# Algebraic De Rham Theorem

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**Definition**

Let $X$ be a topological space. A presheaf $\mathscr{F}$ of abelian groups on $X$ consists of the data

- for every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$, and
- for every inclusion $V \subseteq U$ of open subsets of $X$, a morphism of abelian groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$,

subject to the conditions

1. $\mathscr{F}(\varnothing) = 0$, where $\varnothing$ is the empty set,
2. $\rho_{UU}$ is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$, and
3. if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

# Sheaf Theory

## Definition

A presheaf $\mathscr{F}$ on a topological space $X$ is a sheaf if it satisfies the following supplementary conditions:

1. if $U$ is an open set, if $\{V_i\}$ is an open covering of $U$, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all $i$, then $s = 0$;
2. if $U$ is an open set, if $\{V_i\}$ is an open covering of $U$, and if we have elements $s_i \in \mathscr{F}(V_i)$ for each $i$, with the property that for each $i, j$, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each $i$.

Denote the sheaf of locally constant functions over $\mathbb{C}$ as $\underline{\mathbb{C}}$. $\mathcal{O}_X$ is called the sheaf of regular functions.

# Sheaf Associated to a Module

We had defined the module of relative differentials $\Omega^1_{R/\mathbb{C}}$ over a variety $X$ whose coordinate ring is $R$. It is finitely generated as an $R$ module. And as $R$ is noetherian, it is coherent (take this as the definition).

# Sheaf Associated to a Module

For a given $R$ module $M$ (finitely generated), we define the (coherent) sheaf associated to it as $\tilde{M}(U) = M \otimes_R \mathcal{O}_X(U)$. This defines an $\mathcal{O}_X$ module and

1. $\tilde{M}(D(f)) = M[1/f]$
2. $\tilde{M}(X) = M$
3. $\tilde{\Omega}^1{}_{X/\mathbb{C}}(U) = \Omega^1_{R/\mathbb{C}} \otimes_R \mathcal{O}_X(U)$.
4. $\tilde{\Omega}^k{}_{X/\mathbb{C}}(U) = \Omega^k_{R/\mathbb{C}} \otimes_R \mathcal{O}_X(U)$.

# Algebraic De Rham Complex

We defined the algebraic De Rham complex for the global sections $\tilde{\Omega}^k{}_{X/\mathbb{C}}(X) = \Omega^k_{R/\mathbb{C}}$. We extend it to sheaves as

$$0 \to \mathcal{O}_X \to \Omega^1_{X/\mathbb{C}} \to \ldots$$

We will define the hypercohomology of this complex.

A covariant functor $F$ is called left (right) exact if given $0 \to A \to B \to C \to 0$ a short exact then

$$0 \to F(A) \longrightarrow F(B) \to F(C)$$

is exact.

$$(F(A) \to F(B) \to F(C) \to 0 \text{ exact })$$

$\Gamma(U, )$ functor is left exact.
If we further have $0 \to F(A) \to F(B) \to F(C) \to 0$ exact, then $F$ is called an exact functor.

A sequence of sheaves

$$0 \to A \to B \to C \to \ldots$$

is called exact if it is exact on stalks.

# Resolutions

Let $\mathscr{A}^k$ denote the sheaf of smooth $k$-forms on a manifold $M$.

**Proposition**

$$0 \to \underline{\mathbb{R}} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \longrightarrow \ldots \longrightarrow \mathscr{A}^n \to 0$$

is exact.

# Resolutions

**Definition**

An exact sequence of sheaves

$$0 \to \mathscr{A} \longrightarrow \mathscr{F}^0 \xrightarrow{d} \mathscr{F}^1 \xrightarrow{d} \dots \to \dots$$

on X is called a (right) resolution of $\mathscr{A}$.

**Definition**

A sheaf $\mathscr{F}$ is called injective if $Hom_{Sh(X)}(-, \mathscr{F})$ is exact.

# Injective Resolutions

## Definition

A resolution

$$0 \to \mathscr{A} \xrightarrow{\epsilon} I^0 \xrightarrow{d} I^1 \xrightarrow{d} \ldots$$

is called injective if each $I^i$ is injective.

Every sheaf $\mathscr{F}$ has a canonical injective resolution $C^\bullet(\mathscr{F})$ called Godement resolution.

# Sheaf Cohomology

## Definition

Let X be a topological space and $\mathscr{F}$ a sheaf. Let $0 \to \mathscr{F} \to I^\bullet$ be an injective resolution of $\mathscr{F}$. The i-th sheaf cohomology group $H^i(X, \mathscr{F})$ of $\mathscr{F}$ is defined as

$$H^i(X, \mathscr{F}) := H^i(\Gamma(X, I^\bullet)).$$

# Sheaf Cohomology

**Definition**

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1. Definition is independent of the injective resolution chosen.

### Proposition

$$H^0(X, \mathscr{F}) = \mathscr{F}(X)$$

### Proposition

If $\mathscr{F}$ is injective then $H^i(X, \mathscr{F}) = 0$ for $i > 0$.

# Acyclic Resolutions

**Definition**

A sheaf $\mathscr{L}$ is called acyclic if $H^i(X, \mathscr{L}) = 0$ for $i > 0$. A resolution $0 \to \mathscr{F} \to \mathscr{L}^\bullet$ is called acyclic resolution if each $\mathscr{L}^i$ is acyclic.

**Proposition**

Let $0 \to A \to Q^\bullet$ be an acyclic resolution of A. Then $H^i(X, A) = H^i(\Gamma(X, Q^\bullet))$.

As we saw by Poincare's Lemma

$$0 \to \underline{\mathbb{R}} \longrightarrow \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \longrightarrow \ldots \longrightarrow \mathscr{A}^n \to \text{''}0$$

is exact hence a resolution. Each $\mathscr{A}^k$ is acyclic (partitions of unity). Therefore we have an acyclic resolution of $\underline{\mathbb{R}}$. Using above proposition

$$H^i(X, \underline{\mathbb{R}}) = H^i(\Gamma(X, \mathscr{A}^\bullet)) = H^i_{DR}(X).$$

# Cohomology Sheaves

$$0 \to \mathcal{L}^0 \xrightarrow{d} \mathcal{L}^1 \xrightarrow{d} \dots$$

Define cohomology sheaf as

$$\mathcal{H}^k = \frac{\ker(d : \mathcal{L}^k \to \mathcal{L}^{k+1})}{\mathrm{im}(d : \mathcal{L}^{k-1} \to \mathcal{L}^k)}$$

It can be seen that $\mathcal{H}^k_p(\mathcal{L}^\bullet) = H^k(\mathcal{L}^\bullet_p)$.

