Introduction to Morse Theory

August 28, 2023

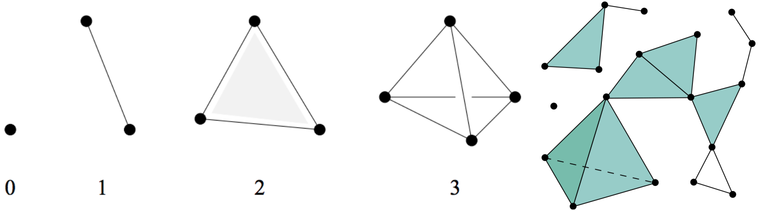
Outline

- Simplicial Complexes and CW-complexes
- Classical Morse Theory
- Discrete Morse Theory

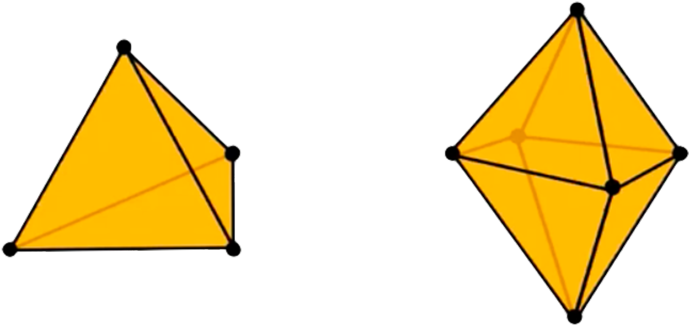
Simplicial Complexes

Definition

An abstract simplicial complex is a set of vertices V, along with a collection K of subsets of V called simplices, which is closed under subsets and contains all singletons. Their geometric realisations are topological spaces.



Different models of S^2



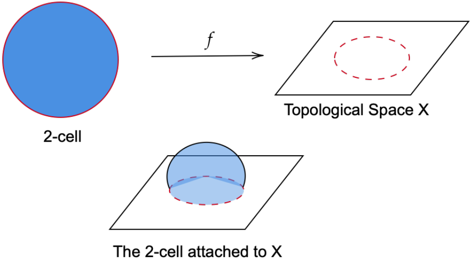
Cell Attachment

Definition

A d-cell is a closed ball of dimension d.

Definition

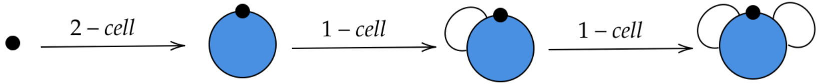
We can attach a d-cell to a topological space X by **gluing** its boundary to X by a continuous map.



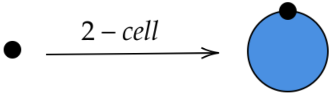
CW complexes

Definition

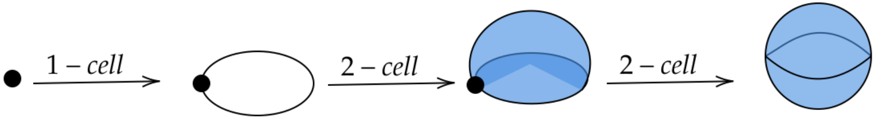
A CW complex is a space built out of smaller spaces, iteratively by a process called attaching cells. A k-cell is a k-dimensional disc. Attaching a k-cell to another space X means, intuitively, forming the union of X and D^k where we glue the boundary of D^k to X.



CW complex structure



S^2 *as a* CW – *complex with a* 0 – *cell and a* 2 – *cell*



S^2 *as a* CW – *complex with a* 0 – *cell,* 1 – *cell and two* 2 – *cell*

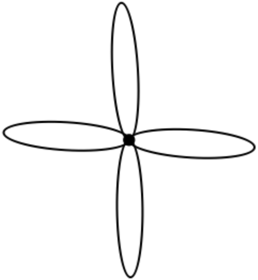
Vedge of d-spheres

Theorem

Let X be a CW-complex obtained by attaching n d-cells to a O-cell. Then X is homotopy equivalent to $\vee^n S^d$

Proposition

The homotopy group $\pi_{d-1}(\vee^n S^d) = \{0\}$



$\vee\, ^4 S^1$

SLOGAN:

A well chosen map $f: M \longrightarrow \mathbb{R}$ can be used to analyse the topology of M.

