

# THE DYNAMICS OF LONG WAVES IN A BAROCLINIC WESTERLY CURRENT

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## ABSTRACT

Previous studies of the long-wave perturbations of the free atmosphere have been based on mathematical models which either fail to take properly into account the continuous vertical shear in the zonal current or else neglect the variations of the vertical component of the earth's angular velocity. The present treatment attempts to supply both these elements and thereby to lead to a solution more nearly in accord with the observed behavior of the atmosphere.

By eliminating from consideration at the outset the meteorologically unimportant acoustic and shearing-gravitational oscillations, the perturbation equations are reduced to a system whose solution is readily obtained.

Exact stability criteria are deduced, and it is shown that the instability increases with shear, lapse rate, and latitude, and decreases with wave length. Application of the criteria to the seasonal averages of zonal wind suggests that the westerlies of middle latitudes are a seat of constant dynamic instability.

The unstable waves are similar in many respects to the observed perturbations: The speed of propagation is generally toward the east and is approximately equal to the speed of the surface zonal current. The waves exhibit thermal asymmetry and a westward tilt of the wave pattern with height. In the lower troposphere the maximum positive vertical velocities occur between the trough and the nodal line to the east in the pressure field.

The distribution of the horizontal mass divergence is calculated, and it is shown that the notion of a fixed level of nondivergence must be replaced by that of a sloping surface of nondivergence.

The Rossby formula for the speed of propagation of the barotropic wave is generalized to a baroclinic atmosphere. It is shown that the barotropic formula holds if the constant value used for the zonal wind is that observed in the neighborhood of 600 mb.

## CONTENTS

1. Introduction.....	135
2. Discussion of results.....	137
3. The atmospheric model.....	139
4. The fundamental equations.....	139
5. The boundary conditions.....	141
6. The steady state.....	141
7. The perturbation equations.....	142
8. Form of the perturbation and definition of stability.....	143
9. The barotropic wave.....	143
10. Reduction of the perturbation equations.....	144
11. Generalization of the Rossby formula.....	146
12. The normal equation for $V$ .....	147
13. The boundary condition for $V$ .....	148
14. Solution of the normal equation.....	148
Case I, $\Lambda = 0$ .....	148
Case II, $\Lambda \neq 0$ .....	149
15. Determination of the wave velocity.....	149
16. The stability criteria.....	153
17. The wave velocity.....	153
Case I, the neutral wave.....	153
Case II, the unstable wave.....	154
18. The structure of the wave.....	155
Case I, the neutral wave.....	155
Case II, the unstable wave.....	156

Acknowledgment.....	158
Appendix.....	158
A. Elimination of density and pressure from the perturbation equations.....	158
B. Reduction of the perturbation equations.....	159
C. Solution of the confluent hypergeometric equation for the case $b = 0$ .....	159
D. Tables of $\psi_1$ and $\psi_2$ .....	161
References.....	162

## 1. Introduction

The large-scale weather phenomena in the extra-tropical zones of the earth are associated with great migratory vortices (cyclones) traveling in the belt of prevailing westerly winds. One of the fundamental problems in theoretical meteorology has been the explanation of the origin and development of these cyclones. The first significant step toward a solution was taken in 1916 by V. Bjerknes [8, p. 785], who advanced the theory, based upon general hydrodynamic considerations, that cyclones originate as dynamically unstable wavelike disturbances in the westerly current. The subsequent discovery of the polar front by J. Bjerknes [2] made possible an empirical confirmation of the theory, for, following this discovery, the synoptic studies of J. Bjerknes and

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H. Solberg [3, 4] revealed that cyclones actually develop from wavelike perturbations on the polar front.

These important discoveries initiated several attempts to construct a mathematical theory of the frontal wave, the most successful of which was the theory presented by Solberg [15, 16]. Assuming a model consisting of two isothermal layers in parallel motion, he demonstrated that unstable waves, similar to young cyclones with respect to wave length and velocity of propagation, can exist in the sloping surface of separation between the two layers.

In 1937 J. Bjerknes [6] studied cyclogenesis from a new approach based on the concept of the upper-air wave as an independent entity. Starting from the principle embodied in the tendency equation that the surface pressure changes are due to the integrated effect of the horizontal mass divergence, he found that the deepening of cyclones can be attributed to the relative horizontal displacement of the upper-air wave with respect to the surface cyclone. This displacement in turn is a consequence of the baroclinicity of the atmosphere in middle latitudes which necessitates a vertical shear of the westerly winds. Accordingly, the responsibility for the intensification of pressure systems is transferred from the shear at the frontal surface to a general shearing motion throughout the troposphere.

The early investigators of the cyclone problem were, however, hindered by the sparsity of observations and consequently were forced to rely primarily upon indirect information. The gradual establishment of more dense observational networks made available additional information concerning the nature of the atmospheric flow patterns. The observations failed to reveal a one-to-one correspondence between the surface frontal perturbations and the major perturbations of the upper atmosphere. It was found instead that the number of surface frontal perturbations greatly exceeds the relatively small number of major waves and vortices at upper levels. Apparently there exists a fundamental difference between the long (3000–6000 km) waves and the frontal waves of length 1000–2000 km studied by Solberg, and, while there is undoubtedly a connection between the two types, it is natural, because of the difference in scale, to attempt to explain the motion of the long waves in terms of the properties of the general westerly flow without reference to frontal surfaces.

In line with this trend of ideas, in 1939 Rossby [14] gave a theoretical treatment of the motion of long waves for the special case of constant zonal motion of a homogeneous incompressible atmosphere. His theory led to the result that the speed of propagation of the waves depends on (a) the strength of the westerlies and (b) the wave length. It was found that the speed

of propagation toward the east decreases with increasing wave length up to the critical wave length at which the waves become stationary and beyond which they become retrograde. The theory was extended in 1940 by Haurwitz [10, 11], who took into account the curvature of the earth and the finite lateral extent of the wave. Finally, in 1944, Holmboe [12] derived a formula analogous to that of Rossby for the more general barotropic atmosphere. The results of these investigations were in agreement with the qualitative conclusions of J. Bjerknes's theory.

The studies of incompressible and barotropic atmospheres with no shear, however, cannot solve the problem of instability. Neither model contains a source of potential energy that can automatically convert itself into the energy of wave motion. It can be shown that waves in an atmosphere without shear are necessarily stable. This serious limitation can be overcome only by the adoption of a baroclinic model.

In 1944 J. Bjerknes and Holmboe [7] presented a theory of wave motion in a baroclinic atmosphere. Their solution is derived from the following principle [7, p. 10]:

The wave will travel with such a speed that the pressure tendencies arising from the displacement of the pressure pattern are in accordance with the field of horizontal divergence.

The field of horizontal divergence is evaluated from the pressure pattern by means of gradient-wind relationships, and on this basis the following relation is established: If  $\bar{u}(z)$  denotes the speed of the westerly current at any height  $z$ ,  $u_c$  an increasing function of wave length, and  $c$  the wave-velocity, then

$$c = \bar{u}(h) - u_c$$

where  $h$  is the height at which the mass divergence in the horizontal velocity field is zero; the wave is unstable provided that  $h$  is sufficiently small.

This work presents a clear physical explanation of instability in the westerlies and establishes necessary criteria, such as the relation above, that any exact mathematical treatment of baroclinic waves must satisfy. However, fundamental problems concerning the dynamics of the waves and their three-dimensional structure cannot be solved by a method of analysis based on semiempirical considerations of the gradient wind.

It is the purpose of the present investigation to present a theoretical solution of these remaining problems. A complete solution to the problem of baroclinic waves can be obtained only by integrating the fundamental equations of motion. Integration of the tendency equation alone could lead to a solution for barotropic waves, in which the motion is independent of height, but it cannot lead to a solution for the more general case of baroclinic waves because the

wave patterns must first be ascertained. It will be demonstrated that integration of the fundamental equations of motion leads to the solution of the following basic problems:

- (a) The determination of the speed of propagation of the wave.
- (b) The establishment of exact stability criteria.
- (c) The determination of the three-dimensional structure of the wave, i.e., particle velocities, pressure pattern, temperature pattern, etc.

## 2. Discussion of results

A description unencumbered by mathematical detail will now be given of the main contents of the investigation in order to set forth more clearly the physical basis of the procedure followed and the results obtained.

Section 3 concerns the construction of a model that corresponds to the observed state of the atmosphere and yet permits a not too cumbersome mathematical treatment. The troposphere is characterized by nearly constant values of vertical lapse rate and horizontal gradient of temperature, and a consequent increase of the zonal wind at a constant rate with height; the stratosphere is assumed to be isothermal with a zonal wind independent of height (see fig. 1).

In sections 4 and 5 the equations of motion and the boundary conditions are formulated for a compressible atmosphere in which the individual changes of pressure and density are adiabatic. It is shown in section 6 that these equations are satisfied by the mean flow prescribed in the model.

The actual flow is considered to be a small perturbation superimposed on the mean flow. The linearized equations of motion for this perturbation are then derived. They admit of a solution in the form of a sinusoidal wave traveling in the west-east direction with constant speed and with an amplitude depending on elevation. The problem is reduced to that of determining the amplitude as a function of height, and the speed of propagation as a function of the wave length and the parameters characterizing the mean state of the atmosphere, namely, the vertical shear of the zonal wind, the surface zonal wind, the vertical lapse rate of temperature, and the mean latitude of the wave. The possibility that both velocity and amplitude of the wave may be complex is not precluded, so that, for certain values of the parameters, the wave may become unstable and the phase of the wave may alter with height. These phenomena are regularly observed on weather maps but are not explainable in terms of a barotropic atmosphere. They may, therefore, be attributed to the vertical shear of the zonal wind, i.e., to the baroclinicity of the atmosphere.

As an application of the general theory, the equations are integrated for the special case of the barotropic atmosphere, and the wave-velocity formula of Rossby and Holmboe is rederived. Since the mode of excitation is not specified, the solution includes both the gravitational wave components, which are propagated by the action of gravity, and the long waves, in which the wave propagating force is predominantly inertial. The two wave types are distinguished by the fact that the wave velocity in the former greatly exceeds that of the latter. As pure gravitational waves have no appreciable influence on large-scale weather phenomena, it is shown, by means of a certain inequality, how these waves might have been eliminated from the outset. Although nothing is gained by this procedure in the study of the barotropic wave, the process of elimination becomes of great value for the more general baroclinic wave since here, were one to attempt to carry through the general solution embracing all wave types, severe analytic difficulties would supervene. Accordingly, in the discussion of the general problem of baroclinic motion, the elimination of inconsequential wave types is carried out and a set of equations obtained which are integrable by known methods. An interesting by-product of the calculation is that the meridional velocity component of the wave perturbation is nearly geostrophic. Indeed, had this been assumed *ab initio*, the simplified equations of motion would have been obtained directly.

Before the integration is carried through, however, a generalization of the Rossby formula is derived. It is shown that the simple formula for the speed of a barotropic wave will apply to the baroclinic wave if the constant value of the zonal wind in the formula is the mean zonal wind averaged with respect to pressure from the top to the bottom of the baroclinic atmosphere. It turns out that this value is the zonal wind in the vicinity of 600 mb, a fact which appears to be in good agreement with experience. A further result is the fact that the magnitude or direction of the zonal wind at very high levels in the stratosphere, say above 20 km, is of little consequence in the determination of the wave velocity. It is hoped that this result will help to clarify the rather vexing question regarding the influence of motions at high levels upon low-level weather phenomena.

It has been pointed out by Holmboe that the formula for the barotropic wave speed is strictly true only at the level of nondivergence in the atmosphere. It follows, therefore, that this level is in the vicinity of 600 mb.

It is of some interest to consider the case of a baroclinic atmosphere in which the zonal wind is constant. This atmosphere differs from the barotropic only by having statical stability. It is shown that this stability alone has no perceptible influence on the motion

of the wave, so that the speed is virtually the same as that given by the formula for the barotropic wave.

The integration of the equations of motion for the general case is accomplished by their reduction to a single second-order linear differential equation of the confluent hypergeometric type. The boundary conditions reduce to a single transcendental equation relating the wave speed to the wave length and physical parameters. In order to solve this equation it is necessary to simplify the model further by supposing that the zonal wind continues to increase with height above the tropopause; it is shown, however, that this expediency leads to no significant change in the stability criteria. This may be seen by comparing the dashed curve in fig. 7 with the solid curve beneath.

For a given wave length, the waves are found to be neutral if the shear of the zonal wind lies below a certain critical value which increases with wave length. Beyond this value the waves are unstable, and the instability becomes more pronounced with increasing shear. The stability of the wave is almost independent of the value of the surface zonal speed (see fig. 7).

For a given value of the surface zonal speed, the speed of the neutral wave increases with the shear of the zonal wind, and, in the vicinity of the critical shear, the wave speed is nearly equal to the surface zonal speed (see fig. 9). This conclusion is in qualitative agreement with the results of Solberg and Godske [5], who find that the incipient cyclone wave's propagation speed, which must be intermediate between the translational speed of the warm layer and that of the cold layer of the model described in the introduction, is much nearer to the translational speed of the warm layer. In the present case, of course, no surface of discontinuity exists, but, if the shallow cold layer is ignored and the theory is applied to the thick overlying warm layer and if the surface zonal wind is taken to be that of the lower part of the warm layer, the results may be interpreted to mean that the incipient cyclone wave moves with approximately the speed of the surface wind in the warm air. However, it should again be emphasized that the waves considered by Solberg and Godske are of a different order of magnitude, and it may not be permissible to force a comparison between the two theories.

A discussion of the properties of the damped or stable baroclinic wave is not attempted, for presumably such components are extinguished as soon as they are formed.

The structure of the neutral baroclinic wave is similar to that of the barotropic wave. The two differ only in that the perturbation fields of velocity in the baroclinic case diminish with increasing height and eventually approach zero, whereas in the barotropic case these fields remain constant. In both cases the

wave in the meridional-velocity field lags  $90^\circ$  behind the wave in the pressure field and the wave in the density field is in phase with the pressure wave. Furthermore, the wave in the latitudinal-velocity field is  $180^\circ$  out of phase with the pressure wave, and the vertical-velocity wave lags  $90^\circ$  behind the pressure wave, and in neither case is there a change of phase with height. (Some of these relationships are shown in fig. 10.)

With the appearance of instability, a thermal asymmetry develops in the baroclinic wave, so that the colder air is found behind the trough in the isobars. This asymmetry results in a tilt of the axes of low and high pressure toward the west, the tilt being most pronounced at low levels and diminishing to zero as height increases. Like the neutral wave, the meridional-velocity wave lags  $90^\circ$  behind the pressure wave. (These relationships are represented in fig. 11.) The waves in the remaining two velocity components show a more complicated relationship to the pressure wave. Vertical cross sections of the fields of vertical velocity, vertical momentum, and horizontal mass divergence are given in figs. 12 and 13. These diagrams show that, at low levels, the maximum vertical component of velocity occurs some distance behind the inflection point in the pressure profile, whereas at high levels it is found to be slightly in advance of the point of inflection.

The existing data on the three-dimensional distribution of the vertical velocity component appear to support these conclusions. Where it can be ascertained, the maximum vertical velocity component at, say, 700 mb is found closer to the trough than to the preceding wedge in the pressure field, while at high levels, although no conclusive data are available, one may cite as evidence that upper clouds are frequently observed to form with west to northwest wind. The maximum absolute magnitude is found at levels above the tropopause. This result is questionable and probably is due to the assumption of a continued increase of the zonal wind above the tropopause. It should be expected that, were the model to provide for a decrease in the zonal wind above the tropopause, the position of the maximum would be brought much lower, so that the change from ascending to descending motion which is often observed to take place near the tropopause would be verified.

Fig. 13 shows that the maximum horizontal mass divergence takes place between the trough and preceding wedge in the pressure pattern at low levels and is replaced by convergence at higher levels. It is also seen that with instability there no longer exists a constant level at which the divergence vanishes; rather the divergence vanishes along an inclined surface. However, with slight instability, the major part of the surface of nondivergence is nearly horizontal

and is found to be between 350 mb and 400 mb. It is proved in section 11 by means of the generalization of the Rossby-Holmboe formula that, when observed zonal winds are used in place of the assumed winds, this level is in the vicinity of 600 mb. The discrepancy here can also be attributed to the lack of correspondence between the model and the observed state of the atmosphere at high levels.

The field of vertical momentum shown in fig. 13 has a maximum along any vertical at the altitude where the horizontal divergence vanishes. This result would have been obtained had the local time rate of change of density as well as the horizontal density advection been ignored in comparison with the remaining terms in the equation of continuity. One may therefore infer that these quantities, at least in the case of the baroclinic wave of small amplitude, can properly be ignored. This conclusion has been verified by those who have calculated vertical velocities by means of the equation of continuity.

It is seen from the preceding discussion that the baroclinic wave model exhibits many of the characteristics of the waves observed on the daily weather maps, both with respect to speed of propagation and internal structure. The theory, moreover, predicts that waves of length less than 6000 km will be unstable when the vertical shear of the zonal wind is greater than about  $1.5 \text{ m sec}^{-1}\text{km}^{-1}$  (see fig. 7). Since this value is usually exceeded in middle latitudes, particularly in the winter months, one may infer that the westerlies are a seat of constant instability. This conclusion is verified by the observed storminess in these regions and also by the fact that the observed wave patterns almost invariably exhibit the tilt with height which, according to the theory, is characteristic of instability.

It should here be remarked that the investigation does not tell what relationship exists, in the generation of cyclones, between the frontal perturbation and the long atmospheric wave. J. Bjerknes and Holmboe [7] adopt the point of view that the initial impulse for wave formation in the free atmosphere is supplied by the frontal perturbation and that, thereafter, the induced upper wave propagates and develops according to its own law of motion, independently, so to speak, of the frontal wave. The two, however, develop and move along together, the upper waves lagging a little behind in phase. There is here a suggestion both of independence and of dependence in the motions of the frontal and upper-air waves. On one hand, it is true that by far the majority of deepening upper-air waves in middle latitudes are associated with frontal perturbations. On the other, it seems equally clear that waves which form on a surface of shearing discontinuity possess essentially different characteristics from the long upper-air waves—the waves differ in

length and also frequently in direction of motion. Waves of short period, having periods of the order of 24 hours and lengths of the order of 1000 km, are often found on the frontal surface, but they are certainly not the same as the long upper-air waves. Nevertheless the two types of wave cannot be treated as independent phenomena, as, for example, the gravitational waves and the long waves in the atmosphere, for, whereas there is no appreciable linkage between the latter pair (see the discussion in section 7), there must be a linkage between the frontal and long wave.

The author wishes to make a final remark concerning the application of the present theory to the problem of wave motion in the tropical easterlies. In this case, the mathematical formulation of the problem is similar to that for the westerlies. Although the solution has not yet been brought to completion, a preliminary analysis indicates that, where the shear of the zonal current is positive, the stability criteria are qualitatively the same as for the westerlies, but, where the normal meridional temperature gradient is reversed so that the shear is negative, the flow is unstable.

### 3. The atmospheric model

We shall adopt, as an approximation to the atmosphere in middle latitudes, a model whose undisturbed state is characterized as follows: (a) the motion is zonal; the speed is constant in each horizontal level, is a linear function of height in the troposphere, and is independent of height in the stratosphere; (b) the lapse rate of temperature is constant in the troposphere and zero in the stratosphere. A comparison, in meridional cross section, of the theoretical model with the observed mean state of the atmosphere is shown in fig. 1. It will be seen that the model corresponds closely with the mean atmosphere in low levels. The deviations are most pronounced at high levels, where they become relatively unimportant because of the exponential decrease of density with height. Thus, in February, only 10 per cent of the atmosphere in middle latitudes lies above 16 km.

