1. Maclaurin and Taylor Polynomials

Background.

When a Taylor series is truncated to a finite number of terms the result is a Taylor polynomial. A Taylor series expanded about $x_0 = 0$, is called a Maclarin series. These Taylor (and Maclaurin) polynomials are used to numerically approximate functions. We attribute much of the founding theory to Brook Taylor (1685-1731), Colin Maclaurin (1698-1746) and Joseph-Louis Lagrange (1736-1813).

Theorem (Taylor Polynomial Approximation). Assume that $f \in C^{n+1}[x_0 - R, x_0 + R]$, then

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ is a polynomial that can be used to approximate $f(x)$, and we write

$$f(x) \approx P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

The remainder term $R_n(x)$ has the form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1},$$

for some value $c = c(x)$ that lies between c and x_0. The formula $R_n(x)$ is referred to as the Lagrange form of the remainder.

Corollary 1. Assume that $f \in C^{n+1}[x_0 - R, x_0 + R]$, and that the Taylor polynomial of degree n for $f(x)$ is $P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$, then

$$P^{(k)}(x_0) = f^{(k)}(x_0) \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Corollary 2. Assume that $f \in C^{n+1}[x_0 - R, x_0 + R]$, and that the Taylor polynomial of degree n for $f(x)$ is $P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$, then

$$\left| R_n(x) \right| \le \frac{M}{(n+1)!} R^{n+1},$$

where $M = \max\{ |f^{(n+1)}(x)| : x_0 - R \le x \le x_0 + R\}$.

Example 1. Find the Maclaurin polynomial for $f[x] = \frac{1}{\sqrt{1-x}}$ expanded about $x_0 = 0$.

Solution 1.

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Example 2. Find the Taylor polynomial for $f[x] = \text{Log}[x]$ expanded about $x_0 = 1$..

Solution 2.

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Example 3. Consider the function $f[x] = \frac{1}{1+x^2}$. Investigate the error term $E_n(x)$ for the Maclaurin polynomial of degree n = 10 over the interval [-0.5, 0.5].

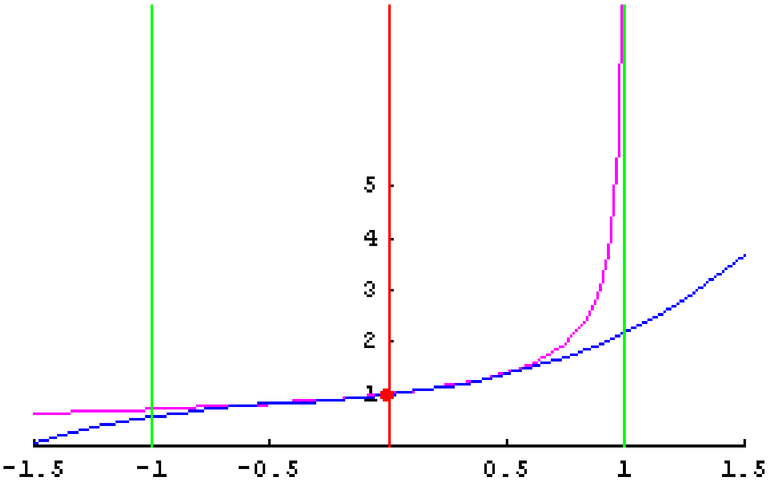
Solution 3.

Example 4. Consider the function $f[x] = \text{Cos}[x]$. Investigate the error term $E_n(x)$ for the Maclaurin polynomial of degree n = 10 over the interval [-2.0, 2.0].

Solution 4.

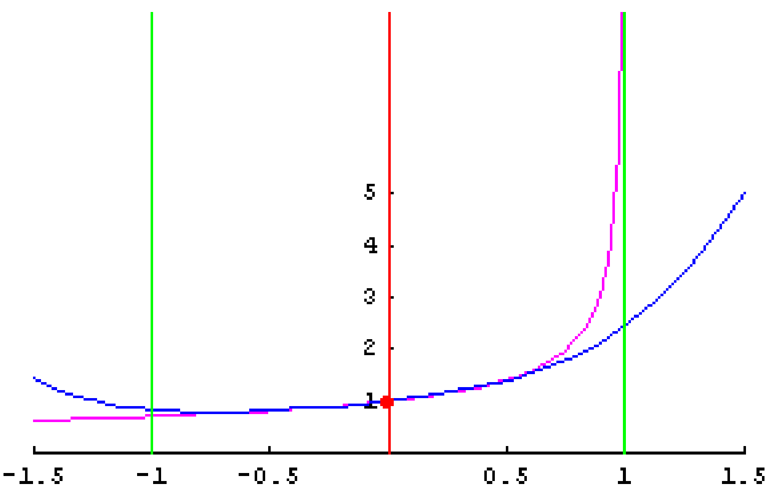
Example 1. Find the Maclaurin polynomial for $f[x] = \frac{1}{\sqrt{1-x}}$ expanded about $x_0 = 0$.

