2. The Power Method for Eigenvectors

Power Method

We now describe the power method for computing the dominant eigenpair. Its extension to the inverse power method is practical for finding any eigenvalue provided that a good initial approximation is known. Some schemes for finding eigenvalues use other methods that converge fast, but have limited precision. The inverse power method is then invoked to refine the numerical values and gain full precision. To discuss the situation, we will need the following definitions.

Definition If λ_1 is an eigenvalue of $\mathbf{A}$ that is larger in absolute value than any other eigenvalue, it is called the dominant eigenvalue. An eigenvector $\mathbf{V}_1$ corresponding to λ_1 is called a dominant eigenvector.

Definition An eigenvector $\mathbf{V}$ is said to be normalized if the coordinate of largest magnitude is equal to unity (i.e., the largest coordinate in the vector $\mathbf{V}$ is the number 1).

Remark. It is easy to normalize an eigenvector $[v_1, v_2, \ldots, v_n]^T$ by forming a new vector $\mathbf{V} = \frac{1}{c}[v_1, v_2, \ldots, v_n]^T$ where $c = v_j$ and $v_j = \max_{1 \le i \le n}\{|v_i|\}$.

Theorem (Power Method) Assume that the $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ and that they are ordered in decreasing magnitude; that is, $|\lambda_1| > |\lambda_2| \ge |\lambda_3| \ge \ldots \ge |\lambda_n|$. If $\mathbf{X}_0$ is chosen appropriately, then the sequences $\left\{\mathbf{X}_k = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)^T\right\}$ and $\{c_k\}$ generated recursively by

$$\mathbf{Y}_k = \mathbf{A}\mathbf{X}_k$$

and

$$\mathbf{X}_{k+1} = \frac{1}{c_{k+1}}\mathbf{Y}_k$$

where $c_{k+1} = x_j^{(k)}$ and $x_j^{(k)} = \max_{1 \le i \le n}\left\{\left|x_i^{(k)}\right|\right\}$, will converge to the dominant eigenvector $\mathbf{V}_1$ and eigenvalue λ_1, respectively. That is,

$$\lim_{k\to\infty}\mathbf{X}_k = \mathbf{V}_1 \quad \text{and} \quad \lim_{k\to\infty} c_k = \lambda_1.$$

Remark. If $\mathbf{X}_0$ is an eigenvector and $\mathbf{X}_0 \ne \mathbf{V}$, then some other starting vector must be chosen.

Speed of Convergence

In the iteration in the theorem uses the equation

$$X_k = \frac{(\lambda_1)^k}{c_1 c_2 \ldots c_k} \left(b_1 V_1 + b_2 \left(\frac{\lambda_2}{\lambda_1}\right)^{k-1} V_2 + \ldots + b_n \left(\frac{\lambda_n}{\lambda_1}\right)^{k-1} V_n \right),$$

and the coefficient of V_j that is used to form X_k goes to zero in proportion to $\left(\frac{\lambda_j}{\lambda_1}\right)^k$. Hence, the speed of convergence of $\{X_k\}$ to V_1 is governed by the terms $\left(\frac{\lambda_2}{\lambda_1}\right)^k$. Consequently, the rate of convergence is linear. Similarly, the convergence of the sequence of constants $\{c_k\}$ to λ_1 is linear. The Aitken Δ^2 method can be used for any linearly convergent sequence $\{p_k\}$ to form a new sequence,

$$\hat{p}_k = p_k - \frac{(\Delta p_k)^2}{\Delta^2 p_k} = p_k - \frac{(p_{k+1} - p_k)^2}{p_{k+2} - 2\, p_{k+1} + p_k},$$

that converges faster. The Aitken Δ^2 can be adapted to speed up the convergence of the power method.

Shifted-Inverse Power Method

We will now discuss the shifted inverse power method. It requires a good starting approximation for an eigenvalue, and then iteration is used to obtain a precise solution. Other procedures such as the **QM** and Givens' method are used first to obtain the starting approximations. Cases involving complex eigenvalues, multiple eigenvalues, or the presence of two eigenvalues with the same magnitude or approximately the same
magnitude will cause computational difficulties and require more advanced methods. Our illustrations will focus on the case where the eigenvalues are distinct. The shifted inverse power method is based on the following three results (the proofs are left as exercises).

