5. Fast Fourier Transform (FFT)

Definition (Piecewise Continuous). The function $f(x)$ is **piecewise continuous** on the closed interval $[a, b]$, if there exists values $x_0, x_1, \ldots, x_n$ with $a = x_0 < x_1 < \ldots < x_n = b$ such that f is continuous in each of the open intervals $x_{k-1} < x < x_k$, for $k = 1, 2, \ldots, n$ and has left-hand and right-hand limits at each of the values x_k, for $k = 0, 1, 2, \ldots, n$.

Definition (Fourier Series). If $f(x)$ is periodic with period $2L$ and is piecewise continuous on $[0, 2L]$, then the Fourier Series $S(x)$ for $f(x)$ is

$$S(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left(a_j \cos\left(j \frac{\pi}{L} x\right) + b_j \sin\left(j \frac{\pi}{L} x\right) \right),$$

where the coefficients a_j and b_j are given by the so-called Euler's formulae:

$$a_j = \frac{1}{L} \int_0^{2L} f(x) \cos\left(j \frac{\pi}{L} x\right) dx \quad \text{for } j = 0, 1, \ldots,$$

and

$$b_j = \frac{1}{L} \int_0^{2L} f(x) \sin\left(j \frac{\pi}{L} x\right) dx \quad \text{for } j = 1, 2, \ldots.$$

Theorem (Fourier Expansion). Assume that $S(x)$ is the Fourier Series for $f(x)$. If $f(x)$ and $f'(x)$ are piecewise continuous on $[0, 2L]$, then $S(x)$ is convergent for all $x \in [0, 2L]$.

The relation $f(x) = S(x)$ holds for all $x \in [0, 2L]$ where $f(x)$ is continuous. If $x = a$ is a point of discontinuity of $f(x)$, then

$$S(a) = \frac{f(a^-) + f(a^+)}{2},$$

where $f(a^-)$ and $f(a^+)$ denote the left-hand and right-hand limits, respectively. With this understanding, we have the Fourier Series expansion:

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \left(a_j \cos\left(j \frac{\pi}{L} x\right) + b_j \sin\left(j \frac{\pi}{L} x\right) \right).$$

Definition (Fourier Polynomial). If $f(x)$ is periodic with period $2L$ and is piecewise continuous on $[0, 2L]$, then the Fourier Polynomial $S(x)$ for $f(x)$ of degree m is

$$S(x) = \frac{a_0}{2} + \sum_{j=1}^{m} \left(a_j \cos\left(j \frac{\pi}{L} x\right) + b_j \sin\left(j \frac{\pi}{L} x\right) \right),$$

where the coefficients a_j and b_j are given by the so-called Euler's formulae:

$$a_j = \frac{1}{L}\int_0^{2L} f(x)\cos\left(j\frac{\pi}{L}x\right)dx \quad \text{for } j = 0, 1, \ldots, m,$$

and

$$b_j = \frac{1}{L}\int_0^{2L} f(x)\sin\left(j\frac{\pi}{L}x\right)dx \quad \text{for } j = 1, 2, \ldots, m.$$

Example 1. Assume that $f(x)$ is periodic with period 2π, i.e. $f(x+2\pi) = f(x)$, and is defined by

$f(x) = |x - \pi|$ for $0 \le x \le 2\pi$.

Find the Fourier polynomial of degree n = 5.

Solution 1.

The Fast Fourier Transform for data.

The FFT is used to find the trigonometric polynomial when only data points are given. We will demonstrate three ways to calculate the FFT. The first method involves computing sums, similar to "numerical integration," the second method involves "curve fitting," the third method involves "complex numbers."

Computing the FFT with sums.

Given data points $\{(x_n, y_n)\}$ where $x_0 = 0$ and $x_n = 2L$ over $[0,2L]$ where $x_k = k\frac{2L}{n}$ for $k = 0, 1, 2, \ldots, n$. Also given that $y_n = y_0$, to that the data is periodic with period $2L$. We shall construct the FFT polynomial over $[0,2L]$ of degree m.

$$P(x) = \frac{a_0}{2} + \sum_{j=1}^{m}\left(a_j\cos\left(j\frac{\pi}{L}x\right) + b_j\sin\left(j\frac{\pi}{L}x\right)\right),$$

The abscissa's form n subintervals of equal width $\Delta x = \frac{2L}{n}$ based on $x_k = k\frac{2L}{n}$. The coefficients are

$$a_j = \frac{2}{n}\sum_{k=0}^{n-1}\mathrm{Cos}\left[j\frac{\pi}{L}x_k\right]y_k \quad \text{for } j = 0, 1, 2, \ldots, m$$

and

$$b_j = \frac{2}{n}\sum_{k=0}^{n-1}\mathrm{Sin}\left[j\frac{\pi}{L}x_k\right]y_k \quad \text{for } j = 1, 2, \ldots, m.$$

The construction is possible provided that $2m + 1 \le n$.

