9. Monte Carlo Integration

Background

We will be discussing how to approximate the value of an integral based on the average of function values. The following concept is useful.

Theorem (Mean Value Theorem for Integrals). If $f(x)$ is continuous over $[a, b]$, then there exists a number c, with $a < c < b$, such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c).$$

This can be written in the equivalent form:

$$\int_a^b f(x)\,dx = (b-a) * f(c).$$

Remark. This computation shows that the area under the curve is the base width $(b-a)$ times the "average height" $f(c)$.

Example 1. Let $f(x) = \sin(x) + \frac{1}{3}\sin(3x)$. Find c so that

$$\int_0^{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x)\right)dx = (\pi - 0) * f(c).$$

Solution 1.

Composite Midpoint Rule

An intuitive method of finding the area under a curve $y = f(x)$ is to approximate that area with a series of rectangles that lie above the intervals $\{[x_{k-1}, x_k]\}_{k=1}^{n}$.

Theorem (Composite Midpoint Rule). Consider $y = f(x)$ over $[a, b]$. Let the interval $[a, b]$ be subdivided into n subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{n}$ of equal width $h = \frac{b-a}{n}$. Form the equally spaced nodes $c_k = a + \left(k - \frac{1}{2}\right)h$ for $k = 1, 2, \ldots, n$. The composite midpoint rule for n subintervals is

$$\int_a^b f(x)\,dx \approx h \sum_{k=1}^{n} f(c_k).$$

This can be written in the equivalent form

$$\int_a^b f(x)\,dx \approx (b-a) * \hat{f}, \quad \text{where} \quad \hat{f} = \frac{1}{n} \sum_{k=1}^{n} f(c_k).$$

Corollary (Remainder term for the Midpoint Rule) The composite midpoint rule $M(f,h) = (b-a)\frac{1}{n}\sum_{k=1}^{n} f(c_k)$ is an numerical approximation to the integral, and

$$\int_a^b f(x)\,dx = M(f,h) + E_M(f,h).$$

Furthermore, if $f(x) \in C^2[a,b]$, then there exists a value c with $a < c < b$ so that the error term $E_M(f,h)$ has the form

$$E_M(f,h) = \frac{(b-a)\,f^{2}(c)}{24}\,h^2.$$

This is expressed using the "big O" notation $E_M(f,h) = O(h^2)$.

Algorithm Composite Midpoint Rule. To approximate the integral

$$\int_a^b f(x)\,dx \approx (b-a)\frac{1}{n}\sum_{k=1}^{n} f(c_k),$$

by sampling $f(x)$ at the n equally spaced points $c_k = a + \left(k - \frac{1}{2}\right)h$ for $k = 1, 2, \ldots, n$, where $h = \frac{b-a}{n}$.

Example 2. Let $f(x) = \sqrt{x}$. Use the midpoint rule to calculate approximations to the integral $\int_0^4 \sqrt{x}\,dx$.

Solution 2.

Monte Carlo Method

Monte Carlo methods can be thought of as statistical simulation methods that utilize a sequences of random numbers to perform the simulation. The name "Monte Carlo" was coined by Nicholas

Constantine Metropolis (1915-1999) and inspired by Stanslaw Ulam (1909-1986), because of the similarity of statistical simulation to games of chance, and because Monte Carlo is a center for gambling and games of chance.

Approximation for an Integral

The Monte Carlo method can be used to numerically approximate the value of an integral. For a function of one variable the steps are:

(i) Pick n randomly distributed points $x_1, x_2, x_3, \ldots, x_n$ in the interval $[a, b]$.

(ii) Determine the average value of the function

$$\hat{f} = \frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

(iii) Compute the approximation to the integral

$$\int_a^b f(x)\,dx \approx (b-a)*\hat{f}.$$

(iv) An estimate for the error is

$$\text{Error} \approx (b-a)\sqrt{\frac{\hat{f^2}-(\hat{f})^2}{n}}, \quad \text{where} \quad \hat{f^2} = \frac{1}{n}\sum_{i=1}^{n} f^2(x_i).$$

Every time a Monte Carlo simulation is made using the same sample size it will come up with a slightly different value. Larger values of n will produce more accurate approximations. The values converge very slowly of the order $O(n^{-1/2})$. This property is a consequence of the Central Limit Theorem.

Example 3. Let $f(x) = \sqrt{x+\sqrt{x}}$. Use the Monte Carlo method to calculate approximations to the integral $\int_0^1 \sqrt{x+\sqrt{x}}\,dx$.

Solution 3.

Approximation for a Double Integral

The Monte Carlo method can be used to numerically approximate the value of a double integral. For a function of two variables the steps are:

(i) Pick n randomly distributed points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$ in the rectangle

$[a, b] \times [c, d]$.

(ii) Determine the average value of the function

$$\hat{f} = \frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i) .$$

(iii) Compute the approximation to the integral

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx \approx (b - a) * (d - c) * \hat{f} .$$

(iv) An estimate for the error is

$$\text{Error} \approx (b - a) * (d - c) \sqrt{\frac{\hat{f^2} - (\hat{f})^2}{n}} , \quad \text{where} \quad \hat{f^2} = \frac{1}{n} \sum_{i=1}^{n} f^2(x_i, y_i) .$$

Example 4. Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. Use the Monte Carlo method to calculate approximations to the double integral

$$\int_0^{5/4} \left(\int_0^{5/4} \sqrt{4 - x^2 - y^2}\, dy \right) dx .$$

Solution 4.

Iterated Integrals in Higher Dimensions

Sometimes we are given integrals which cannot be done analytically, especially in higher dimensions where the standard methods of discretization can become computationally expensive. For example, the error in the composite midpoint rule (and the composite trapezoidal rule) of an d-dimensional integral has the order of convergence $O(n^{-2/d})$. We can apply the inequality

$$O(n^{-1/2}) < O(n^{-2/d}) \quad \text{when} \quad d > 4$$

to see that Monte-Carlo integration will usually converge faster for quintuple multiple integrals and higher, i.e. $\int\int\underset{V}{\int}\int\int f\, dw\, du\, dz\, dy\, dx$, etc.

