8. Monte Carlo Pi

Background

We start the familiar example of finding the area of a circle. Figure 1 below shows a circle with radius $r = 1$ inscribed within a square. The area of the circle is $\pi r^2 = \pi 1^2 = \pi$, and the area of the square is $(2 r)^2 = 2^2 = 4$. The ratio of the area of the circle to the area of the square is

$$\rho = \frac{\text{Area of Circle}}{\text{Area of Square}} = \frac{\pi r^2}{(2 r)^2} = \frac{\pi}{4} = \frac{3.141592653589793 2}{4} = 0.7853981633974483$$

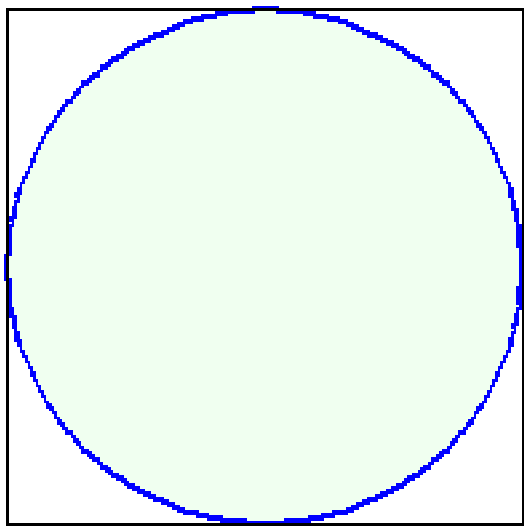


Figure 1. $\rho = \frac{\text{Area of Circle}}{\text{Area of Square}} = \frac{\pi}{4} = 0.7853981633974483$

If we could compute ratio, then we could multiple it by four to obtain the value π. One particularly simple way to do this is to pick lattice points in the square and count how many of them lie inside the circle, see Figure 2. Suppose for example that the points $\left\{-1 + \frac{2 i - 1}{32}, -1 + \frac{2 j - 1}{32}\right\}_{i=1\ j=1}^{32\ \ 31}$ are selected, then there are 812 points inside the circle and 212 points outside the circle and the percentage of points inside the circle is $\rho = \frac{812}{812 + 212} = \frac{812}{1024} = 0.79296875$. Then the area of the circle is approximated with the following calculation

$$\text{Area of Circle} \approx \rho * \text{Area of Square} = \rho * 4 = 0.79296875 * 4 = 3.171875$$

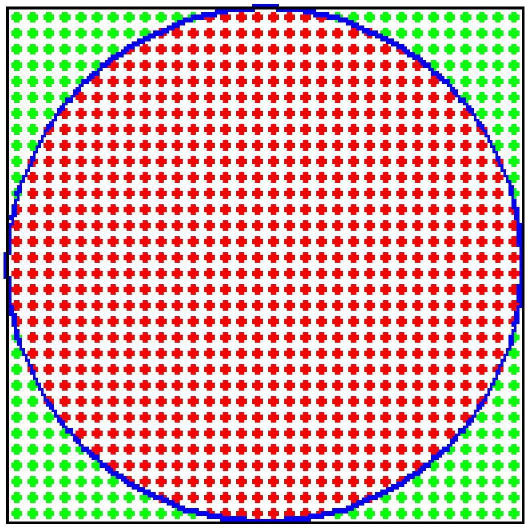


Figure 2. $\text{Circle} \approx \rho\,\text{Square} = \rho * 4 = 3.171875$

Monte Carlo Method for π

Monte Carlo methods can be thought of as statistical simulation methods that utilize a sequences of random numbers to perform the simulation. The name "Monte Carlo" was coined by Nicholas Constantine Metropolis (1915-1999) and inspired by Stanslaw Ulam (1909-1986), because of the similarity of statistical simulation to games of chance, and because Monte Carlo is a center for gambling and games of chance. In a typical process one compute the number of points in a set **A** that lies inside box **R**. The ratio of the number of points that fall inside **A** to the total number of points tried is equal to the ratio of the two areas (or volume in 3 dimensions). The accuracy of the ratio ρ depends on the number of points used, with more points leading to a more accurate value.

A simple Monte Carlo simulation to approximate the value of π could involve randomly selecting points $\{(x_i, y_i)\}_{i=1}^{n}$ in the unit square and determining the ratio $\rho = \frac{m}{n}$, where m is number of points that satisfy $x_i^2 + y_i^2 \le 1$. In a typical simulation of sample size $n = 1000$ there were 787 points satisfying $x_i^2 + y_i^2 \le 1$, shown in Figure 3. Using this data, we obtain

$$\rho = \frac{m}{n} = \frac{787}{1000} = 0.787 \quad \text{and} \quad \pi \approx \rho * 4 = 0.787 * 4 = 3.148$$

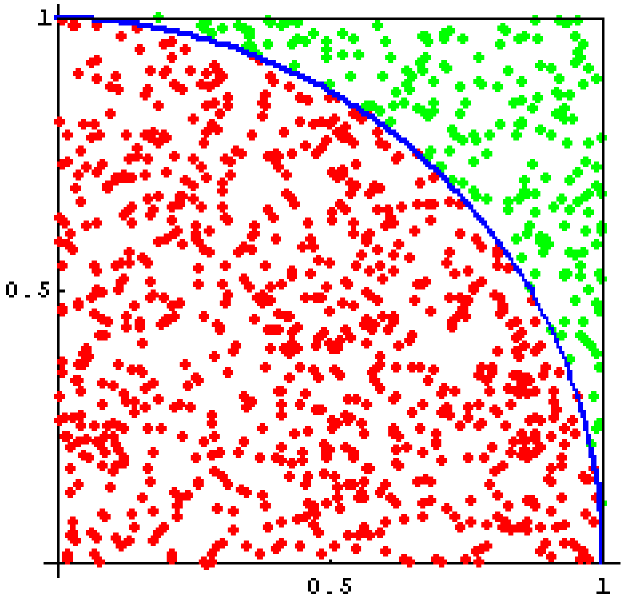


Figure 3. $\pi \approx \rho * 4 = 0.787 * 4 = 3.148$

Every time a Monte Carlo simulation is made using the same sample size n it will come up with a slightly different value. The values converge very slowly of the order $O(n^{-1/2})$. This property is a consequence of the Central Limit Theorem.

Remark. Another interesting simulation for approximating π is known as Buffon's Needle problem. The reader can find many references to it in books, articles and on the internet.

Example 1. Use Monte Carlo simulation to approximate the number π.
Solution 1.

Area Under a Curve

Monte Carlo simulation can be used to approximate the area A under a curve $y = f(x)$ for $a \le x \le b$. First, we must determine the rectangular box R containing A as follows.

$$R = \{(x, y) : a \le x \le b \text{ and } 0 \le y \le d\} \quad \text{where} \quad d = \max_{a \le x \le b} f(x).$$

Second, randomly pick points $\{(x_i, y_i)\}_{i=1}^{n}$ in R, where x_i and y_i are chosen from independent uniformly distributed random variables over $[a, b]$ and $[0, d]$, respectively. Third, calculate the ratio ρ as follows:

$$\rho = \frac{m}{n}, \text{ where } m \text{ is number of points that lie in } A.$$

The area is computed using the approximation, $\text{area}\, A \approx \rho * \text{area}\, R$, and we can use the formula

$$\text{area } \mathbb{A} \approx \rho * (b - a) * d.$$

An "estimate" for the accuracy of the above computation is

$$\Delta \mathbb{A} \approx (b - a) * d * \sqrt{\frac{\rho - \rho^2}{n}}.$$

Caveat. The standard implementation of Monte Carlo integration is done by selecting points only in the domain of the function, for a one dimensional integral we would use random numbers to select points $\{x_i\}_{i=1}^{n}$ in the interval $a \le x \le b$ and then use the approximation

$$\int_a^b f[x]\, dx \approx (b - a) \frac{1}{n} \sum_{i=1}^{n} f[x_i].$$

Example 2. Use Monte Carlo simulation to approximate the integral $\int_0^2 \frac{4}{1 + x^2}\, dx$

Solution 2.

Example 3. Use Monte Carlo simulation to approximate the integral $\int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

Solution 3.

Example 4. Use Monte Carlo simulation to approximate the area of the cardioid defined by the constraint

$$(x^2 + y^2 - 2x)^2 \le 4(x^2 + 4y^2).$$

Solution 4.