### 4. The fundamental equations

We assume for purposes of mathematical simplicity that the curvature of the earth can be neglected. This simplification is permissible when the length of the wave is small compared with the circumference of the zonal circle along which the wave moves. The mean motion can therefore be considered planar, and a rectangular system of coordinates  $x$ ,  $y$ , and  $z$  can conveniently be introduced with  $x$  increasing eastward,  $y$  northward, and  $z$  vertically upward. If the corresponding velocity components are denoted by  $u$ ,  $v$ , and  $w$ , density by  $\rho$ , pressure by  $p$ , angular velocity of the earth by  $\Omega$ , and geographical latitude by

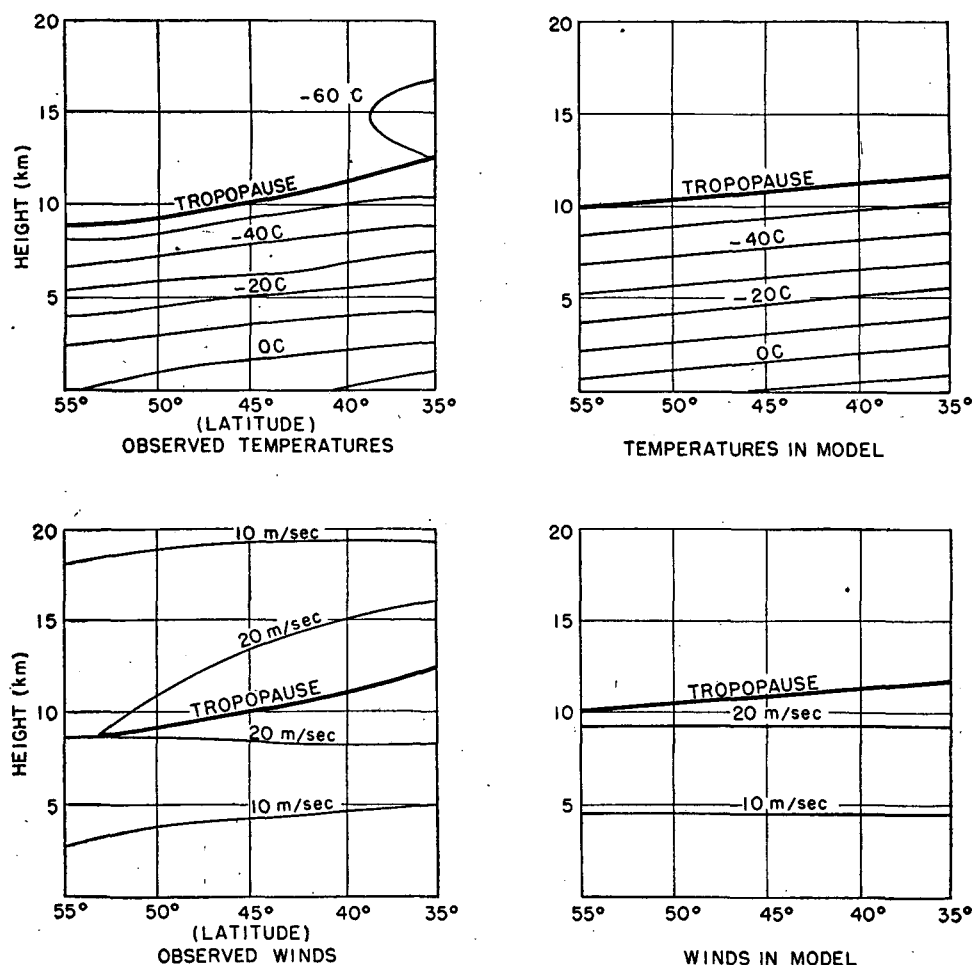


FIG. 1. Comparison of the theoretical model (right) with the mean state of the atmosphere (left) for the month of February. (After V. Bjerknes *et al.* [8, pp. 628, 649].)

$\varphi$ , the Eulerian equations of motion become

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \varphi - 2\Omega w \cos \varphi \quad (1)$$

$$\frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2\Omega u \sin \varphi \quad (2)$$

$$\frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + 2\Omega u \cos \varphi - g. \quad (3)$$

In dealing with the large-scale quasi-horizontal motions of the atmosphere it is customary to omit the vertical components of acceleration and Coriolis force as well as the horizontal component of the Coriolis force involving  $w$ , for these quantities may be shown both empirically and theoretically to be negligible in comparison with the forces of pressure and gravity. We may therefore replace the first equation of motion by

$$\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2\Omega v \sin \varphi \quad (1')$$

and the third by the hydrostatic equation,

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g. \quad (3')$$

A fourth equation is obtained from the law of conservation of mass,

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w), \quad (4)$$

and a fifth equation from the condition that the motion be adiabatic. This condition is expressed by means of the differential relationship

$$\frac{dp}{dt} = \sigma^2 \frac{d\rho}{dt}, \quad (5)$$

where  $\sigma$  is the Laplacian velocity of sound. If  $\epsilon$  is the ratio of the specific heat of air at constant pressure to that at constant volume,  $R$  the gas constant referred to unit mass of dry air, and  $T$  the absolute temperature,  $\sigma$  is given by the equation

$$\sigma^2 = \epsilon RT.$$

The adiabatic hypothesis is valid when the effects of radiation, turbulent heat transfer, and condensation can be ignored. The first two effects are usually regarded as of secondary importance in the free atmosphere, whereas condensation can produce appreciable errors. However, as long as one is concerned with waves of small amplitude, the vertical motions will not be of sufficient magnitude to cause condensation, so that this factor may also be ignored.

### 5. The boundary conditions

The boundary conditions express the following physical properties of the motion: (a) the normal component of the velocity vanishes at the surface of the earth, (b) the momentum vanishes at the limit of the atmosphere, and (c) the variation of the velocity components, pressure, density, and temperature across the tropopause must be zero, i.e., the tropopause is a discontinuity surface of the first order. Condition (a) gives

$$w(x, y, 0, t) = 0. \quad (6)$$

Condition (b) gives

$$\lim_{z \rightarrow \infty} \rho u = \lim_{z \rightarrow \infty} \rho v = \lim_{z \rightarrow \infty} \rho w = 0, \quad (7)$$

and condition (c) gives

$$\Delta u = \Delta v = \Delta w = \Delta p = \Delta \rho = \Delta T = 0, \quad (8)$$

where the symbol  $\Delta$  stands for a variation from one side to the other of the tropopause.

### 6. The steady state

The fundamental equations (1'–5) together with the boundary conditions (6–8) impose the necessary restrictions on the theoretical model. It will be shown that the specifications already given are consistent with these restrictions and are sufficient to determine completely the mean state of the atmosphere.

We adopt the convention of denoting a steady-state quantity by a bar placed over the symbol representing the same quantity in the perturbed state. The condition for zonal flow is then given by the equations

$$\bar{u} = \bar{u}(y, z), \quad \bar{v} = 0, \quad \bar{w} = 0.$$

Since  $\bar{v} = \bar{w} = d\bar{u}/dt = \partial\bar{p}/\partial t = 0$ , (1') and (4) state that  $\bar{p}$  and  $\bar{p}$  are functions of  $y$  and  $z$  only. Equation (2) states that the undisturbed flow must satisfy the condition of geostrophic equilibrium,

$$0 = -\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial y} - f\bar{u}, \quad (9)$$

where the symbol  $f$  is used to denote the Coriolis parameter  $2\Omega \sin \varphi$ . Equation (3') expresses the condi-

tion for hydrostatic equilibrium in the mean state

$$0 = -\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} - g, \quad (10)$$

and (5) is satisfied identically, since  $d\bar{p}/dt$  and  $d\bar{p}/dt$  both vanish. Integration of (10) shows that the pressure field is completely specified by the mass field if the pressure vanishes at  $z = \infty$ . The relationship between the fields of mass and velocity is then brought out by the elimination of pressure from (9). Differentiating with respect to  $z$ , and substituting  $\partial\bar{p}/\partial z$  from the hydrostatic equation, we obtain

$$\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial y} = \frac{f\bar{u}}{g} \left( \frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial z} + \frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial z} \right). \quad (11)$$

The mass field, in turn, is related to the field of temperature through the equation of state

$$\bar{p} = \bar{\rho} R \bar{T}. \quad (12)$$

We may therefore regard  $\bar{u}$  and  $\bar{T}$  as the fundamental variables by means of which all other quantities are determined. The necessary and sufficient relationship between these two fields can be obtained by elimination of  $\bar{\rho}$  and  $\bar{p}$  from equations (9, 10, 12). By this means we obtain the thermal wind equation for zonal motion on a flat earth,

$$-\frac{1}{\bar{T}} \frac{\partial \bar{T}}{\partial y} = \frac{f\bar{u}}{g} \left( -\frac{1}{\bar{T}} \frac{\partial \bar{T}}{\partial z} + \frac{1}{\bar{u}} \frac{\partial \bar{u}}{\partial z} \right). \quad (13)$$

Any distribution of  $\bar{u}$  and  $\bar{T}$  that satisfies (13) and the boundary conditions (6–8) will therefore automatically satisfy the fundamental equations. We shall now show that this is the case for the distribution prescribed in section 3. This distribution is expressed by the equations

$$\bar{u}(z) = \bar{u}_0 + \Lambda z \quad (\bar{u}_0, \Lambda = \text{const}), \quad (14a)$$

$$\bar{T}(y, z) = \bar{T}(y, 0) - \gamma z \quad (\gamma = \text{const}) \quad (14b)$$

in the troposphere, and

$$\partial \bar{u} / \partial z = 0, \quad (15a)$$

$$\bar{T} = \bar{T}_s \quad (\bar{T}_s = \text{const}) \quad (15b)$$

in the stratosphere. Inserting the tropospheric values of  $\bar{u}$  and  $\partial \bar{T} / \partial z$  into (13) we obtain the equation

$$\frac{\partial \bar{T}}{\partial y} + \frac{f\Lambda}{g} \bar{T} = -\frac{f\bar{u}\gamma}{g}, \quad (16)$$

whose solution, subject to the condition (14b), is given by

$$\begin{aligned} \bar{T}(y, z) &= [\bar{T}(0, 0) + \bar{u}_0 \gamma / \Lambda] e^{-f\Lambda y / g} - \bar{u}_0 \gamma / \Lambda - \gamma z \\ &= \bar{T}(y, 0) - \gamma z \end{aligned} \quad (17)$$

and holds throughout the troposphere. Equations (14a) and (14b), therefore, suffice to determine  $\bar{u}$  and

$\bar{T}$  everywhere in the troposphere. Since the stratospheric values of  $\bar{u}$  and  $\bar{T}$  prescribed by (15a) and (15b) satisfy (13) identically, it remains only to show that the boundary conditions are satisfied. The requirement of continuity for  $\bar{T}$  in (8) will be fulfilled if the tropopause is isothermal and has a temperature equal to that of the stratosphere. The equation of the tropopause is obtained therefore by setting  $\bar{T}(y, z)$  in (17) equal to  $\bar{T}_s$ . Differentiation of the resulting equation with respect to  $y$  then gives the slope of the tropopause the value

$$\frac{dz}{dy} = -\frac{f\Lambda}{\gamma g} \bar{T}_s - \frac{f\bar{u}}{g} \approx -\frac{f\Lambda}{\gamma g} \bar{T}_s, \quad (18)$$

and the variation of  $\bar{u}$  on the tropospheric side of the tropopause is obtained from the equation

$$\frac{d\bar{u}}{dy} = \frac{d\bar{u}}{dz} \frac{dz}{dy} \approx -\frac{f\Lambda^2}{\gamma g} \bar{T}_s, \quad (19)$$

which by the condition of continuity in  $\bar{u}$  in (8) also determines the corresponding variation of  $\bar{u}$  with  $y$  on the stratospheric side of the tropopause. This condition coupled with the requirement (15a) completely determines  $\bar{u}$  in the stratosphere. The boundary condition (6) is satisfied since  $\bar{w} = 0$ , and (7) is satisfied since  $\bar{p} \rightarrow 0$ , and therefore also  $\bar{p}\bar{u} \rightarrow 0$ , as  $z \rightarrow \infty$ . Hence the model prescribed in equations (14a-15b, 17, 19) is consistent with the fundamental equations (1'-5) and satisfies the boundary conditions (6-8).

The theoretical cross section shown in fig. 1 is constructed in accordance with these equations by using  $\bar{u}_0 = 0 \text{ m sec}^{-1}$ ,  $\Lambda = 2.2 \text{ m sec}^{-1} \text{ km}^{-1}$ ,  $\bar{T}(0, 0) = 288^\circ \text{C}$ ,  $\bar{T}_s = 213^\circ \text{C}$ , and  $\gamma = 6.5 \text{ C km}^{-1}$ .

## 7. The perturbation equations

The motion may be regarded as a small perturbation with velocity components  $u'$ ,  $v'$ , and  $w'$  superimposed on the steady zonal current  $\bar{u} = \bar{u}(z)$ . Thus

$$\left. \begin{aligned} u &= u'(x, y, z, t) + \bar{u}(z) \\ v &= v'(x, y, z, t) \\ w &= w'(x, y, z, t) \end{aligned} \right\} \quad (20)$$

Similar expressions obtain for the density and pressure in the disturbed state, thus

$$\left. \begin{aligned} \rho &= \rho'(x, y, z, t) + \bar{\rho}(y, z) \\ p &= p'(x, y, z, t) + \bar{p}(y, z) \end{aligned} \right\} \quad (21)$$

We assume that the velocity perturbation is independent of the meridional coordinate. This assumption, introduced by Rossby, reduces the differential equations in the final formulation of the problem from the partial to the ordinary variety and leads to a considerable simplification.

Substituting the perturbed velocity, pressure, and mass fields into the fundamental equations (1'-5), and simplifying by means of the steady-state relations (9-11), we obtain the system

$$\left. \begin{aligned} L(u') - fv' + \Lambda w' + \frac{1}{\bar{\rho}} p'_z &= 0 \\ fu' + L(v') + \frac{f\bar{u}}{\bar{\rho}} \rho' + \frac{1}{\bar{\rho}} p'_y &= 0 \\ g\rho' + p'_z &= 0 \\ u'_z + \frac{f}{g}(s\bar{u} + \Lambda)v' + \frac{1}{\bar{\rho}}(\bar{\rho}w')_z + \frac{1}{\bar{\rho}}L(\rho') &= 0 \\ f\left(k\bar{u} + \frac{\bar{\sigma}^2\Lambda}{g}\right)v' + gkw' - \frac{1}{\bar{\rho}}L(p') + \frac{\bar{\sigma}^2}{\bar{\rho}}L(\rho') &= 0 \end{aligned} \right\} \quad (22)$$

where the abridged notation

$$\left. \begin{aligned} s &= \partial \ln \bar{\rho} / \partial z \\ k &= 1 + \bar{\sigma}^2 s / g \\ L &= \partial / \partial t + \bar{u} \partial / \partial x \end{aligned} \right\} \quad (23)$$

has been adopted, and the subscripts  $x$ ,  $y$ , and  $z$  denote partial differentiation with respect to  $x$ ,  $y$ ,  $z$  respectively. From the definition of  $k$  it follows that

$$k = -\epsilon R(\gamma_d - \gamma) / g, \quad (24)$$

where  $\gamma_d$  is the adiabatic lapse rate. Elimination of  $\rho'$  and  $p'$  from (22) gives

$$\left. \begin{aligned} gu'_{zz} + sL^2(u') + \Lambda L(u'_z) + L^2(u'_z) \\ + f(s\bar{u} + \Lambda)v'_z - sfL(v') - fL(v'_z) \\ + \frac{g}{\bar{\rho}}(\bar{\rho}w')_{zz} + \frac{\Lambda}{\bar{\rho}}L[(\bar{\rho}w')_z] &= 0 \\ kL^2(u') + \frac{\bar{\sigma}^2\Lambda}{g}L(u'_z) + \frac{\bar{\sigma}^2}{g}L^2(u'_z) \\ - kfL(v') + f\left(k\bar{u} + \frac{\bar{\sigma}^2\Lambda}{g}\right)v'_z - \frac{\bar{\sigma}^2 f}{g}L(v'_z) \\ + \frac{\bar{\sigma}^2\Lambda}{g\bar{\rho}}L[(\bar{\rho}w')_z] + \frac{gk}{\bar{\rho}}(\bar{\rho}w')_z + \frac{\Lambda}{\bar{\rho}}L(\bar{\rho}w') &= 0 \\ \left(1 + \frac{\Lambda\bar{u}}{g}\right)u'_z - \frac{\Lambda}{g}L(u') + \frac{\bar{u}}{g}L(u'_z) \\ + f\left(\frac{f\bar{u}}{f^2} + \frac{\Lambda}{g}\right)v' - \frac{f\bar{u}}{g}v'_z + \frac{1}{f}L(v'_z) \\ + \frac{\Lambda\bar{u}}{g\bar{\rho}}(\bar{\rho}w')_z - \frac{\Lambda}{g\bar{\rho}}(s\bar{u} + \Lambda)\bar{\rho}w' &= 0 \end{aligned} \right\} \quad (25)$$

The details of the elimination will be found in appendix A.

The assumption that  $u'$ ,  $v'$ , and  $w'$  are independent of  $y$  is inconsistent with the fact that some of the

coefficients in (25) involve  $y$ . We may overcome this difficulty by a mathematical expedient: following Rossby [14], we replace  $f$ , wherever it occurs in *undifferentiated* form, and  $f_y$  by their values at the mean latitude of the disturbance. Thus, if we are concerned with waves in the zone between  $35^\circ$  and  $55^\circ$ , we replace  $f$  by the constant value  $2\Omega \sin 45^\circ$ , and we replace  $f_y$ , which may be written

$$f_y = \frac{d}{dy} (2\Omega \sin \varphi) = \frac{d}{d\varphi} (2\Omega \sin \varphi) \frac{d\varphi}{dy} = \frac{2\Omega \cos \varphi}{R},$$

where  $R$  is the radius of the earth, by  $(2\Omega \cos 45^\circ)/R$ . The quantities  $\bar{u}$  and  $\bar{T}$ , which are also involved in the coefficients, are treated in a similar manner. Since  $\bar{u}$  is independent of  $y$  in the troposphere, we need only be concerned with its variation in the stratosphere, and this usually is small. For example, if  $\gamma = 6.5 \text{ C km}^{-1}$ ,  $\Lambda = 2 \text{ m sec}^{-1}\text{km}^{-1}$ , and  $\bar{T} = 220^\circ\text{C}$ , equation (19) gives  $\partial\bar{u}/\partial y = 3.0 \text{ m sec}^{-1}(20^\circ \text{ lat.})^{-1}$ ; and, taking the mean value of  $\bar{u}$  at the tropopause to be  $25 \text{ m sec}^{-1}$ , we find that the largest proportional deviation from this value in the zone  $35^\circ$ – $55^\circ$  is only 6 per cent. We may therefore with fair approximation replace  $\bar{u}$  in the stratosphere by its mean value. Finally we assume that, in the troposphere,  $\bar{T}$  may be regarded as a constant, independent of  $y$  and  $z$ , whenever it occurs in *undifferentiated* form. This approximation is similar to that made in the study of motion in an incompressible homogeneous atmosphere moving zonally with constant speed. In this case the condition of geostrophic balance requires the height of the atmosphere to increase on the right of the current, but this height may be assumed with good approximation to be constant as long as it does not appear in differentiated form (Rossby [14]).