Approximation for a Multiple Integral

The Monte Carlo method can be used to numerically approximate the value of a multiple integrals. For a function of d variables the steps are:

(i) Pick n randomly distributed points $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ in the "volume" $V = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$.

(ii) Determine the average value of the function

$$\hat{f} = \frac{1}{n} \sum_{i=1}^{n} f(\vec{X}_i).$$

(iii) Compute the approximation to the integral

$$\iint_V \ldots \int f(\vec{X})\, dV \approx (b_1 - a_1)(b_2 - a_2) \ldots (b_d - a_d) * \hat{f}.$$

(iv) An estimate for the error is

$$\text{Error} \approx (b_1 - a_1)(b_2 - a_2) \ldots (b_d - a_d) \sqrt{\frac{\hat{f^2} - (\hat{f})^2}{n}}, \quad \text{where} \quad \hat{f^2} = \frac{1}{n} \sum_{i=1}^{n} f^2(\vec{X}_i).$$

Example 5. Let $f(x, y, z) = 4 - x^2 - y^2 - z^2$. Use the Monte Carlo method to calculate approximations to the triple integral

$$\int_0^{9/10} \left(\int_0^1 \left(\int_0^{11/10} (4 - x^2 - y^2 - z^2)\, dz \right) dy \right) dx.$$

Solution 5.

Example 1. Let $f(x) = \sin(x) + \frac{1}{3}\sin(3x)$. Find c so that

$$\int_0^{\pi} \left(\sin(x) + \frac{1}{3}\sin(3x)\right) dx = (\pi - 0) * f(c).$$

Solution 1.

$$f[x] = \text{Sin}[x] + \frac{1}{3}\text{Sin}[3x]$$

$$\int f[x]\,dx = -\text{Cos}[x] - \frac{1}{9}\text{Cos}[3x]$$

$$\int_0^{\pi} f[x]\,dx = \int_0^{\pi}(\text{Sin}[x] + \frac{1}{3}\text{Sin}[3x])\,dx = \frac{20}{9}$$

$$\frac{1}{\pi}\int_0^{\pi} f[x]\,dx = \int_0^{\pi}(\text{Sin}[x] + \frac{1}{3}\text{Sin}[3x])\,dx = \frac{20}{9\pi}$$

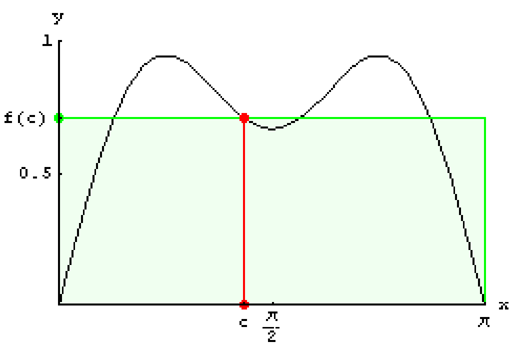
Solve $f[c] = \frac{1}{\pi}\int_0^{\pi} f[x]\,dx$ for c

$$f[c] = \frac{20}{9\pi}$$

$$\text{Sin}[c] + \frac{1}{3}\text{Sin}[3c] == \frac{20}{9\pi}$$

$$c = \text{ArcSin}\left[\frac{9\pi}{\left(\frac{1}{2}\left(-9720\pi^2 + i\sqrt{-94478400\pi^4 + 17006112\pi^6}\right)\right)^{1/3}} + \frac{\left(\frac{1}{2}\left(-9720\pi^2 + i\sqrt{-94478400\pi^4 + 17006112\pi^6}\right)\right)^{1/3}}{18\pi}\right]$$

$$c = 1.36432$$



$$y = f[x] = \text{Sin}[x] + \frac{1}{3}\text{Sin}[3x]$$

The value of the integral is over $[0,\pi]$ is

$$\int_0^\pi f[x]dx = \int_0^\pi (\text{Sin}[x] + \frac{1}{3}\text{Sin}[3x])dx$$

$$\int_0^\pi f[x]dx = \frac{20}{9} = 2.22222222222222$$

The average height of the function is

$$\hat{f} = f[c] = \frac{1}{\pi}\int_0^\pi f[x]dx = \frac{1}{\pi}\int_0^\pi (\text{Sin}[x] + \frac{1}{3}\text{Sin}[3x])dx = 0.707355$$

$$c = 1.36432$$

$$\hat{f} = f[c] = f[1.36432] = 0.707355$$

The area under the curve is the same as

the area of the rectangle of height $f[c] = 0.707355$

$$\int_0^\pi f[x]dx = (b - a)*\hat{f}$$

$$\int_0^\pi f[x]dx = (b - a)*f[c]$$

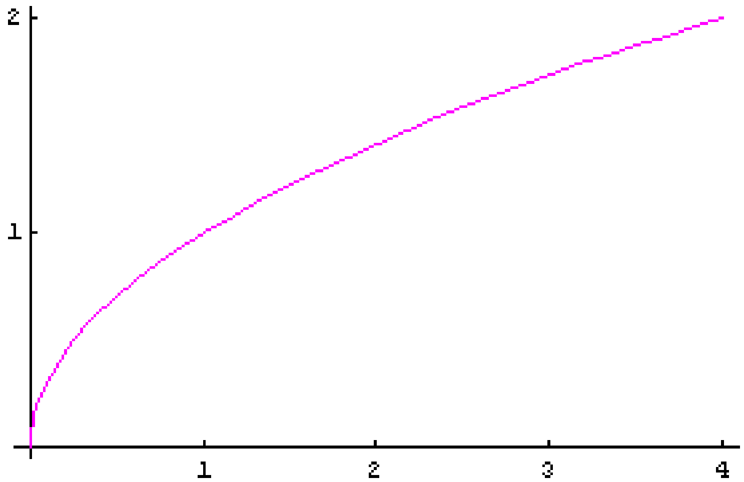
$$\int_0^\pi f[x]dx = (\pi - 0)*f[1.36432]$$

$$\int_0^\pi f[x]dx = (\pi)*0.7073553026306458$$

$$\int_0^\pi f[x]dx = 2.22222222222222$$

Example 2. Let $f(x) = \sqrt{x}$. Use the midpoint rule to calculate approximations to the integral $\int_0^4 \sqrt{x}\, dx$.