Example 5. Use Monte Carlo simulation to approximate the area of the region defined by the constraints

$$-\frac{5}{4} - \frac{3}{3 + x^2} \le y \le \frac{5}{4} + \frac{3}{3 + x^2}$$

and

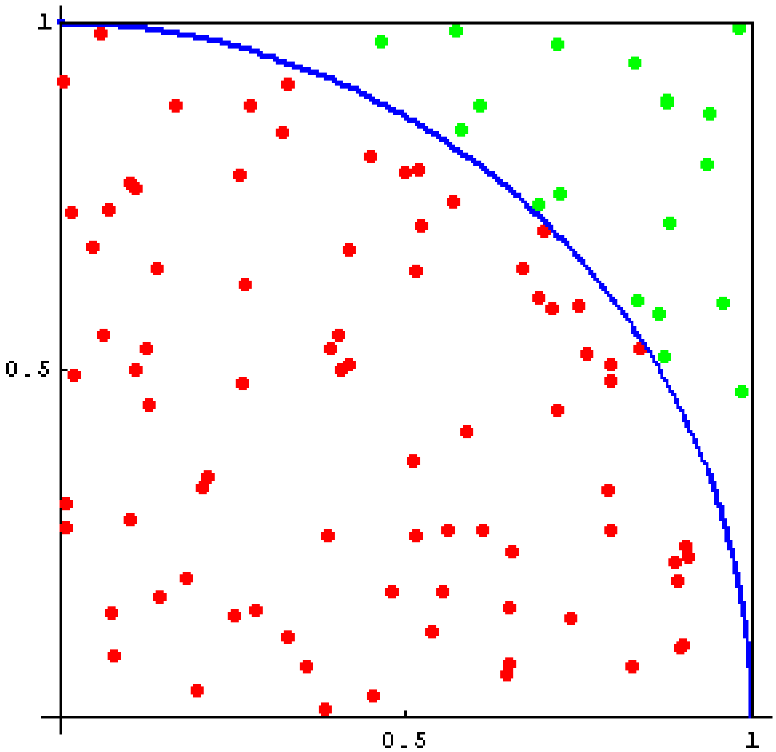
$$-\frac{1}{3} - \frac{5y^2}{6} + \frac{y^4}{6} \le x \le \frac{11}{3} - \frac{2}{3}\sqrt{2}\, |y|^{3/2}$$

Solution 5.

Example 1. Use Monte Carlo simulation to approximate the number π.
Solution 1.

Explore what happens with $n = 100, 400, 1600, 6400,$ and 10000 points.

```
MonteCarloResults[100];
```

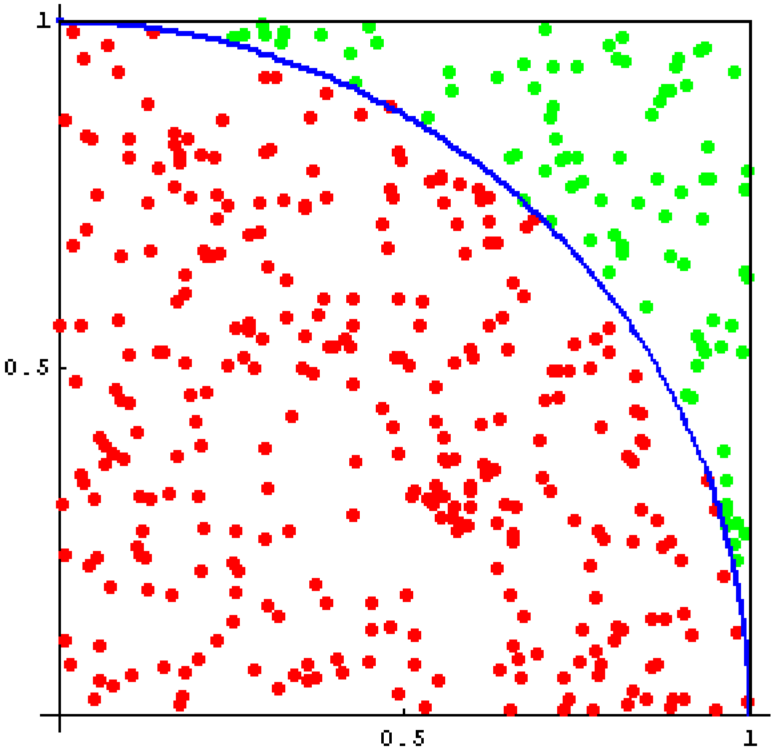



```
Points Generated   n = 100
    Inside Points  m = 81
   Outside Points  k = 19
Approximation to   π ≈ 3.24
```

Order of error $\frac{\pi}{\sqrt{n}} \approx 0.314159$

$|\pi - \text{approx}| \approx 0.0984073$

```
MonteCarloResults[400];
```

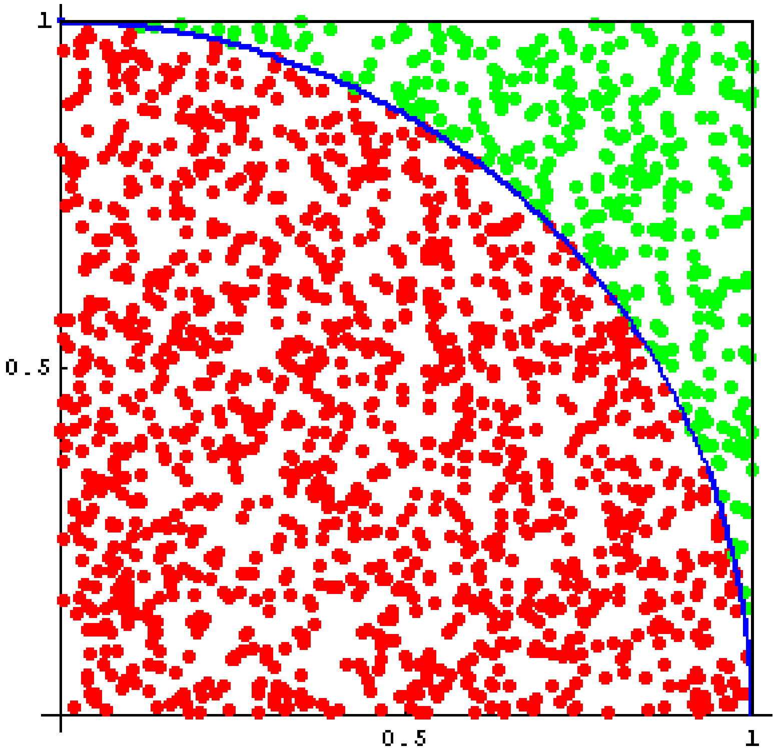



```
Points Generated  n = 400
   Inside Points  m = 311
  Outside Points  k = 89
Approximation to  π ≈ 3.11
```

Order of error $\frac{\pi}{\sqrt{n}} \approx 0.15708$

$|\pi\text{-approx}| \approx 0.0315927$

MonteCarloResults[1600];



