# 8. Picard Iteration

## Introduction

The term "Picard iteration" occurs two places in undergraduate mathematics. In numerical analysis it is used when discussing fixed point iteration for finding a numerical approximation to the equation $x = g(x)$. In differential equations, Picard iteration is a constructive procedure for establishing the existence of a solution to a differential equation $y' = f(x, y)$ that passes through the point $(x_0, y_0)$.

The first type of Picard iteration uses computations to generate a "sequence of numbers" which converges to a solution. We will not present this application, but mention that it involves the traditional role of computer software as a "number cruncher."

The goal of this article is to illustrate the second application of Picard iteration; i. e. how to use a computer to efficiently generate a "sequence of functions" which converges to a solution. We will see that computer software can perform the more sophisticated task of "symbol cruncher." For years, software such as Mathematica, Maple, Scientific Work Place, MuPad, Derive, LiveMath, Macsyma, REDUCE, Mathcad, and Matlab have been introduced in undergraduate mathematics courses to assist in keeping track of equations during complicated symbolic manipulations, see references [3 to 7].

## Background

Most differential equations texts give a proof for the existence and uniqueness of the solution to a first order differential equation. Then exercises are given for performing the laborious details involved in the method of successive approximations. The concept seems straightforward, just repeated integration, but students get bogged down with the details. Now computers can do all the drudgery and we can get a better grasp on how the process works.

**Theorem 1 (Existence Theorem).** If both $f(x, y)$ and $f_y(x, y)$ are continuous on the rectangle $R = \{(x, y) : a \le x \le b, c \le y \le d\}$ and $(x_0, y_0) \in R$, then there exists a unique solution to the initial value problem (I.V.P.)

(1) $\quad y' = f(x, y) \quad \text{with} \quad y(x_0) = y_0$

for all values of $x$ in some (smaller) interval $x_0 - \delta \le x \le x_0 + \delta$ contained in $a \le x \le b$.

## Picard's Method for D.E.'s

The method of successive approximations uses the equivalent integral equation for (1) and an iterative method for constructing approximations to the solution. This is a traditional way to prove (1) and appears in most all differential equations textbooks. It is attributed to the French mathematician Charles Emile Picard (1856-1941).

**Theorem 2 (Successive Approximations - Picard Iteration).** The solution to the I.V.P in (1) is found by constructing recursively a sequence $\{Y_n(x)\}_{n=0}^{\infty}$ of functions

(2)
$$Y_0(x) = y_0, \quad \text{and}$$
$$Y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, Y_n(t))\,dt \quad \text{for} \quad n \geq 0.$$

Then the solution $y(x)$ to (1) is given by the limit:

(3) $$y(x) = \lim_{n\to\infty} Y_n(x).$$

**Example 1.** Use Picard iteration to find the solution of the I.V.P.
$$y' = y - x \text{ with } y(0) = 2.$$
**Solution 1.**

**Example 2.** Use Picard iteration to find the solution of the I.V.P.
$$y' = 1 + y^2 \text{ with } y(0) = 0.$$
**Solution 2.**

**Example 3.** Use Picard iteration to find and plot approximations for the solution of the I.V.P.
$$y' = -2xy \text{ with } y(1) = 1.$$
**Solution 3.**

## Extension to First Order Systems in 2D

Suppose that we want to solve the initial value problem for a system of two differential equations

(7)
$$x' = f(t, x, y) \quad \text{with} \quad x(t_0) = x_0, \quad \text{and}$$
$$y' = g(t, x, y) \quad \text{with} \quad y(t_0) = y_0$$

Picard iteration can be used to generate two sequences $\{X_n(t)\}_{n=0}^{\infty}$ and $\{Y_n(t)\}_{n=0}^{\infty}$ which converge to the solutions $x(t)$ and $y(t)$, respectively, see reference [2]. They are defined recursively by

(8)
$$X_0(t) = x_0,$$
$$Y_0(t) = y_0, \quad \text{and}$$

$$X_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, X_n(s), Y_n(s))\, ds,$$

$$Y_{n+1}(t) = y_0 + \int_{t_0}^{t} g(s, X_n(s), Y_n(s))\, ds \quad \text{for} \quad n \geq 0$$

The sequence of approximations will converge to the solution, i.e.

(9)
$$\lim_{n\to\infty} X_n(t) = x(t)$$
and
$$\lim_{n\to\infty} Y_n(t) = y(t)$$

**Example 4.** Use Picard iteration to find the solution to the system of differential equations

$$x' = f(t, x, y) = x + 2y \quad \text{with} \quad x(0) = x_0 = 1$$
and
$$y' = g(t, x, y) = -x - y \quad \text{with} \quad y(0) = y_0 = 1$$

**Solution 4.**

## Extension to First Order Systems in 3D

Suppose that we want to solve the initial value problem for a system of three differential equations

$$x' = f(t, x, y, z) \quad \text{with} \quad x(0) = x_0$$
and
$$y' = g(t, x, y, z) \quad \text{with} \quad y(0) = y_0$$
and
$$z' = h(t, x, y, z) \quad \text{with} \quad z(0) = z_0$$

**Example 5.** Use Picard iteration to find and plot approximations for the solution of the I.V.P.

$$x' = f(t, x, y, z) = x + 2y - z \quad \text{with} \quad x(0) = x_0 = 1$$

$$y' = g(t, x, y, z) = -2x + y - 2z \quad \text{with} \quad y(0) = y_0 = 1$$

$$y' = g(t, x, y, z) = x + 2y + z \quad \text{with} \quad z(0) = z_0 = 1$$

**Solution 5.**

**Example 1.** Use Picard iteration to find the solution of the I.V.P.

$$y' = y - x \text{ with } y(0) = 2.$$

**Solution 1.**

First define the function $f(x, y)$ and the initial condition $y_0 = 2$ by typing:

Solve the I.V.P.

$$y' = -x + y \quad \text{with} \quad y(0) = 2$$

Picard iteration for generating the first six approximations is:

$$Y_0(x) = 2$$

$$Y_1(x) = 2 + 2x - \frac{x^2}{2}$$

$$Y_2(x) = 2 + 2x + \frac{x^2}{2} - \frac{x^3}{6}$$

$$Y_3(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$$

$$Y_4(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

$$Y_5(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} - \frac{x^6}{720}$$

$$Y_6(x) = 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040}$$

The I.V.P. is

$$y' = -x + y \quad \text{with} \quad y(0) = 2$$

After 6 iterations, we have the approximation

$$y(x) \approx 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} - \frac{x^7}{5040}$$

Techniques from calculus can be used to find the solution $y(x) = 1 + x + e^x$.