Solution 2.



$y = f[x] = \sqrt{x}$

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val1 = MidpointRuleResults[10];
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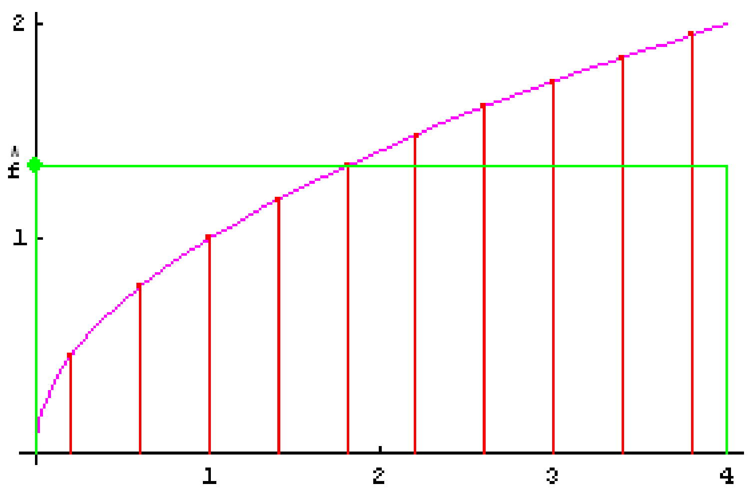


Points Generated n = 10

The average of $\{f[x_i]\}_{i=1}^{10}$ is

$$\hat{f} = \frac{1}{10}\sum_{i=1}^{10} f[x_i] = 1.33677$$

Approximation for the integral

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i]\right)$$

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\frac{1}{10}\sum_{i=1}^{10} f[x_i]$$

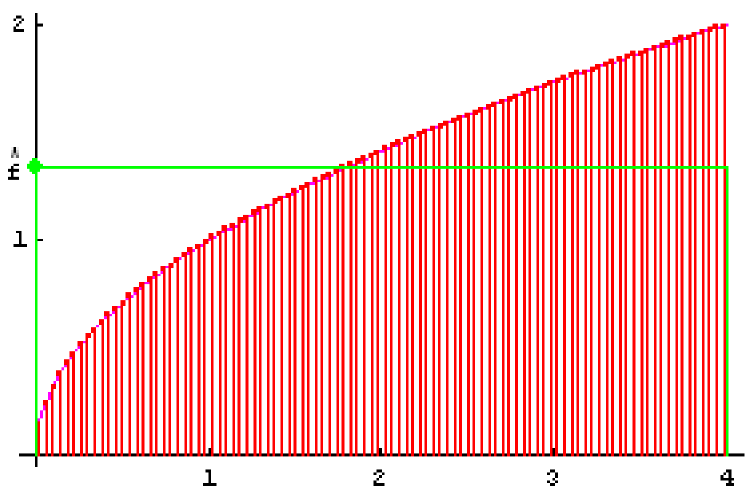
$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\hat{f}$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4-0)*(1.33677)$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4)*(1.33677)$$

$$\int_0^4 (\sqrt{x})\,dx \approx 5.34707$$

```
val2 = MidpointRuleResults[100];
```



Points Generated n = 100

The average of $\{f[x_i]\}_{i=1}^{100}$ is

$$\hat{f} = \frac{1}{100}\sum_{i=1}^{100} f[x_i] = 1.33345$$

Approximation for the integral

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i]\right)$$

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\frac{1}{100}\sum_{i=1}^{100} f[x_i]$$

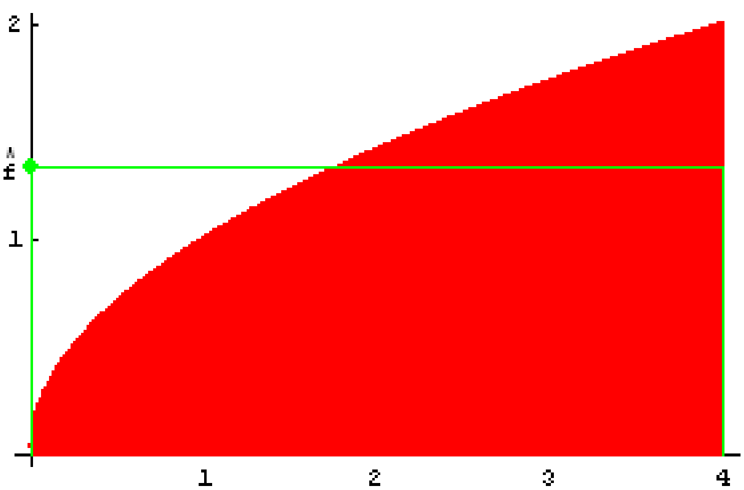
$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\hat{f}$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4 - 0)*(1.33345)$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4)*(1.33345)$$

$$\int_0^4 (\sqrt{x})\,dx \approx 5.3338$$

val3 = MidpointRuleResults[1000];



Points Generated n = 1000

The average of $\{f[x_i]\}_{i=1}^{1000}$ is

$$\hat{f} = \frac{1}{1000}\sum_{i=1}^{1000} f[x_i] = 1.33334$$

Approximation for the integral

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*(\frac{1}{n}\sum_{i=1}^{n} f[x_i])$$

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\frac{1}{1000}\sum_{i=1}^{1000} f[x_i]$$