Solution 1.



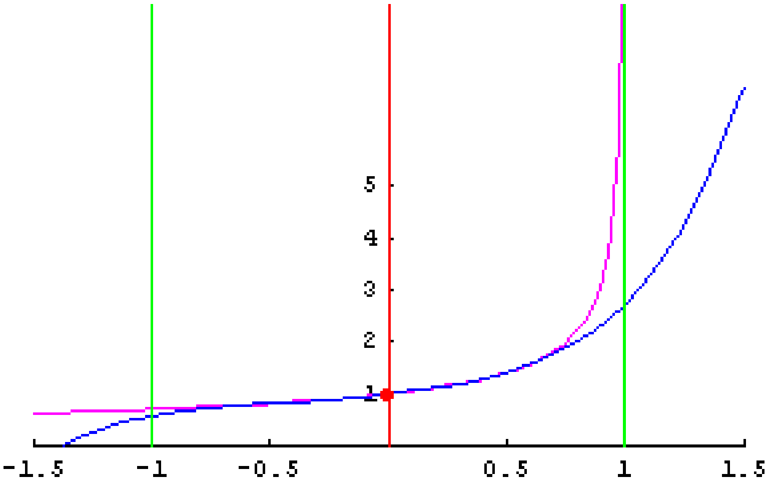
$$f[x] = \frac{1}{\sqrt{1-x}}$$

$$P[x] = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16}$$



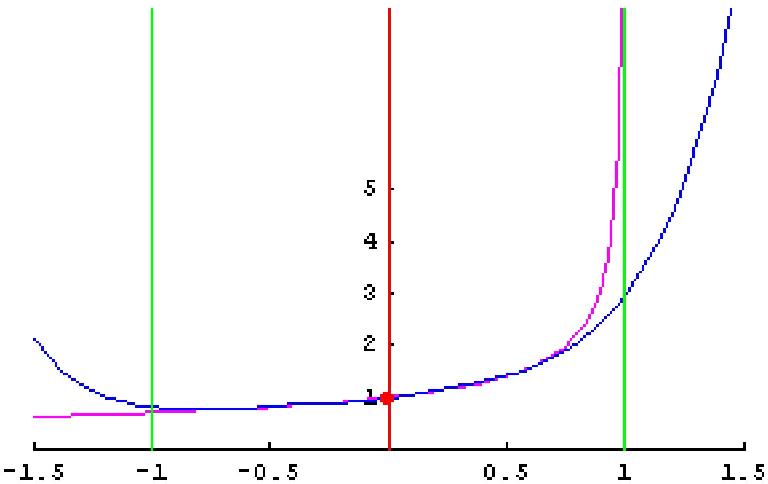
$$f[x] = \frac{1}{\sqrt{1-x}}$$

$$P[x] = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128}$$



$$f[x] = \frac{1}{\sqrt{1-x}}$$

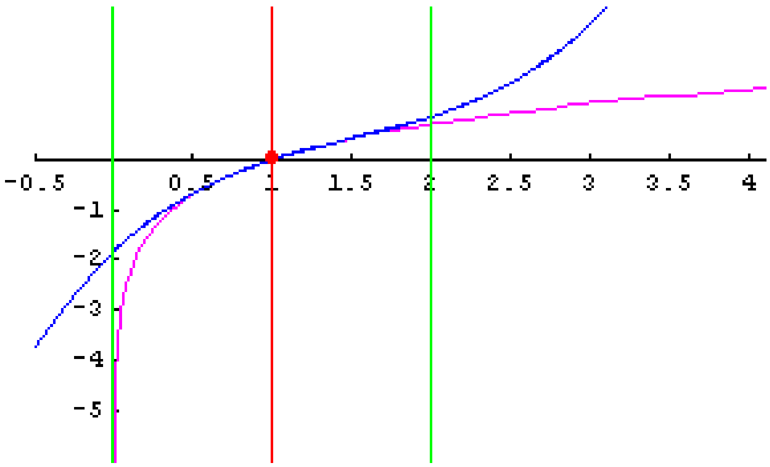
$$P[x] = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128} + \frac{63x^5}{256}$$