Remark. Notice that the sums $\sum_{k=0}^{n-1}$ involve only n ordinates y_k.

Example 2. Given the 12 equally spaced data points

$$(0, \pi),\ \left(\frac{\pi}{6}, \frac{5\pi}{6}\right),\ \left(\frac{\pi}{3}, \frac{2\pi}{3}\right),\ \left(\frac{\pi}{2}, \frac{\pi}{2}\right),\ \left(\frac{2\pi}{3}, \frac{\pi}{3}\right),\ \left(\frac{5\pi}{6}, \frac{\pi}{6}\right),\ (\pi, 0),\ \left(\frac{7\pi}{6}, \frac{\pi}{6}\right),\ \left(\frac{4\pi}{3}, \frac{\pi}{3}\right),$$
$$\left(\frac{3\pi}{2}, \frac{\pi}{2}\right),\ \left(\frac{5\pi}{3}, \frac{2\pi}{3}\right),\ \left(\frac{11\pi}{6}, \frac{5\pi}{6}\right)$$

which can be extended periodically over $[0, 2\pi]$, if we define $(x_n, y_n) = (2\pi, \pi)$.
Find the Fourier polynomial of degree n = 5 for the 12 equally spaced points over interval $[0, 2\pi]$.
Use numerical sums to find the coefficients.
Solution 2.

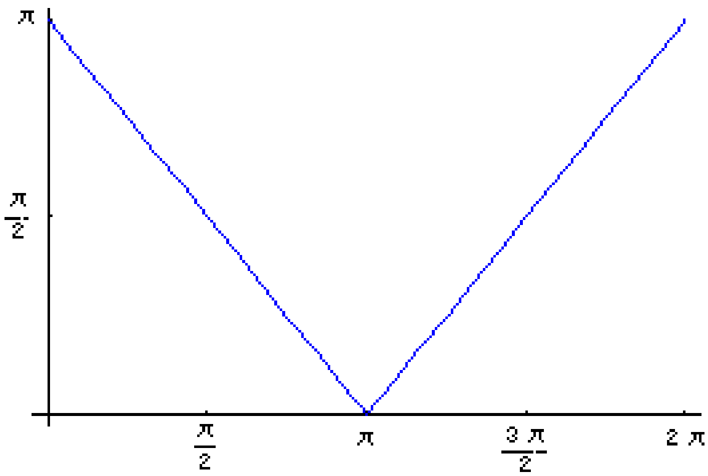
Example 1. Assume that $f(x)$ is periodic with period 2π, i.e. $f(x+2\pi) = f(x)$, and is defined by

$f(x) = |x - \pi|$ for $0 \le x \le 2\pi$.

Find the Fourier polynomial of degree n = 5.

Solution 1.

We need to define this function individually in each sub-interval.



$$f[x] = \text{Abs}[-\pi + x] \quad \text{for } 0 \le x \le 2\pi.$$

$$c_k = \frac{\frac{1}{k^2} - \frac{2\,\text{Cos}[k\,\pi]}{k^2} + \frac{\text{Cos}[2\,k\,\pi] + k\,\pi\,\text{Sin}[2\,k\,\pi]}{k^2}}{\pi}$$

$$c_k = -\frac{2\,(-1 + (-1)^k)}{k^2\,\pi}$$

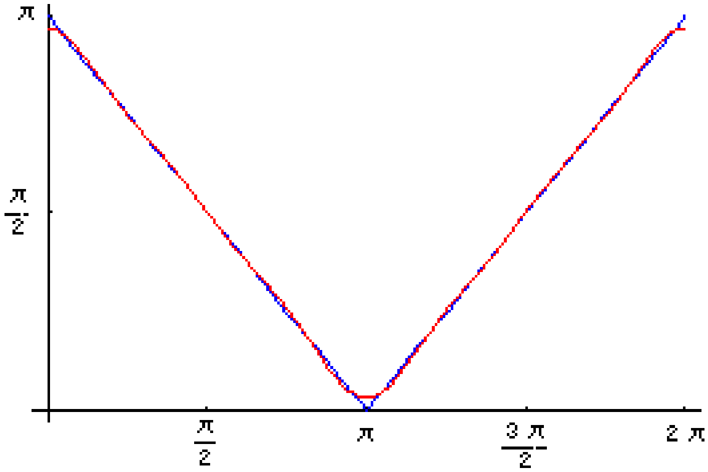
$$d_k = \frac{\frac{\pi}{k} - \frac{2\,\text{Sin}[k\,\pi]}{k^2} + \frac{-k\,\pi\,\text{Cos}[2\,k\,\pi] + \text{Sin}[2\,k\,\pi]}{k^2}}{\pi}$$

