4. Simpson's Rule for Numerical Integration

The numerical integration technique known as "Simpson's Rule" is credited to the mathematician Thomas Simpson (1710-1761) of Leicestershire, England. He also worked in the areas of numerical interpolation and probability theory.

Theorem (Simpson's Rule) Consider $y = f(x)$ over $[x_0, x_2]$, where $x_1 = x_0 + h$, and $x_2 = x_0 + 2h$. Simpson's rule is

$$SR(f, h) = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)).$$

This is an numerical approximation to the integral of $f(x)$ over $[x_0, x_2]$ and we have the expression

$$\int_{x_0}^{x_2} f(x)\,dx \approx SR(f, h).$$

The remainder term for Simpson's rule is $R_{SR}(f, h) = -\frac{1}{90} f^{(4)}(c)\,h^5$, where c lies somewhere between x_0 and x_2, and have the equality

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2)) - \frac{1}{90} f^{(4)}(c)\,h^5.$$

Composite Simpson Rule

Our next method of finding the area under a curve $y = f(x)$ is by approximating that curve with a series of parabolic segments that lie above the intervals $\{[x_{k-1}, x_k]\}_{k=1}^{2m}$. When several parabolas are used, we call it the composite Simpson rule.

Theorem (Composite Simpson's Rule) Consider $y = f(x)$ over $[a, b]$. Suppose that the interval $[a, b]$ is subdivided into $2m$ subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{2m}$ of equal width $h = \frac{b-a}{2m}$ by using the equally spaced sample points $x_k = x_0 + kh$ for $k = 0, 1, 2, \ldots, 2m$. The composite Simpson's rule for $2m$ subintervals is

$$S(f, h) = \frac{h}{3}(f(a) + f(b)) + \frac{2h}{3}\sum_{k=1}^{m-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{m} f(x_{2k-1}).$$

is an numerical approximation to the integral, and

$$\int_a^b f(x)\,dx = S(f, h) + E_S(f, h).$$

Furthermore, if $f(x) \in C^4[a, b]$, then there exists a value c with $a < c < b$ so that the error term $E_3(f, h)$ has the form

$$E_3(f, h) = -\frac{(b-a) f^4(c)}{180} h^4.$$

This is expressed using the "big O" notation $E_3(f, h) = O(h^4)$.

Remark. When the step size is reduced by a factor of $\frac{1}{2}$ the remainder term $E_3(f, h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^4 = 0.0625$.

Example 1. Let f[x] be $\int_0^2 \left(2 + \operatorname{Cos}\left[2\sqrt{x}\right]\right) dx$.

1 (a) Numerically approximate the integral by using Simpson's rule with m = 1, 2, 4, and 8.
1 (b) Find the analytic value of the integral (i.e. find the "true value").
1 (c) Find the error for the Simpson rule approximation.
Solution 1.

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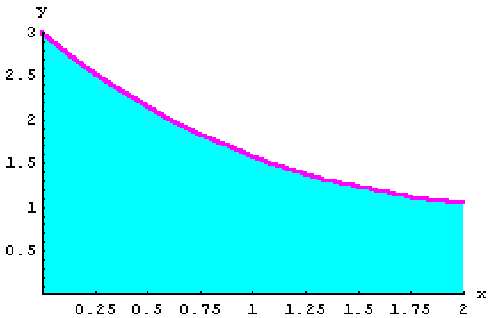
Solution 1 (a).



$f[x] = 2 + \text{Cos}\left[2\sqrt{x}\right]$

We will use simulated hand computations for the solution.

$$\mathbf{f[x\_] = 2 + Cos\left[2\sqrt{x}\right];}$$

$$\mathbf{s1} = \frac{\frac{2-0}{2}}{3}\,(\mathbf{f[0] + 4\,f[1] + f[2]})$$

NumberForm[N[s1], 12]

$$\frac{1}{3}\left(5 + 4\,(2 + \text{Cos}[2]) + \text{Cos}\left[2\sqrt{2}\right]\right)$$

3.4613498419

$$\mathbf{s2} = \frac{\frac{2-0}{4}}{3}\left(\mathbf{f[0] + 4\,f\left[\frac{1}{2}\right] + 2\,f[1] + 4\,f\left[\frac{3}{2}\right] + f[2]}\right)$$

