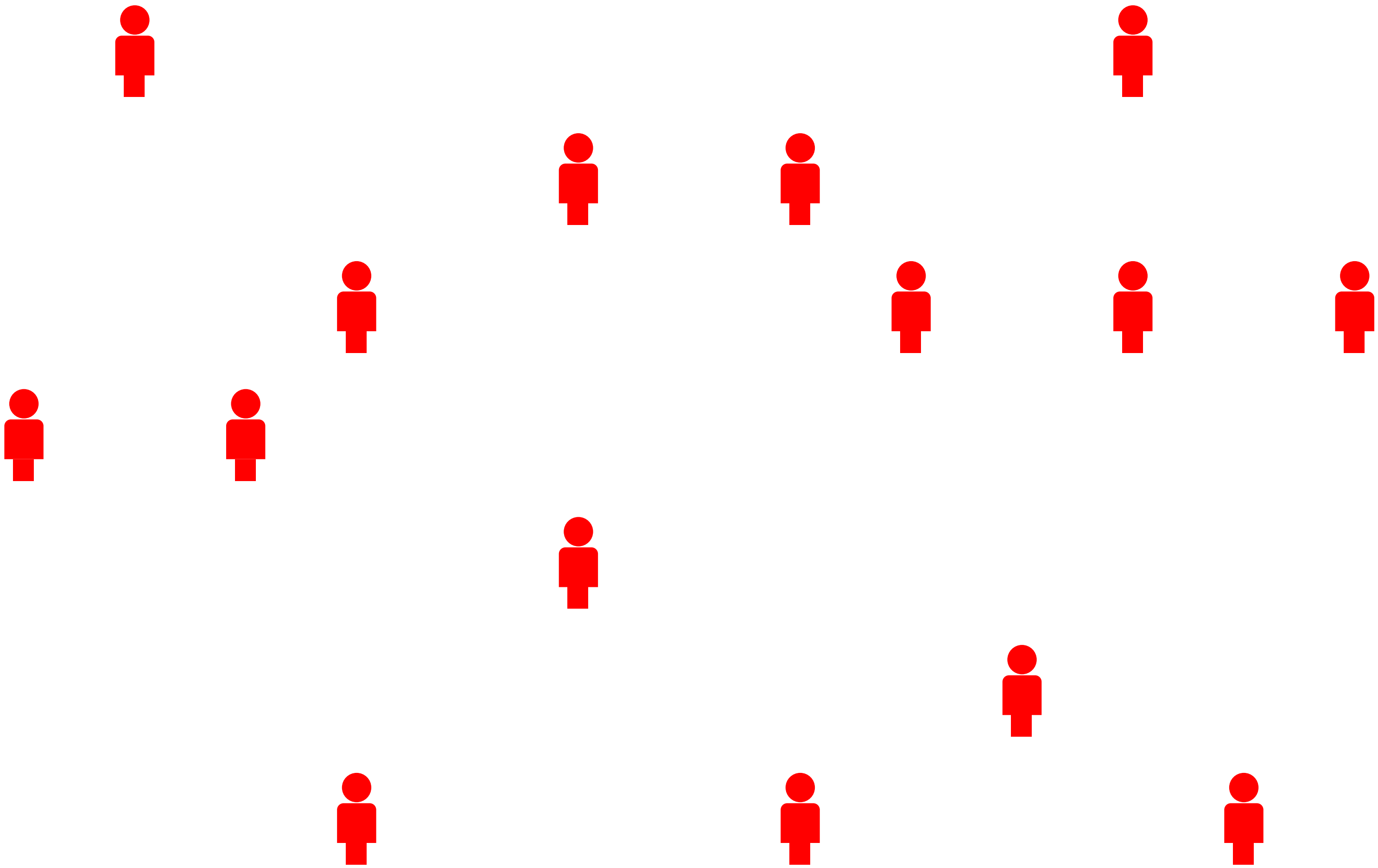
  - The mean of all of those numbers is 83. Is that the true average happiness of Bhutanese people?

Population

Sample



Sample



Collect one (or more) numbers from each person

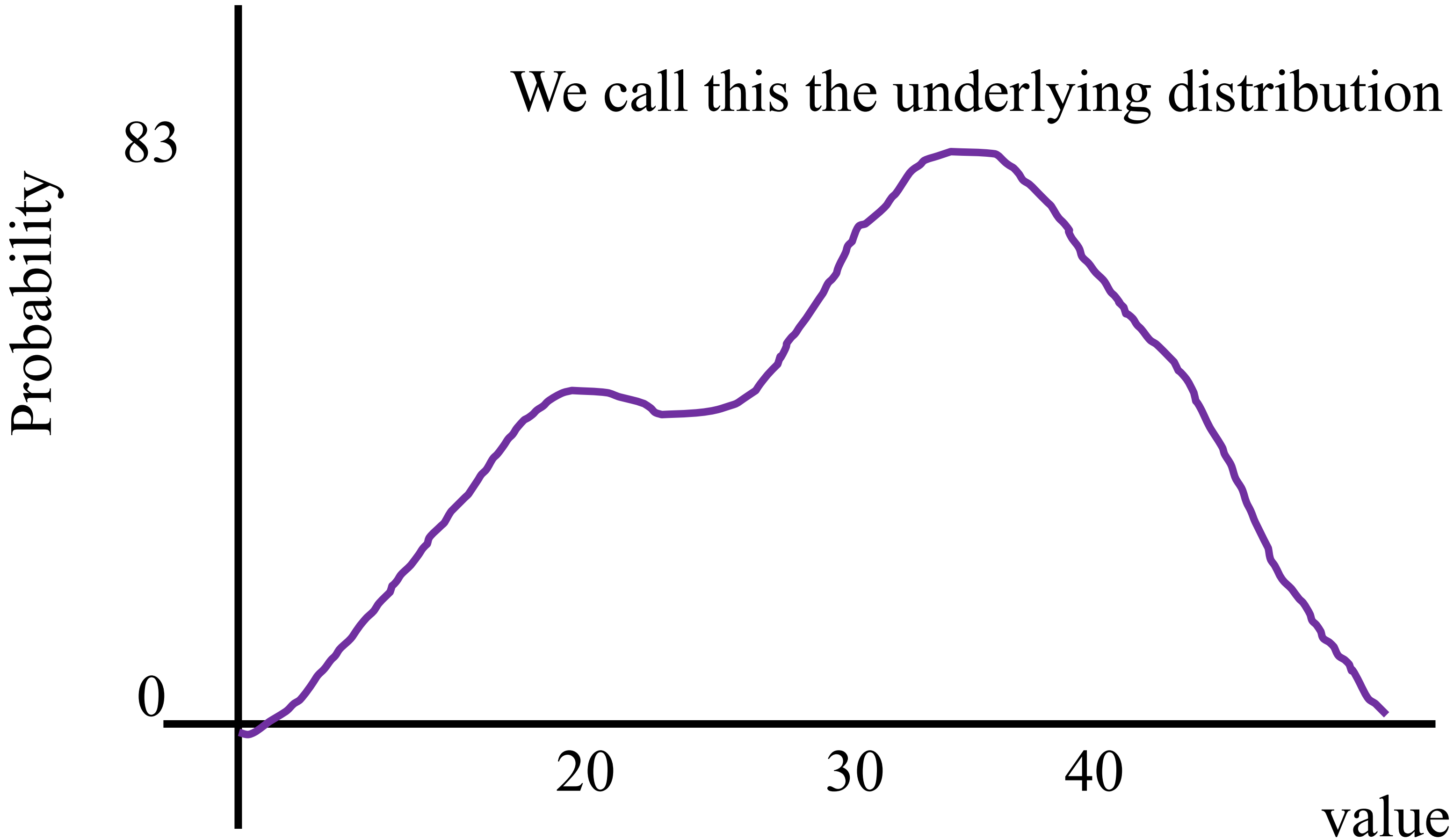
Sample

IID Samples

Consider n IID samples:

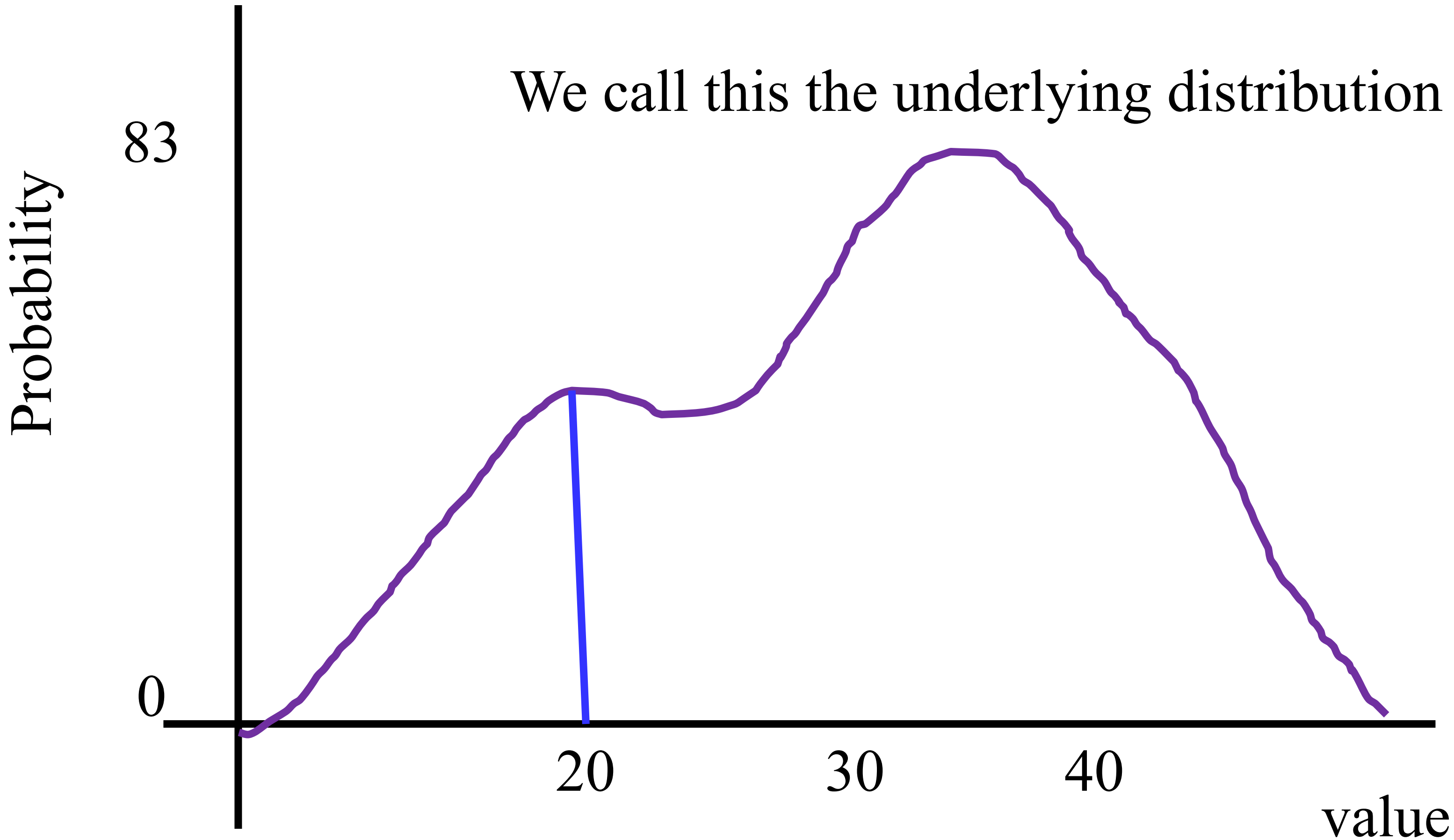
$$X_1, X_2, \ldots, X_n$$

IID Samples



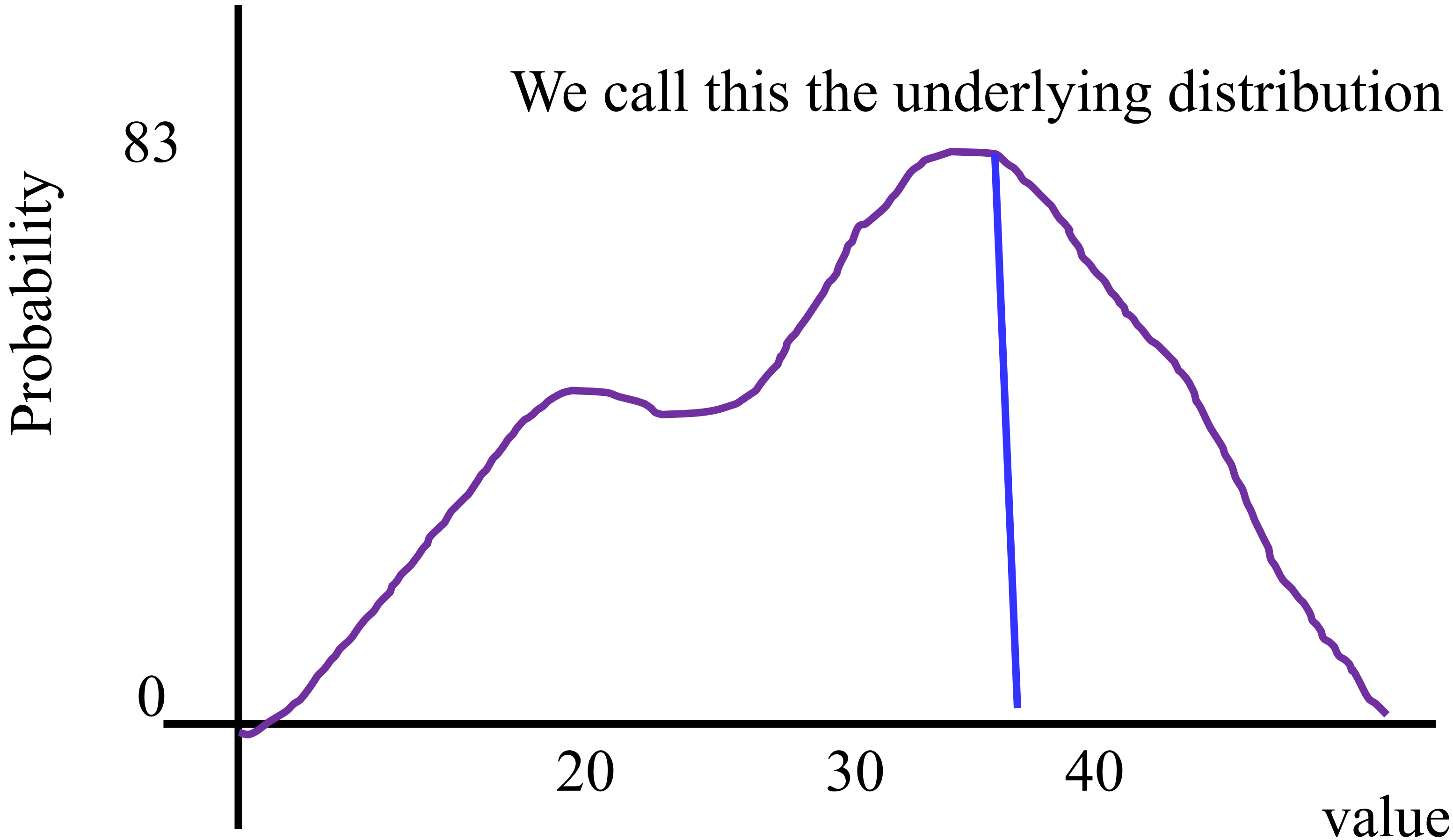
IID Samples = []

IID Samples



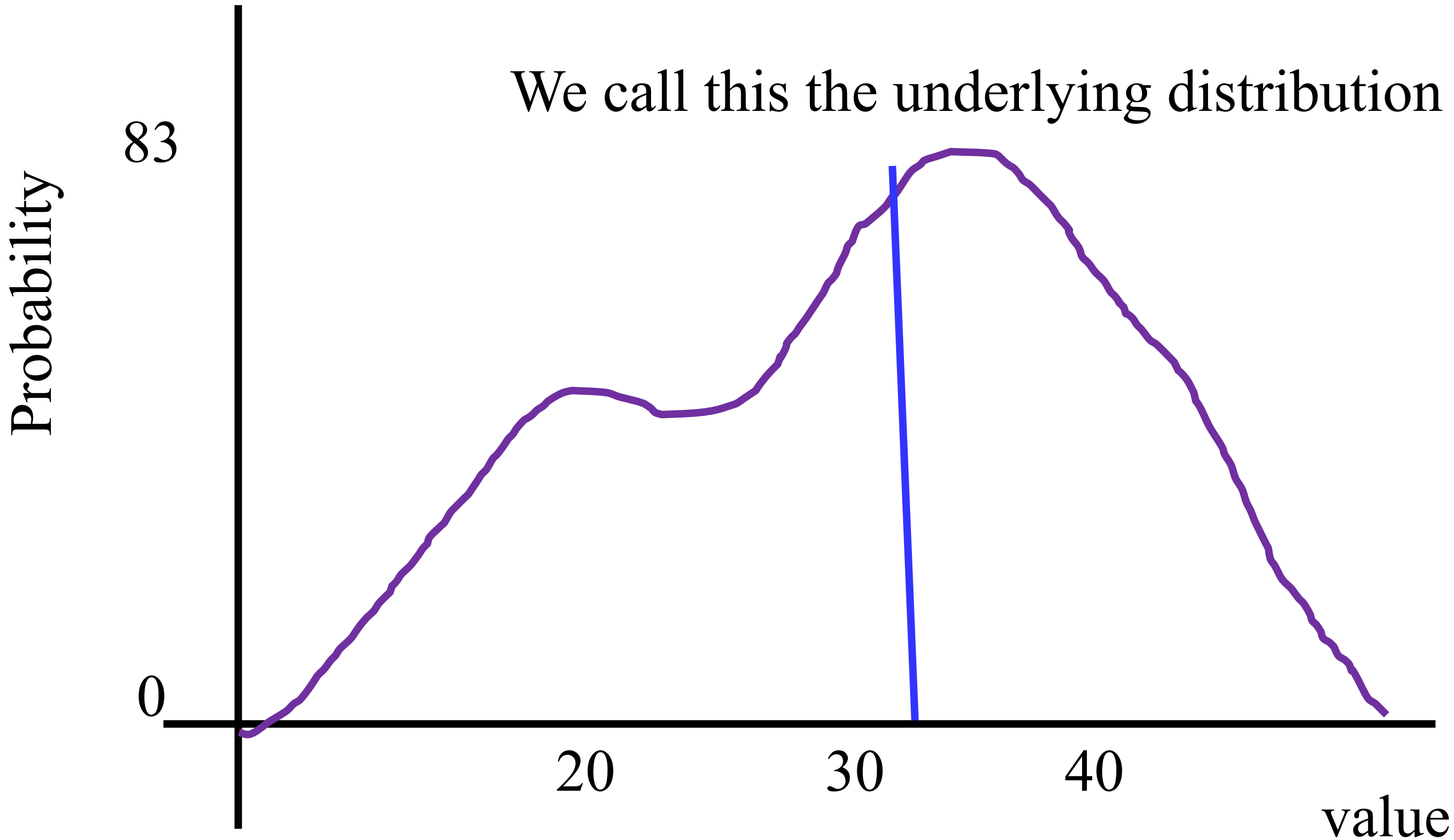
IID Samples = [20]