### 8. Form of the perturbation and definition of stability

The most general expression for the velocity components of a simple harmonic perturbation of infinite lateral extent and wave length  $L$ , traveling in the  $x$ -direction with constant velocity, is

$$\left. \begin{aligned} u' &= U(z) e^{i\mu(x-ct)} \\ v' &= V(z) e^{i\mu(x-ct)} \\ w' &= W(z) e^{i\mu(x-ct)}, \end{aligned} \right\} (26)$$

where  $\mu = 2\pi/L$ . While  $\mu$  is always real and positive,  $c$  may be complex, i.e.,

$$c = c_r + ic_i$$

and the functions  $U$ ,  $V$ , and  $W$  may also be complex. If  $c$  is complex, the exponential factor in (26) becomes

$$\exp [i\mu(x - c_r t)] \exp [\mu c_i t].$$

The first factor represents a sinusoidal wave of con-

stant amplitude traveling with the velocity  $c_r$ . The second factor either increases indefinitely or decreases to zero, according as  $c_i$  is positive or negative. In the first instance the wave is said to be unstable, in the second it is said to be stable, and, if  $c_i = 0$ , the wave is said to be neutral.

### 9. The barotropic wave

As an introduction to the general problem, the solution for the barotropic wave is derived here. From the condition of barotropy

$$\rho = \rho(p),$$

together with equations (5, 9, 10), we obtain

$$k = \Lambda = 0.$$

Simplifying the perturbation equation (25) by means of these relations and introducing the expressions (26) for  $u'$ ,  $v'$ , and  $w'$ , we obtain

$$\left. \begin{aligned} i\mu g[1 + (s/g)(\bar{u} - c)^2]U + sfcV \\ + i\mu(\bar{u} - c)^2 U_z - f(\bar{u} - c)V_z = -(g/\rho)(\bar{\rho}W)_z \\ i\mu(\bar{u} - c)^2 U_z - f(u - c)V_z = 0 \\ i\mu U + (1/f)[f_y - \mu^2(\bar{u} - c)]V \\ + (i\mu\bar{u}/g)(\bar{u} - c)U_z - (f\bar{u}/g)V_z = 0. \end{aligned} \right\} (27)$$

The last two equations show that  $U$  and  $V$  are constant, and elimination of  $U$  between the first and third equations gives

$$\left\{ \frac{\partial \bar{\rho}}{\partial z} \left[ c + \frac{\mu^2}{f^2} (\bar{u} - c - u_c)(\bar{u} - c)^2 \right] + \frac{\mu^2}{f^2} g\bar{\rho}(\bar{u} - c - u_c) \right\} V = -\frac{g}{f} (\bar{\rho}W)_z, \quad (28)$$

where the quantity  $u_c$  is defined by

$$u_c = \frac{f_y}{\mu^2} = \frac{\Omega L^2 \cos \varphi}{2\pi^2 R} \quad (29)$$

and is called the 'critical speed' by Bjerknes and Holmboe [7]. Integrating (28) from 0 to  $z$ , and utilizing the condition  $W(0) = 0$ , we obtain

$$\left\{ (\bar{\rho} - \bar{\rho}_0) \left[ c + \frac{\mu^2}{f^2} (\bar{u} - c - u_c)(\bar{u} - c)^2 \right] - \frac{\mu^2}{f^2} (\bar{\rho} - \bar{\rho}_0)(\bar{u} - c - u_c) \right\} V = -\frac{g\bar{\rho}}{f} W, \quad (30)$$

where  $\bar{\rho}_0$  and  $\bar{p}_0$  are the mean surface density and pressure. Evaluating the terms at  $z = \infty$ , we derive the following equation for the wave velocity:

$$\bar{u} - c - u_c = \frac{f^2}{\mu^2 g H_0} \frac{c}{(\bar{u} - c)^2}, \quad (31)$$

where  $H_0 = R\bar{T}_0/g$  is the height of a homogeneous

atmosphere whose surface temperature  $\bar{T}_0$  is equal to the mean surface temperature of the barotropic atmosphere. This equation was derived by Rossby [14] for an incompressible atmosphere and by Holmboe [12] for a barotropic atmosphere.

Introduction of  $\bar{u} - c - u_c$ , from (31), into (30) gives

$$W = -\frac{fc}{g} \left[ 1 - \frac{gH - (\bar{u} - c)^2}{gH_0 - (\bar{u} - c)^2} \right] V, \quad (32)$$

where  $H = R\bar{T}/g$ . It can be shown that the equations of the present section hold not only for an adiabatic barotropic atmosphere but also for an arbitrary barotropic atmosphere. In particular, we may set  $H = H_0$  in (32) and deduce the interesting conclusion that  $W \equiv 0$  in an isothermal barotropic atmosphere.

Since (31) is of the third degree in  $c$ , it has three roots. Two of the roots can be shown to be nearly equal to the solutions of

$$(\bar{u} - c)^2 - gH_0 = 0, \quad (33)$$

which is Lagrange's equation for gravitational waves in a moving fluid. Writing

$$gH_0 = R\bar{T}_0 = \bar{\sigma}_0^2/\epsilon,$$

we see that the gravitational wave speed is of the order of magnitude of  $\sigma$ , the speed of sound. As we are here concerned only with long waves whose speeds are very small compared with that of sound, we may ignore  $(\bar{u} - c)^2$  in comparison with  $gH_0$  in (31) and so obtain the equation

$$\bar{u} - c - u_c = \frac{f^2 c}{\mu^2 g H_0}, \quad (34)$$

whose solution can be shown to be nearly equal to the third root of equation (31).

The preceding considerations suggest the following principle which will be adopted in the sequel: whenever  $\bar{u}$  or  $c$  occurs in conjunction with  $\sigma$  in a mathematical expression, the expression can always be simplified by means of any of the inequalities

$$1 \gg \frac{\bar{u}^2}{\bar{\sigma}^2}, \quad \frac{\bar{u}(\bar{u} - c)}{\bar{\sigma}^2}, \quad \frac{c(\bar{u} - c)}{\bar{\sigma}^2}, \quad \frac{c^2}{\bar{\sigma}^2}. \quad (35)$$

By means of this principle we can separate out the gravitational and acoustic wave components from our solution and leave only the long wave component.<sup>2</sup>

<sup>2</sup> Strictly speaking, sound waves, whose vertical accelerations are of the same order of magnitude as the vertical pressure forces per unit mass, are excluded by the requirement of quasi-hydrostatic equilibrium. An important exception occurs when the wave fronts are planes perpendicular to the ground. In this case, since sound waves are longitudinal, the vertical acceleration vanishes identically; the force of gravity is exactly balanced by the vertical pressure force, and the sound waves are consequently indistinguishable from long gravitational waves. In order for such waves to exist the atmosphere must be *isothermal*: if the

It will now be shown that  $v'$ , the meridional component of the perturbation velocity, is approximately geostrophic. From (22) and (26) we derive

$$v' - v'_{gs} = v' - \frac{1}{\bar{\rho}f} p'_x = \frac{1}{f} L(u') = \frac{i\mu}{f} (\bar{u} - c) U e^{i\mu(x-ct)},$$

and substituting for  $U$  in terms of  $V$  from the third of equations (27), we obtain

$$\bar{v}' - v'_{gs} = \eta V e^{i\mu(x-ct)} = \eta v', \quad (36)$$

where

$$\eta = (\mu^2/f^2)(\bar{u} - c)(\bar{u} - c - u_c). \quad (37)$$

We now replace  $(\bar{u} - c - u_c)$  by its value from (34) and obtain

$$\eta = \frac{c(\bar{u} - c)}{gH_0} = \frac{\epsilon c(\bar{u} - c)}{\bar{\sigma}_0^2}, \quad (38)$$

which by (35) is much less than 1. Hence from (36)

$$v' \approx v'_{gs}.$$

## 10. Reduction of the perturbation equations

Returning to the problem of the baroclinic wave, we substitute the wave expressions (26) into the perturbations (25) and obtain

$$-(g/\bar{\rho})(\bar{\rho}W)_z = i\mu g U + f(\Lambda + sc)V + i\mu(\bar{u} - c)^2 U_z - f(\bar{u} - c)V_z \quad (39)$$

$$\begin{aligned} \frac{\bar{\sigma}^2 \Lambda}{g\bar{\rho}} (\bar{u} - c)(\bar{\rho}W)_z + \frac{gk}{\bar{\rho}} (\bar{\rho}W) \\ = -i\mu \left[ k(\bar{u} - c)^2 + \frac{\bar{\sigma}^2 \Lambda}{g} (\bar{u} - c) \right] U \\ - f \left( kc + \frac{\bar{\sigma}^2 \Lambda}{g} \right) V - \frac{i\mu \bar{\sigma}^2}{g} (\bar{u} - c)^2 U_z \\ + \frac{\bar{\sigma}^2 f}{g} (\bar{u} - c) V_z \quad (40) \end{aligned}$$

$$\begin{aligned} -\frac{\Lambda \bar{u}}{g\bar{\rho}} (\bar{\rho}W)_z + \frac{\Lambda}{g\bar{\rho}} (su + \Lambda) \bar{\rho}W \\ = i\mu U - f \left( \chi - \frac{\Lambda}{g} \right) V \\ + \frac{i\mu}{g} \bar{u}(\bar{u} - c) U_z - \frac{f\bar{u}}{g} V_z, \quad (41) \end{aligned}$$

wave fronts are to remain perpendicular to the ground, the relative wave speed  $(dp/d\rho)^{1/2}$  must be constant; therefore it follows from (12) that  $\bar{T}$  must be constant and  $(dp/d\rho)^{1/2}$  equal to  $(RT)^{1/2}$ , the Newtonian velocity of sound. Since the relative speed of long gravitational waves in an *arbitrary* barotropic atmosphere depends only on the mean surface temperature (33), we have an explanation for the fact that this speed is always equal to the Newtonian velocity of sound. Furthermore, since (33) applies in the limiting case of constant density if  $H_0$  is interpreted as the actual height, we see why  $RT/g$  must be the height of the homogeneous atmosphere.

where

$$\chi = (\mu^2/f^2)(\bar{u} - c - u_c)$$

and the following inequalities are assumed to hold:

$$1 \gg \left\{ \left| \frac{(\bar{u} - c)^2}{\bar{\sigma}^2} \right|, \left| \frac{u(\bar{u} - c)}{\bar{\sigma}^2} \right|, \left| \frac{c(\bar{u} - c)}{\bar{\sigma}^2} \right|, \left| \frac{\bar{u}^2}{\bar{\sigma}^2} \right| \right\} \sim 10^{-2}-10^{-3} \quad (42)$$

$$1 \gg \left\{ \left| \frac{s(\bar{u} - c)^2}{g} \right|, \left| \frac{s\bar{u}(\bar{u} - c)}{g} \right|, \left| \frac{sc(\bar{u} - c)}{g} \right|, \left| \frac{s\bar{u}^2}{g} \right| \right\} \sim 10^{-2}-10^{-3} \quad (43)$$

$$1 \gg \left\{ \left| \frac{\Lambda(\bar{u} - c)}{g} \right|, \left| \frac{\Lambda\bar{u}}{g} \right| \right\} \sim 10^{-2}-10^{-3}. \quad (44)$$

The justification of the first set follows directly from (35) or may be verified by the substitution of observed values of  $c$ ,  $\bar{u}$ , and  $\bar{\sigma}$ . The orders of magnitude obtained by the latter method are shown at the right. The second set receives a similar justification since  $s/g$  is of the order of magnitude of  $1/\bar{\sigma}^2$ . To show this we write

$$-\frac{s}{g} = -\frac{1}{g\bar{\rho}} \frac{\partial \bar{\rho}}{\partial z} = -\frac{1}{g\bar{\rho}} \frac{\partial \bar{\rho}}{\partial z} + \frac{1}{g\bar{T}} \frac{\partial \bar{T}}{\partial z} = \frac{1 - \gamma R/g}{R\bar{T}} \sim \frac{1}{R\bar{T}} = \frac{\epsilon}{\bar{\sigma}^2}. \quad (45)$$

The last set of inequalities (44) can be demonstrated by exhibiting them in a form similar to that of the first two. For instance, the second inequality in (44) may be written

$$1 \gg \frac{\Lambda\bar{u}}{g} = \frac{\Lambda z\bar{u}}{gz} = \frac{(\bar{u} - \bar{u}_0)\bar{u}}{gz}.$$

Since the expression on the extreme right is an increasing function of  $z$ , it has its maximum value at the tropopause. Hence, if the height of the tropopause is  $z_T$ , we must show that

$$1 \gg \frac{[\bar{u}(z_T) - \bar{u}_0]\bar{u}(z_T)}{gz_T}.$$

But  $gz_T$  is of the same order of magnitude as the dynamic height,  $R\bar{T}_0$ , of a homogeneous atmosphere, so that the last inequality is equivalent to

$$1 \gg \frac{[\bar{u}(z_T) - \bar{u}_0]\bar{u}(z_T)}{R\bar{T}_0} < \frac{\bar{u}^2(z_T)}{R\bar{T}_0} = \frac{\epsilon \bar{u}^2(z_T)}{\bar{\sigma}^2},$$

which was established in (42). Finally, the first of the inequalities (44) can be demonstrated by similar reasoning.

By linear combination of equations (39-41) and

the use of the inequalities (42-44), we obtain (see appendix B)

$$(g\bar{u}/\bar{\rho})(\bar{\rho}W)_z = -i\mu gcU - f(\Lambda c + s\bar{u}c + g\eta)V, \quad (46)$$

$$(gk\bar{u}/\bar{\rho})(\bar{\rho}W) = i\mu\bar{\sigma}^2(\bar{u} - c)U - f(k\bar{u}c + \bar{\sigma}^2\Lambda c/g + \bar{\sigma}^2\eta)V, \quad (47)$$

$$0 = -i\mu U + f\left(\chi - \frac{\Lambda}{g}\right)V - \frac{i\mu}{g}\bar{u}(\bar{u} - c)U_z + \frac{f\bar{u}}{g}V_z, \quad (48)$$

and eliminating  $\bar{\rho}W$  between (46) and (47) we get

$$i\mu[(g\bar{u} + \epsilon R\gamma\bar{u} + \bar{\sigma}^2\Lambda)c - \epsilon g\bar{u}^2]U + f[(\bar{u} + \epsilon R\gamma\bar{u}/g + \bar{\sigma}^2\Lambda/g)\Lambda c + g\eta\bar{u} - \bar{u}^2(\bar{\sigma}^2\eta/\bar{u})_z]V + i\mu\bar{\sigma}^2\bar{u}(\bar{u} - c)U_z - f(kc\bar{u}^2 + \bar{\sigma}^2\Lambda\bar{u}c/g + \bar{\sigma}^2\eta\bar{u})V_z = 0. \quad (49)$$

By linear combination of equations (46-49) and use of inequalities (42-44), we at last arrive at the system,

$$i\mu U = f(\chi - \Lambda/g)V + (1 - \eta)(f\bar{u}/g)V_z, \quad (50)$$

$$g^2kW = -f(gkc + \bar{\sigma}^2\Lambda)V + (1 - \eta)f\bar{\sigma}^2(\bar{u} - c)V_z, \quad (51)$$

$$-(g/\bar{\rho})(\bar{\rho}W)_z = f(cs + g\chi)V + (1 - \eta)fcV_z, \quad (52)$$

where  $U$ ,  $V$ ,  $W$ , and  $(\bar{\rho}W)_z$  are now expressed in terms of  $V$  and  $V_z$ .

The quantity  $\eta$  appearing in (50-52) and defined by (37) was shown to be entirely negligible in the case of the barotropic wave. The following argument will serve to demonstrate that  $\eta$  is always small. Referring to (37) we observe that  $\eta$  is of the same order of magnitude as the quantity  $\mu^2c^2/f^2$ , which may be written

$$\left(\frac{2\pi}{f}\right)^2 \frac{L^2}{c^2} = \frac{P_i^2}{P_1^2},$$

where  $P_i$  is the period of an inertial oscillation at the latitude of the long wave, and  $P_1$  is the period of the long wave. The inertial period is one half of a pendulum day, or approximately 17 hours at  $45^\circ$ . On the other hand, the period of oscillation of the long waves in the atmosphere is of the order of three to four days. The ratio  $P_i^2/P_1^2$  is therefore of the order of 0.05 and may be neglected in comparison with 1.

The approximation  $\eta \ll 1$  has an interpretation similar to that given for the barotropic wave; when taken together with the inequalities (42-44), it is equivalent to the assumption that  $v'$  is nearly geostrophic; for if  $v'$  is assumed to be geostrophic, then without further assumption a system identical to (50-52), without the  $\eta$ -term, will result; and, conversely, if we begin with the system (50-52) without the  $\eta$ -term, it can be shown by retracing the steps that  $v'$  will be geostrophic.

Equations (50-52) can now be replaced by the

system

$$i\mu U = f(\chi - \Lambda/g)V + (f\bar{u}/g)V_z, \quad (53)$$

$$g^2kW = -f(gkc + \bar{\sigma}^2\Lambda)V + f\bar{\sigma}^2(\bar{u} - c)V_z, \quad (54)$$

$$-(g/\bar{\rho})(\bar{\rho}W)_z = f(cs + g\chi)V + fcV_z, \quad (55)$$

which will be taken as the starting point for all future deductions.

We have in effect obtained the system of equations above by ignoring acoustic and shearing-gravitational wave components, both of which are contained in the original equations of motion. We have shown that this procedure is equivalent to assuming  $v'$  to be geostrophic. It would be desirable to have a general principle whereby this assumption could have been introduced a priori. Such a principle would be useful for eliminating what may be called the 'meteorological noises' from the problems of motion and would thereby lead to a considerable simplification of the analysis of these problems.

### 11. Generalization of the Rossby formula

Recalling the definitions of  $s$  and  $\chi$  (in sections 7 and 10 respectively), we may write for (55)

$$-(g/f)(\bar{\rho}W)_z = g\bar{\rho}(\mu/f)^2(\bar{u} - c - u_e)V + c(\bar{\rho}_zV + \bar{\rho}V_z), \quad (55')$$

and integrating from 0 to  $\infty$  we obtain, with the aid of condition (7),

$$g \frac{\mu^2}{f^2} \int_0^\infty (\bar{u} - c - u_e)V \bar{\rho} dz = -c \int_0^\infty (\bar{\rho}V)_z dz, \quad (56)$$

or

$$\int_{\bar{p}_0}^0 (\bar{u} - c - u_e)V d\bar{p} = -\frac{f^2c}{\mu^2} \bar{\rho}_0 V_0, \quad (56')$$

where  $V_0$  is the surface value of  $V$ . The last equation may be regarded as a generalization of the formula of Rossby [14] and Holmboe [12], since it reduces to (34) in the special case of a barotropic atmosphere. To show this we substitute the barotropic conditions  $V = V_0$  and  $\bar{u} = \text{const}$  and obtain

$$\bar{u} - c - u_e = \frac{f^2c\bar{\rho}_0V_0}{\mu^2\bar{\rho}_0V_0} = \frac{f^2c}{\mu^2gH_0},$$

which is identical with (34).

Equation (56'), although not the final solution for the wave velocity  $c$  since it contains the unknown amplitude factor  $V$ , is well adapted to computation from observed data. Both  $V$  and  $\bar{u}$  may be determined empirically as functions of the mean pressure, and  $c$  obtained by a numerical integration. If this is done, the right-hand term is found small in comparison with the individual left-hand terms, never amounting to more than 10 per cent of the latter, and may therefore be ignored.

While, strictly, (56') is derived for a constantly increasing zonal wind, this assumption need not have been made, and by a slight modification of the proof the equation can be shown to hold for any zonal-wind profile whose curvature is not excessive.

