MATH 226 SOLUTIONS MANUAL
PART 1.1: SETS AND COORDINATES

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1. *Decide whether each of the sets below is open, closed, or neither.*
   (a) $\{(x, y) \in \mathbb{R}^2 : \ 0 \leq x < 1, \ 0 \leq y < 2\}$ – neither
   (b) $\{(x, y) \in \mathbb{R}^2 : \ x + y < 2\}$ – open
   (c) $\{(x, y) \in \mathbb{R}^2 : \ 0 < \sqrt{x^2 + y^2} \leq 3\}$ – neither open nor closed
   (d) $\{(x, y, z) \in \mathbb{R}^3 : \ |x + 2y + 3z| \geq 6\}$ – closed

2. *The following conditions describe sets in $\mathbb{R}^3$. Are these sets open, closed, or neither? Specify the boundary of each set.*
   (a) $x^2 + y^2 + z^2 \geq 16$: The boundary is the sphere $x^2 + y^2 + z^2 = 16$. Since this sphere is included in S, the set is closed.
   (b) $x^2 + y^2 \geq 4$, $z \geq 0$: This is the set above the plane $z = 0$ and on the outside of the cylinder $x^2 + y^2 = 4$, with boundary included. The boundary is the union of the cylindrical surface $\{(x, y, z) : \ x^2 + y^2 = 4, \, z \geq 0\}$, and the set $x^2 + y^2 \geq 4$ in the plane $z = 0$. This set is closed.
   (c) $\sqrt{x^2 + y^2} \leq z < 1$: This is the set above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 1$, with the cone surface included and the plane not included. The boundary consists of the conical segment $\{(x, y, z) : \ z = \sqrt{x^2 + y^2}, \, z \leq 1\}$, and the disk $x^2 + y^2 \leq 1$ in the plane $z = 1$. This set is neither closed nor open.
   (d) $0 < x < 1$, $0 < y < 1$, $z = 1$. This is the unit square in the plane $z = 1$, without the edges. The boundary is the closed square $\{(x, y, z) : \ 0 \leq x \leq 1, 0 \leq y \leq 1, \, z = 1\}$. This set is neither closed nor open.
   (e) $\{(x, y, z) \in \mathbb{R}^3 : \ 0 < \sqrt{x^2 + y^2} \leq 3\}$: The set is neither open nor closed. The boundary consists of the cylinder $\sqrt{x^2 + y^2} = 3$ (included in the set) and the line $x = y = 0$ in $\mathbb{R}^3$ (not included in it).
   (f) $\{(x, y, z) \in \mathbb{R}^3 : \ x \geq 0, \ x^2 + y^2 + z^2 \geq 4\}$: The set is closed. The boundary consists of the half-sphere $x \geq 0, \ x^2 + y^2 + z^2 = 4$, and the set $x = 0, \ y^2 + z^2 \geq 4$.
   (g) $\{(x, y, z) \in \mathbb{R}^3 : \ z > \sqrt{x^2 + y^2}, \ x^2 + y^2 + z^2 \leq 2\}$: The set is neither open nor closed. The boundary is

$$\{(x, y, z) \in \mathbb{R}^3 : \ z = \sqrt{x^2 + y^2}, \ x^2 + y^2 + z^2 \leq 2\}$$
$$\cup \, \{(x, y, z) \in \mathbb{R}^3 : \ z > \sqrt{x^2 + y^2}, \ x^2 + y^2 + z^2 = 2\}.$$

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Optional: There are other equivalent ways to express the answer, for example

$$\{(x, y, z) \in \mathbb{R}^3 : \ z = \sqrt{x^2 + y^2}, \ 0 \le z \le 1\}$$
$$\cup \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 2, \ z \ge 1\}.$$

(h) $\{(x, y, z) \in \mathbb{R}^3 : \ x > 0, \ 0 \le y \le 4\}$: The set is neither open nor closed. The boundary is

$$\{(x, y, z) \in \mathbb{R}^3 : \ x > 0, \ y = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 : \ x > 0. \ y = 4\}$$
$$\cup\{(x, y, z) \in \mathbb{R}^3 : \ x = 0, \ 0 \le y \le 4\}.$$