Notice that when the last term in the Picard approximation is dropped, what is left is a Maclaurin (or Taylor) polynomial approximation.

We can express $y(x) = 1 + x + e^x$ as a Maclaurin series and observe that the sequence $\{Y_n(x)\}_{n=0}^{\infty}$ is converging to the solution.

$$y(x) = (1+x) + e^{x}$$

$$y(x) = (1+x) + 1 + x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \frac{1}{4!}x^{4} + \frac{1}{5!}x^{5} + \ldots + \frac{1}{n!}x^{n}\ldots$$

$$y(x) = 2 + 2x + \frac{1}{2!}x^{2} + \frac{1}{3!}x^{3} + \frac{1}{4!}x^{4} + \frac{1}{5!}x^{5} + \ldots + \frac{1}{n!}x^{n}\ldots$$

which is the same as

$$y(x) = 2 + 2x + \frac{x^{2}}{2} + \frac{x^{3}}{6} + \frac{x^{4}}{24} + \frac{x^{5}}{120} + \ldots + \frac{1}{n!}x^{n}\ldots$$

Mathematica can sum the infinite series to obtain the analytic solution.

$$y(x) = 2 + 2x + \sum_{n=2}^{\infty}\frac{1}{n!}x^{n}$$

$$y(x) = 1 + e^{x} + x$$

**Example 2.** Use Picard iteration to find the solution of the I.V.P.

$$y' = 1 + y^2 \text{ with } y(0) = 0.$$

**Solution 2.**

First define the function $f(x, y)$ and the initial condition $y_0 = 2$ by typing:

Solve the I.V.P.

$$y' = 1 + y^2 \quad \text{with} \quad y(0) = 0$$

Picard iteration for generating the first six approximations is:

$$Y_0(x) = 0$$

$$Y_1(x) = x$$

$$Y_2(x) = x + \frac{x^3}{3}$$

$$Y_3(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{x^7}{63}$$

$$Y_4(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{38x^9}{2835} + \frac{134x^{11}}{51975} + \frac{4x^{13}}{12285} + \frac{x^{15}}{59535}$$

$$Y_5(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1142x^{11}}{155925} + \frac{13324x^{13}}{6081075} + \frac{377017x^{15}}{638512875} + \frac{1522814x^{17}}{10854718875} + \frac{24022x^{19}}{820945125} + \frac{29756x^{21}}{5746615875} + \frac{12676238x^{23}}{16962094524375} + \frac{256948x^{25}}{3016973334375} + \frac{100732x^{27}}{14119435204875} + \frac{8x^{29}}{21210236775} + \frac{x^{31}}{109876902975}$$

The I.V.P. is

$$y' = 1 + y^2 \quad \text{with} \quad y(0) = 0$$

After 5 iterations, we have the approximation

$$y(x) \approx x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1142x^{11}}{155925} + \frac{13324x^{13}}{6081075} + \frac{377017x^{15}}{638512875} + \frac{1522814x^{17}}{10854718875} + \frac{24022x^{19}}{820945125} + \frac{29756x^{21}}{5746615875} + \frac{12676238x^{23}}{16962094524375} + \frac{256948x^{25}}{3016973334375} + \frac{100732x^{27}}{14119435204875} + \frac{8x^{29}}{21210236775} + \frac{x^{31}}{109876902975}$$

Techniques from calculus can be used to find the solution $y(x) = \tan(x)$, and it is easy to verify this fact using the rules of differentiation and a trigonometric identity .

$$\begin{aligned} y'(x) &= 1 + (\tan(x))^2 \\ &= (\sec(x))^2 \\ &= \frac{d}{dx}\tan(x) \end{aligned}$$

The first five terms of the Picard approximation are the same as the Maclaurin series for $y(x) = \tan(x)$.

$$y(x) = \mathrm{Tan}[x] = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + O[x]^{10}$$

**Example 3.** Use Picard iteration to find and plot approximations for the solution of the I.V.P.

$y' = -2xy$ with $y(1) = 1$.

**Solution 3.**

Solve the I.V.P.

$y' = -2xy$ with $y(1) = 1$

Picard iteration for generating the first six approximations is:

$$Y_1(x) = 1$$

$$Y_2(x) = 1 - 2(-1+x) - (-1+x)^2$$

$$Y_3(x) = 1 - 2(-1+x) + (-1+x)^2 + 2(-1+x)^3 + \frac{1}{2}(-1+x)^4$$

$$Y_4(x) = 1 - 2(-1+x) + (-1+x)^2 + \frac{2}{3}(-1+x)^3 - \frac{3}{2}(-1+x)^4 - (-1+x)^5 - \frac{1}{6}(-1+x)^6$$

$$Y_5(x) = 1 - 2(-1+x) + (-1+x)^2 + \frac{2}{3}(-1+x)^3 - \frac{5}{6}(-1+x)^4 + \frac{1}{3}(-1+x)^5 + \frac{5}{6}(-1+x)^6 + \frac{1}{3}(-1+x)^7 + \frac{1}{24}(-1+x)^8$$

$$Y_6(x) = 1 - 2(-1+x) + (-1+x)^2 + \frac{2}{3}(-1+x)^3 - \frac{5}{6}(-1+x)^4 + \frac{1}{15}(-1+x)^5 + \frac{1}{6}(-1+x)^6 - \frac{1}{3}(-1+x)^7 - \frac{7}{24}(-1+x)^8 - \frac{1}{12}(-1+x)^9 - \frac{1}{120}(-1+x)^{10}$$

$$Y_7(x) = 1 - 2(-1+x) + (-1+x)^2 + \frac{2}{3}(-1+x)^3 - \frac{5}{6}(-1+x)^4 + \frac{1}{15}(-1+x)^5 + \frac{23}{90}(-1+x)^6 - \frac{1}{15}(-1+x)^7 + \frac{1}{24}(-1+x)^8 + \frac{5}{36}(-1+x)^9 + \frac{3}{40}(-1+x)^{10} + \frac{1}{60}(-1+x)^{11} + \frac{1}{720}(-1+x)^{12}$$

