7. Jacobi and Gauss-Seidel Iteration

Background

Iterative schemes require time to achieve sufficient accuracy and are reserved for large systems of equations where there are a majority of zero elements in the matrix. Often times the algorithms are taylor-made to take advantage of the special structure such as band matrices. Practical uses include applications in circuit analysis, boundary value problems and partial differential equations.

Iteration is a popular technique finding roots of equations. Generalization of fixed point iteration can be applied to systems of linear equations to produce accurate results. The method Jacobi iteration is attributed to Carl Jacobi (1804-1851) and Gauss-Seidel iteration is attributed to Johann Carl Friedrich Gauss (1777-1855) and Philipp Ludwig von Seidel (1821-1896).

Consider that the n×n square matrix $\mathbf{A}$ is split into three parts, the main diagonal $\mathbf{D}$, below diagonal $\mathbf{L}$ and above diagonal $\mathbf{U}$. We have $\mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{U}$.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-2,1} & a_{n-2,2} & a_{n-2,3} & \cdots & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n-2} & a_{n,n-1} & a_{n,n} \end{pmatrix} =$$

$$\begin{pmatrix} a_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{2,2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-2,n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_{n,n} \end{pmatrix} -$$

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_{2,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_{3,1} & -a_{3,2} & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -a_{n-2,1} & -a_{n-2,2} & -a_{n-2,3} & \cdots & 0 & 0 & 0 \\ -a_{n-1,1} & -a_{n-1,2} & -a_{n-1,3} & \cdots & -a_{n-1,n-2} & 0 & 0 \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & \cdots & -a_{n,n-2} & -a_{n,n-1} & 0 \end{pmatrix} -$$

$$\begin{pmatrix} 0 & -a_{1,2} & -a_{1,3} & \cdots & -a_{1,n-2} & -a_{1,n-1} & -a_{1,n} \\ 0 & 0 & -a_{2,3} & \cdots & -a_{2,n-2} & -a_{2,n-1} & -a_{2,n} \\ 0 & 0 & 0 & \cdots & -a_{3,n-2} & -a_{3,n-1} & -a_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2,n-1} & -a_{n-2,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Definition (Diagonally Dominant). The matrix $\mathbf{A}$ is strictly diagonally dominant if

$$|a_{i,i}| > \sum_{j=1}^{i-1} |a_{i,j}| + \sum_{j=i+1}^{n} |a_{i,j}| \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Theorem (Jacobi Iteration). The solution to the linear system $\mathbf{AX} = \mathbf{B}$ can be obtained starting with $\mathbf{P}_0$, and using iteration scheme

$$\mathbf{P}_{k+1} = \mathbf{M}_J \mathbf{P}_k + \mathbf{C}_J$$

where

$$\mathbf{M}_J = \mathbf{D}^{-1}(\mathbf{L} + \mathbf{U}) \quad \text{and} \quad \mathbf{C}_J = \mathbf{D}^{-1}\mathbf{B}.$$

If $\mathbf{P}_0$ is carefully chosen a sequence $\{\mathbf{P}_k\}$ is generated which converges to the solution **P**, i.e. $\mathbf{AP} = \mathbf{B}$.
A sufficient condition for the method to be applicable is that **A** is strictly diagonally dominant or diagonally dominant and irreducible.

Theorem (Gauss-Seidel Iteration). The solution to the linear system $\mathbf{AX} = \mathbf{B}$ can be obtained starting with $\mathbf{P}_0$, and using iteration scheme

$$\mathbf{P}_{k+1} = \mathbf{M}_S \mathbf{P}_k + \mathbf{C}_S$$

where

$$\mathbf{M}_S = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{U} \quad \text{and} \quad \mathbf{C}_S = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{B}.$$

If $\mathbf{P}_0$ is carefully chosen a sequence $\{\mathbf{P}_k\}$ is generated which converges to the solution **P**, i.e. $\mathbf{AP} = \mathbf{B}$.
A sufficient condition for the method to be applicable is that **A** is strictly diagonally dominant or diagonally dominant and irreducible.

Example 1. Use Jacobi iteration to solve the linear system $\begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix}$. Try 10 iterations.

Solution 1.

