MATH 227, PART 1.1: CURVES AND VECTOR-VALUED FUNCTIONS

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1. Lecture Notes

1.1. **Continuity and Derivatives for Vector-Valued Functions.**

In this chapter, we will work with **vector-valued functions**. These are functions $\mathbf{r} : I \to \mathbb{R}^n$ with $n = 2$ or $n = 3$ and $I \subset \mathbb{R}$ an interval. We can always write the function $\mathbf{r} = \mathbf{r}(t)$ in terms of its component functions:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where $x, y, z : I \to \mathbb{R}$.

Definition 1.1. *The vector-valued function $\mathbf{r} : I \to \mathbb{R}^n$ is called **continuous** if and only if all component functions $x(t), y(t)$ and $z(t)$ are continuous on the interval I.*

A useful interpretation is to think of $\mathbf{r}$ as describing the motion of an object in space if the coordinates of that object at time t are $(x(t), y(t), z(t))$. The assumption that $\mathbf{r}$ is continuous guarantees that the object does not make any instantaneous jumps from one location to another.

Definition 1.2. *A **curve** is the set $\{\mathbf{r}(t) : t \in I\} \subset \mathbb{R}^n$ of values attained by a continuous vector-valued function. If the same curve (i.e. set of values) corresponds to several different vector-valued functions $\mathbf{r}_1(t), ..., \mathbf{r}_j(t)$, we refer to the functions $\mathbf{r}_j$ is different **parametrizations** of the curve.*

With our previous interpretation of vector valued functions, we interpret the curve $\{\mathbf{r}(t) : t \in I\}$ as the path travelled by a particle or object over time, but without any information on where the particle was at any given time $t \in I$.

Definition 1.3. *The vector-valued function $\mathbf{r} : I \to \mathbb{R}^n$ is **differentiable** on an open interval I if and only if all component functions $x(t), y(t)$ and $z(t)$ are differentiable on I. In this case, we write*

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) := x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

Following our intepretation, the function $\mathbf{r}'(t) = \mathbf{v}(t)$ is the (instantaneous) velocity of our moving object at time t. You can check that $\mathbf{v}$ is the limit of the **average velocity** on the interval $(t, t + \Delta t)$, defined as

$$\frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \to \mathbf{v}(t) \text{ as } \Delta t \to 0.$$

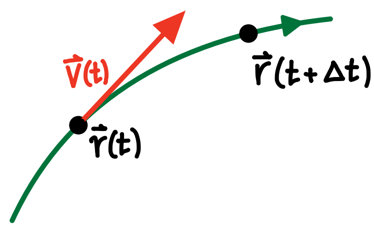

FIGURE 1. The particle at times t and $t+\Delta t$, as well as the instantaneous velocity at time t. The green arrow indicates the direction of motion of the particle along its associated curve.

The function $\mathbf{v}$ is, itself, vector-valued, so that it has both *magnitude* and *direction*. The magnitude of instantaneous velocity, given by $v(t) := |\mathbf{v}(t)|$, is the **speed** at time t. Further, the **acceleration** at time t is given by the formula $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

1.2. **Some Examples.** The following are examples of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, with some associated parametrizations.

Example 1.4. *Let $P_0 := (x_0, y_0, z_0)$ be some point in $\mathbb{R}^3$ and let $\mathbf{v}$ be some (constant) vector. Then, a straight line through P_0 in the direction $\mathbf{v}$ can be parametrized as*

$$\mathbf{r}(t) := \vec{OP_0} + t\mathbf{v} = \mathbf{r}_0 + t\mathbf{v} \quad t \in \mathbb{R},$$

where $\mathbf{r}_0 = \vec{OP_0}$. Notice that, with this parametrization, we have $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{v}$, so that the velocity of $\mathbf{r}$ is constant at all times $t \in \mathbb{R}$.

However, the same line can also be parametrized as:

$$\mathbf{r}_2(t) = \mathbf{r}_0 + 2t\mathbf{v} \tag{1.1}$$

$$\mathbf{r}_3(t) = \mathbf{r}_0 + t^3\mathbf{v} \tag{1.2}$$

Both (1.1) and (1.2) describe the same curve. However, in (1.1), the velocity is now $2\mathbf{v}$ (so double that of our original $\mathbf{r}$); whereas, in (1.2), the velocity is $3t^2\mathbf{v}$, and so is variable in time $t \in \mathbb{R}$.