# Cohomology Sheaves

**Proposition**

If $T$ is an exact functor from sheaves on $X$ to abelian groups. Then

$$T(\mathcal{H}^k(\mathcal{L}^\bullet)) = H^k(T(\mathcal{L}^\bullet))$$

The functor $\Gamma(U, C^p(-))$ is exact as $C^p(-)$ gives injective sheaf. Therefore

$$\Gamma(X, C^p(\mathcal{H}^q(\mathscr{F}^\bullet))) = H^q(\Gamma(X, C^p(\mathscr{F}^\bullet)))$$

# Hypercohomology

We want to define the cohomology for a sequence of sheaves.

$$0 \to \mathcal{L}^0 \to \mathcal{L}^1 \to \dots$$

Using Godement resolution, we can resolve each $\mathcal{L}^i$ as

$$0 \to \mathcal{L}^i \to C^0\mathcal{L}^i \to \dots$$

Applying global sections functor and deleting the first term gives the sequence of groups whose cohomology is the sheaf cohomology of $\mathcal{L}^i$

$$0 \to C^0\mathcal{L}^i(X) \to C^1\mathcal{L}^i(X) \to \dots$$

$$H^k(C^\bullet\mathcal{L}^i(X)) = H^k(X, \mathcal{L}^i)$$

# Hypercohomology

To define it, first, resolve each sequence.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathcal{L}^2 \arrow[r] \arrow[u, dash, dotted] & C^0\mathcal{L}^2 \arrow[r, "\delta"] \arrow[u] & C^1\mathcal{L}^2 \arrow[r, "\delta"] \arrow[u] & C^2\mathcal{L}^2 \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathcal{L}^1 \arrow[r, "\varepsilon_1"'] \arrow[u, "d"] & C^0\mathcal{L}^1 \arrow[r, "\delta"] \arrow[u, "d"'] & C^1\mathcal{L}^1 \arrow[r, "\delta"] \arrow[u, "d"'] & C^2\mathcal{L}^1 \arrow[r] \arrow[u, "d"'] & \cdots \\
0 \arrow[r] & \mathcal{L}^0 \arrow[r, "\varepsilon_0"'] \arrow[u, "d"] & C^0\mathcal{L}^0 \arrow[r, "\delta"] \arrow[u, "d"'] & C^1\mathcal{L}^0 \arrow[r, "\delta"] \arrow[u, "d"'] & C^2\mathcal{L}^0 \arrow[r] \arrow[u, "d"'] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

# Hypercohomology

As we did in sheaf cohomology, apply the global section functor and remove the first column.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & C^0\mathcal{L}^2(X) \arrow[u, "d"] \arrow[r, "\delta"] & C^1\mathcal{L}^2(X) \arrow[u, "d"] \arrow[r, "\delta"] & C^2\mathcal{L}^2(X) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & C^0\mathcal{L}^1(X) \arrow[u, "d"] \arrow[r, "\delta"] & C^1\mathcal{L}^1(X) \arrow[u, "d"] \arrow[r, "\delta"] & C^2\mathcal{L}^1(X) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
0 \arrow[r] & C^0\mathcal{L}^0(X) \arrow[u, "d"] \arrow[r, "\delta"] & C^1\mathcal{L}^0(X) \arrow[u, "d"] \arrow[r, "\delta"] & C^2\mathcal{L}^0(X) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We will define the hypercohomology of the sequence $\mathcal{L}^\bullet$ as the total cohomology of the bigraded complex given above. But what does that mean? We can obtain a single complex from this grid. Consider

$$K^\bullet = \bigoplus K^k = \bigoplus_k \bigoplus_{p+q=k} K^{p,q}$$

where $K^{p,q} = \Gamma(X, C^p\mathcal{L}^q) = C^p\mathcal{L}^q(X)$. We have a horizontal differential $\delta$ induced from Godement resolution and a vertical one $d$ induced from the complex $\mathcal{L}^\bullet$. Now we can define a new differential on the single complex $K^\bullet$ as

$$D : K^k \to K^{k+1},\ D = \delta + (-1)^p d$$

so that $D^2 = 0$.

# Hypercohomology

So we have the following single complex called the total complex.

$$0 \to C^0\mathcal{L}^0(X) \xrightarrow{D} C^0\mathcal{L}^1(X) \oplus C^1\mathcal{L}^0(X) \xrightarrow{D} \dots$$

We define the hypercohomology as

$$\mathbb{H}^k(X, \mathcal{L}^\bullet) := H^k_D(K^\bullet)$$

.

# Hypercohomology

For this to make sense, we need to recover the sheaf cohomology for

$$\mathcal{L}^\bullet = (0 \to \mathscr{F} \to 0 \to \ldots)$$

This is the case because we are only left with the first row and $d = 0$.

A (cohomological) spectral sequence with rightward orientation consists of the following data:

1. bigraded objects $E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$,
2. differentials $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$
3. isomorphism between successive pages

$$E_{r+1}^{p,q} = H_{d_r}(E_r^{p,q})$$

# Spectral Sequqnces

$$\begin{array}{cc} \uparrow & \uparrow \\ A_{01} & A_{11} \\ \uparrow & \uparrow \\ A_{00} & A_{10} \\ \uparrow & \uparrow d_0 \end{array}$$

$E_0$ page

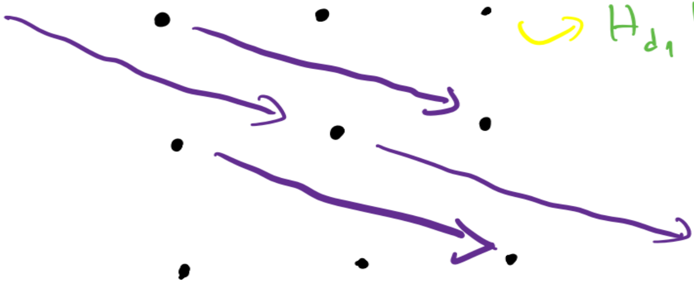
go to other page

$$H_{d_0}(A_{01}) \longrightarrow H_{d_0}(A_{11}) \longrightarrow$$

$$H_{d_0}(A_{00}) \longrightarrow H_{d_0}(A_{10}) \longrightarrow$$

$E_1$ page

# Spectral Sequences



$H_{d_1} H_{d_0}(A_{p,q})$

$E_2$-page

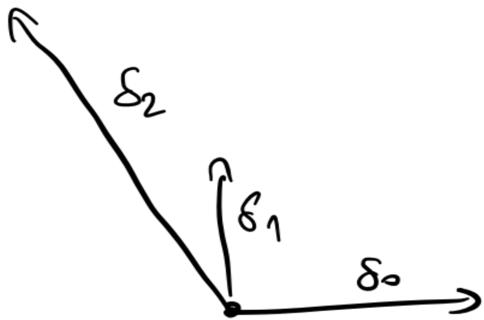
For a first quadrant spectral sequence and for sufficiently large r, the arrow will come from 0 and go to 0. Hence the spectral sequence will stabilize.

$$E_r^{p,q} = E_{r+1}^{p,q} = \ldots \text{ if } r > max\{p, q+1\}$$

We denote this stable page as $E_\infty$.