Let us denote the function  $V$  by

$$V = |V|e^{i\psi(z)},$$

where  $\psi$  is the phase angle between the wave at the level  $z$  and the surface wave. If the displacement of the trough between the upper wave and the surface wave is denoted by  $l$ ,  $\psi$  becomes

$$\psi = 2\pi l/L,$$

and substitution into equation (56') gives

$$c = \frac{\int_{\bar{p}_0}^0 \bar{u}|V|(\cos\psi + i\sin\psi)d\bar{p}}{\int_{\bar{p}_0}^0 |V|(\cos\psi + i\sin\psi)d\bar{p}} - u_e,$$

from which both the real and imaginary parts of  $c$  can be calculated. Ordinarily it will be found that the imaginary part is small in comparison with the real part and that the amplitude remains nearly constant. Under these circumstances we may set  $\sin\psi = 0$ ,  $\cos\psi = 1$ ,  $|V| = \text{const}$ , and obtain the simplified expression,

$$c_r = -\frac{1}{\bar{p}_0} \int_{\bar{p}_0}^0 \bar{u}(\bar{p}) d\bar{p} - u_e = \bar{u}^* - u_e, \quad (56'')$$

where  $\bar{u}^*$  is the mean value of the zonal speed averaged with respect to pressure from the bottom to the top of the atmosphere. Let  $h$  be the level at which  $\bar{u}^*$  is equal to  $\bar{u}$  so that

$$c_r = \bar{u}(h) - u_e.$$

In this form the formula for the wave speed is closely analogous to (34) without the small right-hand term. It implies that the correct value for the wave speed is obtained from (34) if the actual atmosphere is replaced by a barotropic atmosphere whose zonal wind is given by  $\bar{u}(h)$ .

Because of the practical use which has been made of (34) at the University of Chicago and in the 5-day forecasting project of the U. S. Weather Bureau, it is of interest to indicate the proper level  $h$  at which (34) applies. It has been pointed out by Holmboe that this equation is strictly true only at the level of non-divergence in the atmosphere. A determination of  $h$  by means of (56'') will therefore also fix this level.

The quantity  $\bar{u}^*$  is evaluated by calculating the area bounded by the curve  $\bar{u} = \bar{u}(\bar{p})$ , the lines  $\bar{p} = \bar{p}_0$ ,  $\bar{p} = 0$ , and the  $\bar{p}$ -axis from  $\bar{p}_0$  to 0, and dividing this area by  $\bar{p}_0$ . In fig. 2, curves of  $\bar{u}(\bar{p})$  are drawn for the following distributions of zonal speed: (A) That

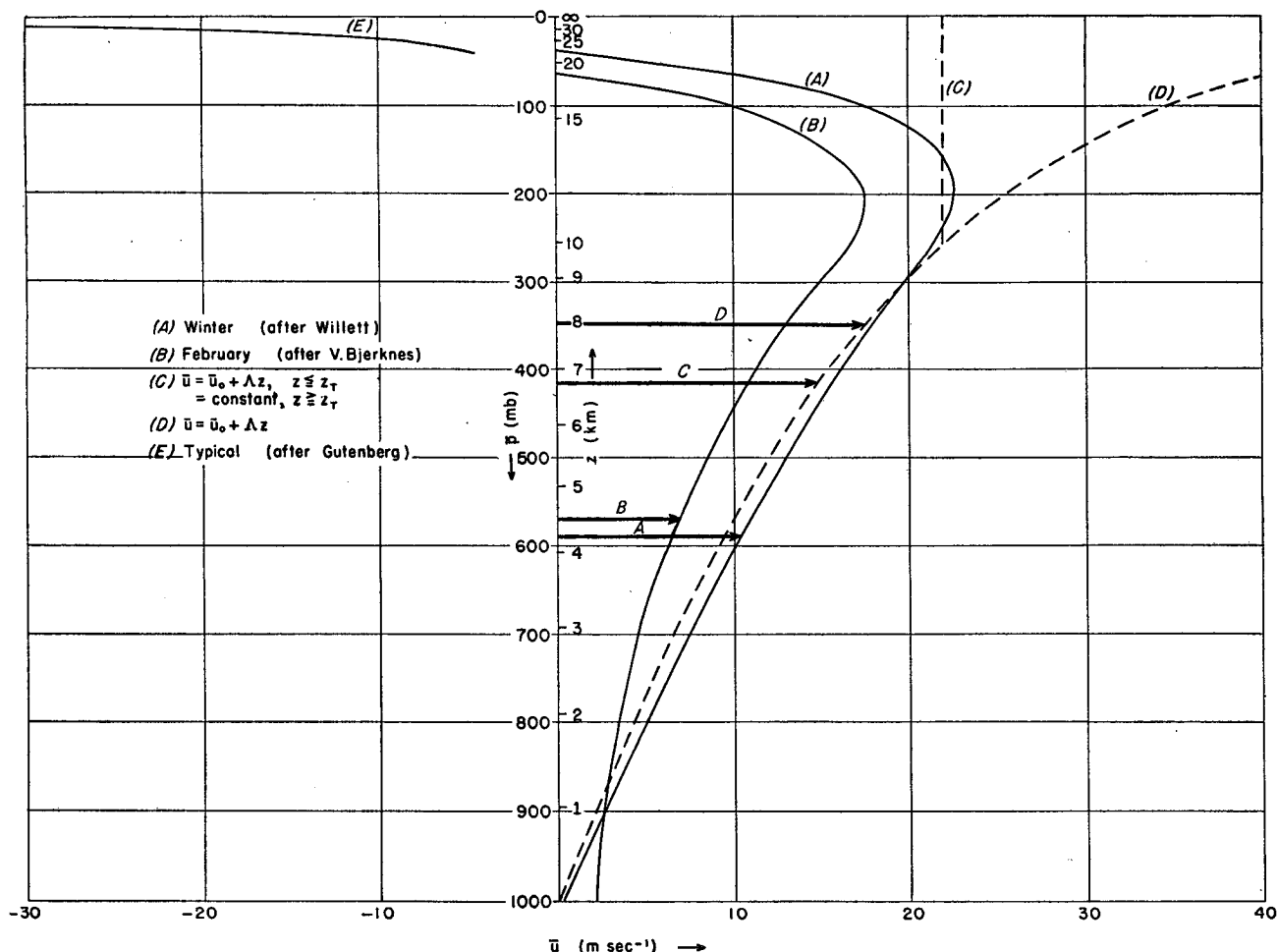


FIG. 2. Illustration of the method of calculation of  $\bar{u}^* = \bar{u}(h)$  for observed and theoretical distributions of zonal velocity.

between  $35^\circ\text{N}$  and  $55^\circ\text{N}$  compiled from mean data for the month of February by V. Bjerknes [8, p. 649]. (B) That for the same zone compiled by Willett [18] from more extensive North American winter data. (C) The distribution prescribed for the model. (D) The distribution corresponding to a constant increase of wind to the top of the atmosphere. (E) The distribution above 20 km compiled by Gutenberg [9] from mean data.

By piecing together curves (A) and (E) and (B) and (E), two curves representing somewhat different versions of the variation of zonal wind in the atmosphere are obtained. The values of  $\bar{u}^*$  are calculated for each of these curves and also for the theoretical distributions (C) and (D). These values are represented by the horizontal arrows A, B, C, and D.

Both Bjerknes's and Willett's data indicate that the zonal wind should be evaluated at a level between 4.0 km and 4.5 km, or 610 mb and 570 mb. This result agrees well with the experience at the University of Chicago, where it is found that the level at which (34) holds best is in the vicinity of 600 mb. The levels at which  $\bar{u}^*$  is evaluated for the curves (C) and (D) are found to be too high. This is to be expected since

neither of these distributions provides for a decrease of zonal velocity above the tropopause.

From the character of curve (E), which represents the variation of the zonal wind above 20 km, it can be seen that, although the velocities become very large, their contribution to the total area is very small. It may therefore be stated that the motion in the stratospheric regions above 20 km has no appreciable influence on motions in the lower troposphere.

## 12. The normal equation for $V$

Elimination of  $W$  from (54) and (55) gives

$$\frac{\bar{\sigma}^2}{g}(\bar{u} - c)V_{zz} - \epsilon(\bar{u} - c)V_z + (\epsilon\Lambda + gk\chi)V = 0. \quad (57)$$

This becomes, if the explicit values of  $\bar{\sigma}^2$ ,  $k$ , and  $\chi$  are substituted,

$$V_{zz} - \frac{1}{H}V_z - a^2 \frac{\bar{u} - c - \bar{u}_c}{\bar{u} - c}V = 0, \quad (58)$$

where

$$a^2 = \mu^2 R(\gamma_d - \gamma)/(f^2 H), \quad (59)$$

and

$$\bar{u}_c = u_c + \Lambda/(a^2 H), \quad (60)$$

and  $\Lambda$  and  $\gamma$  are both understood to be replaced by zero in the stratosphere.

### 13. The boundary condition for $V$

The solution of (58) is subject to the boundary conditions (6–8). Condition (6) states that  $W$  must vanish at the ground. If  $W$  is set equal to zero in (54), the condition to be satisfied by  $V$  at the ground becomes

$$\left(\frac{V_z}{V}\right)_{z=0} = \frac{gkc + \bar{\sigma}^2 \Lambda}{\bar{\sigma}^2(\bar{u} - c)}. \quad (61)$$

Condition (8) states that  $U$  and  $V$  are continuous at the tropopause. Applying this condition to  $U$  and  $V$  in equation (53) we obtain

$$V \Delta \Lambda = \bar{u} \Delta(V_z).$$

$V$  must therefore satisfy the relations

$$\left. \begin{aligned} \Delta V &= 0 \\ \frac{\Delta(V_z)}{V} &= \frac{\Delta(\bar{u}_z)}{\bar{u}} = \frac{\Lambda}{\bar{u}} \end{aligned} \right\} \quad (62)$$

at the tropopause. Condition (7) requires that

$$\lim_{z \rightarrow \infty} \bar{\rho} V = 0 \quad (63)$$

at the top of the atmosphere. Since  $\bar{\rho}$  is proportional to  $\exp[-gz/RT_s]$  in an isothermal atmosphere whose temperature  $T_s$  is equal to that of the stratosphere, this condition may be written

$$\lim_{z \rightarrow \infty} V \exp(-z/H_s) = 0 \quad (63')$$

where  $H_s = RT_s/g$ .

### 14. Solution of the normal equation

Case I,  $\Lambda = 0$

We shall first investigate the simplest case, where  $\Lambda = 0$  throughout the atmosphere. In this case, the coefficients in (58) depend only on  $\bar{T}$ , and, while it is possible to perform the integration of (58) keeping  $\bar{T}$  a linear function of  $z$  in the troposphere, no appreciable error is introduced if  $\bar{T}$  is replaced by a suitable mean value. With this simplification the coefficients become constant, and the integral can immediately be written down as follows:

$$V = Ae^{mz} + Be^{nz}, \quad (64)$$

where  $m$  and  $n$  have the values

$$\left. \begin{aligned} m &= \frac{1}{2H} - \left( \frac{1}{4H^2} + a^2 \frac{\bar{u} - c - u_c}{\bar{u} - c} \right)^{\frac{1}{2}} \\ n &= \frac{1}{2H} + \left( \frac{1}{4H^2} + a^2 \frac{\bar{u} - c - u_c}{\bar{u} - c} \right)^{\frac{1}{2}} \end{aligned} \right\} \quad (65)$$

The constants involved in the expressions for  $m$  and  $n$ , as well as the constants of integration  $A$  and  $B$ , will, in general, have different values in the troposphere and stratosphere. If we denote tropospheric quantities by the subscript T and stratospheric quantities by the subscript S, conditions (61) and (62) become

$$A_T m_T + B_T n_T = gkc[\bar{\sigma}^2(\bar{u} - c)]^{-1}(A_T + B_T), \quad (66)$$

$$A_T \exp(m_T z_T) + B_T \exp(n_T z_T) = A_S \exp(m_S z_T) + B_S \exp(n_S z_T), \quad (67)$$

$$m_T A_T \exp(m_T z_T) + n_T B_T \exp(n_T z_T) = m_S A_S \exp(m_S z_T) + n_S B_S \exp(n_S z_T). \quad (68)$$

Suppose now that  $(\bar{u} - c - u_c)/(\bar{u} - c)$  is positive;<sup>3</sup> then it follows from (65) that  $n_S > 1/H_s$ , and condition (63') gives  $B_S = 0$ . With substitution of  $B_S = 0$ , equations (66–68) become linear and homogeneous in  $A_T$ ,  $B_T$ , and  $A_S$ . A necessary and sufficient condition for their consistency is the vanishing of the determinant of the coefficients. Thus we obtain the following determinantal equation for the wave speed  $c$ :

$$\begin{vmatrix} m_T - \delta & n_T - \delta & 0 \\ \exp(m_T z_T) & \exp(n_T z_T) & \exp(m_S z_T) \\ m_T \exp(m_T z_T) & n_T \exp(n_T z_T) & m_S \exp(m_S z_T) \end{vmatrix} = 0 \quad (69)$$

where  $\delta = gkc/[\bar{\sigma}^2(\bar{u} - c)]$ . Expansion and rearrangement of terms gives

$$m_T - \delta = (n_T - \delta) \frac{m_S - m_T}{m_S - n_T} \exp[(m_T - n_T)z_T]. \quad (70)$$

Substitution of typical values of  $\bar{T}$ ,  $L$ ,  $f$ ,  $\gamma$ ,  $\bar{u}$ , and  $c$  into the right-hand term of the last equation shows that it is small in comparison with  $m_T$  and  $\delta$ . Hence, approximately,

$$m_T - \delta = 0$$

or

$$\frac{1}{2H_T} - \left( \frac{1}{4H_T^2} + a_T^2 \frac{\bar{u} - c - u_c}{\bar{u} - c} \right)^{\frac{1}{2}} = \frac{gkc}{\bar{\sigma}^2(\bar{u} - c)}.$$

Transposing, squaring, and rearranging, we obtain

$$\bar{u} - c - u_c - \frac{f^2 c}{\mu^2 g H_T} \left( 1 - \frac{k}{\epsilon} \frac{c}{\bar{u} - c} \right) = 0 \quad (71)$$

since, by (24) and (59),

$$a^2 = -g^2 \mu^2 k / (f^2 \bar{\sigma}^2).$$

The nondimensional factor  $-k/\epsilon = R(\gamma_d - \gamma)/g$  appearing in (71) measures the extent of the baroclinicity of the atmosphere. It varies from 0.0 in a barotropic

<sup>3</sup> This assumption is required for determinacy of the solution under the condition that  $\bar{\rho} V$  shall approach zero with increasing height. Professor Rossby has called my attention to the fact that negative values of  $(\bar{u} - c - u_c)/(\bar{u} - c)$  correspond to long internal waves having nodal surfaces at finite levels

atmosphere to 0.289 in an isothermal atmosphere. One root of (71) is extraneous and the other can be shown to be very close to the solution of the equation

$$\bar{u} - c - u_0 - \frac{f^2 c}{\mu^2 g H_T} \left( 1 - \frac{k \bar{u} - u_0}{\epsilon u_0} \right) = 0,$$

which, except for the occurrence of

$$\frac{1}{H_T} \left( 1 - \frac{k \bar{u} - u_0}{\epsilon u_0} \right)$$

instead of  $1/H_0$  in the small term  $f^2 c / (\mu^2 g H_0)$ , is identical to (34). Hence there is no significant difference, with respect to the motion of long waves, between a barotropic atmosphere and a baroclinic atmosphere with a constant zonal wind. One may conclude that here, at least, the statical stability of the atmosphere plays no important role.

#### Case II, $\Lambda \neq 0$

We now consider the problem of wave motion in the baroclinic model specified in section 3. We shall solve this problem by determining  $V(z)$ , the amplitude factor of the meridional component of wave velocity. All other quantities can easily be found in terms of  $V$ .

The equations governing  $V$  are (58–60). For the sake of mathematical simplicity, the following approximations are made:  $\bar{T}$  is replaced by its mean value,  $\bar{T}_T$ , in the troposphere and  $z_T$ , the height of the tropopause, is replaced by its value at a mean latitude. The quantities  $H$ ,  $a$ , and  $\bar{u}_0$  then become constant, and the boundary conditions (62) apply at the constant level  $z_T$ . The mathematical problem is now strictly determined; we must find the solution to (58) that satisfies the boundary conditions (61–63') at  $z = 0$ ,  $z_T$ , and  $\infty$ .

In the troposphere  $\Lambda \neq 0$ , and (58) can be reduced to a standard form by the following change of dependent and independent variables:

$$\left. \begin{aligned} \psi &= V \exp [(\bar{a} - \frac{1}{2} H_T^{-1}) z] \\ \xi &= (2\bar{a}/\Lambda)(\bar{u} - c) = 2\bar{a}z + (2\bar{a}/\Lambda)(\bar{u}_0 - c), \end{aligned} \right\} \quad (72)$$

where

$$\bar{a}^2 = a^2 + 1/(4H_T^2). \quad (73)$$

With this change of variables (58) becomes

$$\xi \frac{d^2 \psi}{d\xi^2} - \xi \frac{d\psi}{d\xi} + r\psi = 0, \quad (74)$$

where

$$r = \frac{\bar{a}}{2\Lambda} \left( \frac{a^2}{\bar{a}^2 u_0} + \frac{\Lambda}{\bar{a}^2 H_T} \right) = \frac{a^2 \bar{u}_0}{2\bar{a}\Lambda}. \quad (75)$$

Equation (74) is a special case of the confluent hypergeometric equation

$$\xi \frac{d^2 \psi}{d\xi^2} + (b - \xi) \frac{d\psi}{d\xi} - a\psi = 0, \quad (76)$$

which is satisfied by the functions

$$\psi = M(a, b, \xi) = 1 + \frac{a}{1 \cdot b} \xi + \frac{a(a+1)}{2! b(b+1)} \xi^2 + \dots \quad (77)$$

and

$$\psi = \xi^{1-b} M(a - b + 1, 2 - b, \xi). \quad (78)$$

In the present case only the latter integral is a solution of equation (74), since  $M(a, b, \xi)$  is undefined for  $b = 0$ . Another solution is obtained by contour integration in the complex domain; it is shown in appendix C that the following function satisfies (74):

$$\begin{aligned} \psi &= \frac{\sin \pi a}{\pi} \left\{ a \xi M(a+1, 2, \xi) \right. \\ &\quad \times \left[ \ln \xi + \frac{\Gamma'(a)}{\Gamma(a)} - 2 \frac{\Gamma'(1)}{\Gamma(1)} \right] + 1 \\ &\quad \left. + \sum_{n=1}^{\infty} B_n \frac{a(a+1) \cdots (a+n-1)}{(n-1)! n!} \xi^n \right\}, \quad (79) \end{aligned}$$

where  $a = -r$ , and

$$B_n = \sum_{\nu=0}^{n-1} \left( \frac{1}{a+\nu} - \frac{2}{1+\nu} \right) + \frac{1}{n}.$$

The tabulated values of the two functions (78) and (79) will be found in appendix D. Denoting (79) by  $\psi_1$  and (78) by  $\psi_2$ , we may take, as the general solution of (74), the linear combination

$$\psi = A\psi_1 + B\psi_2. \quad (80)$$

The solution for the stratosphere has already been determined in the investigation of the case  $\Lambda = 0$ . Recalling equation (64) we may write, since  $B_s = 0$ ,

$$V = A_s \exp(m_s z), \quad (81)$$

where  $m_s$  is defined by (65).