$$f[x] = \frac{1}{\sqrt{1-x}}$$

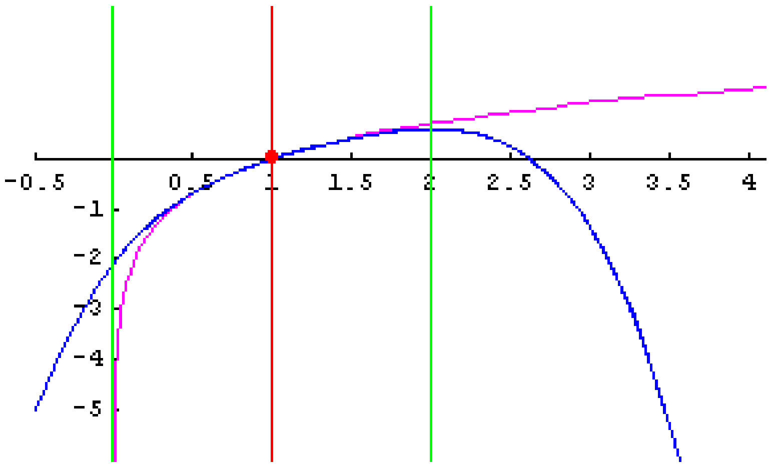
$$P[x] = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \frac{35x^4}{128} + \frac{63x^5}{256} + \frac{231x^6}{1024}$$

Example 2. Find the Taylor polynomial for $f[x] = \text{Log}[x]$ expanded about $x_0 = 1$.
Solution 2.



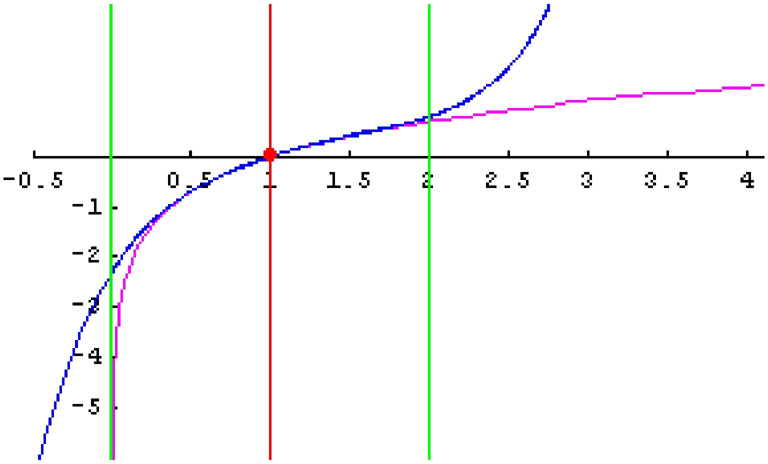
$$f[x] = \text{Log}[x]$$

$$P[x] = -1 - \frac{1}{2}(-1+x)^2 + \frac{1}{3}(-1+x)^3 + x$$



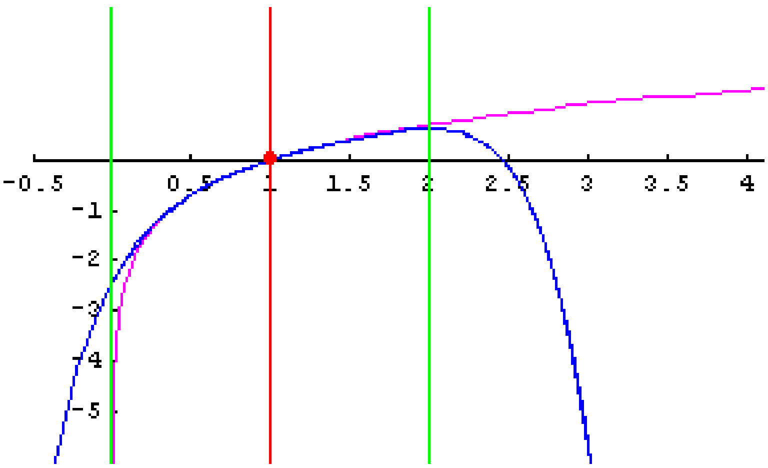
$$f[x] = \text{Log}[x]$$

$$P[x] = -1 - \frac{1}{2}(-1+x)^2 + \frac{1}{3}(-1+x)^3 - \frac{1}{4}(-1+x)^4 + x$$



