ON A PHYSICAL BASIS FOR NUMERICAL PREDICTION OF LARGE-SCALE MOTIONS IN THE ATMOSPHERE

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ABSTRACT

The small-scale "noise" disturbances of the atmosphere create difficulties for the numerical integration of the equations of motion. For example, their existence demands that very small time differences be used in the integration of the finite-difference equations. To eliminate the noise, a filtering method is devised which consists essentially in replacing the primitive hydrodynamical equations by combining the geostrophic and hydrostatic equations with the conservation equations for potential temperature and potential vorticity. In this way a single equation in the pressure is obtained for the motion of the large-scale systems. A method is suggested for its numerical integration.

The spread of data required for a short-period forecast is discussed in terms of the rate of spread of influences or "signal velocity" in the atmosphere. It is shown that a small disturbance is propagated both horizontally and vertically at a finite rate. Estimates are obtained for the maximum signal-velocity components in order to establish bounds for the influence region. It is found that numerical forecasts for periods of one or perhaps two days are now possible for certain areas of the earth but that forecasts for longer periods require a greater spread of observation stations than is available.

A method is given for reducing the three-dimensional forecast problem to a two-dimensional one by construction of an "equivalent-barotropic" atmosphere. The method is applied to the calculation of the 500-mb height tendency, and the results are compared with observation. A rule is derived for determining the positions of the isallohyptic centers from the field of the absolute-vorticity advection.

1. Introduction

A fundamental need in weather prediction is a mathematical or statistical apparatus capable of dealing with the large number of parameters required for describing the meteorologically significant motions of the atmosphere. For want of such an apparatus, the theoretical meteorologist is constantly forced to reduce the number of degrees of freedom of the motion by imposing kinematic constraints in the form of symmetry, periodicity, and stationarity conditions, and by reducing its dimensionality. The synoptic meteorologist, who must also reduce the freedom, does it by substituting for the actual motion the 'gestalt' constructs: pressure system, ridge, trough, air mass, front, wave, jet stream, *etc.* As the success of a weather prediction depends upon the number of relevant parameters the forecaster has at his disposal from which to draw statistical or dynamical inferences, it is not difficult to understand the disappointingly slow progress made in the field of weather prediction.

It is for this reason that recent developments in the design of large-scale digital computing machines have revived the interest of meteorologists in the problem of numerical weather prediction. Promise is given that the purely mechanical difficulties connected with the handling of great quantities of data can be overcome so that the computational time factor will eventually cease to be the unsurmountable obstacle to the practical realization of a program of numerical forecasting. The role of the enormous weather factory envisaged by Richardson (1922) with its thousands of computers will, it may be hoped, be taken over by a completely automatic electronic computing machine.

This note of optimism must, however, be tinctured by the sober realization that there are serious obstacles other than the time factor that still stand between the hope and it fulfillment. There still remains to be answered the basic question: Do we actually know the laws governing the motion of the atmosphere? In the last analysis this question can be answered only by deducing consequences from hypotheses and subjecting them to experimental vertification. Since it is practically impossible to experiment with the atmosphere on a large scale, and since an adequate similarity theory or technique is lacking for model experiments, a theory describing what the atmosphere will do under a given set of circumstances can be tested only by integration of the appropriate equations of motion. In this connection the fundamental importance of high-speed arithmetical devices is readily appreciated. By reducing the mathematical difficulties involved in carrying a physical train of thought to its logical conclusion, the machines will give a greater scope to the

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making and testing of physical hypotheses and so lead to a wider use of inductive methods in meteorology.

The motions described most accurately by the existing observations are the large-scale disturbances of the atmosphere. Just as we achieve a degree of certainty in predicting the motion of a gas by transferring attention from the individual molecules to certain space and time averages, so by restricting ourselves to the major weather-producing motions of the atmosphere whose horizontal scale is of the order of 1000 km, we minimize the random effect of the micrometeorological motions. For the former the laws may be assumed known to a first approximation and to be expressed by the nonviscous hydrodynamical equations and the adiabatic equation. No doubt modifications in the laws will be required as forecast periods are extended and as inadequacies are revealed in the existing equations by comparison of prediction with observation.

An introduction to an important aspect of the numerical forecast problem is afforded by the following simple example. Consider the motion of small perturbations in an incompressible atmosphere of height H moving with constant translation U over a plane earth which rotates with the angular speed Ω. A motion of this sort could conceivably occur near the poles of the earth. If the motion is assumed not to vary across the current, then, in a rectangular coordinate system with x directed along, and y normal to, the current, the perturbation equations become

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x} + fv, \tag{1}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -fu, \tag{2}$$

$$\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} = \frac{fU}{g}v - H\frac{\partial u}{\partial x}, \tag{3}$$

where h is the perturbation height. Let us solve these equations numerically by calculating the successive time increments in u, v, and h. Using centered space differences and uncentered time differences we replace $\partial\alpha/\partial x$ by

$$[\alpha(x + \tfrac{1}{2}\Delta x) - \alpha(x - \tfrac{1}{2}\Delta x)]/\Delta x,$$

and $\partial\alpha/\partial t$ by

$$[\alpha(x + \Delta t - \alpha(x)]/\Delta t.$$

Then if the values of u, v, and h are known at the points $\ldots\, x - \tfrac{1}{2}\Delta x,\ x,\ x + \tfrac{1}{2}\Delta x, \ldots$ along the base AB of the grid triangle of fig. 1, their values can be determined at the apex P by iterative application of the finite-difference analogues to (1–3). It might be thought that the degree of approximation of the finite-difference solution to the actual solution of the equations of motion would increase with diminishing Δx and Δt independently of the manner in which these increments approached zero, but this is not the case. To demonstrate we eliminate u and v from (1–3) to obtain

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 h - gH\frac{\partial^2 h}{\partial x^2}\right] + f^2\frac{\partial h}{\partial t} = 0, \tag{4}$$

an equation which governs the motion of both gravity waves and the meteorologically far more important large-scale disturbances. Now it can be shown by a generalization of Riemann's method for hyperbolic equations (Holmgren, 1904) that the solution of (4) at the point (ξ, τ) is determined by the initial values of h on that part of the x-axis which is intercepted by the characteristic lines $x - \xi = \pm c(t - \tau)$, where $c \equiv \sqrt{gH}$. If Δx and Δt are so chosen that the solution depends on data not covering at least this much of the x-axis, it is obvious that the finite-difference approximation will not converge to the correct solution for $\Delta x,\ \Delta t \to 0$.[2] From the figure it can be seen that the condition for the grid triangle PAB to contain the characteristic triangle PCD is $\tfrac{1}{2}\Delta x/\Delta t > c$, or $\Delta t < \tfrac{1}{2}\Delta x/c$. If H is taken to be about 9 km, the height of the homogeneous atmosphere, c becomes 300 m sec^{-1}, and if $\Delta x = 400$ km we must have $\Delta t < 11$ minutes. Richardson chose approximately the value 400 km for Δx but the value 6 hours for Δt. His computations would not have yielded a correct forecast even with the best possible data.

This situation illustrates a basic shortcoming of the primitive equations; the meteorologically important solution of (4), given to a close approximation by $h = h\,(x - Ut, 0)$, $u = 0$, and $v = f^{-1}g\ \partial h/\partial x$, *i.e.*, by a geostrophically balanced lateral current advected with the zonal wind, can be found only by taking into account the entirely irrelevant gravity motions if the solution is to be obtained by numerical integration.

2. The geostrophic approximation

In an article entitled "On the scale of atmospheric motions"[3] the writer (1948) presented a method for

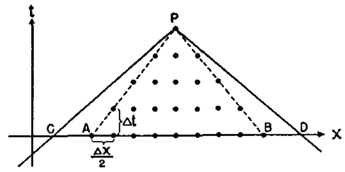

FIG. 1. The initial values of u, v, and h are given at the points along the x-axis ($t = 0$). The finite-difference computation gives the values at the remaining points in the grid triangle PAB. As presented, the grid triangle lies within the characteristic triangle PCD, and the computation fails.

[2] A mathematical proof of the corresponding theorem for hyperbolic equations has been given by Courant, Friedrichs, and Lewy (1928).

[3] Hereafter referred to by (S).

filtering out the meteorologically insignificant "noise" motions from the primitive equations. This method permits a simplified treatment of certain theoretical problems and is also useful for numerical forecasting. By replacing the primitive equations by the simplified equations presented in (S), a method of integration was developed whereby the difficulties encountered in the application of Richardson's method can be overcome. We shall now turn to a discussion of this method.

In a rectangular coordinate system the Eulerian equations of motion for a nonviscous fluid may be written

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin \varphi, \quad (5)$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin \varphi, \quad (6)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \quad (7)$$

where the x-axis points east, the y-axis north, and the z-axis vertically upward. To avoid unnecessary geometrical complications, we assume the earth plane. We also assume that the hydrostatic approximation holds and that the x-component of the Coriolis force involving w is negligible.