#### 15. Determination of the wave velocity

The arbitrary constants in the solutions (80) and (81) must be chosen in such a way that they satisfy the boundary conditions (61–63'). We see from the manner in which the solution for the stratosphere was determined that it automatically satisfies the condition (63'). For the purpose of satisfying condition (62) we change to the new variables  $\psi$  and  $\xi$  defined by (72), and obtain

$$\begin{aligned} \frac{1}{V} \frac{dV}{dz} &= - \left( \bar{a} - \frac{1}{2H_T} \right) + \frac{1}{\psi} \frac{d\psi}{d\xi} \frac{d\xi}{dz} \\ &= -\bar{a} + \frac{1}{2H_T} + \frac{2\bar{a}}{\psi} \frac{d\psi}{d\xi} \end{aligned} \quad (82)$$

in the troposphere and

$$(1/V) dV/dz = m_s \quad (83)$$

in the stratosphere. If these values of  $V_z/V$  are substituted into (62), we obtain

$$\frac{\psi'}{\psi} = \frac{1}{2} - \frac{1}{4\bar{a}H_T} + \frac{m_s}{2\bar{a}} + \frac{\Lambda}{2\bar{a}\bar{u}(z_T)}, \quad (84)$$

where  $\psi' = d\psi/d\xi$ . Let us denote the right-hand side of this equation by  $\lambda$  and the value of  $\xi$  at the tropopause by  $\xi_1$ ; then, if we substitute for  $\psi$  the expression in (80), we derive

$$\frac{A\psi_1'(\xi_1) + B\psi_2'(\xi_1)}{A\psi_1(\xi_1) + B\psi_2(\xi_1)} = \lambda. \quad (85)$$

We satisfy condition (61) by a similar procedure and obtain

$$\frac{A\psi_1'(\xi_0) + B\psi_2'(\xi_0)}{A\psi_1(\xi_0) + B\psi_2(\xi_0)} = \alpha + \frac{\beta}{\xi_0}, \quad (86)$$

where  $\xi_0$  is the value of  $\xi$  at  $z = 0$  and  $\alpha$  and  $\beta$  are defined by the equations

$$\alpha = \frac{1}{2} - gk/(2\bar{a}\bar{\sigma}^2) - 1/(4\bar{a}H_T), \quad (87)$$

$$\beta = 1 + gk\bar{u}_0/(\Lambda\bar{\sigma}^2). \quad (88)$$

Equations (85) and (86) are linear and homogeneous in  $A$  and  $B$ . The determinant of the coefficients of  $A$  and  $B$  must therefore vanish. This gives the following equation for the determination of  $c$ :

$$\frac{\Delta_{11} - \lambda\Delta_{10}}{\Delta_{01} - \lambda\Delta_{00}} = \alpha + \frac{\beta}{\xi_0}, \quad (89)$$

where

$$\Delta_{ij} = \begin{vmatrix} \psi_1^{(i)}(\xi_0) & \psi_2^{(i)}(\xi_0) \\ \psi_1^{(j)}(\xi_1) & \psi_2^{(j)}(\xi_1) \end{vmatrix}$$

and  $\psi_1^{(0)} = \psi_1$ ,  $\psi_2^{(0)} = \psi_2$ ,  $\psi_1^{(1)} = \psi_1'$ ,  $\psi_2^{(1)} = \psi_2'$ .

Equation (89) is of the form

$$F(\xi_0, \xi_1, r, \lambda, \alpha, \beta) = 0. \quad (90)$$

Since  $\alpha$ ,  $\beta$ , and  $r$  are functions of  $\Lambda$ ,  $\bar{u}_0$ , and  $L$ , while  $\xi_0$ ,  $\xi_1$ , and  $\lambda$  involve these quantities as well as  $c$ , we may write instead

$$F(\Lambda, \bar{u}_0, L, c) = 0. \quad (90')$$

This equation expresses the dependency of the wave speed  $c$  on both the wave length and the physical parameters characterizing the mean flow of the atmosphere. As we are not primarily concerned with the variation of the parameters  $\gamma$ ,  $\varphi$ ,  $\bar{T}_0$ ,  $\bar{T}_T$ , and  $\bar{z}_T$ , they are given appropriate constant values. We shall, however, be able to say something about the influence of  $\varphi$  and  $\gamma$  on stability.

Equation (89) is solved by a graphical method. The left- and right-hand sides of the equation are plotted separately as functions of  $\xi_0$ . The graphs consist of two triply infinite families of curves depending on the parameters  $\Lambda$ ,  $\bar{u}_0$ , and  $L$ . The points of intersection of

members corresponding to the same values of  $L$ ,  $\Lambda$ , and  $\bar{u}_0$  give the roots  $\xi_0$ . Since  $\xi_0$  is a function of  $c$ , the value of  $c$  is determined. The points of tangency give the critical values of the parameters at which  $c$  becomes complex. Since (89) has analytic functions of  $\xi_0$  on each side, except at  $\xi_0 = 0$ , the roots of this equation will occur in conjugate complex pairs. According to the definition of stability in section 8, a value of  $c$  with a positive imaginary part corresponds to an amplified wave, and a value of  $c$  with a negative imaginary part corresponds to a damped wave. Thus, if complex roots exist at all, there must be both stable and unstable wave components. Presumably the stable component is damped out as soon as it is formed, so that we may limit our considerations to neutral and unstable waves.

Fig. 3 shows the critical curve for  $\bar{u}_0 = 0$  calculated from (89). Values of  $\Lambda$  and  $L$  at points above the curve

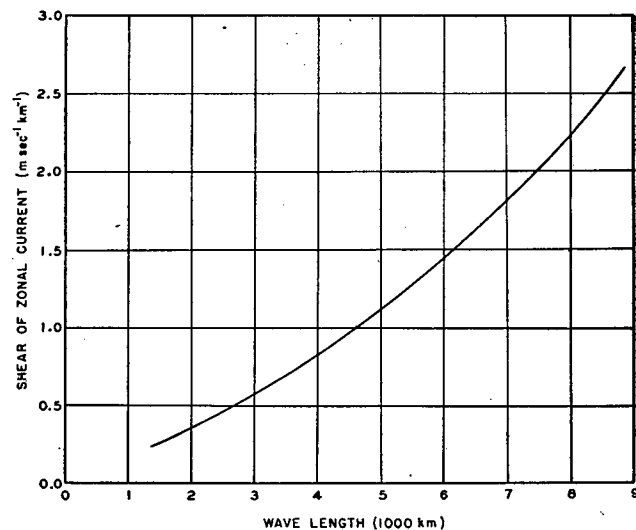


FIG. 3. The critical stability curve for  $\bar{u}_0 = 0$ .

correspond to instability and values at points below correspond to stability. The curve shows that instability increases with shear and diminishes with wave length. Although calculated for a particular value of  $\bar{u}_0$ , the curve represents approximately the stability criteria for a wide range of  $\bar{u}_0$ . A proof of this assertion will be given in section 16. Here, to make the assertion plausible, the explanation may be offered that increases in  $\bar{u}_0$  unaccompanied by variations in  $\Lambda$  merely impart to the atmosphere a slightly greater absolute rate of rotation without changing the relative motion of its parts. Such a change should not be expected to affect the stability beyond increasing slightly the gyroscopic stability that a rotating body has by virtue of the conservation of angular momentum.

In order to avoid the extremely laborious computations involved in a further analysis of (89), we shall assume that the zonal wind in the atmospheric model is defined in the stratosphere as well as in the tropo-

sphere by the function  $\bar{u}_0 + \Lambda z$ . This assumption eliminates the necessity of piecing together separate solutions for the two atmospheric layers, but is of course subject to the criticism that the zonal winds in the stratosphere will here depart considerably from the observed values. However, the influence of this discrepancy upon the wave motion cannot be great, for the exponential decrease of density with height reduces the zonal momentum to a negligible value and nullifies the influence of the increasingly large zonal winds. Thus it is shown in section 11 that the rapid increase of the magnitude of the zonal wind above 20 km has no perceptible influence on the wave speed in the troposphere. The effects produced at lower stratospheric levels by deviations of the postulated from the observed winds are indicated in fig. 2, where it will be seen that the change from a constant to an increasing zonal-wind distribution lifts the mean level of nondivergence 68 mb. On the other hand, it is proved in section 16 that the stability criteria are left virtually unaltered by this change.

The problem now reduces to the integration of the single equation (74). It was shown in case II of section 14 that two independent integrals are given by the functions  $\psi_1$  and  $\psi_2$ . For the present purpose, it is more convenient to employ two other integrals. It is demonstrated in appendix C that two independent integrals of (74) are

$$W_1 = \frac{1}{2\pi i} \int_{\gamma_1} \left(1 - \frac{\xi}{t}\right)^r e^t dt \quad (91)$$

and

$$W_2 = \frac{1}{2\pi i} \int_{\gamma_2} \left(1 - \frac{\xi}{t}\right)^r e^t dt, \quad (92)$$

where the paths of integration  $\gamma_1$  and  $\gamma_2$  are shown in fig. 4. The asymptotic expansions of  $W_1$  and  $W_2$  are given by

$$W_1 \sim [(-\xi)^r / \Gamma(r)] G(-r, 1-r; -\xi), \quad (93)$$

$$W_2 \sim [\xi^{-r} e^{\xi} / \Gamma(-r)] G(1+r, r; \xi), \quad (94)$$

where

$$G(\mu, \nu; \xi) = 1 + \frac{\mu\nu}{1! \xi} + \frac{\mu(\mu+1)\nu(\nu+1)}{2! \xi^2} + \dots \quad (95)$$

The general solution of (74) may be written

$$\psi = A_1 W_1 + A_2 W_2. \quad (96)$$

This function must satisfy the boundary condition (63')

$$\lim_{z \rightarrow \infty} V \exp\left(-\frac{z}{H_s}\right) = 0.$$

The behavior of the functions  $W_1$  and  $W_2$  at infinity is determined by the asymptotic expansions (93) and (94). Transferring back to the variables  $V$  and  $z$ , we

find that

$$V_1 = W_1(\xi) \exp\left[-(\bar{a} - \frac{1}{2}H_T^{-1})z\right] \sim K_1 z^r \exp\left[-(\bar{a} - \frac{1}{2}H_T^{-1})z\right], \quad (93')$$

and

$$V_2 = W_2(\xi) \exp\left[-(\bar{a} - \frac{1}{2}H_T^{-1})z\right] \sim K_2 z^{-r} \exp\left[(\bar{a} + \frac{1}{2}H_T^{-1})z\right], \quad (94')$$

where  $V_1$  and  $V_2$  are particular solutions of (58) corresponding to  $W_1$  and  $W_2$ , and  $K_1$  and  $K_2$  are constants. Since

$$\bar{a} = \left(a^2 + \frac{1}{4H_T^2}\right)^{\frac{1}{2}} > \frac{1}{2H_T} \approx \frac{1}{2H_s},$$

the first of the expansions above shows that

$$V_1 \exp(-z/H_s) \rightarrow 0,$$

whereas the second shows that

$$V_2 \exp(-z/H_s) \rightarrow \infty.$$

Consequently, (96) can satisfy the boundary condition at infinity only if  $A_2 = 0$ , i.e., if  $\psi = A_1 W_1$ . Now it can be shown that the function  $\psi_1$ , defined by (79), is equal to  $(-1)^{-r} W_1$  (see appendix C). Hence the appropriate solution of (74) is given by a constant multiple of  $\psi_1$ .

An equation similar to (86) is obtained from the requirement that  $\psi$  satisfy the surface boundary condition (61). In the present case, instead of two functions

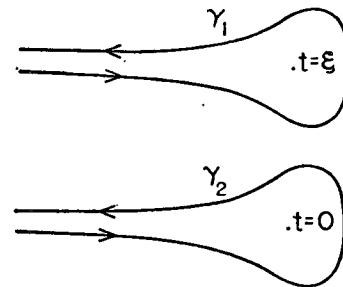


FIG. 4. Paths of integration for the integrals  $W_1$  and  $W_2$ .

$\psi_1$  and  $\psi_2$ , only the single function  $\psi_1$  is involved, and the resulting equation becomes

$$\frac{\psi_1'(\xi_0, r)}{\psi_1(\xi_0, r)} = \alpha + \frac{\beta}{\xi_0}, \quad (97)$$

where, as before,

$$\alpha = \frac{1}{2} - gk/(2\bar{a}\bar{\sigma}^2) - 1/(4\bar{a}H_T) = \frac{1}{2} + \Delta\alpha,$$

$$\beta = 1 + gk\bar{u}_0/(\Lambda\bar{\sigma}^2) = 1 + \Delta\beta,$$

$$r = \frac{\bar{a}}{2\Lambda} \left( \frac{a^2 u_c}{\bar{a}^2} + \frac{\Lambda}{\bar{a}^2 H_T} \right).$$

For convenience in following the ensuing discussion the definitions of the quantities involved in  $\xi_0$ ,  $r$ ,  $\alpha$ , and  $\beta$  are here reproduced:

$\Omega$  = angular velocity of earth,  
 $\varphi$  = latitude,  
 $\gamma$  = lapse rate in troposphere,  
 $\gamma_a$  = adiabatic lapse rate,  
 $R$  = gas constant,  
 $\mathcal{R}$  = radius of earth,  
 $\bar{T}_T$  = mean temperature in troposphere,  
 $L$  = wave length,  
 $\bar{u}_0$  = mean surface zonal speed,  
 $\Lambda$  = shear of mean zonal speed,  
 $f = 2\Omega \sin \varphi$ ,  $\mu = 2\pi/L$ ,  $\bar{u} = \bar{u}_0 + \Lambda z$ ,  
 $\bar{\sigma}^2 = \epsilon R \bar{T}_T$ ,  $H_T = R \bar{T}_T / g$ ,  $k = -(\epsilon R / g)(\gamma_a - \gamma)$ ,  
 $u_c = \frac{\Omega L^2 \cos \varphi}{2\pi^2 \mathcal{R}}$ ,  $a^2 = \frac{\mu^2 R}{f^2 H_T} (\gamma_a - \gamma)$ ,  $\bar{a}^2 = a^2 + \frac{1}{4H_T^2}$

In order to solve (97) for  $c$ , suitable numerical values must be assigned to the constant parameters  $\varphi$ ,  $\gamma$ , and  $\bar{T}_T$ . The following values are selected:

$$\begin{aligned}\varphi &= 45^\circ, \\ \gamma &= 6.5 \text{ C km}^{-1}, \\ \bar{T}_T &= 260 \text{ C}.\end{aligned}$$

With the assignment of these quantities, we may derive numerical expressions for the parameters  $\xi_0$ ,  $r$ ,  $\alpha$ , and  $\beta$  in terms of the fundamental parameters  $L$ ,  $\Lambda$ ,

and  $\bar{u}_0$ . Thus,

$$\begin{aligned}\bar{a} &= (0.48/L^2 + 4.0 \times 10^{-9})^{\frac{1}{2}}, \\ \xi_0 &= 2\bar{a}(\bar{u}_0 - c) \times 10^3/\Lambda, \\ \Delta\alpha &= \alpha - \frac{1}{2} = -2.50 \times 10^{-5}/\bar{a}, \\ \Delta\beta &= \beta - 1 = -\bar{u}_0/(78\Lambda), \\ r &= -\Delta\alpha(2.52 + 3.90/\Lambda),\end{aligned}$$

where  $L$  is measured in km,  $\bar{u}_0$  and  $c$  in m sec<sup>-1</sup>, and  $\Lambda$  in m sec<sup>-1</sup>km<sup>-1</sup>.

Equation (97) is solved for  $\xi_0$ , and hence  $c$ , by graphing the quantities  $\psi_1'/\psi_1$  and  $\alpha + \beta/\xi_0$  as functions of  $\xi_0$ . (The values of  $\psi_1$  and  $\psi_1'$  are given in tables 1 and 2 of the appendix, and the graphs of  $\psi_1'/\psi_1$  are shown in fig. 5.) The intersections of the graphs of  $\psi_1'/\psi_1$  and  $\alpha + \beta/\xi_0$  determine the roots. The critical values of  $\xi_0$  separating real from complex solutions are determined by the points of tangency. The roots are most conveniently represented graphically by plotting  $\xi_0$  as a function of  $r$  for constant values of  $\alpha$  and  $\beta$ . The graphs obtained in this way are shown in fig. 6. The parameters  $\varphi$ ,  $\gamma$ , and  $\bar{T}$  need not have been specified in this type of representation, so that the effect on the wave velocity of varying these parameters can be studied.

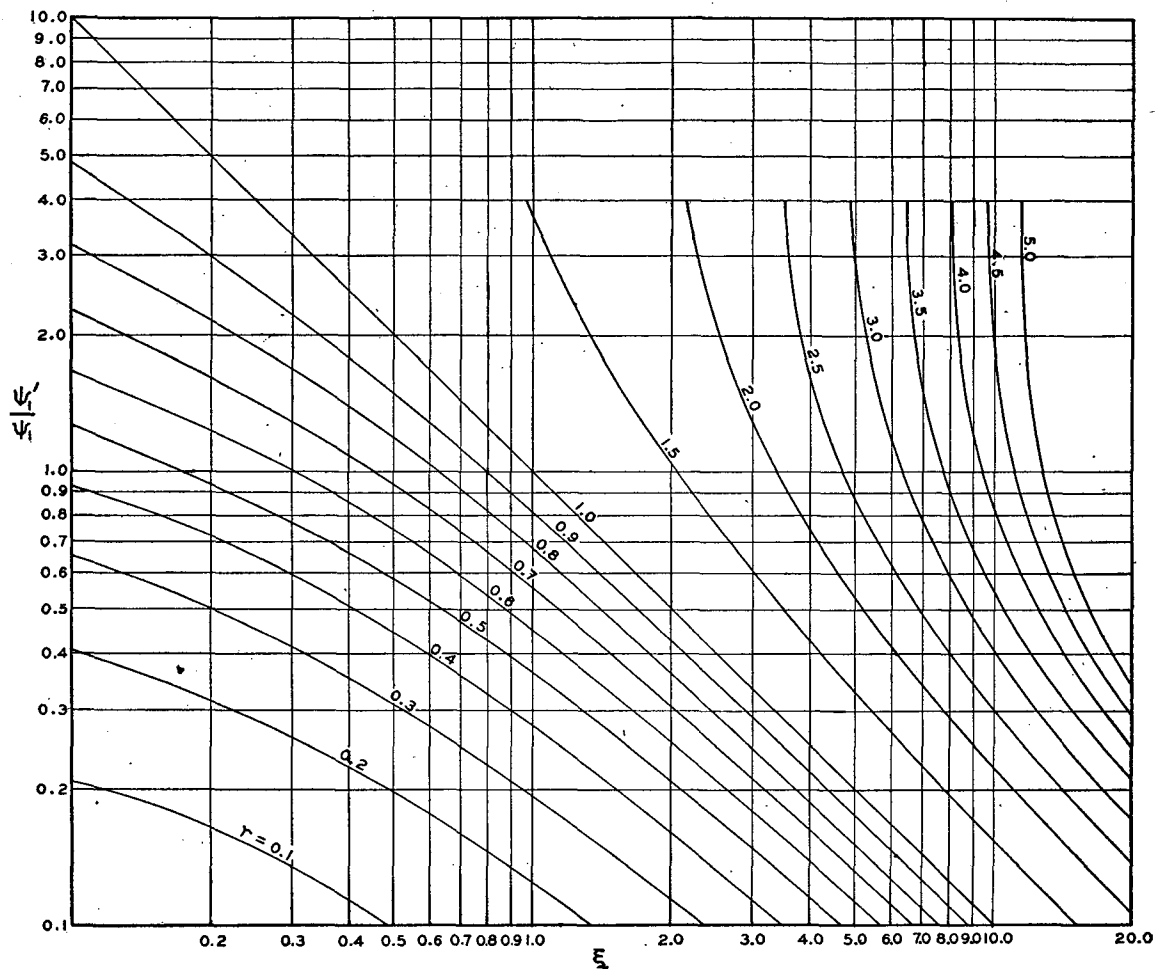
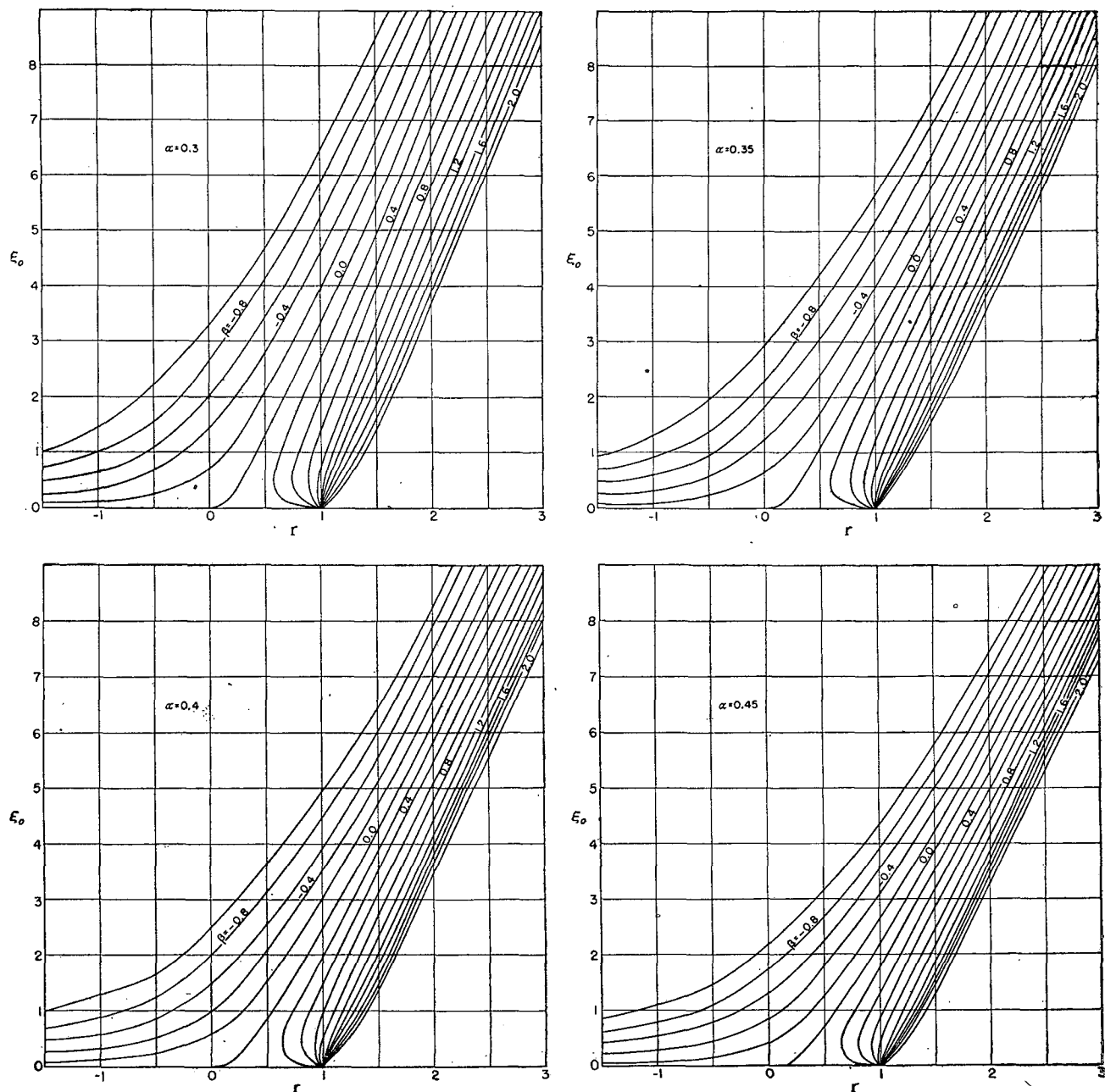