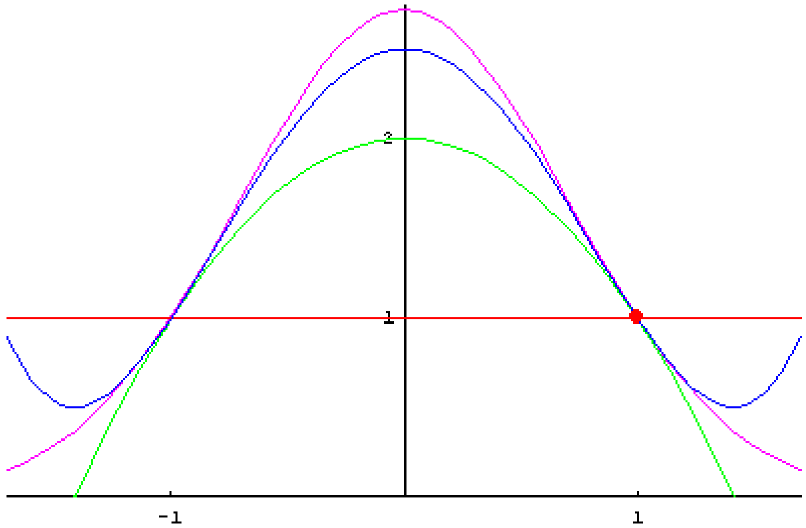
The I.V.P. is

$y' = -2xy$ with $y(1) = 1$

After 6 iterations, we have

$$y(x) \approx 1 - 2(-1+x) + (-1+x)^2 + \frac{2}{3}(-1+x)^3 - \frac{5}{6}(-1+x)^4 + \frac{1}{15}(-1+x)^5 + \frac{23}{90}(-1+x)^6 - \frac{1}{15}(-1+x)^7 + \frac{1}{24}(-1+x)^8 + \frac{5}{36}(-1+x)^9 + \frac{3}{40}(-1+x)^{10} + \frac{1}{60}(-1+x)^{11} + \frac{1}{720}(-1+x)^{12}$$

We can graph the analytic solution and the Picard iterations.




Solve the I.V.P.

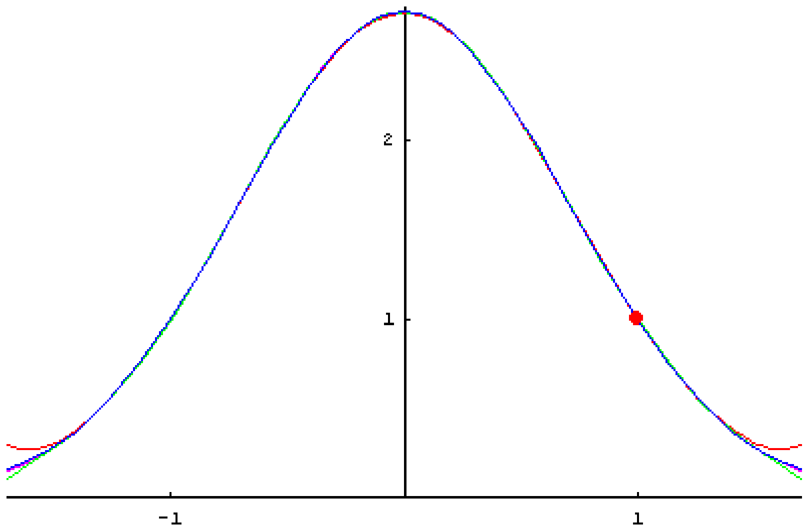
$y' = -2\,x\,y$ with $y(1) = 1$

The analytic solution is known to be

$y(x) = e^{1-x^2}$

The first 3 Picard approximations are

$$y(x) \approx \left\{1,\ 1 - 2\,(-1 + x) - (-1 + x)^2,\ 1 - 2\,(-1 + x) + (-1 + x)^2 + 2\,(-1 + x)^3 + \frac{1}{2}\,(-1 + x)^4\right\}$$



Solve the I.V.P.

$y' = -2\,x\,y$ with $y(1) = 1$

The analytic solution is known to be

$y(x) = e^{1-x^2}$

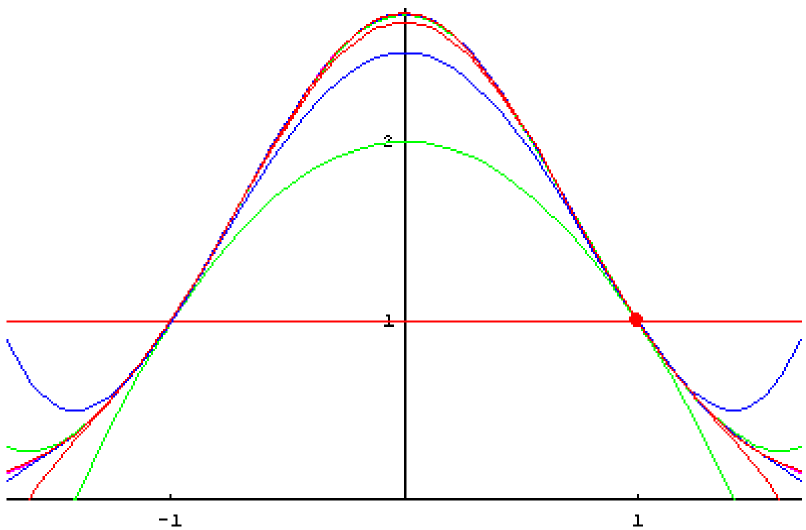
The next 3 Picard approximations are

$$y(x) \approx \Big\{1-2(-1+x)+(-1+x)^2+\frac{2}{3}(-1+x)^3-\frac{5}{6}(-1+x)^4+\frac{1}{3}(-1+x)^5+\frac{5}{6}(-1+x)^6+\frac{1}{3}(-1+x)^7+\frac{1}{24}(-1+x)^8,$$

