6. The Finite Difference Method for Boundary Value Problems

Background

Theorem (Boundary Value Problem). Assume that $f(t, x, y)$ is continuous on the region $\mathbf{R} = \{(t, x, y) : a \le t \le b, -\infty < x < \infty, -\infty < y < \infty\}$ and that $\frac{\partial f}{\partial x} = f_x(t, x, y)$ and $\frac{\partial f}{\partial y} = f_y(t, x, y)$ are continuous on $\mathbf{R}$. If there exists a constant $M > 0$ for which f_x and f_y satisfy

$f_x(t, x, y) > 0$ for all $(t, x, y) \in \mathbf{R}$
and
$|f_y(t, x, y)| < M$ for all $(t, x, y) \in \mathbf{R}$,

then the boundary value problem

$x'' = f(t, x, x')$ with $x(a) = \alpha$ and $x(b) = \beta$

has a unique solution $x = x(t)$ for $a \le t \le b$.

The notation $y = x'(t)$ has been used to distinguish the third variable of the function $f(t, x, x')$. Finally, the special case of linear differential equations is worthy of mention.

Corollary (Linear Boundary Value Problem). Assume that $f(t, x, y)$ in the theorem has the form $f(t, x, y) = p(t)y + q(t)x + r(t)$ and that f and its partial derivatives $\frac{\partial f}{\partial x} = q(t)$ and $\frac{\partial f}{\partial y} = p(t)$ are continuous on $\mathbf{R}$. If there exists a constant $M > 0$ for which p(t) and q(t) satisfy

$q(t) > 0$ for all $t \in [a, b]$
and
$|p(t)| < M = \max_{a \le t \le b}\{|p(t)|\}$,

then the linear boundary value problem

$x'' = p(t)x'(t) + q(t)x(t) + r(t)$ with $x(a) = \alpha$ and $x(b) = \beta$

has a unique solution $x = x(t)$ over $a \le t \le b$.

Finite-Difference Method

Methods involving difference quotient approximations for derivatives can be used for solving certain second-order boundary value problems. Consider the linear equation

(1) $\quad x'' = p(t)\,x'(t) + q(t)\,x(t) + r(t)$

over $[a,b]$ with $x(a) = \alpha$ and $x(b) = \beta$. Form a partition of $[a, b]$ using the points $a = t_0 < t_1 < ... < t_n = b$, where $h = \frac{b-a}{n}$ and $t_j = a + jh$ for $j = 0, 1, 2, ..., n$. The central-difference formulas discussed in Chapter 6 are used to approximate the derivatives

(2) $$x'(t_j) = \frac{x(t_{j+1}) - x(t_{j-1})}{2h} + O(h^2)$$

and

(3) $$x''(t_j) = \frac{x(t_{j+1}) - 2x(t_j) + x(t_{j-1})}{h^2} + O(h^2)$$

Use the notation x_j for the terms $x(t_j)$ on the right side of (2) and (3) and drop the two terms $O(h^2)$. Also, use the notations $p_j = p(t_j)$, $q_j = q(t_j)$, and $r_j = r(t_j)$ this produces the difference equation

$$\frac{x_{j+1} - 2x_j + x_{j-1}}{h^2} = p_j \frac{x_{j+1} - x_{j-1}}{2h} + q_j x_j + r_j$$

which is used to compute numerical approximations to the differential equation (1). This is carried out by multiplying each side by h^2 and then collecting terms involving x_{j-1}, x_j, and x_{j+1} and arranging them in a system of linear equations:

$$\left(\frac{-h}{2} p_j - 1\right) x_{j-1} + (2 + h^2 q_j)\, x_j + \left(\frac{h}{2} p_j - 1\right) x_{j+1} = -h^2 r_j$$

for $j = 1, 2, ..., n-1$, where $x_0 = \beta$ and $x_n = \beta$. This system has the familiar tridiagonal form.

Example 1. Solve $x'' + \frac{1}{t} x' + \left(1 - \frac{1}{4t^2}\right) x = \sqrt{t}\cos(t)$ over $[1, 6]$ with $x(1) = 1.0$ and $x(6) = -0.5$.
Use the finite difference method with 25 subintervals (total of 26 points).
Solution 1.

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Use the finite difference method with 25 subintervals (total of 26 points).
Solution 1.

Enter the function p(t), q(t) and r(t).

The D. E. we wish to solve is

$$x'' + \left(\frac{1}{t}\right)x' + \left(1 - \frac{1}{4t^2}\right)x = \sqrt{t}\ \mathrm{Cos}[t]$$

with $x(1.) = \alpha = 1.$
and $x(6.) = \beta = -0.5$

over the interval [1., 6.]

Construct the vectors for the tri-diagonal system.