Example 1.5. *The function $\mathbf{r}(t) + \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ gives a parametrization of the unit circle $x^2 + y^2 = 1$ in the xy-plane. The velocity and acceleration are*

$$\mathbf{v}(t) := -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$$

$$\mathbf{a}(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}$$

Example 1.6. *The equation*

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k},$$

Date: January 2, 2024.

describes a helix curve. This curve wraps around the z-axis, and is contained in the surface of the cylinder $x^2 + y^2 = 1$. The velocity, speed and acceleration are given by

$$\mathbf{v}(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}$$

$$v(t) := |\mathbf{v}(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1^2} = \sqrt{2}$$

$$\mathbf{a}(t) := -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}$$

Example 1.7. *The function $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t^3\mathbf{k}$ parametrizes a "deformed" helix, where the loops (corresponding to full revolutions about the z-axis) are compressed in the z-direction for small $|z|$ and elongated for large values of $|z|$.*

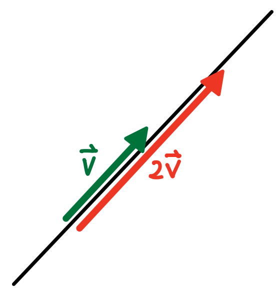

FIGURE 2. Two different velocities along the same line, discussed in Example 1.4

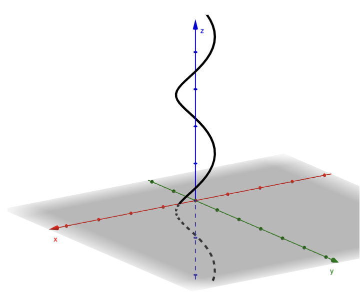

FIGURE 3. The helical curve described in Example 1.6

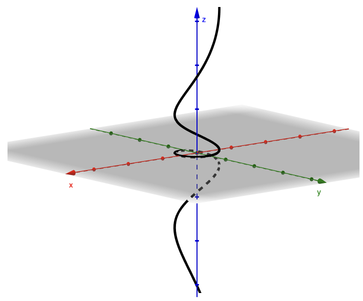

FIGURE 4. The deformed helix of Example 1.7

We have the following differentiation rules for vector-valued functions. These follow easily from the single-variable differentiation rules applied to the component functions.

Proposition 1.8. *Suppose that* $\mathbf{u}, \mathbf{v} : I \to \mathbb{R}^3$ *are vector-valued functions and* $\lambda : I \to \mathbb{R}$ *is scalar-valued, all of whom are differentiable. Then,*

$$\begin{aligned} (\mathbf{u}+\mathbf{v})' &= \mathbf{u}' + \mathbf{v}' \\ (\lambda \mathbf{u})' &= \lambda' \mathbf{u} + \lambda \mathbf{u}' \\ (\mathbf{u} \cdot \mathbf{v})' &= \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}' \\ (\mathbf{u} \times \mathbf{v})' &= \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}' \end{aligned}$$

Further, if $s : J \to I$*, we have the following* ***vector-valued Chain Rule***

$$\frac{d}{dt}\big[\mathbf{u}(s(t))\big] = s'(t)\mathbf{u}'(s(t)).$$

We often use this identity when ***re-parametrizing curves*** *(more to come).*

2. Worked Problems

In Questions 1 and 2, we parametrize curves at the intersection of two surfaces. In each case, it helps that one of those surfaces projects to a simple curve (circle or line) in the xy-plane, and so does the 3-dimensional curve we need to parametrize. Therefore we can parametrize $x(t)$ and $y(t)$ first (we already know how to do it for both the line and the circle), and then we use the equation of the other surface to find $z(t)$. In general, if no such simplifications are available, parametrizing an intersection curve can be significantly more difficult.

Questions 3 and 4 are important – we will use the formula in Question 3 and (especially) the equivalence in Question 4 many times in this class.

1. *Parametrize the curve of intersection of the cylinder* $x^2 + y^2 = 9$ *and the plane* $z = x + y$.

   Let $x(t) = 3\cos t$ and $y(t) = 3\sin t$, so that

$$x^2 + y^2 = 9(\cos^2 t + \sin^2 t) = 9.$$

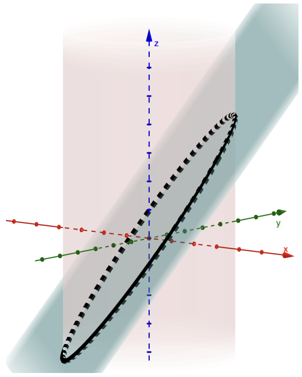

FIGURE 5. The curve described in Worked Problem no. 1

Then, for any $t, z \in \mathbb{R}$, the point $(x(t), y(t), z)$ lies on the cylinder $x^2 + y^2 = 9$. For the point to also lie in the plane $z = x + y$, we let

$$z(t) = x(t) + y(t) = 3(\cos(t) + \sin(t)).$$

Our parametrization is then,

$$\mathbf{r}(t) := 3\cos(t)\mathbf{i} + 3\sin(t)\mathbf{j} + 3(\cos(t) + \sin(t))\mathbf{k},$$

and we choose $t \in [0, 2\pi]$, which corresponds to one full revolution of the curve.

