6. Halley's Method

Background

Definition (Order of a Root) Assume that $f(x)$ and its derivatives $f'(x), \ldots, f^{(m)}(x)$ are defined and continuous on an interval about $x = p$. We say that $f(x) = 0$ has a root of order m at $x = p$ if and only if

$$f(p) = 0, f'(p) = 0, f''(p) = 0, \ldots, f^{(m-1)}(p) = 0, f^{(m)}(p) \neq 0.$$

A root of order $m = 1$ is often called a simple root, and if $m > 1$ it is called a multiple root. A root of order $m = 2.$ is sometimes called a double root, and so on.

Definition (Order of Convergence) Assume that p_n converges to p, and set $E_n = p - p_n$ for $n \geq 0.$ If two positive constants $A \neq 0$ and $R > 0$ exist, and

$$\lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^R} = A,$$

then the sequence is said to converge to p with order of convergence R. The number A is called the asymptotic error constant. The cases $R = 1, 2, 3$ are given special consideration.

(i) If $R = 1$, the convergence of $\{p_k\}_{k=0}^{\infty}$ is called linear.

(ii) If $R = 2$, the convergence of $\{p_k\}_{k=0}^{\infty}$ is called quadratic.

(ii) If $R = 3$, the convergence of $\{p_k\}_{k=0}^{\infty}$ is called cubic.

Halley's Method

The Newton-Raphson iteration function is

(1) $$g(x) = x - \frac{f(x)}{f'(x)}.$$

It is possible to speed up convergence significantly when the root is simple. A popular method is attributed to Edmond Halley (1656-1742) and uses the iteration function:

(2) $$g(x) = x - \frac{f(x)}{f'(x)} \left[1 - \frac{f(x)f''(x)}{2(f'(x))^2}\right]^{-1},$$

The term in brackets shows where Newton-Raphson iteration function is changed.

Theorem (Halley's Iteration). Assume that $f \in C^3[a, b]$ and there exists a number $p \in [a, b]$, where $f(p) = 0$. If $f'(p) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$p_{k+1} = p_k - \frac{f[p_k]}{f'[p_k]}\left(1 - \frac{f[p_k]\,f''[p_k]}{2\,(f'[p_k])^2}\right)^{-1} \quad \text{for} \quad k = 0, 1, \ldots$$

will converge to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Furthermore, if p is a simple root, then $\{p_{k+1}\}$ will have order of convergence $R = 3$, i.e.
$\lim_{n\to\infty} \frac{|p - p_{n+1}|}{|p - p_n|^3} = \lim_{n\to\infty} \frac{|E_{n+1}|}{|E_n|^3} = A$.

Square Roots

The function $f(x) = x^2 - a$ where $a > 0$ can be used with (1) and (2) to produce iteration formulas for finding $\sqrt{a}$. If it is used in (1), the result is the familiar Newton-Raphson formula for finding square roots:

(3) $$g(x) = x - \frac{x^2 - a}{2x}.$$

When it is used in (2) the resulting Halley formula is:

$$g(x) = x - \left(\frac{x^2 - a}{2x}\right)\left(1 - \frac{(x^2 - a)\,2}{2\,(2x)^2}\right)^{-1}$$

(4) or

$$g(x) = \frac{x^3 + 3ax}{3x^2 + a}$$

This latter formula is a third-order method for computing $\sqrt{a}$. Because of the rapid convergence of the sequences generated by (3) and (4), the iteration usually converges to machine accuracy in a few iterations. Multiple precision arithmetic is needed to demonstrate the distinction between second and third order convergence.

Example 1. Consider the function $f(x) = x^3 - 3x + 2$, which has a root at $x = -2$.
1 (a). Use the Newton-Raphson formula to find the root. Use the starting value $p_0 = -2.2$
1 (b). Use Halley's formula to find the root. Use the starting value $p_0 = -2.2$
Solution 1.

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Solution 1(a).

Form the Newton-Raphson iteration function $g(x)$.

$$f[x] = 2 - 3x + x^3$$

$$g[x] = x - \frac{f[x]}{f'[x]}$$

$$g[x] = x - \frac{2 - 3x + x^3}{-3 + 3x^2}$$

$$g[x] = \frac{2(1 + x + x^2)}{3(1 + x)}$$

We start the iteration with $p_0 = -2.2$.

k	p_k	$E_k = p - p_k$	$\frac{\lvert E_{k+1} \rvert}{(\lvert E_k \rvert)^2}$
0	-2.2000000000000000	0.20000000000000000	0.5555555555555556
1	-2.0222222222222222	0.022222222222222222	0.6521739130434783
2	-2.0003220611916264	0.00032206119162640902	0.6664520283322601
3	-2.0000000691266777	$6.9126677747673110 \times 10^{-8}$	0.6666666205822180
4	-2.0000000000000032	$3.1856648307393369 \times 10^{-15}$	0.6666666666666645
5	-2.0000000000000000	$6.7656402758729706 \times 10^{-30}$	0.6666666666666667
6	-2.0000000000000000	$3.0515925561676324 \times 10^{-59}$	

Verify the convergence rate. At the simple root $p = -2$ we can explore the ratio $\frac{|E_{k+1}|}{(|E_k|)^2}$.
Therefore, the Newton-Raphson iteration is converging quadratically.

Solution 1(b).

Form the Halley iteration function $h(x)$.

$$f[x] = 2 - 3x + x^3$$

$$h[x] = x - \frac{f[x]}{f'[x]}\left(1 - \frac{f[x]\, f''[x]}{2\,(f'[x])^2}\right)^{-1}$$

$$h[x] = x - \frac{2 - 3x + x^3}{(-3 + 3x^2)\left(1 - \frac{3x(2-3x+x^3)}{(-3+3x^2)^2}\right)}$$

$$h[x] = \frac{2 + 4x + 2x^2 + x^3}{3 + 4x + 2x^2}$$

We start the iteration with $p_0 = -2.2$.

k	p_k	$E_k = p - p_k$	$\frac{|E_{k+1}|}{(|E_k|)^3}$
0	-2.2000000000000000	3.6142135623730950	0.072362146701574997
1	-2.0020618556701031	3.4162754180431981	0.085631205027722069
2	-2.0000000029138018	3.4142135652868968	0.085786437407266078
3	-2.0000000000000000	3.4142135623730950	0.085786437626904951
4	-2.0000000000000000	3.4142135623730950	

Therefore, since the Halley iteration is converging cubically, we can conclude that Halley's method is faster than Newton's method.