$$1-2(-1+x)+(-1+x)^2+\frac{2}{3}(-1+x)^3-\frac{5}{6}(-1+x)^4+\frac{1}{15}(-1+x)^5+\frac{1}{6}(-1+x)^6-\frac{1}{3}(-1+x)^7-\frac{7}{24}(-1+x)^8-\frac{1}{12}(-1+x)^9-\frac{1}{120}(-1+x)^{10},$$

$$1-2(-1+x)+(-1+x)^2+\frac{2}{3}(-1+x)^3-\frac{5}{6}(-1+x)^4+\frac{1}{15}(-1+x)^5+\frac{23}{90}(-1+x)^6-$$

$$\frac{1}{15}(-1+x)^7+\frac{1}{24}(-1+x)^8+\frac{5}{36}(-1+x)^9+\frac{3}{40}(-1+x)^{10}+\frac{1}{60}(-1+x)^{11}+\frac{1}{720}(-1+x)^{12}\Big\}$$



Solve the I.V.P.

$y' = -2\,x\,y \quad \text{with} \quad y(1) = 1$

The analytic solution is known to be

$y(x) = e^{1-x^2}$

The first 6 Picard approximations are

$$y(x) \approx \left\{1,\ 1-2\,(-1+x)-(-1+x)^2,\ 1-2\,(-1+x)+(-1+x)^2+2\,(-1+x)^3+\frac{1}{2}\,(-1+x)^4,\ 1-2\,(-1+x)+(-1+x)^2+\frac{2}{3}\,(-1+x)^3-\frac{3}{2}\,(-1+x)^4-\right.$$
$$(-1+x)^5-\frac{1}{6}\,(-1+x)^6,\ 1-2\,(-1+x)+(-1+x)^2+\frac{2}{3}\,(-1+x)^3-\frac{5}{6}\,(-1+x)^4+\frac{1}{3}\,(-1+x)^5+\frac{5}{6}\,(-1+x)^6+\frac{1}{3}\,(-1+x)^7+\frac{1}{24}\,(-1+x)^8,$$
$$1-2\,(-1+x)+(-1+x)^2+\frac{2}{3}\,(-1+x)^3-\frac{5}{6}\,(-1+x)^4+\frac{1}{15}\,(-1+x)^5+\frac{1}{6}\,(-1+x)^6-\frac{1}{3}\,(-1+x)^7-\frac{7}{24}\,(-1+x)^8-\frac{1}{12}\,(-1+x)^9-\frac{1}{120}\,(-1+x)^{10},$$
$$1-2\,(-1+x)+(-1+x)^2+\frac{2}{3}\,(-1+x)^3-\frac{5}{6}\,(-1+x)^4+\frac{1}{15}\,(-1+x)^5+\frac{23}{90}\,(-1+x)^6-$$
$$\left.\frac{1}{15}\,(-1+x)^7+\frac{1}{24}\,(-1+x)^8+\frac{5}{36}\,(-1+x)^9+\frac{3}{40}\,(-1+x)^{10}+\frac{1}{60}\,(-1+x)^{11}+\frac{1}{720}\,(-1+x)^{12}\right\}$$

**Example 4.** Use Picard iteration to find the solution to the system of differential equations

$$x' = f(t, x, y) = x + 2y \quad \text{with} \quad x(0) = x_0 = 1$$

and

$$y' = g(t, x, y) = -x - y \quad \text{with} \quad y(0) = y_0 = 1$$

**Solution 4.**

First define the functions $f(t, x, y)$, $g(t, x, y)$ and the initial conditions $x_0 = 1$, $y_0 = 1$ by typing:

Solve the I.V.P.

$$x' = x + 2y \quad \text{with} \quad x(0) = 1$$

$$y' = -x - y \quad \text{with} \quad y(0) = 1$$

Picard iteration for generating the first six approximations is:

$$X_0(t) = 1$$

$$Y_0(t) = 1$$

$$X_1(t) = 1 + 3t$$

$$Y_1(t) = 1 - 2t$$

$$X_2(t) = 1 + 3t - \frac{t^2}{2}$$

$$Y_2(t) = 1 - 2t - \frac{t^2}{2}$$

$$X_3(t) = 1 + 3t - \frac{t^2}{2} - \frac{t^3}{2}$$

$$Y_3(t) = 1 - 2t - \frac{t^2}{2} + \frac{t^3}{3}$$

$$X_4(t) = 1 + 3t - \frac{t^2}{2} - \frac{t^3}{2} + \frac{t^4}{24}$$

$$Y_4(t) = 1 - 2t - \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{24}$$

$$X_5(t) = 1 + 3t - \frac{t^2}{2} - \frac{t^3}{2} + \frac{t^4}{24} + \frac{t^5}{40}$$

$$Y_5(t) = 1 - 2t - \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{24} - \frac{t^5}{60}$$

$$X_6(t) = 1 + 3t - \frac{t^2}{2} - \frac{t^3}{2} + \frac{t^4}{24} + \frac{t^5}{40} - \frac{t^6}{720}$$

$$Y_6(t) = 1 - 2t - \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{24} - \frac{t^5}{60} - \frac{t^6}{720}$$