Theorem (Shifting Eigenvalues) Suppose that $\lambda, \mathbf{V}$ is an eigenpair of $\mathbf{A}$. If α is any constant, then $\lambda - \alpha, \mathbf{V}$ is an eigenpair of the matrix $\mathbf{A} - \alpha \mathbf{I}$.

Theorem (Inverse Eigenvalues) Suppose that $\lambda, \mathbf{V}$ is an eigenpair of $\mathbf{A}$. If $\lambda \neq 0$, then $\frac{1}{\lambda}, \mathbf{V}$ is an eigenpair of the matrix $\mathbf{A}^{-1}$.

Theorem (Shifted-Inverse Eigenvalues) Suppose that $\lambda, \mathbf{V}$ is an eigenpair of $\mathbf{A}$. If $\alpha \neq \lambda$, then $\frac{1}{\lambda - \alpha}, \mathbf{V}$ is an eigenpair of the matrix $(\mathbf{A} - \alpha \mathbf{I})^{-1}$.

Theorem (Shifted-Inverse Power Method) Assume that the $n \times n$ matrix $\mathbf{A}$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and consider the eigenvalue λ_j. Then a constant α can be chosen so that

$\mu_1 = \frac{1}{\lambda_j - \alpha}$ is the dominant eigenvalue of $(\mathbf{A} - \alpha\mathbf{I})^{-1}$. Furthermore, if $\mathbf{X}_0$ is chosen appropriately, then the sequences $\left\{\mathbf{X}_k = \left(x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}\right)^T\right\}$ and $\{c_k\}$ generated recursively by

$$\mathbf{Y}_k = (\mathbf{A} - \alpha\mathbf{I})^{-1}\mathbf{X}_k$$

and

$$\mathbf{X}_{k+1} = \frac{1}{c_{k+1}}\mathbf{Y}_k$$

where $c_{k+1} = x_j^{(k)}$ and $x_j^{(k)} = \max_{1 \le i \le n}\left\{\left|x_i^{(k)}\right|\right\}$, will converge to the dominant eigenpair $\mu_1, \mathbf{V}_j$ of the matrix $(\mathbf{A} - \alpha\mathbf{I})^{-1}$. Finally, the corresponding eigenvalue for the matrix $\mathbf{A}$ is given by the calculation

$$\lambda_j = \frac{1}{\mu_1} + \alpha$$

Remark. For practical implementations of this Theorem, a linear system solver is used to compute $\mathbf{Y}_k$ in each step by solving the linear system $(\mathbf{A} - \alpha\mathbf{I})\mathbf{Y}_k = \mathbf{X}_k$.

Example 1. Use the power method to find the dominant eigenvalue and eigenvector for the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}.$$

Solution 1.

Example 2. Use the power method to find the dominant eigenvalue and eigenvector for the matrix

$$\mathbf{A} = \begin{pmatrix} 87 & 270 & -12 & -49 & -276 & 40 \\ -14 & -45 & 6 & 10 & 46 & -4 \\ -50 & -156 & 4 & 25 & 162 & -25 \\ 94 & 294 & -5 & -47 & -306 & 49 \\ 1 & 1 & 3 & 1 & 0 & 2 \\ 16 & 48 & 1 & -6 & -48 & 8 \end{pmatrix}.$$

Solution 2.

Shifted Inverse Power Method

If a good approximation to an eigenvalue is known, then the shifted inverse power method can be used and convergence is faster. Other methods such as the QM method or Givens method are used to obtain approximate starting values.

Example 3. Find the dominant eigenvalue and eigenvector for the matrix $\mathbf{A} = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}$.

Use the shift $\alpha = 4.2$ in the shifted inverse power method.
Solution 3.

Application to Markov Chains

In the study of Markov chains the elements of the transition matrix are the probabilities of moving from any state to any other state. A Markov process can be described by a square matrix whose entries are all positive and the column sums are all equal to 1. For example, a 3×3 transition matrix looks like

$$\mathbf{A} = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix}$$

where $p_{1,1} + p_{2,1} + p_{3,1} = 1$, $p_{1,2} + p_{2,2} + p_{3,2} = 1$ and $p_{1,3} + p_{2,3} + p_{3,3} = 1$. The initial state vector is $\mathbf{P}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}$.