$$f[x] = \text{Log}[x]$$

$$P[x] = -1 - \frac{1}{2}(-1+x)^2 + \frac{1}{3}(-1+x)^3 - \frac{1}{4}(-1+x)^4 + \frac{1}{5}(-1+x)^5 + x$$



$$f[x] = \text{Log}[x]$$

$$P[x] = -1 - \frac{1}{2}(-1+x)^2 + \frac{1}{3}(-1+x)^3 - \frac{1}{4}(-1+x)^4 + \frac{1}{5}(-1+x)^5 - \frac{1}{6}(-1+x)^6 + x$$

Example 3. Consider the function $f[x] = \frac{1}{1+x^2}$. Investigate the error term $E_n(x)$ for the Maclaurin polynomial of degree n = 10 over the interval [-0.5, 0.5].

Solution 3.

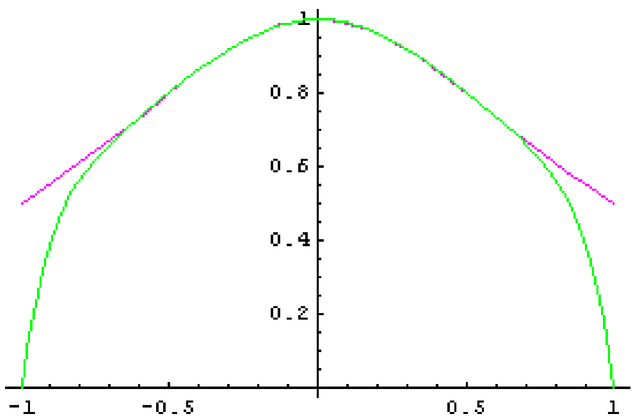
Find the terms up to x^{10} in the Maclaurin series for f[x].

$$f[x] = \frac{1}{1+x^2}$$
$$s[x] = 1 - x^2 + x^4 - x^6 + x^8 - x^{10}$$

Remark. If you just find the "Series" it will include a "Big O" term, which cannot be used in either evaluations or graphing, we eliminate it with the command "Normal." The "Big O term lets us know the power of x in the "remainder.

$$1 - x^2 + x^4 - x^6 + x^8 - x^{10} + O[x]^{11}$$

Now graph f[x] and the Maclaurin polynomial s[x] over the interval [-1, 1].



$$f[x] = \frac{1}{1+x^2}$$
$$s[x] = 1 - x^2 + x^4 - x^6 + x^8 - x^{10}$$

Notice that there is a significant amount of error near $x = \pm 1$. How close are were the two curves? An "error" function e [x] can be defined as follows:

$$\mathbf{f[x] - s[x]}$$
$$-1 + x^2 - x^4 + x^6 - x^8 + x^{10} + \frac{1}{1+x^2}$$

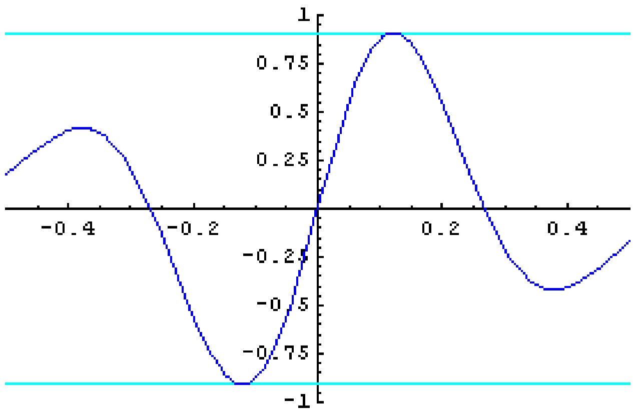
The Lagrange form of the error is $R[x, c] = \frac{f^{(11)}[c]}{11!} x^{11}$ where c is known to exist and lies somewhere between 0 and x.

