Definition of vector-based functions

Vector-valued functions $\underset{\sim}{r} : I \to \mathbb{R}^n$ $(n=2 \text{ or } n=3)$ $(I \subset \mathbb{R})$

For example, $\underset{\sim}{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$. $(x, y, z : I \to \mathbb{R})$

Definition of continuity of vector-based functions

It's continuous iff $x(t), y(t), z(t)$ are continuous on I.

Analogy

The motion of an object in 3D space

Definition of curve

A curves is the set $\{\underset{\sim}{r}(t) : t \in I\} \subset \mathbb{R}^n$ of values attained by a continuous vector-based function.

If the same curve set of values corresponds to several $\underset{\sim}{r}(t)$, $\underset{\sim}{r}_j$ is the different parameterizations of the curve.

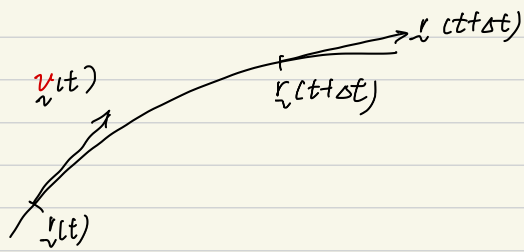
Definition of differentiability

is differentiable on an open interval I iff $x(t), y(t), z(t)$ differentiable on I

$$\frac{d\underset{\sim}{r}}{dt} = \underset{\sim}{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}.$$

Average velocity

$$\frac{\underset{\sim}{r}(t+\Delta t) - \underset{\sim}{r}(t)}{\Delta t} \to \underset{\sim}{v}(t) \text{ as } \Delta t \to 0$$



speed $= v(t) := |\underset{\sim}{v}(t)|$

acceleration $= \underset{\sim}{a}(t) = \underset{\sim}{v}'(t) = \underset{\sim}{r}''(t)$

Examples

1-4

$p_0 := (x_0, y_0, z_0)$ in $\mathbb{R}^3$, let $\underset{\sim}{v}$ be a constant vector. Then a straight line through p_0:

$$\underset{\sim}{r}(t) = \overrightarrow{OP_0} + t\underset{\sim}{v} = \underset{\sim}{r_0} + t\underset{\sim}{v}.$$

$$\underset{\sim}{v}(t) = \underset{\sim}{r}'(t) = \underset{\sim}{v}$$

$$\begin{cases} \underset{\sim}{r_2}(t) = \underset{\sim}{r_0} + \boxed{2t}\underset{\sim}{v} \rightarrow 2\underset{\sim}{v} \\ \underset{\sim}{r_3}(t) = \underset{\sim}{r_0} + \boxed{t^3}\underset{\sim}{v} \rightarrow 3t^2\underset{\sim}{v} \end{cases}$$

velocity

1-5 ✓

1-6

Helix curve:

$$\underset{\sim}{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j} + t\hat{k}.$$

$$\underset{\sim}{v}(t) = -\sin(t)\,\hat{i} + \cos(t)\,\hat{j} + \hat{k}$$

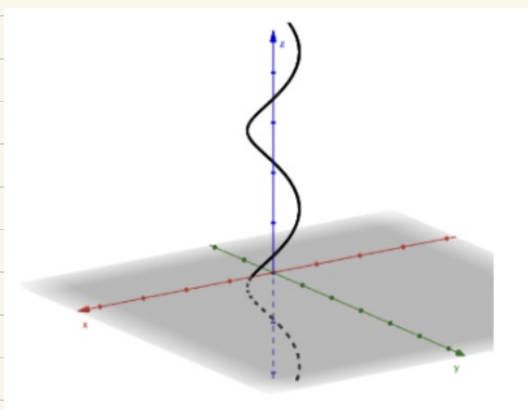
$$\underset{\sim}{a}(t) = -\cos(t)\,\hat{i} - \sin(t)\,\hat{j} + \boxed{?}$$