The computation $\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p_{1,1} x_0 \\ p_{2,1} x_0 \\ p_{3,1} x_0 \end{pmatrix} = x_0 \begin{pmatrix} p_{1,1} \\ p_{2,1} \\ p_{3,1} \end{pmatrix}$ shows how the x_0 is redistributed in the next state. Similarly we see that

$\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ y_0 \\ 0 \end{pmatrix} = \begin{pmatrix} p_{1,2} y_0 \\ p_{2,2} y_0 \\ p_{3,2} y_0 \end{pmatrix} = y_0 \begin{pmatrix} p_{1,2} \\ p_{2,2} \\ p_{3,2} \end{pmatrix}$ shows how the x_0 is redistributed in the next state.
and
$\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix} = \begin{pmatrix} p_{1,3} z_0 \\ p_{2,3} z_0 \\ p_{3,3} z_0 \end{pmatrix} = z_0 \begin{pmatrix} p_{1,3} \\ p_{2,3} \\ p_{3,3} \end{pmatrix}$ shows how the x_0 is redistributed in the next state.

Therefore, the distribution for the next state is

$$\mathbf{P}_1 = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} \\ p_{2,1} & p_{2,2} & p_{2,3} \\ p_{3,1} & p_{3,2} & p_{3,3} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \mathbf{A}\,\mathbf{P}_0$$

A recursive sequence is generated using the general rule

$$\mathbf{P}_{k+1} = \mathbf{A}\,\mathbf{P}_k \qquad \text{for } k = 0, 1, 2, \ldots .$$

We desire to know the limiting distribution $\mathbf{P} = \lim_{k\to\infty} \mathbf{P}_k$. Since we will also have $\lim_{k\to\infty} \mathbf{P}_{k+1} = \mathbf{P}$ we obtain the relation

$$\mathbf{P} = \lim_{k\to\infty} \mathbf{P}_{k+1} = \mathbf{A}\left(\lim_{k\to\infty} \mathbf{P}_k\right) = \mathbf{A}\,\mathbf{P}$$

From which it follows that

$$\mathbf{P} = \mathbf{A}\,\mathbf{P}$$

Therefore the limiting distribution $\mathbf{P}$ is the eigenvector corresponding to the dominant eigenvalue $\lambda_1 = 1$. The following subroutine reminds us of the iteration used in the power method.

Example 4. Let $\mathbf{P}_0 = (x^{(0)}, y^{(0)}, z^{(0)})^T$ record the number of people in a certain city who use brands X, Y, and Z, respectively.

Each month people decide to keep using the same brand or switch brands.

The probability that a user of brand X will switch to brand Y or Z is 0.3 and 0.3, respectively.

The probability that a user of brand Y will switch to brand X or Z is 0.3 and 0.2, respectively.

The probability that a user of brand Z will switch to brand X or Y is 0.1 and 0.3, respectively.

The transition matrix for this process is $\mathbf{P}_{k+1} = \mathbf{A}\,\mathbf{P}_k$ or

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix}$$

Assume that the initial distribution $\mathbf{P}_0 = (2000, 6000, 4000)^T$.

4 (a). Find the first few terms in the sequence $\{\mathbf{P}_k\}$.

4 (b). Verify that $\lambda_1 = 1$ is the dominant eigenvector of **A**.

4 (c). Verify that a corresponding eigenvector is $\mathbf{V}_1 = (3000, 4500, 4500)^T$.

4 (d). Conclude that the limiting distribution is $\lim_{k\to\infty} \mathbf{P}_k = \mathbf{V}_1$.

Solution 4.

Example 1. Use the power method to find the dominant eigenvalue and eigenvector for the matrix

$$A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}.$$

Solution 1.