IID Samples



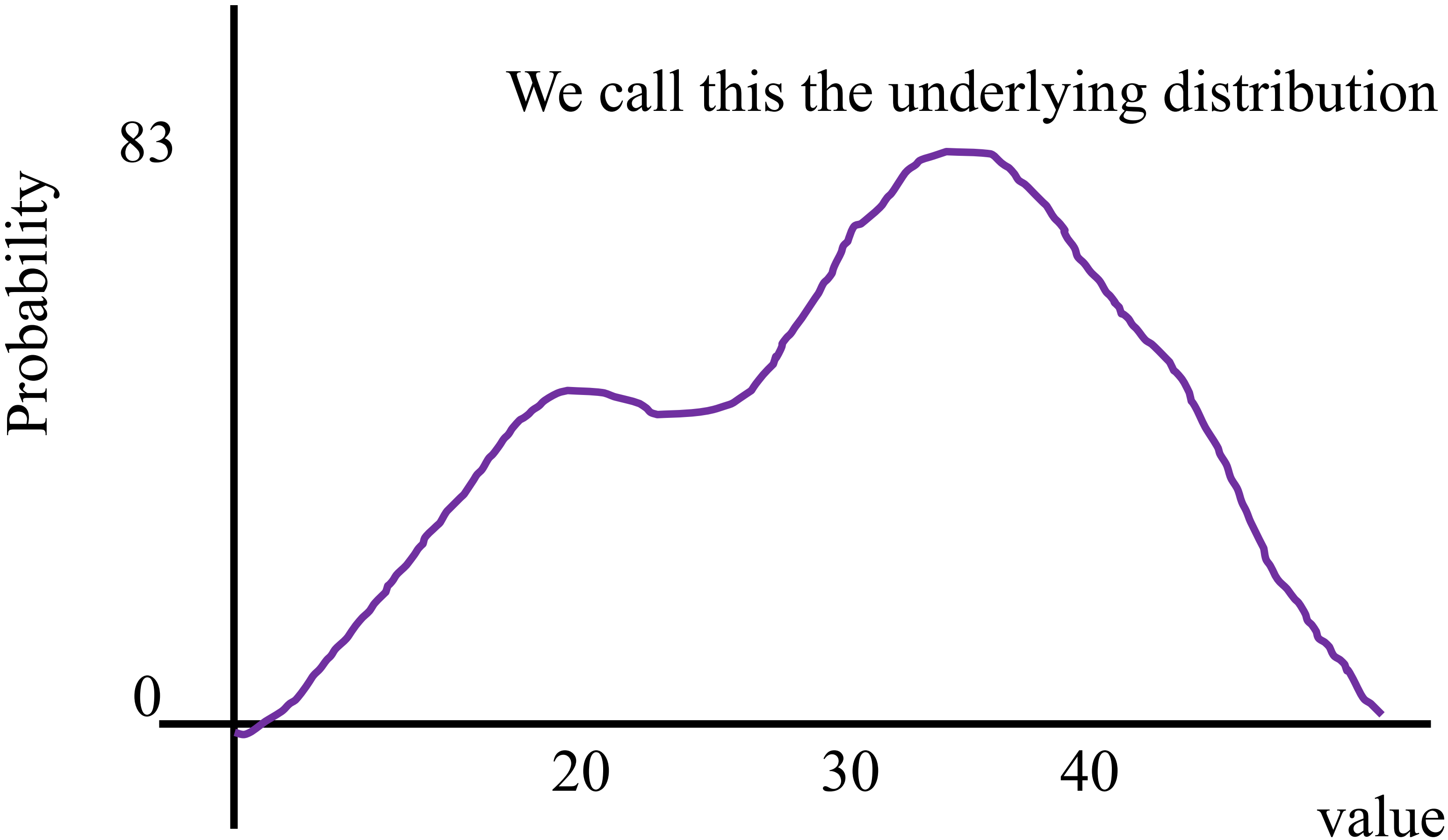
IID Samples = [20, 38]

IID Samples



IID Samples = [20, 38, 32]

IID Samples



IID Samples = [20, 38, 32, …, 38]

X_1 X_2 X_n

Sample Mean

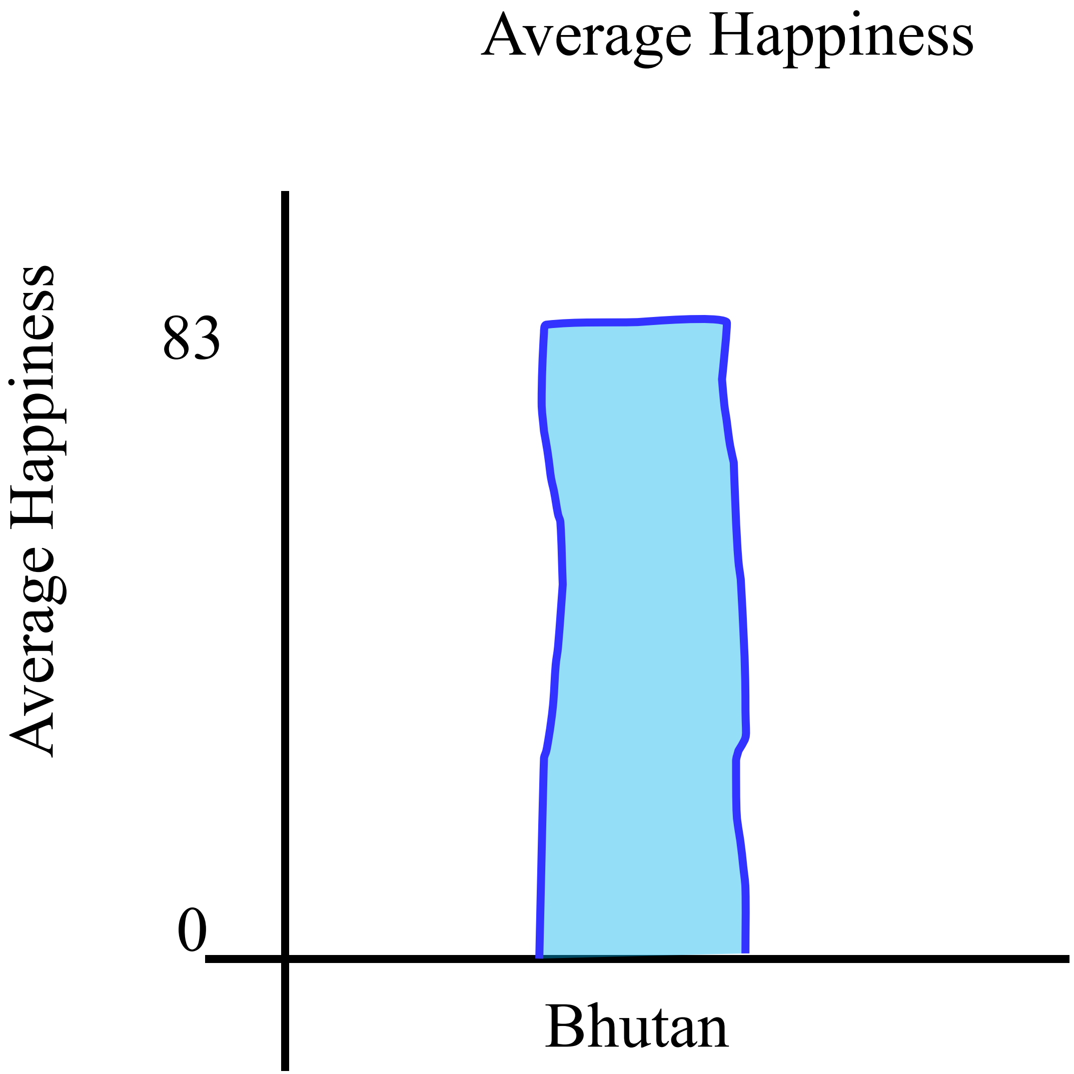
- Consider *n* random variables $X_1, X_2, \ldots X_n$
  - X_i are all independently and identically distributed (I.I.D.)
  - Have same distribution function *F* and $E[X_i] = \mu$
  - We call sequence of X_i a **sample** from distribution *F*
  - Sample mean: $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$
  - Compute $E[\bar{X}]$

$$E[\bar{X}] = E\left[\sum_{i=1}^{n} \frac{X_i}{n}\right] = \frac{1}{n} E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} E[X_i] = \frac{1}{n} \sum_{i=1}^{n} \mu = \frac{1}{n} n\mu = \mu$$

  - $\bar{X}$ is "unbiased" estimate of μ $(E[\bar{X}] = \mu)$

Sample Mean



Sample Mean:

ith sample

$$\overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}$$

Size of the sample

Sample Variance

- Consider *n* I.I.D. random variables $X_1, X_2, \ldots X_n$
  - X_i have distribution *F* with $E[X_i] = \mu$ and $Var(X_i) = \sigma^2$
  - We call sequence of X_i a **sample** from distribution *F*
  - Recall sample mean: $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$ where $E[\bar{X}] = \mu$
  - Sample deviation: $\bar{X} - X_i$ for i = 1, 2, ..., n
  - Sample variance: $S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}$

  - What is $E[S^2]$?
  - $E[S^2] = \sigma^2$
  - We say S^2 is “unbiased estimate” of σ^2

I Believe What I See

Intuition that E[S^2] = σ^2

Population variance

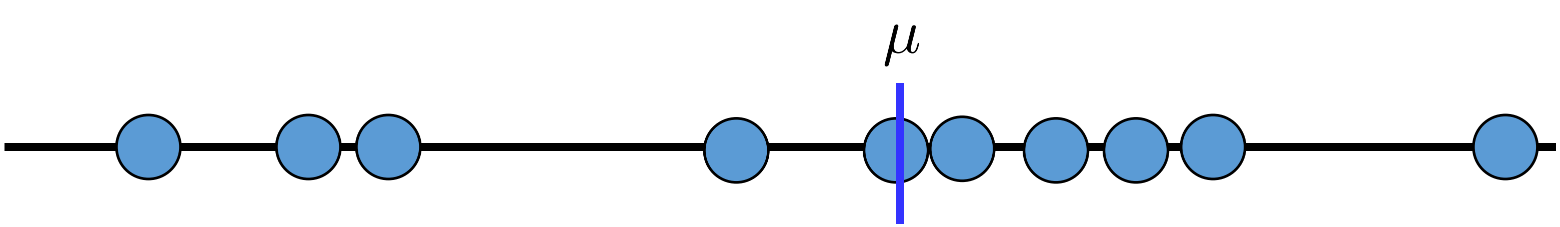
$$\sigma^2 = \sum_{i=1}^{N} \frac{(X_i - \mu)^2}{N}$$

This is the actual mean

Unbiased sample variance

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

This is the sample mean



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Population variance

$$\sigma^2 = \sum_{i=1}^{N} \frac{(X_i - \mu)^2}{N}$$

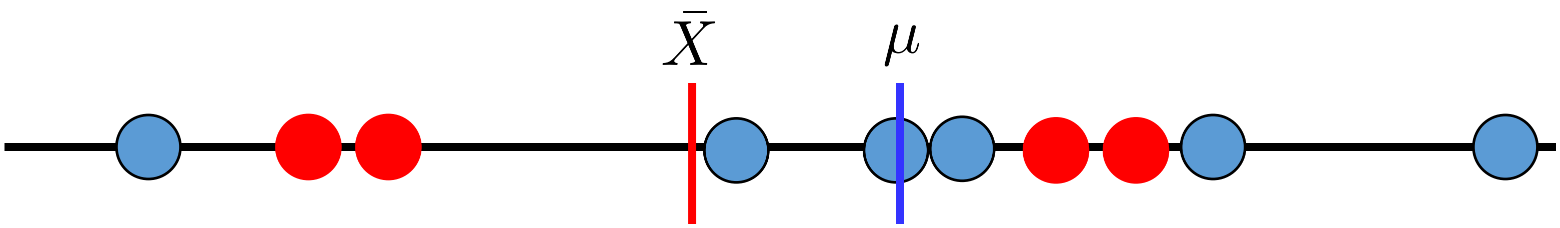
This is the actual mean

Unbiased sample variance

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}$$

This is the sample mean

The variance of the sample mean? Related to population variance



Proof that E[S^2] = σ^2 (just for reference)

$$E[S^2] = E\left[\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{n-1}\right] \Rightarrow (n-1)E[S^2] = E\left[\sum_{i=1}^{n} (X_i - \bar{X})^2\right]$$

$$(n-1)E[S^2] = E\left[\sum_{i=1}^{n} (X_i - \bar{X})^2\right] = E\left[\sum_{i=1}^{n} ((X_i - \mu) + (\mu - \bar{X}))^2\right]$$

$$= E\left[\sum_{i=1}^{n} (X_i - \mu)^2 + \sum_{i=1}^{n} (\mu - \bar{X})^2 + 2\sum_{i=1}^{n} (X_i - \mu)(\mu - \bar{X})\right]$$

$$= E\left[\sum_{i=1}^{n} (X_i - \mu)^2 + n(\mu - \bar{X})^2 + 2(\mu - \bar{X})\sum_{i=1}^{n} (X_i - \mu)\right]$$

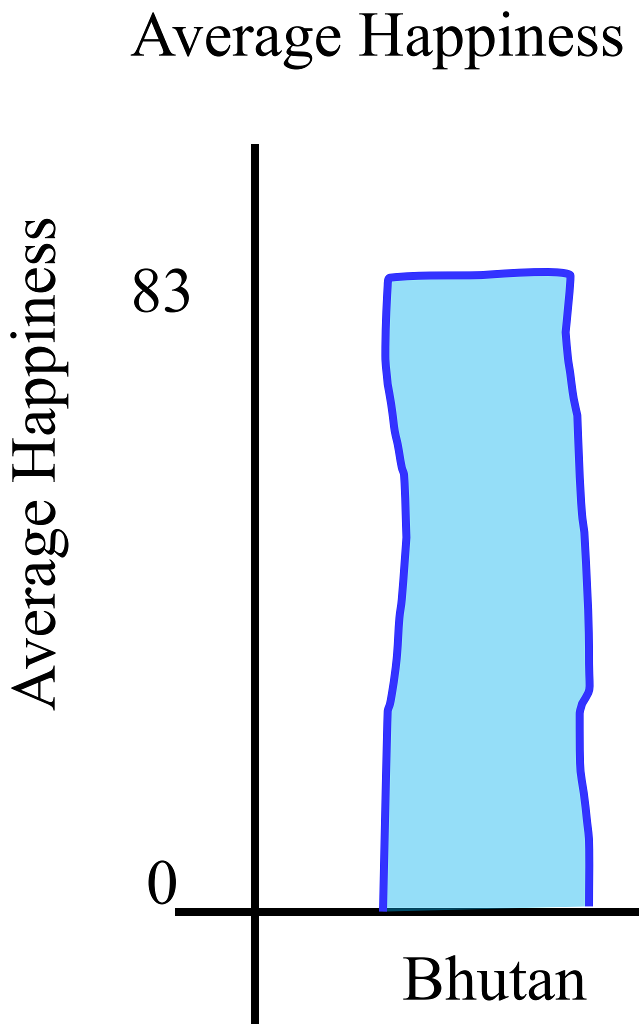
$$= E\left[\sum_{i=1}^{n} (X_i - \mu)^2 + n(\mu - \bar{X})^2 + 2(\mu - \bar{X})n(\bar{X} - \mu)\right]$$

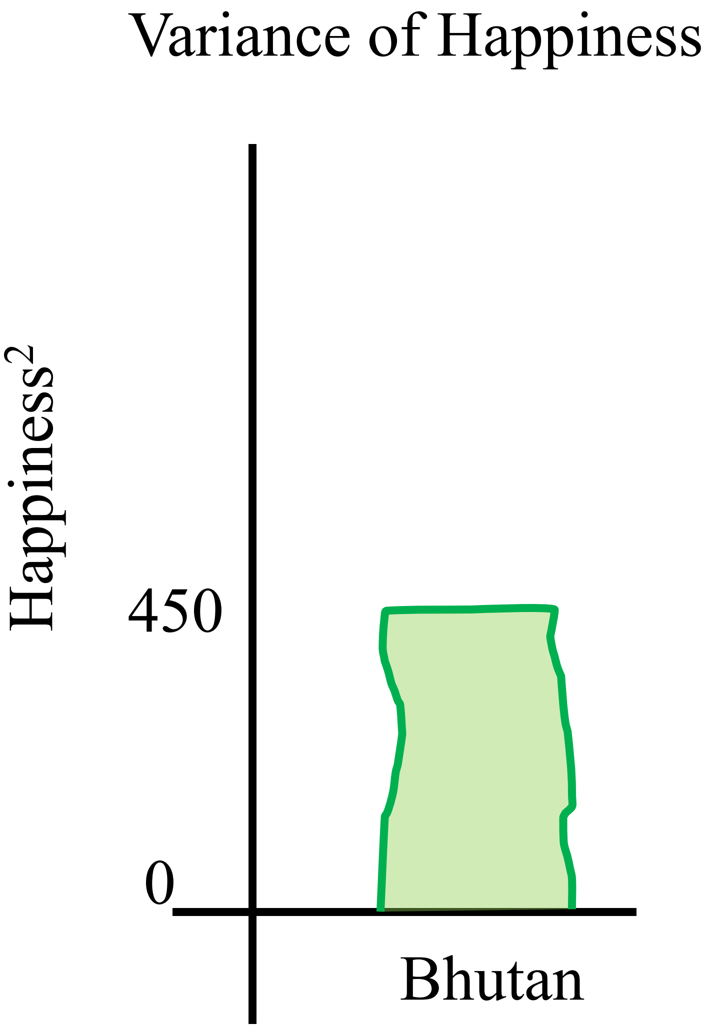
$$= E\left[\sum_{i=1}^{n} (X_i - \mu)^2 - n(\mu - \bar{X})^2\right] = \sum_{i=1}^{n} E[(X_i - \mu)^2] - nE[(\mu - \bar{X})^2]$$

$$= n\sigma^2 - n\mathrm{Var}(\bar{X}) = n\sigma^2 - n\frac{\sigma^2}{n} = n\sigma^2 - \sigma^2 = (n-1)\sigma^2$$

- So, E[S^2] = σ^2

Sample Mean


Average Happiness
Average Happiness
83
0
Bhutan


Variance of Happiness
Happiness^2
450
0
Bhutan

Sample Variance:

Sample mean

$$S^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n-1}$$

Makes it "unbiased"

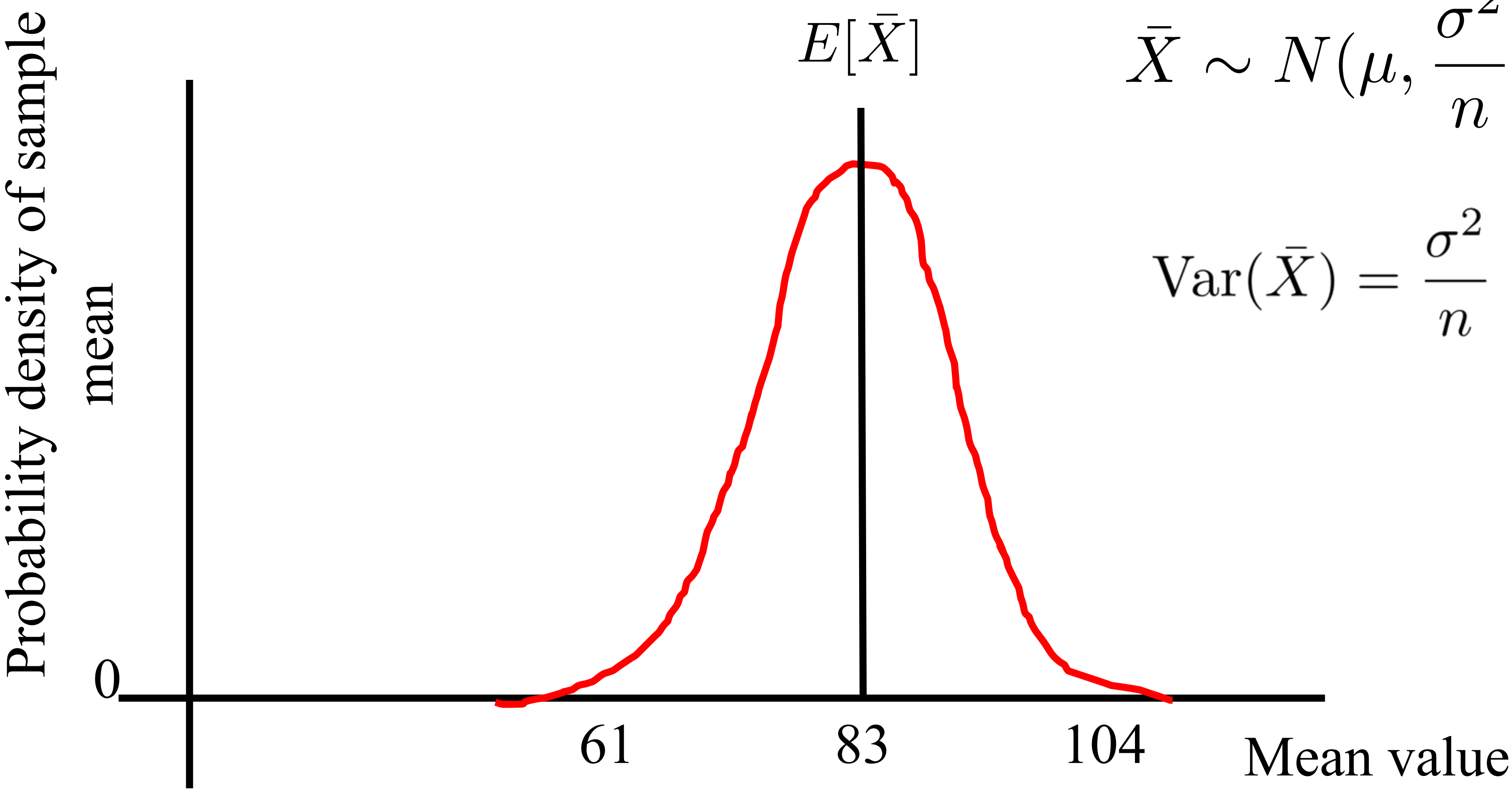
No Error Bars ☹

Variance of Sample Mean

By central limit theorem:

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$$



Variance of Sample Mean

- Consider *n* **I.I.D.** random variables $X_1, X_2, \ldots X_n$
  - X_i have distribution *F* with $E[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$
  - We call sequence of X_i a **sample** from distribution *F*
  - Recall sample mean: $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$ where $E[\bar{X}] = \mu$
  - What is $\mathrm{Var}(\bar{X})$?