Points Generated $n = 1600$

Inside Points $m = 1268$

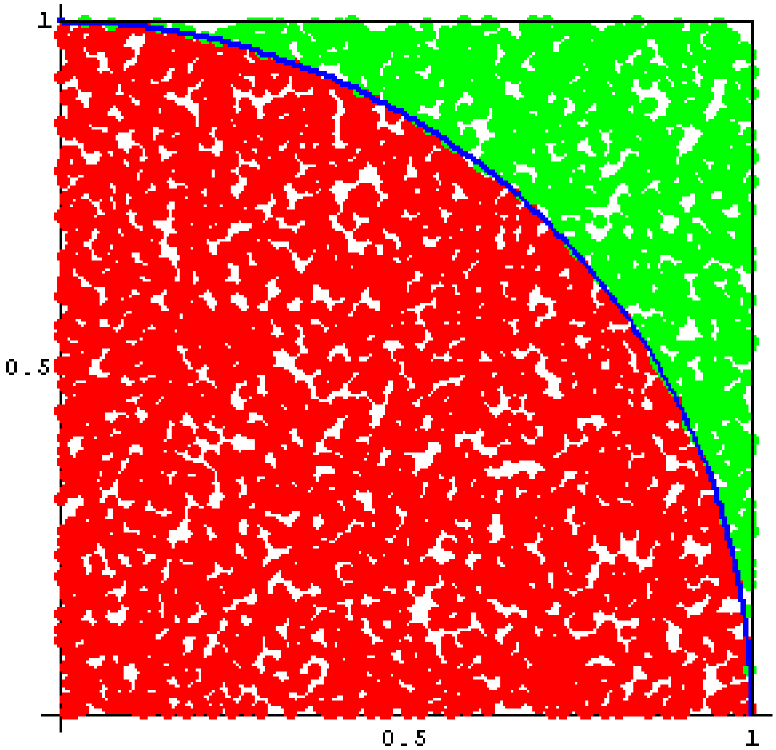
Outside Points $k = 332$

Approximation to $\pi \approx 3.17$

Order of error $\dfrac{\pi}{\sqrt{n}} \approx 0.0785398$

$|\pi\text{-approx}| \approx 0.0284073$

```
MonteCarloResults[6400];
```

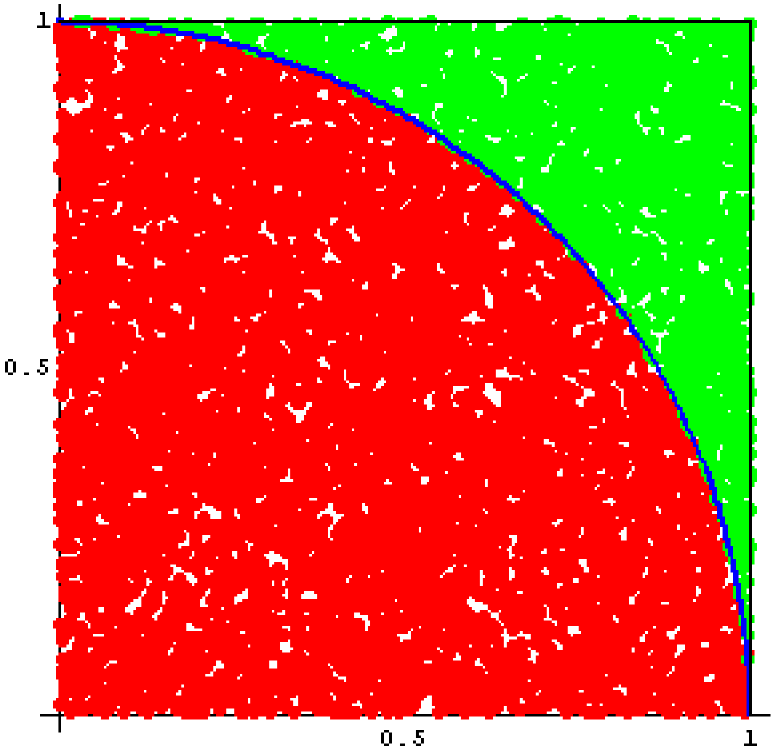



```
Points Generated  n = 6400
   Inside Points  m = 5016
  Outside Points  k = 1384
Approximation to  π ≈ 3.135
```

Order of error $\frac{\pi}{\sqrt{n}} \approx 0.0392699$

$|\pi\text{-approx}| \approx 0.00659265$

MonteCarloResults[10000];



Points Generated $n = 10000$

Inside Points $m = 7845$

Outside Points $k = 2155$

Approximation to $\pi \approx 3.138$

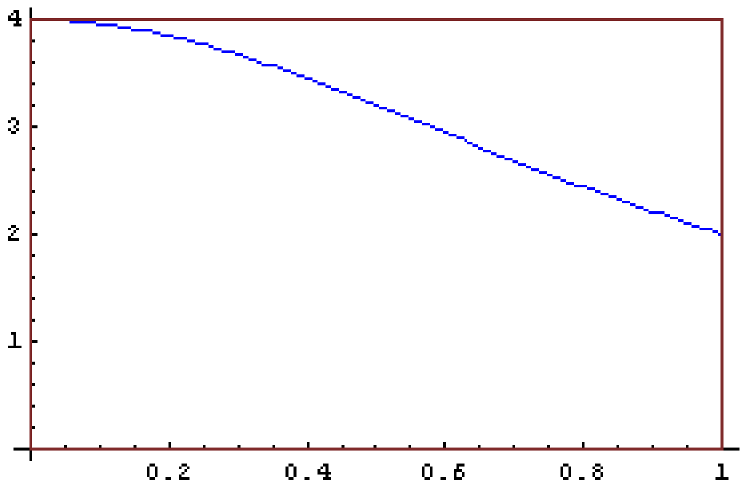
Order of error $\dfrac{\pi}{\sqrt{n}} \approx 0.0314159$

$|\pi\text{-approx}| \approx 0.00359265$

Example 2. Use Monte Carlo simulation to approximate the integral $\int_0^1 \frac{4}{1+x^2}\,dx$

Solution 2.

Set up the function and the rectangular box enclosing the area.



To find the area under the curve

$y = \frac{4}{1+x^2}$ over the interval [a,b] = [0,1]

It appears that a rectangle containing the area is

R = { (x,y): $0 \le x \le 1$ and $0 \le y \le 4$ }

If this is correct, then Mathematica will proceed using

a = 0

b = 1

c = 0

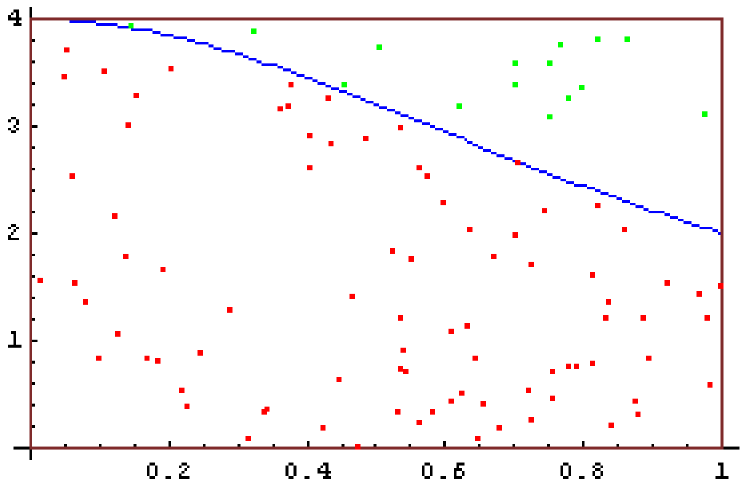
d = 4

If not, then enter the correct value (s) of a, b, d.

Then invoke the subroutine 'MonteCarloSimulation'

Explore what happens with $n = 100, 1000,$ and 10000 points.

```
val1 = MonteCarloSimulation[100];
```



To find the area under the curve

$y = \dfrac{4}{1 + x^2}$ over the interval [0,1]

Points generated $n = 100$

Points under curve $m = 85$

Points above curve $k = 15$

The ratio $\rho = \dfrac{m}{n} = 85/100$

$$\rho = \frac{17}{20} = 0.85$$

BoxArea = 4

Area under curve

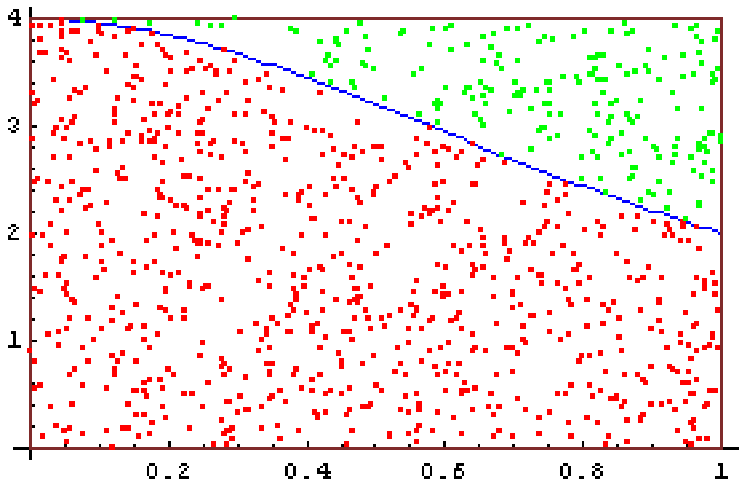
$\rho * \text{BoxArea} = \dfrac{17}{20} * 4 = 3.4$

Monte Carlo Approximation

The area under the curve $A \approx 3.4$

Estimate for the error $\Delta A \approx 0.142829$

```
val2 = MonteCarloSimulation[1000];
```



To find the area under the curve

$y = \frac{4}{1 + x^2}$ over the interval [0,1]