$$d_k = 0$$

$$S[x] = \frac{\pi}{2} + \frac{4\,\text{Cos}[x]}{\pi} + \frac{4\,\text{Cos}[3\,x]}{9\,\pi} + \frac{4\,\text{Cos}[5\,x]}{25\,\pi}$$

$$s[x] = 1.5708 + 1.27324\,\text{Cos}[x] + 0.141471\,\text{Cos}[3\,x] + 0.0509296\,\text{Cos}[5\,x]$$

Now plot the function and the Fourier polynomial.

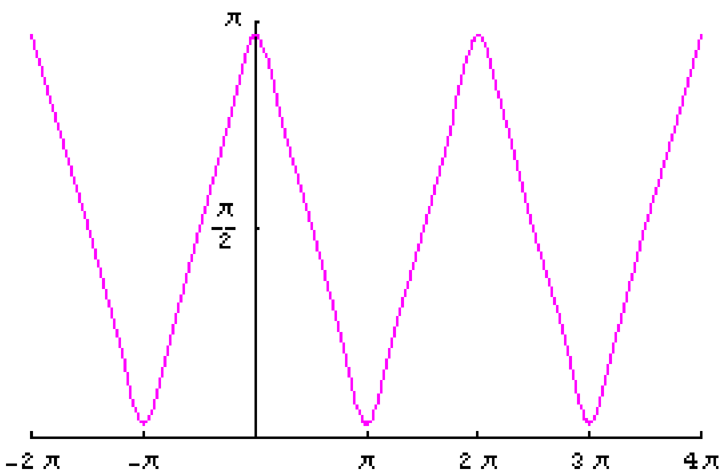


$$f[x] = \text{Abs}[-\pi + x] \quad \text{for } 0 \le x \le 2\pi.$$

$$S[x] = \frac{\pi}{2} + \frac{4\,\text{Cos}[x]}{\pi} + \frac{4\,\text{Cos}[3\,x]}{9\,\pi} + \frac{4\,\text{Cos}[5\,x]}{25\,\pi}$$

$$s[x] = 1.5708 + 1.27324\,\text{Cos}[x] + 0.141471\,\text{Cos}[3\,x] + 0.0509296\,\text{Cos}[5\,x]$$

Remark. Observe that the Fourier polynomial has period $2L$.



$$S[x] = \frac{\pi}{2} + \frac{4\,\text{Cos}[x]}{\pi} + \frac{4\,\text{Cos}[3\,x]}{9\,\pi} + \frac{4\,\text{Cos}[5\,x]}{25\,\pi}$$

$$s[x] = 1.5708 + 1.27324\,\text{Cos}[x] + 0.141471\,\text{Cos}[3\,x] + 0.0509296\,\text{Cos}[5\,x]$$

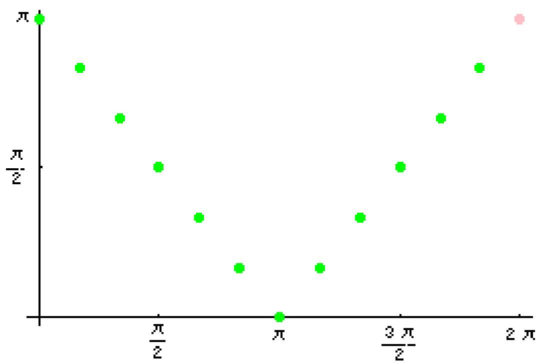
Example 2. Given the 12 equally spaced data points

$(0, \pi)$, $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{2\pi}{3}, \frac{\pi}{3}\right)$, $\left(\frac{5\pi}{6}, \frac{\pi}{6}\right)$, $(\pi, 0)$, $\left(\frac{7\pi}{6}, \frac{\pi}{6}\right)$, $\left(\frac{4\pi}{3}, \frac{\pi}{3}\right)$, $\left(\frac{3\pi}{2}, \frac{\pi}{2}\right)$, $\left(\frac{5\pi}{3}, \frac{2\pi}{3}\right)$, $\left(\frac{11\pi}{6}, \frac{5\pi}{6}\right)$

which can be extended periodically over $[0, 2\pi]$, if we define $(x_n, y_n) = (2\pi, \pi)$.
Find the Fourier polynomial of degree n = 5 for the 12 equally spaced points over interval $[0, 2\pi]$.
Use numerical sums to find the coefficients.