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) = \left(\frac{1}{n}\right)^2 \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \mathrm{Var}(X_i) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \sigma^2 = \left(\frac{1}{n}\right)^2 n\sigma^2$$

$$= \frac{\sigma^2}{n}$$

Standard Error of the Mean

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) = \left(\frac{1}{n}\right)^2 \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{\sigma^2}{n}$$

$$\mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

$$= \frac{S^2}{n}$$

Since S_2 is an unbiased estimate

$$\mathrm{Std}(\bar{X}) = \sqrt{\frac{S^2}{n}}$$

Change variance to standard deviation

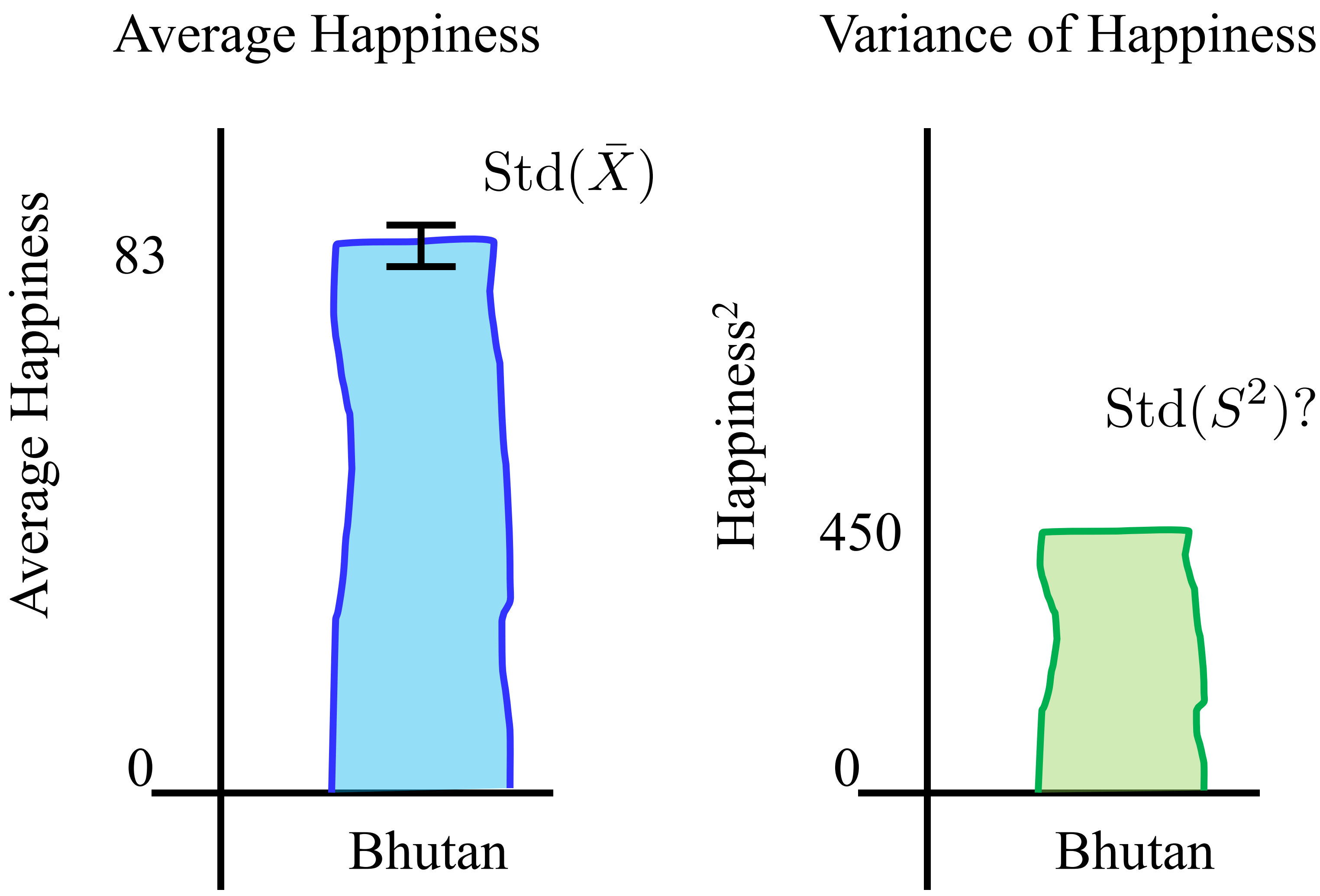
$$= \sqrt{\frac{450}{200}}$$

The numbers for our Bhutanese poll

$$= 1.5$$

Bhutanese standard error of the mean

Sample Mean



Claim: The average happiness of Bhutan is 83 ± 2

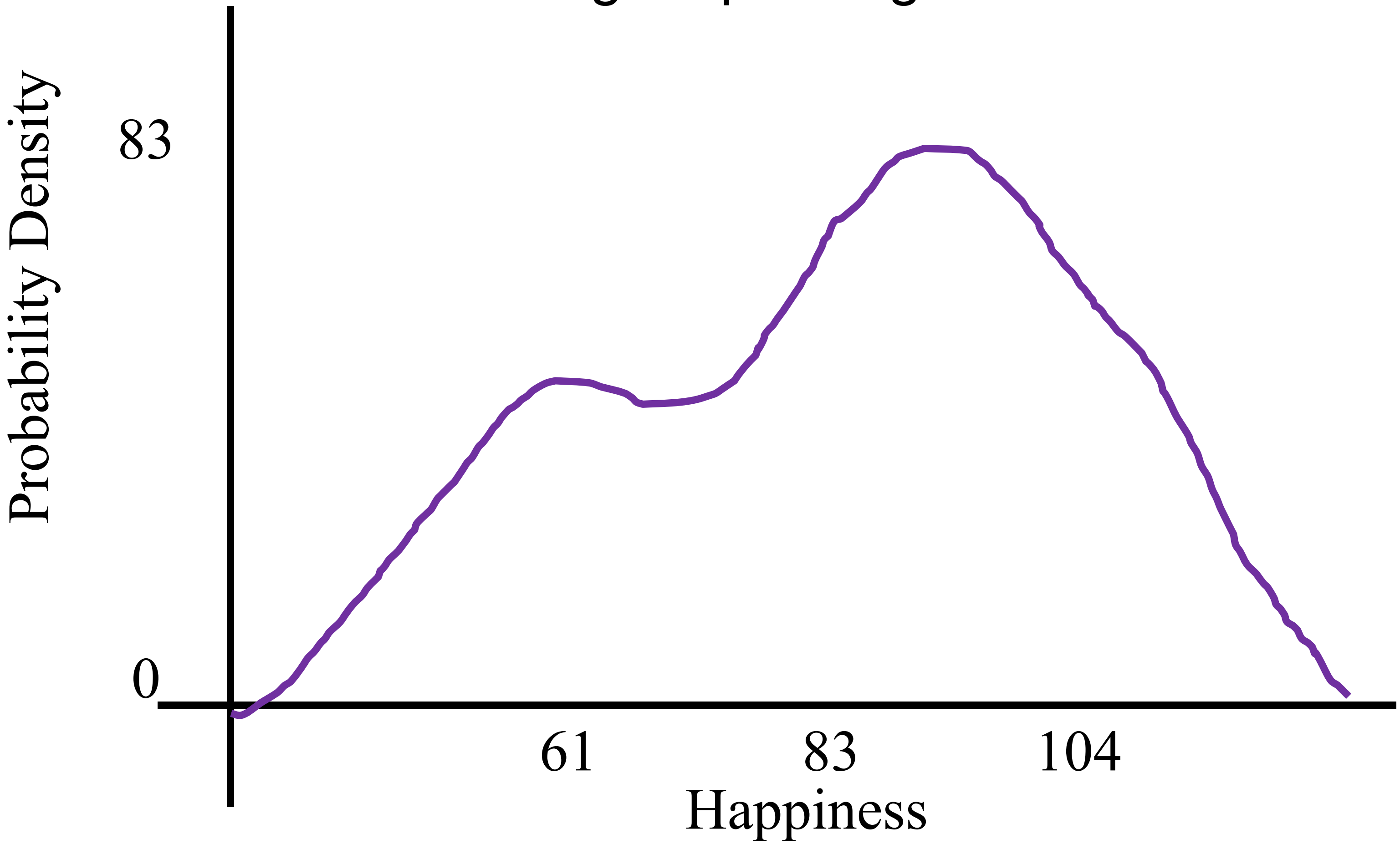
Bootstrap:
Probability for Computer Scientists

Bootstraping allows you to:

- Know the distribution of statistics
- Calculate p values

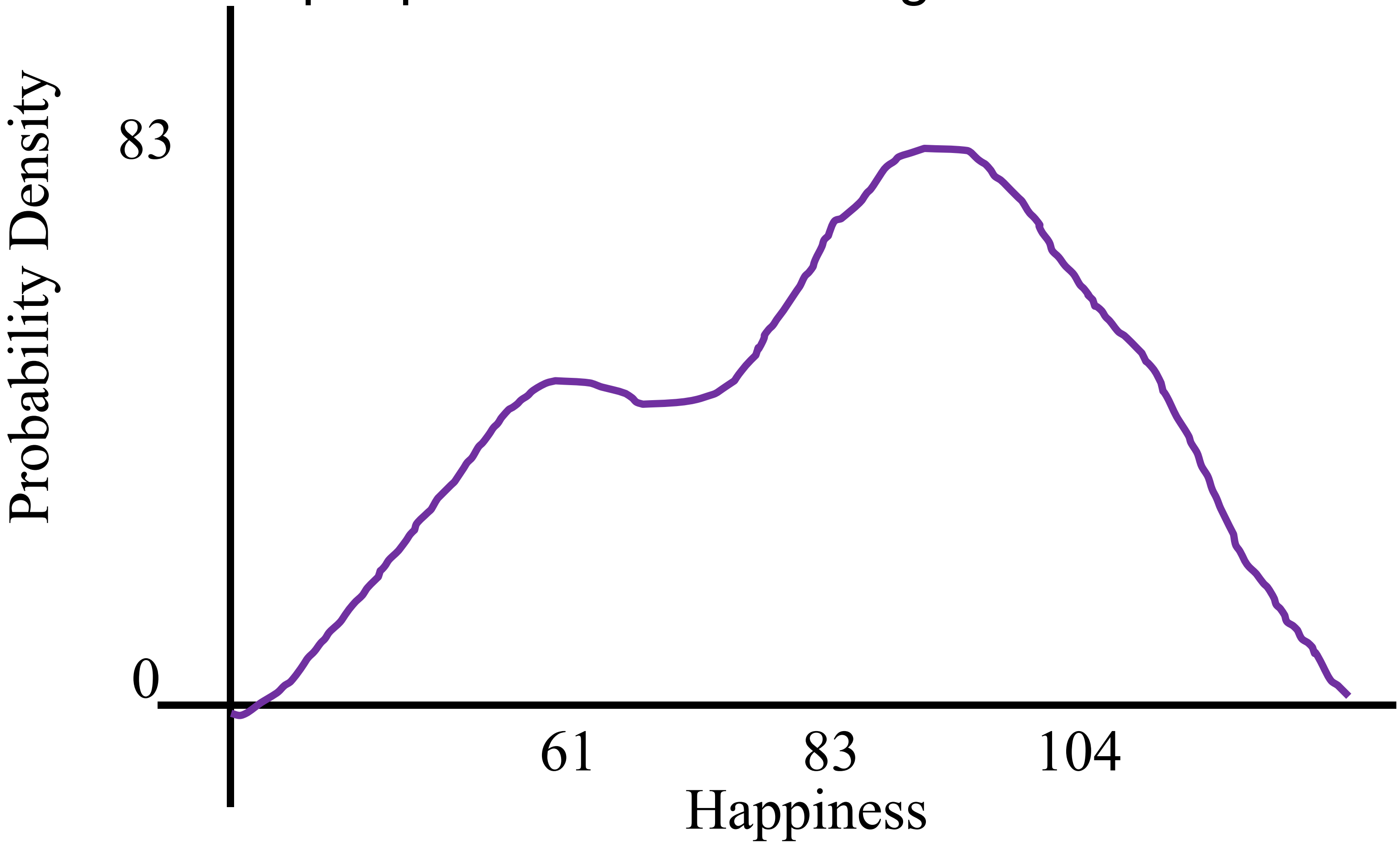
Hypothetical

What is the probability that a Bhutanese peep is just straight up loving life?



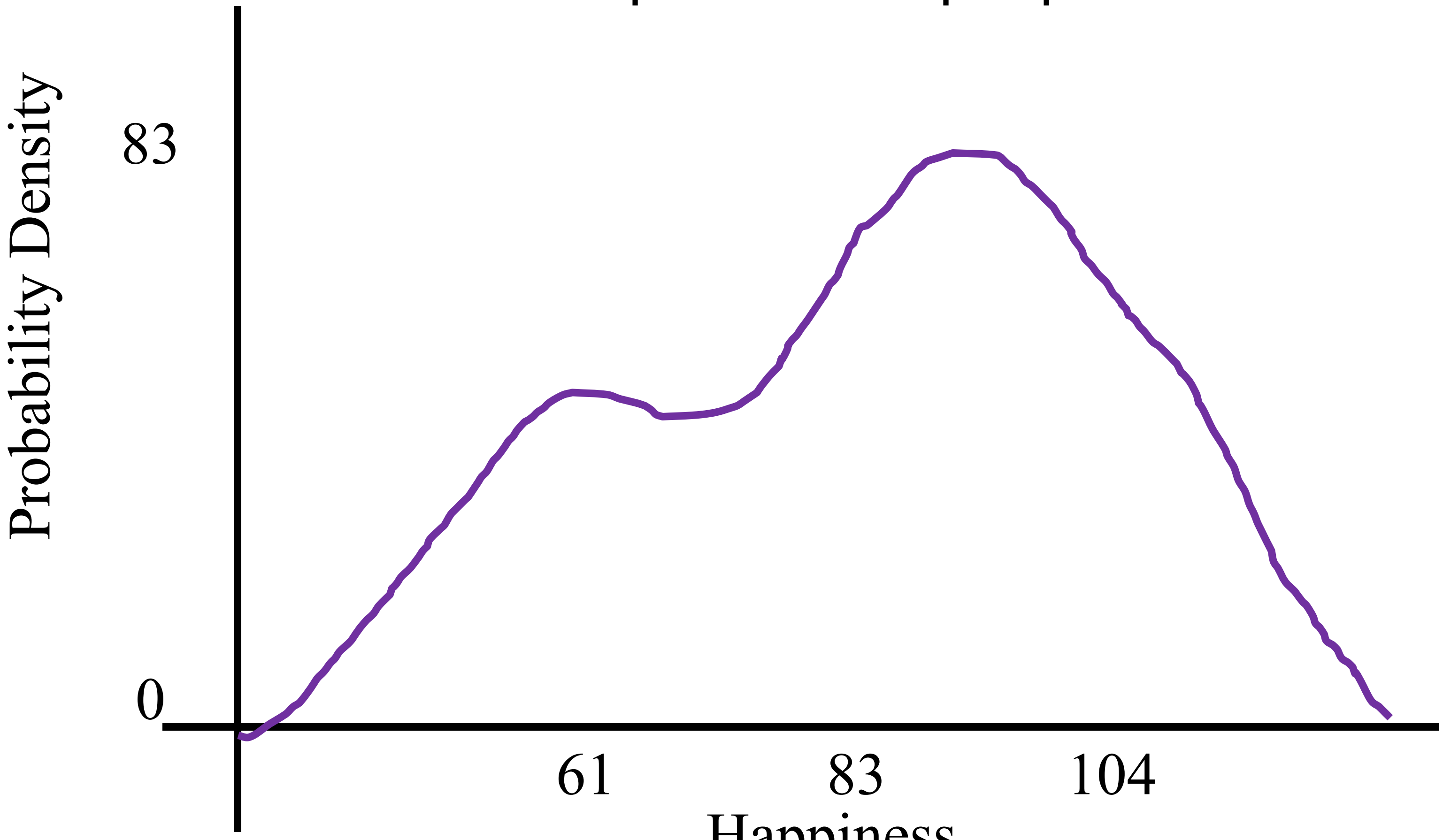
Hypothetical

What is the probability that the mean of a sample of 200 people is within the range 81 to 85?