FIG. 5. Graph of the function  $\psi_1'(\xi, r)/\psi_1(\xi, r)$ .

FIG. 6. Representation of  $\xi_0$  as a function of  $r$ ,  $\alpha$ , and  $\beta$ .

### 16. The stability criteria

The upper and lower curves in fig. 7 represent the critical values of  $L$  and  $\Lambda$  for  $\bar{u}_0 = 0$  and  $\bar{u}_0 = 20$  m sec<sup>-1</sup>, calculated by means of the approximative process described in the last section. The middle curve is a reproduction of fig. 3 and represents the critical values of  $L$  and  $\Lambda$  for  $\bar{u}_0 = 0$  calculated without approximation directly from (89). The accuracy of the approximation is attested by the close correspondence of the two curves for  $\bar{u}_0 = 0$ . It will also be seen that, as anticipated, the influence of the variation in  $\bar{u}_0$  on stability is small; a very slight increase in stability accompanies a large increase in  $\bar{u}_0$ . As before, the

curves show a slightly greater than linear increase of the critical shear with increasing wave length.

The influence of lapse rate on stability can be ascertained from the representation in fig. 6 of  $\xi_0$  as a function of  $r$ ,  $\alpha$ , and  $\beta$ . It is found that instability increases both with lapse rate and with latitude.

### 17. The wave velocity

#### Case I, the neutral wave

The real solutions of (97) correspond to neutral waves, and one result of a consideration of these solutions is that the relative zonal velocity,  $u_0 - c$ , is

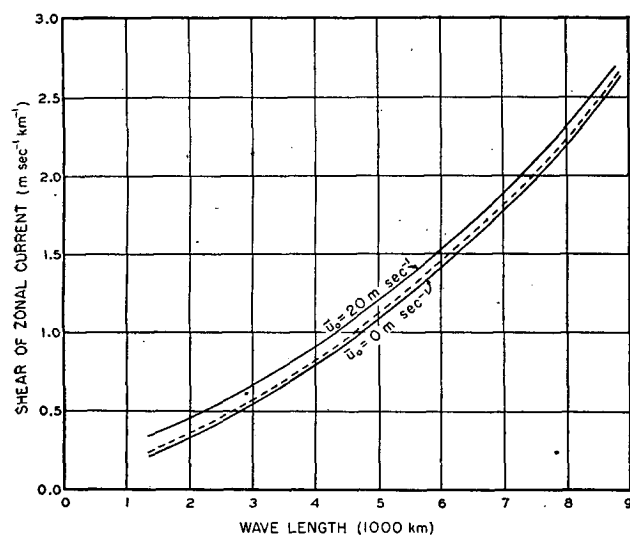


FIG. 7. Approximate critical stability curves for  $\bar{u}_0 = 0$  and  $\bar{u}_0 = 20 \text{ m sec}^{-1}$ . The dashed curve is a duplicate of the exact curve in fig. 3.

always positive. This is a consequence of the fact that  $\ln \xi_0$  appears in the expression for both  $\psi_1(\xi_0)$  and  $\psi_1'(\xi_0)$ . If  $\xi_0$  were real and negative,  $\ln \xi_0$  would be complex, and the left-hand side of (97) then would also be complex; but this is an impossibility because the right-hand side is always real for real values of  $\xi_0$ . Since  $\xi_0 = 2\bar{a}(\bar{u}_0 - c)/\Lambda$ , we conclude that  $\bar{u}_0 - c$  is always positive.

The speed of the neutral wave is a function of the parameters  $L$ ,  $\Lambda$ , and  $\bar{u}_0$ . The dependency on  $L$  and  $\Lambda$  when  $\bar{u}_0 = 0$  is illustrated in fig. 8. Changes in  $\bar{u}_0$  have a small effect upon the character of the curves. The exact nature of this effect is illustrated in fig. 9, where  $\bar{u}_0 - c$  is plotted as a function of  $\Lambda$  for different values of  $\bar{u}_0$  and for  $L = 4150 \text{ km}$ . Referring to fig. 8 we note that the limiting curve corresponding to  $\Lambda = 0$  virtually coincides with the graph of

$$\bar{u} - c - u_c = f^2 c / (\mu^2 g H_0).$$

This result was anticipated when it was shown in section 14, case I, that waves in a baroclinic atmosphere with zero shear have virtually the same velocity as waves in the corresponding barotropic atmosphere.

The curves in fig. 8 show that the wave velocity, at constant wave length and constant surface zonal speed, increases rapidly with increasing shear; and that for  $\Lambda > 0$  we always have

$$0 < \bar{u} - c < u_c.$$

#### Case II, the unstable wave

In order to calculate the speed of the unstable wave it would be necessary to evaluate the complex roots of (97), and this is an extremely difficult process. However, one may make several inferences concerning this speed from the real solutions already obtained.

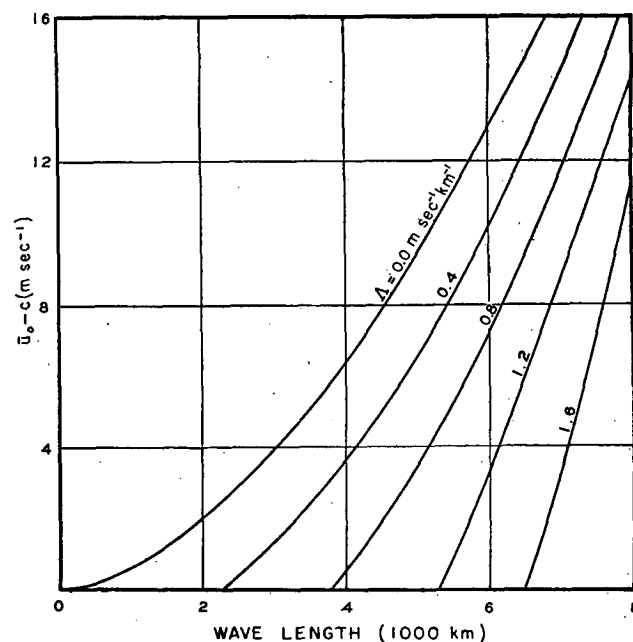


FIG. 8. The speed of the neutral wave as a function of  $\Lambda$  and of wave length  $L$ .

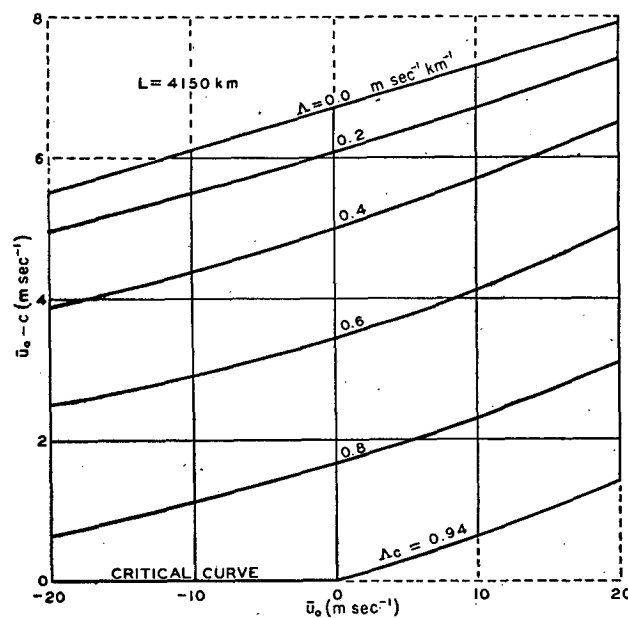


FIG. 9. The speed of the neutral wave as a function of  $\Lambda$  and  $\bar{u}_0$ .

Let us consider what happens to the speed of a neutral wave when the shear is gradually increased while  $L$  and  $\bar{u}_0$  remain constant. An inspection of fig. 9 shows that  $\bar{u}_0 - c$  decreases from its value in a barotropic atmosphere to its value at the point where  $\Lambda$  is equal to  $\Lambda_c$ , the critical shear corresponding to the given values of  $L$  and  $\bar{u}_0$ . It can be seen that for  $\bar{u}_0 > 0$  the value of  $\bar{u}_0 - c$  at  $\Lambda = \Lambda_c$  is slightly greater than zero, whereas for  $\bar{u}_0 < 0$  it is equal to zero. In either case  $\bar{u}_0 - c$  is small. In the case of incipient instability, where the shear is only slightly greater than its critical value, we may employ the argument

of continuity to deduce that the real part of  $\xi_0$  does not vary greatly from its value at the critical shear. The value of  $\bar{u}_0 - c$  therefore remains small, and we may infer that the speed of the incipient unstable wave does not differ greatly from the surface zonal speed.

### 18. The structure of the wave

#### Case I, the neutral wave

(a) *Phase.* The relation between the amplitude  $V$  of the meridional wave component of the velocity and the function  $\psi_1$  may be written

$$V(z) = A \psi_1(\xi, r) \exp \left[ - \left( \bar{a} - \frac{1}{2} H_T^{-1} \right) z \right], \quad (98)$$

or, if we define  $V_0 = V(0)$ ,

$$V(z) = V_0 \frac{\psi_1(\xi, r)}{\psi_1(\xi_0, r)} \exp \left[ - \left( \bar{a} - \frac{1}{2 H_T} \right) z \right]. \quad (99)$$

Since this function is real for the neutral wave, a consequence of the relation

$$v' = V e^{i\mu(x-ct)}$$

is that the phase of the wave cannot change with height.

(b) *Amplitude.* It can be shown by means of (99) that, as height increases,  $V$  may either increase or decrease initially but finally tends toward zero.

(c) *Pressure pattern.* It was proved in section 10 that the  $y$ -component of the velocity is very nearly geostrophic, i.e.,

$$p'_x \approx \bar{\rho} f v'.$$

Hence, from

$$v' = V e^{i\mu(x-ct)}$$

we obtain the equation

$$p' = -i(\bar{\rho} f V / \mu) e^{i\mu(x-ct)} = (\bar{\rho} f V / \mu) e^{i\mu(x-ct) - \frac{1}{2} i\pi}, \quad (100)$$

which shows that the wave in the field of meridional velocity lags  $90^\circ$  behind the wave in the field of  $p'$ . Since by condition (63)  $\bar{\rho} V \rightarrow 0$  as  $z \rightarrow \infty$ , it follows also that the amplitude of the pressure wave approaches zero with increasing height.

(d) *Density pattern.* The density perturbation is obtained from the pressure perturbation by means of the third of equations (22). Substituting the expression for  $p'$  in (100), we obtain

$$\rho' = i \frac{f \bar{\rho} V}{\mu} \left( s + \frac{V_z}{V} \right) e^{i\mu(x-ct)}, \quad (101)$$

and, introducing the value of  $V_z/V$  given by (82), we obtain

$$\rho' = i \frac{f \bar{\rho} V}{\mu} \left( s - \bar{a} + \frac{1}{2 H_T} + 2\bar{a} \frac{\psi_1'}{\psi_1} \right) e^{i\mu(x-ct)}. \quad (102)$$

Examination of the coefficient of  $e^{i\mu(x-ct)}$  in (102) re-

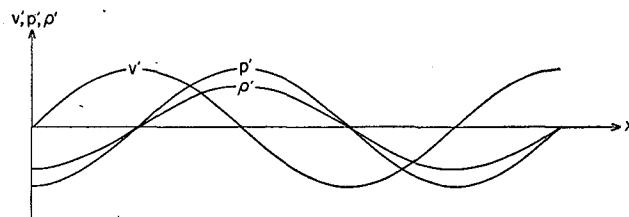


FIG. 10. Pressure, density, and meridional velocity component in the neutral baroclinic wave.

veals that the wave in the density field at the ground may be either  $180^\circ$  out of phase with, or in phase with, the pressure field, but that at sufficiently high levels it is always in phase with the pressure field. Since the amplitude of the pressure wave decreases to zero, the last property can also be deduced as a consequence of the rule derived from the hydrostatic equation that cold highs and warm lows decrease in intensity with height. The upper-level density field together with the fields of pressure and velocity are shown in fig. 10.

(e) *Pattern of vertical velocity component.* The amplitude  $W$  of the  $w'$ -perturbation is expressed in terms of  $V$  by (54), thus

$$\begin{aligned} \frac{g^2 k}{f \bar{\sigma}^2} W &= - \left( \frac{g k c}{\bar{\sigma}^2} + \Lambda \right) V + (\bar{u} - c) V_z \\ &= (\bar{u} - c) V \left( \frac{V_z}{V} - \frac{g k c / \bar{\sigma}^2 + \Lambda}{\bar{u} - c} \right) \\ &= 2\bar{a}(\bar{u} - c) V \left( \frac{\psi_1'}{\psi_1} - \alpha - \frac{\beta}{\xi} \right). \end{aligned} \quad (103)$$

By (97) the last parenthesized quantity equals zero when  $z = 0$ , and, since, as can be shown,  $\psi_1'/\psi_1$  decreases faster than  $\alpha + \beta/\xi$ , the parenthesized expression is negative. From this, together with the fact that  $k$  is negative and  $\bar{u} - c$  positive, it follows that  $W$  has the same sign as  $V$ , and the  $w'$ -perturbation is always in phase with the  $v'$ -perturbation.

(f) *The horizontal divergence.* The horizontal mass divergence  $D$  is defined by

$$D = (\rho u)_x + (\rho v)_y. \quad (104)$$

In terms of the perturbations  $u'$  and  $v'$ ,  $D$  may be written, with the aid of (11),

$$\begin{aligned} D &= \bar{\rho} u'_x + (\bar{\rho} f / g)(s u + \Lambda) v' \\ &= \bar{\rho} [i \mu U + (f / g)(s \bar{u} + \Lambda) V] e^{i\mu(x-ct)}, \end{aligned} \quad (104')$$

whence by (53) we obtain

$$\begin{aligned} D &= \bar{\rho} f \left[ \left( \chi + \frac{s \bar{u}}{g} \right) V + \frac{\bar{u}}{g} V_z \right] e^{i\mu(x-ct)} \\ &= \bar{\rho} f V \frac{\mu^2}{2} \left( \bar{u} - c - u_c + \frac{f^2 s \bar{u}}{\mu^2 g} \right. \\ &\quad \left. + \frac{f^2 \bar{u}}{\mu^2 g} \frac{V_z}{V} \right) e^{i\mu(x-ct)}. \end{aligned} \quad (104'')$$

An evaluation of the orders of magnitude of

$$\frac{f^2 s \bar{u}}{\mu^2 g} \quad \text{and} \quad \frac{f^2 \bar{u}}{\mu^2 g} \frac{V_z}{V}$$

in this equation shows that these terms are negligible in comparison with the remaining parenthetical terms. Therefore, we may write

$$D \approx (\bar{\rho} V \mu^2 / f)(\bar{u} - c - u_c) e^{i\mu(x-ct)}. \quad (105)$$

The level where  $D = 0$  is called the level of nondivergence. If the height of this level is denoted by  $h$ , we derive from the preceding equation the result that

$$\bar{u}(h) - c - u_c \approx 0. \quad (106)$$

Since at the ground  $\bar{u} - c$  is less than  $u_c$ , the result follows from (105) that the field of  $D$  is  $180^\circ$  out of phase with the field of  $v'$  at low levels; but, at levels above the level of nondivergence,  $D$  is in phase with the field of  $v'$ .