For illustration purposes we will set the maximum number of iterations to be 50 and $\epsilon = 0.000001$.

$$A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

```
12. , { 0.500000, 0.666667, 1.000000}
5.33333 , { 0.437500, 0.625000, 1.000000}
4.5 , { 0.416667, 0.611111, 1.000000}
4.22222 , { 0.407895, 0.605263, 1.000000}
4.10526 , { 0.403846, 0.602564, 1.000000}
4.05128 , { 0.401899, 0.601266, 1.000000}
4.02532 , { 0.400943, 0.600629, 1.000000}
4.01258 , { 0.400470, 0.600313, 1.000000}
4.00627 , { 0.400235, 0.600156, 1.000000}
4.00313 , { 0.400117, 0.600078, 1.000000}
4.00156 , { 0.400059, 0.600039, 1.000000}
4.00078 , { 0.400029, 0.600020, 1.000000}
4.00039 , { 0.400015, 0.600010, 1.000000}
4.0002 , { 0.400007, 0.600005, 1.000000}
4.0001 , { 0.400004, 0.600002, 1.000000}
4.00005 , { 0.400002, 0.600001, 1.000000}
4.00002 , { 0.400001, 0.600001, 1.000000}
4.00001 , { 0.400000, 0.600000, 1.000000}
4.00001 , { 0.400000, 0.600000, 1.000000}
4. , { 0.400000, 0.600000, 1.000000}
4. , { 0.400000, 0.600000, 1.000000}
4. , { 0.400000, 0.600000, 1.000000}
```

$$A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}$$

The \`dominant\` eigenpair is

$$\lambda = 4., \quad X = \begin{pmatrix} 0.4 \\ 0.6 \\ 1. \end{pmatrix}$$

That is $\in$ close to the dominant eigenvalue $\lambda = 4$ and corresponding eigenvector $\mathbf{X} = \begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \\ 1 \end{pmatrix}$.

Example 2. Use the power method to find the dominant eigenvalue and eigenvector for the matrix

$$A = \begin{pmatrix} 87 & 270 & -12 & -49 & -276 & 40 \\ -14 & -45 & 6 & 10 & 46 & -4 \\ -50 & -156 & 4 & 25 & 162 & -25 \\ 94 & 294 & -5 & -47 & -306 & 49 \\ 1 & 1 & 3 & 1 & 0 & 2 \\ 16 & 48 & 1 & -6 & -48 & 8 \end{pmatrix}.$$

Solution 2.

For illustration purposes we will set the maximum number of iterations to be 50 and $\epsilon = 0.000002$.

$$A = \begin{pmatrix} 87 & 270 & -12 & -49 & -276 & 40 \\ -14 & -45 & 6 & 10 & 46 & -4 \\ -50 & -156 & 4 & 25 & 162 & -25 \\ 94 & 294 & -5 & -47 & -306 & 49 \\ 1 & 1 & 3 & 1 & 0 & 2 \\ 16 & 48 & 1 & -6 & -48 & 8 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

```
79.     , { 0.759494, -0.012658, -0.506329, 1.000000, 0.101266, 0.240506}
4.      , { 0.351266,  0.148734, -0.658228, 1.000000, 0.177215, 0.525316}
4.55063 , { 0.377608,  0.107789, -0.620306, 1.000000, 0.126565, 0.394993}
3.91238 , { 0.323676,  0.097938, -0.651262, 1.000000, 0.105937, 0.429435}
4.10132 , { 0.330892,  0.075366, -0.644015, 1.000000, 0.079657, 0.385022}
3.9727  , { 0.322268,  0.059422, -0.654201, 1.000000, 0.061484, 0.383741}
4.0235  , { 0.326023,  0.045320, -0.654447, 1.000000, 0.046370, 0.366110}
3.99283 , { 0.325637,  0.034599, -0.658622, 1.000000, 0.035118, 0.360754}
4.00577 , { 0.327791,  0.026170, -0.659906, 1.000000, 0.026431, 0.352657}
3.99818 , { 0.328630,  0.019769, -0.661876, 1.000000, 0.019899, 0.348528}
4.00143 , { 0.329861,  0.014886, -0.662891, 1.000000, 0.014951, 0.344422}
3.99954 , { 0.330593,  0.011199, -0.663905, 1.000000, 0.011231, 0.341824}
4.00036 , { 0.331292,  0.008414, -0.664549, 1.000000, 0.008431, 0.339625}
3.99989 , { 0.331768,  0.006319, -0.665096, 1.000000, 0.006327, 0.338096}
4.00009 , { 0.332163,  0.004743, -0.665477, 1.000000, 0.004747, 0.336886}
3.99997 , { 0.332447,  0.003560, -0.665779, 1.000000, 0.003562, 0.336009}
4.00002 , { 0.332670,  0.002671, -0.665998, 1.000000, 0.002672, 0.335335}
3.99999 , { 0.332833,  0.002004, -0.666166, 1.000000, 0.002004, 0.334837}
4.00001 , { 0.332959,  0.001503, -0.666291, 1.000000, 0.001503, 0.334460}
4.      , { 0.333052,  0.001127, -0.666385, 1.000000, 0.001127, 0.334179}
4.      , { 0.333122,  0.000846, -0.666455, 1.000000, 0.000846, 0.333967}
4.      , { 0.333175,  0.000634, -0.666508, 1.000000, 0.000634, 0.333809}
4.      , { 0.333214,  0.000476, -0.666548, 1.000000, 0.000476, 0.333690}
4.      , { 0.333244,  0.000357, -0.666577, 1.000000, 0.000357, 0.333601}
4.      , { 0.333266,  0.000268, -0.666600, 1.000000, 0.000268, 0.333534}
4.      , { 0.333283,  0.000201, -0.666617, 1.000000, 0.000201, 0.333484}
4.      , { 0.333296,  0.000151, -0.666629, 1.000000, 0.000151, 0.333446}
4.      , { 0.333305,  0.000113, -0.666638, 1.000000, 0.000113, 0.333418}
```