It was shown in (S) that the orders of magnitude of the horizontal acceleration and the horizontal Coriolis force satisfy the relation

$$\frac{\text{horizontal acceleration}}{\text{horizontal Coriolis force}} \sim \frac{C/S}{f}, \quad (8)$$

where f is the Coriolis parameter $2\Omega \sin \varphi$; C is a characteristic mean speed of propagation of the horizontal streamline pattern, and S a characteristic horizontal length parameter. The ratio C/S is a kind of characteristic frequency of the motion, and f is the frequency of a horizontal inertial oscillation. The relation (8) states that the winds are near-geostrophic providing the characteristic frequency of the motion is small compared to the horizontal inertial frequency. For the major pressure systems of the atmosphere C/S is of the order 10^{-5} sec^{-1}, whereas f is of the order 10^{-4} sec^{-1}. Hence the fractional deviation of the wind from the geostrophic is of the order 10^{-1}, and the large-scale wind systems are quasigeostrophic.

This property of the wind implies a corresponding property of the horizontal divergence. If the variabilities of f and ρ are ignored, elimination of p from equations (5) and (6) by cross differentiation with respect to x and y gives

$$\frac{\partial}{\partial x}\left(\frac{dv}{dt}\right) - \frac{\partial}{\partial y}\left(\frac{du}{dt}\right) + f\frac{\partial u}{\partial x} + f\frac{\partial v}{\partial y} = 0, \quad (9)$$

and it follows from (5), (6), and (8) that the magnitude of each of the first two terms in this equation is one order less than that of the last two. Hence the sum of the terms $\partial u/\partial x$ and $\partial v/\partial y$ must be smaller by one order of magnitude than the terms themselves. As shown in (S), the inclusion of terms arising from the variability of f and ρ does not alter this conclusion. We may therefore state that the horizontal divergence is small to the same extent to which the winds are geostrophic.

The near-geostrophic and near-incompressible properties of the motion have the following destructive consequences. If it were desired to determine the local time derivatives of u and v from the fields u, v, w, and p, it would be necessary to measure u and v with an error no greater than one per cent in order to obtain $\partial u/\partial t$ and $\partial v/\partial t$ with an error not exceeding ten per cent. But such accuracy is unattainable, not only because of the inadequacy of present measuring techniques, but because the values of the geostrophic deviation associated with the smaller scale motions, for which S is small, may be as large as those pertaining to the major motions or even larger. Whereas the first difficulty conceivably can be overcome by a refinement in observing techniques, the second cannot. A similar difficulty is encountered in the application of the tendency equation where an accurate evaluation of the horizontal divergence is required. Here again the noise level is far too high; the small-scale divergences are as great in magnitude as the large-scale or greater. Thus, constant-level horizontal divergence charts show that the scale of the predominant horizontal divergence pattern is perhaps one-fifth that of the major pressure patterns, and in consequence, the large-scale horizontal divergence patterns are largely obscured.

The hydrodynamical noise effect may be further illustrated by an example from another branch of hydrodynamics. Suppose it were required to determine the two-dimensional tidal motion of an ocean. Since all except sound motions may be regarded as incompressible, the natural choice of dependent variable is the stream function. A computational scheme requiring a knowledge of the divergence $\partial u/\partial x + \partial w/\partial z$ would be unsuitable because this quantity is more sensitive to sound waves than, say, to surface gravitational waves. In practice, therefore, one filters out the sound waves by substituting the derivatives $-\partial\psi/\partial z$ and $\partial\psi/\partial x$ for u and w respectively in the vorticity equation.

In a similar way, the knowledge that $\mathrm{div}_2\, \boldsymbol{v}$ is small should be used as a directive for substituting the geostrophic wind components for u and v respectively in the equation for the vertical vorticity component, taking care first to eliminate the horizontal divergence. It is necessary to perform this elimination because the geostrophic wind components can be used to evaluate the horizontal divergence no more than the expressions $-\partial\psi/\partial z$, $\partial\psi/\partial x$ can be used to evaluate $\partial u/\partial x + \partial w/\partial z$. For the purpose of eliminating the horizontal divergence we suppose that there exists a conservative quantity, σ, in the atmosphere depending only on p and ρ:

$$d\sigma/dt = 0. \qquad (10)$$

With the aid of this law we may eliminate the horizontal divergence between the vorticity equation

$$\frac{d}{dt}(2\boldsymbol{\Omega} + \nabla \times \boldsymbol{v}) + (2\boldsymbol{\Omega} + \nabla \times \boldsymbol{v})\nabla\cdot\boldsymbol{v} = (2\boldsymbol{\Omega} + \nabla \times \boldsymbol{v})\cdot\nabla\boldsymbol{v} + \nabla(\rho^{-1}) \times \nabla p, \qquad (11)$$

and the continuity equation,

$$\frac{d\rho}{dt} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \qquad (12)$$

to derive the conservation equation

$$\frac{d}{dt}[\rho^{-1}\nabla\sigma\cdot(2\boldsymbol{\Omega} + \nabla \times \boldsymbol{v})] = 0, \qquad (13)$$

where $2\boldsymbol{\Omega}$ is the earth's vorticity. If, as a first approximation, isentropic motion is assumed so that σ may be the potential temperature θ, we obtain essentially Rossby's equation for the conservation of "potential vorticity" (see Rossby, 1940). Since the isentropic surfaces are quasi-horizontal in the large-scale systems (13) may be written

$$\frac{d}{dt}\left[\frac{1}{\rho}\frac{\partial\theta}{\partial z}(\zeta + f)\right] = 0, \qquad (14)$$

where ζ is the relative vertical vorticity component $\partial v/\partial x - \partial u/\partial y$.

It is now permissible to introduce the geostrophic and hydrostatic approximations for u, v, and ρ in terms of p. If then w is eliminated between the equation of conservation of θ and (14), the following equation in the pressure results:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{f(f+\zeta)}{s} \times \left(\frac{\partial^2}{\partial z^2} + \alpha\frac{\partial}{\partial z} + \beta\right)\right]\frac{\partial p}{\partial t} = \gamma, \qquad (15)$$

where $s = g\,\partial(\ln\theta)/\partial z$ and α, β, and γ are functions of p and its space derivatives. We assume

$$\zeta \approx \frac{1}{\rho f}\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right), \qquad (16)$$

an approximate expression for the geostrophic vorticity derived in (S). From the fact that (15) is of the first order in the time, one may conclude that the motion is determined merely by a knowledge of the initial pressure field. Further, from the manner in which it was derived, we can expect that it will be insensitive to the small-scale noises of the atmosphere and therefore suitable for numerical computation. Thus it was shown in (S) that all motions whose period is not greater than a pendulum day are filtered out by this equation.[4]

A necessary attribute of a meteorological theory is that it express those factors which are consciously or unconsciously used by the forecaster, since his skill is essentially positive. The fact that such terms as $\text{div}_2\,\boldsymbol{v}$ and $d\boldsymbol{v}_2/dt$ are never considered by the forecaster may be taken as an indication that they are not representative of the motions with which he deals. One might have predicted, solely on the basis of the behavior of the forecaster, that an equation governing the motions of only the large-scale systems would have the property that its solution is determined by the initial pressure field alone.

Equation (15) has certain disadvantages. Some of these are easily eliminable by an appropriate modification of the underlying assumptions, whereas others are more basic. It was not necessary, for example, to assume the motion adiabatic. To provide for condensation effects the law of conservation of wet-bulb potential temperature could have been used for (10). Other nonadiabatic energy changes, such as radiative transfer, could also have been taken into account by a suitable modification of (10). However, in a first attempt at numerical forecasting of the present sort, it has not been considered advisable to deal with effects which, from the available evidence, appear to be secondary. Among these are included eddy viscosity. In support of the view that the role of friction is secondary, we may cite the work of Haurwitz (1941) who, on the assumption that virtually all the frictional dissipation of kinetic energy takes place in the friction layer, showed that the kinetic energy of a hypothetical atmosphere moving with a uniform speed of 10 m sec^{-1}, equal to that of the wind at the top of the friction layer, would be dissipated in 72 hours. It is now known that the actual kinetic energy of the atmosphere exceeds that of this hypothetical atmosphere by a factor of 4 or more. The dissipation time is therefore closer to two weeks, and it appears that surface friction may be safely ignored for forecast intervals of a day or two.

What is of more concern is that in applying the geostrophic approximation, all motions whose periods are smaller than or of the order of a pendulum day are filtered out of the equations as noise. In doing so, motions of considerable importance for forecasting small-period weather changes may be excluded. For the present, it appears that one can only hope to forecast the major pressure patterns and to use these as steering currents for the smaller motions. The filtering

[4] Other means of eliminating the noise sensitivity which involve arithmetical smoothing processes are also being investigated by the writer and his colleagues. However, the method given here appears to be physically the most natural.

process then results in a certain distortion in the large-scale motions owing to the fact that the small- and large-scale motions are not linearly superposable. This amounts, as Reynolds has shown, to the introduction of a set of turbulent stresses. The tentative assumption is that these stresses will not be important for short-range forecasts since the smaller scale motions are confined to a relatively thin surface layer of the atmosphere.

3. Method of integration and initial-data requirements

We note that (15) is a second order partial differential equation of the elliptic type in $\partial p/\partial t$, since $(f+\zeta)/s$ is positive for large-scale motions. If at any moment, t, the field of p is known, it can be determined for the time $t+\Delta t$ by solving for $\partial p/\partial t$. An n-fold iteration of this process then gives the solution for the time $t+n\Delta t$.