Points generated $n = 1000$

Points under curve $m = 797$

Points above curve $k = 203$

The ratio $\rho = \frac{m}{n} = 797/1000$

$$\rho = \frac{797}{1000} = 0.797$$

BoxArea = 4

Area under curve

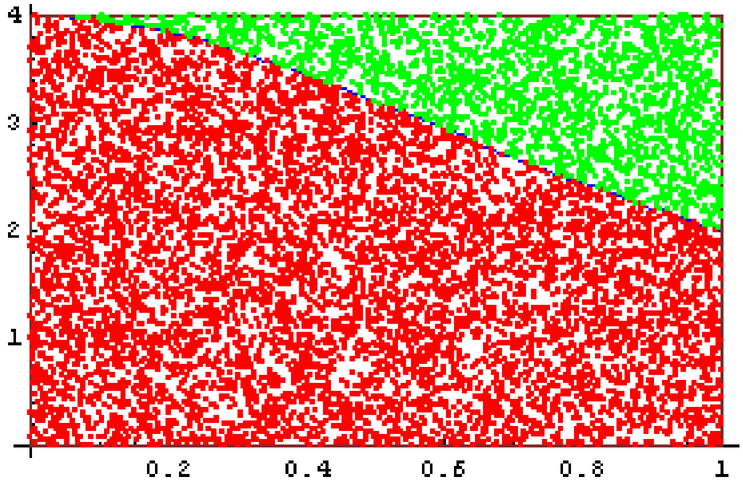
$$\rho * \text{BoxArea} = \frac{797}{1000} * 4 = 3.188$$

Monte Carlo Approximation

The area under the curve $A \approx 3.188$

Estimate for the error $\Delta A \approx 0.0508788$

```
val3 = MonteCarloSimulation[10000];
```



To find the area under the curve

$y = \frac{4}{1 + x^2}$ over the interval $[0,1]$

Points generated $n = 10000$

Points under curve $m = 7810$

Points above curve $k = 2190$

The ratio $\rho = \frac{m}{n} = 7810/10000$

$$\rho = \frac{781}{1000} = 0.781$$

BoxArea = 4

Area under curve

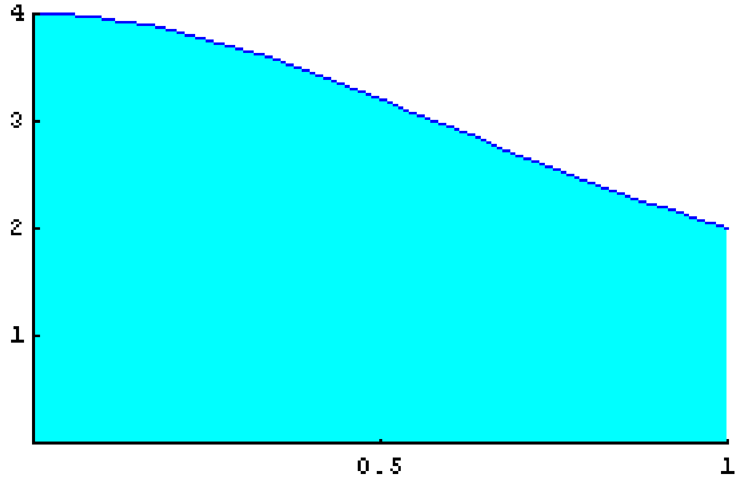
$\rho * \text{BoxArea} = \frac{781}{1000} * 4 = 3.124$

Monte Carlo Approximation

The area under the curve $A \approx 3.124$

Estimate for the error $\Delta A \approx 0.0165427$

Aside. The analytic value of the integral can be found.



$$f[x] = \frac{4}{1+x^2}$$

$$F[x] = \int \left(\frac{4}{1+x^2}\right) dx$$

$$F[x] = 4\,\mathrm{ArcTan}[x]$$

$$\int_0^1 \left(\frac{4}{1+x^2}\right) dx = F[1] - F[0]$$

$$\int_0^1 \left(\frac{4}{1+x^2}\right) dx = (\pi) - (0)$$

Final answer

$$\int_0^1 \left(\frac{4}{1+x^2}\right) dx = \pi$$

$$\int_0^1 \left(\frac{4}{1+x^2}\right) dx = 3.14159$$

The approximations obtained with Monte Carlo simulation:

Using $n = 100$ area ≈ 3.4

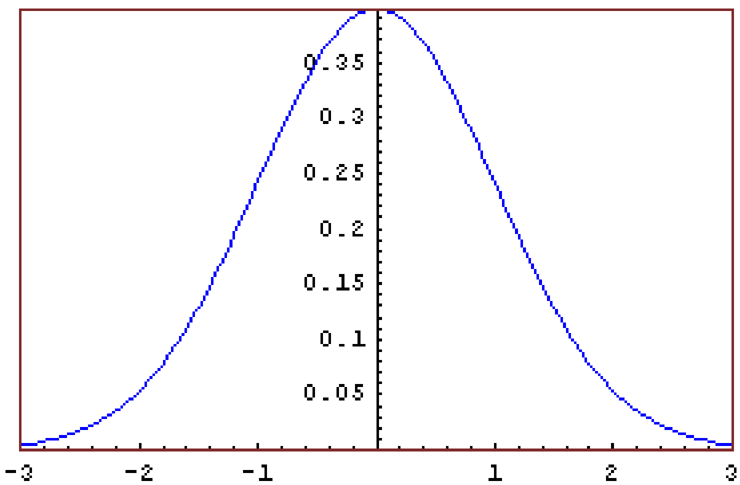
Using $n = 1000$ area ≈ 3.188

Using $n = 10000$ area ≈ 3.124

Example 3. Use Monte Carlo simulation to approximate the integral $\int_{-3}^{3} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$.

Solution 3.

Set up the function and the rectangular box enclosing the area.



To find the area under the curve

$y = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ over the interval $[a,b] = [-3,3]$

It appears that a rectangle containing the area is

$R = \{(x,y): \ -3 \le x \le 3 \ \text{ and } \ 0 \le y \le 0.398942\ \}$

If this is correct, then Mathematica will proceed using

$a = -3$

$b = 3$

$c = 0$

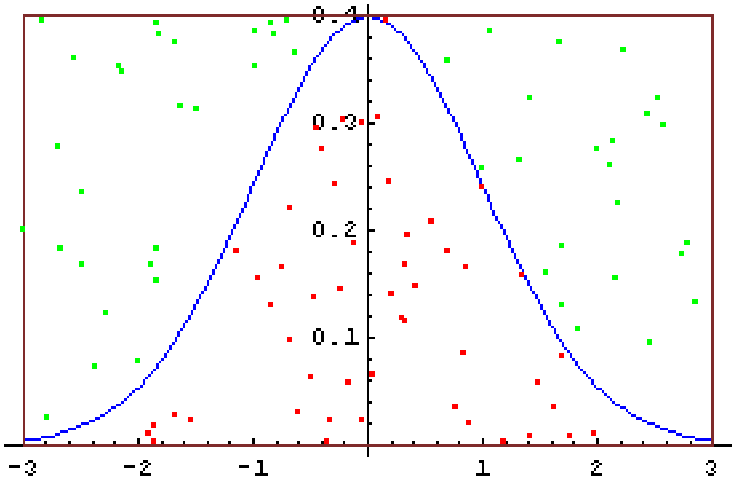
$d = 0.398942$

If not, then enter the correct value (s) of a, b, d.

Then invoke the subroutine 'MonteCarloSimulation'

Explore what happens with $n = 100$, 1000, and 10000 points.

```
val1 = MonteCarloSimulation[100];
```



To find the area under the curve

$$y = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$ over the interval [-3,3]

Points generated $n = 100$

Points under curve $m = 50$

Points above curve $k = 50$

The ratio $\rho = \frac{m}{n} = 50/100$

$\rho = \frac{1}{2} = 0.5$

BoxArea = 2.39365

Area under curve

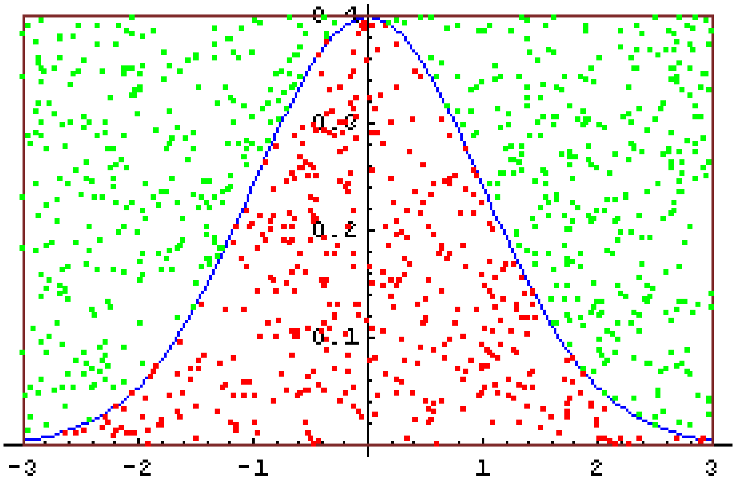
$\rho * \text{BoxArea} = \frac{1}{2} * 2.39365 = 1.19683$

Monte Carlo Approximation

The area under the curve $A \approx 1.19683$

Estimate for the error $\Delta A \approx 0.119683$

val2 = MonteCarloSimulation[1000];