3. *Prove directly from the definition that the set* $S = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 < 4\}$ *is an open set in* $\mathbb{R}^3$. *(Given a point* $P = (x, y, z) \in S$, *find a neigbourhood of* P *which is contained in* S.*)*

   Let $P = (x, y, z) \in S$. By the definition of S, we have $x^2 + y^2 < 4$, so that if we choose $\varepsilon = 2 - \sqrt{x^2 + y^2}$ then $\varepsilon > 0$. We claim that $B_\varepsilon(P) \subset S$. To see this, let $Q = (a, b, c) \in B_\varepsilon(P)$. Let P', Q' be the projections of P, Q onto the xy-plane, respectively. That is, let $P' = (x, y, 0)$ and $Q' = (a, b, 0)$. Then $|PQ| < \varepsilon$ by construction and thus

$$\begin{aligned} |P'Q'| &= \sqrt{(x-a)^2 + (y-b)^2} \\ &\le \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} \\ &= |PQ| \\ &< \varepsilon. \end{aligned}$$

   By the triangle inequality, $|OQ'| \le |OP'| + |P'Q'|$, where $O = (0, 0, 0)$ denotes the origin. This yields

$$\begin{aligned} \sqrt{a^2 + b^2} &\le \sqrt{x^2 + y^2} + |P'Q'| \\ &< \sqrt{x^2 + y^2} + \varepsilon \\ &= 2. \end{aligned}$$

   This is equivalent to $a^2 + b^2 < 4$, which shows that $Q \in S$, as required.

4. *Let* $S' = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 < 4, \ z \ge 0\}$. *Is* S' *an open set in* $\mathbb{R}^3$*? Is it closed? Prove your answers (again, using points and neighbourhoods).*

   S' is not an open set. For example, $P = (0, 0, 0)$ belongs to S', but has no neighbourhood contained in S'. To check that last part, let $B_r(P)$ be any neighbourhood of P in $\mathbb{R}^3$. Then $B_r(P)$ contains the point $Q = (0, 0, -r/2)$, but $Q \notin S'$ since the z-coordinate of Q is negative.

   S' is not a closed set, either. For this we must show that its complement $(S')^c$ is not open. Let $\tilde{P} = (2, 0, 0)$. Then $\tilde{P} \in (S')^c$, but there is no neighbourhood of $\tilde{P}$ contained in $(S')^c$. To see this, let $B_r(\tilde{P})$ be any neighbourhood of $\tilde{P}$ in $\mathbb{R}^3$. Then $B_r(\tilde{P})$ contains the point $\tilde{Q} = (2 - \frac{\min(r,4)}{2}, 0, 0)$, but $\tilde{Q} \notin (S')^c$ since $\tilde{Q} \in S'$.

5. *Prove directly from the definition that the set $S = \{(x, y) \in \mathbb{R}^2 : \ x + y > 0\}$ is an open set in $\mathbb{R}^2$. (Given a point $P = (x, y) \in S$, find an $r > 0$ such that the neigbourhood $B_r(P)$ is contained in S.)*

   Let $P = (x, y) \in S$, and choose $r = (x + y)/2$. By the definition of S, we have $x + y > 0$, so that $r > 0$. We claim that $B_r(P) \subset S$. To see this, let $Q = (x', y') \in B_r(P)$. Then

$$|PQ| = \sqrt{(x - x')^2 + (y - y')^2} < r,$$

   so that

$$|x' - x| \le |PQ| < r = \frac{x + y}{2}, \quad |y' - y| \le |PQ| < r = \frac{x + y}{2}.$$

   In particular, $x' - x > -\frac{x+y}{2}$ and $y' - y > -\frac{x+y}{2}$, so that

$$\begin{aligned} x' + y' &= (x + y) + (x' - y) + (y' - y) \\ &> (x + y) - \frac{x + y}{2} - \frac{x + y}{2} \\ &= 0. \end{aligned}$$

   Therefore $Q \in S$. Since $Q \in B_r(P)$ was arbitrary, we have $B_r(P) \subset S$, as claimed.