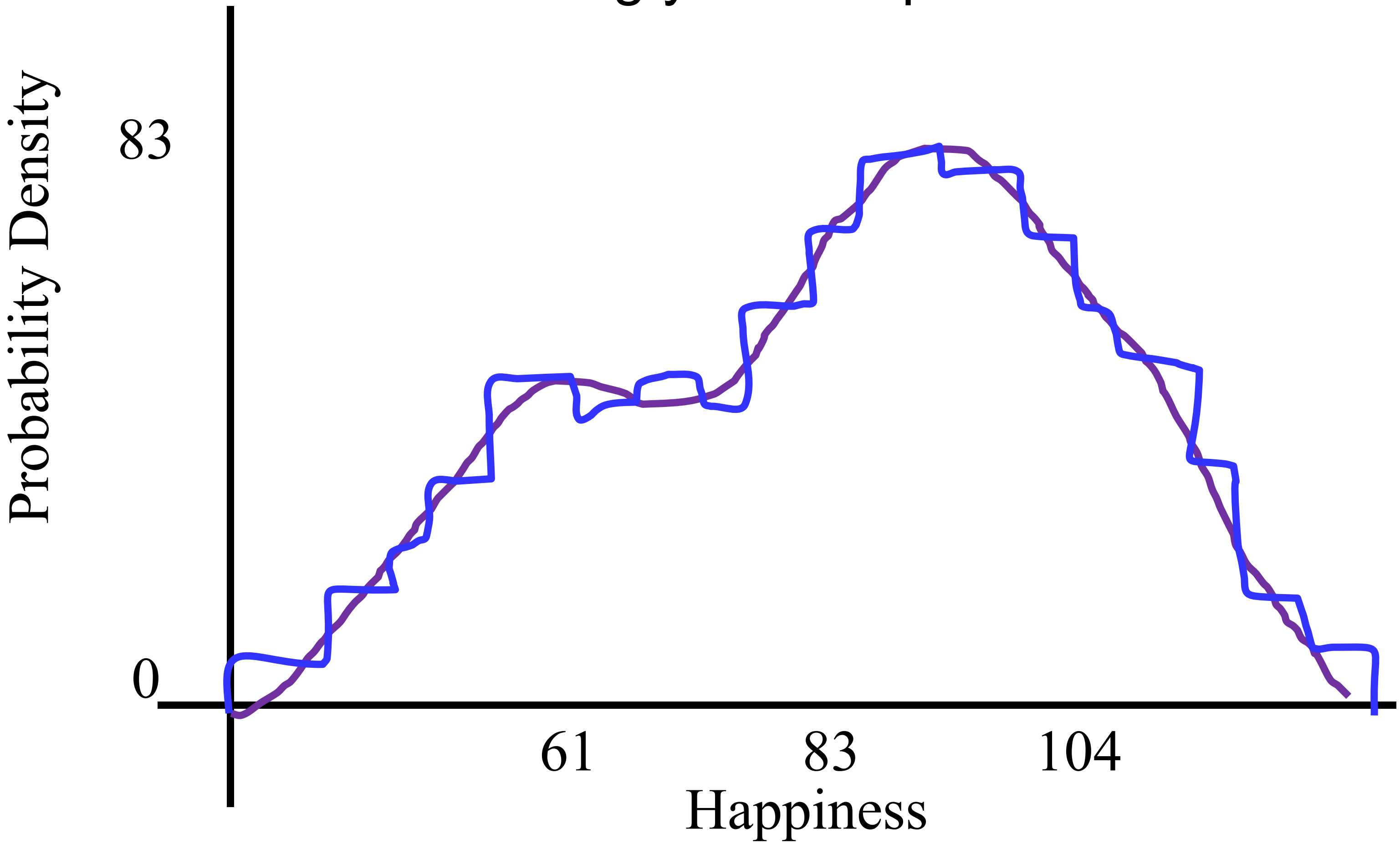
Hypothetical

What is the variance of the sample variance of subsamples of 200 people?



Key Insight

You can estimate the PMF of the underlying distribution, using your sample.*



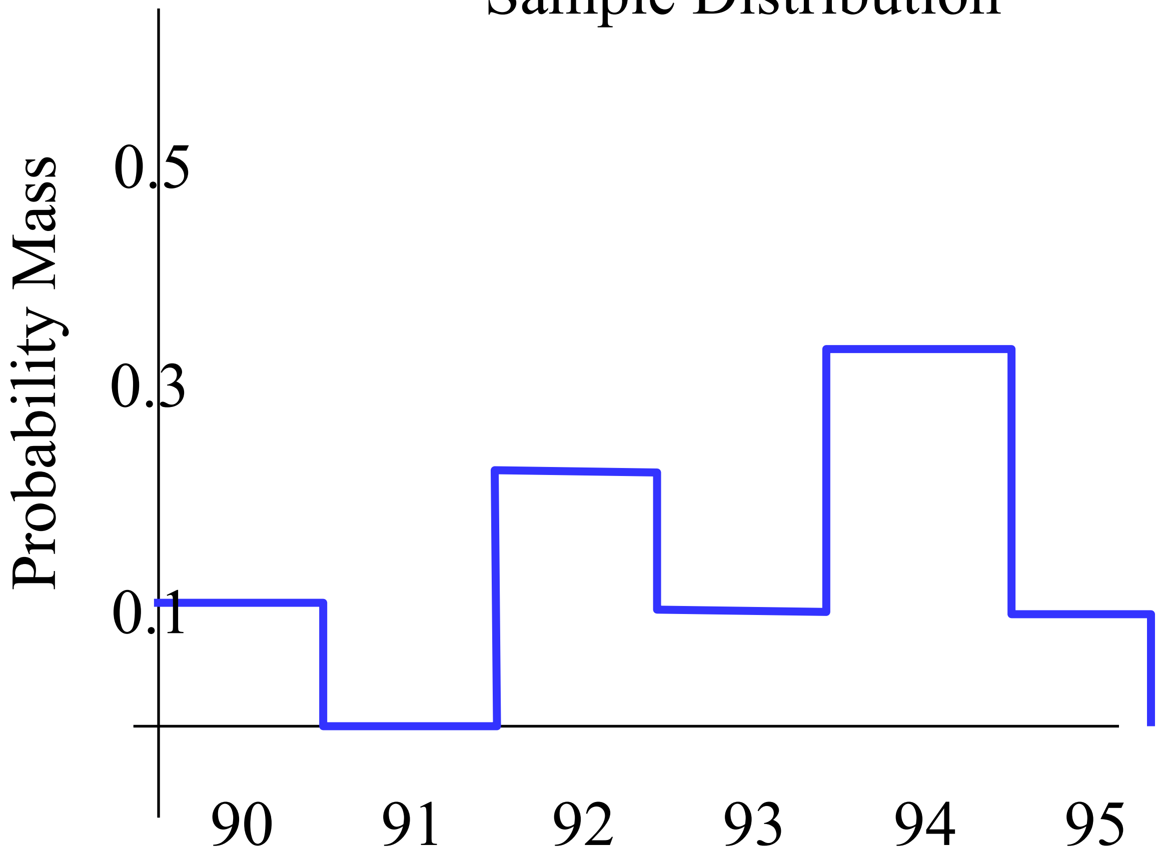
* This is just a histogram of your data!!

Key Insight

IID Samples

90,
92,
92,
93,
94,
94,
94,
95,

Sample Distribution



Bootstrapping Assumption

$$F \approx \hat{F}$$

The underlying distribution

The sample distribution

(aka the histogram of your data)

Algorithm

Bootstrap Algorithm (sample):

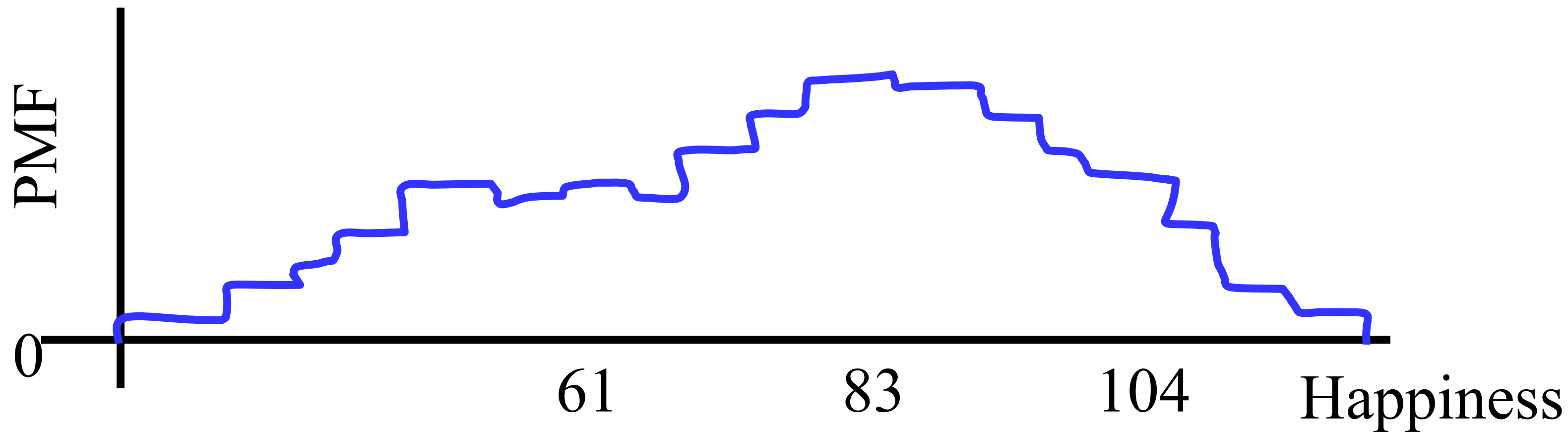
1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF
   b. Recalculate the stat on the resample
3. You now have a **distribution of your stat**

Bootstrap of Means

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Draw **sample.size**() new samples from PMF
   b. Recalculate the mean on the resample
3. You now have a **distribution of your means**

Bootstrap of Means



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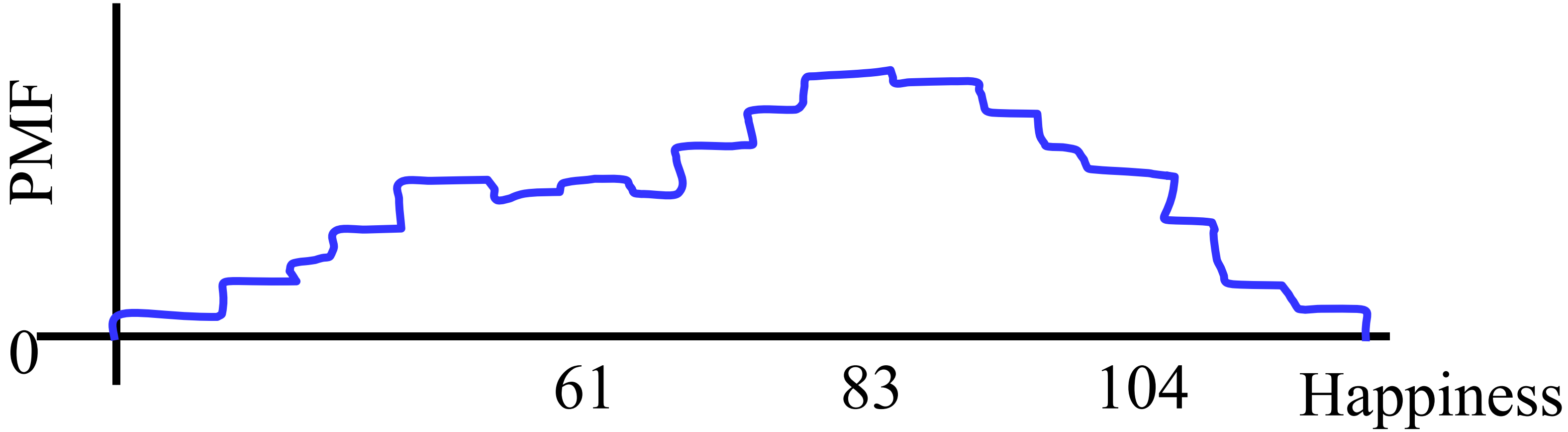

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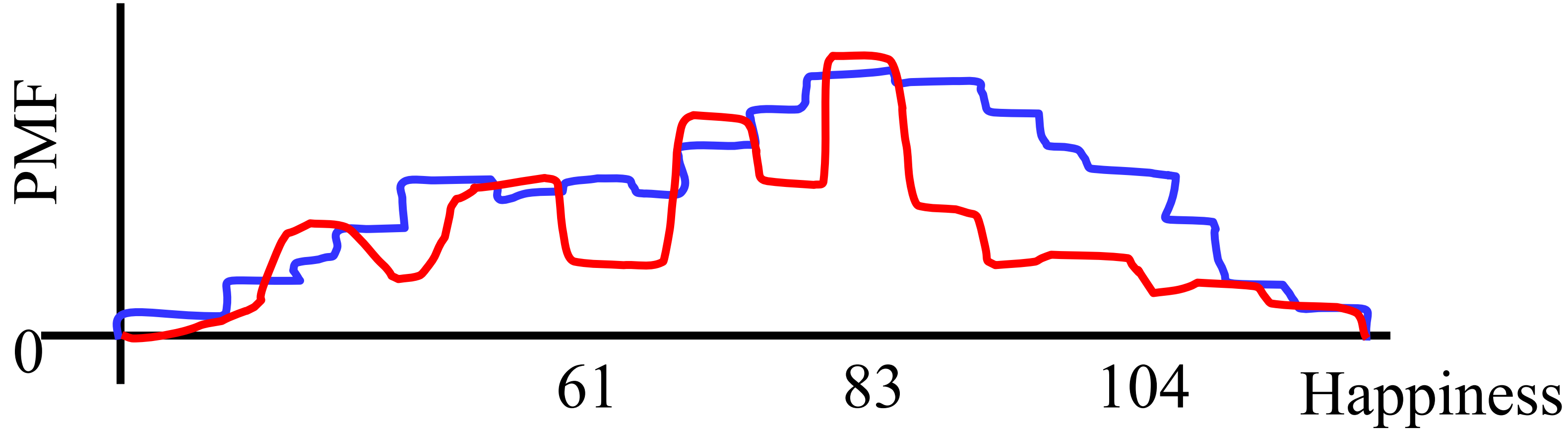

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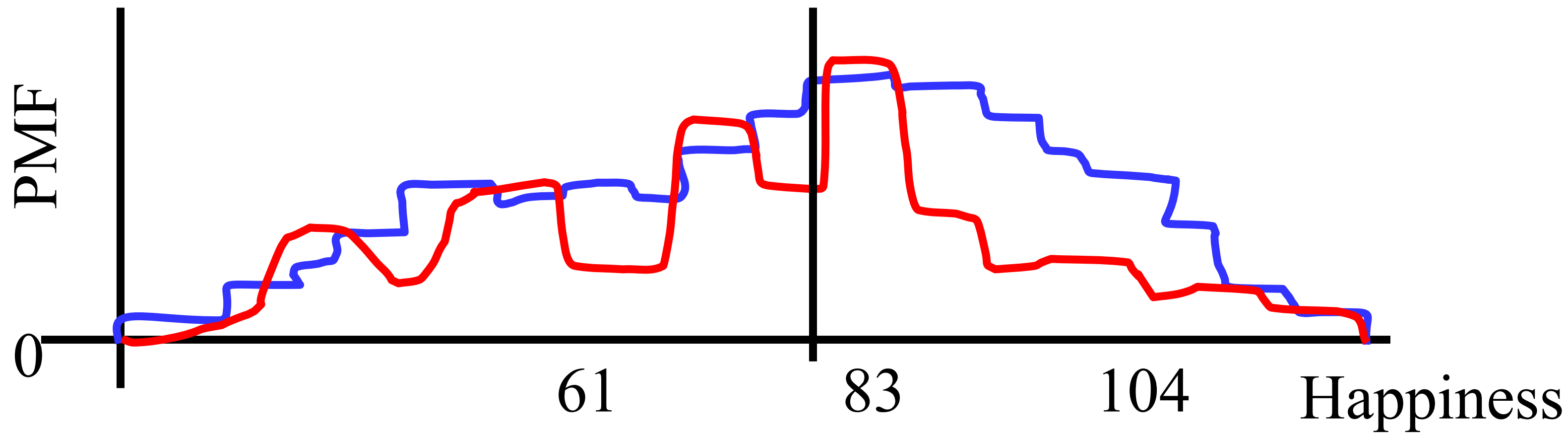

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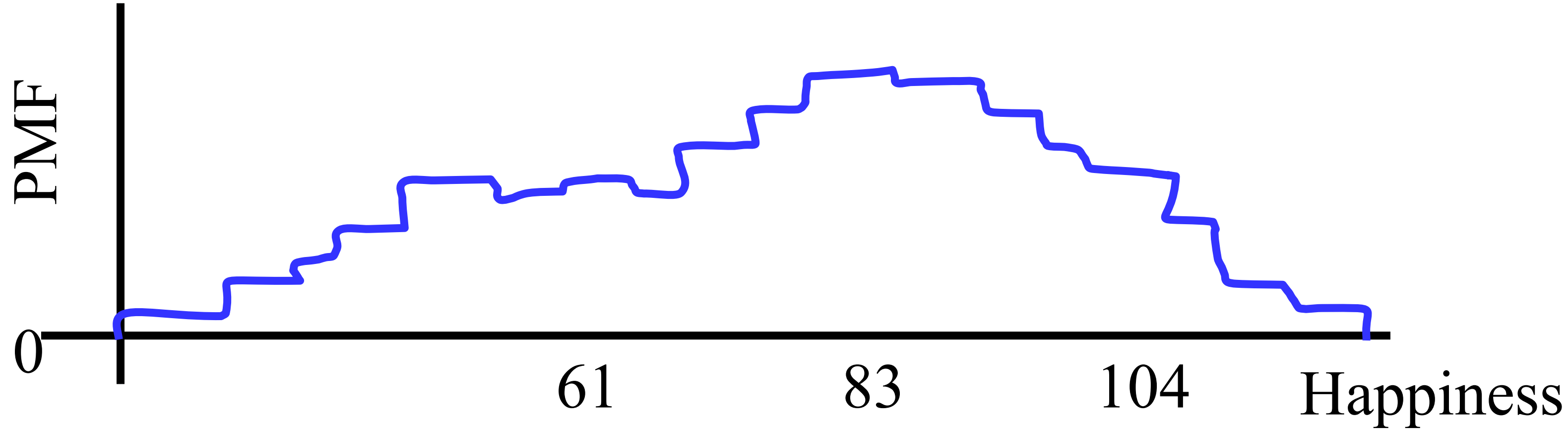
Means = [82.7]

Bootstrap of Means



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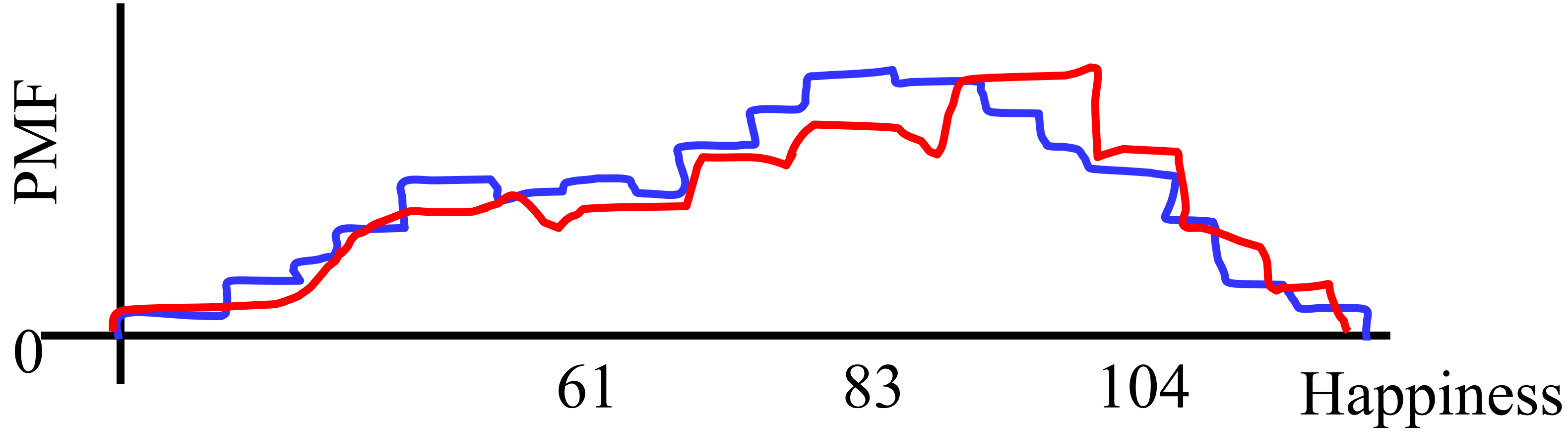

