4. The Matrix Inverse

Theorem (Inverse Matrix) Assume that A is an $n \times n$ nonsingular matrix. Form the augmented matrix $M = [A \mid I_{n,n}]$ of dimension $n \times 2n$. Use Gauss-Jordan elimination to reduce the matrix M so that the identity $I_{n,n}$ is in the first n columns. Then the inverse A^{-1} is located in columns $n, n+1, \ldots, 2n$. The augmented matrix $M = [A \mid I_{n,n}]$ looks like:

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & 1 & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & 0 & 1 & 0 & \cdots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & & \vdots & & \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example 1. Use Gauss-Jordan elimination to find the inverse of the matrix $A = \begin{pmatrix} 4 & 8 & 4 & 0 \\ 1 & 4 & 7 & 2 \\ 1 & 5 & 4 & -3 \\ 1 & 3 & 0 & -2 \end{pmatrix}$.

Solution 1.

Example 1. Use Gauss-Jordan elimination to find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 4 & 8 & 4 & 0 \\ 1 & 4 & 7 & 2 \\ 1 & 5 & 4 & -3 \\ 1 & 3 & 0 & -2 \end{pmatrix}$.

Solution 1. Form the augmented matrix $\mathbf{M} = [\mathbf{A}, \mathbf{I}_{4\times4}]$. using the following steps.

$$\mathbf{M} = \begin{pmatrix} 4 & 8 & 4 & 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & 7 & 2 & 0 & 1 & 0 & 0 \\ 1 & 5 & 4 & -3 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & -2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then perform Gauss-Jordan elimination.

$$\begin{pmatrix} 4 & 8 & 4 & 0 & 1 & 0 & 0 & 0 \\ 1 & 4 & 7 & 2 & 0 & 1 & 0 & 0 \\ 1 & 5 & 4 & -3 & 0 & 0 & 1 & 0 \\ 1 & 3 & 0 & -2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 2 & 6 & 2 & -\frac{1}{4} & 1 & 0 & 0 \\ 0 & 3 & 3 & -3 & -\frac{1}{4} & 0 & 1 & 0 \\ 0 & 1 & -1 & -2 & -\frac{1}{4} & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 2 & \frac{5}{12} & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 1 & -1 & -\frac{1}{12} & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 4 & 4 & -\frac{1}{12} & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & -2 & -1 & -\frac{1}{6} & 0 & -\frac{1}{3} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 3 & \frac{19}{48} & \frac{1}{4} & -\frac{5}{6} & 0 \\ 0 & 1 & 0 & -2 & -\frac{1}{16} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{48} & \frac{1}{4} & -\frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & -\frac{5}{24} & \frac{1}{2} & -\frac{2}{3} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{49}{48} & -\frac{5}{4} & \frac{7}{6} & -3 \\ 0 & 1 & 0 & 0 & -\frac{23}{48} & \frac{3}{4} & -\frac{5}{6} & 2 \\ 0 & 0 & 1 & 0 & \frac{3}{16} & -\frac{1}{4} & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 1 & -\frac{5}{24} & \frac{1}{2} & -\frac{2}{3} & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{49}{48} & -\frac{5}{4} & \frac{7}{6} & -3 \\ -\frac{23}{48} & \frac{3}{4} & -\frac{5}{6} & 2 \\ \frac{3}{16} & -\frac{1}{4} & \frac{1}{2} & -1 \\ -\frac{5}{24} & \frac{1}{2} & -\frac{2}{3} & 1 \end{pmatrix}$$