The I.V.P. is

$x' = x + 2y$ with $x(0) = 1$

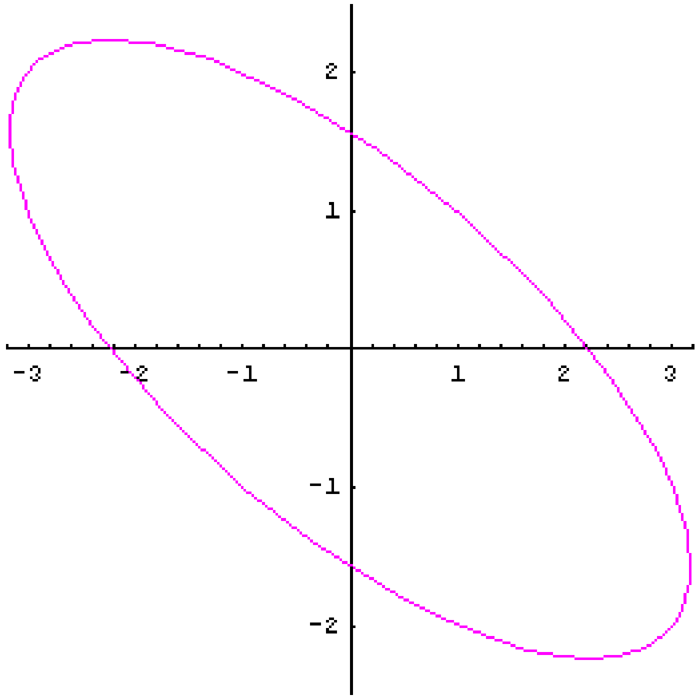
$y' = -x - y$ with $y(0) = 1$

After 6 iterations, we have the approximation

$$x(t) \approx 1 + 3t - \frac{t^2}{2} - \frac{t^3}{2} + \frac{t^4}{24} + \frac{t^5}{40} - \frac{t^6}{720}$$

$$y(t) \approx 1 - 2t - \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{24} - \frac{t^5}{60} - \frac{t^6}{720}$$

Mathematica can find the analytic solution.




Solve the I.V.P.

$x' = x + 2y$ with $x(0) = 1$

$y' = -x - y$ with $y(0) = 1$

The analytic solution is known to be

$x(t) = \mathrm{Cos}[t] + 3\,\mathrm{Sin}[t]$

$y(t) = \mathrm{Cos}[t] - 2\,\mathrm{Sin}[t]$

**Example 5.** Use Picard iteration to find and plot approximations for the solution of the I.V.P.

$$x' = f(t, x, y, z) = x + 2y - z \quad \text{with} \quad x(0) = x_0 = 1$$

$$y' = g(t, x, y, z) = -2x + y - 2z \quad \text{with} \quad y(0) = y_0 = 1$$

$$y' = g(t, x, y, z) = x + 2y + z \quad \text{with} \quad z(0) = z_0 = 1$$

**Solution 5.**

Solve the I.V.P.

$x' = x + 2y - z \quad \text{with} \quad x(0) = 1$

$y' = -2x + y - 2z \quad \text{with} \quad y(0) = 1$

$y' = x + 2y + z \quad \text{with} \quad z(0) = 1$

Picard iteration for generating the first six approximations is:

$X_0(t) = 1$

$Y_0(t) = 1$

$Z_0(t) = 1$

$X_2(t) = 1 + 2t$

$Y_2(t) = 1 - 3t$

$Z_2(t) = 1 + 4t$

$X_3(t) = 1 + 2t - 4t^2$

$Y_3(t) = 1 - 3t - \frac{15t^2}{2}$

$Z_3(t) = 1 + 4t$

$X_4(t) = 1 + 2t - 4t^2 - \frac{19t^3}{3}$

$Y_4(t) = 1 - 3t - \frac{15t^2}{2} + \frac{t^3}{6}$

$Z_4(t) = 1 + 4t - \frac{19t^3}{3}$

$X_5(t) = 1 + 2t - 4t^2 - \frac{19t^3}{3} + \frac{t^4}{12}$

$Y_5(t) = 1 - 3t - \frac{15t^2}{2} + \frac{t^3}{6} + \frac{51t^4}{8}$

$Z_5(t) = 1 + 4t - \frac{19t^3}{3} - \frac{37t^4}{12}$

$X_6(t) = 1 + 2t - 4t^2 - \frac{19t^3}{3} + \frac{t^4}{12} + \frac{191t^5}{60}$

$Y_6(t) = 1 - 3t - \frac{15t^2}{2} + \frac{t^3}{6} + \frac{51t^4}{8} + \frac{99t^5}{40}$

$Z_6(t) = 1 + 4t - \frac{19t^3}{3} - \frac{37t^4}{12} + \frac{39t^5}{20}$

$$X_7(t) = 1 + 2t - 4t^2 - \frac{19t^3}{3} + \frac{t^4}{12} + \frac{191t^5}{60} + \frac{371t^6}{360}$$

$$Y_7(t) = 1 - 3t - \frac{15t^2}{2} + \frac{t^3}{6} + \frac{51t^4}{8} + \frac{99t^5}{40} - \frac{187t^6}{144}$$

$$Z_7(t) = 1 + 4t - \frac{19t^3}{3} - \frac{37t^4}{12} + \frac{39t^5}{20} + \frac{121t^6}{72}$$

The I.V.P. is

$x' = x + 2y - z$ with $x(0) = 1$

$y' = -2x + y - 2z$ with $y(0) = 1$

$z' = x + 2y + z$ with $z(0) = 1$

After 6 iterations, we have the approximation

$$x(t) \approx 1 + 2t - 4t^2 - \frac{19t^3}{3} + \frac{t^4}{12} + \frac{191t^5}{60} + \frac{371t^6}{360}$$

$$y(t) \approx 1 - 3t - \frac{15t^2}{2} + \frac{t^3}{6} + \frac{51t^4}{8} + \frac{99t^5}{40} - \frac{187t^6}{144}$$

$$z(t) \approx 1 + 4t - \frac{19t^3}{3} - \frac{37t^4}{12} + \frac{39t^5}{20} + \frac{121t^6}{72}$$

**Aside.** *Mathematica* can find the analytic solution.

Solve the I.V.P.