Solution 2.

Note. The data are computed using the function `f[x] = Abs[x - π]`, which is the function that was used in examples 1. Construct the data points to be used.



n = 12

$\left\{\{0, \pi\}, \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}, \left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}, \left\{\frac{\pi}{2}, \frac{\pi}{2}\right\}, \left\{\frac{2\pi}{3}, \frac{\pi}{3}\right\}, \left\{\frac{5\pi}{6}, \frac{\pi}{6}\right\}, \{\pi, 0\}, \left\{\frac{7\pi}{6}, \frac{\pi}{6}\right\}, \left\{\frac{4\pi}{3}, \frac{\pi}{3}\right\}, \left\{\frac{3\pi}{2}, \frac{\pi}{2}\right\}, \left\{\frac{5\pi}{3}, \frac{2\pi}{3}\right\}, \left\{\frac{11\pi}{6}, \frac{5\pi}{6}\right\}\right\}$

The ordinates y_k for $k = 0, 1, 2, \ldots, n-1$ are used in computing the sums for constructing the coefficients $\{a_j\}$, $\{b_j\}$ and the Fourier polynomial.

$\left\{\pi, \frac{5\pi}{6}, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}\right\}$

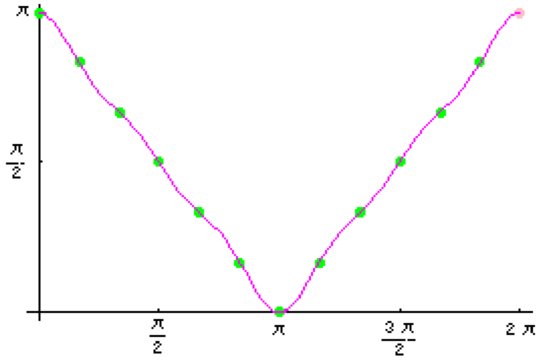
We can adjust the subscript so that the math y_k for $k = 0, 1, 2, \ldots, n-1$ are used in computing the sums for constructing the coefficients $\{a_j\}$, $\{b_j\}$.

The coefficients are
$a_j = \{3.14159, 1.30273, 0, 0.174533, 0, 0.0935319\}$
$b_j = \{0, 0, 0, 0, 0\}$

Remark. Notice that precisely 12 data points are used in computing the coefficients, and a point corresponding to $x_n = 2\pi$ is not used.

Construct the Fourier polynomial using the coefficients $\{a_j\}$ and $\{b_j\}$.

`P[x] = 1.5708 + 1.30273 Cos[x] + 0.174533 Cos[3 x] + 0.0935319 Cos[5 x]`



Using the method of numerical sums, the FFT trig. poly. is

$P[x] = 1.5708 + 1.30273\, Cos[x] + 0.174533\, Cos[3\, x] + 0.0935319\, Cos[5\, x]$

Caveat. Since the data has period $2\,L$, if data points were used at both end of the interval $[0, 2\,L]$ then the "numerical sums" would weight the endpoints twice, whereas all other function values would have weight 1. If 13 data points were used which included the right end point, then a wrong answer will result, and spurious terms appear in the trigonometric polynomial. Look at the wrong answer.

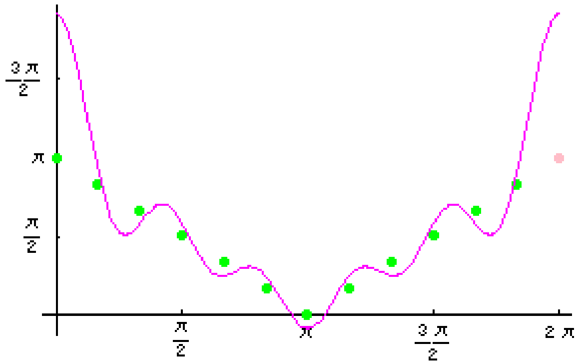
Wrong Answer.

Observe. The upper limit of summation has been changed from $n - 1$ to the wrong value n .

The coefficients are

$a_j = \{3.66519, 1.82633, 0.523599, 0.698132, 0.523599, 0.617131\}$

$b_j = \{0, 0, 0, 0, 0\}$



$p[x] = 1.8326 + 1.82633\, Cos[x] + 0.523599\, Cos[2\, x] + 0.698132\, Cos[3\, x] + 0.523599\, Cos[4\, x] + 0.617131\, Cos[5\, x]$

This is not the right answer, because
data at both endpoints cannot be used.