9. Padé Approximation

Background.

A Padé rational approximation to f(x) on [a,b] is the quotient of two polynomials $P_n(x)$ and $Q_m(x)$ of degrees n and m, respectively. We use the notation $R_{n,m}(x)$ to denote this quotient:

$$R_{n,m}(x) = \frac{P_n(x)}{Q_m(x)}.$$

We attribute much of the founding theory to Henri Eugène Padé (1863-1953).

Theorem (Padé Approximation). Assume that $f \in C^{n+m}$, and that $f(x)$ Maclaurin polynomial expansion of degree at least $n+m$. Then

$$f(x) \approx R_{n,m}(x) = \frac{P_n(x)}{Q_m(x)},$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of degree n and m, respectively.

Example 1. Find the Padé approximation $R_{2,2}(x)$ for $f[x] = e^x$.
Solution 1.

Example 2. Find the Padé approximation $R_{4,4}(x)$ for $f[x] = \mathrm{Cos}[x]$.
Solution 2.

Example 1. Find the Padé approximation $R_{2,2}(x)$ for $f[x] = e^x$.
Solution 1.

First, set up the equation $f(x)\,Q_m(x) - P_n(x) = 0$.

$$P_n(x) = p_0 + x\,p_1 + x^2\,p_2$$
$$Q_m(x) = 1 + x\,q_1 + x^2\,q_2$$
$$f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + O[x]^5$$

Form $f(x)Q_m(x) - P_n(x) = 0$

$$(1-p_0) + (1-p_1+q_1)\,x + \left(\frac{1}{2} - p_2 + q_1 + q_2\right)x^2 + \left(\frac{1}{6} + \frac{q_1}{2} + q_2\right)x^3 + \left(\frac{1}{24} + \frac{q_1}{6} + \frac{q_2}{2}\right)x^4 + O[x]^5 == 0$$

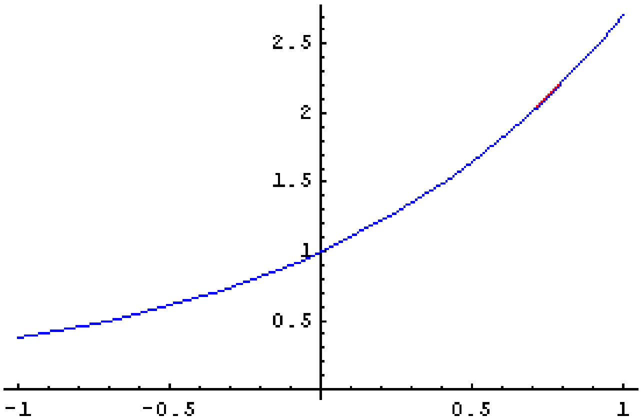
Second, solve the equation $f(x)\,Q_m(x) - P_n(x) = 0$.

$$1 - p_0 == 0$$
$$1 - p_1 + q_1 == 0$$
$$\frac{1}{2} - p_2 + q_1 + q_2 == 0$$
$$\frac{1}{6} + \frac{q_1}{2} + q_2 == 0$$
$$\frac{1}{24} + \frac{q_1}{6} + \frac{q_2}{2} == 0$$

$$p_0 \to 1$$
$$p_1 \to \frac{1}{2}$$
$$p_2 \to \frac{1}{12}$$
$$q_1 \to -\frac{1}{2}$$
$$q_2 \to \frac{1}{12}$$

$$f[x] = e^x$$
$$R[x] = \frac{12 + 6x + x^2}{12 - 6x + x^2}$$

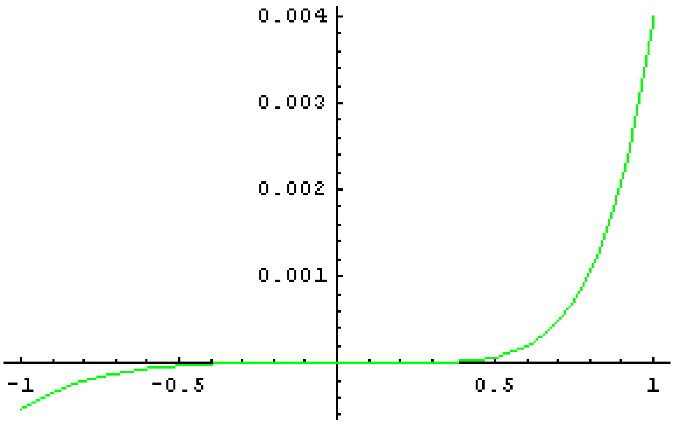
Plot graphs of the function and its Pade approximation over the interval [-1,1].



$$f[x] = e^x$$

$$r[x] = \frac{12 + 6x + x^2}{12 - 6x + x^2}$$

Find the error over the interval [-1,1].