$x' = x[t] + 2y[t] - z[t]$ with $x(0) = 1$

$y' = -2x[t] + y[t] - 2z[t]$ with $y(0) = 1$

$z' = x[t] + 2y[t] + z[t]$ with $z(0) = 1$

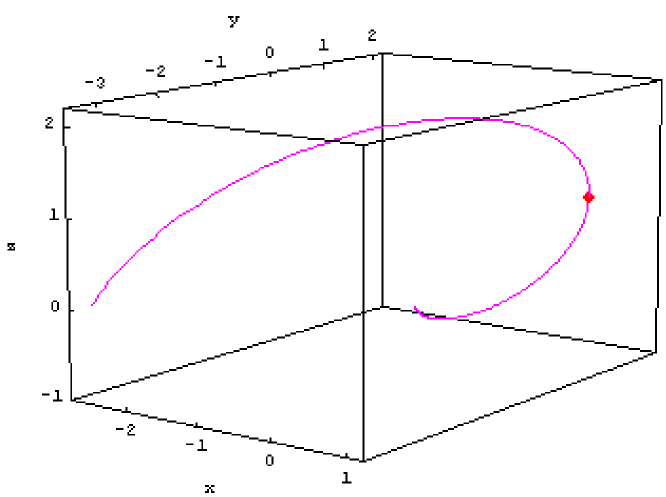
Get

$$x[t] \to -\frac{2e^t}{9} + \frac{11}{9}e^t \mathrm{Cos}[3t] + \frac{1}{3}e^t \mathrm{Sin}[3t]$$

$$y[t] \to \frac{e^t}{9} + \frac{8}{9}e^t \mathrm{Cos}[3t] - \frac{4}{3}e^t \mathrm{Sin}[3t]$$

$$z[t] \to \frac{2e^t}{9} + \frac{7}{9}e^t \mathrm{Cos}[3t] + e^t \mathrm{Sin}[3t]$$

We can plot the analytic solution.



Solve the I.V.P.

$x' = x[t] + 2y[t] - z[t]$ with $x(0) = 1$

$y' = -2x[t] + y[t] - 2z[t]$ with $y(0) = 1$

$z' = x[t] + 2y[t] + z[t]$ with $z(0) = 1$

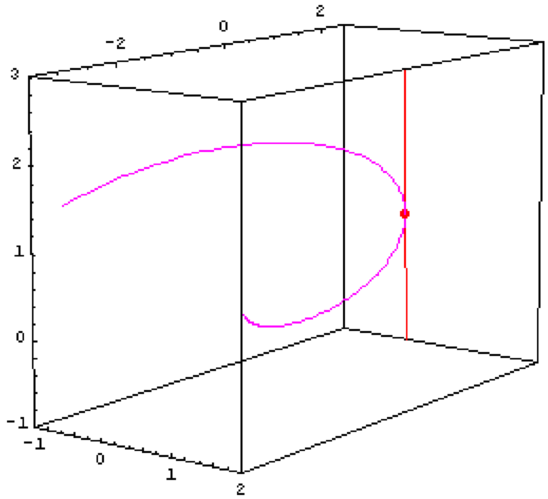
The analytic solution is

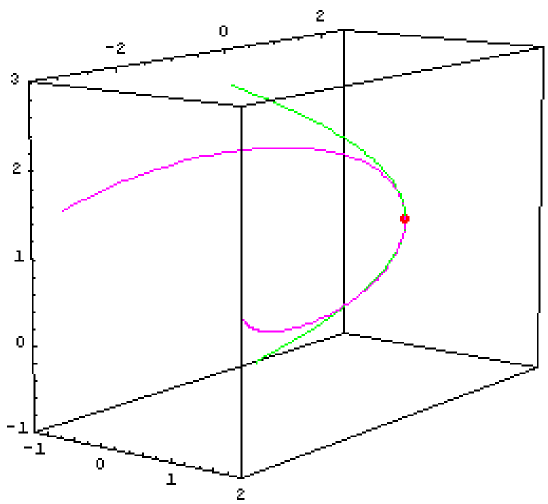
$$x(t) = -\frac{2e^t}{9} + \frac{11}{9}e^t \mathrm{Cos}[3t] + \frac{1}{3}e^t \mathrm{Sin}[3t]$$

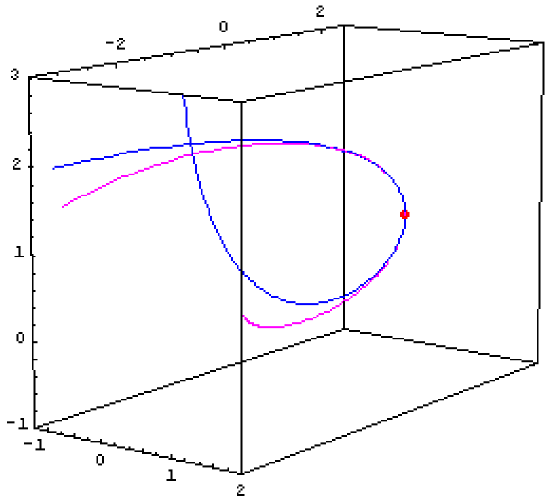
$$y(t) = \frac{e^t}{9} + \frac{8}{9}e^t \mathrm{Cos}[3t] - \frac{4}{3}e^t \mathrm{Sin}[3t]$$

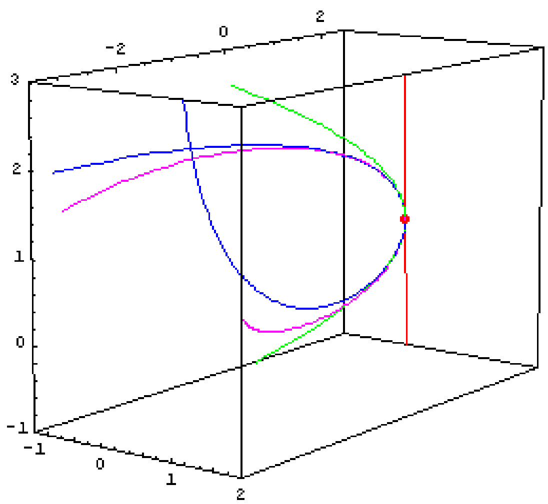
$$z(t) = \frac{2e^t}{9} + \frac{7}{9}e^t \mathrm{Cos}[3t] + e^t \mathrm{Sin}[3t]$$

We can plot the analytic solution and the first three Picard approximations.