We could change the definition by $r \mapsto -r$ and get the negative orientation.

$$\delta_r : E_r^{p,q} \to E^{p-r,q+r+1}$$




We call the first one the usual spectral sequence and denote by $'E_r$, and this one is just the second spectral sequence denoted as $''E_r$.

**Theorem**

*Let $M$ be a double complex with total complex Tot $M$. Then there exist two spectral sequences ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ such that*

$$
\begin{aligned}
{}'E_0^{p,q} &= M^{p,q}, \quad {}'E_1^{p,q} = H_\delta^q\left(M^{p,\cdot}\right), \quad {}'E_2^{p,q} = H_d^p\left(H_\delta^q(M)\right),\\
{}''E_0^{p,q} &= M^{q,p}, \quad {}''E_1^{p,q} = H_d^q\left(M^{\cdot,p}\right), \quad {}''E_2^{p,q} = H_\delta^p\left(H_d^q(M)\right).
\end{aligned}
$$

*Further, if $M$ is a first or third quadrant double complex, then both ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ converge to $H^{p+q}(\mathrm{Tot}\, M)$.*

Therefore, we can compute the hypercohomology of a first quadrant double complex via two spectral sequences.

$$\bigoplus_{p+q=k} {}'E_\infty^{p,q} \cong \bigoplus_{p+q=k} {}''E_\infty^{p,q} \cong \mathbb{H}^k(X, \mathcal{L}^\bullet)$$

# Our Case

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & C^0\mathcal{L}^2(X) \arrow[r, "\delta"'] \arrow[u, "d"] & C^1\mathcal{L}^2(X) \arrow[r, "\delta"'] \arrow[u, "d"] & C^2\mathcal{L}^2(X) \arrow[r, "\delta"'] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & C^0\mathcal{L}^1(X) \arrow[r, "\delta"'] \arrow[u, "d"] & C^1\mathcal{L}^1(X) \arrow[r, "\delta"'] \arrow[u, "d"] & C^2\mathcal{L}^1(X) \arrow[r, "\delta"'] \arrow[u, "d"] & \cdots \\
0 \arrow[r] & C^0\mathcal{L}^0(X) \arrow[r, "\delta"'] \arrow[u, "d"] & C^1\mathcal{L}^0(X) \arrow[r, "\delta"'] \arrow[u, "d"] & C^2\mathcal{L}^0(X) \arrow[r, "\delta"'] \arrow[u, "d"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Usual spectral gives $E_1 = H_d$ and $E_2 = H_\delta H_d$, and the other spectral reverses the roles.

# Our case

We know that $T = \Gamma(X, C^p(-))$ is exact so

$$h^q(T(\mathcal{L}^\bullet)) = T(\mathcal{H}^q(\mathcal{L}^\bullet)).$$

The usual spectral sequence gives

$$E_1^{p,q} = H_d^{p,q} = h^q(\Gamma(X, C^p(\mathcal{L}^\bullet))) = \Gamma(X, C^p\mathcal{H}^q)$$
$$E_2^{p,q} = H_\delta^{p,q}(E_1) = h_\delta^p(\Gamma(X, C^\bullet\mathcal{H}^q)) = H^p(X, \mathcal{H}^q).$$

The other one gives

$$E_1^{p,q} = H^p(X, \mathcal{L}^q)\ ,\ E_2^{p,q} = h_d^q(H^p(X, \mathcal{L}^\bullet)).$$

**Theorem**

*A quasi isomorphism $\mathscr{F}^\bullet \to \mathcal{G}^\bullet$ of complexes of sheaves induces canonical isomorphism*

$$\mathbb{H}^k(X, \mathscr{F}^\bullet) \cong \mathbb{H}^k(X, \mathcal{G}^\bullet).$$

**Proof.**

(Sketch) By definition, quasi isomorphism means that we have $\mathcal{H}(\mathscr{F}^\bullet) \cong \mathcal{H}(\mathcal{G}^\bullet)$. Therefore, the second pages of the usual spectral sequences are isomorphic. This induces an isomorphism on $E_\infty$ pages, and we get the isomorphism of hypercohomologies. $\square$

THEOREM. If $\mathcal{L}^\bullet$ is a complex of acyclic sheaves of abelian groups on a topological space $X$, then the hypercohomology of $\mathcal{L}^\bullet$ is isomorphic to the cohomology of the complex of global sections of $\mathcal{L}^\bullet$ :

$$\mathbb{H}^k\left(X, \mathcal{L}^\bullet\right) \simeq h^k\left(\mathcal{L}^\bullet(X)\right)$$

where $\mathcal{L}^\bullet(X)$ denotes the complex

$$0 \to \mathcal{L}^0(X) \to \mathcal{L}^1(X) \to \mathcal{L}^2(X) \to \cdots$$

# Acyclic Sheaves

Using the second spectral sequence, we get $E_1^{p,q} = H^p(X, \mathcal{L}^q)$, using the fact that $\mathcal{L}^k$'s are acyclic we find

$$E_1 = H_\delta = \begin{array}{c|c|c|c|c} q & & & & \\ & \mathcal{L}^2(X) & 0 & 0 & \\ & \mathcal{L}^1(X) & 0 & 0 & \\ & \mathcal{L}^0(X) & 0 & 0 & \\ \hline & 0 & 1 & 2 & p \end{array}$$

The second page gives

$$E_2^{p,q} = \begin{cases} h^q(\mathcal{L}^\bullet(X)) & p = 0 \\ 0 & p > 0 \end{cases}$$

So the spectral sequence degenerates at $E_2$, and we get

$$\mathbb{H}^k(X, \mathcal{L}^\bullet) = E_2^{0,k} = h^k(\mathcal{L}^\bullet(X)).$$

# Acyclic Resolutions

Let

$$0 \to \mathscr{F} \to \mathcal{L}^0 \to \mathcal{L}^1 \to \mathcal{L}^2 \to \dots$$

be an acyclic resolution of $\mathscr{F}$, which means that $H^p(X, \mathcal{L}^q) = 0$ for $p > 0$.