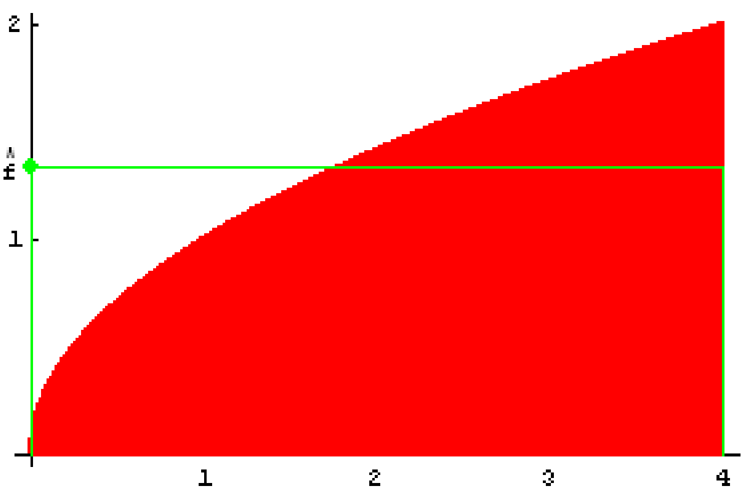
$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\hat{f}$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4 - 0)*(1.33334)$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4)*(1.33334)$$

$$\int_0^4 (\sqrt{x})\,dx \approx 5.33335$$

```
val4 = MidpointRuleResults[10000];
```



Points Generated $n = 10000$

The average of $\{f[x_i]\}_{i=1}^{10000}$ is

$$\hat{f} = \frac{1}{10000}\sum_{i=1}^{10000} f[x_i] = 1.33333$$

Approximation for the integral

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i]\right)$$

$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\frac{1}{10000}\sum_{i=1}^{10000} f[x_i]$$

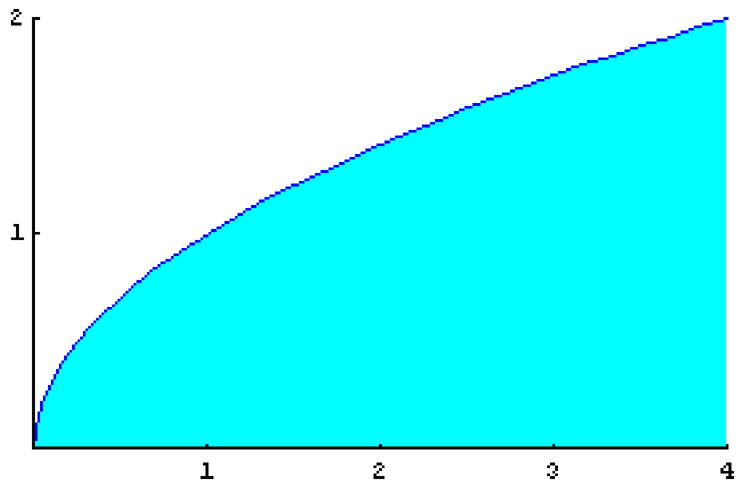
$$\int_0^4 (\sqrt{x})\,dx \approx (b-a)*\hat{f}$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4-0)*(1.33333)$$

$$\int_0^4 (\sqrt{x})\,dx \approx (4)*(1.33333)$$

$$\int_0^4 (\sqrt{x})\,dx \approx 5.33333$$

Aside. The analytic value of the integral can be found.



$$f[x] = \sqrt{x}$$

$$F[x] = \int (\sqrt{x})\,dx$$

$$F[x] = \frac{2x^{3/2}}{3}$$

$$\int_0^4 (\sqrt{x})\,dx = F[4] - F[0]$$

$$\int_0^4 (\sqrt{x})\,dx = \left(\frac{16}{3}\right) - (0)$$

Final answer

$$\int_0^4 (\sqrt{x})\,dx = \frac{16}{3}$$

$$\int_0^4 (\sqrt{x})\,dx = 5.333333333$$

The approximations obtained with the midpoint rule:

Using n = 10 area ≈ 5.347070729

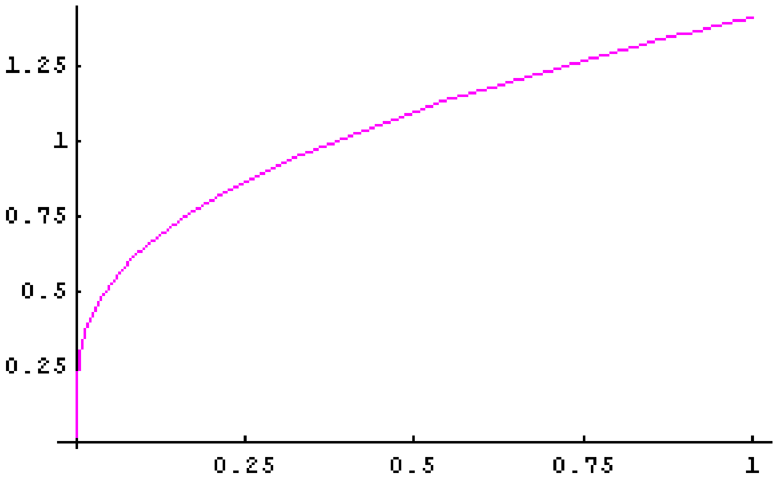
Using n = 100 area ≈ 5.333803774

Using n = 1000 area ≈ 5.33334857

Using n = 10000 area ≈ 5.333333819

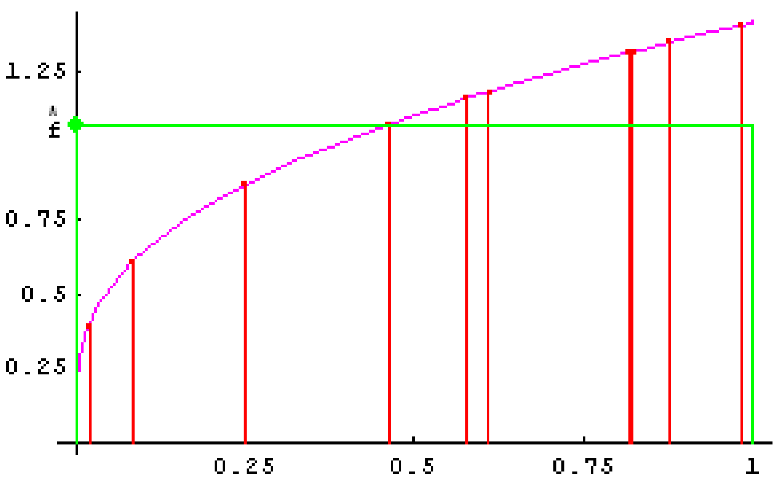
Example 3. Let $f(x) = \sqrt{x + \sqrt{x}}$. Use the Monte Carlo method to calculate approximations to the integral $\int_0^1 \sqrt{x + \sqrt{x}}\, dx$.