It would appear from the elliptic character of (15) that the initial pressure field must be known throughout the atmosphere, since the only surfaces where the boundary conditions are known as functions of time are the surface of the earth and the top of the atmosphere. Notwithstanding this there is, for *practical* purposes, a finite rate at which influences propagate in the atmosphere, *i.e.*, a point forecast can be made with a knowledge of the initial pressure field within a limited region surrounding the point. This will be demonstrated in the next two sections.

4. The horizontal signal velocity

The concept of the speed of propagation of a hydrodynamical influence, or "signal velocity," in the atmosphere is an important one for meteorology. It is used to determine the dimensions of the region through which the initial data are needed in forecasting for a prescribed area, and more generally, it enters in any investigation of the causal connection between one part of the atmosphere and another.

A limit to the speed of propagation of a hydrodynamical influence, or "signal velocity," would be the velocity of sound were it not for the fact that the use of the hydrostatic approximation filters out pure sound waves and therefore introduces an infinite speed of propagation in the vertical. Although horizontally moving sound waves are also eliminated in this process, the horizontal signal velocity remains finite because of the possibility of vertical accommodation for a horizontal displacement. In this case the horizontal signal velocities would be limited essentially by the speed of gravity waves, but the introduction of the geostrophic approximation as an additional artificial constraint eliminates gravity waves as well and causes these velocities also to become mathematically infinite. This is the explanation for the observation that the solution of equation (15) for $\partial p/\partial t$ is mathematically determinate when only the initial pressure field is known for the whole atmosphere.

If one is concerned only with the horizontal propagation of effects in large-scale atmospheric systems, it is sufficient to study the propagation in a barotropic atmosphere with velocity independent of height, because such an atmosphere can always be chosen to approximate the mean horizontal motion of the real atmosphere.[5] A set of filtering equations equivalent to (15) for a barotropic atmosphere was found in (S) to be

$$\frac{d}{dt}\left(\frac{\zeta+f}{p}\right)=0, \qquad u=-\frac{1}{\rho f}\frac{\partial p}{\partial y}, \qquad v=\frac{1}{\rho f}\frac{\partial p}{\partial x}, \tag{17}$$

where p and ρ are the surface values of pressure and density. The earth is assumed plane and the motion a small perturbation independent of y on a constant zonal current of strength U. We then obtain

$$\frac{\partial^3 p}{\partial x^2\partial t}+U\frac{\partial^3 p}{\partial x^3}-\lambda^2\frac{\partial p}{\partial t}+\beta\frac{\partial p}{\partial x}=0, \tag{18}$$

where p is now understood to be the perturbation pressure, $\beta=df/dy$, and $\lambda^2=f^2/RT$ with T the mean surface temperature. It is convenient to take the day as the unit of time and the radius of the latitude circle as the unit of distance. In these units $\Omega=2\pi$ and $\beta=4\pi\cos^2\varphi$. At 45° latitude, $\beta=2\pi$ and $\lambda^2=2.5$. Unless otherwise stated it will always be assumed that $\varphi=45°$.

For greater simplicity the problem is reduced to one in which $U=0$ by referring the motion to a coordinate system moving parallel to the x-axis with the speed U. If we neglect the small error arising from the fact that the earth's surface deviates slightly from a geopotential surface in this system, an approximation equivalent to the assumption that the ground has the (negligible) slope of an isobaric surface, equation (18) takes the simple form

$$\left(\frac{\partial^2}{\partial x^2}-\lambda^2\right)\frac{\partial p}{\partial t}+\beta\frac{\partial p}{\partial x}=0. \tag{19}$$

The exact equation in the moving system is formally the same as (19) if β is replaced by $\beta'=\beta+\lambda^2U$. For $U=15$ m sec^{-1} the correction λ^2U is only 10 per cent of β and can be ignored. We shall, however, take it into account in calculations in which U is in the vicinity of 15 m sec^{-1} by increasing β to 1.1β—from 2π to 7.0 at 45° latitude.

Equation (19) has the wave solution

$$p=e^{i(kx-\nu t)} \tag{20}$$

[5] See section 6.

provided

$$\nu = -\beta k/(k^2 + \lambda^2), \tag{21}$$

where k is related to the wave length L by the formula $k = 2\pi/L$. The phase and group velocities c and c_g are given by the formulas

$$c = \nu/k = -\beta/(k^2 + \lambda^2), \tag{22}$$

$$c_g = \frac{d\nu}{dk} = \frac{\beta(k^2 - \lambda^2)}{(k^2 + \lambda^2)^2}, \tag{23}$$

and are represented in fig. 2 as functions of k.[6]

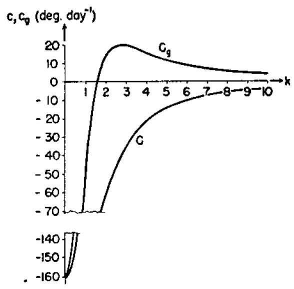

FIG. 2. Phase and group velocity in a resting barotropic atmosphere represented as a function of wave number k. The length unit is a radian of longitude at latitude 45°, so that $k = 1$ corresponds to a wave length equal to the circumference of the 45° latitude circle.

Let us consider the disturbance emitted by a point[7] source at a certain instant of time. This disturbance will consist of a series of waves spreading out from the point along the positive and negative x-axis, preceded in each direction by a non-wavelike forerunner containing but little of the total energy. Within the main body of the disturbance, k and ν are slowly varying functions of distance satisfying (21) and for which the group velocity (23) can be defined. It can be shown (see, for example, Jeffreys, 1946, p. 482) that the kinetic energy contained between two points each moving with the local group velocity is constant; hence the maximum velocity of propagation of a point disturbance is effectively limited by the maximum group velocity, and the minimum velocity by the minimum group velocity. Since an arbitrary disturbance can be regarded as a collection of point sources, the result follows that the pressure at a given locality will remain virtually unaffected by a disturbance whose distance away is either greater than the maximum group velocity times the forecast time or less than the minimum group velocity times the forecast time.

The mathematical argument is based on the method of stationary phase (see Jeffreys, 1946, p. 474). Write the initial function $p(x, 0)$ as the Fourier integral,

$$p(x, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} p(\alpha, 0)e^{ik(x-\alpha)}\, d\alpha, \tag{24}$$

[6] The importance of the group-velocity concept for dispersive atmospheric motions was first recognized by Rossby (1945); see also Yeh (1949).

[7] Strictly, a plane source.

whence, in virtue of (13) and (14), the pressure distribution

$$p(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} p(\alpha, 0)e^{i[k(x-\alpha)-\nu t]}\, d\alpha \tag{25}$$

satisfies both the initial condition and the equation of motion (19). Now subtract (24) from (25) and interchange the order of integration:

$$p(x, t) = p(x, 0) + \int_{-\infty}^{\infty} I(x - \alpha, t)p(\alpha, 0)\, d\alpha, \tag{26}$$

where

$$I(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (e^{-i\nu t} - 1)e^{ikx}\, dk. \tag{27}$$

In case $U \neq 0$, the formula corresponding to (26) is

$$p(x, t) = p(x - Ut, 0) + \int_{-\infty}^{\infty} I(x - Ut - \alpha, t)p(\alpha, 0)\, d\alpha.$$

Equation (25) states that the pressure may be regarded as a sum of sine waves of varying length each moving with the wave speed given by (22). In contrast (26) states that the pressure may be regarded as a sum of point-source disturbances.

The function I plays the role of a Green's function and is called the "influence function" since it determines the influence of an initial disturbance at a given point at the time t. Physically, $I(x - \alpha, t)$ is the change in the value of p at the point x in time t which is caused by a unit point disturbance originating at the point α at the time 0.[8]

The principle of stationary phase states in application that the integral I will annihilate itself by destructive interference except for those values of $x - \alpha$ for which the phase $k(x - \alpha) - \nu t$ is stationary, *i.e.*, except where

$$x - \alpha = c_g t.$$

Hence, if α satisfies either of the inequalities

$$x - \alpha > (\max c_g)t, \qquad x - \alpha < (\min c_g)t,$$

the phase cannot be stationary, and I is negligible. We may therefore replace the lower and upper limits of integration in (26) by $x - (\max c_g)t$ and $x - (\min c_g)t$

[8] The notion of a meteorological influence function has been used by Ertel (1941; 1944) to prove the *impossibility* of forecasting for a limited region of the atmosphere. He argues that because the influence function is generally different from zero over the whole earth there will be a basic indeterminacy in the forecast if the initial data are known for only a part of the atmosphere. This reasoning is *formally* correct, providing sound signals are excluded, just as it is correct to say (Rayleigh (1909)) that a disturbance of the surface of an ocean is propagated instantaneously, if the ocean is incompressible. But for *practical* purposes there is an effective limit to the speed of propagation of a disturbance in both cases, and this limit is given approximately by the maximum group velocity.

respectively, so that the pressure at a point x is determined by a knowledge of the initial disturbance between these two limits alone.