**Theorem**

*For $0 \to \mathscr{F} \to \mathcal{L}^\bullet$, an acylic resolution. We have the following isomorphism*

$$H^k(X, \mathscr{F}) \cong h^k(\mathcal{L}^\bullet(X))$$

The resolution may be viewed as a quasi-isomorphism of two complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathcal{L}^0 \arrow[r] & \mathcal{L}^1 \arrow[r] & \mathcal{L}^2 \arrow[r] & \cdots ,
\end{tikzcd}$$

by a theorem above, we get $\mathbb{H}^k(X, \mathscr{F}^\bullet) \cong \mathbb{H}^k(X, \mathcal{L}^\bullet)$. The first one is just the sheaf cohomology and we proved that $\mathbb{H}^k(X, \mathcal{L}^\bullet) = h^k(\mathcal{L}^\bullet(X))$. Therefore

$$H^k(X, \mathscr{F}) = h^k(\mathcal{L}^\bullet(X)).$$

# Statement for Affine Case

**Theorem**

*(Serre) A coherent algebraic sheaf $\mathscr{F}$ on an affine variety X is acyclic on X.*

For an affine variety, hypercohomology reduces to usual cohomology. So if $X$ is affine, then our statement is

$$H^k(X_{an}, \underline{\mathbb{C}}) \cong h^k(\Omega^{\bullet}_{X/\mathbb{C}}(X)).$$

# Analytic De Rham Theorem

Let M be a complex manifold and $\Omega^k_{an}$ the sheaf of holomorphic k-forms on M, locally, the forms are $\sum a_I dz_I$. In complex coordinates $d = \partial + \bar{\partial}$.

**Theorem**

*(Holomorphic Poincare) On a complex manifold M of complex dimension n the sequence*

$$0 \to \underline{\mathbb{C}} \longrightarrow \Omega^0_{an} \xrightarrow{d} \Omega^1_{an} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{an} \to 0$$

*of sheaves is exact.*

We can view this as quasi isomorphism between $\underline{\mathbb{C}}^\bullet$ and $\Omega^\bullet_{an}$ and get the analytic de Rham theorem

$$H^k(M, \underline{\mathbb{C}}) \cong \mathbb{H}^k(M, \Omega^\bullet_{an}).$$

# Algebraic De Rham For a Projective Variety

## Theorem

*(Algebraic de Rham for a projective variety) If $X$ is a smooth complex projective variety, then*

$$H^k(X_{an}, \mathbb{C}) \cong \mathbb{H}^k(X, \Omega^\bullet_{X/\mathbb{C}}).$$

$0 \to \Omega^0_{X/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}} \to$

We would be done if we had a Poincare lemma for $\Omega^\bullet_{X/\mathbb{C}}$. Unfortunately, there is no such lemma, so we need higher machinery.

## Theorem

*(GAGA) For a smooth complex projective variety $X$, we have the following*

$$H^p(X, \Omega^q_{X/\mathbb{C}}) \cong H^p(X_{an}, \Omega^q_{an}).$$

# The Proof

Consider the two second spectral sequences converging to $\mathbb{H}^*(X_{an}, \Omega^\bullet_{an})$ and $\mathbb{H}^*(X, \Omega^\bullet_{X/\mathbb{C}})$. As we calculated before, the first pages are exactly the ones given in GAGA's statement. Therefore, the two pages are isomorphic. Isomorphism in $E_1$ pages induces isomorphism in hypercohomology, hence

$$\longrightarrow \mathbb{H}^*(X_{an}, \Omega^\bullet_{an}) \cong \mathbb{H}^*(X, \Omega^\bullet_{X/\mathbb{C}}).$$

Using the analytic de Rham theorem, we get the desired result.

Let $\mathfrak{U} = \{U_\alpha\}$ be an open cover of $X$ and $\mathscr{F}$ a presheaf. Define the group of Cech p-cochains on $\mathfrak{U}$ with values in the presheaf $\mathscr{F}$ to be the direct product

$\Omega^k$

$$\check{C}^p(\mathfrak{U}, \mathscr{F}) = \prod_{\alpha_0 < \ldots < \alpha_p} \mathscr{F}(U_{\alpha_0 \ldots \alpha_p}).$$

$= U_{\alpha_0} \cap U_{\alpha_1} \ldots$

Define the Cech coboundary operator as

$$(\delta \omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_p} \Big|_{\alpha_0, \ldots \alpha_{p+1}}$$

The Cech cohomology of the complex $\check{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = \frac{\ker \delta_p}{\operatorname{im} \delta_{p-1}}.$$

# Cech Cohomology

$$0 \to \mathcal{F}(X) \to \prod \mathcal{F}(U_\alpha) \xrightarrow{\delta} \prod \mathcal{F}(U_{\alpha\beta})$$

**Proposition**

For any sheaf $\mathscr{F}$
$$\check{H}^0(\mathfrak{U}, \mathscr{F}) = \Gamma(X, \mathscr{F}).$$

For $f \in \check{C}^0$ with $\delta(f) = 0$ we get that $f$ agrees on double intersections. From the axioms for a sheaf, it follows that all these local sections can be glued together to give a global section.

$$0 \to \prod \mathcal{F}(U_\alpha) \xrightarrow{\delta} \prod \mathcal{F}(U_{\alpha\beta})$$

$j_*(j^*\mathcal{F})$ $\qquad\qquad$ $j : U_\alpha \to X$

**Proposition**

The following sequence (augmented Cech) is exact

$$0 \to \mathscr{F}(\cancel{X}) \to \prod \mathscr{F}(U_\alpha) \to \prod \mathscr{F}(U_{\alpha\beta}) \xrightarrow{\delta} \dots \prod \mathcal{F}|U_{\alpha\beta\gamma})$$

$|U_\alpha$

At stalks, fix $x \in U_\gamma$. For $f_x \in \check{C}^i(\mathfrak{U}, \mathscr{F})_x$, $i \geq 0$, choose a section $f$ of this sheaf on some neighborhood $V \subset U_\gamma$ of $x$. Define

$$(\tilde{\theta} f)_{\alpha_0 \dots \alpha_{i-1}} = f_{\gamma\alpha_0 \dots \alpha_{i-1}}$$

$\delta\theta f$

and define $\theta f$ as the image of $\tilde{\theta} f$ in the stalk at $x$. We have

$$(\tilde{\theta}\delta f)_{\alpha_0 \dots \alpha_i} = (\delta f)_{\gamma\alpha_0 \dots \alpha_i} = f_{\alpha_0 \dots \alpha_i} - \sum (-1)^j f_{\gamma \dots \hat{\alpha}_j \dots \alpha_i}$$

$i > 0$

Which shows that $\theta\delta + \delta\theta = 1$ meaning that $H^i(id) = 0$.