$\ne 0$

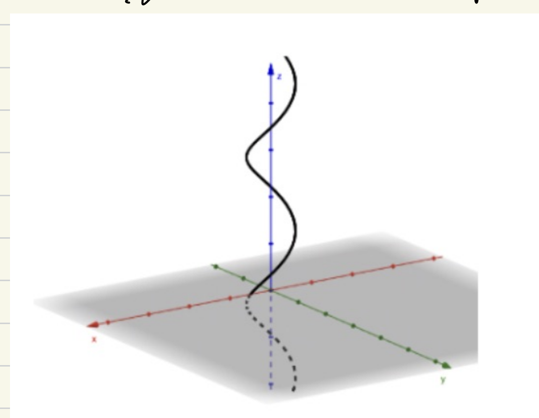
1-7

Imagine the following two helix curves?

$\underset{\sim}{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j} + t\hat{k}$



$\underset{\sim}{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j} + t^3\hat{k}$



Worked problems

Proposition 1.8

$\underset{\sim}{u}, \underset{\sim}{v} : I \to \mathbb{R}^3$, $\quad \lambda : I \to \mathbb{R}$

$$(\underset{\sim}{u} + \underset{\sim}{v})' = u' + v'$$

$$(\lambda \underset{\sim}{u})' = \lambda' \underset{\sim}{u} + \lambda \underset{\sim}{u}'$$

$$(\underset{\sim}{u} \cdot \underset{\sim}{v})' = \underset{\sim}{u}' \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{v}'$$

$$(\underset{\sim}{u} \times \underset{\sim}{v})' = \underset{\sim}{u}' \times \underset{\sim}{v} + \underset{\sim}{u} \times \underset{\sim}{v}'$$

If $s : J \to I$, **vetor-valued chain rule:**

$$\frac{d}{dt}\left[\underset{\sim}{u}(s(t))\right] = s'(t)\, \underset{\sim}{u}'(s(t))$$

to re-paramterizing curves.

Q1.

1. *Parametrize the curve of intersection of the cylinder $x^2 + y^2 = 9$ and the plane $z = x + y$.*

Sol. $x^2 + y^2 = 9 = 9(\cos^2\theta + \sin^2\theta)$

$\longrightarrow \underset{\sim}{r}(t) = \cancel{9\cos^2\theta\, \hat{i} + 9\sin^2\theta\, \hat{j}}$

$= 3\cos\theta\, \hat{i} + 3\sin\theta\, \hat{j}$

$z = x + y$

$\longrightarrow$ further: $z = 9\cos^2\theta + 9\sin^2\theta$

$\hookrightarrow \underset{\sim}{r}(t) = 3\cos\theta\, \hat{i} + 3\sin\theta\, \hat{j} + 3(\cos\theta + \sin\theta)\hat{k}$.

Q2.

2. *Parametrize the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $x = 2y$.*

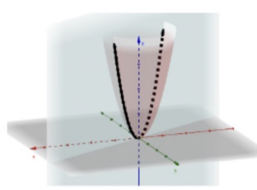


FIGURE 6. The curve described in Worked Problem no. 2

Sol. $\underset{\sim}{r} = \langle 2t, t, 5t^2 \rangle$ ✓

Q3

3. *Find a formula for* $\frac{d}{dt}|\mathbf{u}(t)|$.