$4.\ ,\ \{0.333312,\ 0.000085, -0.666646,\ 1.000000,\ 0.000085,\ 0.333397\}$
$4.\ ,\ \{0.333317,\ 0.000063, -0.666651,\ 1.000000,\ 0.000063,\ 0.333381\}$
$4.\ ,\ \{0.333321,\ 0.000048, -0.666655,\ 1.000000,\ 0.000048,\ 0.333369\}$
$4.\ ,\ \{0.333324,\ 0.000036, -0.666658,\ 1.000000,\ 0.000036,\ 0.333360\}$
$4.\ ,\ \{0.333327,\ 0.000027, -0.666660,\ 1.000000,\ 0.000027,\ 0.333353\}$
$4.\ ,\ \{0.333328,\ 0.000020, -0.666662,\ 1.000000,\ 0.000020,\ 0.333348\}$
$4.\ ,\ \{0.333330,\ 0.000015, -0.666663,\ 1.000000,\ 0.000015,\ 0.333345\}$
$4.\ ,\ \{0.333331,\ 0.000011, -0.666664,\ 1.000000,\ 0.000011,\ 0.333342\}$
$4.\ ,\ \{0.333331,\ 8.475670\times 10^{-6}, -0.666665,\ 1.000000,\ 8.475670\times 10^{-6},\ 0.333340\}$
$4.\ ,\ \{0.333332,\ 6.356750\times 10^{-6}, -0.666665,\ 1.000000,\ 6.356750\times 10^{-6},\ 0.333338\}$
$4.\ ,\ \{0.333332,\ 4.767570\times 10^{-6}, -0.666665,\ 1.000000,\ 4.767570\times 10^{-6},\ 0.333337\}$
$4.\ ,\ \{0.333332,\ 3.575670\times 10^{-6}, -0.666666,\ 1.000000,\ 3.575670\times 10^{-6},\ 0.333336\}$

$$A = \begin{pmatrix} 87 & 270 & -12 & -49 & -276 & 40 \\ -14 & -45 & 6 & 10 & 46 & -4 \\ -50 & -156 & 4 & 25 & 162 & -25 \\ 94 & 294 & -5 & -47 & -306 & 49 \\ 1 & 1 & 3 & 1 & 0 & 2 \\ 16 & 48 & 1 & -6 & -48 & 8 \end{pmatrix}$$

The \`dominant\` eigenpair is

$$\lambda = 4.\ ,\quad X = \begin{pmatrix} 0.333332 \\ 3.57567\times 10^{-6} \\ -0.666666 \\ 1. \\ 3.57567\times 10^{-6} \\ 0.333336 \end{pmatrix} \approx \begin{pmatrix} 0.333332 \\ 0 \\ -0.666666 \\ 1. \\ 0 \\ 0.333336 \end{pmatrix}$$

That is 2ϵ close to the dominant eigenvalue $\lambda = 4$ and corresponding eigenvector

$$\mathbf{X} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ -\frac{2}{3} \\ 1 \\ 0 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0.333333 \\ 0. \\ -0.666667 \\ 1. \\ 0. \\ 0.333333 \end{pmatrix}.$$

Example 3. Find the dominant eigenvalue and eigenvector for the matrix $A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}$.