The Lagrange form of the remainder is

$R[x,c] = \frac{f^{(11)}[c]}{11!} x^{11}$ where c lies somewhere between 0 and x.

$$R[x,c] = -\frac{4\left(-3c + 55c^3 - 198c^5 + 198c^7 - 55c^9 + 3c^{11}\right)x^{11}}{\left(1+c^2\right)^{12}}$$

First we need to bound the size of the term $\frac{f^{(11)}[c]}{11!}$ for values of c in the interval $-0.5 \le c \le 0.5$. This can easily be done graphically, but to do it analytically with derivatives is quite messy. We choose to look at the following graph to see what is happening.



$$\frac{f^{(11)}[c]}{11!} = -\frac{4\left(-3c + 55c^3 - 198c^5 + 198c^7 - 55c^9 + 3c^{11}\right)}{\left(1+c^2\right)^{12}}$$

How big does $\frac{f^{(11)}[c]}{11!}$ get ? Looking at the graph we can estimate it to be $\left|\frac{f^{(11)}[c]}{11!}\right| \le 0.909253$.

How big does the error $R[x, c] = \frac{f^{(11)}[c]}{11!} x^{11}$ get ? Notice that $|R[x, c]| = \left|\frac{f^{(11)}[c]}{11!}\right| |x^{11}|$.
We will use the bound the first portion $\left|\frac{f^{(11)}[c]}{11!}\right| \le 0.91$ and then bound the portion $|x^{11}|$ over the interval [-0.5, 0.5] by evaluating it at $x = \pm 0.5$

$$|x^{11}| \le 0.000488281$$

Now multiply the two numbers together to find the error bound for Lagrange's remainder formula.

$$|R[x, c]| \le 0.0004881 * 0.909253 = 0.000443806$$

This is a little larger than the actual maximum error we found.

$$e[0.5] = 0.000195313$$
$$e[-0.5] = 0.000195313$$

Example 4. Consider the function $f[x] = Cos[x]$. Investigate the error term $E_n(x)$ for the Maclaurin polynomial of degree n = 10 over the interval [-2.0, 2.0].
Solution 4.

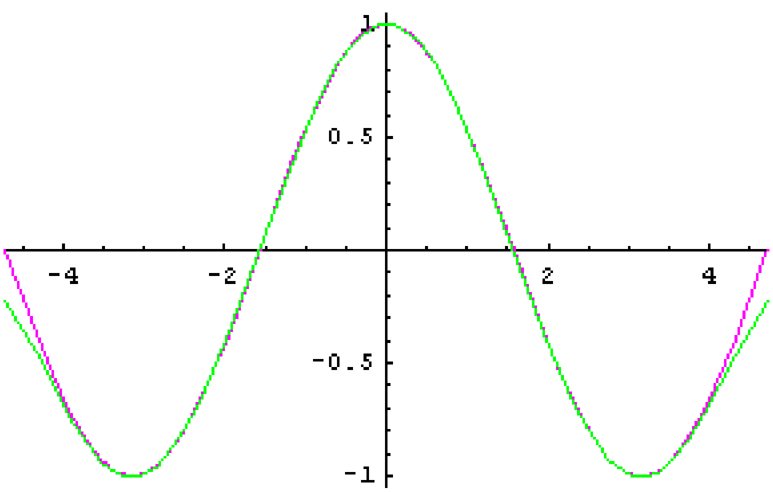
Find the terms up to x^{10} in the Maclaurin series for f[x].

$$f[x] = Cos[x]$$
$$s[x] = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800}$$

Remark. If you just find the "Series" it will include a "Big O" term, which cannot be used in either evaluations or graphing, we eliminate it with the command "Normal." The "Big O term lets us know the power of x in the "remainder.