#### Case II, the unstable wave

The velocities of the unstable waves correspond to complex roots of (97), whose evaluation presents great difficulties. However, a qualitative description of the structure of the unstable wave can be given without such an evaluation. For this purpose, we approximate  $\xi_0$  in the vicinity of its critical values by means of an expansion in Taylor's series in the following manner: Consider the variation in  $\xi_0$  produced by a small increase of  $r$  from its critical value  $r_c$ . We may study this variation in one of the diagrams of fig. 6 by noting the intersections of the vertical line for constant  $r$  with the curve for constant  $\beta$ . If we confine our attention to the case  $\bar{u}_0 > 0$ ,  $\Delta\beta$  is less than zero, and we need only consider the curves for  $\beta < 1$ . Denote the critical values of  $r$ ,  $\alpha$ , and  $\beta$  by  $r_c$ ,  $\alpha_c$ , and  $\beta_c$ ; then the line corresponding to  $r = r_c$  is tangent to the curve corresponding to  $\beta = \beta_c$  in the diagram for  $\alpha = \alpha_c$ . At the point of tangency,  $\partial r / \partial \xi_0 = 0$  and  $\partial^2 r / \partial \xi_0^2 > 0$ . The relationship between  $\xi_0$ ,  $r$ ,  $\alpha$ , and  $\beta$  is expressed by

$$F = \frac{\psi_1'(\xi_0, r)}{\psi_1(\xi_0, r)} - \alpha - \frac{\beta}{\xi_0} = 0. \quad (97)$$

If  $F$  is analytic, this equation defines  $r$  as an analytic function of  $\xi_0$  in the vicinity of its critical value. An inspection of the expression for  $\xi_1$  in (79) shows that  $F$  is analytic whenever  $\xi_0$  is greater than zero. As this condition is assured by the provision that  $\bar{u}_0$  be greater than zero,  $r$  may be expanded in Taylor's series about  $\xi_0$ ; thus,

$$r = r_c + \left( \frac{\partial r}{\partial \xi_0} \right)_c (\xi_0 - \xi_{0c}) + \frac{1}{2!} \left( \frac{\partial^2 r}{\partial \xi_0^2} \right)_c (\xi_0 - \xi_{0c})^2 + \dots, \quad (107)$$

where  $\xi_{0c}$ ,  $(\partial r / \partial \xi_0)_c$ , and  $(\partial^2 r / \partial \xi_0^2)_c$  denote critical values. Recalling that  $(\partial r / \partial \xi_0)_c = 0$  and  $(\partial^2 r / \partial \xi_0^2)_c > 0$ , we obtain

$$\xi_0 - \xi_{0c} \approx \pm \left( 2 \frac{r - r_c}{(\partial^2 r / \partial \xi_0^2)_c} \right)^{\frac{1}{2}}, \quad (108)$$

if terms of third and higher orders are neglected. Since  $r - r_c$  is negative when  $\Lambda - \Lambda_c$  is positive, we may write

$$\xi_0 = \xi_{0c} + i\xi_i, \quad (109)$$

where

$$\xi_i = - \left( - 2 \frac{r - r_c}{(\partial^2 r / \partial \xi_0^2)_c} \right)^{\frac{1}{2}},$$

the minus sign in (108) being selected since  $c_i$  is greater than zero and therefore  $\xi_i$  is less than zero in the unstable wave.

Equation (109) gives the surface value of  $\xi$ . The corresponding value of  $\xi$  at any height  $z$  is

$$\begin{aligned} \xi &= (2\bar{a}/\Lambda)(\bar{u}_0 - c) + 2\bar{a}z \\ &= \xi_0 + 2\bar{a}z = \xi_{0c} + 2\bar{a}z + i\xi_i, \end{aligned}$$

and, if the quantity  $\xi_{0c} + 2\bar{a}z$  is denoted by  $\xi_c$ , we have

$$\xi = \xi_c + i\xi_i. \quad (110)$$

Expanding the function  $\psi_1$  about  $\xi_c$  we obtain

$$\psi_1(\xi) = \psi_1(\xi_c) + i\xi_i \psi_1'(\xi_c) \quad (111)$$

if terms of second and higher orders are neglected; or, in polar coordinates,

$$\begin{aligned} \psi_1(\xi) &= |\psi_1(\xi)| e^{i\mu\Phi(z)} \\ &= \{\psi_1^2(\xi_c) + \xi_i^2 [\psi_1'(\xi_c)]^2\}^{\frac{1}{2}} e^{i\mu\Phi(z)}, \end{aligned} \quad (111')$$

where

$$\Phi(z) = \frac{1}{\mu} \tan^{-1} \left[ \xi_i \frac{\psi_1'(\xi_c)}{\psi_1(\xi_c)} \right], \quad (112)$$

and, if this expression is substituted into equation (99), we obtain

$$\begin{aligned} V &= \left\{ V_0 \exp \left[ - \left( \bar{a} - \frac{1}{2H_T} \right) z \right] \right\} \\ &\quad \times \left| \frac{\psi_1(\xi)}{\psi_1(\xi_0)} \right| e^{i\mu[\Phi(z) - \Phi(0)]}. \end{aligned} \quad (98')$$

Since  $v' = V e^{i\mu(x-ct)}$ , we may infer that the phase of the unstable wave changes with height and that the equation of the trough line at the time  $t = 0$  is given by

$$x + \Phi(z) - \Phi(0) = 0. \quad (113)$$

Since  $\xi_i$  is negative and  $\psi_1'(\xi_c)/\psi_1(\xi_c)$  is a positive, monotonically decreasing function of  $\xi_c$ , the trough line slopes toward the west at first rapidly and then less rapidly with height and eventually approaches the vertical. (See fig. 12.)

The expressions (98–105) for the fields of pressure,

density, vertical velocity component, and horizontal divergence derived for the neutral wave hold as well for the unstable wave. The real parts of the functions involving  $\xi_0$  and  $\xi$  are sensibly unaltered by the change from  $\xi_{00}$  and  $\xi_0$ , since  $\xi_i$  is assumed to be small compared with  $\xi_0$ , but we have now to consider the imaginary parts.

(a) *The pressure pattern.* Since  $\xi$  is not involved explicitly in the equation between  $p'$  and  $v'$ , the phase relationship between these two fields remains the same as for the neutral wave.

(b) *The density pattern.* The nature of the density perturbation may be deduced from (102). If we define the function  $Q(\xi)$  by

$$Q(\xi) = \frac{\psi_1'(\xi)}{\psi_1(\xi)} - \frac{1}{2} + \frac{s}{2\bar{a}} + \frac{1}{4\bar{a}H_T}$$

and recall the relation (100) between  $p'$  and  $v'$ , we may write for equation (102)

$$\rho' = -2\bar{a} Q(\xi) p'. \quad (102')$$

Approximating  $Q$  by a terminating Taylor's series about  $\xi = \xi_0$  we obtain

$$Q(\xi) = Q(\xi_0 + i\xi_i) \approx Q(\xi_0) + i\xi_i Q'(\xi_0),$$

or in polar coordinates

$$\begin{aligned} Q(\xi) &\approx [Q^2(\xi_0) + \xi_i^2 Q'^2(\xi_0)]^{1/2} \\ &\quad \times \exp \{i \tan^{-1} [\xi_i Q'(\xi_0)/Q(\xi_0)]\} \\ &\approx Q(\xi_0) \exp \{i \tan^{-1} [\xi_i Q'(\xi_0)/Q(\xi_0)]\}. \end{aligned}$$

It can be shown that, depending on the values of  $\Lambda$ ,  $L$ , and  $\bar{u}_0$ , (a')  $Q$  may be positive or negative at the ground and (b')  $Q$  decreases monotonically with height and approaches a negative value. These properties imply that the wave in the density field precedes the pressure wave by a phase angle  $|\tan^{-1} (\xi_i Q'/Q)|$ , which increases with increasing  $|\xi_i|$  and therefore with increasing instability. Thus, with the appearance of instability, the density wave develops, with respect to the pressure wave, an asymmetry of such a nature that the coldest air is in advance of the wedge in the isobars. It is this asymmetry which accounts for the previously noted tilt in the pressure and meridional velocity waves. The phase relationship between the waves  $\rho'$ ,  $p'$ , and  $v'$  is illustrated in fig. 11.

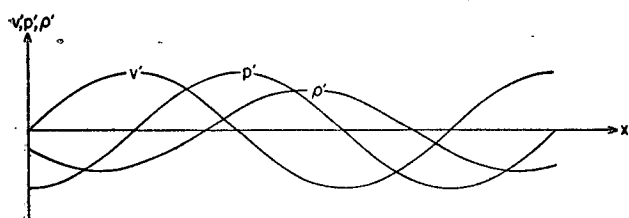


FIG. 11. Pressure, density, and meridional velocity component in the unstable baroclinic wave.

(c) *The pattern of the vertical velocity component.* The nature of the field of vertical velocity may be deduced from (103). If we replace  $2\bar{a}(\bar{u} - c)$  by  $\Lambda\xi$  and multiply through by  $e^{i\mu(x-ct)}$ , this equation becomes

$$w' = -[f\bar{\sigma}^2\Lambda/(g^2k)] M(\xi) v',$$

where

$$M(\xi) = -\xi \left[ \frac{\psi_1'(\xi)}{\psi_1(\xi)} - \alpha - \frac{\beta}{\xi} \right].$$

It is not possible to approximate  $M$  at the ground by the first terms of a Taylor's series because  $M$  vanishes there. However, for large values of  $\xi$ , the approximation is valid and we may write

$$\begin{aligned} M(\xi) &\approx M(\xi_0) + i\xi_i M'(\xi_0) \\ &\approx M(\xi_0) \exp \{i \tan^{-1} [\xi_i M'(\xi_0)/M(\xi_0)]\}, \quad (103') \end{aligned}$$

from which it can be shown that (a')  $|M|$  increases continuously with height and is asymptotic to  $\alpha\xi_0$ , and (b') the argument of  $M$  is negative and approaches zero with increasing height. Since, by (93'),  $\xi_0 V \rightarrow 0$  as  $z \rightarrow \infty$ , it follows from (103') and (a') that  $w'$  finally approaches zero with increasing height. The conclusion (b') together with (103') leads to the result that the  $w'$ -wave precedes the  $v'$ -wave by the angle  $|\tan^{-1} (\xi_i M'/M)|$  at high levels.

Near the ground  $w'$  is evaluated indirectly by numerical integration of (55) with the result that the amplitude of the  $w'$ -wave increases with height and the wave lags, in low levels, behind the  $v'$ -wave. Since the  $v'$ -wave lags  $90^\circ$  behind the wave in the pressure field, we may summarize the statements above by saying that in low levels the maximum vertical velocities occur between the point of inflection and the trough to the west in the pressure profile whereas at high levels the maximum vertical velocities occur between the point of inflection and the wedge to the east.

Fig. 12 contains a schematic representation of the field of vertical velocity in the unstable wave. The corresponding horizontal patterns of pressure and vertical velocity at a constant level are also indicated in the lower part of the diagram.

(d) *The horizontal divergence.* If the real and imaginary parts of the wave velocity are denoted by  $c_r$  and  $c_i$  respectively, (105) becomes

$$\begin{aligned} D &\approx (\bar{\rho}\mu^2/f)(\bar{u} - c_r - u_0) V e^{i\mu(x-ct)} \\ &\quad - i(\bar{\rho}\mu^2/f)c_i V e^{i\mu(x-ct)}, \quad (114) \end{aligned}$$

and it follows that the field of horizontal divergence consists of two wave components: the first in phase with, or  $180^\circ$  out of phase with, the  $v'$ -wave according as  $\bar{u} - c_r - u_0 \geq 0$ ; the second preceding the  $v'$ -wave by  $90^\circ$  since  $c_i$  is positive.

No definition of a level of nondivergence can be given, for there is no constant level at which both wave components vanish simultaneously. If, however, the

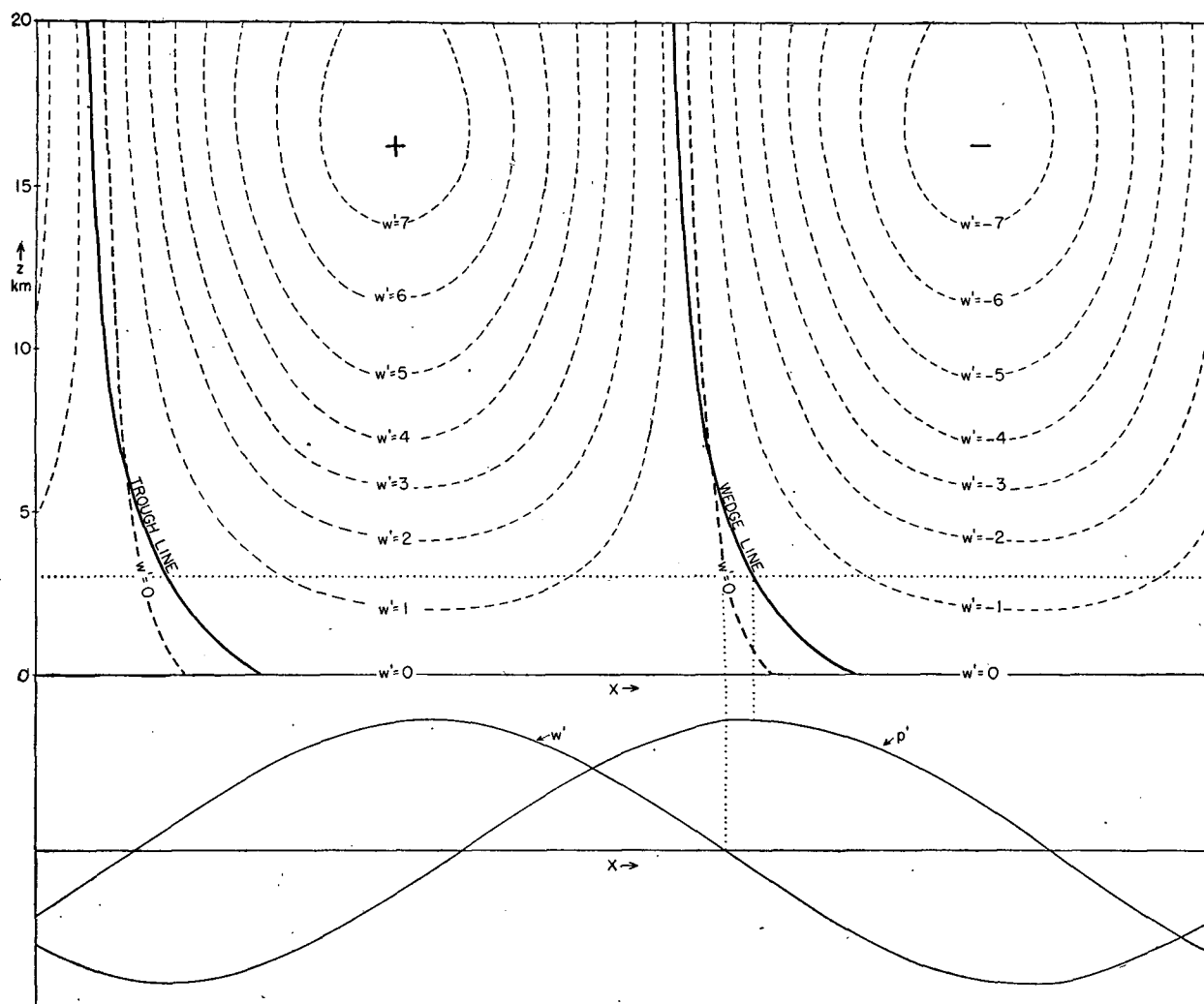


FIG. 12. Schematic representation of the field of vertical velocity component in the unstable baroclinic wave in vertical cross section (upper diagram). The dashed lines in the upper diagram are isopleths of vertical velocity component, the unit being  $\text{cm sec}^{-1}$ , and the solid lines indicate the positions of the trough and wedge in the pressure field. The position of the vertical-velocity wave relative to the pressure wave at the level indicated by the dotted line is shown in the lower diagram.

wave is only slightly unstable,  $c_i$  will be small in comparison with  $c_r$ , and the second component will be negligible. In this case the level of nondivergence is approximately determined by  $u - c_r - u_0 = 0$ . In general, it is possible to define a surface of nondivergence by setting the magnitude of  $D$  in (114) equal to zero. For this purpose let  $V = |V|e^{i\mu\Phi(z)}$ ,  $\Phi(z)$  being defined by (112), and obtain

$$D \approx (\bar{\rho}\mu^2/f)(\bar{u} - c_r - u_0)|V|e^{i\mu(x - ct + \Phi)} + (\bar{\rho}\mu^2/f)c_i|V|e^{i\mu(x - ct + \Phi) - i\pi/2}, \quad (114')$$

and, when  $|D| = 0$ ,

$$(\bar{u} - c_r - u_0) \sin [\mu(x - ct + \Phi)] - c_i \cos [\mu(x - ct + \Phi)] = 0, \quad (115)$$

or

$$x - c_r t = -\Phi + \frac{1}{\mu} \tan^{-1} \frac{c_i}{\bar{u} - c_r - u_0}. \quad (116)$$

(We note that the same result could have been obtained by a polar representation of (114).) A vertical

cross section showing the field of divergence, calculated from (114'), and the field of vertical momentum, calculated from the vertical velocities of fig. 12, is given in fig. 13. The numerical values used in figs. 12 and 13 are correct only in order of magnitude and do not necessarily correspond to an actual situation.

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## APPENDIX

A. Elimination of density and pressure from the perturbation equations. Elimination of  $p'_y$  by differentiation of the first of equations (22) with respect to

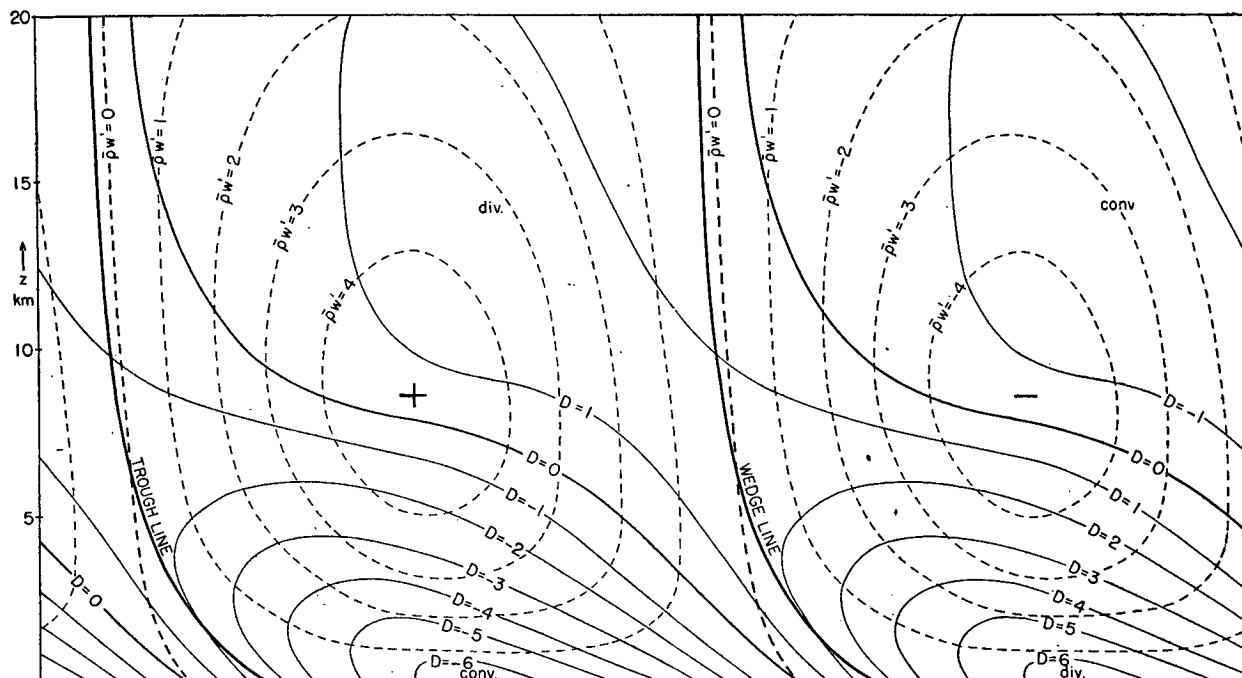


FIG. 13. Schematic representation of the fields of horizontal mass divergence and vertical momentum component in the unstable baroclinic wave. The dashed lines are isopleths of vertical momentum component, the unit being  $10^{-3} \text{ g cm}^{-2} \text{ sec}^{-1}$ . The light solid lines are isopleths of horizontal mass divergence, the unit being  $10^{-3} \text{ g cm}^{-2} \text{ sec}^{-1}$ .

y and the second with respect to x gives

$$fu'_x + f_y v' + L(v'_x) + \frac{f}{g\bar{\rho}} (s\bar{u} + \Lambda) p'_x + \frac{f\bar{u}}{\bar{\rho}} \rho'_x = 0, \quad (1A)$$

and elimination of  $p'_x$  by differentiation of the first of equations (22) with respect to z and the third with respect to x gives

$$\Lambda u'_x + sL(u') - sfv' + L(u'_z) - fv'_z + \frac{\Lambda}{\bar{\rho}} (\bar{\rho}w')_z - \frac{g}{\bar{\rho}} \rho'_x = 0. \quad (2A)$$

The last two equations together with the first, fourth, and fifth of (22) constitute a system equivalent to (22).