Use the shift $\alpha = 4.2$ in the shifted inverse power method.

Solution 3.

$$A = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix}, \quad X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The `shift` is $\alpha = 4.2$

```
4.1568627451,  { 0.411764705882,  0.607843137255,  1.000000000000}
4.01331114809,  { 0.400998336106,  0.600665557404,  1.000000000000}
4.00120282664,  { 0.400090211998,  0.600060141332,  1.000000000000}
4.00010928812,  { 0.400008196609,  0.600005464406,  1.000000000000}
4.00000993479,  { 0.400000745109,  0.600000496740,  1.000000000000}
4.00000090316,  { 0.400000067737,  0.600000045158,  1.000000000000}
4.00000008211,  { 0.400000006158,  0.600000004105,  1.000000000000}
4.00000000746,  { 0.400000000560,  0.600000000373,  1.000000000000}
4.00000000068,  { 0.400000000051,  0.600000000034,  1.000000000000}
4.00000000006,  { 0.400000000005,  0.600000000003,  1.000000000000}
```

The eigenvalue is $\lambda = 4.$

The eigenvector is $X = \begin{pmatrix} 0.4 \\ 0.6 \\ 1. \end{pmatrix}$

$$A\,X = \begin{pmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.6 \\ 1. \end{pmatrix} = \begin{pmatrix} 1.6 \\ 2.4 \\ 4. \end{pmatrix}$$

$$\lambda\,X = 4. \begin{pmatrix} 0.4 \\ 0.6 \\ 1. \end{pmatrix} = \begin{pmatrix} 1.6 \\ 2.4 \\ 4. \end{pmatrix}$$

That is ϵ close to the dominant eigenvalue $\lambda = 4$ and corresponding eigenvector $\mathbf{X} = \begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \\ 1 \end{pmatrix}$.

Example 4. Let $P_0 = (x^{(0)}, y^{(0)}, z^{(0)})^T$ record the number of people in a certain city who use brands X, Y, and Z, respectively.

Each month people decide to keep using the same brand or switch brands.

The probability that a user of brand X will switch to brand Y or Z is 0.3 and 0.3, respectively.

The probability that a user of brand Y will switch to brand X or Z is 0.3 and 0.2, respectively.

The probability that a user of brand Z will switch to brand X or Y is 0.1 and 0.3, respectively.

The transition matrix for this process is $P_{k+1} = A\,P_k$ or

$$\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \\ z^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \\ z^{(k)} \end{pmatrix}$$

Assume that the initial distribution $P_0 = (2000, 6000, 4000)^T$.

4 (a). Find the first few terms in the sequence $\{P_k\}$.

4 (b). Verify that $\lambda_1 = 1$ is the dominant eigenvector of **A**.

4 (c). Verify that a corresponding eigenvector is $V_1 = (3000, 4500, 4500)^T$.

4 (d). Conclude that the limiting distribution is $\lim_{k\to\infty} P_k = V_1$.

Solution 4.

4 (a). Enter the matrix **A** and vector P_0 and use the subroutine Markov to find the first few terms in the sequence $\{P_k\}$.

$$A = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}$$

$$P_0 = \begin{pmatrix} 2000 \\ 6000 \\ 4000 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 3000. \\ 4800. \\ 4200. \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 3060. \\ 4560. \\ 4380. \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 3030. \\ 4512. \\ 4458. \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 3011.4 \\ 4502.4 \\ 4486.2 \end{pmatrix}$$

$$P_5 = \begin{pmatrix} 3003.9 \\ 4500.48 \\ 4495.62 \end{pmatrix}$$

$$P_6 = \begin{pmatrix} 3001.27 \\ 4500.1 \\ 4498.64 \end{pmatrix}$$

$$P_7 = \begin{pmatrix} 3000.4 \\ 4500.02 \\ 4499.58 \end{pmatrix}$$

$$P_8 = \begin{pmatrix} 3000.12 \\ 4500. \\ 4499.87 \end{pmatrix}$$

$$P_9 = \begin{pmatrix} 3000.04 \\ 4500. \\ 4499.96 \end{pmatrix}$$

$$P_{10} = \begin{pmatrix} 3000.01 \\ 4500. \\ 4499.99 \end{pmatrix}$$

$$P_{11} = \begin{pmatrix} 3000. \\ 4500. \\ 4500. \end{pmatrix}$$

4 (b). Verify that $\lambda_1 = 1$ is an eigenvector of **A**.