To find the area under the curve

$$y = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$ over the interval [-3,3]

Points generated $n = 1000$

Points under curve $m = 406$

Points above curve $k = 594$

The ratio $\rho = \frac{m}{n} = 406/1000$

$$\rho = \frac{203}{500} = 0.406$$

BoxArea = 2.39365

Area under curve

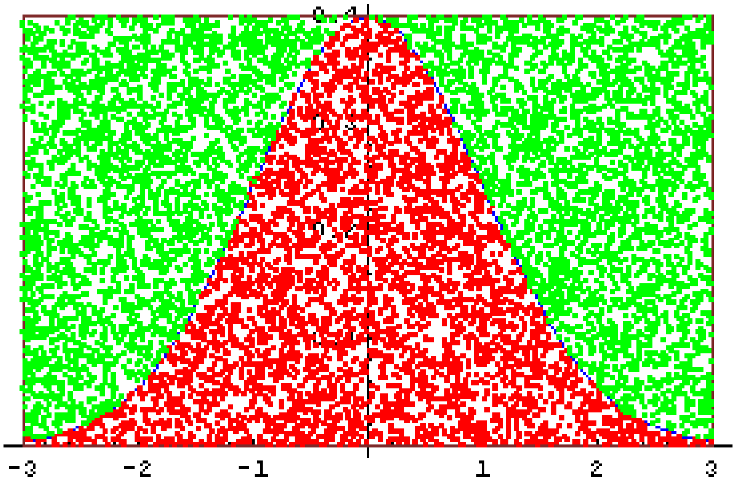
$$\rho * \text{BoxArea} = \frac{203}{500} * 2.39365 = 0.971823$$

Monte Carlo Approximation

The area under the curve $A \approx 0.971823$

Estimate for the error $\Delta A \approx 0.0371721$

```
val3 = MonteCarloSimulation[10000];
```



To find the area under the curve

$$y = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \quad \text{over the interval} \quad [-3,3]$$

Points generated $n = 10000$

Points under curve $m = 4177$

Points above curve $k = 5823$

The ratio $\rho = \frac{m}{n} = 4177/10000$

$$\rho = \frac{4177}{10000} = 0.4177$$

BoxArea = 2.39365

Area under curve

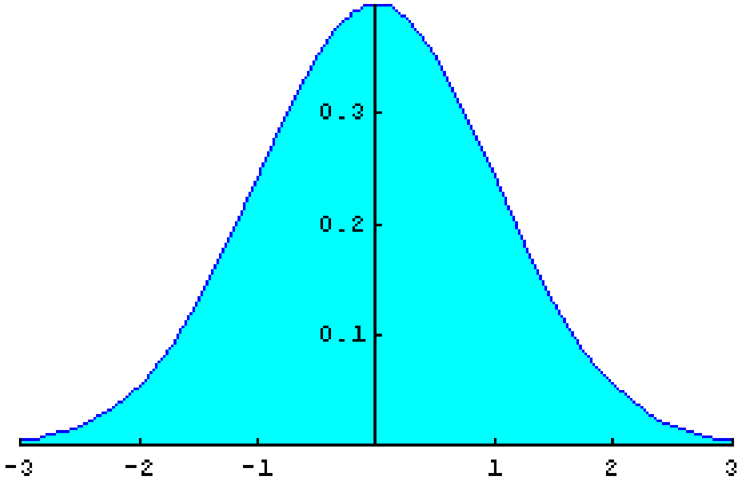
$$\rho * \text{BoxArea} = \frac{4177}{10000} * 2.39365 = 0.999829$$

Monte Carlo Approximation

The area under the curve $A \approx 0.999829$

Estimate for the error $\Delta A \approx 0.011805$

Aside. The analytic value of the integral can be found.



$$f[x] = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$F[x] = \int \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx$$

$$F[x] = \frac{1}{2}\,\mathrm{Erf}\left[\frac{x}{\sqrt{2}}\right]$$

$$\int_{-3}^{3} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx = F[3] - F[-3]$$

$$\int_{-3}^{3} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx = \left(\frac{1}{2}\,\mathrm{Erf}\left[\frac{3}{\sqrt{2}}\right]\right) - \left(-\frac{1}{2}\,\mathrm{Erf}\left[\frac{3}{\sqrt{2}}\right]\right)$$

Final answer

$$\int_{-3}^{3} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx = \mathrm{Erf}\left[\frac{3}{\sqrt{2}}\right]$$

$$\int_{-3}^{3} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx = 0.9973$$

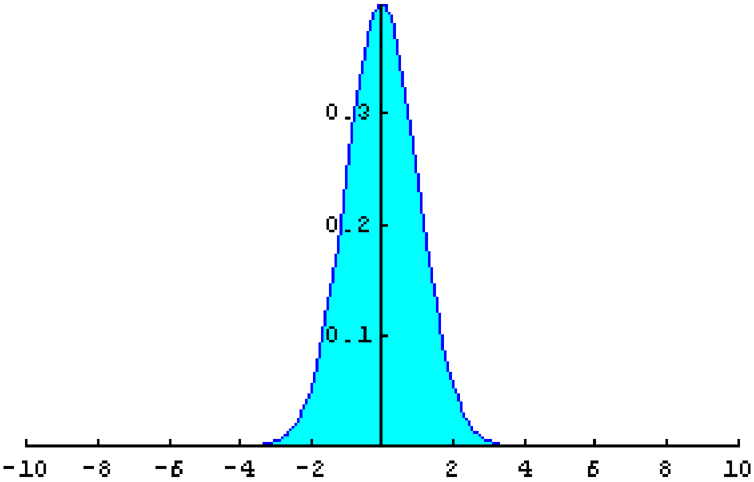
The approximations obtained with Monte Carlo simulation:

Using $n = 100$ area ≈ 1.19683

Using $n = 1000$ area ≈ 0.971823

Using $n = 10000$ area ≈ 0.999829

Aside. This is the standard normal distribution function. Suppose that the interval of integration is $[-\infty, \infty]$.



$$f[x] = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$F[x] = \int \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx$$

$$F[x] = \frac{1}{2}\,\mathrm{Erf}\left[\frac{x}{\sqrt{2}}\right]$$

$$\int_{-\infty}^{\infty} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx = F[\infty] - F[-\infty]$$

$$\int_{-\infty}^{\infty} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx = \left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)$$

Final answer

$$\int_{-\infty}^{\infty} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\right) dx = 1$$

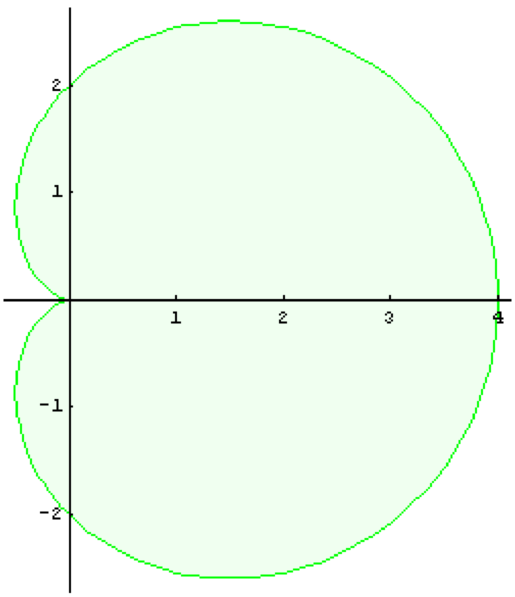
This is the Gaussian distribution.

Example 4. Use Monte Carlo simulation to approximate the area of the cardioid defined by the constraint

$$(x^2 + y^2 - 2x)^2 \leq 4(x^2 + 4y^2) \,.$$

Solution 4.

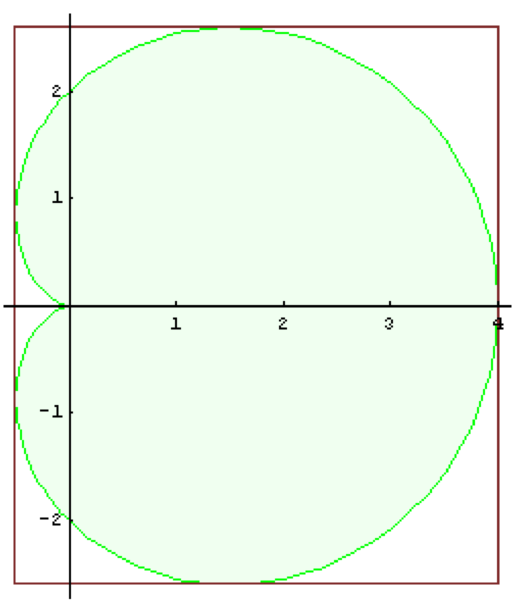
Set up the constraint for the cardioid.



