4. Newton's Method

If $f(x)$, $f'(x)$, and $f''(x)$ are continuous near a root p, then this extra information regarding the nature of $f(x)$ can be used to develop algorithms that will produce sequences $\{p_k\}$ that converge faster to p than either the bisection or false position method. The Newton-Raphson (or simply Newton's) method is one of the most useful and best known algorithms that relies on the continuity of $f'(x)$ and $f''(x)$. The method is attributed to Sir Isaac Newton (1643-1727) and Joseph Raphson (1648-1715).

Theorem (Newton-Raphson Theorem). Assume that $f \in C^2[a, b]$ and there exists a number $p \in [a, b]$, where $f(p) = 0$. If $f'(p) \neq 0$, then there exists a $\delta > 0$ such that the sequence $\{p_k\}_{k=0}^{\infty}$ defined by the iteration

$$p_{k+1} = g(p_k) = p_k - \frac{f(p_k)}{f'(p_k)} \quad \text{for} \quad k = 0, 1, \ldots$$

will converge to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Example 1. Use Newton's method to find the three roots of the cubic polynomial $f[x] = 4x^3 - 15x^2 + 17x - 6$.
Determine the Newton-Raphson iteration formula $g[x] = x - \frac{f[x]}{f'[x]}$ that is used. Show details of the computations for the starting value $p_0 = 3$.
Solution 1.

Definition (Order of a Root) Assume that f(x) and its derivatives $f'(x), \ldots, f^{(m)}(x)$ are defined and continuous on an interval about x = p. We say that f(x) = 0 has a root of order m at x = p if and only if

$$f(p) = 0,\ f'(p) = 0,\ f''(p) = 0,\ \ldots,\ f^{(m-1)}(p) = 0,\ f^{(m)}(p) \neq 0.$$

A root of order m = 1 is often called a **simple root**, and if m > 1 it is called a **multiple root**. A root of order m = 2 is sometimes called a **double root**, and so on. The next result will illuminate these concepts.

Definition (Order of Convergence) Assume that p_n converges to p, and set $E_n = p - p_n$ for $n \geq 0$. If two positive constants $A \neq 0$ and $R > 0$ exist, and

$$\lim_{n \to \infty} \frac{|p - p_{n+1}|}{|p - p_n|^R} = \lim_{n \to \infty} \frac{|E_{n+1}|}{|E_n|^R} = A,$$

then the sequence is said to converge to p with **order of convergence R**. The number A is called the **asymptotic error constant**. The cases $R = 1, 2$ are given special consideration.

(i) If $R = 1$, the convergence of $\{p_k\}_{k=0}^{\infty}$ is called **linear**.

(ii) If $R = 2$, the convergence of $\{p_k\}_{k=0}^{\infty}$ is called **quadratic**.

Theorem (Convergence Rate for Newton-Raphson Iteration) Assume that Newton-Raphson iteration produces a sequence $\{p_k\}_{k=0}^{\infty}$ that converges to the root p of the function $f(x)$.

If p is a simple root, then convergence is quadratic and

$$\left| E_{k+1} \right| \approx \frac{|f''(p)|}{2\,|f'(p)|} (|E_k|)^2$$ for k sufficiently large.

If p is a multiple root of order m, then convergence is linear and

$$\left| E_{k+1} \right| \approx \frac{m-1}{m} \left| E_k \right|$$ for k sufficiently large.

Example 2. Use Newton's method to find the roots of the cubic polynomial $f[x] = x^3 - 3x + 2$.
2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.4$
2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.2$
Solution 2.

Example 3. Fast Convergence Find the solution to $3\,\text{Exp}[x] - 4\,\text{Cos}[x] = 0$. Use the starting approximation $p_0 = 1.0$.
Solution 3.