Example 2. Use Gauss-Seidel iteration to attempt solving the linear system

$$\begin{pmatrix} 2 & 8 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 7 & -2 & 1 & 2 \\ -1 & 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

. Try 10 iterations.

Solution 2.

Example 1. Use Jacobi iteration to solve the linear system $\begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix}$. Try 10 iterations.

Solution 1.

$$P_0 = \{0, -1, 1, 1\}$$
$$P_1 = \{-0.285714, -0.75, 0.6, 1.75\}$$
$$P_2 = \{-0.371429, -0.622321, 0.242857, 1.525\}$$
$$P_3 = \{-0.219643, -0.438839, 0.315714, 1.37188\}$$
$$P_4 = \{-0.133878, -0.484967, 0.407321, 1.29835\}$$
$$P_5 = \{-0.139136, -0.53157, 0.453885, 1.34431\}$$
$$P_6 = \{-0.172236, -0.553462, 0.434447, 1.37926\}$$
$$P_7 = \{-0.185698, -0.542266, 0.41385, 1.38534\}$$
$$P_8 = \{-0.181295, -0.531937, 0.408723, 1.3746\}$$
$$P_9 = \{-0.174541, -0.529772, 0.413903, 1.36815\}$$
$$P_{10} = \{-0.172821, -0.532597, 0.417832, 1.36836\}$$

Determine if the method has converged.

$$A\,X = \begin{pmatrix} 3.01 \\ -1.98456 \\ 4.9987 \\ 3.99042 \end{pmatrix} \approx \begin{pmatrix} 3 \\ -2 \\ 5 \\ 4 \end{pmatrix} = B$$

Example 4 Use Gauss-Seidel iteration to attempt solving the linear system

$$\begin{pmatrix} 2 & 8 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 7 & -2 & 1 & 2 \\ -1 & 0 & 5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \\ 5 \end{pmatrix}$$

. Try 10 iterations.

Solution 4.

$P_0 = \{-1, 1, 0, 1\}$

$P_1 = \{-5.5, 0., 39.5, -99.\}$

$P_2 = \{-10.75, 219.75, 715.75, -1792.25\}$

$P_3 = \{-1057.5, 3944.38, 18878.8, -47723.1\}$

$P_4 = \{-20235.1, 104888., 446870., -1.12729 \times 10^6\}$

$P_5 = \{-526211., 2.47802 \times 10^6, 1.08941 \times 10^7, -2.74983 \times 10^7\}$

$P_6 = \{-1.2504 \times 10^7, 6.04437 \times 10^7, 2.63412 \times 10^8, -6.64783 \times 10^8\}$

$P_7 = \{-3.04502 \times 10^8, 1.46127 \times 10^9, 6.38363 \times 10^9, -1.61113 \times 10^{10}\}$

$P_8 = \{-7.36487 \times 10^9, 3.54144 \times 10^{10}, 1.54606 \times 10^{11}, -3.90196 \times 10^{11}\}$

$P_9 = \{-1.78468 \times 10^{11}, 8.57696 \times 10^{11}, 3.74506 \times 10^{12}, -9.45188 \times 10^{12}\}$

$P_{10} = \{-4.32243 \times 10^{12}, 2.07763 \times 10^{13}, 9.07134 \times 10^{13}, -2.28945 \times 10^{14}\}$

$$A\ X = \begin{pmatrix} 2.00761 \times 10^{14} \\ -9.64939 \times 10^{14} \\ -4.38986 \times 10^{14} \\ 5.00879 \end{pmatrix} \approx \begin{pmatrix} -2 \\ 4 \\ 3 \\ 5 \end{pmatrix} = B$$

Was a solution found ? Why ?

$$A = \begin{pmatrix} 2 & 8 & 3 & 1 \\ 0 & 2 & -1 & 4 \\ 7 & -2 & 1 & 2 \\ -1 & 0 & 5 & 2 \end{pmatrix}$$

The matrix **A** is not diagonally dominant! If you rearrange **A**,

$$\mathbf{A} = \begin{pmatrix} 7 & -2 & 1 & 2 \\ 2 & 8 & 3 & 1 \\ -1 & 0 & 5 & 2 \\ 0 & 2 & -1 & 4 \end{pmatrix};$$

Dominant[A];

then the matrix will be strictly diagonally dominant. Now Jacobi iteration should converge.