Vt = {1.2, 1.4, 1.6, 1.8, 2., 2.2, 2.4, 2.6, 2.8, 3., 3.2, 3.4, 3.6, 3.8, 4., 4.2, 4.4, 4.6, 4.8, 5., 5.2, 5.4, 5.6, 5.8}

Va = {-0.928571, -0.9375, -0.944444, -0.95, -0.954545, -0.958333, -0.961538, -0.964286, -0.966667, -0.96875, -0.970588, -0.972222, -0.973684, -0.975, -0.97619, -0.977273, -0.978261, -0.979167, -0.98, -0.980769, -0.981481, -0.982143, -0.982759}

Vc = {-1.08333, -1.07143, -1.0625, -1.05556, -1.05, -1.04545, -1.04167, -1.03846, -1.03571, -1.03333, -1.03125, -1.02941, -1.02778, -1.02632, -1.025, -1.02381, -1.02273, -1.02174, -1.02083, -1.02, -1.01923, -1.01852, -1.01786}

Vd = {1.96694, 1.9651, 1.96391, 1.96309, 1.9625, 1.96207, 1.96174, 1.96148, 1.96128, 1.96111, 1.96098, 1.96087, 1.96077, 1.96069, 1.96063, 1.96057, 1.96052, 1.96047, 1.96043, 1.9604, 1.96037, 1.96034, 1.96032, 1.9603}

Vb = {0.900789, -0.00804431, 0.00147739, 0.0121929, 0.0235408, 0.0349155, 0.0456946, 0.0552677, 0.0630656, 0.0685887, 0.0714322, 0.0713075, 0.0680592, 0.0616752, 0.0522915, 0.0401894, 0.0257867, 0.00962161, -0.00766803, -0.0253715, -0.0427353, -0.0589957, -0.073413, -0.593925}

Now solve the tri-diagonal system for the ordinates that are strictly inside the interval.

Vt = {1.2, 1.4, 1.6, 1.8, 2., 2.2, 2.4, 2.6, 2.8, 3., 3.2, 3.4, 3.6, 3.8, 4., 4.2, 4.4, 4.6, 4.8, 5., 5.2, 5.4, 5.6, 5.8}

Xt = {1.0568, 1.08727, 1.08577, 1.04617, 0.962608, 0.830202, 0.645789, 0.408541, 0.120491, -0.21309, -0.583505, -0.978659, -1.3833, -1.77948, -2.14729, -2.46569, -2.71357, -2.87088, -2.91985, -2.84614, -2.63995, -2.29696, -1.81909, -1.21494}

We need to put the vectors together to form points.

points = {{1.2, 1.0568}, {1.4, 1.08727}, {1.6, 1.08577}, {1.8, 1.04617}, {2., 0.962608}, {2.2, 0.830202}, {2.4, 0.645789}, {2.6, 0.408541}, {2.8, 0.120491}, {3., -0.21309}, {3.2, -0.583505}, {3.4, -0.978659}, {3.6, -1.3833}, {3.8, -1.77948}, {4., -2.14729}, {4.2, -2.46569}, {4.4, -2.71357}, {4.6, -2.87088}, {4.8, -2.91985}, {5., -2.84614}, {5.2, -2.63995}, {5.4, -2.29696}, {5.6, -1.81909}, {5.8, -1.21494}}

We need to append the first and last boundary points to the list.

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points = {{1.2, 1.0568}, {1.4, 1.08727}, {1.6, 1.08577}, {1.8, 1.04617}, {2., 0.962608}, {2.2, 0.830202}, {2.4, 0.645789}, {2.6, 0.408541},
  {2.8, 0.120491}, {3., -0.21309}, {3.2, -0.583505}, {3.4, -0.978659}, {3.6, -1.3833}, {3.8, -1.77948}, {4., -2.14729}, {4.2, -2.46569},
  {4.4, -2.71357}, {4.6, -2.87088}, {4.8, -2.91985}, {5., -2.84614}, {5.2, -2.63995}, {5.4, -2.29696}, {5.6, -1.81909}, {5.8, -1.21494}, {6, -0.5}}

points = {{1., 1.}, {1.2, 1.0568}, {1.4, 1.08727}, {1.6, 1.08577}, {1.8, 1.04617}, {2., 0.962608}, {2.2, 0.830202}, {2.4, 0.645789},
  {2.6, 0.408541}, {2.8, 0.120491}, {3., -0.21309}, {3.2, -0.583505}, {3.4, -0.978659}, {3.6, -1.3833}, {3.8, -1.77948}, {4., -2.14729}, {4.2, -2.46569},
  {4.4, -2.71357}, {4.6, -2.87088}, {4.8, -2.91985}, {5., -2.84614}, {5.2, -2.63995}, {5.4, -2.29696}, {5.6, -1.81909}, {5.8, -1.21494}, {6, -0.5}}
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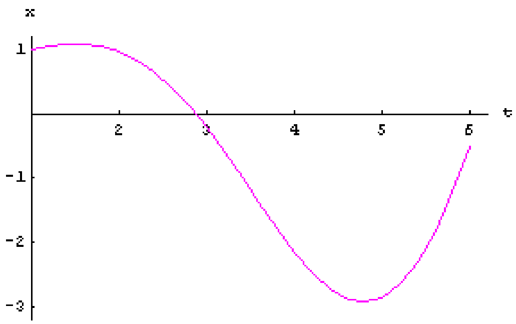
Now we have the solution.



The solution to

$$x'' + \left(\frac{1}{t}\right)x' + \left(1 - \frac{1}{4t^2}\right)x = \sqrt{t}\,\mathrm{Cos}[t]$$

$x(1.) = 1.$
$x(6.) = -0.5$