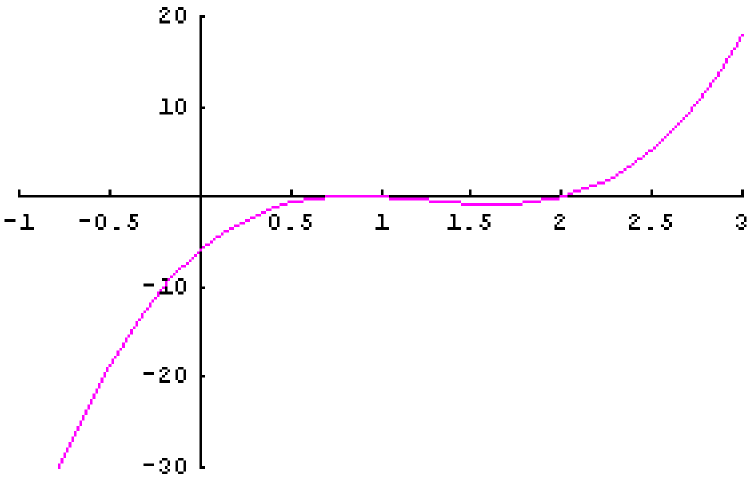
Example 4. NON Convergence, Cycling Find the solution to $x^3 - x + 3 = 0$. Use the starting approximation $p_0 = 0.0$.
Solution 4.

Example 1. Use Newton's method to find the three roots of the cubic polynomial $f[x] = 4x^3 - 15x^2 + 17x - 6$.

Determine the Newton-Raphson iteration formula $g[x] = x - \frac{f[x]}{f'[x]}$ that is used. Show details of the computations for the starting value $p_0 = 3$.

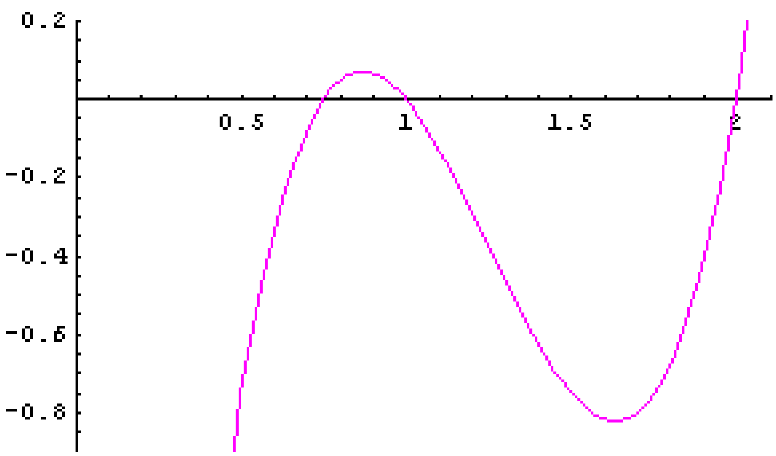
Solution 1.

Graph the function.



$f[x] = -6 + 17x - 15x^2 + 4x^3$

How many real roots are there?



$f[x] = -6 + 17x - 15x^2 + 4x^3$

The Newton-Raphson iteration formula g[x] is

$$g[x] = x - \frac{-6 + 17x - 15x^2 + 4x^3}{17 - 30x + 12x^2}$$

$$g[x] = \frac{6 - 15x^2 + 8x^3}{17 - 30x + 12x^2}$$

Starting with $p_0 = 3$, Use the Newton-Raphson method to find a numerical approximation to the root. First, do the iteration one step at a time.

$p_0 = 3.0000000000000000, \quad f[p_0] = 18.$
$p_1 = 2.4857142857142860, \quad f[p_1] = 5.010192419825074$
$p_2 = 2.1834197620337600, \quad f[p_2] = 1.244567116269891$
$p_3 = 2.0404526629830990, \quad f[p_3] = 0.2172558662514135$
$p_4 = 2.0026544732145300, \quad f[p_4] = 0.01333585694116124$
$p_5 = 2.0000125925878950, \quad f[p_5] = 0.00006296436663433269$
$p_6 = 2.0000000002854240, \quad f[p_6] = 1.427117979346804 \times 10^{-9}$
$p_7 = 2.0000000000000000, \quad f[p_7] = 0.$
$p = 2.$
$\Delta p = \pm 2.85424 \times 10^{-10}$
$f[p] = 0.$

From the second graph we see that there are two other real roots, use the starting values 0.0 and 1.4 to find them.
First, use the starting value $p_0 = 0.0$.