The area A defined by

$$(-2x + x^2 + y^2)^2 \leq 4(x^2 + y^2)$$

Set up the rectangular box enclosing the area.



To find the area A we must have a rectalgle containing it.

$R = \{(x,y):\ -\frac{1}{2} \le x \le 4 \ \text{ and } \ -\frac{3\sqrt{3}}{2} \le y \le \frac{3\sqrt{3}}{2}\ \}$

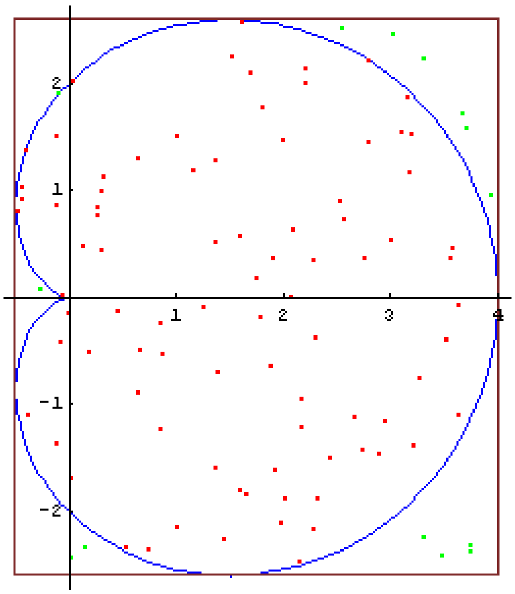
If this is correct, then Mathematica will proceed using it.

If not, then enter the correct value (s) of a, b, c, d.

Then invoke the subroutine ' MonteCarloArea'

Explore what happens with $n = 100,\ 1000,$ and 10000 points.

val1 = MonteCarloArea[100];



To find the area satisfying the constraint(s)

$(-2x + x^2 + y^2)^2 \le 4(x^2 + y^2)$

Points generated n = 100

Points inside the area m = 86

Points outside the area k = 14

The ratio $\rho = \frac{m}{n} = 86/100$

$\rho = \frac{43}{50} = 0.86$

BoxArea $= \frac{27\sqrt{3}}{2}$

Calculation for the area

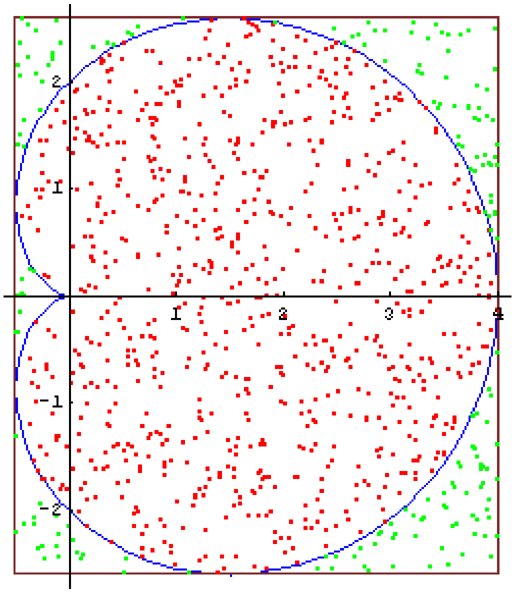
$\rho * \text{BoxArea} = \frac{43}{50} * \frac{27\sqrt{3}}{2} = 20.1091$

Monte Carlo Approximation

The estimate for the area $A \approx 20.1091$

Estimate for the error $\Delta A \approx 0.811349$

val2 = MonteCarloArea[1000];



To find the area satisfying the constraint(s)

$(-2x + x^2 + y^2)^2 \leq 4(x^2 + y^2)$

Points generated $n = 1000$

Points inside the area $m = 820$

Points outside the area $k = 180$

The ratio $\rho = \frac{m}{n} = 820/1000$

$$\rho = \frac{41}{50} = 0.82$$

$$\text{BoxArea} = \frac{27\sqrt{3}}{2}$$

Calculation for the area

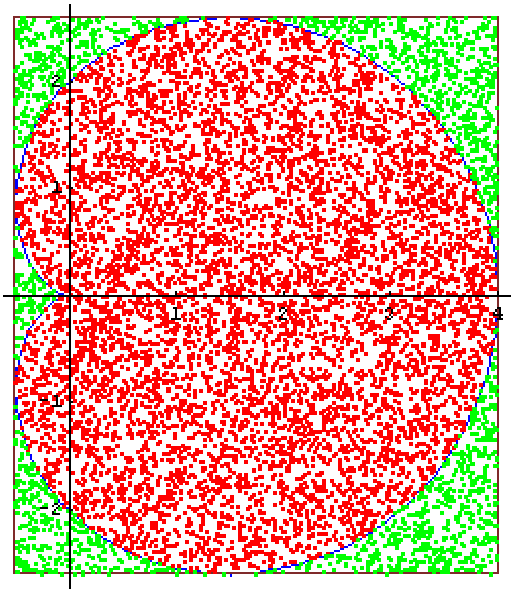
$$\rho * \text{BoxArea} = \frac{41}{50} * \frac{27\sqrt{3}}{2} = 19.1738$$

Monte Carlo Approximation

The estimate for the area $A \approx 19.1738$

Estimate for the error $\Delta A \approx 0.284078$

val3 = MonteCarloArea[10000];



To find the area satisfying the constraint(s)

$(-2x + x^2 + y^2)^2 \le 4(x^2 + y^2)$

Points generated $n = 10000$

Points inside the area $m = 8043$

Points outside the area $k = 1957$

The ratio $\rho = \frac{m}{n} = 8043/10000$

$$\rho = \frac{8043}{10000} = 0.8043$$

$$\text{BoxArea} = \frac{27\sqrt{3}}{2}$$

Calculation for the area

$$\rho * \text{BoxArea} = \frac{8043}{10000} * \frac{27\sqrt{3}}{2} = 18.8067$$

Monte Carlo Approximation

The estimate for the area $A \approx 18.8067$

Estimate for the error $\Delta A \approx 0.0927681$

Aside. The analytic value of the area can be found using polar coordinates.

$$\text{Area} = \int_0^{2\pi}\left(\int_0^{2(1+\text{Cos}[\theta])} r\, dr\right) d\theta$$

$$f_1[r] = r$$

$$f_2[r] = \int f_1[r]\, dr$$

$$f_2[r] = \frac{r^2}{2}$$

$$\int_0^{2(1+\text{Cos}[\theta])} r\, dr = f_2[2(1+\text{Cos}[\theta])] - f_2[0]$$

$$\int_0^{2(1+\text{Cos}[\theta])} r\, dr = 2(1+\text{Cos}[\theta])^2$$

$$\text{Area} = \int_0^{2\pi}(2(1+\text{Cos}[\theta])^2) d\theta$$

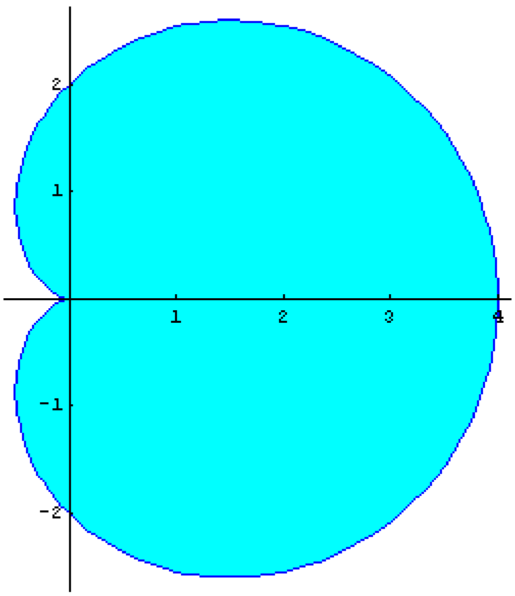
$$f_3[\theta] = 2(1+\text{Cos}[\theta])^2$$

$$f_4[\theta] = \int f_3[\theta]\, d\theta$$

$$f_4[\theta] = 2\left(\frac{3}{8}\theta(1+\text{Cos}[\theta])^2\text{Sec}\left[\frac{\theta}{2}\right]^4 + \frac{1}{2}(1+\text{Cos}[\theta])^2\text{Sec}\left[\frac{\theta}{2}\right]^4\text{Sin}[\theta] + \frac{1}{16}(1+\text{Cos}[\theta])^2\text{Sec}\left[\frac{\theta}{2}\right]^4\text{Sin}[2\theta]\right)$$

$$\text{Area} = f_4[2\pi] - f_4[0]$$

$$\text{Area} = 6\pi$$



$$\text{Area} = \int_0^{2\pi}\left(\int_0^{2(1+\text{Cos}[\theta])} r\, dr\right) d\theta = 6\pi = 18.8496$$