Fig. 2 shows that max c_g $(= \beta/8\lambda^2) = 19.8$ deg day^{-1} and min c_g $(= -\beta/\lambda^2) = 158.4$ deg day^{-1} at 45° latitude. The excessively large magnitude of min c_g is contrary to forecasting experience and can be discounted as the discrepancy arises only from the artificial assumption of an infinite plane earth. If one takes account of the fact that the maximum wave length at latitude 45° is limited to the circumference of the latitude circle ($k \geqq 1$), it is seen from fig. 2 that the minimum group velocity takes the more reasonable value -49.3 deg day^{-1}. Moreover, even this negative value may be discounted, for negative group velocities occur only in a small range of k, corresponding to waves with lengths greater than 18,000 km, and these are associated with little of the total energy. Thus it may be anticipated that $I(x - \alpha, t)$ will be appreciable only between the limits $\alpha = x$ and $\alpha = x - (\max c_g)t$.

However, too much confidence cannot be placed in these values, first because the method of stationary phase gives only an asymptotic approximation for large t to the integral I in (26), and $t = 1$ is not "large," and second, because the application of the method presupposes a continuous variation in k, whereas in fact k can only assume integral values, corresponding to wave lengths equal to integral fractions of the circumference of a latitude circle. We therefore turn to a more accurate method for getting at the values of the signal velocity.

In accordance with the requirement that k have only positive integral values the motion will be described in a cylindrical coordinate system with x measured in radians of longitude at the latitude φ and t in days. To solve (19) it is then necessary to use Fourier series in place of the Fourier integral. The solution analogous to (26) becomes

$$p(x, t) = p(x, 0) = \int_{-\pi}^{\pi} I_{\lambda^2}(x - \alpha, t) p(\alpha, 0)\, d\alpha, \quad (28)$$

where

$$I_{\lambda^2}(x, t) = \frac{1}{2\pi} \sum_{-\infty}^{+\infty} (e^{-i\nu t} - 1) e^{ikx}. \quad (29)$$

The form analogous to (27) for a nonzero zonal current is

$$p(x, t) = p(x - Ut, 0) + \int_{-\pi}^{\pi} I_{\lambda^2}(x - Ut - \alpha, t) p(\alpha, 0)\, d\alpha. \quad (30)$$

As no analytic expression for the influence function I_{λ^2} could be obtained, the series (29) was evaluated numerically for $t = 1$. The resulting function is graphed in fig. 3. We note that I_{λ^2} is small outside the region $0 < x < 27°$ longitude with boundaries corresponding to the signal velocities 0 deg day^{-1} and 27 deg day^{-1} respectively. These values are to be compared with the values 0 deg day^{-1} and 19.8 deg day^{-1} for the *effective* group velocities given previously.

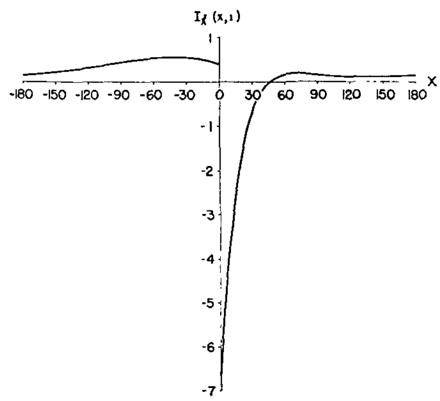

FIG. 3. The influence function $I_{\lambda^2}(x, t)$ for $t = 1$ day. This function represents the 24-hour change in p at the point x produced by a unit point disturbance at the origin at $t = 0$.

As a check on the accuracy of the signal-velocity determination, the influence function I_{λ^2} was used to forecast the actual distribution of v, the meridional geostrophic wind component at 45°N measured from the observed height profile at 500 mb,[9] using data taken from the U. S. Air Force Air Weather Service Historical Map Series. The actual map is shown in fig. 4. Since v also satisfies (19), the forecast equation is identical to (30) with v substituted for p.

The v distribution was calculated first by integrating over the entire range of x and next by integrating only from 0 to 27° longitude. The mean zonal speed at 45°N was measured as 15.6 deg day^{-1}. The results are shown in fig. 5. Curve I is the observed distribution for 0400 GCT 12 January 1944, curve II is the twenty-four hour forecast obtained using the entire range, and curve III by using the restricted range. Curve IV represents the observed distribution of v for 0400 GCT 13 January 1946. The fact that curves II and III coincide practically within the limits of observational error is a verification of the conclusions concerning the finiteness of the signal velocities.

The correspondence between the forecast and the observed distributions has seemed sufficiently close, particularly over North America and the Atlantic Ocean, to justify a further investigation with a view toward practical application. The discussion of these results has appeared in another publication (Charney and Eliassen, 1949). In this work it was found that the lateral variation of a disturbance has a significant

[9] The choice of the 500-mb level is justified in section 6.

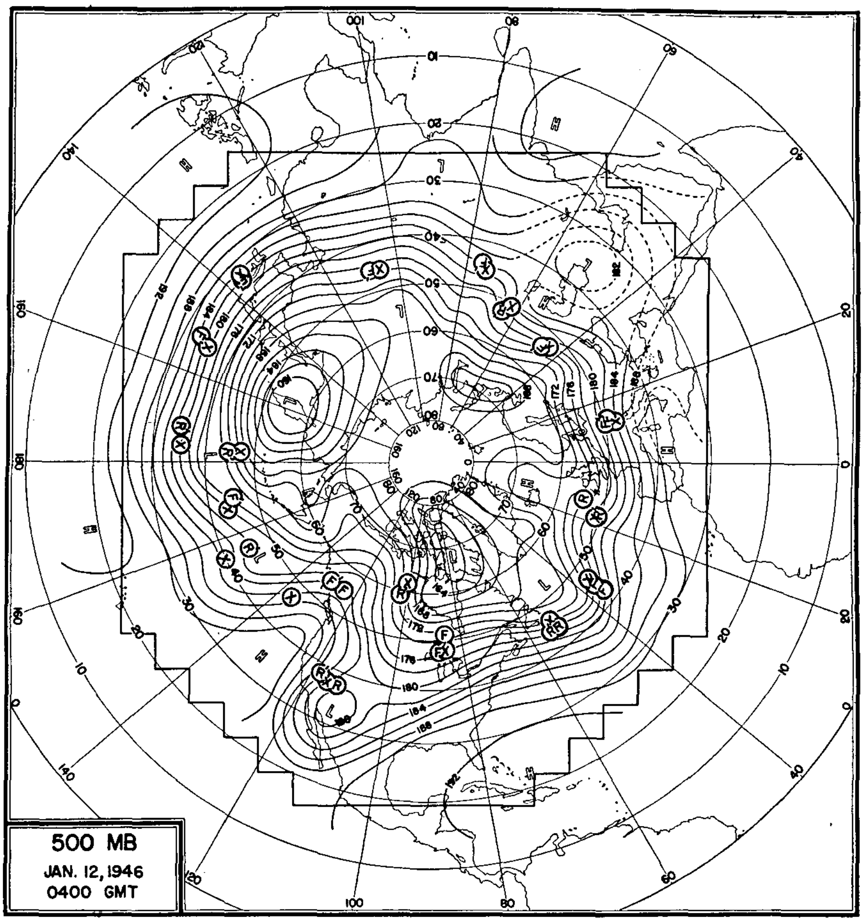

FIG. 4. The 500-mb chart for 0400 GCT 12 January 1946. The letters R and F are located respectively at the points of maximum and minimum height change for the 12-hr period following the time of the map. The letters R′ and F′ have a corresponding meaning for the instantaneous computed change, and the crosses locate the extreme points in the field of the vorticity advection.

effect on its motion, especially if the scale of the disturbance is large. To complete the discussion of the zonal signal velocities we shall now take up the effect of these variations.

If we reintroduce the y-dependency, equation (17) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \lambda^2\right)\frac{\partial p}{\partial t} + \beta\frac{\partial p}{\partial x} = 0 \qquad (31)$$

in a coordinate system translating with the mean zonal

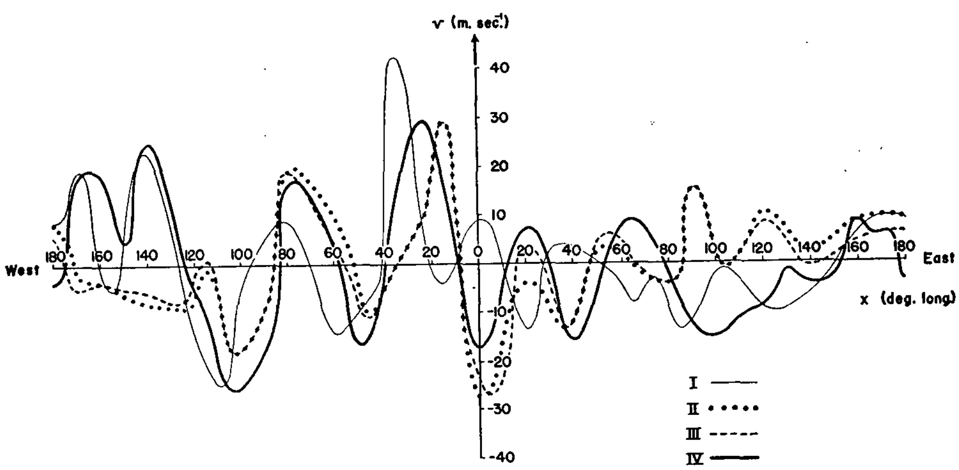

FIG. 5. Observed and predicted 500-mb distribution of v at 45°N. Curve I is the distribution observed 0400 GCT 12 January 1946. Curve II is the distribution predicted for 0400 GCT 13 January 1946, using initial data covering the entire latitude circle at 45°N, and curve III is the predicted distribution using initial data covering only the calculated influence interval ($x - Ut - 27°$ to $x - Ut$). Curve IV is the distribution observed 0400 GCT 13 January 1945.