6. *Let $S' = \{(x, y) \in \mathbb{R}^2 : \ x + y > 0, \ x - y \le 0\}$. Is S' an open set in $\mathbb{R}^2$? Prove your answer (again, using points and neighbourhoods).*

   S' is not an open set. For example, $P = (1, 1)$ belongs to S' since $1 + 1 = 2 > 0$ and $1 - 1 = 0$. However, P has no neighbourhood contained in S'. To check that last part, let $B_r(P)$ be any neighbourhood of P in $\mathbb{R}^2$. Then $B_r(P)$ contains the point $Q = (1, 1 - r/2)$, but $Q \notin S'$ since $1 - (1 - r/2) = r/2 > 0$.

7. *Prove from the definition that the set $S := \{(x, y) : x^2 + 4y^2 < 1\}$ is open in $\mathbb{R}^2$. As above, this means: for each point $P \in S$, find an $r > 0$ such that $B_r(P) \subset S$.*

   The set S is the region enclosed by an ellipse with centre at the origin. The ellipse S has major axis in the horizontal with length 1; its minor axis is vertical with length 1/2. Notice for any $P = (x, y) \in S$, we necessarily have $1 - x^2 - 4y^2 > 0$.

   Now, suppose $P = (x, y) \in S$ be given to us. Let $r = c(1 - x^2 - 4y^2)$ where $0 < c < 1$ is some large. Then $r > 0$. Moreover, if $Q = (u, v) \in B_r(P)$, we know that $|PQ| < r$. In particular, $|u - x| \le |PQ| < r$ and $|v - y| \le |PQ| < r$. By the triangle inequality, this gives,

$$|u| \le |x| + |u - x| < |x| + r \qquad \text{and} \qquad |v| \le |y| + |v - y| < |y| + r. \tag{1}$$

   We now need to show that $Q \in B_r(P)$ also satisfies $Q \in S$. Since Q was an arbitrary element of $B_r(P)$, this would imply that $B_r(P) \subset S$. Since $P \in S$ was also arbitrary, this would show that S is open.

Now, $Q = (u, v)$ is an element of S if and only if $u^2 + 4v^2 < 1$. We will need some intermediate inequalities to show that this is, in fact, true. Notice that by (1)

$$\begin{aligned} u^2 + 4v^2 &\leq (|x| + r)^2 + 4(|y| + r)^2 \\ &= |x|^2 + 2|x|r + r^2 + 4|y|^2 + 8|y|r + 4r^2 \\ \text{(using that } |x|, |y| \leq 1 \text{ for } (x, y) \in S) &\leq |x|^2 + 2r + r^2 + 4|y|^2 + 8r + 4r^2 \\ &= |x|^2 + 4|y|^2 + 10r + 5r^2 \end{aligned}$$

Now, we examine $r = c(1 - x^2 - 4y^2)$. Notice, since $c < 1$, we must have $r < 1$. So, in particular: $r^2 = r(r) < 1 \cdot r = 1$. Let's continue our inequalities

$$\begin{aligned} u^2 + 4v^2 &\leq |x|^2 + 4|y|^2 + 10r + 5r^2 \\ &\leq x^2 + 4y^2 + 15r. \end{aligned}$$

Now substitute the definition of r, to obtain

$$\begin{aligned} u^2 + 4v^2 &\leq x^2 + 4y^2 + 15c(1 - x^2 - 4y^2) \\ &= 1 - (1 - x^2 - 4y^2) + 5c(1 - x^2 - 4y^2) \\ &= 1 - (1 - 15c)\underbrace{(1 - x^2 - 4y^2)}_{>0}. \end{aligned}$$

In particular, if we choose $c < 1/15$, then

$$1 - (1 - 15c)\underbrace{(1 - x^2 - 4y^2)}_{>0} < 1 - (1 - 15 \cdot \frac{1}{15})(1 - x^2 - 4y^2) = 1.$$

So, with this choice of $r = c(1 - x^2 - 4y^2)$, we see that $Q = (x, y) \in S$. This concludes the proof that S is open.

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