6. Cholesky, Doolittle and Crout Factorization

Definition (LU-Factorization). The nonsingular matrix **A** has an LU-factorization if it can be expressed as the product of a lower-triangular matrix **L** and an upper triangular matrix **U**:

$$\mathbf{A} = \mathbf{L}\mathbf{U}.$$

When this is possible we say that **A** has an **LU**-decomposition. It turns out that this factorization (when it exists) is not unique. If **L** has 1's on it's diagonal, then it is called a Doolittle factorization. If **U** has 1's on its diagonal, then it is called a Crout factorization. When $\mathbf{U} = \mathbf{L}^{T}$ (or $\mathbf{L} = \mathbf{U}^{T}$), it is called a Cholesky decomposition.

Doolittle Factorization. If **A** has a Doolittle factorization **A** = **LU**.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & a_{4,n} \\ & & & & & \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ l_{2,1} & 1 & 0 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & 1 & 0 & \cdots & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & 1 & \cdots & 0 \\ & & & & & \\ l_{n,1} & l_{n,2} & l_{n,3} & l_{n,4} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} & \cdots & u_{2,n} \\ 0 & 0 & u_{3,3} & u_{3,4} & \cdots & u_{3,n} \\ 0 & 0 & 0 & u_{4,4} & \cdots & u_{4,n} \\ & & & & & \\ 0 & 0 & 0 & 0 & \cdots & u_{n,n} \end{pmatrix}$$

Crout Factorization. If **A** has a Crout factorization **A** = **LU**.

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & a_{4,n} \\ & & & & & \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} l_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} & 0 & \cdots & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & l_{4,4} & \cdots & 0 \\ & & & & & \\ l_{n,1} & l_{n,2} & l_{n,3} & l_{n,4} & \cdots & l_{n,n} \end{pmatrix} \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & u_{1,4} & \cdots & u_{1,n} \\ 0 & 1 & u_{2,3} & u_{2,4} & \cdots & u_{2,n} \\ 0 & 0 & 1 & u_{3,4} & \cdots & u_{3,n} \\ 0 & 0 & 0 & 1 & \cdots & u_{4,n} \\ & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Cholesky Factorization. If **A** is real, symmetric and positive definite matrix, then it has a Cholesky factorizationhas a Cholesky factorization $\mathbf{A} = \mathbf{U}^T\mathbf{U}$, where **U** an upper triangular matrix ($\mathbf{L} = \mathbf{U}^T$).

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & a_{4,n} \\ & & & & & \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} u_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ u_{1,2} & u_{2,2} & 0 & 0 & \cdots & 0 \\ u_{1,3} & u_{2,3} & u_{3,3} & 0 & \cdots & 0 \\ u_{1,4} & u_{2,4} & u_{3,4} & u_{4,4} & \cdots & 0 \\ & & & & & \\ u_{1,n} & u_{2,n} & u_{3,n} & u_{4,n} & \cdots & u_{n,n} \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} & \cdots & u_{2,n} \\ 0 & 0 & u_{3,3} & u_{3,4} & \cdots & u_{3,n} \\ 0 & 0 & 0 & u_{4,4} & \cdots & u_{4,n} \\ & & & & & \\ 0 & 0 & 0 & 0 & \cdots & u_{n,n} \end{pmatrix}$$

Or if you prefer to write the Cholesky factorization as $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ where **L** is a lower triangular matrix ($\mathbf{U} = \mathbf{L}^T$)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots & a_{2,n} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots & a_{3,n} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots & a_{4,n} \\ & & & & & \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} l_{1,1} & 0 & 0 & 0 & \cdots & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} & 0 & \cdots & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & l_{4,4} & \cdots & 0 \\ & & & & & \\ l_{n,1} & l_{n,2} & l_{n,3} & l_{n,4} & \cdots & l_{n,n} \end{pmatrix} \begin{pmatrix} l_{1,1} & l_{2,1} & l_{3,1} & l_{4,1} & \cdots & l_{n,1} \\ 0 & l_{2,2} & l_{3,2} & l_{4,2} & \cdots & l_{n,2} \\ 0 & 0 & l_{3,3} & l_{4,3} & \cdots & l_{n,3} \\ 0 & 0 & 0 & l_{4,4} & \cdots & l_{n,4} \\ & & & & & \\ 0 & 0 & 0 & 0 & \cdots & l_{n,n} \end{pmatrix}$$

Theorem (A = LU; Factorization with NO Pivoting). Assume that **A** has a Doolittle, Crout or Cholesky factorization. The solution **X** to the linear system $\mathbf{AX} = \mathbf{B}$, is found in three steps:

1. Construct the matrices $\mathbf{L}$ and $\mathbf{U}$, if possible.
2. Solve $\mathbf{LY} = \mathbf{B}$ for $\mathbf{Y}$ using forward substitution.
3. Solve $\mathbf{UX} = \mathbf{Y}$ for $\mathbf{X}$ using back substitution.

Example 1. Find the **A** = **LU** factorization for the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$. Use the Doolittle method.

Solution 1.

Example 2. Find the **A** = **LU** factorization for the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$. Use the Crout method.

Solution 2.

Example 3. Find the **A** = **LU** factorization for the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$. Use the Cholesky method.

Solution 3.

Example 1. Find the **A** = **LU** factorization for the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$. Use the Doolittle method.

Solution 1.

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{3} & 0 & 1 & 0 \\ 1 & 0 & \frac{4}{7} & -\frac{6}{7} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 0 & \frac{3}{2} & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 7 & 0 & 4 \\ 0 & 0 & 0 & \frac{7}{3} & -2 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Example 2. Find the $\mathbf{A} = \mathbf{LU}$ factorization for the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$. Use the Crout method.

Solution 2.

$$L = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & \frac{3}{2} & 0 & 0 & 0 \\ 1 & \frac{3}{2} & 7 & 0 & 0 \\ 3 & -\frac{1}{2} & 0 & \frac{7}{3} & 0 \\ 2 & 0 & 4 & -2 & 2 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{4}{7} \\ 0 & 0 & 0 & 1 & -\frac{6}{7} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 3. Find the **A** = **LU** factorization for the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 3 & 2 \\ 1 & 2 & 2 & 1 & 1 \\ 1 & 2 & 9 & 1 & 5 \\ 3 & 1 & 1 & 7 & 1 \\ 2 & 1 & 5 & 1 & 8 \end{pmatrix}$. Use the Cholesky method.

Solution 3.

$$L = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}} & \sqrt{7} & 0 & 0 \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & 0 & \sqrt{\frac{7}{3}} & 0 \\ \sqrt{2} & 0 & \frac{4}{\sqrt{7}} & -2\sqrt{\frac{3}{7}} & \sqrt{2} \end{pmatrix}$$

$$U = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & \sqrt{2} \\ 0 & \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{7} & 0 & \frac{4}{\sqrt{7}} \\ 0 & 0 & 0 & \sqrt{\frac{7}{3}} & -2\sqrt{\frac{3}{7}} \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{pmatrix}$$