Critical Point

Let M be a compact subset of $\mathbb{R}^n$. Given a C^∞-function $f : M \longrightarrow \mathbb{R}$ the set of critical points of f is defined as

$$Crit(f) \colon = \{x \in M \mid df(x) = 0\}$$

f is said to be a *Morse* if every critical point is non-degenerate. That is to say that for all $x \in Crit(f)$, the Hessian matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{1 \leq i,j \leq n}$ has non-zero determinant.

Non-degenerate critical points

Notice on a neighborhood of a non-degenerate critical point a, the function f can be written as (Taylor Expansion)

$$\begin{aligned} T(x) &= f(a) + (x-a)^t df(a)(x-a) + \frac{1}{2!}(x-a)^t d^2 f(a)(x-a) + \dots \\ &= f(a) + + \frac{1}{2!}(x-a)^t d^2 f(a)(x-a) + \dots \end{aligned}$$

Notice that the second term is a quadratic form. We get the following lemma for Morse functions.

Morse Lemma

Suppose M has dimension d, then on an open neigborhood U of a critical point x there exists a diffeomorphism $\phi \colon U \longrightarrow \mathbb{R}^d$ such that

$$f \circ \phi^{-1}(y_1, \dots, y_d) = f(x) - (y_1^2 + \cdots + y_k^2) + (y_{k+1}^2 + \dots y_d^2)$$

(Türkçesi: f fonksiyonu x'in komşuluğunda yukarıdaki gibi bir kuadratik form olarak yazılabilir.)

We call k the Morse index of f at x.

Morse Index

By $c_k(f)$ we denote the number of critical points of a Morse function f with Morse index k, and by $b_k(M)$ the k-th Betti number of M, the rank of the k-th homology group $H_k(M; Z)$. Morse theory relates these quantites, they are called the Morse inequalities.

A Theorem

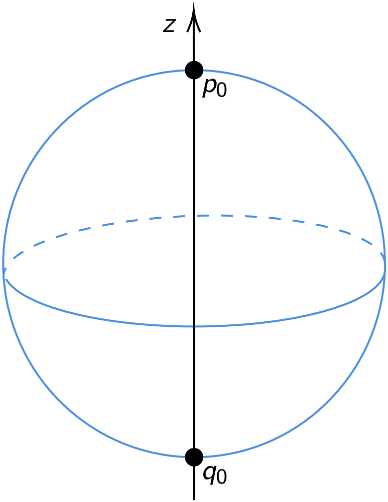
Suppose M is a closed surface and f a Morse function having a critical point of index 2 and a critical point of index 0, then M is diffeomorphic to S^2.

This gives us a clue that using Morse function on spaces we can **collapse** those unnecessary parts of the space that don't carry any relevant topological data. This is what we exactly do with **Discrete Morse Theory** on simplicial complexes.

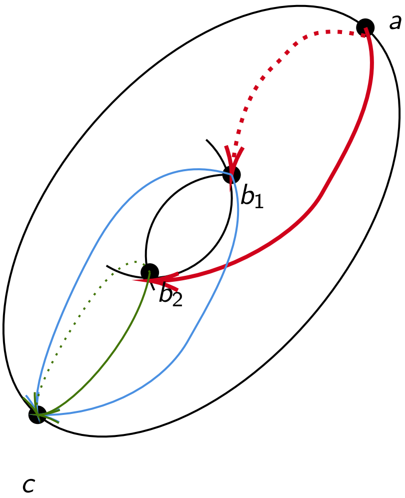
Computing Homology

For a Morse function f on M, we define C_k to be the free abelian group generated by critical points of index k. We need to define a boundary map that goes from points of index k to those that are of index $k-1$. This is a very technical process. We instead give some intuiton by example.