wind. Assuming a solution of the form

$$p = e^{i(kx+\mu y-\nu t)}$$

we find

$$\nu = -\beta k/(k^2+\mu^2+\lambda^2) \tag{32}$$

and

$$c_g = \frac{\partial \nu}{\partial k} = \beta \frac{k^2-\mu^2-\lambda^2}{(k^2+\mu^2+\lambda^2)^2}. \tag{33}$$

These formulas become identical to (22) and (23) provided λ^2 is replaced by $\mu^2+\lambda^2$. In the aforementioned article the most representative value of $a^2=\mu^2+\lambda^2$ was found to be about 18 corresponding to a lateral wave length of about 7200 km. (We note for future reference that the quantity λ^2 is no longer important, since its value (2.5) is small compared to a^2. This means that it is possible to introduce the non-divergence assumption since this is equivalent to setting $\lambda^2=0$.) The maximum and minimum group velocities now assume the much smaller values $\beta/8a^2 = 2.5$ deg day^{-1} and $-\beta/a^2 = -20$ deg day^{-1}, respectively. (The difference between β and $\beta'(=1.1\beta)$ is here ignored.)

Turning again to the question of the distinction between group and signal velocities we replace $\partial^2/\partial y^2$ by $-\mu^2$ in (31), so that equation of motion becomes identical to (17) with λ^2 replaced by a^2. The solution can therefore be represented in the same form as (28):

$$p(x,t) = p(x,0) + \int_{-\pi}^{\pi} I_{a^2}(x-\alpha,t)p(\alpha,0)\,d\alpha, \tag{34}$$

where $I_a{}^2$ has a meaning analogous to $I_\lambda{}^2$. The function $I_{a^2}(x,t)$ has been computed by the method described in the article by Charney and Eliassen for $t = 1, 2, 3, 4, 5, 6, 7$ days and is tabulated in table 1. The values of I_0 used to evaluate I_{a^2} for $t = 2$, 3, and 4 were calculated from data given by Forsythe in a paper referred to in this article. The values of I_{a^2} for $t = 5$, 6, and 7 were calculated from the formula[10]

$$I_{a^2}(x, t_1+t_2) = I_{a^2}(x,t_1) + I_{a^2}(x,t_2)$$
$$\int_{-\pi}^{\pi} I_{a^2}(x-\alpha,t_1)\,I_{a^2}(\alpha,t_2)\,d\alpha.$$

To arrive at a rational means for determining where I_{a^2} can be neglected, *i.e.*, to calculate the signal velocity, we suppose that $I_{a^2}(x)$ is identically zero for $-\pi \leqq x \leqq -x_2$ and $\pi \geqq x \geqq x_1$ and call the resulting function $\hat{I}_{a^2}(x)$. The error incurred by replacing I_{a^2} by $\hat{I}_{a^2}$ is then, from (34),

$$q(x) = \int_{-\pi}^{\pi} [I_{a^2}(x-\alpha) - \hat{I}_{a^2}(x-\alpha)]p(\alpha,0)\,d\alpha.$$

Let us define the quantity Q^2 by the equation

$$Q^2 = \int_{-\pi}^{\pi} q^2(x)\,dx \Big/ \int_{-\pi}^{\pi} p^2(x,0)\,dx$$

so that Q^2 measures the ratio of the mean square deviation of p from its true value to the mean square value of p itself. The criterion that $\hat{I}_{a^2}(x)$ shall be an acceptable approximation to $I_{a^2}(x)$ is that Q^2 be less than some definite value, determined by the kind of accuracy desired for the forecast. Making use of the inequality

$$\int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi} G(x-\alpha)p(\alpha,0)\,d\alpha\right\}^2 dx$$
$$\leqq \pi \int_{-\pi}^{\pi} p^2(x,0)\,dx \int_{-\pi}^{\pi} G^2(x)\,dx,$$

which holds for arbitrary periodic functions $G(x)$ and $p(x)$ when one of their mean values is zero, we find

$$Q^2 \leqq \pi \int_{-\pi}^{\pi} [I_a{}^2(x) - \hat{I}_{a^2}(x)]^2\,dx.$$

[10] This formula has been obtained independently by C. C. Koo (personal communication).

TABLE 1. Values of the influence function $I_{18}(x,t)$ at latitude 45° for $t = 1, 2, 3, 4, 5, 6, 7$ days; x is expressed in degrees longitude.

x	$I_{18}(x,1)$	$I_{18}(-x,1)$	$I_{18}(x,2)$	$I_{18}(-x,2)$	$I_{18}(x,3)$	$I_{18}(-x,3)$	$I_{18}(x,4)$	$I_{18}(-x,4)$	$I_{18}(x,5)$	$I_{18}(-x,5)$	$I_{18}(x,6)$	$I_{18}(-x,6)$	$I_{18}(x,7)$	$I_{18}(-x,7)$	x
0	−4.301	1.999	−10.604	1.997	−18.121	0.780	−25.885	−0.684	−33.100	−1.598	−39.335	−1.533	−44.821	−0.719	0
5*			−5.495		−7.518		−7.983		−6.535		−3.501		0.417		5
10	−1.506	1.239	−2.549	1.701	−2.484	1.285	−1.057	0.312	1.577	−0.732	4.344	−1.274	6.481	−1.123	10
15*			−1.072		−0.365		1.056		2.624		3.634		3.460		15
20	−0.529	0.777	−0.476	1.358	0.194	1.417	1.169	0.902	1.643	0.041	1.482	−0.728	0.457	−1.065	20
25*			−0.134		0.345		0.896		0.899		0.336		−0.698		25
30	−0.167	0.457	0.000	0.968	0.333	1.240	0.538	1.082	0.359	0.531	−0.172	−0.215	−0.923	−0.807	30
40	−0.048	0.275	0.066	0.695	0.181	1.061	0.152	1.159	−0.071	0.914	−0.310	0.275	−0.411	−0.376	40
50	−0.016	0.161	0.031	0.469	0.042	0.817	−0.028	1.035	−0.111	0.979	−0.148	0.603	−0.070	0.033	50
60	0.002	0.091	0.028	0.311	0.023	0.619	−0.012	0.898	−0.029	0.979	−0.006	0.811	0.061	0.386	60
70	−0.003	0.057	0.001	0.209	−0.009	0.455	−0.024	0.729	−0.027	0.925	−0.006	0.931	0.023	0.666	70
80	0.003	0.028	0.006	0.125	0.000	0.312	−0.005	0.561	0.008	0.782	0.016	0.898	0.024	0.790	80
90	−0.001	0.020	0.000	0.089	−0.002	0.230	0.003	0.439	−0.001	0.669	0.002	0.852	0.006	0.867	90
100	0.001	0.008	−0.001	0.048	−0.005	0.144	−0.007	0.307	−0.001	0.522	−0.006	0.730	0.006	0.834	100
110	0.001	0.007	0.003	0.035	0.005	0.108	0.011	0.238	0.008	0.425	0.015	0.617	0.029	0.773	110
120	−0.002	0.004	−0.004	0.019	−0.007	0.065	−0.007	0.157	−0.006	0.309	−0.006	0.500	0.013	0.674	120
130	0.002	0.001	0.005	0.012	0.007	0.044	0.010	0.115	0.011	0.218	0.027	0.392	0.049	0.568	130
140	−0.002	0.003	−0.004	0.010	−0.004	0.031	−0.001	0.079	0.000	0.176	0.012	0.316	0.042	0.481	140
150	0.002	−0.002	0.003	0.001	0.003	0.014	0.006	0.048	0.016	0.113	0.037	0.223	0.078	0.371	150
160	−0.001	0.002	−0.001	0.007	0.001	0.019	0.010	0.042	0.015	0.100	0.048	0.186	0.097	0.310	160
170	0.001	−0.001	−0.001	−0.002	0.000	0.001	0.005	0.015	0.025	0.061	0.067	0.131	0.131	0.233	170
180	0.001	0.001	0.003	0.003	0.009	0.009	0.024	0.024	0.049	0.049	0.105	0.105	0.192	0.192	180

* The values of the influence function for $x = +5°$, $+15°$, and $+25°$ are included in order to give a better definition of the function, as the variation is particularly rapid in the range 0° to +30°.

The right-hand expression therefore serves as a means for placing a bound on the fractional mean square error incurred by "cutting off" the influence function at the points x_1 and $-x_2$. For a given value of Q^2 the points x_1 and $-x_2$ are chosen to give the smallest influence region consistent with the above inequality. The results of the computation are shown in the graph in fig. 6 where x_1 and x_2 are plotted as functions of t for different values of Q^2.