Solution 3.



$y = f[x] = \sqrt{\sqrt{x} + x}$

```
val1 = MonteCarloResults1D[10];
```



Points Generated $n = 10$

The average of $\{f[x_i]\}_{i=1}^{10}$ is

$$\hat{f} = \frac{1}{10}\sum_{i=1}^{10} f[x_i] = 1.06496$$

Approximation for the integral

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i]\right)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\frac{1}{10}\sum_{i=1}^{10} f[x_i]$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\hat{f}$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1 - 0)*(1.06496)$$

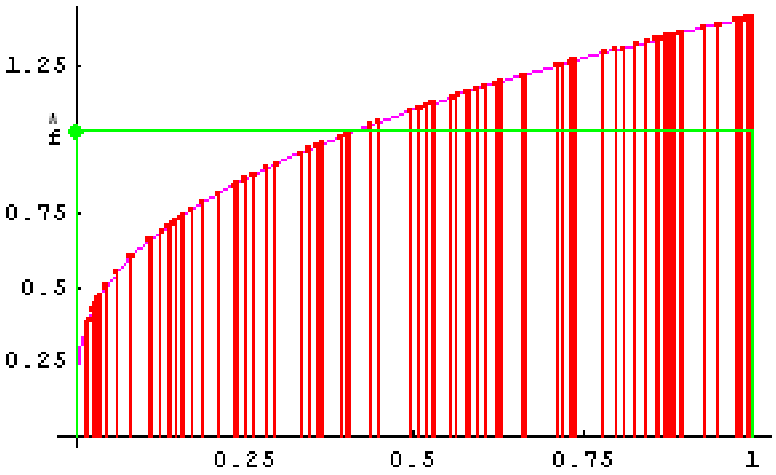
$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1)*(1.06496)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx 1.06496$$

The 'error estimate' $\approx$ 0.101991

Actual |Area-approx| $\approx$ 0.0196588

```
val2 = MonteCarloResults1D[100];
```



Points Generated n = 100

The average of $\{f[x_i]\}_{i=1}^{100}$ is

$$\hat{f} = \frac{1}{100}\sum_{i=1}^{100} f[x_i] = 1.02576$$

Approximation for the integral

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i]\right)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\frac{1}{100}\sum_{i=1}^{100} f[x_i]$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\hat{f}$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1 - 0)*(1.02576)$$

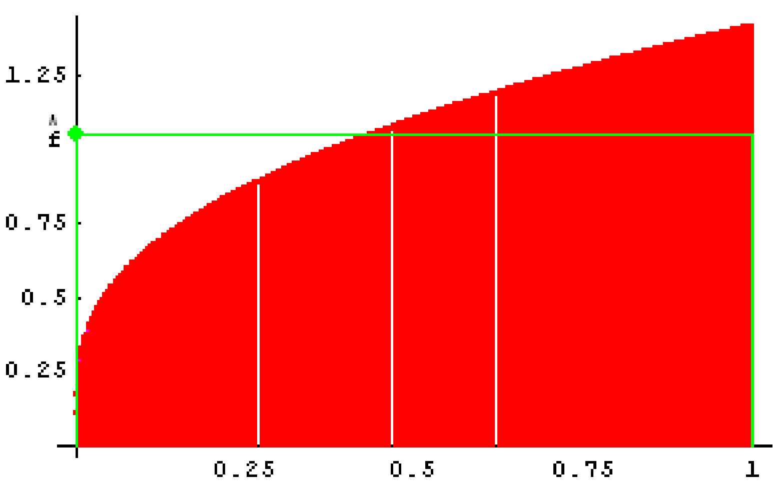
$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1)*(1.02576)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx 1.02576$$

The 'error estimate' ≈ 0.0301419

Actual |Area-approx| ≈ 0.019539

val3 = MonteCarloResults1D[1000];



Points Generated $n = 1000$

The average of $\{f[x_i]\}_{i=1}^{1000}$ is

$$\hat{f} = \frac{1}{1000}\sum_{i=1}^{1000} f[x_i] = 1.05106$$

Approximation for the integral

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i]\right)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\frac{1}{1000}\sum_{i=1}^{1000} f[x_i]$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\hat{f}$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1 - 0)*(1.05106)$$

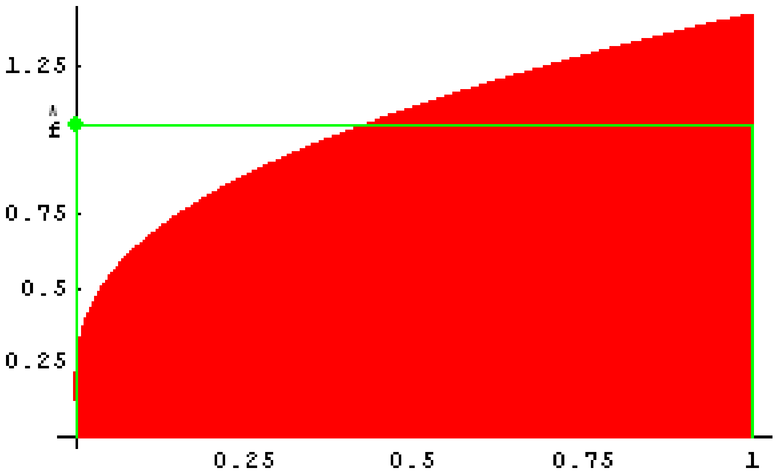
$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1)*(1.05106)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx 1.05106$$