Homology of S^2 using height function

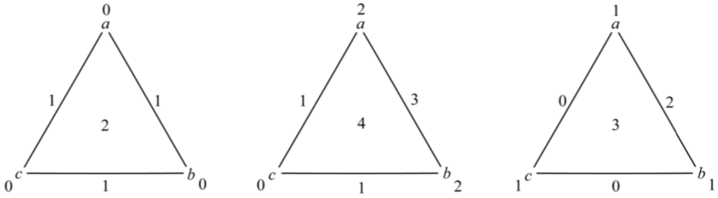


Homology of Torus



Definition

A discrete Morse function on a simplicial complex K is a function $f : K$ such that for any p-simplex $\alpha \in K$, it takes every $(p+1)$-simplex that contains α, except for at most one, to a value strictly greater than $f(\alpha)$. Similarly, f takes every $(p-1)$-simplex that is contained in α, except for at most one, to a value strictly smaller than $f(\alpha)$.


0
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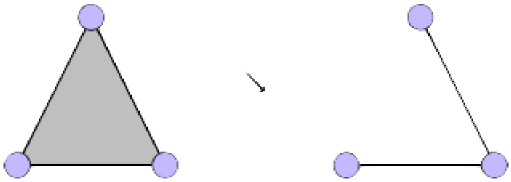
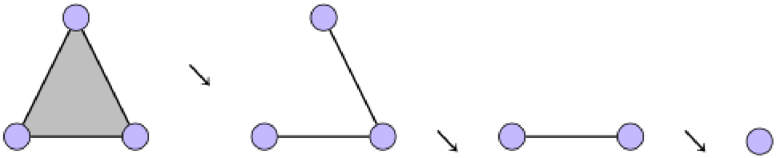
1

Definition

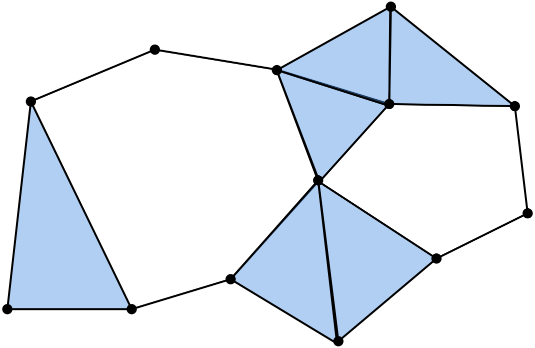
$\alpha \in K$ is a critical p-simplex of the Morse function f. if f takes every $(p+1)$-simplex that contains α, to a value strictly larger than $f(\alpha)$ and takes every $(p-1)$-simplex that is contained in α, to a value strictly smaller than $f(\alpha)$

Elementary Collapse





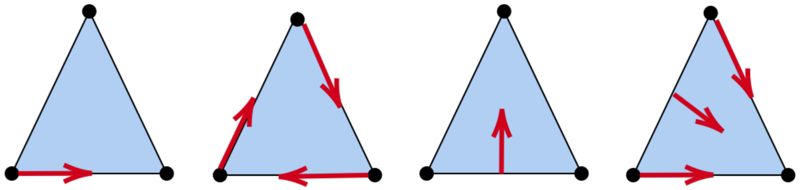
Collapse



Discrete Vector Field

Let K be a simplicial complex. A discrete vector field V on K is a matching of the simplices of K satisfying

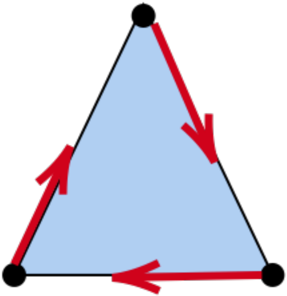
$$V = \{(\sigma^{p-1}, \tau^p) : \sigma \subset \tau, \text{each simplex in at most one pair}\}$$



Let V be a discrete vector field on K. A **V-path** is a sequence of simplices

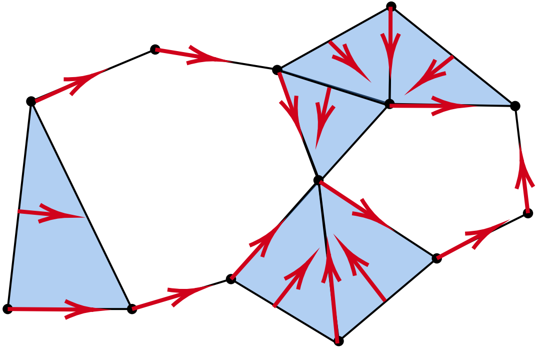
$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \cdots, \tau_{k-1}^{(p+1)}, \sigma_k^{(p)}$$

such that $(\sigma_i^{(p)}, \tau_i^{p+1}) \in V$ and $\tau_{i-1}^{(p+1)} > \sigma_i^{(p)}$. If $\sigma_0^{(p)} = \sigma_k^{(p)}$, the V-path is said to be **closed**