$p_0 = 0.0000000000000000, \quad f[p_0] = -6.$
$p_1 = 0.3529411764705882, \quad f[p_1] = -1.692652147364137$
$p_2 = 0.5670227828549363, \quad f[p_2] = -0.4541102868356983$
$p_3 = 0.6850503150510711, \quad f[p_3] = -0.1075938266370042$
$p_4 = 0.7367776746893981, \quad f[p_4] = -0.01758613258850938$
$p_5 = 0.7492433382396966, \quad f[p_5] = -0.0009492641553616 73$
$p_6 = 0.7499972689032857, \quad f[p_6] = -3.413915646177657 \times 10^{-6}$
$p_7 = 0.7499999999641980, \quad f[p_7] = -4.475264603343021 \times 10^{-11}$
$p_8 = 0.7500000000000001, \quad f[p_8] = -1.110223024625157 \times 10^{-15}$
$p = 0.7500000000000001$
$\Delta p = \pm 3.58021 \times 10^{-11}$
$f[p] = -1.110223024625157 \times 10^{-15}$

Then use the starting value $p_0 = 1.4$.

$p_0 = 1.4000000000000000, \quad f[p_0] = -0.6240000000000006$
$p_1 = 0.9783783783783790, \quad f[p_1] = 0.02017870609835537$
$p_2 = 1.0017155262247010, \quad f[p_2] = -0.001724335120004028$

$$p_3 = 1.0000086994622160, \quad f[p_3] = -8.69968925432119 \times 10^{-6}$$
$$p_4 = 1.0000000002270260, \quad f[p_4] = -2.27026397681129 3\times 10^{-10}$$
$$p_5 = 1.0000000000000000, \quad f[p_5] = 8.88178419700125 \times 10^{-16}$$
$$p = 1.$$
$$\Delta p = \pm 2.27026 \times 10^{-10}$$
$$f[p] = 8.88178419700125 \times 10^{-16}$$

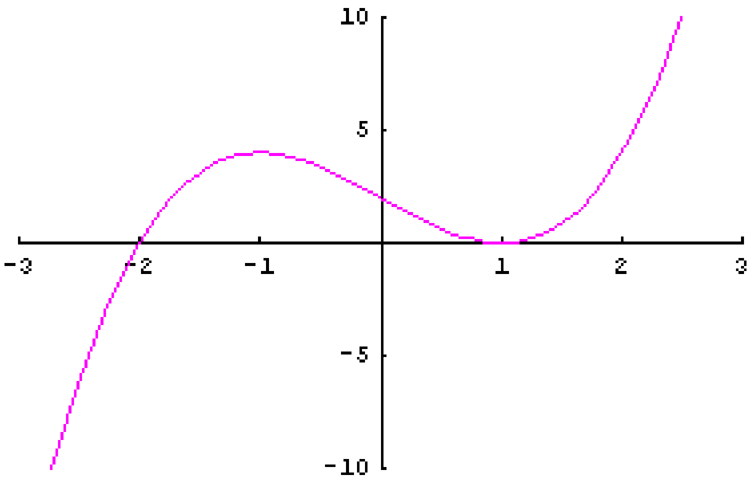
Real roots can also be found symbolically.

$$f[x] = (-2 + x)(-1 + x)(-3 + 4x)$$

$$x \to \frac{3}{4}$$
$$x \to 1$$
$$x \to 2$$

Example 2. Use Newton's method to find the roots of the cubic polynomial $f[x] = x^3 - 3x + 2$.
2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.4$
2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.2$
Solution 2.

Graph the function.



$$f[x] = 2 - 3x + x^3$$

The Newton-Raphson iteration formula g[x] is

$$g[x] = x - \frac{2 - 3x + x^3}{-3 + 3x^2}$$
$$g[x] = \frac{2(1 + x + x^2)}{3(1 + x)}$$

2 (a) Fast Convergence. Investigate quadratic convergence at the simple root $p = -2$, using the starting value $p_0 = -2.4$
First, do the iteration one step at a time.
Notice that convergence is fast and the sequence is converging to the simple root $p = -2$

$p_0 = -2.4000000000000000$, $f[p_0] = -4.623999999999999$
$p_1 = -2.0761904761904760$, $f[p_1] = -0.7209865025375244$
$p_2 = -2.0035960106756570$, $f[p_2] = -0.03244173033865572$
$p_3 = -2.0000085899722210$, $f[p_3] = -0.00007731019271695061$
$p_4 = -2.0000000000491910$, $f[p_4] = -4.427214150837244 \times 10^{-10}$

$p_5 = -2.000000000000000, \quad f[p_5] = 0.$
$p_6 = -2.000000000000000, \quad f[p_6] = 0.$
$p_7 = -2.000000000000000, \quad f[p_7] = 0.$
$p = -2.$
$\Delta p = \pm 0.$
$f[p] = 0.$

At the simple root $p = -2$ we can explore the relationship
$\left| E_{k+1} \right| \approx \frac{|f''(p)|}{2|f'(p)|}(|E_k|)^2$ for k sufficiently large.