The approximations obtained with Monte Carlo simulation:

Using n = 100 area ≈ 20.1091

Using n = 1000 area ≈ 19.1738

Using n = 10000 area ≈ 18.8067

Example 5. Use Monte Carlo simulation to approximate the area of the region defined by the constraints

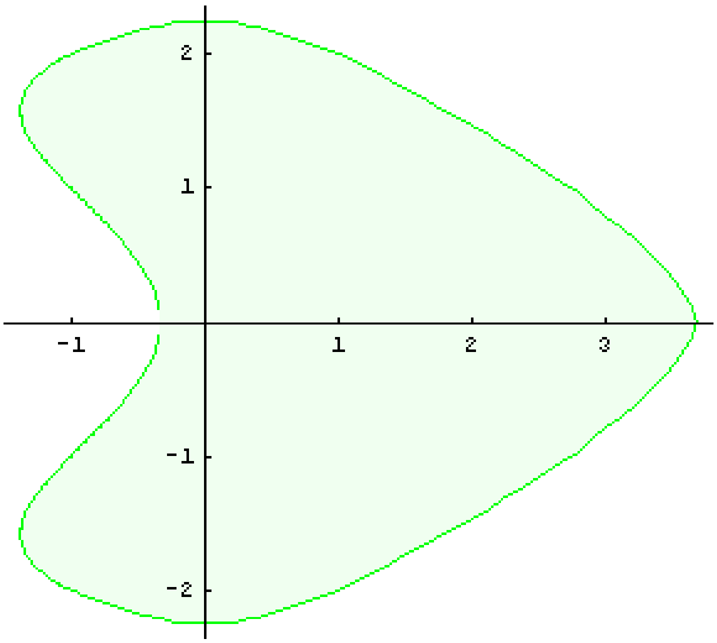
$$-\frac{5}{4} - \frac{3}{3+x^2} \le y \le \frac{5}{4} + \frac{3}{3+x^2}$$

and

$$-\frac{1}{3} - \frac{5y^2}{6} + \frac{y^4}{6} \le x \le \frac{11}{3} - \frac{2}{3}\sqrt{2}\,|y|^{3/2}$$

Solution 5.

Set up the constraints for the region.

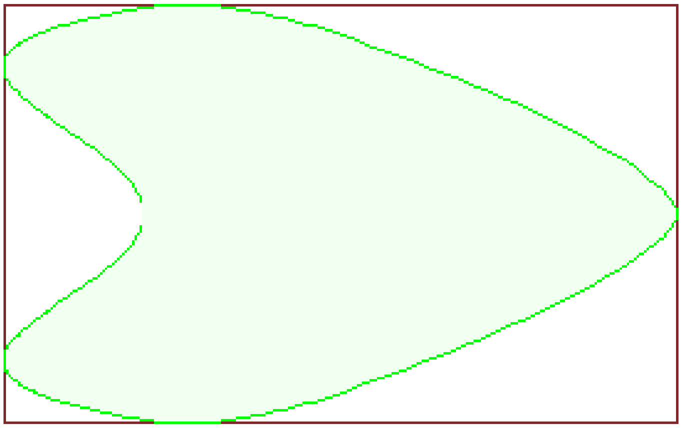


The Area A defined by

$$-\frac{5}{4} - \frac{3}{3+x^2} \le y \le \frac{5}{4} + \frac{3}{3+x^2}$$

$$-\frac{1}{3} - \frac{5y^2}{6} + \frac{y^4}{6} \le x \le \frac{11}{3} - \frac{2}{3}\sqrt{2}\,\text{Abs}[y]^{3/2}$$

Set up the rectangular box enclosing the area.



To find the area of the region we must have a rectalgle containing the region.

$$R = \{(x,y):\ -\frac{11}{8} \le x \le \frac{11}{3} \quad \text{and} \quad -\frac{9}{4} \le y \le \frac{9}{4}\ \}$$

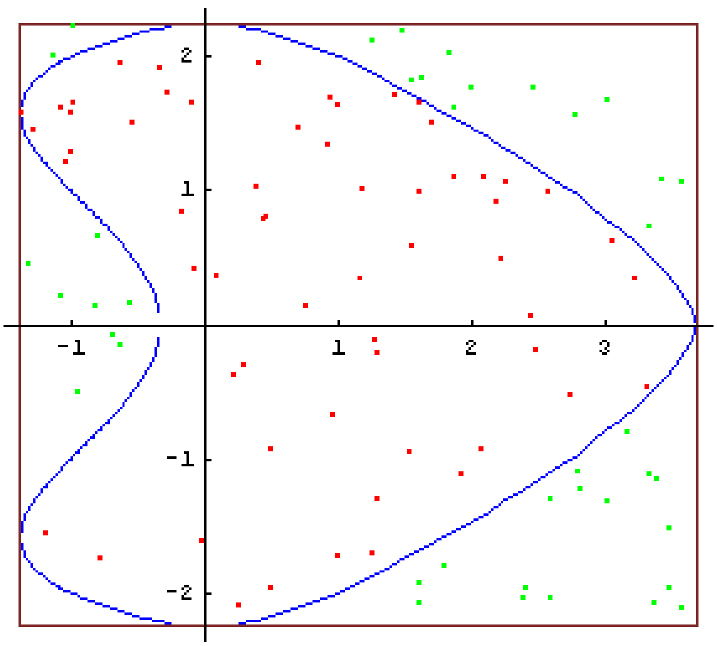
If this is correct, then Mathematica will proceed using

If not, then enter the correct value (s) of a, b, c, d.

Then invoke the subroutine 'MonteCarloArea'

Use Monte Carlo simulation and explore what happens with $n = 100, 1000,$ and 10000 points.

```
val1 = MonteCarloArea[100];
```



To find the area satisfying the constraint(s)

$$-\frac{5}{4} - \frac{3}{3+x^2} \le y \le \frac{5}{4} + \frac{3}{3+x^2}$$

$$-\frac{1}{3} - \frac{5y^2}{6} + \frac{y^4}{6} \le x \le \frac{11}{3} - \frac{2}{3}\sqrt{2}\,\mathrm{Abs}[y]^{3/2}$$

Points generated $n = 100$

Points inside the area $m = 60$

Points outside the area $k = 40$

The ratio $\rho = \frac{m}{n} = 60/100$

$$\rho = \frac{3}{5} = 0.6$$

$$\text{BoxArea} = \frac{363}{16}$$

Calculation for the area

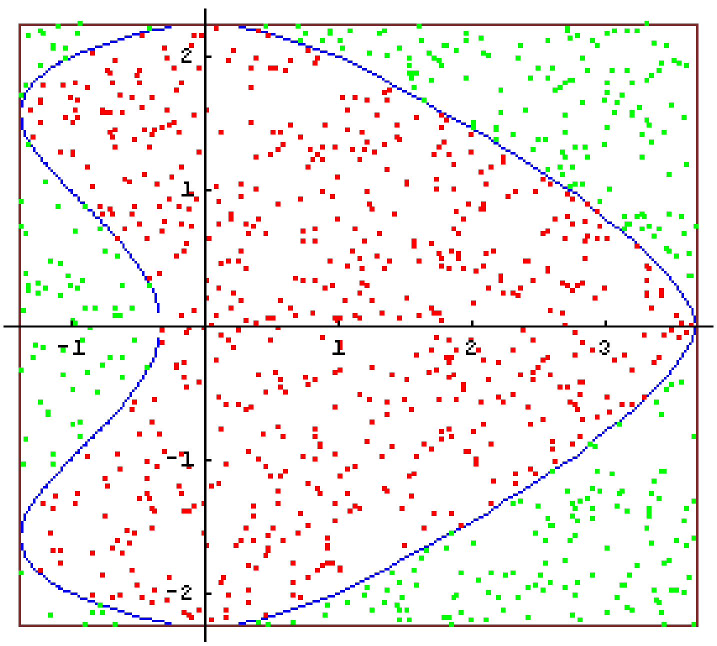
$$\rho * \text{BoxArea} = \frac{3}{5} * \frac{363}{16} = 13.6125$$

Monte Carlo Approximation

The estimate for the area $A \approx 13.6125$

Estimate for the error $\Delta A \approx 1.11146$

```
val2 = MonteCarloArea[1000];
```



