3. The Trapezoidal Rule for Numerical Integration

Theorem (Trapezoidal Rule) Consider $y = f(x)$ over $[x_0, x_1]$, where $x_1 = x_0 + h$. The trapezoidal rule is

$$TR(f, h) = \frac{h}{2}(f(x_0) + f(x_1)).$$

This is an numerical approximation to the integral of $f(x)$ over $[x_0, x_1]$ and we have the expression

$$\int_{x_0}^{x_1} f(x)\,dx \approx TR(f, h).$$

The remainder term for the trapezoidal rule is $R_{TR}(f, h) = -\frac{1}{12} f''(c) h^3$, where c lies somewhere between x_0 and x_1, and have the equality

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f(x_0) + f(x_1)) - \frac{1}{12} f''(c) h^3.$$

Composite Trapezoidal Rule

An intuitive method of finding the area under a curve y = f(x) is by approximating that area with a series of trapezoids that lie above the intervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$. When several trapezoids are used, we call it the composite trapezoidal rule.

Theorem (Composite Trapezoidal Rule) Consider $y = f(x)$ over $[a, b]$. Suppose that the interval $[a, b]$ is subdivided into m subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{m}$ of equal width $h = \frac{b-a}{m}$ by using the equally spaced nodes $x_k = x_0 + k h$ for $k = 1, 2, ..., m$. The composite trapezoidal rule for m subintervals is

$$T(f, h) = \frac{h}{2}(f(a) + f(b)) + h \sum_{k=1}^{m} f(x_k)$$

is an numerical approximation to the integral, and

$$\int_a^b f(x)\,dx = T(f, h) + E_T(f, h).$$

Furthermore, if $f(x) \in C^2[a, b]$, then there exists a value c with a < c < b so that the error term $E_T(f, h)$ has the form

$$E_T(f, h) = -\frac{(b-a) f^2(c)}{12} h^2.$$

This is expressed using the "big O" notation $E_T(f, h) = O(h^2)$.

Remark. When the step size is reduced by a factor of $\frac{1}{2}$ the error term $E_T(f, h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^2 = 0.25$.

Example 1. Let $f[x]$ be $\int_0^2 \left(2 + \mathrm{Cos}\left[2\sqrt{x}\right]\right) dx$.

1 (a) Numerically approximate the integral by using the trapezoidal rule with m = 1, 2, 4, 8, and 16 subintervals.

1 (b) Find the analytic value of the integral (i.e. find the "true value").

1 (c) Find the error for the trapezoidal rule approximations

Solution 1.

Recursive Integration Rules

Theorem (Successive Trapezoidal Rules) Suppose that $j \geq 1$ and the points $\{x_k = a + k h\}$ subdivide $[a, b]$ into $2^j = 2m$ subintervals equal width $h = \frac{b - a}{2^j}$. The trapezoidal rules $T(f, h)$ and $T(f, 2h)$ obey the relationship

$$T(f, h) = \frac{T(f, 2h)}{2} + h \sum_{k=1}^{m} f(x_{2k-1}).$$

Definition (Sequence of Trapezoidal Rules) Define $T(0) = \frac{h}{2}(f(a) + f(b))$, which is the trapezoidal rule with step size $h = b - a$. Then for each $j \geq 1$ define $T(0) = T(f, h)$, where $T(f, h)$ is the trapezoidal rule with step size $h = \frac{b - a}{2^j}$.

Corollary (Recursive Trapezoidal Rule) Start with $T(0) = \frac{h}{2}(f(a) + f(b))$. Then a sequence of trapezoidal rules $\{T(j)\}$ is generated by the recursive formula

$$T(j) = \frac{T(j-1)}{2} + h \sum_{k=1}^{m} f(x_{2k-1}) \quad \text{for } j = 1, 2, \ldots.$$

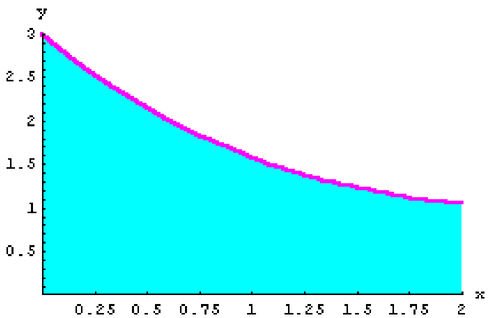
where $h = \frac{b - a}{2^j}$ and $\{x_k = a + k h\}$.

The recursive trapezoidal rule is used for the Romberg integration algorithm.

Example 1. Let f[x] be $\int_0^2 \left(2 + \mathrm{Cos}\left[2\sqrt{x}\right]\right) dx$.

1 (a) Numerically approximate the integral by using the trapezoidal rule with m = 1, 2, 4, 8, and 16 subintervals.
1 (b) Find the analytic value of the integral (i.e. find the "true value").
1 (c) Find the error for the trapezoidal rule approximations

Solution 1 (a).