$id_H$ ''

# Cech Cohomology and Sheaf Cohomology

Every algebraic variety $X$ has an affine open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$. We will compute the hypercohomology for a projective variety using this cover and Cech cohomology. First, resolve the Cech cochain using Godement resolution.

$$
\begin{tikzcd}
 & \uparrow & & \uparrow & \uparrow & \\
0 \arrow[r] & C^1\mathcal{F}(X) \arrow[r] \arrow[u] & \prod C^1\mathcal{F}(U_\alpha) \arrow[r] \arrow[u] & \prod C^1\mathcal{F}(U_{\alpha\beta}) \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & C^0\mathcal{F}(X) \arrow[r] \arrow[u] & \prod C^0\mathcal{F}(U_\alpha) \arrow[r] \arrow[u] & \prod C^0\mathcal{F}(U_{\alpha\beta}) \arrow[r] \arrow[u] & {} \\
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[u] & \prod \mathcal{F}(U_\alpha) \arrow[r] \arrow[u, "\epsilon"'] & \prod \mathcal{F}(U_{\alpha\beta}) \arrow[r] \arrow[u, "\epsilon"'] & {}
\end{tikzcd}
$$

$q$, $p$

$U_\alpha$

(2.7

0

Each row is exact.

# Acyclicity

## Definition

A sheaf $\mathscr{F}$ of abelian groups on a topological space $X$ is acyclic on an open cover $\mathfrak{U} = \{U_\alpha\}$ of X if the sheaf cohomology

$$H^k(U_{\alpha_0 \ldots \alpha_p}, \mathscr{F}) = 0$$

for $k > 0$ and all finite intersections.

## Theorem

*If a sheaf $\mathscr{F}$ of abelian groups is acyclic on an open cover $\mathfrak{U} = \{U_\alpha\}$ of a topological space $X$, then*

$$\check{H}^k(\mathfrak{U}, \mathscr{F}) \cong H^k(X, \mathscr{F}).$$

Use spectral sequences to prove.

$$\prod \mathcal{F}(U_\alpha) \longrightarrow \prod \mathcal{F}(U_{\alpha\beta}) \longrightarrow \prod \mathcal{F}(U_{\alpha\beta\gamma})$$

$$H^k(X,\mathcal{F}) \simeq \check{H}^k(\mathcal{U},\mathcal{F})$$

# Cech Cohomology of Complex of Sheaves

Now consider $\mathcal{L}^{\bullet}$ as a complex of sheaves. The double complex $K^{p,q} = \check{C}^{p}(\mathfrak{U}, \mathcal{L}^{q})$ is called Cech-sheaf double complex. The Cech cohomology $\check{H}^{k}(\mathfrak{U}, \mathcal{L}^{\bullet})$ of this complex is defined as the cohomology of $K^{\bullet}$ where $K^{k} = \bigoplus_{p+q=k} \check{C}^{p}(\mathfrak{U}, \mathcal{L}^{q})$ with $d_K = \delta + (-1)^{p} d_{\mathcal{L}}$.

$\Omega^k_{X/\mathbb{C}}$

**Theorem**

*If $\mathcal{L}^\bullet$ is a complex of sheaves on $X$ such that each $\mathcal{L}^q$ is acyclic on the open cover $\mathfrak{U} = \{U_\alpha\}$ of $X$, then there is an isomorphism*

$$\check{H}^k(\mathfrak{U}, \mathcal{L}^\bullet) \cong \mathbb{H}^k(X, \mathcal{L}^\bullet).$$

In our case, an algebraic variety $X$ has an affine open cover $\mathfrak{U}$, and we know by a theorem of Serre that $\Omega^k_{X/\mathbb{C}}$ is acyclic on affine varieties. Therefore, we can use this cover and Cech cohomology to compute the hypercohomology.

Thm: For Coherent algebraic sheaf $\mathcal{F}$ on affine $X$

$$H^k(X, \mathcal{F}) = 0$$

# Example Computation

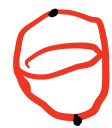
$$\mathcal{H}^k(\mathbb{P}^1, \Omega^\bullet_{\mathbb{P}^1})$$

$$z_2 = \frac{x_0}{x_1}$$

Consider $\mathbb{P}^1 = \mathbb{A}^1_1 \cup \mathbb{A}^1_2$. $\Omega^k_{\mathbb{P}^1}$ is acyclic on affine sets, and as the intersection of affine sets is again affine, we see that $\Omega^k_{\mathbb{P}^1}$ is acyclic on the given cover. Hence we have

$$z_1 = \frac{1}{z_2}$$

$$\check{H}^k(\{\mathbb{A}^1_1, \mathbb{A}^1_2\}, \Omega^\bullet_{\mathbb{P}^1/\mathbb{C}}) \cong \mathbb{H}^k(\mathbb{P}^1, \Omega^\bullet_{\mathbb{P}^1/\mathbb{C}}) \cong H^k_{sing}(\mathbb{P}^1, \mathbb{C}).$$



So, the Cech cohomology should give

$$H^k_{sing}(\mathbb{P}^1, \mathbb{C}) = \begin{cases} \mathbb{C} & k = 0, 2 \\ 0 & k = 1 \end{cases}$$

$$z_1 = \frac{x_1}{x_0}$$

$$\mathbb{A}_1 = \{[x_0 : x_1] \mid x_0 \neq 0\} = \{[1 : z_1]\}$$

$$\mathbb{A}_2 = \{[x_0 : x_1] \mid x_1 \neq 0\} = \{[z_2 : 1]\}$$

## Example Computation

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & \Omega_{\mathbb{P}^1}(\mathbb{A}_1) \oplus \Omega_{\mathbb{P}^1}(\mathbb{A}_2) \arrow[r] \arrow[u] & \Omega_{\mathbb{P}^1}(\mathbb{A}_1 \cap \mathbb{A}_2) \arrow[u] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}_2) \arrow[r, "\delta"'] \arrow[u, "d"'] & \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}_1 \cap \mathbb{A}_2) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$$\boxed{D = d + \delta}$$

This is the cech complex and its cohomology is the cohomology of the corresponding single complex.