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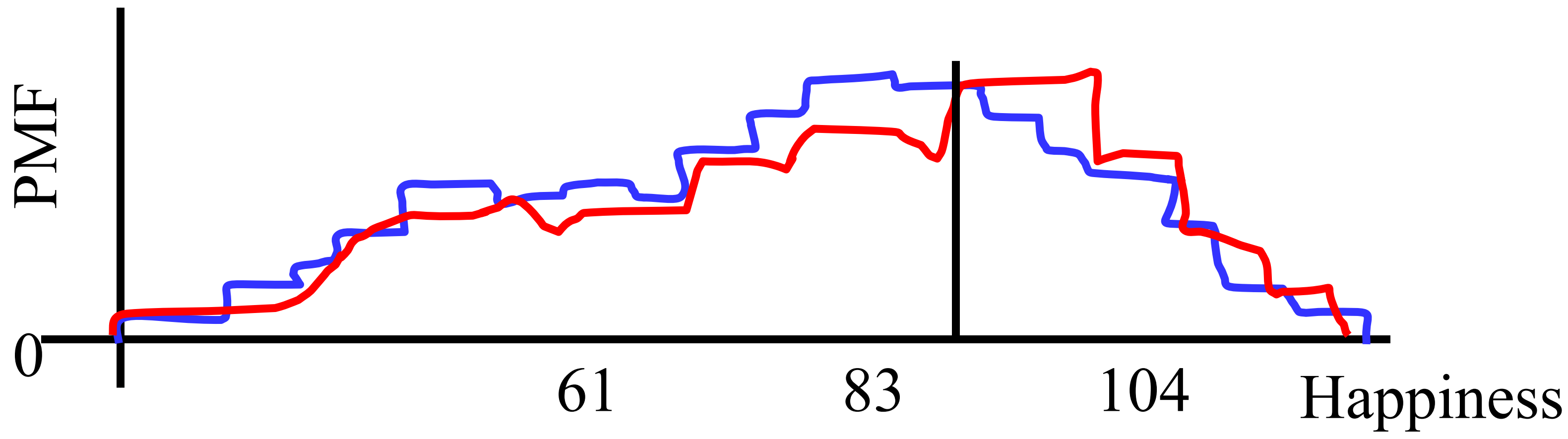

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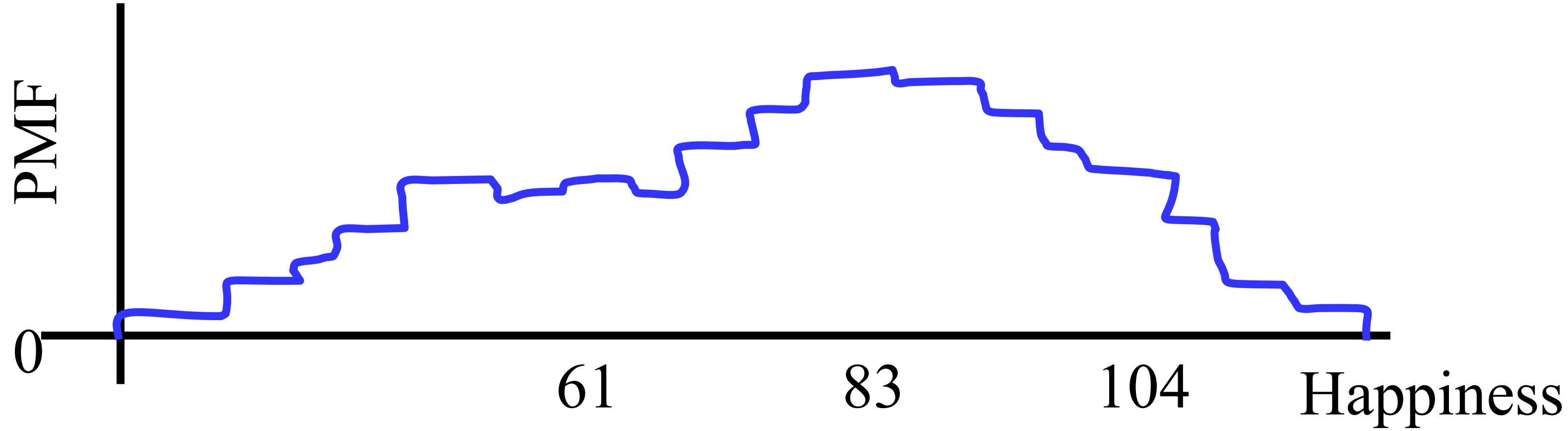
Means = [82.7, 83.4]

Bootstrap of Means



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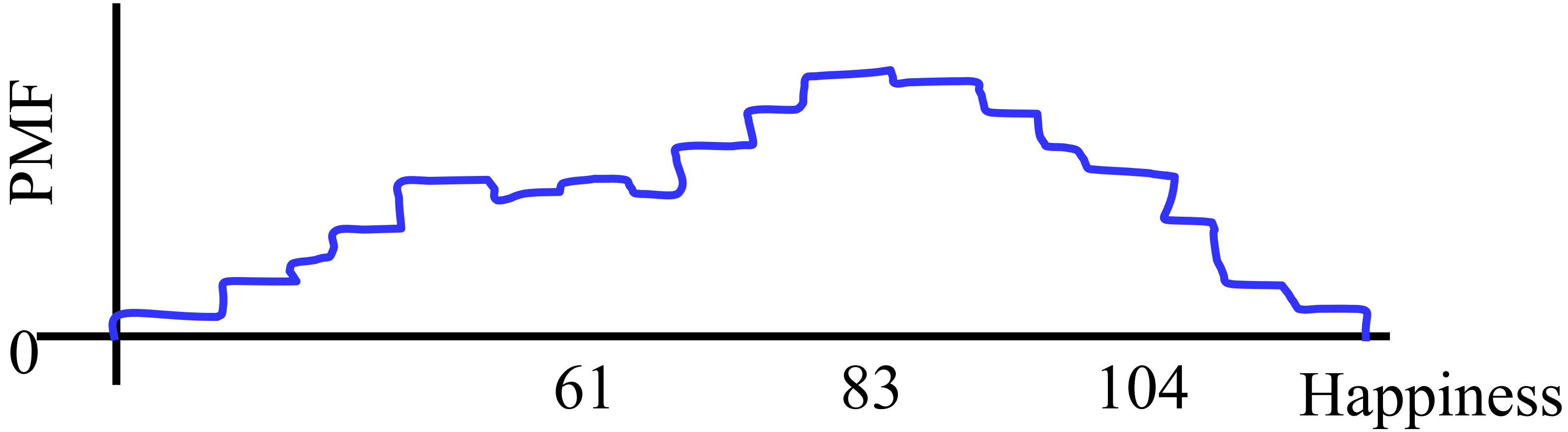

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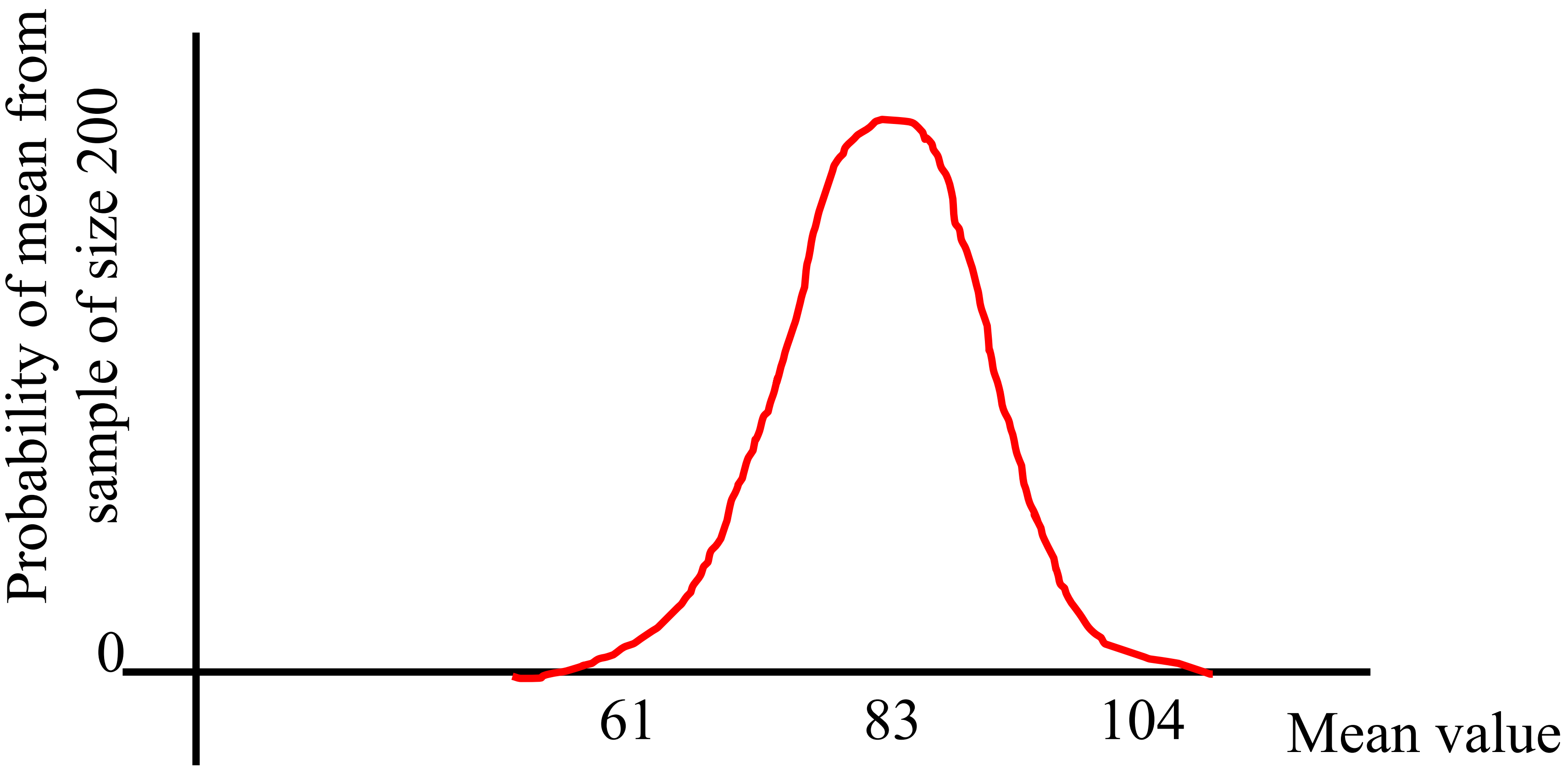
3. You now have a **distribution of your means**

Means = [82.7, 83.4, 82.9, 91.4, 79.3, 82.1, …, 81.7]

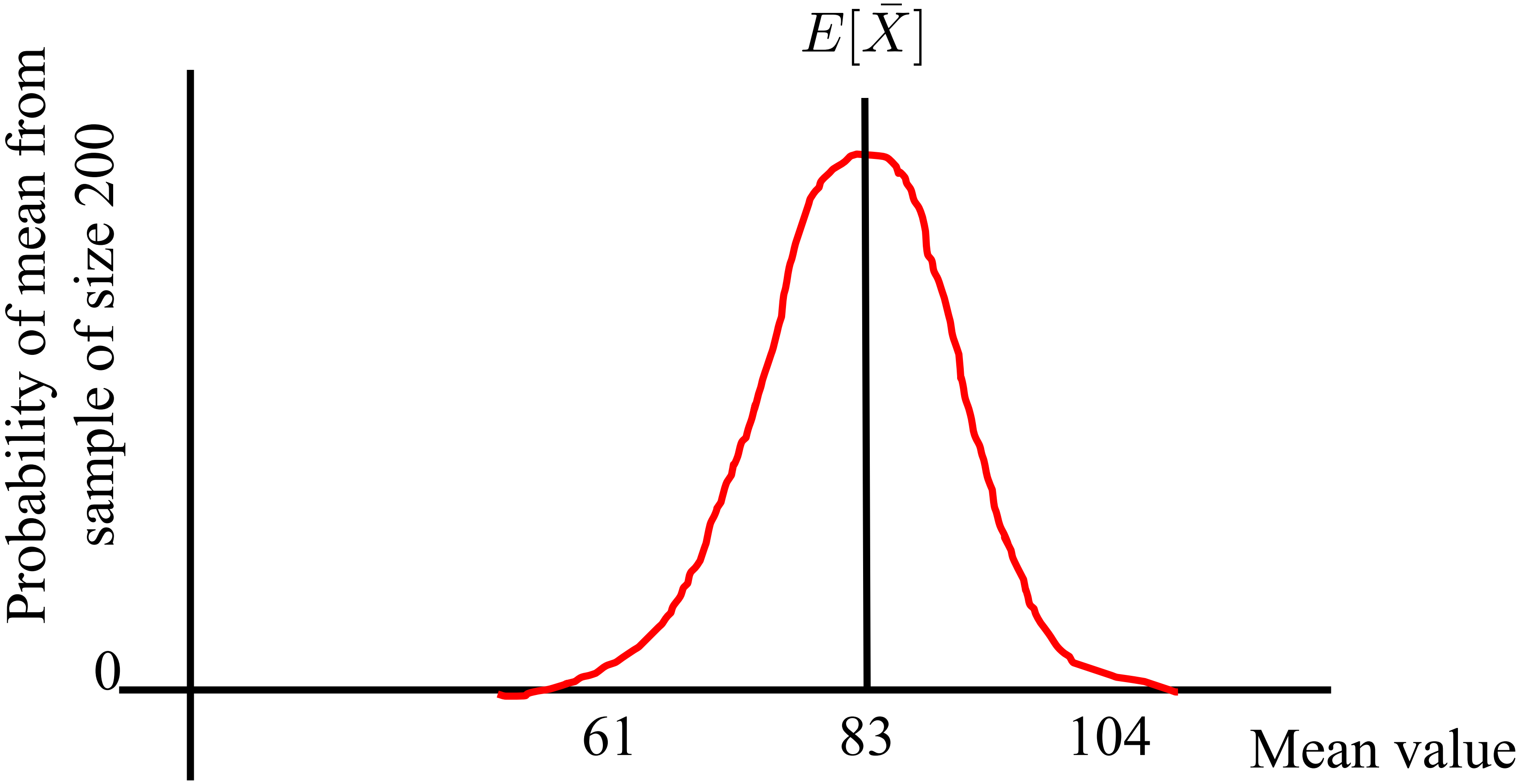
Bootstrap of Means

Means = [82.7, 83.4, 82.9, 91.4, 79.3, 82.1, …, 81.7]



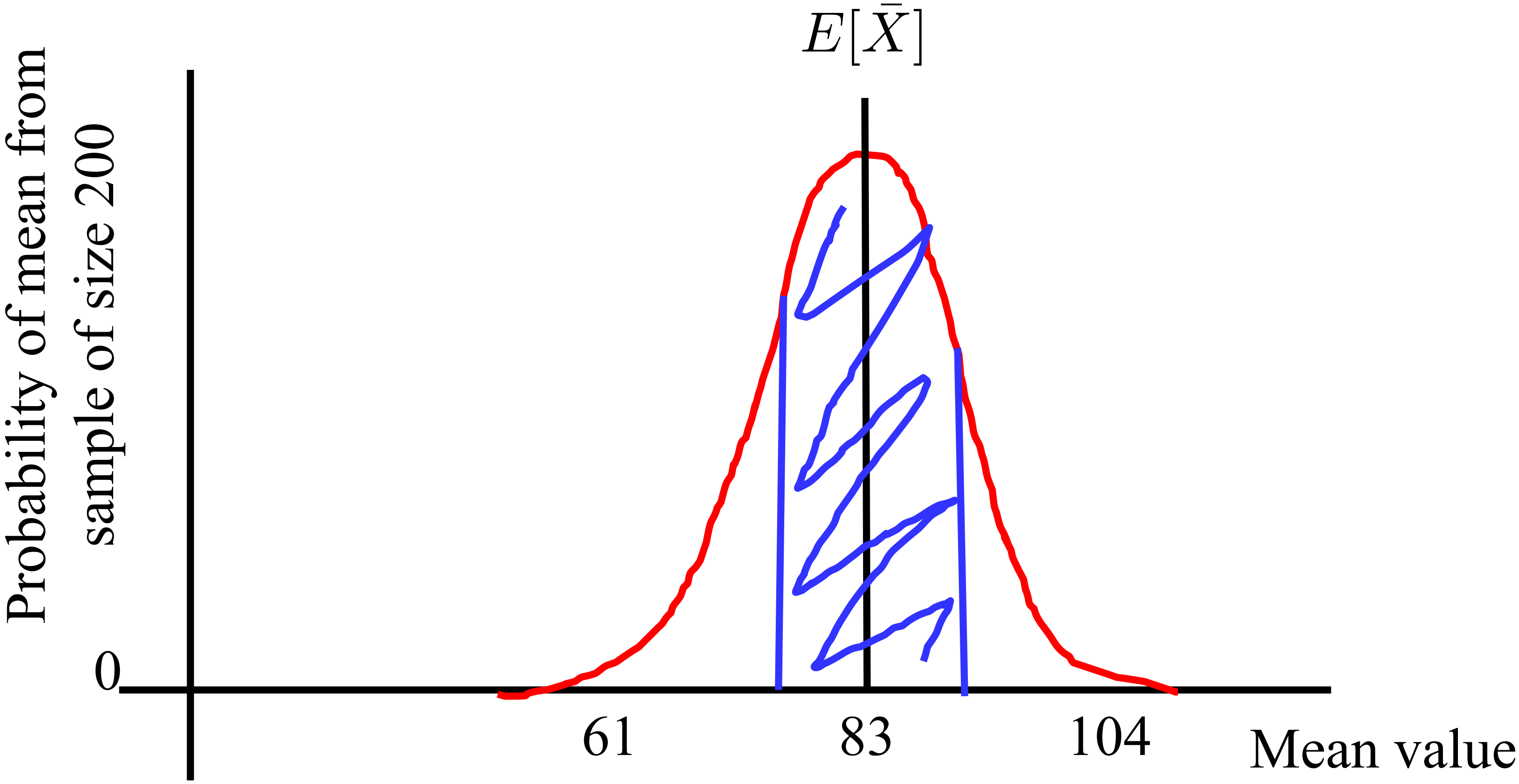
Bootstrap of Means

Means = [82.7, 83.4, 82.9, 91.4, 79.3, 82.1, …, 81.7]