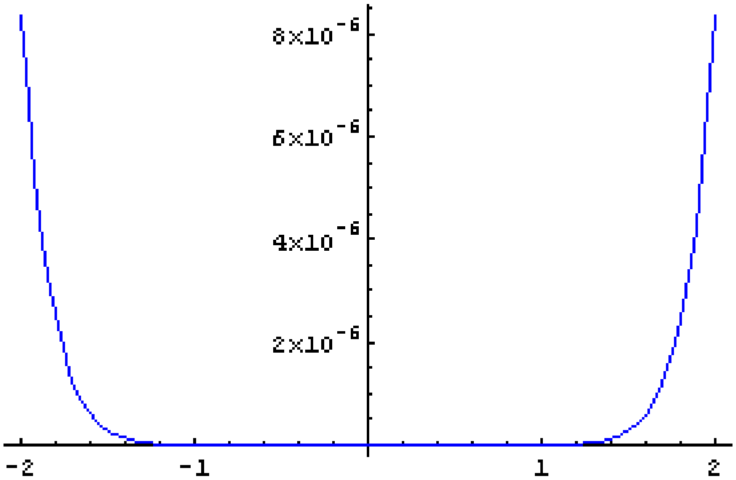
$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + O[x]^{11}$$

Now graph f[x] and the Maclaurin polynomial s[x] over the interval [-4, 4], iin order to show that the curves are distinct.



$$f[x] = Cos[x]$$
$$s[x] = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800}$$

Now look closely at the "error" when the series is used to approximate the function. How close are were the two curves in part (a) ?



$$e[x] = -1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \frac{x^8}{40320} + \frac{x^{10}}{3628800} + \mathrm{Cos}[x]$$

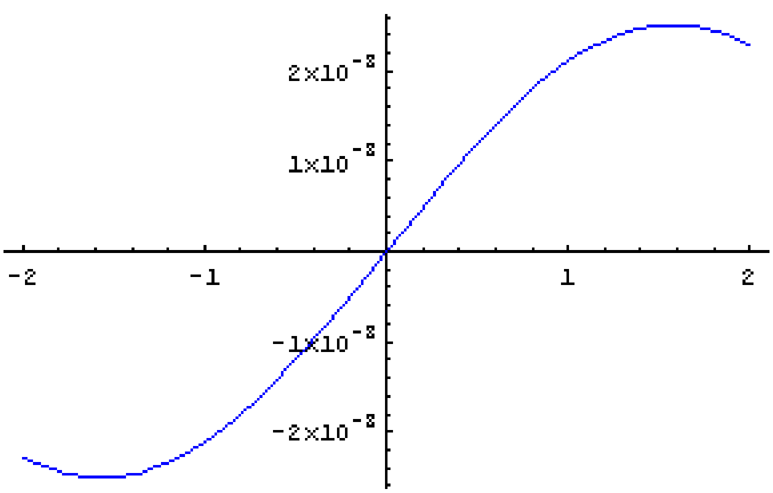
The Lagrange form of the error is $R[x, c] = \frac{f^{(21)}[c]}{21!} x^{21}$ where c is known to exist and lies somewhere between 0 and x.

The Lagrange form of the remainder is

$R[x,c] = \frac{f^{(11)}[c]}{11!} x^{11}$ where c lies somewhere between 0 and x.

$$R[x,c] = \frac{x^{11}\,\mathrm{Sin}[c]}{39916800}$$

First we need to bound the size of the term $\frac{f^{(11)}[c]}{11!}$ for values of c in the interval $-2.0 \le c \le 2.0$. This can easily be done graphically, but to do it analytically with derivatives is quite messy. We choose to look at the following graph to see what is happening.



$$\frac{f^{(11)}[c]}{11!} = \frac{\text{Sin}[c]}{39916800}$$

How big does $\frac{f^{(11)}[c]}{11!}$ get ? Looking at the graph we can estimate it to be

$\left|\frac{f^{(11)}[c]}{11!}\right| \le \frac{1}{11!} = 2.50521\times10^{-8}$.

How big does the error $R[x, c] = \frac{f^{(11)}[c]}{11!}x^{11}$ get ? Notice that $|R[x, c]| = \left|\frac{f^{(11)}[c]}{11!}\right|$ $|x^{11}|$.

We will use the bound the first portion $\left|\frac{f^{(11)}[c]}{11!}\right| \le 2.50521\times10^{-8}$ and then bound the portion $|x^{11}|$ over the interval [-2.0, 2.0] by evaluating it at $x = \pm2.0$

$$|x^{11}| \le 2048.$$

Now multiply the two numbers together to find the error bound for Lagrange's remainder formula.

$$|R[x, c]| \le 2.50521\times10^{-8} * 2048 = 0.0000513067 = 5.13067\times10^{-5}$$

This is a larger than the actual maximum error we found.

$$e[2.0] = 8.36627\times10^{-6}$$
$$e[-2.0] = 8.36627\times10^{-6}$$