11. Row Reduced Echelon Form

Background

An important technique for solving a system of linear equations $AX = B$ is to form the augmented matrix $M = [A, B]$ and reduce M to reduced row echelon form.

Definition (Reduced Row Echelon Form). A matrix is said to be in row-reduced echelon form provided that
(i) In each row that does not consist of all zero elements, the first non-zero element in this row is a 1. (called. a "leading 1).
(ii) In each column that contains a leading 1 of some row, all other elements of this column are zeros.
(iii) In any two successive rows with non-zero elements, the leading 1 of the the lower row occurs farther to the right than the leading 1 of the higher row.
(iv) If there are any rows contains only zero elements then they are grouped together at the bottom.

Theorem (Reduced Row Echelon Form). The reduced row echelon form of a matrix is unique.

Definition (Rank). The number of nonzero rows in the reduced row echelon form of a matrix A is called the rank of A and is denoted by $\mathrm{rank}(A)$.

Theorem. Consider the $m \times n$ linear system $AX = B$, where $M = [A, B]$ is the augmented matrix.
(i) If $\mathrm{rank}(A) = \mathrm{rank}(M) = n$ then the system has a unique solution.
(ii) If $\mathrm{rank}(A) = \mathrm{rank}(M) < n$ then the system has an infinite number of solution.
(iii) If $\mathrm{rank}(A) < \mathrm{rank}(M)$ then the system is inconsistent and has no solution.

Example 1. Solve the linear system of equations

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 1 \\ 3x_1 + 2x_2 + 4x_3 &= -4 \\ 4x_1 + 3x_2 + 3x_3 &= -4 \end{aligned}$$

Solution 1.

Example 2. Solve the linear system of equations

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 1 \\ 3x_1 + 2x_2 + 4x_3 &= -4 \\ 4x_1 + 3x_2 + 2x_3 &= -4 \end{aligned}$$

Solution 2.

Example 3. Solve the linear system

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 3x_2 + 4x_3 &= -3 \\ 3x_1 + 4x_2 + 2x_3 &= 2 \end{aligned}$$

Solution 3.

Example 4. Solve the linear system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 3x_5 &= 1 \\ 2x_1 + 4x_2 + 6x_3 + 2x_4 + 6x_5 &= 2 \\ 3x_1 + 6x_2 + 18x_3 + 9x_4 + 9x_5 &= -6 \\ 4x_1 + 8x_2 + 12x_3 + 10x_4 + 12x_5 &= 4 \\ 5x_1 + 10x_2 + 24x_3 + 11x_4 + 15x_5 &= -4 \end{aligned}$$

Solution 4.

Example 1. Solve the linear system of equations

$$x_1 + x_2 - 2x_3 = 1$$
$$3x_1 + 2x_2 + 4x_3 = -4$$
$$4x_1 + 3x_2 + 3x_3 = -4$$

Solution 1.

Find the reduced row echelon form of the augmented matrix **M** = [**A**, **B**].

The augmented matrix M = [A,B] is

$$M = \begin{pmatrix} 1 & 1 & -2 & 1 \\ 3 & 2 & 4 & -4 \\ 4 & 3 & 3 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 10 & -7 \\ 0 & -1 & 11 & -8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & 1 & -10 & 7 \\ 0 & -1 & 11 & -8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 8 & -6 \\ 0 & 1 & -10 & 7 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

The 3×3 identity matrix appears in the left 3 columns of **M**, and the solution vector is the fourth column of **M**.

The solution is $X = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$

The row reduced echelon form of A is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The row reduced echelon form of M is

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Looking at the above calculations we see that $\mathrm{rank}(\mathbf{A}) = 3$ and $\mathrm{rank}(\mathbf{M}) = 3$.
Since $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{M}) = 3$ the theorem guarantees a unique solution.

Example 2. Solve the linear system of equations

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 1 \\ 3x_1 + 2x_2 + 4x_3 &= -4 \\ 4x_1 + 3x_2 + 2x_3 &= -4 \end{aligned}$$

Solution 2.

Let us investigate the reduced row echelon form of the augmented matrix **M** = [**A**, **B**].

The augmented matrix M = [A,B] is

$$M = \begin{pmatrix} 1 & 1 & -2 & 1 \\ 3 & 2 & 4 & -4 \\ 4 & 3 & 2 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 & 1 \\ 0 & -1 & 10 & -7 \\ 0 & -1 & 10 & -8 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & 0 & 8 & -6 \\ 0 & 1 & -10 & 7 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 8 & -6 \\ 0 & 1 & -10 & 7 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This linear system is equivalent to:

$$\begin{aligned} \mathbf{x_1} \quad\quad + 8\,\mathbf{x_3} &= -6 \\ \mathbf{x_2} - 10\,\mathbf{x_3} &= 7 \\ 0 &= 1 \end{aligned}$$

The third equation states that $0 = 1$ which is a contradiction.
In this case we say that the system of equations is inconsistent and there is no solution.