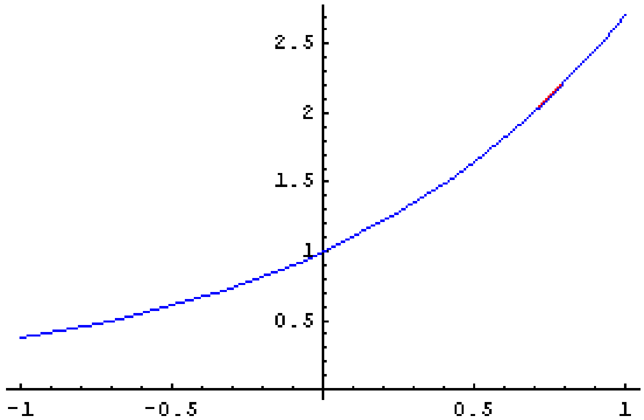
$$f[x]-r[x] = e^x - \frac{12 + 6x + x^2}{12 - 6x + x^2}$$

The maximum error is

$$|f[x]-r[x]| \leq 0.00399611$$

Compare with the error in a 4^{th} degree Maclaurin polynomial over the interval $[-1, 1]$.

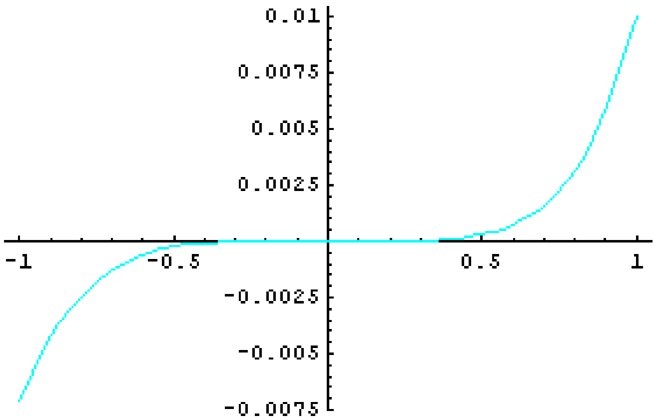
$$s[x] = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$



$$f[x] = e^x$$

$$s[x] = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

Find the error over the interval [-1,1].



$$f[x]-s[x] = -1 + e^x - x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24}$$

The maximum error is

$$|f[x]-s[x]| \le 0.0099485$$

Example 2. Find the Padé approximation $R_{4,4}(x)$ for $f[x] = Cos[x]$.
Solution 2.

First, set up the equation $f(x)\,Q_m(x) - P_n(x) = 0$.

$$P_n(x) = p_0 + x\,p_1 + x^2\,p_2 + x^3\,p_3 + x^4\,p_4$$
$$Q_m(x) = 1 + x\,q_1 + x^2\,q_2 + x^3\,q_3 + x^4\,q_4$$
$$f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} + 0[x]^9$$

Form $f(x)Q_m(x) - P_n(x) = 0$

$$(1-p_0) + (-p_1+q_1)\,x + \left(-\frac{1}{2} - p_2 + q_2\right)x^2 + \left(-p_3 - \frac{q_1}{2} + q_3\right)x^3 + \left(\frac{1}{24} - p_4 - \frac{q_2}{2} + q_4\right)x^4 + \left(\frac{q_1}{24} - \frac{q_3}{2}\right)x^5 + \left(-\frac{1}{720} + \frac{q_2}{24} - \frac{q_4}{2}\right)x^6 + \left(-\frac{q_1}{720} + \frac{q_3}{24}\right)x^7 + \left(\frac{1}{40320} - \frac{q_2}{720} + \frac{q_4}{24}\right)x^8 + 0[x]^9 == 0$$