$$A = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}$$

The eigenvalues are $\{1., 0.3, 0.2\}$

4 (c). Verify that $\lambda_1 = 1$ is an eigenvector of **A** and a corresponding eigenvector is $\mathbf{V}_1 = (3000, 4500, 4500)^T$.

$$A = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}$$

The eigenvalue is $\lambda = 1$

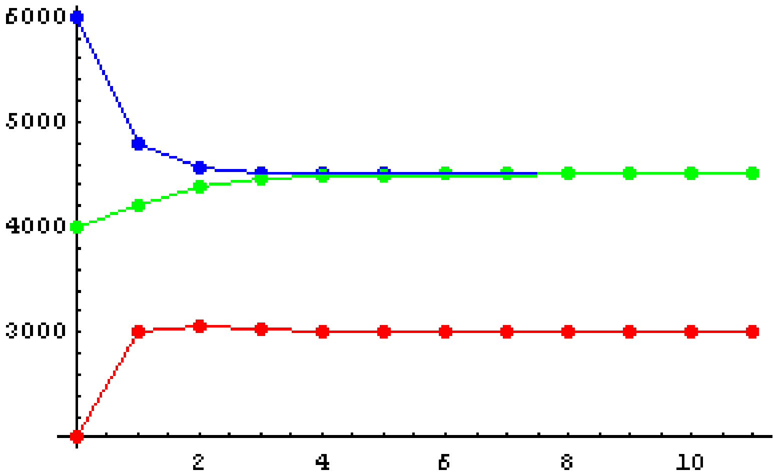
An eigenvector is $X = \begin{pmatrix} 3000 \\ 4500 \\ 4500 \end{pmatrix}$

$$A\,X = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 3000 \\ 4500 \\ 4500 \end{pmatrix} = \begin{pmatrix} 3000. \\ 4500. \\ 4500. \end{pmatrix}$$

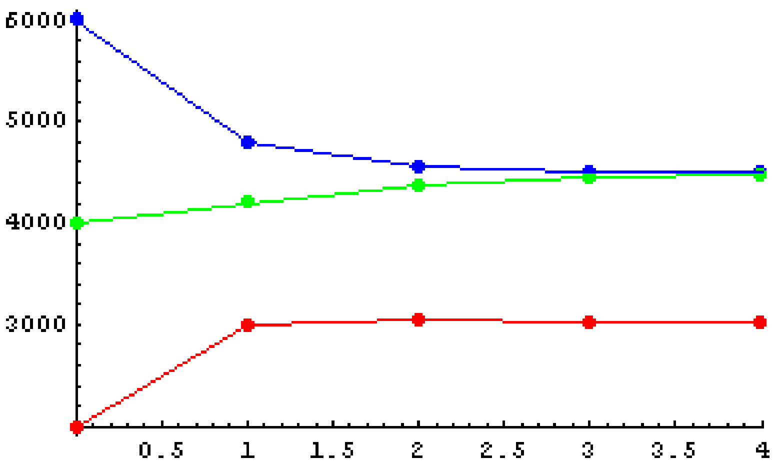
$$\lambda X = \begin{pmatrix} 3000 \\ 4500 \\ 4500 \end{pmatrix} = \begin{pmatrix} 3000 \\ 4500 \\ 4500 \end{pmatrix}$$

4 (d). The iteration in part (a) appears to be converging to $\mathbf{V}_1 = (3000, 4500, 4500)^T$.

Aside. We can graph the situation.



$$P_0 = \begin{pmatrix} 2000 \\ 6000 \\ 4000 \end{pmatrix}, \quad \lim_{k\to\infty} P_k = P = \begin{pmatrix} 3000 \\ 4500 \\ 4500 \end{pmatrix}$$



$$P_0 = \begin{pmatrix} 2000 \\ 6000 \\ 4000 \end{pmatrix}, \quad \lim_{k\to\infty} P_k = P = \begin{pmatrix} 3000 \\ 4500 \\ 4500 \end{pmatrix}$$