The row reduced echelon form of A is

$$\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & -10 \\ 0 & 0 & 0 \end{pmatrix}$$

The row reduced echelon form of M is

$$\begin{pmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Looking at the above calculations we see that $\mathrm{rank}(\mathbf{A}) = 2$ and $\mathrm{rank}(\mathbf{M}) = 3$.
Since $\mathrm{rank}(\mathbf{A}) < \mathrm{rank}(\mathbf{M})$ the theorem states that the system is inconsistent and has no solution.

Example 3. Solve the linear system

$$
\begin{aligned}
x_1 + x_2 - 2x_3 &= 5\\
2x_1 + 3x_2 + 4x_3 &= -3\\
3x_1 + 4x_2 + 2x_3 &= 2
\end{aligned}
$$

Solution 3.

Let us investigate the reduced row echelon form of the augmented matrix **M** = [**A**, **B**].

The augmented matrix M = [A,B] is

$$M = \begin{pmatrix} 1 & 1 & -2 & 5 \\ 2 & 3 & 4 & -3 \\ 3 & 4 & 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & -2 & 5 \\ 0 & 1 & 8 & -13 \\ 0 & 1 & 8 & -13 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -10 & 18 \\ 0 & 1 & 8 & -13 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The equation form for this matrix is

$$
\begin{aligned}
x_1 \qquad - 10x_3 &= 18\\
x_2 + 8x_3 &= -13
\end{aligned}
$$

There is one free variable which we choose to be $x_3 = t$. It is used in computing x_2 and x_1. Solve the previous equations for x_1 and x_2.

$$
\begin{aligned}
x_1 &= 18 + 10x_3\\
x_2 &= -13 - 8x_3
\end{aligned}
$$

Make the substitution $x_3 = t$.

$$
\begin{aligned}
x_1 &= 18 + 10t\\
x_2 &= -13 - 8t
\end{aligned}
$$

The solution vector $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is

$$X = \begin{pmatrix} 18 + 10\,t \\ -13 - 8\,t \\ t \end{pmatrix};$$

Looking at the above calculations we see that $\text{rank}(A) = 2$ and $\text{rank}(M) = 2$.
Since $\text{rank}(A) = \text{rank}(M) = 2 < 3$ the theorem guarantees that there will be an infinite number of solutions.

Example 4. Solve the linear system

$$\begin{aligned}
x_1 + 2x_2 + 3x_3 + 4x_4 + 3x_5 &= 1\\
2x_1 + 4x_2 + 6x_3 + 2x_4 + 6x_5 &= 2\\
3x_1 + 6x_2 + 18x_3 + 9x_4 + 9x_5 &= -6\\
4x_1 + 8x_2 + 12x_3 + 10x_4 + 12x_5 &= 4\\
5x_1 + 10x_2 + 24x_3 + 11x_4 + 15x_5 &= -4
\end{aligned}$$

Solution 4.

Form the augmented matrix **M** = [**A**, **B**] and find its reduced row echelon form.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 3\\ 2 & 4 & 6 & 2 & 6\\ 3 & 6 & 18 & 9 & 9\\ 4 & 8 & 12 & 10 & 12\\ 5 & 10 & 24 & 11 & 15 \end{pmatrix}$$

$$B = \begin{pmatrix} 1\\ 2\\ -6\\ 4\\ -4 \end{pmatrix}$$

The augmented matrix M = [A,B] is

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 & 3 & 1\\ 2 & 4 & 6 & 2 & 6 & 2\\ 3 & 6 & 18 & 9 & 9 & -6\\ 4 & 8 & 12 & 10 & 12 & 4\\ 5 & 10 & 24 & 11 & 15 & -4 \end{pmatrix}$$

The row reduced echelon form of M is

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 3 & 4\\ 0 & 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The row reduced echelon form of A is

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The equation form for this matrix is

$$\begin{aligned} x_1 + 2x_2 \quad + 3x_5 &= 4 \\ x_3 &= -1 \\ x_4 &= 0 \end{aligned}$$

There are two free variables which we choose to be $x_2 = s$ and $x_5 = t$. They is used in computing x_1.

Solve the previous equations for x_1.

$$x_1 + 2x_2 + 3x_5 = 4$$

Make the substitutions $x_2 = s$ and $x_5 = t$.

$$x_1 + 2s + 3t = 4$$

Get

$$x_1 = 4 - 2s - 3t$$

$$\text{The solution is } X = \begin{pmatrix} 4-2s-3t \\ s \\ -1 \\ 0 \\ t \end{pmatrix}$$

Looking at the above calculations we see that $\operatorname{rank}(\mathbf{A}) = 3$ and $\operatorname{rank}(\mathbf{M}) = 3$.

Since $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{M})\ 3 < 5$ the theorem guarantees that there will be an infinite number of solutions.