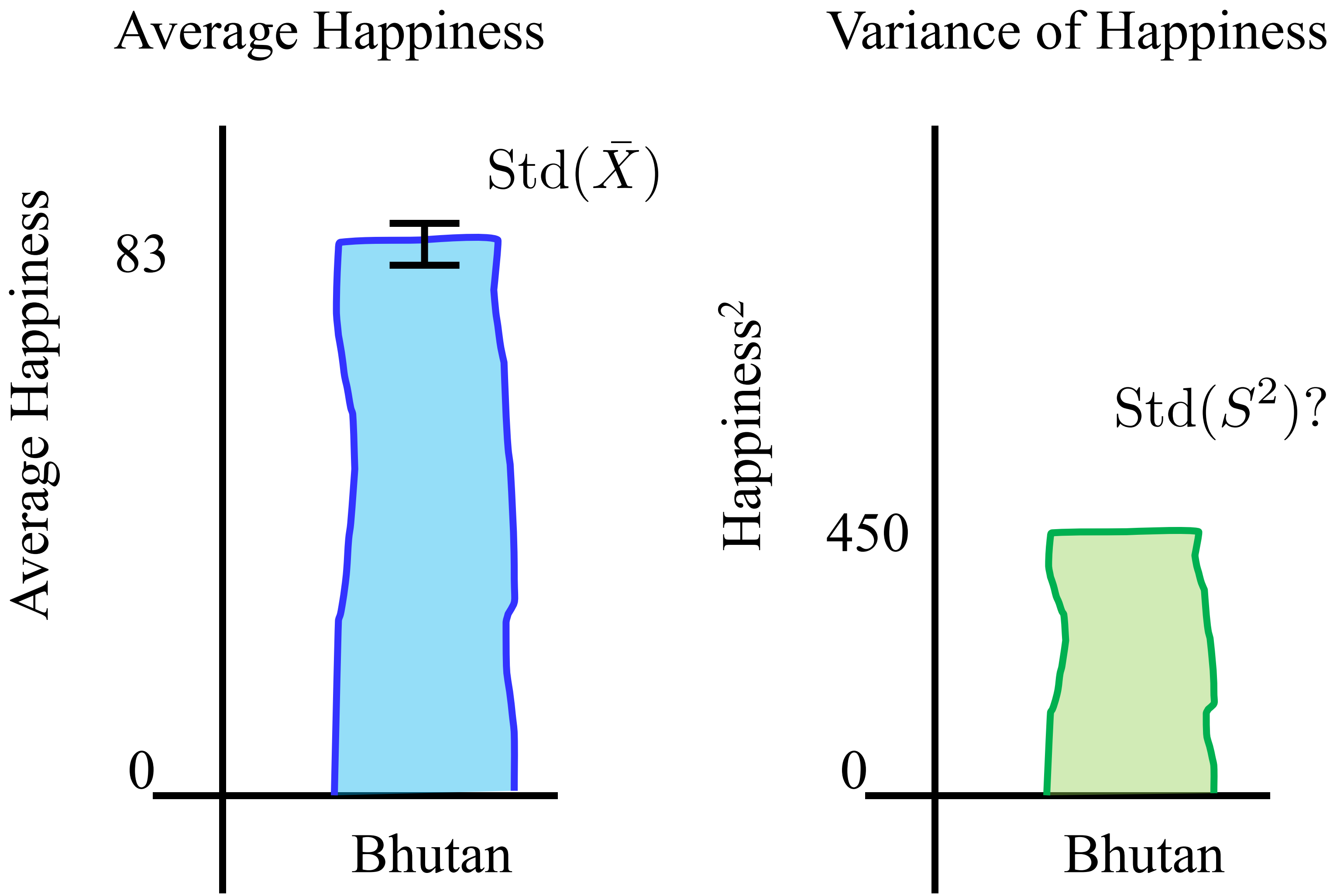


Bootstrap of Means

What is the probability that the mean is in the range 81 to 85?



Sample Mean



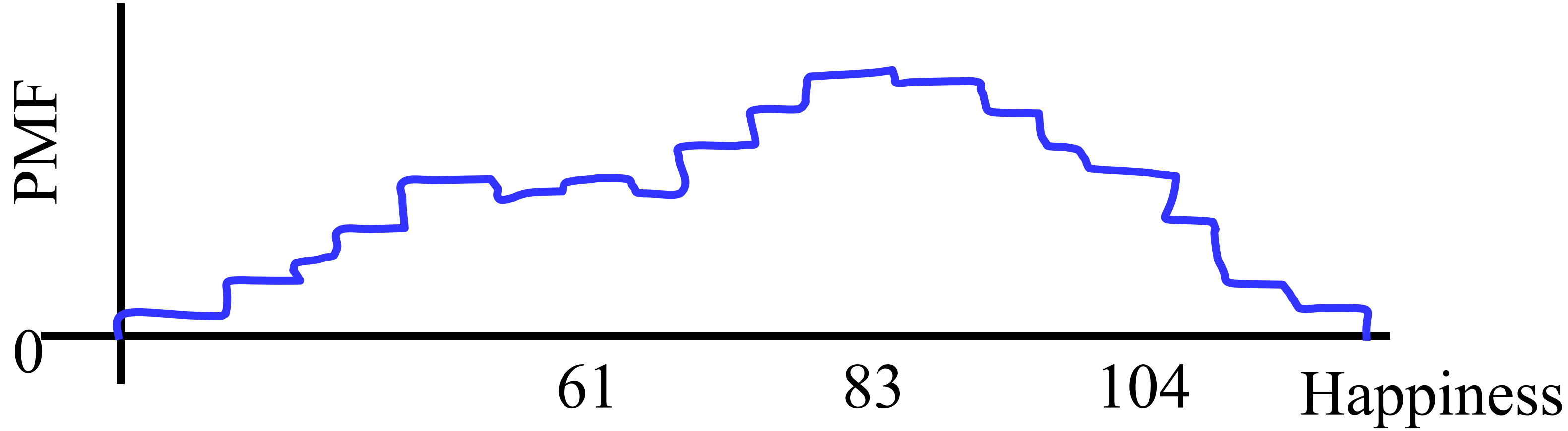
Claim: The average happiness of Bhutan is 83 ± 2

Bootstrap of Variance

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Draw **sample.size**() new samples from PMF
   b. Recalculate the variance on the resample
3. You have a **distribution of your variances**

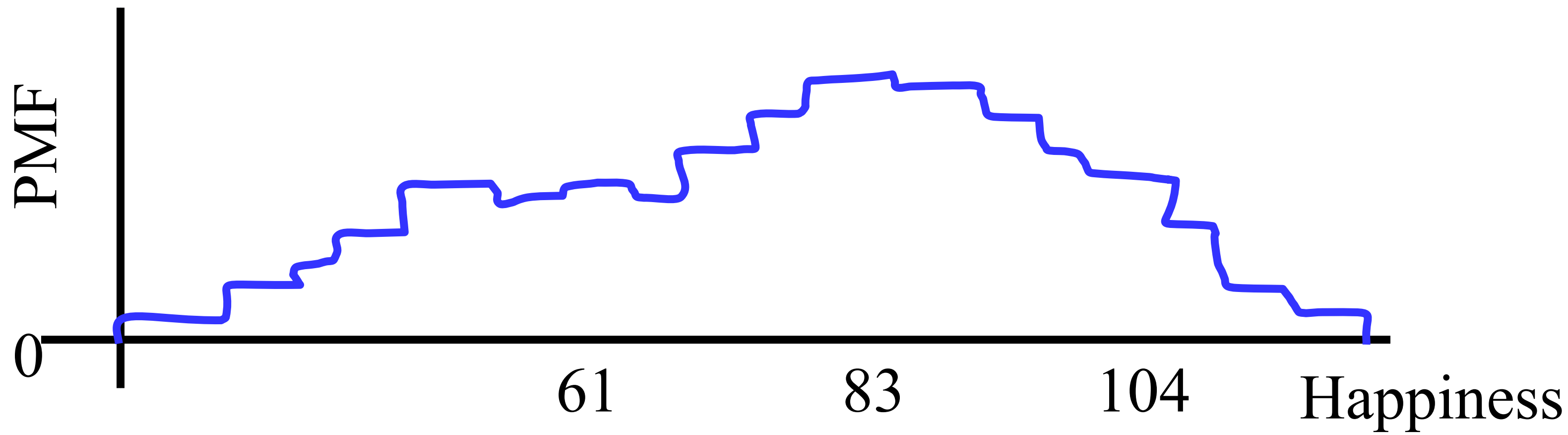
Bootstrap of Variance



Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Draw **sample.size**() new samples from PMF
   b. Recalculate the var on the resample
3. You now have a **distribution of your vars**

Bootstrap of Variance



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2. Repeat **10,000** times:
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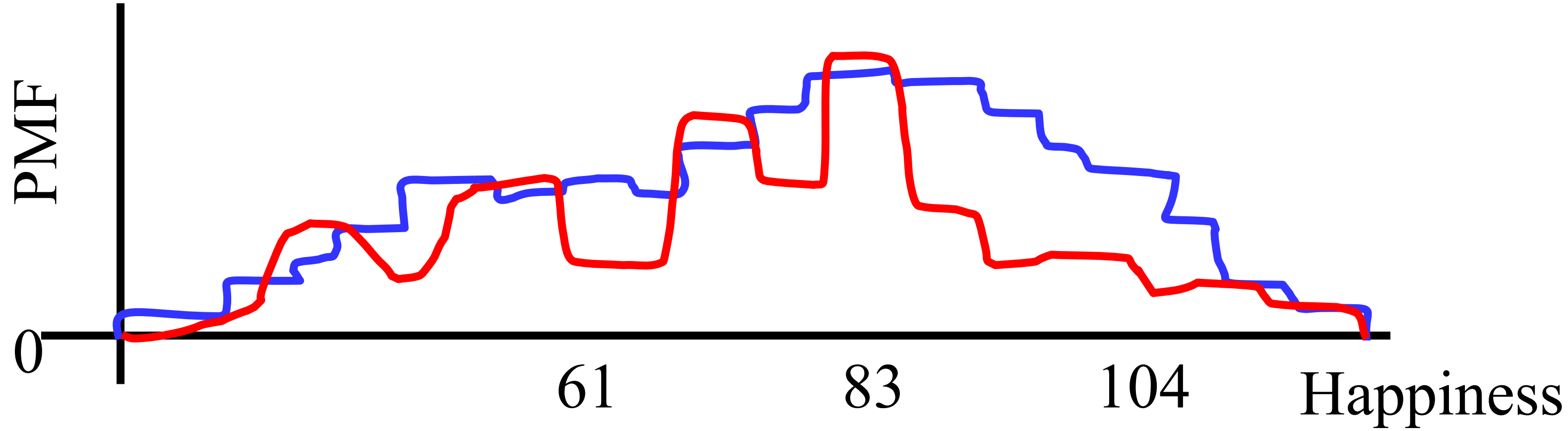
3. You now have a **distribution of your vars**

Bootstrap of Variance



Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
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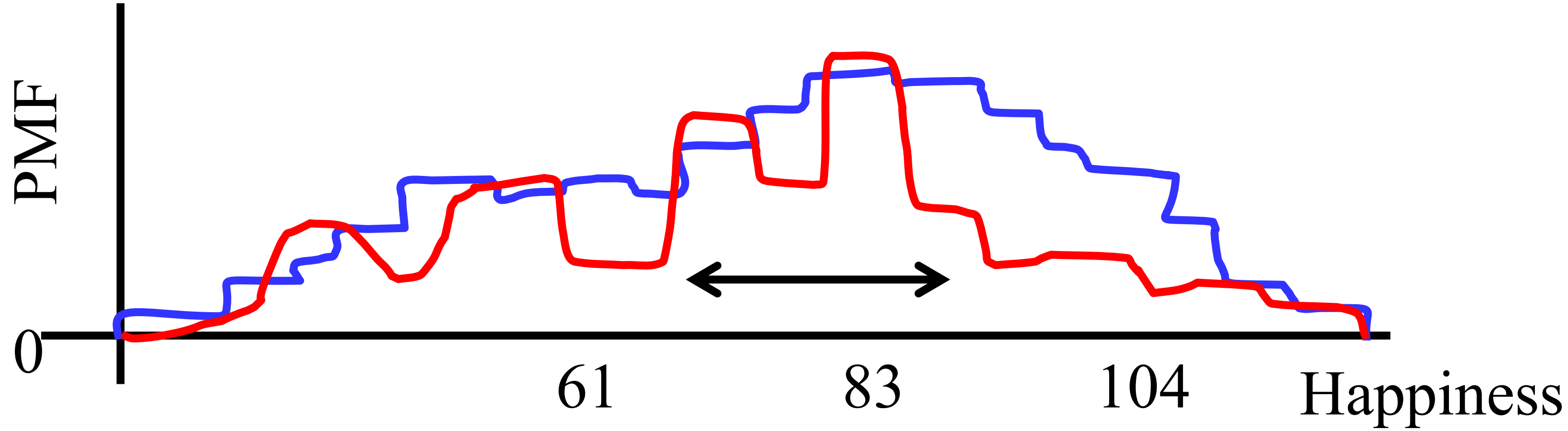
   b. Recalculate the var on the resample
3. You now have a **distribution of your vars**

Bootstrap of Variance



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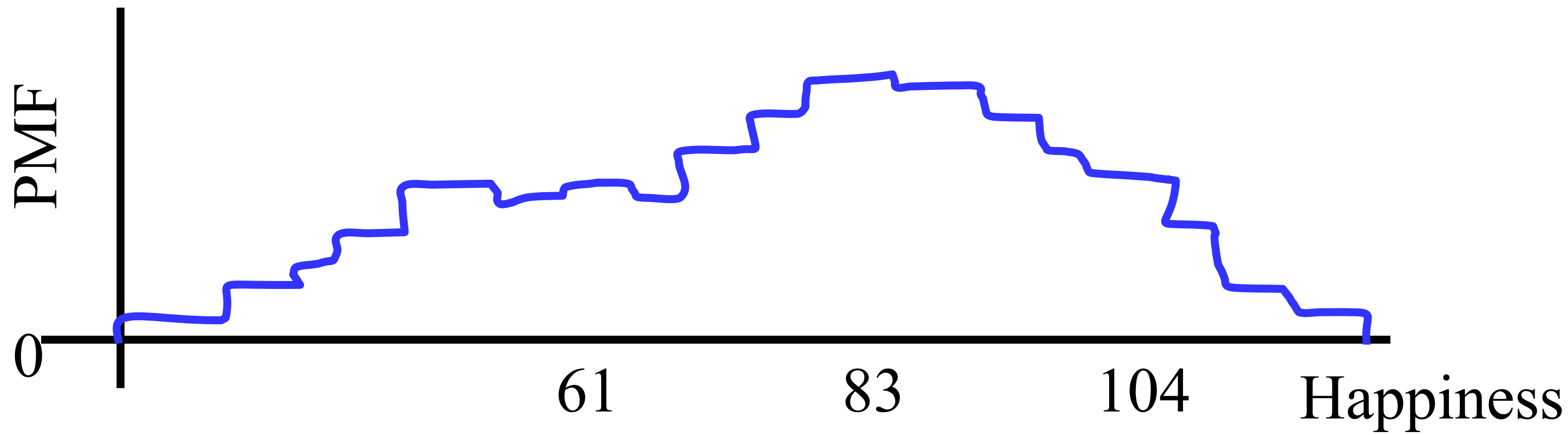
1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Draw **sample.size**() new samples from PMF
   b. Recalculate the vars on the resample
3. You now have a **distribution of your vars**

Vars = [472.7]

Bootstrap of Variance

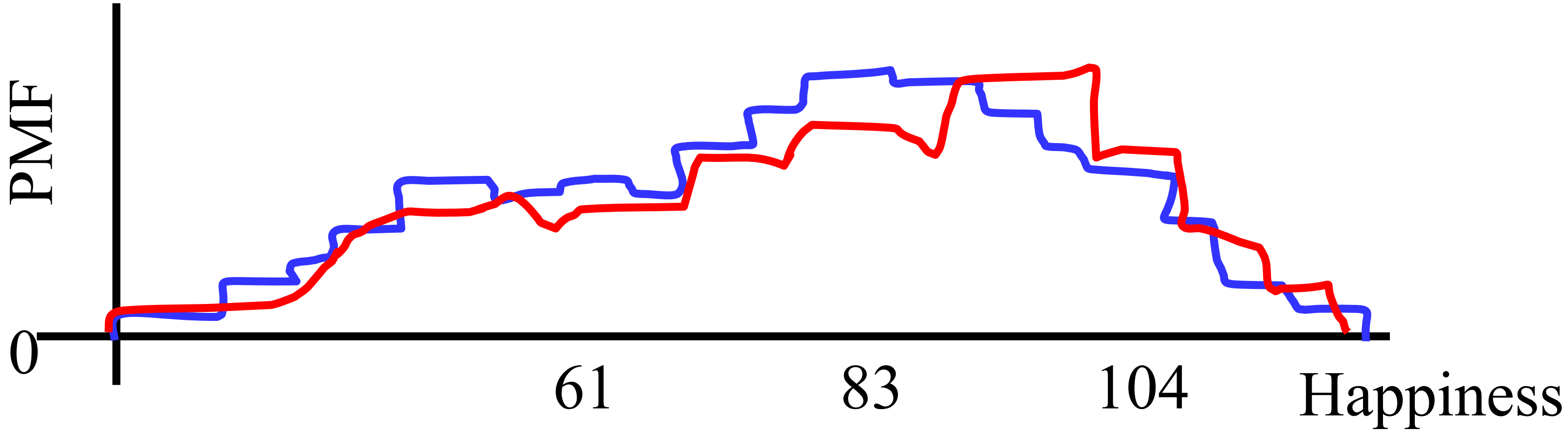


Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Draw **sample.size**() new samples from PMF
   b. Recalculate the var on the resample
3. You now have a **distribution of your vars**

Vars = [472.7]

Bootstrap of Variance



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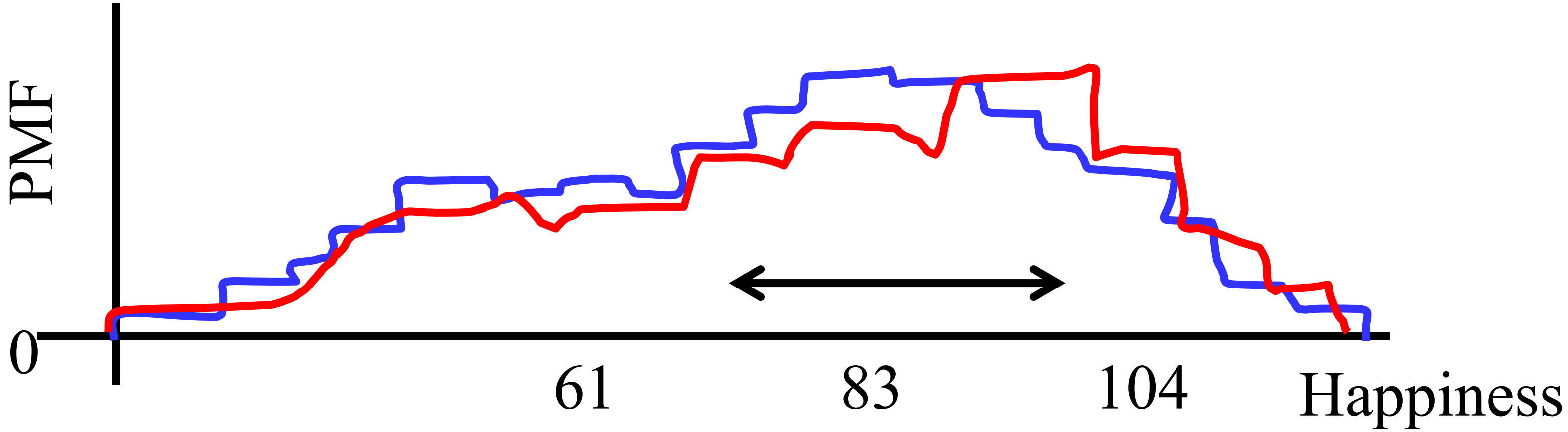