$\left( j : \mathbb{A}^1 \hookrightarrow \mathbb{P}^1 , \quad j_* \Omega^p_{\mathbb{A}^1} = \Omega^p_{\mathbb{P}^1}(* D) \right)$

$$\Omega^p_{\mathbb{A}^1}(\mathbb{A}^1) = \Omega^p_{\mathbb{P}^1}(\mathbb{A}^1)$$

$$D = \delta + (-1)^p d$$

$$0 \to \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1_2) \xrightarrow{D} \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1_1 \cap \mathbb{A}^1_2) \oplus \Omega^1_{\mathbb{P}^1}(\mathbb{A}^1_1) \oplus \Omega^1_{\mathbb{P}^1}(\mathbb{A}^1_2) \xrightarrow{D} \Omega_{\mathbb{P}^1}(\mathbb{A}^1_1 \cap \mathbb{A}^1_2) \to 0$$

$$f, g \longmapsto \left( (f-g)\big|_{A_1 \cap A_2}, \; df, \; dg \right)$$

$\mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1_1)$ $\quad \mathbb{C}[z_1]$ $\qquad (\varphi, \omega, \eta) \longmapsto d\varphi - (\omega - \eta)\big|_{\mathbb{A}^1_1 \cap \mathbb{A}^1_2}$

$$\check{H}^0 = \ker D = \{ (f, g) \in K^0 : (f-g)\big|_{\mathbb{A}^1 \cap \mathbb{A}^2} = 0 \text{ and } df = dg = 0 \}$$

Choose coordinates $z_1$ for $\mathbb{A}_1$ and $z_2$ for $\mathbb{A}_2$ with $\frac{1}{z_1} = z_2$ on intersection

Then $f \in \mathbb{C}[z_1]$ and $g \in \mathbb{C}[z_2]$ polynomials.

If $f(z_1) = g(z_2)$ on intersection, we can replace $z_2$ with

# Continued

$\frac{1}{z_1}$ and get $f(z_1) = g(\frac{1}{z_1})$ meaning that

$$\sum_{k=0}^{n} a_k z_1^k = \sum_{k=0}^{n} b_k \left(\frac{1}{z_1}\right)^k \Longrightarrow a_k = b_k = 0 \text{ for } k > 0 \text{ and } a_0 = b_0$$

Therefore $f$ and $g$ should be constant (some constant)

$df = dg = 0$ already satisfied so we get

$$\check{H}^0(\mathcal{U}, \Omega^{\cdot}_{\mathbb{P}^1}) = \mathbb{C}.$$

$$\check{H}^1(\mathcal{U}, \Omega^{\cdot}_{\mathbb{P}^1}) = \frac{\{(\varphi, \omega, \eta) \in K^1 : d\varphi = (-\eta + \omega)\}}{\{(\varphi, \omega, \eta) \in K^1 : \varphi = f - g \text{ where } f \text{ extends to a polynomial in } \mathbb{C}[z_1],\ g \text{ extends to a polynomial in } \mathbb{C}[z_2],\ df = \omega,\ dg = \eta\}}$$

We claim that this is zero.

To see this, write $\varphi$ as $\varphi(z_1) = \sum_{-m}^{n} a_k z_1^k$

$d\varphi = \sum_{-m}^{n} k a_k z_1^{k-1} dz_1$. Notice that there is no $\frac{dz_1}{z_1}$ term.

So, $d\varphi = (\eta(z_2)dz_2 - \omega(z_1)dz_1)\big|_{A_1 \cap A_2} = \left[\frac{-\eta(\frac{1}{z_1})}{z_1^2} - \omega(z_1)\right] dz_1$

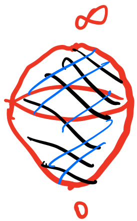
Write $\eta(z_2) = \sum_{k=0}^{n-2} b_k z_2^k$ and $\omega(z_1) = \sum_{k=0}^{n} c_k z_1^k$

See that for $f = \sum_{k=0}^{n} \frac{c_k z_1^{k+1}}{k+1}$ $df = \omega$ on $A_1$ and for

$g = \sum_{k=0}^{n-2} \frac{b_k z_2^{k+1}}{k+1}$ $dg = \eta$ on $A_2$. Therefore we

can write $\varphi = (f-g)\big|_{A_1 \cap A_2}$ where $f$ is in $\mathbb{C}[z_1]$

and $g \in \mathbb{C}[z_2]$. Hence, $\check{H}^1(\mathcal{U}, \Omega^1_{\mathbb{P}^1}) = 0$




$$\mathcal{O}(\mathbb{A}_1 \cap \mathbb{A}_2)$$

$$f(z_1) = \sum_{-n}^{m} (z_1)^k a_k$$

Finally, $\check{H}_2(\mathcal{U}, \Omega^\cdot_{alg}) = \dfrac{\Omega_{lpt}(A_1 \cap A_2)}{\{\alpha \in \Omega_{lp}^1(A_1 \cap A_2) \text{ s.t. } \alpha = d\phi + (w - \eta)\}}$

$\eta(z_2)dz_2 - w(z_1)dz_1$, where $\phi \in \mathcal{O}(A_1 \cap A_2)$, $w \in \Omega(A_2)$ and $\eta \in \Omega(A_1)$ $\}$

$\dfrac{dz_1}{z_1}$

$\boxed{\dfrac{dz_1}{z_1}} \in \Omega^1_{lpt}(A_1 \cap A_2) = \Omega^1_{\mathbb{C}[z_1]/\mathbb{C}} \otimes (\mathcal{O}(A_1))_0$ localize at 0.

as above $\dfrac{dz_1}{z_1}$ does not come from $d\phi$ of $\phi \in \mathcal{O}(A_1 \cap A_2)$

Also if $w \in \Omega(A_1)$ then $w = w(z_1)dz_1$ and

$\eta = \eta(z_2)dz_2$ on intersection

$(w - \eta)|_{A_1 \cap A_2} = \left(w(z_1) - \dfrac{\eta(\frac{1}{z_1})}{z_1^2}\right) dz_1$ $\quad \dfrac{dz_1}{z_1^2}$

no $\dfrac{dz_1}{z_1}$ term. Everything else is obtainable so

$\check{H}^2 = \langle \dfrac{dz_1}{z_1} \rangle$ Therefore $\check{H}^2(\mathcal{U}, \Omega^\cdot_{cpl}) = \mathbb{C}$