The curves are extended to $t = 0$, since there is a finite influence region for the pressure tendency. To see this differentiate (34) with respect to t and set $t = 0$. One obtains

$$\frac{\partial p}{\partial t}(x, 0) = \int_{-\pi}^{\pi} K(x - \alpha) p(\alpha, 0)\, d\alpha,$$

where

$$K(x) = \frac{i\beta}{2\pi} \sum_{-\infty}^{\infty} \frac{k e^{ikx}}{k^2 + a^2} = -\tfrac{1}{2}\beta \operatorname{sgn} x \frac{\sinh a(\pi - |x|)}{\sinh a\pi}.$$

The error made in cutting off $K(x)$ at the points $x = \pm s$ is measured by the quotient

$$Q^2 = \frac{\displaystyle\int_{-\pi}^{\pi} \left\{ \int_{-\pi}^{\pi} [K(x - \alpha) - \hat{K}(x - \alpha)] p(\alpha, 0)\, d\alpha \right\}^2 dx}{\displaystyle\int_{-\pi}^{\pi} \left(\frac{\partial p}{\partial t}(x, 0) \right)^2 dx},$$

or

$$Q^2 \leqq \frac{\pi \left\{ \displaystyle\int_{-\pi}^{\pi} p^2(x, 0)\, dx \right\} \left\{ \displaystyle\int_{-\pi}^{\pi} [K(x) - \hat{K}(x)]^2\, dx \right\}}{\displaystyle\int_{-\pi}^{\pi} \left(\frac{\partial p}{\partial t}(x, 0) \right)^2 dx},$$

where

$$\hat{K}(x) = \begin{cases} K(x), & x \leqq |s| \\ 0, & x > |s|. \end{cases}$$

As one may show that

$$\int_{-\pi}^{\pi} p^2(x, t)\, dx \quad \text{and} \quad \int_{-\pi}^{\pi} \left(\frac{\partial p}{\partial t}(x, t) \right)^2 dx$$

are invariants of the motion, the bound on Q^2 is dependent solely on

$$\int_{-\pi}^{\pi} [K(x) - \hat{K}(x)]^2\, dx.$$

These integrals are determined empirically from an observed distribution of $p(x, t)$ at any time t. Specifically, the data for 0400 GCT 12 January 1946 were used.

The straight lines in fig. 6 are the group-velocity curves:

$$x = (\max c_g)t = (\beta/8a^2)t \quad \text{and} \quad x = |\min c_g| t = (\beta/a^2)t.$$

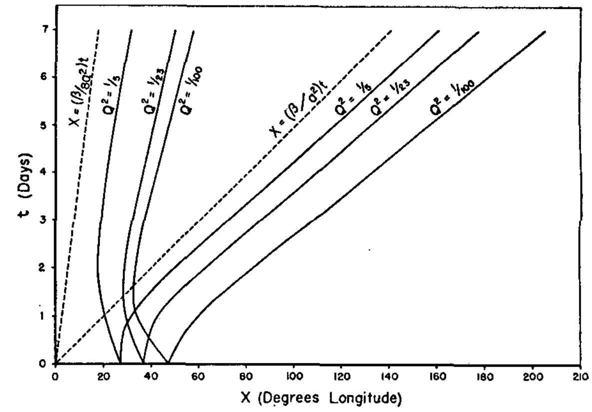

FIG. 6. Group- and signal-velocity curves for $a^2 = 18$ and $U = 0$.

The signal-velocity curves approximately parallel the group-velocity curves for large values of t. This conforms with the notion that the group velocity approaches the signal velocity for large t. However, for small t, say $t < 2$ days, the group velocity gives a very inaccurate estimate of the signal velocity, and for $t \ll 1$, in particular for $t = 0$, it gives no indication whatsoever, since there is a finite influence region for $t = 0$.

The last remark may at first seem paradoxical, but the following consideration will indicate that no basic physical principles are violated. While the instantaneour pressure change becomes a local property of the motion as soon as sound waves are permitted, the change that one obtains in this way bears no relation to what one means by the *meteorological* pressure tendency. The ideal barograph, *viz.*, one that records all pressure variations, traces a continuous but virtually nowhere differentiable curve; a series of extremely small-scale microbarographic fluctuations are found to be superimposed on the uniform macroscopic pressure curve, among which are included the sound fluctuations. Thus, even if an "instantaneous" time derivative of the pressure could be measured it would give no indication of the meteorologically significant trend. It is clear then that one is never concerned with the instantaneous change but only with the change during a time interval that, while small, is large enough to permit the high frequency fluctuations to be averaged out. But in such an interval there will be time for the effects of disturbances at finite distances from a point to make themselves felt, for these effects signal their arrival with the velocity of sound. If the flow is in quasigeostrophic adjustment, the effective region of influence for this time interval must be precisely the one already determined from $K(x)$.

It is possible to obtain an estimate of the *lateral* signal velocity by again using group-velocity considerations, although in view of what was found in preceding paragraphs the estimates can only be rough.

If a plane wave disturbance of the type

$$e^{i(kx \pm \mu y - \nu t)}$$

exists, where $\nu = \nu(k, \mu)$, we define the quantities $c_{gx} = \partial\nu/\partial k$ and $c_{gy} = \partial\nu/\partial\mu$ to be the group-velocity components in the x- and y-directions respectively. These quantities have the kinematic and dynamic properties of the one-dimensional group velocity. Thus it can be proved by a method similar to the one used by Jeffreys (1946), employing the method of stationary phase, that the kinetic energy E of a point-source disturbance obeys the conservation law

$$\frac{\partial E}{\partial t} + \frac{\partial c_{gx}E}{\partial x} + \frac{\partial c_{gy}E}{\partial y} = 0.$$

This law states that the energy of the disturbance associated with a given area in the x, y-plane does not change when each point of the area moves with the local group velocity. If the group-velocity components c_{gx} and c_{gy} are bounded by max c_{gx} and max c_{gy} respectively, we may assert that the energy cannot spread at a rate exceeding that determined by these maximum values. Since any two-dimensional disturbance can be expressed as a sum of point-source disturbances, we obtain the general result that the propagation of energy, and therefore the propagation of signals, is limited in speed by the maximum group velocity.

By differentiating (32) with respect to μ we find

$$c_{gy} = 2\beta k\mu/(k^2 + \mu^2 + \lambda^2)^2.$$

If k and μ are not restricted the extreme values of c_{gy} are $\pm\beta/4\lambda^2$ or $\pm\pi/5$ radians day^{-1}. These occur for $\mu^2 = \frac{1}{2}\lambda^2$, corresponding to a lateral wave length of 25,300 km, which is unrealistically large. But here as before one must place some restriction on the kinematics of the motion. The observed wave-like perturbations in the atmosphere can be said to have a nodal line just north of the subtropical high cells, about 25°N in winter, and a nodal point at the pole. Hence one may say that the lateral wave length cannot exceed $2 \times 65 = 130$ degrees of latitude, which means that μ cannot be less than 2. For a given μ the maximum value of c_{gy} is $\frac{1}{8}3\sqrt{3}\beta\mu(\mu^2 + \lambda^2)^{-\frac{3}{2}}$ and occurs for $k^2 = \frac{1}{3}(\mu^2 + \lambda^2)$. This expression has a maximum at $\mu^2 = \frac{1}{2}\lambda^2$ and thereafter decreases monotonically with increasing μ. Hence its maximum value, compatible with the condition $\mu \geqq 2$, occurs for $\mu = 2$ and is equal to 0.48 rad day^{-1}, or 28 deg day^{-1}. The corresponding minimum value is -28 deg day. It should of course be mentioned that the approximation of taking β and f constant in the two-dimensional case is very crude, so that the computed signal velocities can be considered only as very rough estimates.

We note that identical results are achieved by applying the method of stationary phase directly to the general solution of (31) expressed by the Fourier integral,

$$p(x, y, t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\alpha, \gamma, 0) \times e^{i[k(x-\alpha)+\mu(y-\gamma)-\nu t]}\, d\alpha\, d\gamma\, dk\, d\mu. \quad (35)$$

One has here to consider the stationarity of the phase function $k(x - \alpha) + \mu(y - \gamma) - \nu t$, and is thereby led to consider the equations

$$\partial\nu/\partial k = c_{gx} = (x - \alpha)/t,$$
$$\partial\nu/\partial\mu = c_{gy} = (y - \gamma)/t.$$

The generalization from the one- to the two-dimensional case is obvious.

Recalling that the zonal speed U must always be added to the calculated zonal signal velocities and taking this to be 18 deg day^{-1}, we may give a rough estimate of the size of the influence region surrounding a point for a one-day forecast. Fig. 6 shows that the distance to the west is about 35 + 18 or 53 degrees and the distance to the east about 50 − 18 or 32 degrees. The distance north and south, estimated from the group velocities, is 28 degrees. It is probable that these estimates are somewhat too large. This is because no restriction has been placed on the scale of the motion except that $k \geqq 1$ and $\mu \geqq 2$, whereas the energy spectrum of the traveling disturbances of the atmosphere shows a maximum for disturbances having a k value of between 6 and 9 and a value of μ between 4 and 7. For such motions the group velocities are less than those calculated; one may presume the same to be true for the signal velocities. The estimates nevertheless serve to establish a safe margin for error in first attempts at numerical forecasting. The experience gained from such attempts will undoubtedly lead to better estimates.