The 'error estimate' $\approx$ 0.00863834

Actual |Area-approx| $\approx$ 0.00575848

val4 = MonteCarloResults1D[10000];



Points Generated $n = 10000$

The average of $\{f[x_i]\}_{i=1}^{10000}$ is

$$\hat{f} = \frac{1}{10000}\sum_{i=1}^{10000} f[x_i] = 1.04646$$

Approximation for the integral

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i]\right)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\frac{1}{10000}\sum_{i=1}^{10000} f[x_i]$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (b-a)*\hat{f}$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1 - 0)*(1.04646)$$

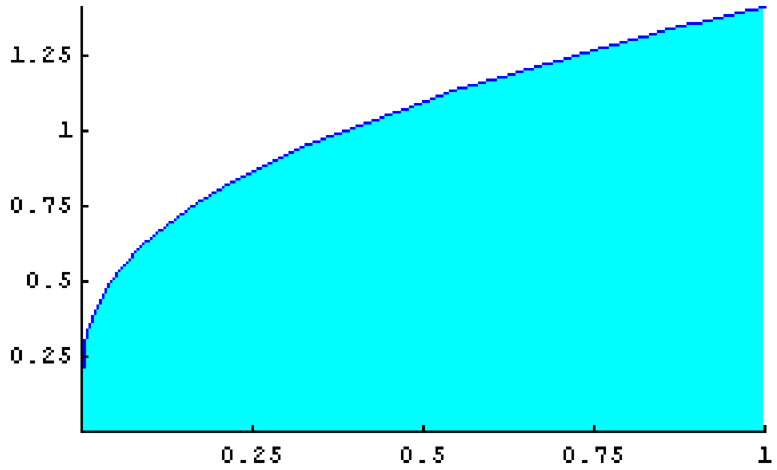
$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx (1)*(1.04646)$$

$$\int_0^1 \left(\sqrt{\sqrt{x} + x}\right)dx \approx 1.04646$$

The 'error estimate' $\approx$ 0.00272404

Actual |Area-approx| $\approx$ 0.00115429

Aside. The analytic value of the integral can be found.



$$f[x] = \sqrt{\sqrt{x}+x}$$

$$F[x] = \int (\sqrt{\sqrt{x}+x})\,dx$$

$$F[x] = \frac{\sqrt{\sqrt{x}+x}\left(-\frac{x^{1/4}}{4}+\frac{x^{3/4}}{6}+\frac{2x^{5/4}}{3}\right)}{x^{1/4}} + \frac{\sqrt{\sqrt{x}+x}\ \mathrm{ArcSinh}[x^{1/4}]}{4\sqrt{1+\sqrt{x}}\ x^{1/4}}$$

$$\int_0^1 (\sqrt{\sqrt{x}+x})\,dx = F[1] - F[0]$$

$$\int_0^1 (\sqrt{\sqrt{x}+x})\,dx = \left(\frac{7}{6\sqrt{2}} + \frac{\mathrm{ArcSinh}[1]}{4}\right) - (0)$$

Final answer

$$\int_0^1 (\sqrt{\sqrt{x}+x})\,dx = \frac{7}{6\sqrt{2}} + \frac{\mathrm{ArcSinh}[1]}{4}$$

$$\int_0^1 (\sqrt{\sqrt{x}+x})\,dx = 1.045301308$$

The approximations obtained with Monte Carlo simulation:

Using n = 10 area ≈ 1.064960098

Using n = 100 area ≈ 1.025762291

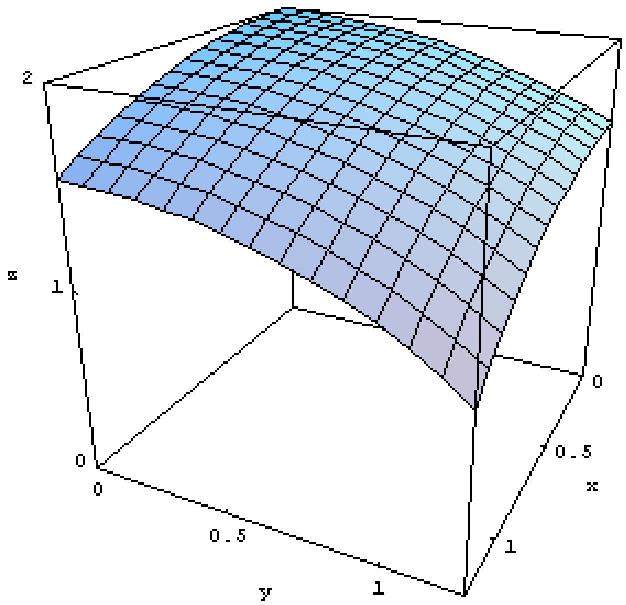
Using n = 1000 area ≈ 1.051059788

Using n = 10000 area ≈ 1.0464556

Example 4. Let $f(x, y) = \sqrt{4 - x^2 - y^2}$. Use the Monte Carlo method to calculate approximations to the double integral

$$\int_0^{5/4}\left(\int_0^{5/4}\sqrt{4-x^2-y^2}\,dy\right)dx.$$

Solution 4.