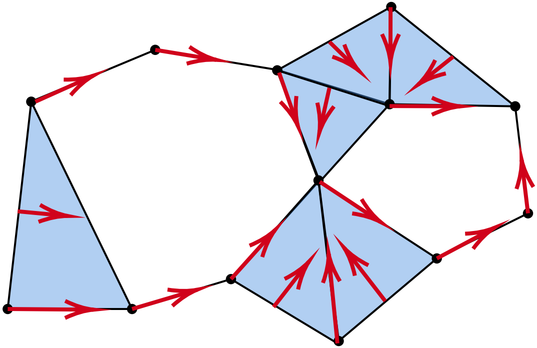


Vector Field Example

A discrete vector field with no closed $V-$paths is said to be a **gradient vector field.**



A discrete vector field with no closed $V-$paths is said to be a **gradient vector field.** A simplex that is not gradient vector field is **critical.**

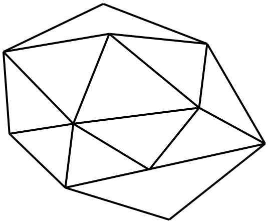


Weak discrete Morse inequalities (Forman)

Let K be a simplicial complex, dim(K) =n, and f a discrete Morse function (or V a gradient vector field) with m_i; critical simplices of dimension i on K. Then

- $b_i \leq m_i$ for all $i = 0, 1, \cdots, n$
- $m_0 - m_1 + m_2 - \cdots + (-1)^n m_n = \chi(K)$

Example



Main Theorem of Discrete Morse Theory

Theorem

Suppose K is a simplicial complex with a discrete Morse function or gradient vector field. Then K is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical simplex of dimension p.

Morse Complex

Let X be a simplicial complex with discrete Morse function f. Let C_k denote the simplical k-chains of X. Define the subspace M_k of C_k be the space of critical k-chains. We write M_* as the space of these Morse chains. Since homotopic spaces have the same homology, if we define the boundary map

$$\tilde{\partial}\colon M_{p+1} \longrightarrow M_p$$

correctly, we must have

$$H_k(M_*, \tilde{\partial}) \cong H_k(C_k, \mathbb{Z}).$$

Boundary Map $\tilde{\partial}$

Theorem

Choose an orientation for each simplex. Then for any critical point $p+1$-simplex β, set

$$\tilde{\partial}\beta = \sum_{\text{critical } \alpha^{(p)}} c_{\alpha,\beta}\alpha$$

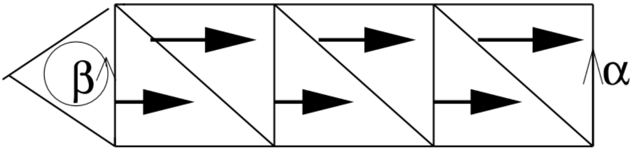
where

$$c_{\alpha,\beta} = \sum_{\gamma \in \Gamma(\alpha,\beta)} m(\gamma)$$

where $\Gamma(\alpha, \beta)$ is the set of gradient paths which go from a maximal face of β to α. The multiplicity $m(\gamma)$ of any gradient path γ is equal to ± 1 depending on whether, given γ, the orientation that β gives to α is the same as the orientation chosen.

Example

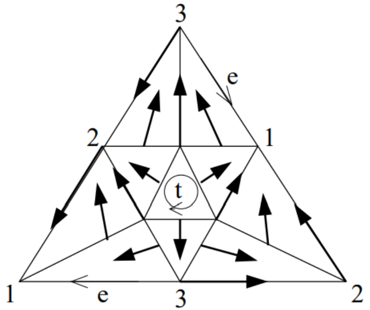
On the example below, how do we decide the orientation that β gives to α and how do we calculate $m(\gamma)$?



A gradient path from the boundary of β to α.

$$\tilde{\partial}(\beta) = -\alpha$$

Homology of $\mathbb{RP}^2$



A gradient vector field on the real projective plane.