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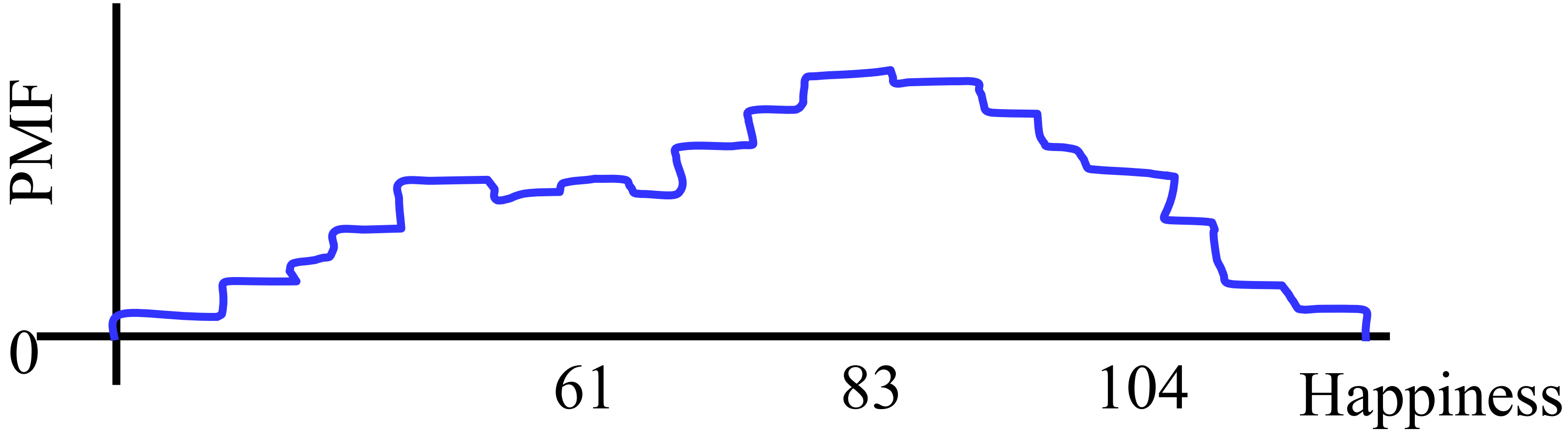
Vars = [472.7, 478.4]

Bootstrap of Variance

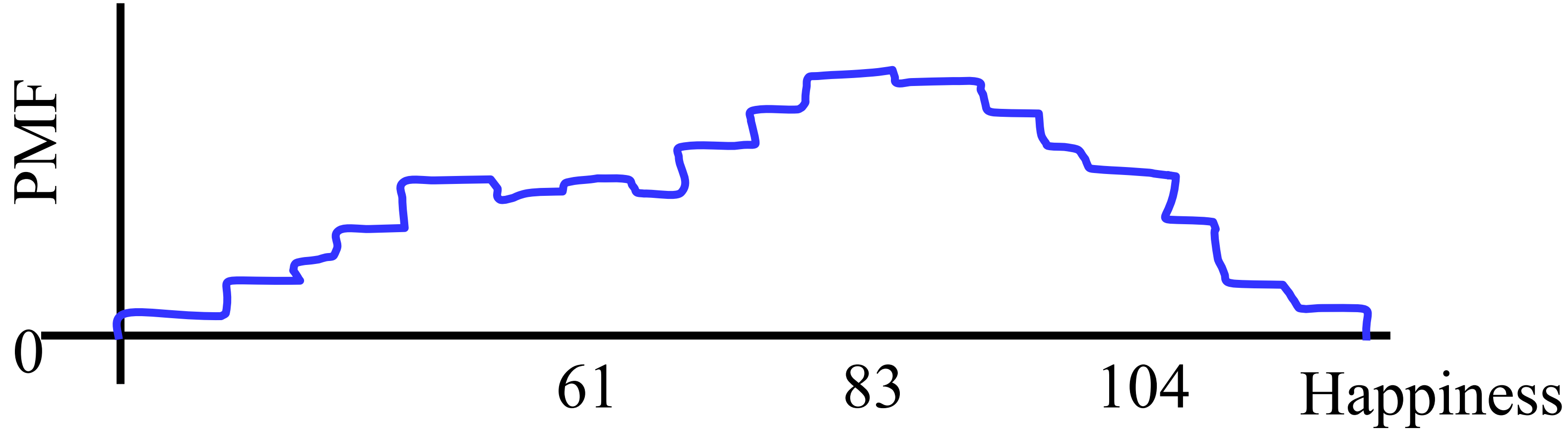


Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Draw **sample.size**() new samples from PMF
   b. Recalculate the var on the resample
3. You now have a **distribution of your vars**

Vars = [472.7, 478.4]

Bootstrap of Variance



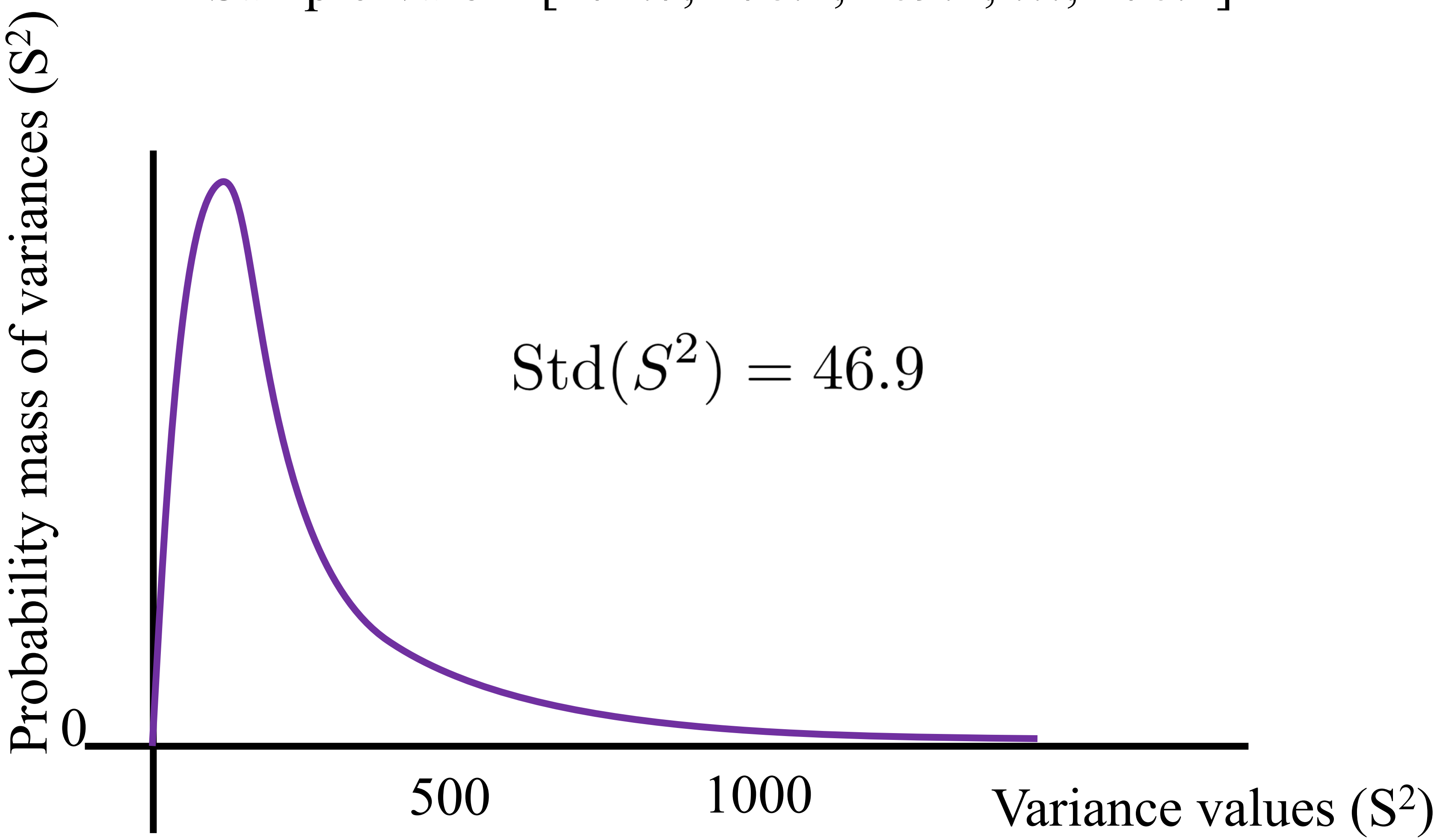
Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Draw **sample.size**() new samples from PMF
   b. Recalculate the var on the resample
3. You now have a **distribution of your vars**

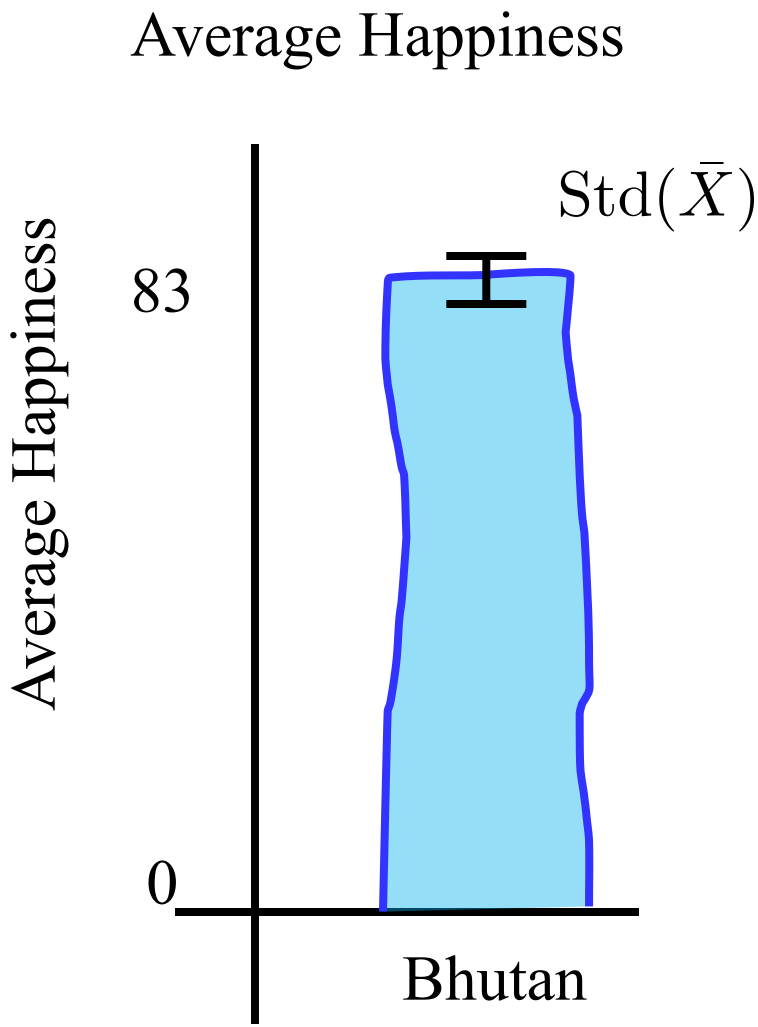
Vars = [472.7, 478.4, 469.2, …, 476.2]

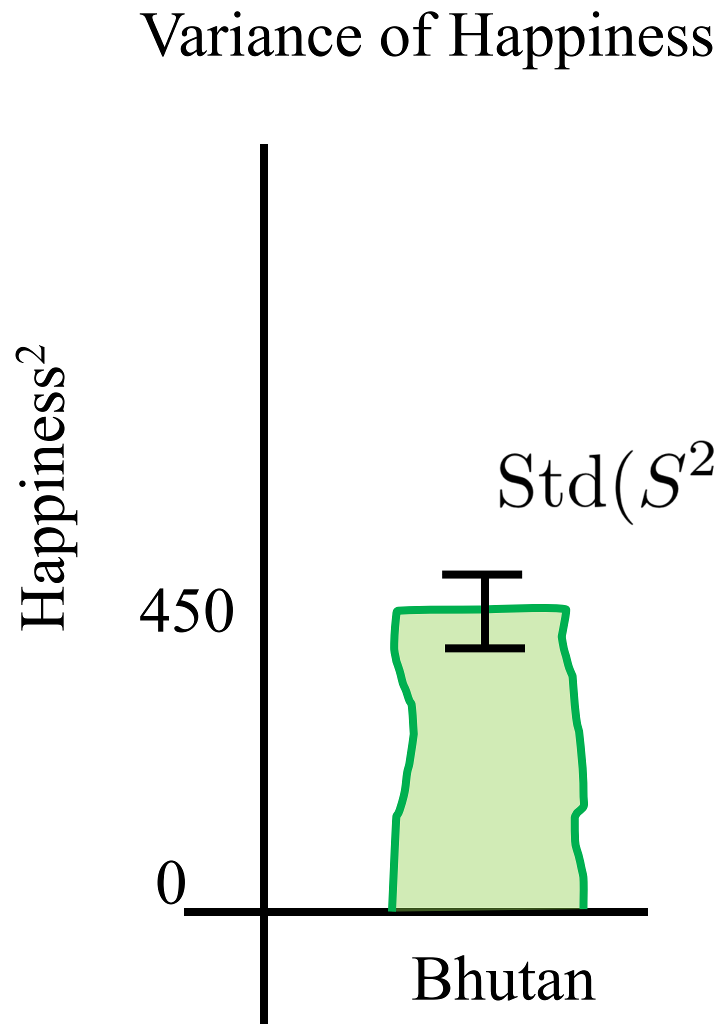
Bootstrap of Variance

Sample Vars = [472.7, 478.4, 469.2, …, 476.2]



Sample Mean

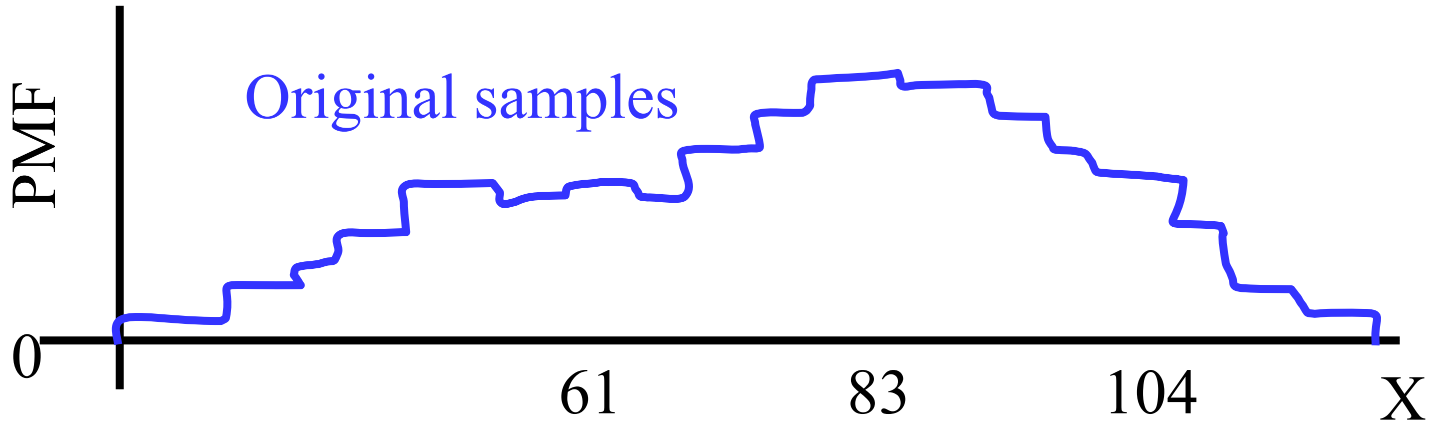




Claim: The average happiness of Bhutan is 83 ± 2

Algorithm in Practice

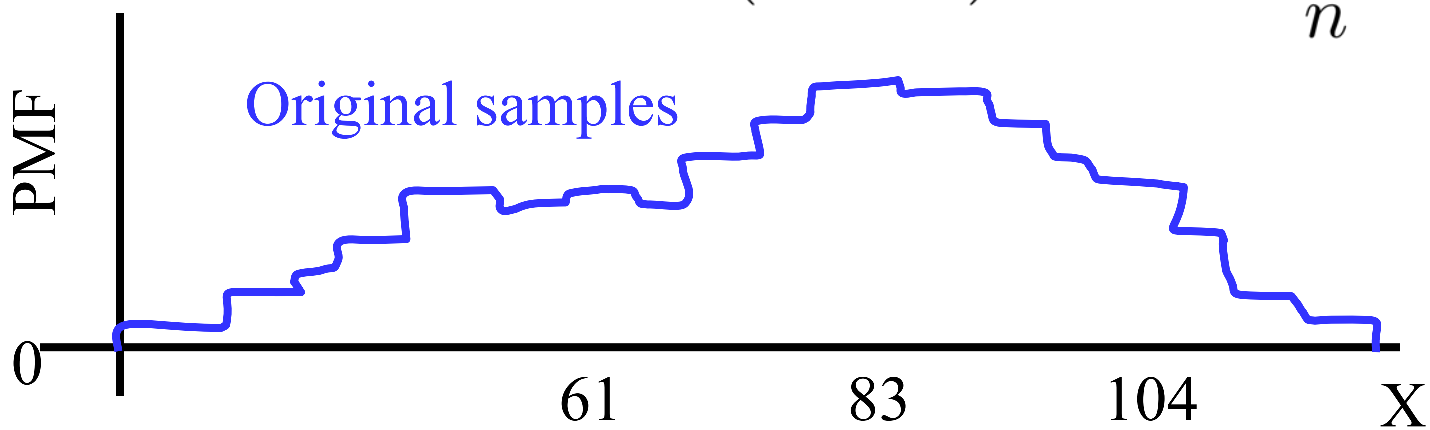
```python
def resample(samples):
    # Estimate the PMF using the samples
    # Draw K new samples from the PMF
```



Algorithm in Practice

```python
def resample(samples):
    # Estimate the PMF using the samples
    # Draw K new samples from the PMF
    return random.sample(samples, K)
```

$$P(X = k) = \frac{\text{count}(X = k)}{n}$$



Algorithm

Bootstrap Algorithm (sample):

1. Estimate the **PMF** using the sample
2. Repeat **10,000** times:
   a. Resample **sample.size**() from PMF
   b. Recalculate the stat on the resample
3. You now have a **distribution of your stat**

Algorithm in Practice

Bootstrap Algorithm (sample):

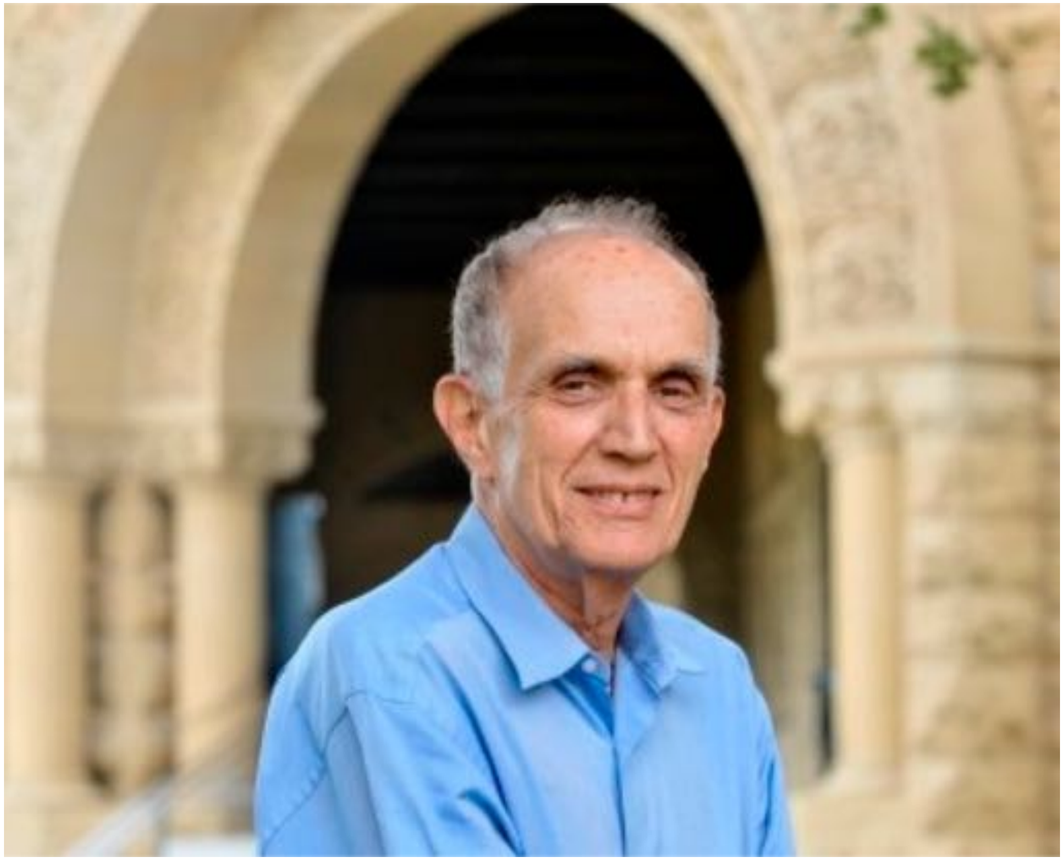
1. Repeat **10,000** times:
   a. Choose sample.size elems from sample, with replacement
   b. Recalculate the stat on the resample
2. You now have a **distribution of your stat**

To the code!