$$z = f[\{x,y\}] = \sqrt{4-x^2-y^2}$$

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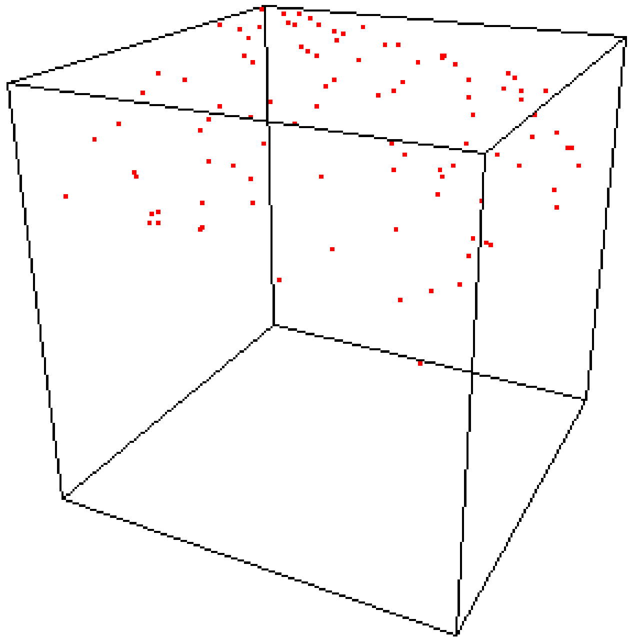
val1 = MonteCarloResults2D[100];
```



Points Generated $n = 100$

The average of $\{f[x_i]\}_{i=1}^{100}$ is

$$\hat{f} = \frac{1}{100}\sum_{i=1}^{100} f[x_i] = 1.71156$$

Approximation for the integral

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx (b-a)*(d-c)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i, y_i]\right)$$

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx (b-a)*(d-c)*\frac{1}{100}\sum_{i=1}^{100} f[x_i]$$

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx (b-a)*(d-c)*\hat{f}$$

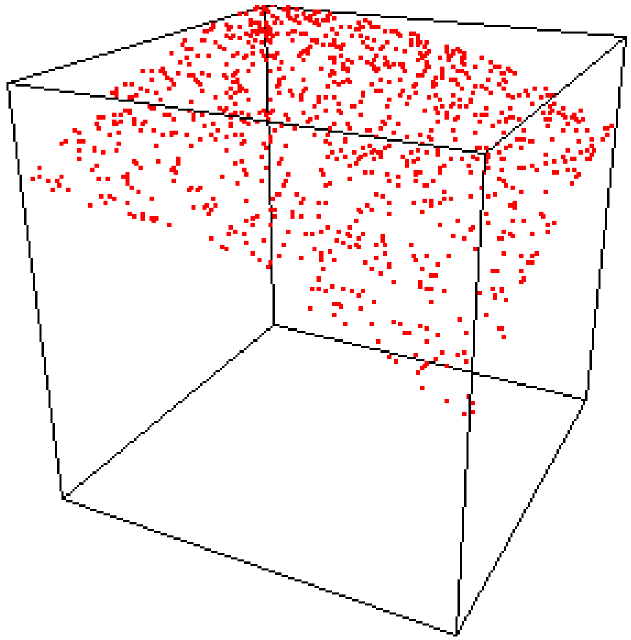
$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx \left(\frac{25}{16}\right)*(1.71156)$$

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx 2.67432$$

The 'error estimate' $\approx$ 0.0299215

Actual |Volume-approx| $\approx$ 0.00526343

val2 = MonteCarloResults2D[1000];



Points Generated $n = 1000$

The average of $\{f[x_i]\}_{i=1}^{1000}$ is

$$\hat{f} = \frac{1}{1000}\sum_{i=1}^{1000} f[x_i] = 1.69926$$

Approximation for the integral

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx \approx (b-a)*(d-c)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i, y_i]\right)$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx \approx (b-a)*(d-c)*\frac{1}{1000}\sum_{i=1}^{1000} f[x_i]$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx \approx (b-a)*(d-c)*\hat{f}$$

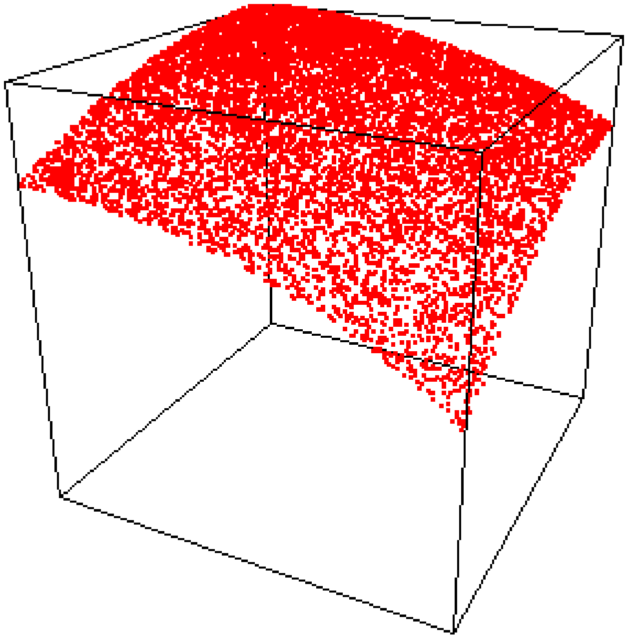
$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx \approx \left(\frac{25}{16}\right)*(1.69926)$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx \approx 2.6551$$

The 'error estimate' $\approx$ 0.0101795

Actual |Volume-approx| $\approx$ 0.0139563