With the present estimates one may say that the horizontal extent of the existing network of meteorological stations is adequate for predicting the motion over certain areas of the globe, *viz.*, for the eastern United States, the Atlantic Ocean, and Europe in middle latitudes. For periods much in excess of twenty-four hours, however, it is likely that influences spreading from uncharted areas will render accurate forecasts impossible.

5. Vertical signal velocities

It is important to know how influences are propagated vertically as well as horizontally. A forecast is possible only if disturbances above the regions in which data are available produce negligible effects at lower levels. This may be because of the small energy available at great heights or because the vertical signal velocity is so small that influences do not propagate into the forecast area within the forecast time interval. We shall investigate the latter possibility.

While it is true that the hydrostatic assumption implies mathematically infinite vertical signal velocities—just as the geostrophic approximation implies mathematically infinite horizontal signal velocities—it is by no means necessary that significant changes be propagated instantaneously. It cannot be argued, for example, that an increase or decrease of mass above a given level causes an instantaneous pressure change at the ground. Such a change can take place only if compensating changes do not occur simultaneously below the level to annul the effect; the hydrostatic approximation does not necessarily imply a rigid connection between one part of a vertical column and another. We shall show, in fact, that appreciable effects are propagated slowly in a statically stable atmosphere. For this purpose the following simple baroclinic model is adopted.

We consider a resting[11] incompressible atmosphere with mean density decreasing exponentially with height at a rate corresponding to the decrease in a compressible atmosphere with a constant mean temperature T_m; thus, $\rho = \rho_0 \exp(-z/H)$, where $H = RT_m/g$. At the same time, to simulate the stability characteristics of the actual atmosphere, we suppose that the static stability $-\partial(\ln \rho)/\partial z$, where it occurs in the process equation, $d\rho/dt = 0$, may be given the value $1/h$, corresponding to a constant average value of the observed stability, $\partial(\ln \theta)/\partial z$. This procedure has the advantage of mathematical simplicity and at the same time leaves the motion, in its essential aspects, similar to the motion in a stable compressible atmosphere with adiabatic changes of state. The motion is also assumed to be independent of the y coordinate.

A stratified atmosphere permits a doubly infinite set of plane internal wave motions, corresponding to a doubly infinite set of horizontal and vertical wave numbers. An arbitrary initial disturbance can be regarded as a linear superposition of such internal waves. For a given x, waves will be propagated in the vertical direction and will be reflected by the ground. A vertical group velocity can be defined in much the same manner as the meridional group velocity for the barotropic model. The demonstration that this group, or signal velocity, is limited will now be given.

The equation governing the motion is derived from the two conservation laws

$$\frac{d}{dt}\left[\frac{1}{\rho}\frac{\partial \rho}{\partial z}(\zeta + f)\right] = 0, \qquad d\rho/dt = 0, \tag{36}$$

together with the hydrostatic and geostrophic relations. On the basis of the assumptions concerning the stability, we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 p}{\partial x^2} + \frac{f^2 h}{g}\left(\frac{\partial^2 p}{\partial z^2} + \frac{1}{H}\frac{\partial p}{\partial z}\right)\right] + \beta\frac{\partial p}{\partial x} = 0. \tag{37}$$

The surface boundary condition in p is found by setting w equal to 0 in the second equation in (36). The resulting equation states that the density at the ground is advected with the wind, or, since the mean wind is zero,

$$0 = \frac{\partial \rho}{\partial t} = -\frac{1}{g}\frac{\partial}{\partial t}\frac{\partial p}{\partial z} \tag{38}$$

at the ground. At the upper boundary of the atmosphere we must have $p = 0$.

The system (37, 38) can be solved as follows. Introduce the dependent variable, $\rho = -g^{-1}\partial p/\partial z$, and make the substitutions

$$\rho = qe^{-z/2H}, \qquad \eta = (g/f^2H)^{\frac{1}{2}}z.$$

The equations become

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial \eta^2} - \omega^2 q\right) + \beta\frac{\partial q}{\partial x} = 0, \tag{39}$$

$$\partial q/\partial t = 0, \qquad (\eta = 0) \tag{40}$$

where $\omega^2 = f^2h/4gH^2$.

Equation (39) is satisfied by the two plane-wave solutions $\exp[i(kx \pm \mu\eta - \nu t)]$, provided

$$\nu = -\beta k/(k^2 + \mu^2 + \omega^2).$$

The general solution is expressed by the Fourier integral

$$q(x, \eta, t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, d\mu \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(\alpha, \gamma, 0) \times e^{i[k(x-\alpha)+\mu(\eta-\gamma)-\nu t]}\, d\alpha\, d\gamma. \tag{41}$$

If we define $q(x, -\eta, 0) = -q(x, \eta, 0)$ the boundary condition is also satisfied.

We may now apply the same consideration to (41) as to (35), with η corresponding to y. The vertical group velocity is found to have the extreme values

$$\frac{dz}{dt} = \frac{dz}{d\eta}\frac{d\eta}{dt} = \pm\sqrt{\frac{f^2H}{g}}\frac{\beta}{4\omega^2} = \pm\frac{\beta H^2}{fh}\sqrt{gH}.$$

If h is taken as 10^5 m so that the stability $-\partial(\ln \rho)/\partial z$ corresponds to a mean lapse rate of 7C km^{-1}, the extreme group velocities are ± 4.5 km day^{-1} at 45° latitude. Thus influences above 16 km, the maximum height at which data are available, will not reach the ground within a 48-hr period. From this point of view, it is likely that the extent of data now at our disposal will be adequate for the preparation of low-level forecasts for periods of from 24 to 48 hours.

The smallness of the vertical signal velocity is not the only reason for expecting that upper-atmosphere

[11] As before, the results to be obtained permit an easy generalization to the case of constant zonal motion.

conditions will not influence lower level motions for short periods. Since the initial function

$$q(x, z, 0) = \rho(x, z, 0)e^{z/2H}$$

damps out quickly with height, we have an additional reason for replacing the infinite γ-limits of integration in (41) by finite values within the region of available data.

6. The equivalent-barotropic atmosphere

The amount and the quality of data are not the only factors that must be considered in the preparation of a numerical forecast. The method of integration itself is of the highest importance and may mean success or failure. Since there does not exist any adequate theory for the numerical integration of the nonlinear equations encountered in meteorology, one must proceed by performing a series of numerical experiments. With respect to the order of these experiments, it was decided that the greatest economy of labor could probably best be achieved by treating in turn each of a hierarchy of models embodying successively more and more of the physical and numerical aspects of the general forecast problem. The one- and two-dimensional small-perturbation models have already been discussed; we shall now turn to the next step and treat a nonlinear two-dimensional model.

The choice of a suitable, yet practical model was the first problem. In view of the success with which Rossby and others had applied small perturbation theory in the barotropic model to explain a variety of atmospheric phenomena, it was deemed worthwhile to extend those studies to finite amplitude motions. Rossby's studies and that of Charney and Eliassen (1949) strongly indicated that those aspects of the observed motions which involve horizontal dispersion —rather than vertical transport—of energy could be explained as essentially barotropic phenomena. Furthermore it was felt that the procedural experience gained in this study would provide an excellent preparation for the eventual attack on the baroclinic case.

The basis of the correspondence between the barotropic and baroclinic atmospheres lies in the notion of the "equivalent-barotropic atmosphere," a barotropic atmosphere in which the horizontal motion approximates the actual motion of the atmosphere at a particular level, called the "equivalent-barotropic level." In a previous article (1947), the writer gave a rationale for the choice of the equivalent-barotropic level in the case of small-amplitude perturbations in a baroclinic zonal current. In the following discussion, the concept of equivalent-barotropic atmosphere will be extended to apply to finite-amplitude motions. One essential feature making it possible to define such an atmosphere is the observed fact that the winds in the large-scale systems vary little in direction with height above a rather shallow surface layer. This property may also be deduced from the thermal wind equation as a necessary consequence of the fact that the horizontal isolines of pressure and temperature nearly coincide in the large-scale motions. Another feature is that the variation of wind speed with height is similar along all verticals. These two properties are expressed through the equations

$$u = A(p)u'(x, y)$$
$$v = A(p)v'(x, y).$$

Let us insert these expressions for u and v into the horizontal equations of motion (5) and (6) and, after multiplying by $-\rho g$, integrate with respect to z from the top to the bottom of the atmosphere. If the small terms involving w in the convective parts of the acceleration components are ignored, we get

$$\frac{\partial \bar{u}}{\partial t} + K\bar{u}\frac{\partial \bar{u}}{\partial x} + K\bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{1}{p_0}\frac{\partial Q}{\partial x} + f\bar{v},$$
$$\frac{\partial \bar{v}}{\partial t} + K\bar{u}\frac{\partial \bar{v}}{\partial x} + K\bar{v}\frac{\partial \bar{v}}{\partial y} = -\frac{1}{p_0}\frac{\partial Q}{\partial y} - f\bar{u},$$

where

$$Q = \int_{z_0}^{0} p\, dz, \qquad K = \int_{p_0}^{0} A^2\, dp \Big/ \left(\int_{p_0}^{0} A\, dp \right)^2,$$

and $\bar{u}$, $\bar{v}$ are the values of u, v at the level $\bar{p}$ where $A(p)$ is equal to its vertical pressure average, *i.e.*,

$$p_0 A(\bar{p}) = \int_0^{p_0} A(p)\, dp.$$

Denoting the vorticity of the flow $\bar{u}$, $\bar{v}$ by $\bar{\zeta}$, we get by cross-differentiation

$$\begin{aligned}\frac{\partial \bar{\zeta}}{\partial t} &+ K\bar{u}\frac{\partial \bar{\zeta}}{\partial x} + K\bar{v}\frac{\partial \bar{\zeta}}{\partial y} + \frac{df}{dy}\bar{v} \\ &+ (f + K\bar{\zeta})\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right) \\ &= \frac{1}{p_0^2}\left(\frac{\partial p_0}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial p_0}{\partial y}\frac{\partial Q}{\partial x}\right) \\ &= \frac{1}{p_0^2}\int_{z_0}^{0}\left(\frac{\partial p_0}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial p_0}{\partial y}\frac{\partial p}{\partial x}\right) dz = 0.\end{aligned}$$