Sol:

$$\frac{d}{dt}|\underset{\sim}{u}(t)| = \frac{d}{dt}\sqrt{\underset{\sim}{u}\cdot\underset{\sim}{u}}$$

$$= \frac{d}{dt}(\underset{\sim}{u}^2)^{\frac{1}{2}}$$

$$= \frac{1}{2(\underset{\sim}{u}^2)^{\frac{1}{2}}}\cdot \cancel{2\underset{\sim}{u}'}$$

$$= \frac{1}{2(\underset{\sim}{u}^2)^{\frac{1}{2}}}\frac{d}{dt}(\underset{\sim}{u}^2)$$

$$= \frac{1}{2(\underset{\sim}{u}^2)^{\frac{1}{2}}}\cdot\frac{d}{dt}(\underset{\sim}{u}\cdot\underset{\sim}{u})$$

$$= \frac{1}{2\sqrt{\underset{\sim}{u}^2}}\cdot(\underset{\sim}{u}\cdot\underset{\sim}{u}' + \underset{\sim}{u}'\cdot\underset{\sim}{u})$$

$$= \frac{1}{2\sqrt{\underset{\sim}{u}^2}}(2\underset{\sim}{u}\cdot\underset{\sim}{u}') = \frac{\underset{\sim}{u}\cdot\underset{\sim}{u}'}{\sqrt{\underset{\sim}{u}^2}}$$

$$= \frac{\underset{\sim}{u}\cdot\underset{\sim}{u}'}{|\underset{\sim}{u}|}$$

Q4.

meaning velocity vector

4. *Suppose* $\mathbf{r} : I \to \mathbb{R}^n$ *is differentiable, where* I *is an interval. Prove that* $|\mathbf{r}|$ *is constant if and only if* $\mathbf{v}\cdot\mathbf{r} = 0$ *for all* $t \in I$.

Sol.

$\underset{\sim}{r} : I \to \mathbb{R}^n$

interval

??

For $\forall a \in [0,\infty)$, $|\underset{\sim}{r}| = a \Leftrightarrow |\underset{\sim}{r}|^2 = a^2$

$\Leftrightarrow \underset{\sim}{r}\cdot\underset{\sim}{r} = a^2$

So $|\underset{\sim}{r}|$ is constant iff $\underset{\sim}{r}\cdot\underset{\sim}{r}$ is constant,

intermediate!

iff $\frac{d}{dt}(\underset{\sim}{r}\cdot\underset{\sim}{r}) = 0$

However, $\frac{d}{dt}(\underset{\sim}{r}\cdot\underset{\sim}{r}) = \underset{\sim}{r}'\cdot\underset{\sim}{r} + \underset{\sim}{r}\cdot\underset{\sim}{r}' = 2\underset{\sim}{r}'\cdot\underset{\sim}{r} = 2\underset{\sim}{v}\cdot\underset{\sim}{r}$

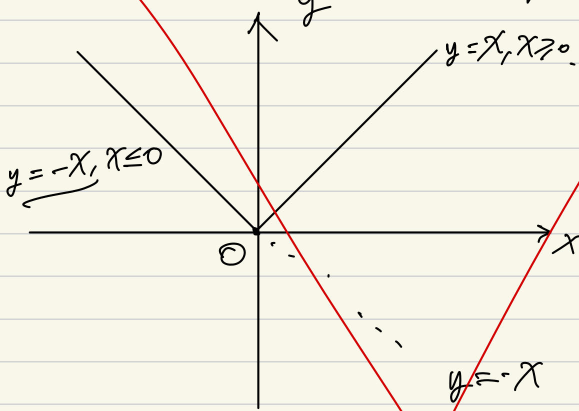
we also have, $\underset{\sim}{r}\cdot\underset{\sim}{r} > 0$

< 0

$= 0$ ...

Practice problems

1. Let C be the curve in the xy-plane that consists of the half-line $y = -x$, $x \leq 0$, and the half-line $y = x$, $x \geq 0$. Give an example of a vector-valued finction $\mathbf{r} : \boxed{\mathbb{R} \to \mathbb{R}^2}$ which parametrizes C and such that $\boxed{\mathbf{r}(t)}$ is differentiable for all t.