$f[x] = 2 + \mathrm{Cos}\left[2\sqrt{x}\right]$

We will use simulated hand computations for the solution.

$\mathbf{f[x\_] = 2 + Cos\left[2\sqrt{x}\right];}$

$\mathbf{t1 = \frac{\frac{2-0}{1}}{2}\,(f[0] + f[2])}$

N[t1]

$5 + \mathrm{Cos}\left[2\sqrt{2}\right]$

4.04864

$\mathbf{t2 = \frac{\frac{2-0}{2}}{2}\,(f[0] + 2\,f[1] + f[2])}$

N[t2]

$\frac{1}{2}\left(5 + 2\,(2 + \mathrm{Cos}[2]) + \mathrm{Cos}\left[2\sqrt{2}\right]\right)$

3.60817

$\mathbf{t4 = \frac{\frac{2-0}{4}}{2}\left(f[0] + 2\,f\left[\frac{1}{2}\right] + 2\,f[1] + 2\,f\left[\frac{3}{2}\right] + f[2]\right)}$

N[t4]

$\frac{1}{4}\left(5 + 2\,(2 + \mathrm{Cos}[2]) + 2\left(2 + \mathrm{Cos}\left[\sqrt{2}\right]\right) + \mathrm{Cos}\left[2\sqrt{2}\right] + 2\left(2 + \mathrm{Cos}\left[\sqrt{6}\right]\right)\right)$

3.4971

$$\mathbf{t8} = \frac{\frac{2-0}{8}}{2}\left(f[0] + 2\,f\left[\frac{1}{4}\right] + 2\,f\left[\frac{1}{2}\right] + 2\,f\left[\frac{3}{4}\right] + 2\,f[1] + 2\,f\left[\frac{5}{4}\right] + 2\,f\left[\frac{3}{2}\right] + 2\,f\left[\frac{7}{4}\right] + f[2]\right)$$

N[t8]

$$\frac{1}{8}\left(5 + 2\,(2 + \mathrm{Cos}[1]) + 2\,(2 + \mathrm{Cos}[2]) + 2\left(2 + \mathrm{Cos}\left[\sqrt{2}\right]\right) + \mathrm{Cos}\left[2\sqrt{2}\right] + 2\left(2 + \mathrm{Cos}\left[\sqrt{3}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{5}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{6}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{7}\right]\right)\right)$$

3.46928

$$\mathbf{t16} = \frac{\frac{2-0}{16}}{2}\left(f[0] + 2\,f\left[\frac{1}{8}\right] + 2\,f\left[\frac{1}{4}\right] + 2\,f\left[\frac{3}{8}\right] + 2\,f\left[\frac{1}{2}\right] + 2\,f\left[\frac{5}{8}\right] + 2\,f\left[\frac{3}{4}\right] + 2\,f\left[\frac{7}{8}\right] + 2\,f[1] + 2\,f\left[\frac{9}{8}\right] + 2\,f\left[\frac{5}{4}\right] + 2\,f\left[\frac{11}{8}\right] + 2\,f\left[\frac{3}{2}\right] + 2\,f\left[\frac{13}{8}\right] + 2\,f\left[\frac{7}{4}\right] + 2\,f\left[\frac{15}{8}\right] + f[2]\right)$$

N[t16]

$$\frac{1}{16}\left(5 + 2\,(2 + \mathrm{Cos}[1]) + 2\,(2 + \mathrm{Cos}[2]) + 2\left(2 + \mathrm{Cos}\left[\sqrt{\frac{3}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\frac{1}{\sqrt{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\frac{3}{\sqrt{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{2}\right]\right) + \mathrm{Cos}\left[2\sqrt{2}\right] + 2\left(2 + \mathrm{Cos}\left[\sqrt{\frac{5}{2}}\right]\right) + \right.$$
$$\left. 2\left(2 + \mathrm{Cos}\left[\sqrt{3}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{\frac{7}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{5}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{\frac{11}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{6}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{\frac{13}{2}}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{7}\right]\right) + 2\left(2 + \mathrm{Cos}\left[\sqrt{\frac{15}{2}}\right]\right)\right)$$

3.46232

Solution 1 (b).

$$\mathbf{val} = \int_0^2 \left(2 + \mathrm{Cos}\left[2\sqrt{x}\right]\right) dx$$

$$\frac{7}{2} + \frac{1}{2}\,\mathrm{Cos}\left[2\sqrt{2}\right] + \sqrt{2}\,\mathrm{Sin}\left[2\sqrt{2}\right]$$

N[val]

3.46

NumberForm[N[val] , 12]

3.45999767217

Solution 1 (c).

val - t16

-0.00232232783