This will be done by investigating the ratio $\frac{|E_{k+1}|}{(|E_k|)^2} \approx \frac{|f''(p)|}{2|f'(p)|}$ for k sufficiently large.

k	p_k	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{(\lvert E_k\rvert)^2}$
0	-2.4	0.4	0.476190476190475
1	-2.07619047619048	0.0761904761904759	0.619469026548684
2	-2.00359601067566	0.00359601067565674	0.664277916217952
3	-2.00000858997222	$8.5899722215288 \times 10^{-6}$	0.666654809469126
4	-2.0000000004919	$4.91908735966717 \times 10^{-11}$	

Evaluate the quantity $\frac{|f''(p)|}{2|f'(p)|}$ at the root $p = -2$.

Which is close to the computed value $\frac{|E_3|}{(|E_3|)^2} = 0.666654809469126$

2 (b) Slow Convergence. Investigate linear convergence at the double root $p = 1$, using the starting value $p_0 = 1.2$
First, do the iteration one step at a time.
Notice that convergence is slow, but the sequence is converging to the double root $p = 1$

$p_0 = 1.200000000000000, \quad f[p_0] = 0.1280000000000001$
$p_1 = 1.103030303030303030, \quad f[p_1] = 0.03293942176586806$
$p_2 = 1.052356417197916 0, \quad f[p_2] = 0.00836710238417515$
$p_3 = 1.026400814055368 0, \quad f[p_3] = 0.002109410394502964$
$p_4 = 1.013257733871906 0, \quad f[p_4] = 0.0005296328010917506$
$p_5 = 1.006643417772674 0, \quad f[p_5] = 0.0001326982063480919$
$p_6 = 1.003325374626461 0, \quad f[p_6] = 0.00003321112159881956$
$p_7 = 1.001663607293284 0, \quad f[p_7] = 8.30737186041652 \times 10^{-6}$

$p_8 = 1.0008320340873970, \quad f[p_8] = 2.077418169044165 \times 10^{-6}$
$p_9 = 1.0004160747097450, \quad f[p_9] = 5.194265224606198 \times 10^{-7}$
$p_{10} = 1.0002080517783450, \quad f[p_{10}] = 1.298656329140613 \times 10^{-7}$
$p_{11} = 1.0001040294960360, \quad f[p_{11}] = 3.246753399466229 \times 10^{-8}$
$p_{12} = 1.0000520156497570, \quad f[p_{12}] = 8.1170243859674 \times 10^{-9}$
$p_{13} = 1.0000260080497260, \quad f[p_{13}] = 2.029273638015638 \times 10^{-9}$
$p_{14} = 1.0000130040806280, \quad f[p_{14}] = 5.073206299499589 \times 10^{-10}$
$p_{15} = 1.0000065020532280, \quad f[p_{15}] = 1.268303240209434 \times 10^{-10}$
$p_{16} = 1.0000032510311470, \quad f[p_{16}] = 3.170774753868955 \times 10^{-11}$
$p_{17} = 1.0000016255111940, \quad f[p_{17}] = 7.926992395823618 \times 10^{-12}$
$p_{18} = 1.0000008127426750, \quad f[p_{18}] = 1.981748098955904 \times 10^{-12}$
$p_{19} = 1.0000004063517890, \quad f[p_{19}] = 4.953815135877448 \times 10^{-13}$
$p_{20} = 1.0000002031692980, \quad f[p_{20}] = 1.239008895481675 \times 10^{-13}$
$p_{21} = 1.0000001015292060, \quad f[p_{21}] = 3.064215547965432 \times 10^{-14}$
$p_{22} = 1.0000000512281560, \quad f[p_{22}] = 7.993605777301127 \times 10^{-15}$
$p_{23} = 1.0000000252216060, \quad f[p_{23}] = 1.998401444325282 \times 10^{-15}$
$p_{24} = 1.0000000120159870, \quad f[p_{24}] = 6.661338147750939 \times 10^{-16}$
$p_{25} = 1.0000000027764380, \quad f[p_{25}] = -2.220446049250313 \times 10^{-16}$
$p = 1.000000002776438$
$\Delta p = \pm 9.23955 \times 10^{-9}$
$f[p] = -2.220446049250313 \times 10^{-16}$

At the double root $p = 1$ we can explore the relationship $\left|E_{k+1}\right| \approx \frac{1}{2}\left|E_k\right|$ for k sufficiently large.