To find the area satisfying the constraint(s)

$$-\frac{5}{4} - \frac{3}{3+x^2} \le y \le \frac{5}{4} + \frac{3}{3+x^2}$$

$$-\frac{1}{3} - \frac{5\,y^2}{6} + \frac{y^4}{6} \le x \le \frac{11}{3} - \frac{2}{3}\sqrt{2}\ \mathrm{Abs}[y]^{3/2}$$

Points generated $n = 1000$

Points inside the area $m = 636$

Points outside the area $k = 364$

The ratio $\rho = \frac{m}{n} = 636/1000$

$$\rho = \frac{159}{250} = 0.636$$

$$\text{BoxArea} = \frac{363}{16}$$

Calculation for the area

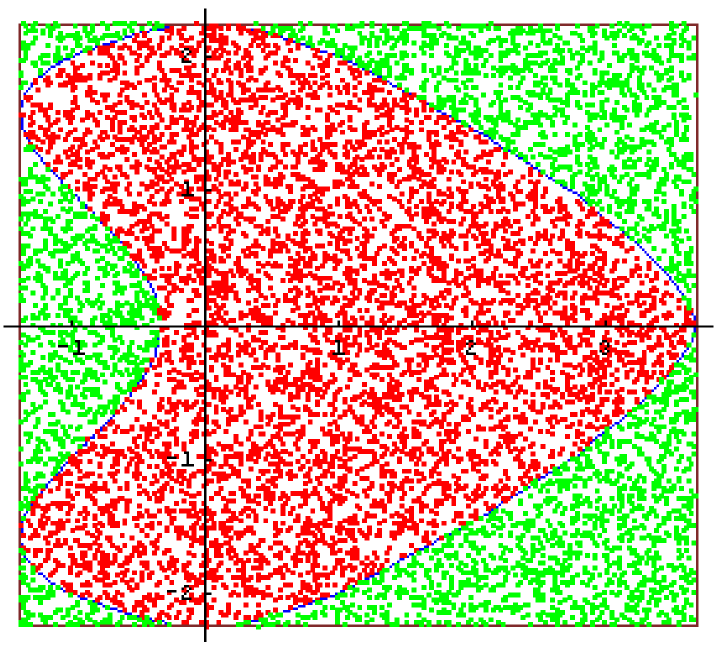
$$\rho * \text{BoxArea} = \frac{159}{250} * \frac{363}{16} = 14.4293$$

Monte Carlo Approximation

The estimate for the area $A \approx 14.4293$

Estimate for the error $\Delta A \approx 0.345196$

val3 = MonteCarloArea[10000];



To find the area satisfying the constraint(s)

$$-\frac{5}{4} - \frac{3}{3+x^2} \le y \le \frac{5}{4} + \frac{3}{3+x^2}$$

$$-\frac{1}{3} - \frac{5y^2}{6} + \frac{y^4}{6} \le x \le \frac{11}{3} - \frac{2}{3}\sqrt{2}\,\mathrm{Abs}[y]^{3/2}$$

Points generated $n = 10000$

Points inside the area $m = 6440$

Points outside the area $k = 3560$

The ratio $\rho = \frac{m}{n} = 6440/10000$

$$\rho = \frac{161}{250} = 0.644$$

$$\mathrm{BoxArea} = \frac{363}{16}$$

Calculation for the area

$$\rho * \mathrm{BoxArea} = \frac{161}{250} * \frac{363}{16} = 14.6108$$

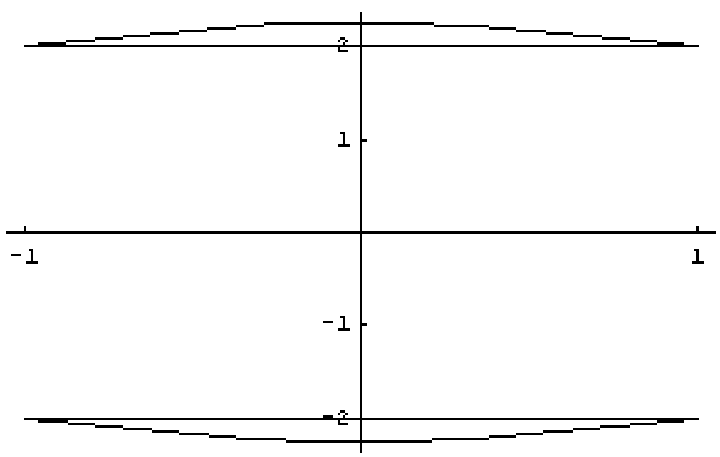
Monte Carlo Approximation

The estimate for the area $A \approx 14.6108$

Estimate for the error $\Delta A \approx 0.108631$

Aside. The analytic value of the area can be found by splitting it up into three parts.

The curved portion at the top from $(-1, 2)$ to $(1, 2)$ and the curved portion at the bottom from $(-1, 2)$ to $(1, 2)$ are given by the following formulas.



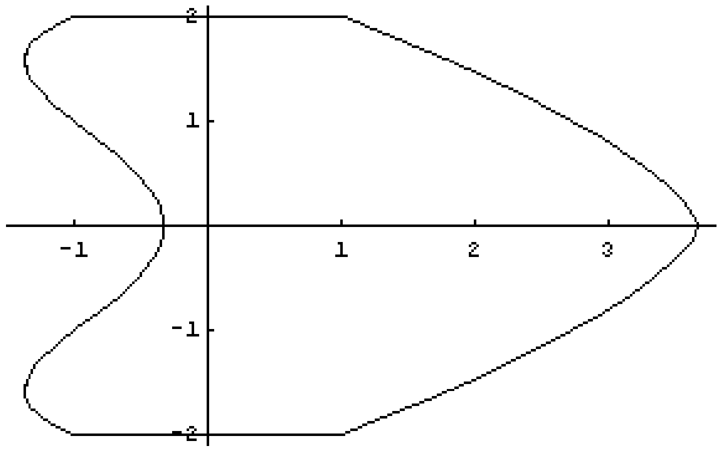
The area above y = 2, and below $y = f_2[x] = \frac{5}{4} + \frac{3}{3+x^2}$ is

$$\text{area}_1 = \int_{-1}^{1} (f_2[x]-2)\,dx = \frac{1}{2}\left(-3 + \frac{2\pi}{\sqrt{3}}\right)$$

The area below y = 2, and above $y = f_1[x] = -\frac{5}{4} - \frac{3}{3+x^2}$ is

$$\text{area}_2 = \int_{-1}^{1} (-2-f_1[x])\,dx = \frac{1}{2}\left(-3 + \frac{2\pi}{\sqrt{3}}\right)$$

The curved portion at the left from $(-1, -2)$ to $(-1, 2)$ and the curved portion at the right from $(1, -2)$ to $(1, 2)$ are given by the following formulas.

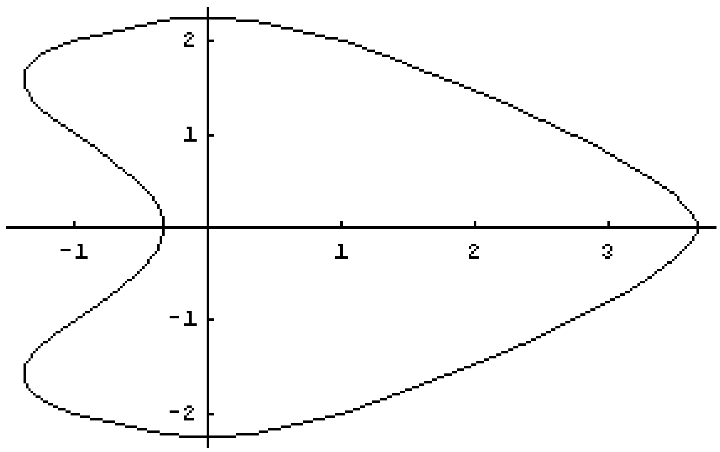


The area to the right of $x = g_1[x] = -\frac{1}{3} - \frac{5x^2}{6} + \frac{x^4}{6}$

that lies to the left of $x = g_2[x] = \frac{11}{3} - \frac{2}{3}\sqrt{2}\,\text{Abs}[x]^{3/2}$

and in the range $-2 \le y \le 2$ is

$$\text{area}_3 = \int_{-2}^{2} (g_2[y]-g_1[y])\,dy = \frac{632}{45}$$



$$area = area_1 + area_2 + area_3$$

$$area = \frac{497}{45} + \frac{2\pi}{\sqrt{3}}$$

$$area = 14.672$$

The approximations obtained with Monte Carlo simulation :

Using $n = 100$ $\quad area \approx 13.6125$

Using $n = 1000$ $\quad area \approx 14.4293$

Using $n = 10000$ $\quad area \approx 14.6108$