2. *Parametrize the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $x = 2y$.*

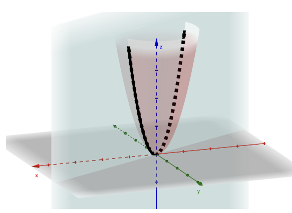

FIGURE 6. The curve described in Worked Problem no. 2

Set $y(t) = t$. Then from the equation of the plane, $x(t) = 2y(t) = 2t$, and from the equation of the paraboloid, $z(t) = x^2(t) + y^2(t) = 5t^2$. Our parametrization is then

$$\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + 5t^2\mathbf{k}, \quad t \in \mathbb{R}.$$

3. *Find a formula for $\frac{d}{dt}|\mathbf{u}(t)|$.*

By the Chain Rule for scalar functions, we have

$$\begin{aligned}
\frac{d}{dt}|\mathbf{u}(t)| &= \frac{d}{dt}\sqrt{\mathbf{u}\cdot\mathbf{u}} = \frac{1}{2\sqrt{\mathbf{u}\cdot\mathbf{u}}}\cdot\frac{d}{dt}(\mathbf{u}\cdot\mathbf{u}) \\
&= \frac{1}{2\sqrt{\mathbf{u}\cdot\mathbf{u}}}\big(\mathbf{u}'\cdot\mathbf{u}+\mathbf{u}\cdot\mathbf{u}'\big) \\
&= \frac{2\mathbf{u}\cdot\mathbf{u}'}{2\sqrt{\mathbf{u}\cdot\mathbf{u}}} \\
&= \frac{\mathbf{u}\cdot\mathbf{u}'}{|\mathbf{u}|}
\end{aligned}$$

4. *Suppose $\mathbf{r} : I \to \mathbb{R}^n$ is differentiable, where I is an interval. Prove that $|\mathbf{r}|$ is constant if and only if $\mathbf{v}\cdot\mathbf{r} = 0$ for all $t \in I$.*

   (Geometrically, the condition that $|\mathbf{r}| = a$ for some constant $a \geq 0$ means that the trajectory of $\mathbf{r}(t)$ must lie on the sphere of radius a centered at the origin. The condition that $\mathbf{v}\cdot\mathbf{r} = 0$ means that $\mathbf{r} \perp \mathbf{v}$ – here and below, we use the convention that the zero vector is perpendicular to any vector.)

   For any $a \in [0, \infty)$, we have the chain of implications:

$$\begin{aligned}
|\mathbf{r}| = a \quad &\Leftrightarrow \quad |\mathbf{r}|^2 = a^2 \\
&\Leftrightarrow \quad \mathbf{r}\cdot\mathbf{r} = a^2
\end{aligned}$$

   So, $|\mathbf{r}|$ is constant if and only if $\mathbf{r}\cdot\mathbf{r}$ is constant, if and only if $\frac{d}{dt}\big(\mathbf{r}\cdot\mathbf{r}\big) = 0$. However,

$$\frac{d}{dt}\big(\mathbf{r}\cdot\mathbf{r}\big) = \mathbf{r}'\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{r}' = 2\mathbf{r}'\cdot\mathbf{r} = 2\mathbf{v}\cdot\mathbf{r}$$

   Combining the above, we get that $|\mathbf{r}|$ is constant if and only if $\mathbf{v}\cdot\mathbf{r} = 0$ for all $t \in I$.

Remark 2.1. *(a) We may also apply similar logic to show that $|\mathbf{v}|$ is constant if and only if $\mathbf{a}\cdot\mathbf{v} = 0$ for all t.*

(b) If instead of $\mathbf{r}\cdot\mathbf{v} = 0$ we have $\mathbf{r}\cdot\mathbf{v} > 0$ for all t, then

$$\frac{d}{dt}\big(|\mathbf{r}|^2\big) = 2\mathbf{r}\cdot\mathbf{v} > 0$$

and so $|\mathbf{r}|$ is increasing in t. Similarly, if $\mathbf{r}\cdot\mathbf{v} < 0$, then $|\mathbf{r}|$ is decreasing in t.

3. Practice Problems

1. Let C be the curve in the xy-plane that consists of the half-line $y = -x$, $x \leq 0$, and the half-line $y = x$, $x \geq 0$. Give an example of a vector-valued finction $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ which parametrizes C and such that $\mathbf{r}(t)$ is differentiable for all t.

2. Suppose that $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a parametrized curve such that $\mathbf{r}$ and $\mathbf{r}'$ are continuous on $[a, b]$ and $\mathbf{r}'(t) \neq 0$. Does there have to exist a $t_0 \in (a, b)$ such that $\mathbf{r}(b) - \mathbf{r}(a) = (b - a)\mathbf{r}'(t_0)$? If yes, prove it. If no, give a counterexample.

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