Bootstrap provides a way to calculate probabilities of statistics using code.

Bootstrap

Bradley Efron



Invented bootstrapping in 1979

Still a professor at Stanford

Won a National Science Medal

Works for any statistic*

*as long as your samples are IID and the underlying distribution doesn't have a long tail

Null Hypothesis Test

Population 1
4.44
3.36
5.87
2.31
...
3.70

Population 2
2.15
3.01
2.02
1.43
...
1.83

$\mu_1 = 3.1$

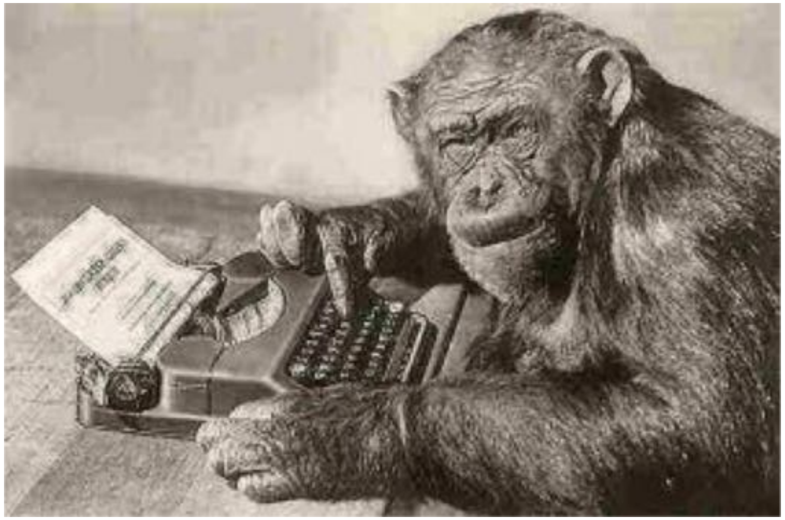
$\mu_2 = 2.4$

Claim: Population 1 and population 2 are different distributions with a 0.7 difference of means

Midterm

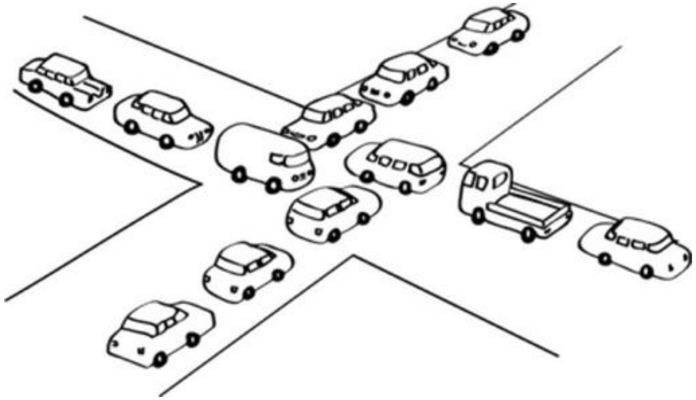
Midterm (part 1)

1



3

2



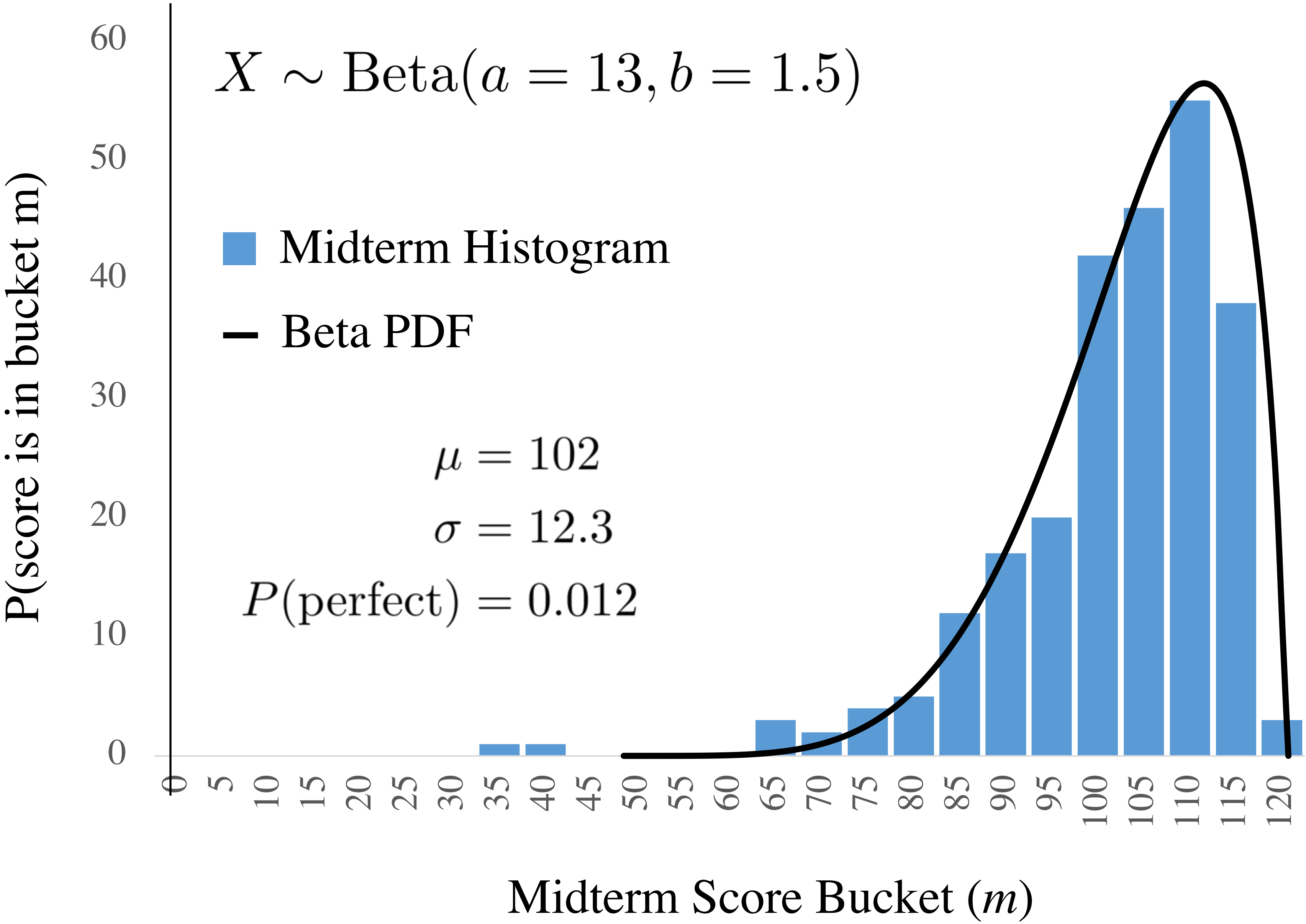
Midterm (part 2)

4

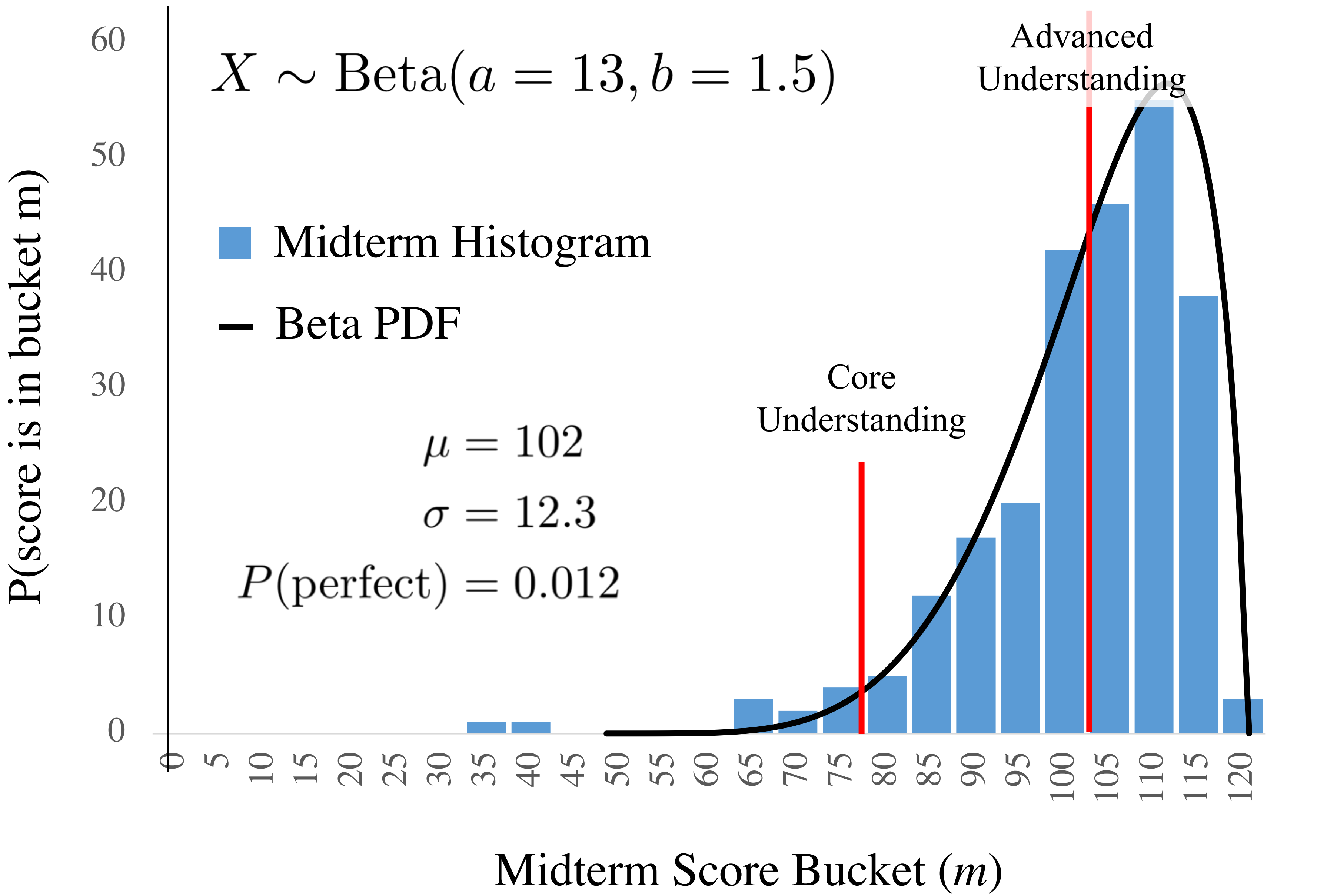
5

6

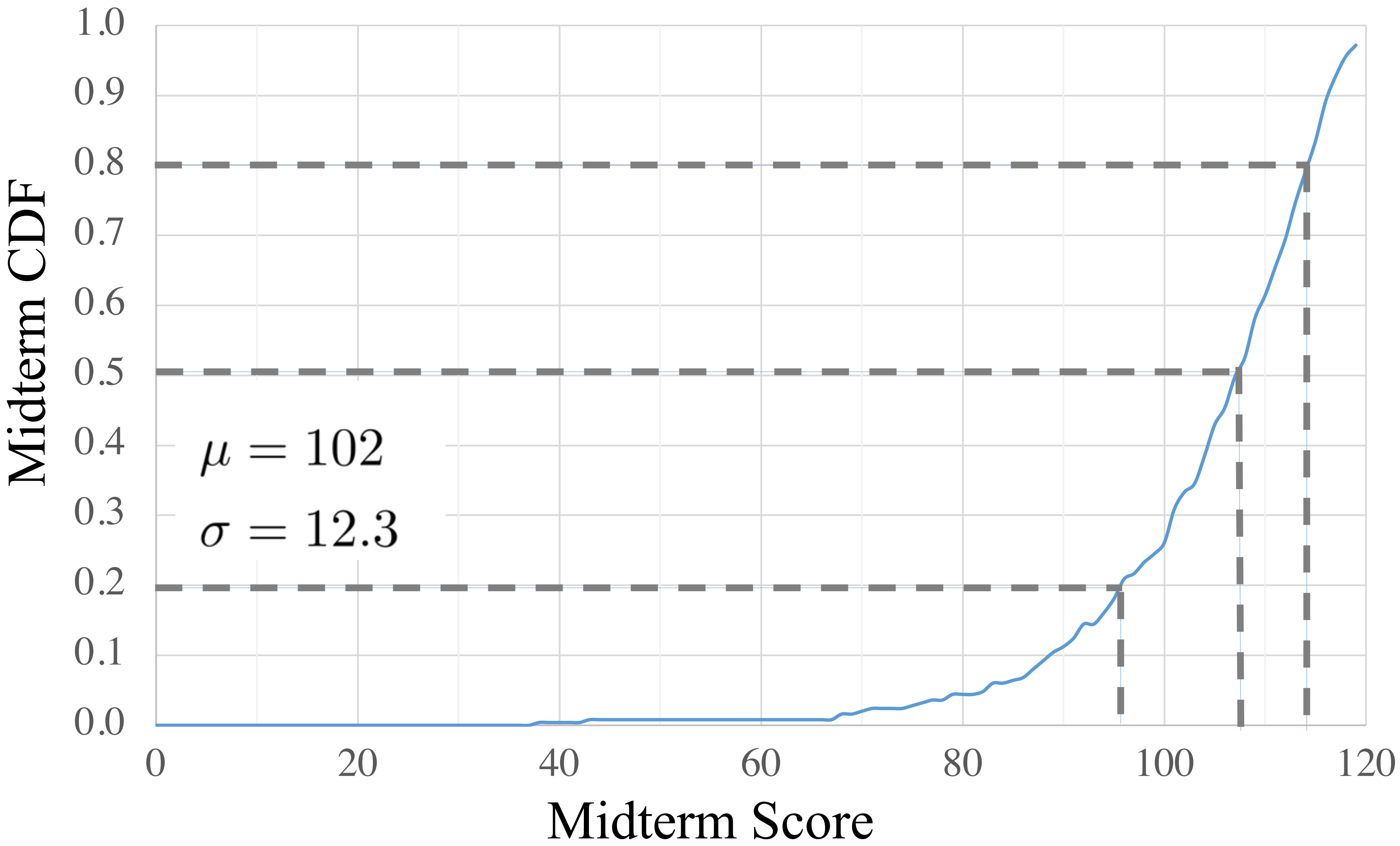
Midterm Distribution



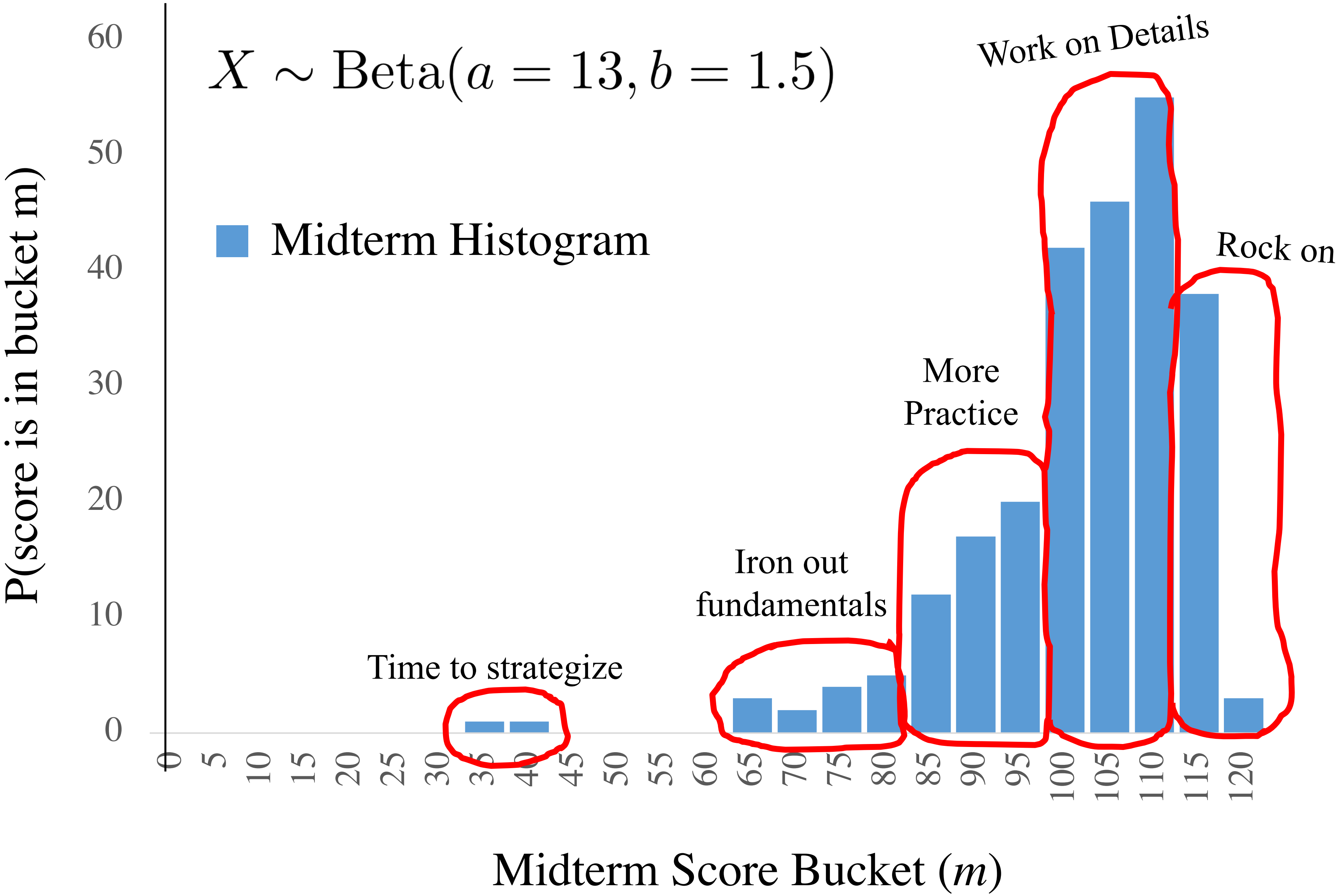
Midterm Distribution



Midterm Cumulative Density



Midterm Distribution



Midterm Correlation