This will be done by investigating the ratio $\frac{|E_{k+1}|}{|E_k|} \approx \frac{1}{2}$ for k sufficiently large.

k	p_k	$E_k = p - p_k$	$\frac{\lvert E_{k+1}\rvert}{\lvert E_k\rvert}$
0	1.2	-0.2	0.515151515151515
1	1.1030303030303	-0.103030303030303	0.508165225744475
2	1.05235641719792	-0.0523564171979156	0.504251732038285
3	1.02640081405537	-0.026400814055368	0.502171404415847
4	1.01325773387191	-0.0132577338719058	0.501097535737631
5	1.00664341777268	-0.0066434177726773	0.500551785277665
6	1.00332537462646	-0.00332537462645899	0.500276654562209
7	1.00166360729329	-0.00166360729329096	0.500138518720712
8	1.0008320340874	-0.000832034087399514	0.500069307340907

9	1.00041607470977	-0.000416074709769898	0.500034665680735
10	1.0002080517784	-0.000208051778398	0.500017335845333
11	1.00010402949595	-0.000104029495952451	0.50000866867469
12	1.00005201564977	-0.0000520156497740842	0.500004334526333
13	1.00002600805035	-0.0000260080503502458	0.500002167304141
14	1.00001300408154	-0.0000130040815424781	0.500001083672986
15	1.00000650205486	$-6.50205486341093\times10^{-6}$	0.500000541788362
16	1.00000325103095	$-3.25103095444312\times10^{-6}$	0.500000270842663
17	1.00000162551636	$-1.62551635773944\times10^{-6}$	0.500000135438333
18	1.0000008127584	$-8.12758399026947\times10^{-7}$	0.500000067616699
19	1.00000040637925	$-4.06379254469513\times10^{-7}$	0.500000033603446
20	1.00000020318964	$-2.031896408905\times10^{-7}$	0.500000016938321
21	1.00000010159482	$-1.01594823886941\times10^{-7}$	0.500000007649564
22	1.00000005079741	$-5.07974127206268\times10^{-8}$	0.500000006556769
23	1.00000002539871	$-2.53987066933803\times10^{-8}$	0.5
24	1.00000001269935	$-1.26993533466902\times10^{-8}$	

Real roots can also be found symbolically.

$$f[x] = (-1+x)^2(2+x)$$

$x \to -2$

$x \to 1$

$x \to 1$

Example 3. Fast Convergence Find the solution to `3 Exp[x] - 4 Cos[x] = 0`. Use the starting approximation $p_0 = 1.0$.

Solution 3.