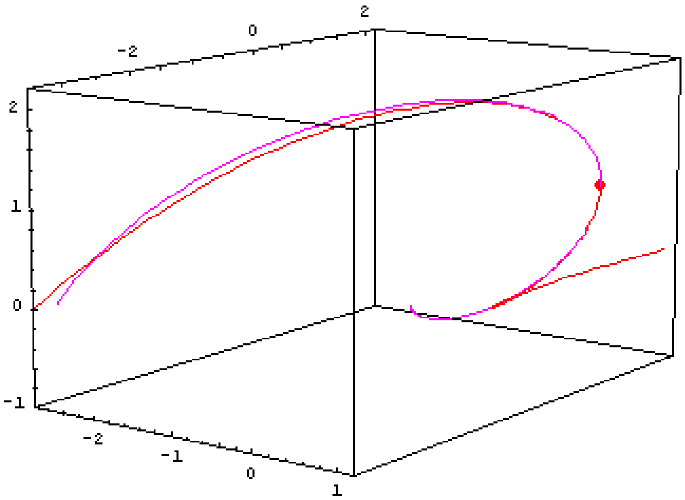

The analytic solution is

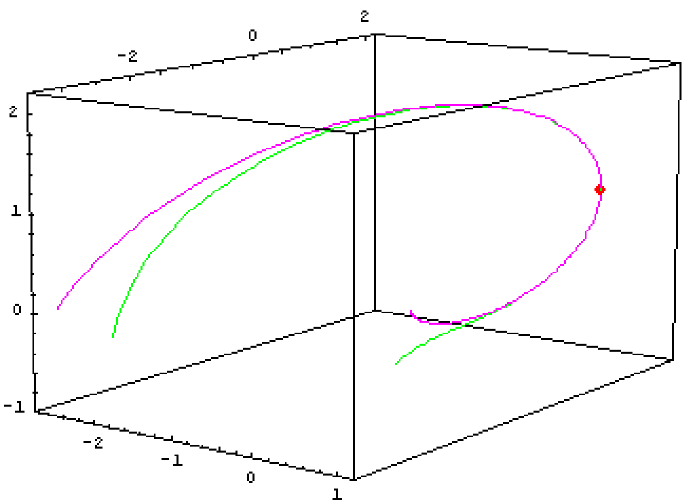
$$\begin{pmatrix} x[t] \\ y[t] \\ z[t] \end{pmatrix} = \begin{pmatrix} -\frac{2e^t}{9} + \frac{11}{9} e^t \mathrm{Cos}[3t] + \frac{1}{3} e^t \mathrm{Sin}[3t] \\ \frac{e^t}{9} + \frac{8}{9} e^t \mathrm{Cos}[3t] - \frac{4}{3} e^t \mathrm{Sin}[3t] \\ \frac{2e^t}{9} + \frac{7}{9} e^t \mathrm{Cos}[3t] + e^t \mathrm{Sin}[3t] \end{pmatrix}$$

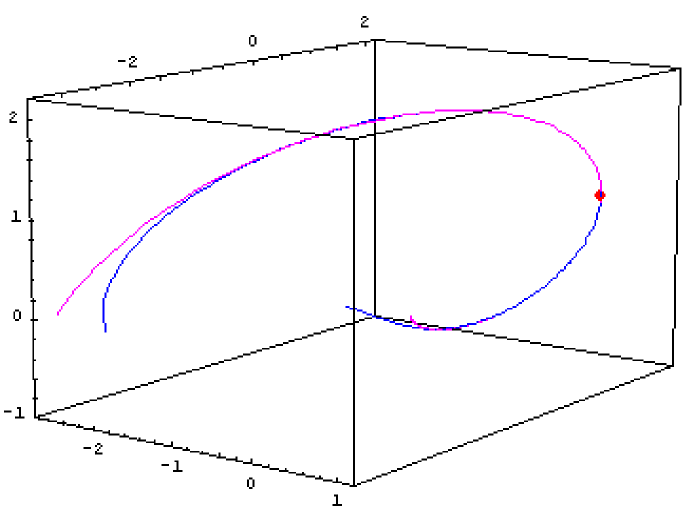
The first 3 Picard approximations are

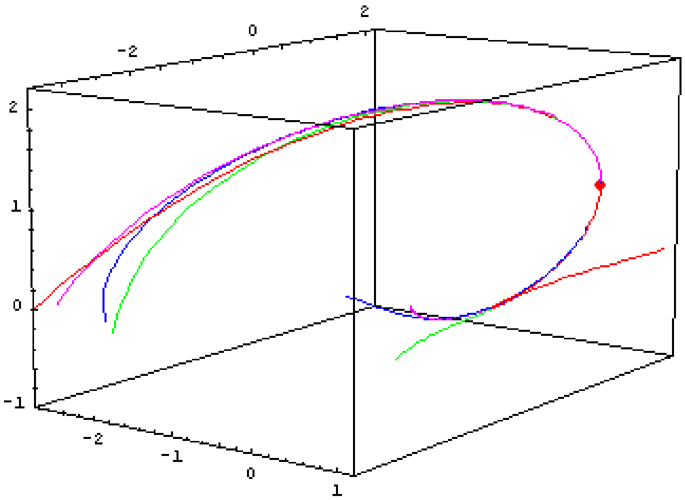
$$\begin{pmatrix} x[t] \\ y[t] \\ z[t] \end{pmatrix} \approx \begin{pmatrix} 1+2t \\ 1-3t \\ 1+4t \end{pmatrix}, \begin{pmatrix} 1+2t-4t^2 \\ 1-3t-\frac{15t^2}{2} \\ 1+4t \end{pmatrix}, \begin{pmatrix} 1+2t-4t^2-\frac{19t^3}{3} \\ 1-3t-\frac{15t^2}{2}+\frac{t^3}{6} \\ 1+4t-\frac{19t^3}{3} \end{pmatrix}$$

We can plot the analytic solution and the second three Picard approximations.