$\underset{\sim}{r} : \mathbb{R} \to \mathbb{R}^2.$

$\underset{\sim}{r}(t) = (\quad)\hat{i} + (\quad)\hat{j}$

2. Suppose that $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a parametrized curve such that $\mathbf{r}$ and $\mathbf{r}'$ are continuous on $[a, b]$ and $\mathbf{r}'(t) \neq 0$. Does there have to exist a $t_0 \in (a, b)$ such that $\mathbf{r}(b) - \mathbf{r}(a) = (b - a)\mathbf{r}'(t_0)$? If yes, prove it. If no, give a counterexample.

$\underset{\sim}{r} : [a,b] \to \mathbb{R}^3$ $\qquad$ $\underset{\sim}{r}, \underset{\sim}{r}'$ are continuous on $[a,b]$

& $\underset{\sim}{r}'(t) \neq 0.$

$t_0 \in (a,b)$ $\qquad$ $\underset{\sim}{r}(b) - \underset{\sim}{r}(a) = (b-a)\underset{\sim}{r}'(t_0)$

Lagrange theorem

Suppose $\underset{\sim}{r} = f(t)\hat{i} + g(t)\hat{j}.$

$\underset{\sim}{r}(b) - \underset{\sim}{r}(a) = [f(b) - f(a)]\hat{i} + [g(b) - g(a)]\hat{j}$

$r'(t_0) = f'(t_0)\hat{i} + g(t_0)\hat{j}.$

$(b-a)\underset{\sim}{r}'(t_0) = (b-a)(f'(t_0)\hat{i} + g'(t_0)\hat{j})$

Because Lagrange also dealt with it in 2D then what we're supposed to do.

Practice problems

Sol.

?

1. Let C be the curve in the xy-plane that consists of the half-line $y = -x$, $x \leq 0$, and the half-line $y = x$, $x \geq 0$. Give an example of a vector-valued fINction $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ which parametrizes C and such that $\mathbf{r}(t)$ is differentiable for all t.

We claim that $\mathbf{r}(t) = \langle t^3, |t^3| \rangle$ works

Bijection from $\mathbb{R} \to C$.

The components of $x(t) = t^3$ is differentiable for all t.
and $y(t) = |t^3|$ is differentiable for all $t \neq 0$.

Next we check the differentiability at $t=0$.

$$\lim_{x\to 0^+} \frac{|t^3|}{t} = \lim_{x\to 0^+} \frac{t^3}{t} = \lim_{x\to 0^+} t^2 = 0$$

$$\lim_{x\to 0^-} \frac{|t^3|}{t} = \lim_{x\to 0^-} \frac{-t^3}{t} = \lim_{x\to 0^-} -t^2 = 0.$$

is differentiable at $t=0$

So derivatives from either sides equal $\to y(t)$

2. Suppose that $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a parametrized curve such that $\mathbf{r}$ and $\mathbf{r}'$ are continuous on $[a, b]$ and $\mathbf{r}'(t) \neq 0$. Does there have to exist a $t_0 \in (a, b)$ such that $\mathbf{r}(b) - \mathbf{r}(a) = (b - a)\mathbf{r}'(t_0)$? If yes, prove it. If no, give a counterexample.

$\mathbf{r} : [a,b] \to \mathbb{R}^3$ $\qquad$ $\mathbf{r}, \mathbf{r}'$ are continuous on $[a,b]$

$\mathbf{r}'(t) \neq 0$

a $t_0 \in (a,b)$ $\qquad$ $\mathbf{r}(b) - \mathbf{r}(a) = (b-a)\mathbf{r}'(t_0)$

Answer is No.

Counterexample:

$\mathbf{r} = \cos t\, \hat{i} + \sin t\, \hat{j} + t\hat{k}$ and $[a,b] = [0, 2\pi]$

So that

$\mathbf{r}(b) - \mathbf{r}(a) = \hat{i} + 2\pi\hat{k} - \hat{i} = 2\pi\hat{k}$.

But $\mathbf{r}'(t) = -\sin t\, \hat{i} + \cos t\, \hat{j} + \hat{k}$

never parallel to the x-axis