## Proof For Affine Case

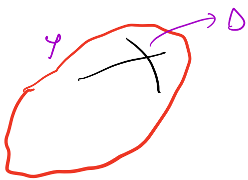
Theorem: Let $X$ be a smooth affine variety over $\mathbb{C}$ and $(\Omega^\bullet_{X/\mathbb{C}}, d)$ the complex of algebraic differential forms on $X$. If $X_{an}$ is the complex manifold corresponding to $X$, then

$$H^k_{sing}(X_{an}, \mathbb{C}) \simeq h^k(\Omega^\bullet_{X/\mathbb{C}}(X)) \simeq \mathcal{H}^k(\underline{\Omega}^\bullet_X)$$

$$0 \to \Omega^0_{X/\mathbb{C}}(X) \xrightarrow{d} \Omega^1_{X/\mathbb{C}}(X) \to \cdots$$

$$0 \to R \xrightarrow{d} \Omega^1_{R/\mathbb{C}} \xrightarrow{d} \Omega^2_{R/\mathbb{C}} \to \cdots$$

A normal crossing divisor on a smooth algebraic variety is a divisor that is locally the zero set of an equation of the form $z_1 \cdots z_k = 0$ where $z_1, \ldots, z_n$ are local coordinates.




Let $X$ be given by $f_i(z_1, \ldots, z_n) = 0$ in $\mathbb{C}^N$

Its projective closure $\bar{X}$ is defined by

$$f_i\left(\frac{Z_1}{Z_0}, \ldots, \frac{Z_N}{Z_0}\right) = 0$$

where $Z_0, \ldots, Z_N$ are homogeneous coordinates of $\mathbb{CP}^N$

and $z_i = \dfrac{Z_i}{Z_0}$. This homogenization may give a singular variety $\bar{X}$.

For example consider $X = V(xy^2-1) \subset \mathbb{C}^2$

it is smooth because $\nabla(xy^2-1) = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix} = 0 \Rightarrow y=0$

but $(x,0)$ is not on the variety. It's projective

closure is $\bar{X} = V(xy^2 - z^3) \subset \mathbb{CP}^2$. It is singular

because $\nabla(xy^2 - z^3) = \begin{pmatrix} y^2 \\ 2xy \\ -3z^2 \end{pmatrix} = 0 \Rightarrow \begin{matrix} y=0 \\ z=0 \end{matrix}$ so $[1:0:0]$

is a singular point which is on the curve $\bar{X}$. $y^2 - z^3 = 0$

Hironaka's resolution of singularities says that

there is a surjective regular map $\pi : Y \to \bar{X}$ from a smooth projective variety $Y$ to $\bar{X}$ such that $\pi^{-1}(\bar{X} - X)$ is a normal crossing divisor $D$ in $Y$ and $\pi|_{Y-D} : Y - D \to X$ is an isomorphism. Thus we may assume $X = Y - D$, via

$j : X \to Y$ inclusion.

Let $\Omega^k_{Y_{an}}(*D)$ be the sheaf of meromorphic $k$-forms on $Y_{an}$ that are holomorphic on $X_{an}$ with poles of any order $\geq 0$ along $D_{an}$.

Let $\mathcal{A}^k_{X_{an}}$ : sheaf of $C^\infty$ complex valued $k$-forms on $X_{an}$. and $j : X_{an} \hookrightarrow Y_{an}$

$$(j_* \mathcal{A}^k_{X_{an}})(V) = \mathcal{A}^k_{X_{an}}(V \cap X_{an}) \quad \text{for all } V \underset{\text{open}}{\subset} Y_{an}.$$

$V \cap X_{an} \subset X_{an}$

A section of $\Omega^k_{Y_{an}}(*D)$ is holomorphic on

so $\Omega^k_{Y_{an}}(*D)$ is a subsheaf of of $j_* \mathcal{A}^k_{X_{an}}$.

**Main Ingredient (Hodge–Atiyah)**

$\phi$ $\Omega^\bullet_{Y_{an}}(*D) \hookrightarrow j_* \mathcal{A}^\bullet_{X_{an}}$ is a quasi-isomorphism.

Therefore,

$$\mathcal{H}^k(Y_{an}, \Omega^\bullet_{Y_{an}}(*D)) \simeq \mathcal{H}^k(Y_{an}, j_* \mathcal{A}^\bullet_{X_{an}})$$

$$\begin{array}{ccc} \wr| & & \wr| \\ h^k(\Omega^\bullet_X(X)) & \longleftrightarrow & H^k(X_{an}, \mathbb{C}) \end{array}$$

The aim is to show the red isomorphisms

RHS in general : Let $M$ be a complex manifold and $U$ an open submanifold, with $j : U \to M$. Denote the sheaf of smooth $\mathbb{C}$-valued $k$-forms on $U$ by $\mathcal{A}^k_U$. Then,

$$\mathcal{H}^k(M, j_* \mathcal{A}^\bullet_U) \simeq H^k(U, \mathbb{C})$$

$\mathcal{A}$

# Proof of RHS

**Proof :** $\mathcal{A}^0$ is the sheaf of smooth $\mathbb{C}$-valued functions on $M$.
For any open $V \subset M$, there is an $\mathcal{A}^0(V)$-module structure on $(j_*\mathcal{A}^k_U)(V) = \mathcal{A}^k_U(U \cap V)$

$$\mathcal{A}^0(V) \times \mathcal{A}^k_U(U \cap V) \longrightarrow \mathcal{A}^k_U(U \cap V)$$

$$(f, w) \longmapsto f.w$$

This allows us to construct partitions of unity on $j_*\mathcal{A}^k_U$ so it is acyclic.

$$\mathcal{H}^k(M, j_*\mathcal{A}^\bullet_U) \simeq h^k((j_*\mathcal{A}^\bullet_U)(M)) = h^k(\mathcal{A}^\bullet_U(U)) = H^k(U, \mathbb{C})$$

smooth de Rham

$\boxtimes$

Let $M = Y_{an}$ and $U = X_{an}$ then we have the RHS.

## LHS

We need to compare meromorphic forms with rational forms.

**D**: normal crossing divisor

**X**: smooth complex affine variety

$X = Y - D$

$\Omega^q_{Y_{an}}(nD)$: sheaf of meromorphic $q$-forms on $Y_{an}$ that are holomorphic on $X_{an}$ with poles of order $\leq n$ along $D_{an}$. If $n = \cancel{\infty}$ then any order.

