6. Boole's Rule for Numerical Integration

Theorem (Boole's Rule) Consider $y = f(x)$ over $[x_0, x_4]$, where $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$, and $x_4 = x_0 + 4h$. Boole's rule is

$$BR(f, h) = \frac{2h}{45}(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)).$$

This is an numerical approximation to the integral of $f(x)$ over $[x_0, x_4]$ and we have the expression

$$\int_{x_0}^{x_4} f(x)\,dx \approx BR(f, h).$$

The remainder term for Boole's rule is $R_{BR}(f, h) = -\frac{8}{945} f^{(6)}(c)\, h^7$, where c lies somewhere between x_0 and x_4, and have the equality

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)) - \frac{8}{945} f^{(6)}(c)\, h^7.$$

Composite Boole's Rule

Our next method of finding the area under a curve $y = f(x)$ is by approximating that curve with a series of quartic segments that lie above the intervals $\{[x_{k-1}, x_k]\}_{k=1}^{4m}$. When several parabolas are used, we call it the composite Boole's rule.

Theorem (Composite Boole's Rule) Consider $y = f(x)$ over $[a, b]$. Suppose that the interval $[a, b]$ is subdivided into $4m$ subintervals $\{[x_{k-1}, x_k]\}_{k=1}^{4m}$ of equal width $h = \frac{b-a}{4m}$ by using the equally spaced sample points $x_k = x_0 + kh$ for $k = 0, 1, 2, \ldots, 4m$. The composite Boole's rule for $4m$ subintervals is

$$B(f, h) = \frac{2h}{45}\sum_{k=1}^{m}(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)).$$

is an numerical approximation to the integral, and

$$\int_a^b f(x)\,dx = B(f, h) + E_B(f, h).$$

Furthermore, if $f(x) \in C^6[a, b]$, then there exists a value c with $a < c < b$ so that the error term $E_B(f, h)$ has the form

$$E_B(f,h) = -\frac{2(b-a)f^{6}(c)}{945}h^{6}.$$

This is expressed using the "big O" notation $E_B(f,h) = O(h^6)$.

Remark. When the step size is reduced by a factor of $\frac{1}{2}$ the remainder term $R_B(f,h)$ should be reduced by approximately $\left(\frac{1}{2}\right)^6 = 0.015625$.

Example 1. Let f[x] be $\int_0^2 \left(2 + \mathrm{Cos}\left[2\sqrt{x}\right]\right) dx$.

1 (a) Numerically approximate the integral by using Boole's rule with m = 1 and 2.
1 (b) Find the analytic value of the integral (i.e. find the "true value").
1 (c) Find the error for the Boole's rule approximations.
Solution 1.

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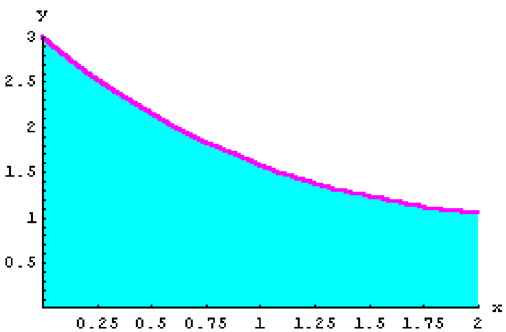
Solution 1 (a).



$$f[x] = 2 + \text{Cos}\left[2\sqrt{x}\right]$$

We will use simulated hand computations for the solution.

$$\mathbf{f[x\_] = 2 + Cos\left[2\sqrt{x}\right];}$$

$$\mathbf{b1} = \frac{2\,\frac{2-0}{4}}{45}\left(7\,f[0] + 32\,f\left[\frac{1}{2}\right] + 12\,f[1] + 32\,f\left[\frac{3}{2}\right] + 7\,f[2]\right)$$

NumberForm[N[b1], 12]

$$\frac{1}{45}\left(21 + 12\,(2 + \text{Cos}[2]) + 32\left(2 + \text{Cos}\left[\sqrt{2}\right]\right) + 7\left(2 + \text{Cos}\left[2\sqrt{2}\right]\right) + 32\left(2 + \text{Cos}\left[\sqrt{6}\right]\right)\right)$$

3.459998021

$$\mathbf{b2} = \frac{2\,\frac{2-0}{8}}{45}\left(7\,f[0] + 32\,f\left[\frac{1}{4}\right] + 12\,f\left[\frac{1}{2}\right] + 32\,f\left[\frac{3}{4}\right] + 14\,f[1] + 32\,f\left[\frac{5}{4}\right] + 12\,f\left[\frac{3}{2}\right] + 32\,f\left[\frac{7}{4}\right] + 7\,f[2]\right)$$

NumberForm[N[b2], 12]

$$\frac{1}{90}\left(21 + 32\,(2 + \text{Cos}[1]) + 14\,(2 + \text{Cos}[2]) + 12\left(2 + \text{Cos}\left[\sqrt{2}\right]\right) + 7\left(2 + \text{Cos}\left[2\sqrt{2}\right]\right) + 32\left(2 + \text{Cos}\left[\sqrt{3}\right]\right) + 32\left(2 + \text{Cos}\left[\sqrt{5}\right]\right) + 12\left(2 + \text{Cos}\left[\sqrt{6}\right]\right) + 32\left(2 + \text{Cos}\left[\sqrt{7}\right]\right)\right)$$

3.45999767763

Solution 1 (b).

The integral of $f[x] = 2 + \text{Cos}\left[2\sqrt{x}\right]$ can be determined.

$$\int \left(2 + \mathbf{Cos}\left[2\sqrt{x}\right]\right) dx$$

$$2\,x + \frac{1}{2}\,\text{Cos}\left[2\,\sqrt{x}\,\right] + \sqrt{x}\,\,\text{Sin}\left[2\,\sqrt{x}\,\right]$$

The value of the definite integral

val = $\int_0^2 \left(2 + \text{Cos}\left[2\,\sqrt{x}\,\right]\right) dx$

$$\frac{7}{2} + \frac{1}{2}\,\text{Cos}\left[2\,\sqrt{2}\,\right] + \sqrt{2}\,\,\text{Sin}\left[2\,\sqrt{2}\,\right]$$

N[val]

3.46

NumberForm[N[val], 17]

3.459997672170804

Solution 1 (c).

val - t16

-0.000000005459196