-2
0
2
2
1
0
-1
-2
-1
0
1


-2
0
2
2
1
0
-1
-2
-1
0
1


-2
0
2
2
1
0
-1
-2
-1
0
1



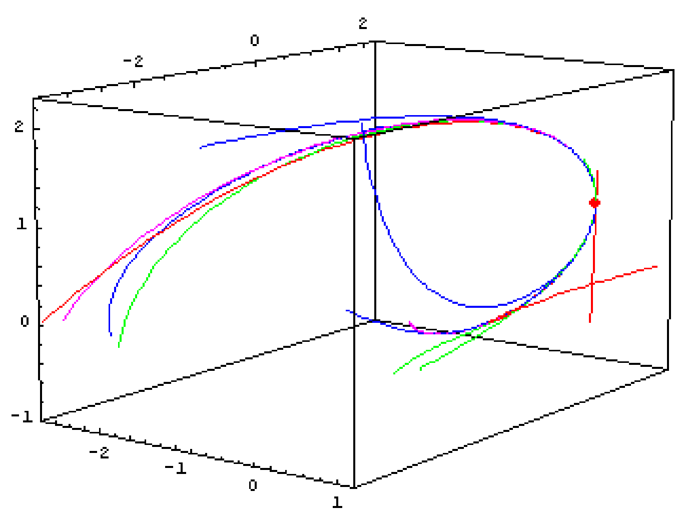

The analytic solution is

$$\begin{pmatrix} x[t] \\ y[t] \\ z[t] \end{pmatrix} = \begin{pmatrix} -\frac{2 e^t}{9} + \frac{11}{9} e^t \mathrm{Cos}[3\,t] + \frac{1}{3} e^t \mathrm{Sin}[3\,t] \\ \frac{e^t}{9} + \frac{8}{9} e^t \mathrm{Cos}[3\,t] - \frac{4}{3} e^t \mathrm{Sin}[3\,t] \\ \frac{2 e^t}{9} + \frac{7}{9} e^t \mathrm{Cos}[3\,t] + e^t \mathrm{Sin}[3\,t] \end{pmatrix}$$

The second 3 Picard approximations are

$$\begin{pmatrix} x[t] \\ y[t] \\ z[t] \end{pmatrix} \approx \begin{pmatrix} 1 + 2\,t - 4\,t^2 - \frac{19\,t^3}{3} + \frac{t^4}{12} \\ 1 - 3\,t - \frac{15\,t^2}{2} + \frac{t^3}{6} + \frac{51\,t^4}{8} \\ 1 + 4\,t - \frac{19\,t^3}{3} - \frac{37\,t^4}{12} \end{pmatrix}, \begin{pmatrix} 1 + 2\,t - 4\,t^2 - \frac{19\,t^3}{3} + \frac{t^4}{12} + \frac{191\,t^5}{60} \\ 1 - 3\,t - \frac{15\,t^2}{2} + \frac{t^3}{6} + \frac{51\,t^4}{8} + \frac{99\,t^5}{40} \\ 1 + 4\,t - \frac{19\,t^3}{3} - \frac{37\,t^4}{12} + \frac{39\,t^5}{20} \end{pmatrix}, \begin{pmatrix} 1 + 2\,t - 4\,t^2 - \frac{19\,t^3}{3} + \frac{t^4}{12} + \frac{191\,t^5}{60} + \frac{371\,t^6}{360} \\ 1 - 3\,t - \frac{15\,t^2}{2} + \frac{t^3}{6} + \frac{51\,t^4}{8} + \frac{99\,t^5}{40} - \frac{187\,t^6}{144} \\ 1 + 4\,t - \frac{19\,t^3}{3} - \frac{37\,t^4}{12} + \frac{39\,t^5}{20} + \frac{121\,t^6}{72} \end{pmatrix}$$

We can plot the analytic solution and the first six Picard approximations.



The analytic solution is

$$\begin{pmatrix} x[t] \\ y[t] \\ z[t] \end{pmatrix} = \begin{pmatrix} -\frac{2 e^t}{9} + \frac{11}{9} e^t \text{Cos}[3 t] + \frac{1}{3} e^t \text{Sin}[3 t] \\ \frac{e^t}{9} + \frac{8}{9} e^t \text{Cos}[3 t] - \frac{4}{3} e^t \text{Sin}[3 t] \\ \frac{2 e^t}{9} + \frac{7}{9} e^t \text{Cos}[3 t] + e^t \text{Sin}[3 t] \end{pmatrix}$$

The first 6 Picard approximations are

$$\begin{pmatrix} x[t] \\ y[t] \\ z[t] \end{pmatrix} \approx \begin{pmatrix} 1 + 2 t \\ 1 - 3 t \\ 1 + 4 t \end{pmatrix}, \begin{pmatrix} 1 + 2 t - 4 t^2 \\ 1 - 3 t - \frac{15 t^2}{2} \\ 1 + 4 t \end{pmatrix}, \begin{pmatrix} 1 + 2 t - 4 t^2 - \frac{19 t^3}{3} \\ 1 - 3 t - \frac{15 t^2}{2} + \frac{t^3}{6} \\ 1 + 4 t - \frac{19 t^3}{3} \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 + 2 t - 4 t^2 - \frac{19 t^3}{3} + \frac{t^4}{12} \\ 1 - 3 t - \frac{15 t^2}{2} + \frac{t^3}{6} + \frac{51 t^4}{8} \\ 1 + 4 t - \frac{19 t^3}{3} - \frac{37 t^4}{12} \end{pmatrix}, \begin{pmatrix} 1 + 2 t - 4 t^2 - \frac{19 t^3}{3} + \frac{t^4}{12} + \frac{191 t^5}{60} \\ 1 - 3 t - \frac{15 t^2}{2} + \frac{t^3}{6} + \frac{51 t^4}{8} + \frac{99 t^5}{40} \\ 1 + 4 t - \frac{19 t^3}{3} - \frac{37 t^4}{12} + \frac{39 t^5}{20} \end{pmatrix}, \begin{pmatrix} 1 + 2 t - 4 t^2 - \frac{19 t^3}{3} + \frac{t^4}{12} + \frac{191 t^5}{60} + \frac{371 t^6}{360} \\ 1 - 3 t - \frac{15 t^2}{2} + \frac{t^3}{6} + \frac{51 t^4}{8} + \frac{99 t^5}{40} - \frac{187 t^6}{144} \\ 1 + 4 t - \frac{19 t^3}{3} - \frac{37 t^4}{12} + \frac{39 t^5}{20} + \frac{121 t^6}{72} \end{pmatrix}$$