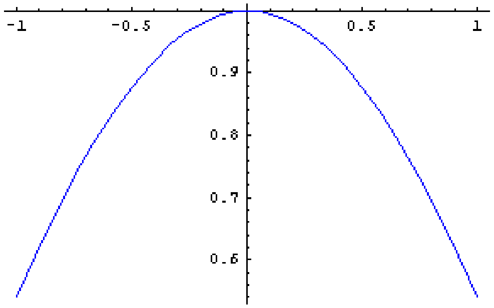
Second, solve the equation $f(x)\,Q_m(x) - P_n(x) = 0$.

$$1 - p_0 == 0$$
$$-p_1 + q_1 == 0$$
$$-\frac{1}{2} - p_2 + q_2 == 0$$
$$-p_3 - \frac{q_1}{2} + q_3 == 0$$
$$\frac{1}{24} - p_4 - \frac{q_2}{2} + q_4 == 0$$
$$\frac{q_1}{24} - \frac{q_3}{2} == 0$$
$$-\frac{1}{720} + \frac{q_2}{24} - \frac{q_4}{2} == 0$$
$$-\frac{q_1}{720} + \frac{q_3}{24} == 0$$
$$\frac{1}{40320} - \frac{q_2}{720} + \frac{q_4}{24} == 0$$

$$p_0 \to 1$$
$$p_1 \to 0$$
$$p_2 \to -\frac{115}{252}$$
$$p_3 \to 0$$
$$p_4 \to \frac{313}{15120}$$
$$q_1 \to 0$$
$$q_2 \to \frac{11}{252}$$
$$q_3 \to 0$$
$$q_4 \to \frac{13}{15120}$$

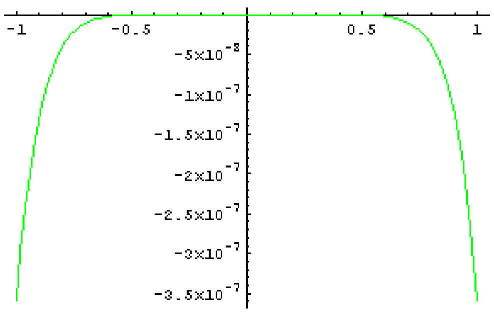
$$f[x] = Cos[x]$$
$$R[x] = \frac{15120 - 6900\,x^2 + 313\,x^4}{15120 + 660\,x^2 + 13\,x^4}$$

Plot graphs of the function and its Pade approximation over the interval [-1, 1].



$$f[x] = \text{Cos}[x]$$
$$r[x] = \frac{15120 - 6900x^2 + 313x^4}{15120 + 660x^2 + 13x^4}$$

Find the error over the interval [-1, 1].



$$f[x]-r[x] = -\frac{15120 - 6900x^2 + 313x^4}{15120 + 660x^2 + 13x^4} + \text{Cos}[x]$$

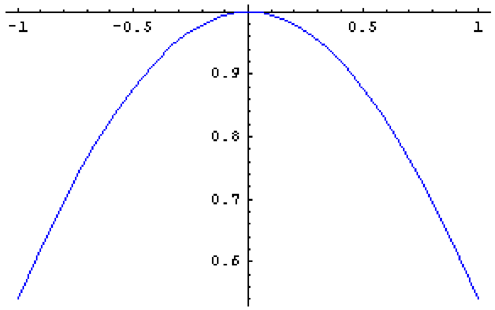
The maximum error is

$$|f[x]-r[x]| \le 3.5987 \times 10^{-7}$$

Compare with the error in a 7^{th} degree Maclaurin polynomial over the interval [-1, 1].
Remark. The coefficient of x^7 is zero, but that's o.k.

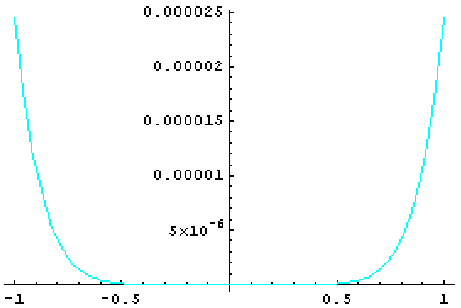
$$s[x] = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

Plot graphs of the function and its Pade approximation over the interval [-1, 1].



$$f[x] = \mathrm{Cos}[x]$$

$$s[x] = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$



$$f[x]-s[x] = -1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} + \mathrm{Cos}[x]$$

The maximum error is

$$|f[x]-s[x]| \leq 0.0000245281$$