The first of equations (25) is now obtained by elimination of  $\rho'$  from (2A) and the fourth of (22), the second by elimination of  $\rho'$  and  $p'$  from (2A) and the first and fifth of (22), and the last by elimination of  $\rho'$  and  $p'$  from (1A), (2A), and the first of (22).

B. *Reduction of the perturbation equations.* Equation (48) is obtained by elimination of  $\bar{\rho}W$  and  $(\bar{\rho}W)_z$  between (39), (40), and (41); equation (47) is obtained by elimination of  $U_z$ ,  $V_z$ , and  $(\bar{\rho}W)_z$  between (39), (40), and (48); and equation (46) is obtained by elimination of  $U_z$ ,  $V_z$ , and  $\bar{\rho}W$  between (39) and (48), when, after each elimination, the resulting expressions are simplified by means of the inequalities (42-44).

Applying the same inequalities, we obtain equation (50) by elimination of  $U_z$  between (48) and (49), equa-

tion (51) by elimination of  $U$  between (47) and (50), and equation (56) by elimination of  $U$  between (46) and (50).

C. *Solution of the confluent hypergeometric equation for the case  $b = 0$ .* The functions

$$M(a, b, \xi) \quad \text{and} \quad \xi^{1-b} M(a - b + 1, 2 - b, \xi)$$

fail to yield two independent integrals of (76) when  $b$  is an integer, for  $M(a, b, \xi)$  is then undefined. Two different independent integrals may be derived by the method of Laplace: thus Mott and Massey [13] prove that the integrals

$$W_1 = \frac{1}{2\pi i} \int_{\gamma_1} \left(1 - \frac{\xi}{t}\right)^{-a} e^t dt, \\ W_2 = \frac{1}{2\pi i} \int_{\gamma_2} \left(1 - \frac{\xi}{t}\right)^{-a} e^t dt$$

satisfy (76) when  $b = 0$ . The paths of integration  $\gamma_1$  and  $\gamma_2$  are shown in fig. 4. The first integral may be expanded in an infinite series as follows: if we express  $\psi_1 = (-1)^a W_1$  as a contour integral of the Mellin-Barnes type, it can be shown, by a method employed by Archibald [1], that

$$\psi_1 = \frac{\xi^{-a}}{2\pi i \Gamma(-a)} \times \int_{-\infty-i}^{\infty-i} \frac{\Gamma(s) \Gamma(-s+a) \Gamma(-s+a+1)}{\Gamma(a) \Gamma(a+1)} \xi^s ds,$$

where the contour has loops if necessary so that the poles of  $\Gamma(s)$  and those of  $\Gamma(-s+a) \Gamma(-s+a+1)$

TABLE 1. Values of the function  $\psi_1(\xi, r)$ .

$$\psi_1(\xi, r) = \frac{\sin \pi a}{\pi} \left\{ a\xi M(a+1, 2, \xi) \left[ \ln \xi + \frac{\Gamma'(a)}{\Gamma(a)} - 2 \frac{\Gamma'(1)}{\Gamma(1)} \right] + 1 + \sum_{n=1}^{\infty} B_n \frac{a(a+1) \cdots (a+n-1)}{(n-1)! n!} \xi^n \right\}^*$$

$$B_n = \sum_{\nu=0}^{n-1} \left( \frac{1}{a+\nu} - \frac{2}{1+\nu} \right) + \frac{1}{n}; a = -r.$$

The function  $\psi_1$  satisfies the confluent hypergeometric differential equation  $\xi\psi'' - \xi\psi' + r\psi = 0$ .

$\frac{r}{\xi}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0	1	1	1	1	1	1	1	1
0.1	1.031	1.065	1.105	1.152	1.210	1.287	1.400	1.598
0.2	1.049	1.106	1.173	1.254	1.357	1.497	1.708	2.090
0.4	1.077	1.167	1.277	1.415	1.595	1.847	2.239	2.972
0.6	1.098	1.215	1.360	1.546	1.793	2.147	2.709	3.781
0.8	1.115	1.255	1.431	1.659	1.968	2.418	3.142	4.544
1.0	1.130	1.290	1.494	1.760	2.128	2.667	3.548	5.274
2.0	1.184	1.421	1.735	2.164	2.782	3.726	5.334	8.622
3.0	1.222	1.515	1.914	2.475	3.302	4.607	6.884	11.68
4.0	1.251	1.589	2.060	2.732	3.750	5.388	8.282	14.50
5.0	1.275	1.652	2.184	2.962	4.150	6.096	9.606	17.16
6.0	1.295	1.706	2.293	3.164	4.513	6.751	10.84	19.75
7.0	1.313	1.754	2.392	3.349	4.851	7.367	12.03	22.27
8.0	1.328	1.798	2.482	3.518	5.164	7.951	13.17	24.71
9.0	1.343	1.837	2.564	3.677	5.461	8.507	14.25	27.09
10.0	1.356	1.873	2.642	3.827	5.740	9.040	15.31	29.43
$\frac{r}{\xi}$	0.9	1.5	2.5	3.5	4.5	5.5	-0.5	-1.5
0	1	1	1	1	1	1	1	1
0.1	2.133	1.296	1.295	1.242	1.154	1.039	0.8820	0.7304
0.2	3.144	1.351	1.164	0.883	0.553	0.198	0.8202	0.6114
0.4	5.048	1.175	0.442	-0.361	-1.124	-1.786	0.7390	0.4742
0.6	6.863	0.752	-0.588	-1.788	-2.681	-3.216	0.6815	0.3878
0.8	8.622	0.149	-1.750	-3.089	-3.722	-3.693	0.6381	0.3290
1.0	10.34	-0.603	-2.932	-4.088	-4.076	-3.158	0.6035	0.2865
2.0	18.53	-6.053	-6.816	-2.126	3.779	8.004	0.4882	0.1468
3.0	26.33	-13.64	-3.302	10.99	15.86	10.15	0.4270	0.1138
4.0	33.84	-22.89	11.51	27.48	12.04	-14.10	0.3840	0.0890
5.0	41.18	-33.56	40.59	33.56	-21.41	-52.59	0.3600	0.1700
6.0	48.38	-45.46	86.15	11.00	-83.01	-66.34	0.3300	0.1070
7.0	55.47	-58.52	151.20	-62.44	-151.20	-4.76	0.3080	0.0770
8.0	62.46	-72.59	235.77	-210.33	-175.68	17.13	0.2940	0.1280
9.0	69.35	-87.65	345.19	-464.65	-79.68	464.65	0.2740	0.0190
10.0	76.18	-103.61	477.79	-851.97	252.47	795.86	0.2900	0.6400

\* See table 3 for definition of  $M$ .

are on opposite sides of it. The integrand has a simple pole at  $s = a$  and double poles at  $s = a + n$ , where  $n = 1, 2, 3, \dots$ . Therefore (see Whittaker and Watson [17])

$$\psi_1 = - \frac{\xi^{-a}}{\Gamma(-a) \Gamma(a) \Gamma(a+1)} \left( R_0 + \sum_{n=1}^{\infty} R_n \right),$$

where  $R_0$  denotes the residue of

$$f(s) = \Gamma(s) \Gamma(-s+a) \Gamma(-s+a+1) \xi^s$$

at the simple pole  $s = a$ , and  $R_n$  the residue at the double pole  $s = a + n$ . Since

$$\Gamma(z) = \Psi(z) + \frac{1}{0!z} - \frac{1}{1!(z+1)} + \frac{1}{2!(z+2)} - \dots,$$

where  $\Psi$  is an integral function, the residue of  $\Gamma(-s+a)$  at  $s = a$  is  $-1$ . Therefore

$$R_0 = -\Gamma(a) \xi^a.$$

To evaluate  $R_n$  we proceed as follows: By means of the relation

$$\Gamma(x) \Gamma(1-x) = \pi / (\sin \pi x)$$

we may express  $f(s)$  as

$$f(s) = -\pi^2 \{\sin^2 [\pi(s-a)]\}^{-1} \varphi(s),$$

where

$$\varphi(s) = \frac{\Gamma(s) \xi^s}{\Gamma(1-a+s) \Gamma(-a+s)}.$$

Now, utilizing the relation

$$\frac{\pi^2}{\sin^2 [\pi(s-a)]} = \sum_{n=-\infty}^{\infty} \frac{1}{(s-a+n)^2}$$

and the Taylor expansion of  $\varphi(s)$  about  $s = a + n$ , we find that the residue of  $f(s)$  at the pole  $a + n$  is  $-\varphi'(a+n)$ . But

$$\begin{aligned} \varphi'(a+n) &= \frac{d}{d\sigma} \left[ \frac{\Gamma(\sigma+a) \xi^{\sigma+a}}{\Gamma(\sigma+1) \Gamma(\sigma)} \right]_{\sigma=a+n} \\ &= \frac{\Gamma(n+a) \xi^{n+a}}{\Gamma(n+1) \Gamma(n)} \left[ \frac{\Gamma'(n+a)}{\Gamma(n+a)} \right. \\ &\quad \left. - \frac{\Gamma'(n+1)}{\Gamma(n+1)} - \frac{\Gamma'(n)}{\Gamma(n)} + \ln \xi \right], \end{aligned}$$

TABLE 2. Values of the function  $\psi_1' = d\psi_1/d\xi$ .

$\frac{r}{\xi}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0								
0.1	0.213	0.432	0.725	1.071	1.525	2.136	3.192	5.066
0.2	0.174	0.344	0.586	0.903	1.269	1.826	2.767	4.535
0.4	0.121	0.262	0.450	0.719	1.061	1.588	2.440	4.161
0.6	0.082	0.213	0.374	0.614	0.930	1.417	2.248	3.894
0.8	0.066	0.182	0.326	0.539	0.836	1.294	2.118	3.726
1.0	0.055	0.161	0.291	0.484	0.772	1.197	1.987	3.570
2.0	0.048	0.102	0.198	0.346	0.579	0.958	1.627	3.121
3.0	0.034	0.076	0.155	0.282	0.489	0.838	1.466	2.873
4.0	0.028	0.065	0.130	0.235	0.428	0.744	1.358	2.726
5.0	0.019	0.053	0.100	0.210	0.374	0.683	1.278	2.591
6.0	0.014	0.044	0.094	0.180	0.343	0.621	1.214	2.508
7.0	0.010	0.042	0.088	0.164	0.306	0.582	1.107	2.427
8.0	0.009	0.040	0.079	0.148	0.289	0.541	1.106	2.446
9.0	0.007	0.033	0.079	0.151	0.295	0.519	1.040	2.438
10.0	0.007	0.028	0.074	0.149	0.298	0.533	1.072	2.443
<hr/>								
$\frac{r}{\xi}$	0.9	1.5	2.5	3.5	4.5	5.5	-0.5	-1.5
0								
0.1	10.30						-0.691	-1.460
0.2	9.495						-0.459	-0.849
0.4	9.036						-0.314	-0.472
0.6	8.613						-0.242	-0.329
0.8	8.536	-1.851					-0.198	-0.255
1.0	8.478	-9.517					-0.171	-0.207
2.0	7.952	-6.265					-0.081	-0.071
3.0	7.635	-8.386					-0.043	-0.005
4.0	7.445	-9.724	18.76				-0.035	
5.0	7.289	-11.07	36.12				-0.027	
6.0	7.160	-12.09	54.36				-0.021	
7.0	7.100	-13.22	73.63	-92.41			-0.016	
8.0	6.996	-14.16	95.02	-194.34			-0.012	
9.0	6.727	-15.16	121.16	-322.93				
10.0	6.780	-16.06	145.72	-477.10	585.73			

and

$$\frac{\Gamma'(n+a)}{\Gamma(n+a)} = \sum_{\nu=0}^{n-1} \frac{1}{a+\nu} + \frac{\Gamma'(a)}{\Gamma(a)},$$

$$\frac{\Gamma'(n+1)}{\Gamma(n+1)} = \sum_{\nu=0}^{n-1} \frac{1}{1+\nu} + \frac{\Gamma'(1)}{\Gamma(1)},$$

$$\frac{\Gamma'(n)}{\Gamma(n)} = \sum_{\nu=0}^{n-2} \frac{1}{1+\nu} + \frac{\Gamma'(1)}{\Gamma(1)}.$$

Combining these results we obtain

$$\psi_1 = \frac{\xi^{-a}}{\Gamma(-a)\Gamma(a)\Gamma(a+1)} \left\{ \Gamma(a)\xi^a \right.$$

$$+ \ln \xi \sum_{n=1}^{\infty} \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(n)} \xi^{n+a}$$

$$+ \sum_{n=1}^{\infty} B_n \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(n)} \xi^{n+a}$$

$$\left. + \left[ \frac{\Gamma'(a)}{\Gamma(a)} - 2 \frac{\Gamma'(1)}{\Gamma(1)} \right] \sum_{n=1}^{\infty} \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(n)} \xi^{n+a} \right\},$$

where

$$B_n = \sum_{\nu=0}^{n-1} \left( \frac{1}{a+\nu} - \frac{2}{1+\nu} \right) + \frac{1}{n}.$$

From the definition of  $M$  in (77) it follows that

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+a)}{\Gamma(n+1)\Gamma(n)} \xi^n = a \Gamma(a) \xi M(a+1, 2, \xi),$$

whence we obtain for  $\psi_1$  the expression

$$\psi_1 = \frac{\sin \pi a}{\pi} \left\{ a \xi M(a+1, 2, \xi) \right.$$

$$\times \left[ \ln \xi + \frac{\Gamma'(a)}{\Gamma(a)} - 2 \frac{\Gamma'(1)}{\Gamma(1)} \right] + 1$$

$$\left. + \sum_{n=1}^{\infty} B_n \frac{a(a+1) \cdots (a+n-1)}{(n-1)! n!} \xi^n \right\}.$$

The asymptotic expansions of  $W_1$  and  $W_2$  given in (93) and (94) are demonstrated by Mott and Massey [13, p. 39].

D. *Tables of  $\psi_1$  and  $\psi_2$ .* The function  $\psi_1(\xi, r)$  was evaluated for  $\xi \leq 4$  by means of the infinite series expansion (79), and for  $\xi \geq 4$  by means of the asymptotic expansion (93). Some of the calculations were facilitated by use of the recursion formula

$$r \psi_1(\xi, r-1) + r \psi_1(\xi, r+1) = -(\xi - 2r) \psi_1(\xi, r).$$

The tabulations are given in table 1. The derivatives of  $\psi_1$ , evaluated graphically, are tabulated in table 2.

The function  $\psi_2(\xi, r)$ , defined by

$$\psi_2 = \xi M(a+1, 2, \xi) = \xi \left[ 1 + \frac{a+1}{1! 2!} \xi \right.$$

$$\left. + \frac{(a+1)(a+2)}{2! 3!} \xi^2 + \cdots \right], \quad a = -r,$$

was evaluated directly from the infinite series, and its derivatives were calculated graphically. The tabulated values are given in tables 3 and 4.

TABLE 3. Values of the function  $\psi_2(\xi, r) = \xi M(a+1, 2, \xi)$ , where  $a = -r$  and

$$M(a, b, \xi) = 1 + \frac{a}{1!b} \xi + \frac{a(a+1)}{2!b(b+1)} \xi^2 + \dots$$

The function  $\psi_2$  satisfies the confluent hypergeometric differential equation  $\xi\psi'' - \xi\psi' + r\psi = 0$ .

$\xi \backslash r$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	-0.01046	-0.02082	-0.03108	-0.04123	-0.05128	-0.06123	-0.07107	-0.08082
0.2	-0.02192	-0.04340	-0.06445	-0.08507	-0.10526	-0.12503	-0.14439	-0.16334
0.4	-0.04821	-0.09450	-0.13889	-0.18144	-0.22218	-0.26114	-0.29838	-0.33401
0.6	-0.07978	-0.15480	-0.22520	-0.29112	-0.35269	-0.41005	-0.46334	-0.51304
0.8	-0.11776	-0.22618	-0.32564	-0.41649	-0.49909	-0.57378	-0.64090	-0.70166
1.0	-0.16349	-0.31086	-0.44291	-0.56040	-0.66410	-0.75471	-0.83295	-0.90130
2.0	-0.58020	-1.05048	-1.42153	-1.70335	-1.90526	-2.03598	-2.10352	-2.13381
3.0	-1.66374	-2.88290	-3.72026	-4.23255	-4.47087	-4.48115	-4.30451	-4.05807
4.0	-4.515	-7.538	-9.338	-10.151	-10.181	-9.607	-8.586	-7.491

TABLE 3.—Continued.

$\xi \backslash r$	0.9	1.5	2.5
0	0.00000	0.00000	0.00000
0.1	-0.09046	-0.14622	-0.23141
0.2	-0.18187	-0.28474	-0.42626
0.4	-0.36777	-0.53789	-0.71017
0.6	-0.55819	-0.75770	-0.85963
0.8	-0.75370	-0.94222	-0.88284
1.0	-0.95495	-1.08929	-0.78830
2.0	-2.07882	-1.16727	1.12710
3.0	-3.53321	0.21402	4.47090
4.0	-5.6716	3.89358	7.58565
5.0	-9.320	11.52548	8.17913
6.0	-16.37	26.51661	2.85015
7.0	-31.3	56.14203	-13.82990
8.0	-65	116.40504	-51.15560

TABLE 4. Values of the function  $\psi_2' = d\psi_2/d\xi$ .

$\xi \backslash r$	0.7	0.8	0.9
0	-0.700	-0.788	-0.839
0.1	-0.721	-0.821	-0.900
0.2	-0.732	-0.836	-0.916
0.4	-0.795	-0.872	-0.949
0.6	-0.851	-0.916	-0.966
0.8	-0.916	-0.983	-0.993
1.0	-0.993	-0.997	-1.002
2.0	-1.529	-1.442	-1.192
3.0	-2.965	-2.426	-1.757
4.0	-7.11	-4.950	-2.550

TABLE 5. Values of the function

$$X = \xi \left[ \ln \xi + \sum_{n=1}^{\infty} \frac{\xi^n}{(n+1)!n} \right] - 1$$

and of its derivative.

$\xi$	$X$	$dX/d\xi$
0.0	-1.000	$\infty$
0.1	-1.225	9.83
0.2	-1.301	4.81
0.4	-1.280	2.69
0.6	-1.106	1.93
0.8	-0.809	1.97
1.0	-0.400	2.32
2.0	3.362	7.05
3.0	10.98	17.68
4.0	25.62	39.28
5.0	54.60	84.28
6.0	114.7	181.9
7.0	248	405.

When  $r$  is a positive integer, both  $\psi_1$  and  $\psi_2$  are undefined. In this case it is necessary to derive the solutions of (74) anew. In particular, when  $r = 1$ , we have

$$\xi\psi'' - \xi\psi' + \psi = 0.$$

Two solutions of this equation are found to be  $\xi$  and

$$X = \xi \left[ \ln \xi + \sum_{n=1}^{\infty} \frac{\xi^n}{(n+1)!n} \right] - 1.$$

The function  $X$  and its derivative are tabulated in table 5.

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