```
val3 = MonteCarloResults2D[10000];
```



Points Generated $n = 10000$

The average of $\{f[x_i]\}_{i=1}^{10000}$ is

$$\hat{f} = \frac{1}{10000}\sum_{i=1}^{10000} f[x_i] = 1.70826$$

Approximation for the integral

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx (b-a)*(d-c)*\left(\frac{1}{n}\sum_{i=1}^{n} f[x_i,y_i]\right)$$

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx (b-a)*(d-c)*\frac{1}{10000}\sum_{i=1}^{10000} f[x_i]$$

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx (b-a)*(d-c)*\hat{f}$$

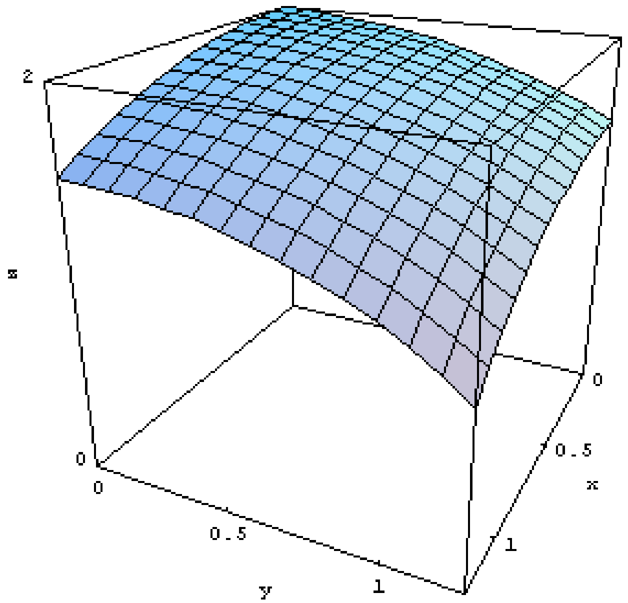
$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx \left(\frac{25}{16}\right)*(1.70826)$$

$$\int_0^{\frac{5}{4}}\left(\int_0^{\frac{5}{4}}\left(\sqrt{4-x^2-y^2}\right)dy\right)dx \approx 2.66916$$

The 'error estimate' $\approx$ 0.00314594

Actual |Volume-approx| $\approx$ 0.0001046

Aside. The analytic value of the double integral can be found.


2
z
0
0
0.5
y
1
0
0.5
x
1

$$z = f[\{x,y\}] = \sqrt{4 - x^2 - y^2}$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx = \int_0^{\frac{5}{4}} \left(\left(\frac{1}{2}\, y \sqrt{4 - x^2 - y^2} - \frac{1}{2}\,(4 - x^2)\, \text{ArcTan}\left[\frac{y\sqrt{4 - x^2 - y^2}}{-4 + x^2 + y^2}\right]\right) \Big|_{y==0}^{y==\frac{5}{4}} \right) dx$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx = \int_0^{\frac{5}{4}} \left(\frac{1}{2}\left(\frac{5}{4}\sqrt{\frac{39}{16} - x^2} - 4\, \text{ArcTan}\left[\frac{5\sqrt{\frac{39}{16} - x^2}}{4\left(-\frac{39}{16} + x^2\right)}\right] + x^2\, \text{ArcTan}\left[\frac{5\sqrt{\frac{39}{16} - x^2}}{4\left(-\frac{39}{16} + x^2\right)}\right]\right)\right) dx$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx = \left(\frac{1}{2}\left(\frac{5}{6}\, x \sqrt{\frac{39}{16} - x^2} + \frac{835}{192}\, \text{ArcSin}\left[\frac{4x}{\sqrt{39}}\right] + \right.\right.$$

$$\left.\left. \frac{1}{3}\, x\,(-12 + x^2)\, \text{ArcTan}\left[\frac{5\sqrt{\frac{39}{16} - x^2}}{4\left(-\frac{39}{16} + x^2\right)}\right] + \frac{8}{3}\, \text{ArcTan}\left[\frac{4\,(-39 + 32x)\sqrt{\frac{39}{16} - x^2}}{5\,(-39 + 16x^2)}\right] + \frac{8}{3}\, \text{ArcTan}\left[\frac{4\,(39 + 32x)\sqrt{\frac{39}{16} - x^2}}{5\,(-39 + 16x^2)}\right]\right)\right) \Big|_{x==0}^{x==\frac{5}{4}}$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx = \frac{1}{384}\left(50\sqrt{14} + 835\, \text{ArcSin}\left[\frac{5}{\sqrt{39}}\right] - 512\, \text{ArcTan}\left[\frac{1}{5\sqrt{14}}\right] + 835\, \text{ArcTan}\left[\frac{5}{\sqrt{14}}\right] - 512\, \text{ArcTan}\left[\frac{79}{5\sqrt{14}}\right]\right)$$

$$\int_0^{\frac{5}{4}} \left(\int_0^{\frac{5}{4}} \left(\sqrt{4 - x^2 - y^2}\right) dy\right) dx = 2.66905414$$

The approximations obtained with Monte Carlo simulation:

Using n = 100 volume ≈ 2.674317568

Using n = 1000 volume ≈ 2.65509782

Using n = 10000 volume ≈ 2.669158741

Example 5. Let $f(x, y, z) = 4 - x^2 - y^2 - z^2$. Use the Monte Carlo method to calculate approximations to the triple integral

$$\int_0^{9/10}\left(\int_0^1\left(\int_0^{11/10}(4 - x^2 - y^2 - z^2)\,dz\right)dy\right)dx.$$

Solution 5.

$f[\{x,y,z\}] = 4 - x^2 - y^2 - z^2$

$$\int_0^{9/10}(\int_0^1(\int_0^{11/10}(4-x^2-y^2-z^2)dz)dy)dx = \frac{14817}{5000} = 2.9634$$

Mathematica's approximation using 'NIntegrate' is

$$\int_0^{9/10}(\int_0^1(\int_0^{11/10}(4-x^2-y^2-z^2)dz)dy)dx \approx 2.9634$$

$f[\{x,y,z\}] = 4 - x^2 - y^2 - z^2$

$$\int_0^{9/10}(\int_0^1(\int_0^{11/10}(4-x^2-y^2-z^2)dz)dy)dx = \frac{14817}{5000}$$

Mathematica's approximation using 'NIntegrate' is

$$\int_0^{9/10}(\int_0^1(\int_0^{11/10}(4-x^2-y^2-z^2)dz)dy)dx \approx 2.9634$$

The approximations obtained with Monte Carlo simulation:

Using n = 100 integral $\approx$ 2.937775047

Using n = 1000 integral $\approx$ 2.946929467

Using n = 10000 integral $\approx$ 2.968812089