$$f[x] = 3e^x - 4\,\mathrm{Cos}[x]$$

$$g[x] = x - \frac{f[x]}{f'[x]}$$

$$g[x] = x - \frac{3e^x - 4\,\mathrm{Cos}[x]}{3e^x + 4\,\mathrm{Sin}[x]}$$

$$g[x] = x + \frac{-3e^x + 4\,\mathrm{Cos}[x]}{3e^x + 4\,\mathrm{Sin}[x]}$$

$p_0 =$ 1.0000000000000000, $f[p_0] =$ 5.993636261904577

$p_1 =$ 0.4797520156057185, $f[p_1] =$ 1.298583433809675

$p_2 =$ 0.2857383591311282, $f[p_2] =$ 0.1544175142133715

$p_3 =$ 0.2555769004716556, $f[p_3] =$ 0.003548454019948188

$p_4 =$ 0.2548504777343278, $f[p_4] = 2.042950823177847 \times 10^{-6}$

$p_5 =$ 0.2548500590289893, $f[p_5] = 6.785683126508957 \times 10^{-13}$

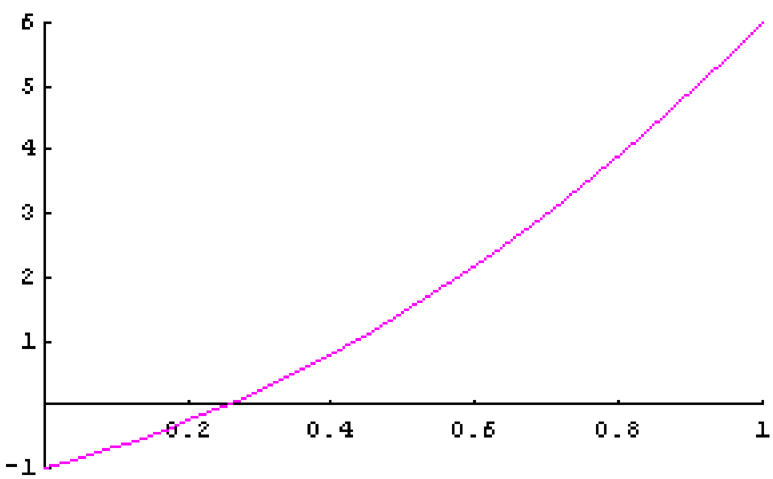
$p_6 =$ 0.2548500590288503, $f[p_6] =$ 0.

$$f[x] = 3e^x - 4\,\mathrm{Cos}[x]$$

$p =$ 0.2548500590288503

$\Delta p = \pm 1.39055 \times 10^{-13}$

$f[p] =$ 0.



Example 4. NON Convergence, Cycling Find the solution to $x^3 - x + 3 = 0$. Use the starting approximation $p_0 = 0.0$.

Solution 4.

$$f[x] = 3 - x + x^3$$

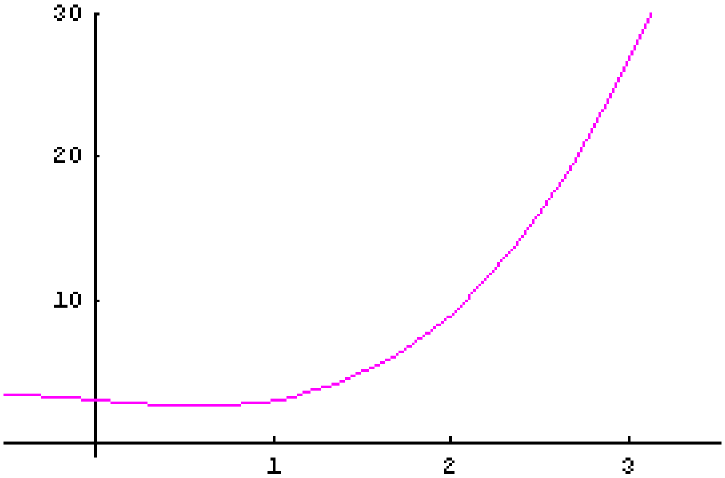
$$g[x] = x - \frac{f[x]}{f'[x]}$$

$$g[x] = x - \frac{3 - x + x^3}{-1 + 3x^2}$$

$$g[x] = \frac{3 - 2x^3}{1 - 3x^2}$$

$p_0 = 0.0000000000000000,\quad f[p_0] = 3.$
$p_1 = 3.0000000000000000,\quad f[p_1] = 27.$
$p_2 = 1.9615384615384610,\quad f[p_2] = 8.58574192080109$
$p_3 = 1.1471759614035470,\quad f[p_3] = 3.362522157362049$
$p_4 = 0.0065793714807121,\quad f[p_4] = 2.99342091332797$
$p_5 = 3.0003890740712320,\quad f[p_5] = 27.01011728831863$
$p_6 = 1.9618181756663240,\quad f[p_6] = 8.58869137914838$
$p_7 = 1.1474302284816020,\quad f[p_7] = 3.363271968902603$
$p_8 = 0.0072562475524216,\quad f[p_8] = 2.992744134511713$
$p_9 = 3.0004731887732160,\quad f[p_9] = 27.0123049233781$
$p_{10} = 1.9618786463602410,\quad f[p_{10}] = 8.58932913619802$
$p_{11} = 1.1474851932167660,\quad f[p_{11}] = 3.363434113639871$
$p_{12} = 0.0074025013328707,\quad f[p_{12}] = 2.992597904302187$
$p_{13} = 3.0004924429169550,\quad f[p_{13}] = 27.01280569846047$
$p_{14} = 1.9618924882463070,\quad f[p_{14}] = 8.58947512636186$
$p_{15} = 1.1474977745445400,\quad f[p_{15}] = 3.363471231200353$
$p_{16} = 0.0074359752567048,\quad f[p_{16}] = 2.992564435906089$

$f[x] = 3 - x + x^3$
$p = 0.00743597525670480 8$
$\Delta p = \pm 1.14006$
$f[p] = 2.992564435906089$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