MATH 227 SOLUTIONS MANUAL
PART 1.1 : CURVES AND VECTOR-VALUED FUNCTIONS

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1. *Let C be the curve in the xy-plane that consists of the half-line $y = -x$, $x \leq 0$, and the half-line $y = x$, $x \geq 0$. Give an example of a vector-valued finction $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$ which parametrizes C and such that $\mathbf{r}(t)$ is differentiable for all t.*

   We are looking for a differentiable parametrization of the absolute value curve. We claim that $\mathbf{r}(t) = (t^3, |t^3|)$ works. This parametrization is clearly a bijection from $\mathbb{R}$ to C, the component $x(t) = t^3$ is differentiable for all t, and $y(t) = |t^3|$ is differentiable for all $t \neq 0$. It remains to check that $y(t)$ is differentiable at $t = 0$:

$$\lim_{x\to 0^+} |t^3|/t = \lim_{x\to 0^+} t^3/t = \lim_{x\to 0^+} t^2 = 0,$$
$$\lim_{x\to 0^-} |t^3|/t = \lim_{x\to 0^-} -t^3/t = \lim_{x\to 0^+} -t^2 = 0.$$

   Since the one-sided derivatives exist and are equal, $y(t)$ is differentiable at $t = 0$.

2. *Suppose that $\mathbf{r} : [a, b] \to \mathbb{R}^3$ is a parametrized curve such that $\mathbf{r}$ and $\mathbf{r}'$ are continuous on $[a, b]$ and $\mathbf{r}'(t) \neq 0$. Does there have to exist a $t_0 \in (a, b)$ such that $\mathbf{r}(b) - \mathbf{r}(a) = (b - a)\mathbf{r}'(t_0)$? If yes, prove it. If no, give a counterexample.*

   The answer is no. Let $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ and $[a, b] = [0, 2\pi]$, so that $\mathbf{r}(b) - \mathbf{r}(a) = \mathbf{i} + 2\pi\mathbf{k} - \mathbf{i} = 2\pi\mathbf{k}$. But $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$ is never parallel to the z axis.

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