NumberForm[N[s2] , 12]

$$\frac{1}{6}\left(5 + 2\,(2 + \text{Cos}[2]) + 4\left(2 + \text{Cos}\left[\sqrt{2}\right]\right) + \text{Cos}\left[2\sqrt{2}\right] + 4\left(2 + \text{Cos}\left[\sqrt{6}\right]\right)\right)$$

3.46008250981

$$\mathbf{s4} = \frac{\frac{2-0}{8}}{3}\left(\mathbf{f[0] + 4\,f\left[\frac{1}{4}\right] + 2\,f\left[\frac{1}{2}\right] + 4\,f\left[\frac{3}{4}\right] + 2\,f[1] + 4\,f\left[\frac{5}{4}\right] + 2\,f\left[\frac{3}{2}\right] + 4\,f\left[\frac{7}{4}\right] + f[2]}\right)$$

NumberForm[N[s4], 12]

$$\frac{1}{12}\left(5 + 4\,(2 + \text{Cos}[1]) + 2\,(2 + \text{Cos}[2]) + 2\left(2 + \text{Cos}\left[\sqrt{2}\right]\right) + \text{Cos}\left[2\sqrt{2}\right] + 4\left(2 + \text{Cos}\left[\sqrt{3}\right]\right) + 4\left(2 + \text{Cos}\left[\sqrt{5}\right]\right) + 2\left(2 + \text{Cos}\left[\sqrt{6}\right]\right) + 4\left(2 + \text{Cos}\left[\sqrt{7}\right]\right)\right)$$

3.46000297964

$$s8 = \frac{\frac{2-0}{16}}{3}\left(f[0] + 4 f\left[\frac{1}{8}\right] + 2 f\left[\frac{1}{4}\right] + 4 f\left[\frac{3}{8}\right] + 2 f\left[\frac{1}{2}\right] + 4 f\left[\frac{5}{8}\right] + 2 f\left[\frac{3}{4}\right] + 4 f\left[\frac{7}{8}\right] + 2 f[1] + 4 f\left[\frac{9}{8}\right] + 2 f\left[\frac{5}{4}\right] + 4 f\left[\frac{11}{8}\right] + 2 f\left[\frac{3}{2}\right] + 4 f\left[\frac{13}{8}\right] + 2 f\left[\frac{7}{4}\right] + 4 f\left[\frac{15}{8}\right] + f[2]\right)$$

NumberForm[N[s8], 12]

$$\frac{1}{24}\left(5 + 2\,(2 + \mathrm{Cos}[1]) + 2\,(2 + \mathrm{Cos}[2]) + 4\left(2 + \mathrm{Cos}\left[\sqrt{\frac{3}{2}}\right]\right) + 4\left(2 + \mathrm{Cos}\left[\frac{1}{\sqrt{2}}\right]\right) + 4\left(2 + \mathrm{Cos}\left[\frac{3}{\sqrt{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{2}\right]\right) + \mathrm{Cos}\left[2\sqrt{2}\right] + 4\left(2 + \mathrm{Cos}\left[\sqrt{\frac{5}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{3}\right]\right) + 4\left(2 + \mathrm{Cos}\left[\sqrt{\frac{7}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{5}\right]\right) + 4\left(2 + \mathrm{Cos}\left[\sqrt{\frac{11}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{6}\right]\right) + 4\left(2 + \mathrm{Cos}\left[\sqrt{\frac{13}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{7}\right]\right) + 4\left(2 + \mathrm{Cos}\left[\sqrt{\frac{15}{2}}\right]\right)\right)$$

3.45999800397

Solution 1 (b).

The integral of $f[x] = 2 + \mathrm{Cos}\left[2\sqrt{x}\right]$ can be determined.

$$\int \left(2 + \mathrm{Cos}\left[2\sqrt{x}\right]\right) dx$$

$$2x + \frac{1}{2}\mathrm{Cos}\left[2\sqrt{x}\right] + \sqrt{x}\,\mathrm{Sin}\left[2\sqrt{x}\right]$$

The value of the definite integral

$$val = \int_0^2 \left(2 + \mathrm{Cos}\left[2\sqrt{x}\right]\right) dx$$

$$\frac{7}{2} + \frac{1}{2}\mathrm{Cos}\left[2\sqrt{2}\right] + \sqrt{2}\,\mathrm{Sin}\left[2\sqrt{2}\right]$$

N[val]

3.46

NumberForm[N[val], 17]

3.459997672170804

Solution 1 (c).

val - t16

-0.000000331799196