The right-hand side is set equal to zero since the assumption of horizontal barotropy implies that the isobars are parallel at all levels. This equation may be combined with the tendency equation,

$$\begin{aligned}\frac{\partial p_0}{\partial t} - \rho_0 g w_0 &= -g\int_{z_0}^{z}\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}\right) dz \\ &= p_0\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y}\right),\end{aligned}$$

to yield

$$\frac{1}{f+K\bar{\zeta}}\left(\frac{\partial\bar{\zeta}}{\partial t}+K\bar{u}\frac{\partial\bar{\zeta}}{\partial x}+K\bar{v}\frac{\partial\zeta}{\partial y}+\frac{df}{dy}\bar{v}\right)+\frac{1}{p_0}\left(\frac{\partial p_0}{\partial t}-\rho_0 g w_0\right)=0,$$

which is equivalent to Rossby's potential-vorticity equation for a barotropic atmosphere,

$$\frac{d}{dt}\left(\frac{\bar{\zeta}+f}{p_0}\right)=0,$$

provided the geostrophic approximation $\bar{u}\,\partial p_0/\partial x + \bar{v}\,\partial p_0/\partial y = 0$ is made, implying

$$\frac{dp_0}{dt}=\frac{\partial p_0}{\partial t}+w_0\frac{\partial p_0}{\partial z}=\frac{\partial p_0}{\partial t}-\rho_0 g w_0,$$

and provided $K \approx 1$.

This last assumption was tested empirically. The ratio of A to its maximum value A^* was found to be quite well approximated by the empirical formula

$$A = A^*\sigma e^{1-\sigma},$$

where σ is the ratio of p to its value at the level where $A = A^*$. This formula gives $K = 5/4$, so that no great error is made by setting $K \approx 1$. The formula also gives the value 550 mb for $\bar{p}$, the pressure at the equivalent-barotropic level, if the tropopause is taken at 250 mb and the ground at 1000 mb. This value agrees fairly well with the values 570 mb and 610 mb obtained in the previous article (1947) from two different mean zonal wind profiles. In a particular weather situation (0400 GCT 12 January 1946) a small systematic variation of $\bar{p}$ with latitude was observed, but none with longitude. The over-all mean for some 40 points selected in the range 20°–90°E, 25°–65°N, was 602 mb, with a standard deviation of 78 mb. The pressure at the equivalent-barotropic level would thus appear to lie between 550 mb and 600 mb.

In the following we shall suppose that the surface of the earth is horizontal and that $\partial p_0/\partial t$ is negligible. The latter assumption is equivalent to the non-divergence assumption which, as indicated in section 4, is probably valid for motions whose scale is not too great. Using the geostrophic approximation, we may then write

$$(\partial/\partial t + \mathbf{v}_g\cdot\nabla)(\zeta_g + f) = 0 \qquad (42)$$

for the equation governing the motion of the equivalent-barotropic atmosphere.

The numerical solutions of the corresponding equivalent-barotropic equation for a spherical earth can be obtained in the following way. On a spherical earth we have

$$\zeta = \frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}-\frac{1}{a}\frac{\partial u}{\partial\varphi}+\frac{u}{a}\tan\varphi,$$

where φ is the latitude, λ the longitude, and a the radius of the earth. Since it is now common to operate with constant-pressure maps it will be convenient to introduce the height z of an isobaric surface as the dependent variable. The geostrophic relations then take the form

$$u_g = -\frac{g}{fa}\frac{\partial z}{\partial\varphi}, \qquad v_g = \frac{g}{fa\cos\varphi}\frac{\partial z}{\partial\lambda}.$$

Whence, upon ignoring small terms arising from the horizontal variation of f,

$$\zeta_g = \frac{g}{f}\Delta_s p = \frac{g}{f}\left[\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial z}{\partial\varphi}\cos\varphi\right)+\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 z}{\partial\lambda^2}\right],$$

where Δ_s is the expression for the Laplacian operator in surface spherical coordinates. Equation (42) may be written

$$\Delta_s\partial z/\partial t = J_s(\zeta_g + f, z), \qquad (43)$$

where J_s is the Jacobian operator

$$\frac{1}{a^2\cos\varphi}\left(\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\varphi}-\frac{\partial}{\partial\varphi}\frac{\partial}{\partial\lambda}\right).$$

In accordance with the general integration procedure outlined in section 3, we regard (43) as a partial differential equation in the height tendency $\partial z/\partial t$. The solution can be immediately written

$$\frac{\partial z}{\partial t}(\varphi,\lambda) = \int\int G_s(\sigma)J_s(\varphi,\lambda;\varphi',\lambda')a^2\cos\varphi'\,d\lambda'\,d\varphi', \qquad (44)$$

where σ is the great circle distance between the fixed point φ, λ and the variable point φ', λ', and $G_s(\sigma) = (2\pi)^{-1}\ln 2\sin\frac{1}{2}\sigma$ (Courant–Hilbert, 1931), the double integral being extended over the entire sphere. Since f vanishes at the equator the right-hand side ceases to be defined. We can escape this difficulty by arbitrarily assigning some constant value to J_s in the vicinity of the equator. Since a finite time is required for effects to propagate from this vicinity to northern latitudes no errors in the computed motion at these latitudes are introduced provided the forecast time interval is not great.

The nature of the Green's function G_s illustrates clearly the dependence of the signal velocity on scale. G_s decreases so slowly with increasing σ that apparently the motion over at least an entire hemisphere must be taken into account in evaluating the tendency at some fixed point. This seems to contradict the conclusions already reached regarding the smallness of the influence area for the pressure tendency, but it must be remembered that a certain limited scale was predicated in arriving at this area for the influence region. It appears that the effects of small-scale circulations at large distances cancel themselves out, whereas those of the large-scale circulations do not.

The task of numerical integration is somewhat simplified if (43) is transformed into a differential equation in the plane by conformally mapping the spherical earth onto a plane, say by a stereographic projection from the south pole. The equation becomes

$$\nabla^2 \partial z/\partial t = J(m^2 f^{-1} g \nabla^2 z + f, z), \qquad (45)$$

where m is the magnification factor $\sec^2(\frac{1}{4}\pi - \frac{1}{2}\varphi)$, while J is the plane Jacobian and ∇^2 the plane Laplacian.

For hand computation (45) can best be solved by the method of relaxation (Southwell, 1946). Using this method, the 500-mb height tendency was computed for the map shown in fig. 4. The solution was obtained for the interior of the indicated polygonal area on the assumption of constant z along the boundary.

It would have been interesting as a check on the barotropic model to compare the calculated with observed tendencies for the entire area. Since upper-air tendencies are not measured we have recorded the position of the centers of rise and fall in the height change from 0400 GCT 12 January to 1600 GCT 12 January by the letters R and F, and, for comparison, the calculated centers by the letters R′ and F′. The comparison can only be made from 130°W to 20°E longitude since the 1600 GCT map extends only over this region. The agreement of the observed with the calculated positions has seemed sufficiently good to warrant a continued effort to exploit the barotropic model. The results of this work will be presented in a later publication.

Finally we show that the location of the centers of maximum and minimum height tendency can be deduced solely from the properties of the field of J, *i.e.*, of the field of absolute vorticity advection. Let us consider the height tendency at a point where J is a maximum. Since $\partial z/\partial t$ is effectively determined from a knowledge of the values of J within a relatively small circular area surrounding this point, its values on the periphery may be chosen as constant. Then (45) may be interpreted as the equation for the displacement $(\partial z/\partial t)$ of a stretched membrane acted upon by the normal force field J. If the membrane is fixed along a level, nearly circular curve, and J has a maximum at the center and is symmetrically distributed about the center, it is physically evident that the displacement will be a maximum at the center. We may therefore state the rule that *the centers of rise and fall are located approximately where J takes on its minimum and maximum values respectively*, or, what is the same, where the absolute vorticity advection $\mathbf{v}\cdot\nabla(\zeta + f)$ is respectively a maximum or a minimum. To illustrate this rule the positions of the maximum and minimum points in the field of J have been entered in the map of fig. 4 as crosses. The rule seems to be well substantiated.

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