$\Omega^q_Y(nD)$: sheaf of rational $q$-forms on $Y$ that are regular on $X$ with "poles" of order $\leq n$ along $D$.

$$\varphi_{ij} \quad \Omega^q_Y(iD) \hookrightarrow \Omega^q_Y(jD) \quad \text{for } i \leq j$$

$$\varphi_{ii} = id \qquad \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \quad \text{for } i \leq j \leq k$$

$$\varinjlim \Omega^q_Y(iD) = \bigsqcup_i \Omega^q_Y(iD) \Big/ \sim$$

if $w_i \in \Omega^q_Y(iD)$ and $w_j \in \Omega^q_Y(jD)$ then

$w_i \sim w_j \iff$ for some $k$ with $i \leq k$ and $j \leq k$

$$\varphi_{ik}(w_i) = \varphi_{jk}(w_j)$$

$$\boxed{\Omega^q_{Y_{an}}(*D) = \varinjlim_n \Omega^q_{Y_{an}}(nD)}$$ and $$\Omega^q_Y(*D) = \varinjlim_n \Omega^q_Y(nD)$$

As $Y - D = X$ we can pushforward a regular form on $X$ to a form on $Y$ which has poles along $D$.

$$j_* \Omega^q_X = \Omega^q_Y(*D)$$

This is not true for $X_{an}$ and $Y_{an}$ (essential singularity)

* Affine and projective varieties are noetherian, i.e. any descending chain $Y_1 \supset Y_2 \supset \dots$ of closed sets must terminate after finitely many steps.

**PROPOSITION 2.10.2** (Commutativity of direct limit with cohomology) *Let $(\mathcal{F}_\alpha)$ be a direct system of sheaves on a topological space $Z$. The natural map*

$$\varinjlim H^k(Z, \mathcal{F}_\alpha) \to H^k(Z, \varinjlim \mathcal{F}_\alpha)$$

*is an isomorphism if*

*(i) $Z$ is compact; or*

*(ii) $Z$ is noetherian.*

In our case $X$ and $Y$ are noetherian and $Y_{an}$ is compact as $Y_{an} \subset \mathbb{C}P^N$ and it is closed.

Proposition 1 $\mathbb{H}^*(Y, \Omega^\bullet_Y(*D)) \simeq \mathbb{H}^*(Y_{an}, \Omega^\bullet_{Y_{an}}(*D)).$

proof: $Y$ is projective and each $\Omega^\bullet_Y(nD)$ is coherent.

To see this consider $D$ defined by $z_1 \cdots z_k = 0$

We denote $\Omega^1_Y(1 \cdot D) = \Omega^1_Y(\log D)$

$$\Omega^1_Y(D) = \mathcal{O}_Y \frac{dz_1}{z_1} \oplus \mathcal{O}_Y \frac{dz_2}{z_2} \oplus \cdots \oplus \mathcal{O}_Y \frac{dz_k}{z_k} \oplus \mathcal{O}_Y dz_{k+1} \oplus \cdots \oplus \mathcal{O}_Y dz_n$$

So it is locally finitely generated. We can use

Gaga $\Longrightarrow$ $H^p(Y, \Omega^q_Y(nD)) \simeq H^q(Y_{an}, \Omega^q_{Y_{an}}(nD))$

Take direct limit of both sides and use the above proposition to get

$$H^p(Y, \Omega^q_Y(*D)) \simeq H^p(Y_{an}, \Omega^q_{Y_{an}}(*D))$$

Those are the $E_1$ terms of second spectral sequences which induce the result. ∎

Proposition 2 $\mathbb{H}^k(Y, \Omega_Y^\bullet(*D)) \simeq \mathbb{H}^k(X, \Omega_X^\bullet)$

proof: For $V \subset Y$ open affine, $V$ is noetherian so

$$H^p(V, \Omega_Y^q(*D)) = H^p(V, \varinjlim_n \Omega_Y^q(nD)) \simeq \lim H^p(V, \Omega_Y^q(nD)) = 0 \quad \text{for } p>0$$

affine coherent

because $V$ is affine and $\Omega_Y^q(nD)$ is coherent which means $\Omega_Y^\bullet(*D)$ is acyclic on any affine open cover $\mathcal{U} = \{U_\alpha\}$. Therefore we have

$$\mathbb{H}^k(Y, \Omega_Y^\bullet(*D)) \simeq \check{H}^k(\mathcal{U}, \Omega_Y^\bullet(*D))$$

We know that for $j: X \to Y$ inclusion then $\Omega_Y^\bullet(*D) = j_* \Omega_X^\bullet$

$$K^{p,q} = \check{C}^p(\mathcal{U}, \Omega^q_Y(*D)) = \check{C}^p(\mathcal{U}, j_*\Omega^q_X)$$

$$= \prod_{\alpha_0 < \dots < \alpha_p} \Omega^q(U_{\alpha_0,\dots,\alpha_p} \cap X)$$

$\mathcal{U} = \{U_\alpha\}$ is open affine cover of $Y$

$\mathcal{U}|_X := \{U_\alpha \cap X\}$ is an affine open cover of $X$.

$\Omega^q_X$ is coherent and $U_\alpha \cap X$ is affine open

Serre vanishing $\Rightarrow H^p(U_\alpha \cap X, \Omega^q_X) = 0 \quad \forall p > 0$

Therefore

Thus $\Omega^\bullet_X$ is acyclic on $\mathcal{U}|_X$.

$$\mathbb{H}^k(X, \Omega^\bullet_X) \simeq \check{\mathbb{H}}^k(\mathcal{U}|_X, \Omega^\bullet_X)$$

If we show that $\check{\mathbb{H}}^k(\mathcal{U}|_X, \Omega^\bullet_X) \simeq \check{\mathbb{H}}^k(\mathcal{U}, \Omega^\bullet_Y(*D))$

we are done.

LHS is the cohomology of $K^{p,q} = \check{C}^p(\mathcal{U}|_X, \Omega_X^q)$

$$= \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(V_{\alpha_0 \ldots \alpha_p} \cap X)$$

which is the same complex of RHS. Therefore

$$\mathbb{H}^k(Y, \Omega_Y^\bullet(*D)) \simeq \mathbb{H}^k(X, \Omega_X^\bullet) \qquad \boxed{\checkmark}$$

Finally $\mathbb{H}^k(X, \Omega_X^\bullet) \simeq h^k(\Omega_X^\bullet(X))$ as $\Omega_X^\bullet$ is acyclic ($X$ affine) combining results we have the proof of algebraic de Rham for affine variety.