15. Fixed Point Iteration and Newton's Method in 2D and 3D

Background

Iterative techniques will now be introduced that extend the fixed point and Newton methods for finding a root of an equation. We desire to have a method for finding a solution for the system of nonlinear equations

(1) $$\begin{aligned} f_1(x, y) &= 0 \\ f_2(x, y) &= 0. \end{aligned}$$

Each equation in (1) implicitly defines a curve in the plane and we want to find their points of intersection. Our first method will be be fixed point iteration and the second one will be Newton's method.

Definition (Jacobian Matrix). Assume that $f_1(x, y)$ and $f_2(x, y)$ are functions of the independent variables x and y, then their Jacobian matrix $\mathbf{J}(x, y)$ is

$$\mathbf{J}(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

Similarly, if $f_1(x, y, z)$, $f_2(x, y, z)$ and $f_3(x, y, z)$ are functions of the independent variables x, y and z, then their Jacobian matrix $\mathbf{J}(x, y, z)$ is

$$\mathbf{J}(x, y, z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}.$$

Generalized Differential

For a function of several variables, the differential is used to show how changes of the independent variables affect the change in the dependent variables. Suppose that we have

$$u = f_1(x, y, z)$$

$$v = f_2(x, y, z)$$

$$w = f_3(x, y, z)$$

Suppose that the values of these functions in are known at the point (x_0, y_0, z_0) and we wish to predict their value at a nearby point (x, y, z). Let du, dv and dw, denote differential changes in the dependent variables and , and dx, dy and dz denote differential changes in the independent variables. These changes obey the relationships

$$du = \frac{\partial}{\partial x} f_1(x_0, y_0, z_0)\, dx + \frac{\partial}{\partial z} f_1(x_0, y_0, z_0)\, dy + \frac{\partial}{\partial z} f_1(x_0, y_0, z_0)\, dz,$$

$$dv = \frac{\partial}{\partial x} f_2(x_0, y_0, z_0)\, dx + \frac{\partial}{\partial z} f_2(x_0, y_0, z_0)\, dy + \frac{\partial}{\partial z} f_2(x_0, y_0, z_0)\, dz,$$

$$dw = \frac{\partial}{\partial x} f_3(x_0, y_0, z_0)\, dx + \frac{\partial}{\partial z} f_3(x_0, y_0, z_0)\, dy + \frac{\partial}{\partial z} f_3(x_0, y_0, z_0)\, dz.$$

If vector notation is used, then this can be compactly written by using the Jacobian matrix. The function changes are $d\mathbf{F}$ and the changes in the variables are denoted $d\mathbf{X}$.

$$\begin{pmatrix} du \\ dv \\ dw \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

or

$$d\mathbf{F} = \mathbf{J}(x_0, y_0, z_0)\, d\mathbf{X}.$$

Convergence near a Fixed Point

In solving for solutions of a system of equations, iteration is used. We now turn to the study of sufficient conditions which will guarantee convergence.

Definition (Fixed Point). A fixed point for the system of two equations

$$x = g_1(x, y),$$
$$y = g_2(x, y).$$

is a point (p, q) such that $p = g_1(p, q)$ and $q = g_2(p, q)$. Similarly, in three dimensions a fixed point for the system of three equations

$$x = g_1(x, y, z),$$
$$y = g_2(x, y, z),$$
$$z = g_3(x, y, z).$$

is a point (p, q, r) such that $p = g_1(p, q, r)$, $q = g_2(p, q, r)$ and $r = g_3(p, q, r)$.

Definition (Fixed Point Iteration). For a system of two equations, fixed point iteration is

$$p_{k+1} = g_1(p_k, q_k), \text{ and}$$
$$q_{k+1} = g_2(p_k, q_k), \text{ for } k = 0, 1, \ldots$$

Similarly, for a system of three equations, fixed point iteration is

$$p_{k+1} = g_1(p_k, q_k, r_k),$$
$$q_{k+1} = g_2(p_k, q_k, r_k), \text{ and}$$
$$r_{k+1} = g_3(p_k, q_k, r_k), \text{ for } k = 0, 1, \ldots$$

Theorem (Fixed-Point Iteration). Assume that all the functions and their first partial derivatives are continuous on a region that contains the fixed point (p, q) or (p, q, r), respectively. If the starting point is chosen sufficiently close to the fixed point, then one of the following cases apply.

Case (i) Two dimensions. If (p_0, q_0) is sufficiently close to (p, q) and if $f_1(x, y)$

$$\left| \frac{\partial g_1}{\partial x}(p, q) \right| + \left| \frac{\partial g_1}{\partial y}(p, q) \right| < 1,$$

$$\left| \frac{\partial g_2}{\partial x}(p, q) \right| + \left| \frac{\partial g_2}{\partial y}(p, q) \right| < 1,$$

then fixed point iteration will converge to the fixed point (p, q).

Case (ii) Three dimensions. If (p_0, q_0, r_0) is sufficiently close to (p, q, r) and if

$$\left| \frac{\partial g_1}{\partial x}(p, q, r) \right| + \left| \frac{\partial g_1}{\partial y}(p, q, r) \right| + \left| \frac{\partial g_1}{\partial y}(p, q, r) \right| < 1,$$

$$\left| \frac{\partial g_2}{\partial x}(p, q, r) \right| + \left| \frac{\partial g_2}{\partial y}(p, q, r) \right| + \left| \frac{\partial g_2}{\partial y}(p, q, r) \right| < 1,$$

$$\left| \frac{\partial g_3}{\partial x}(p, q, r) \right| + \left| \frac{\partial g_3}{\partial y}(p, q, r) \right| + \left| \frac{\partial g_3}{\partial y}(p, q, r) \right| < 1,$$

then fixed point iteration will converge to the fixed point (p, q, r).

If these conditions are not met, the iteration might diverge, which is usually the case.

Algorithm (Fixed Point Iteration for Non-Linear Systems) In two dimensions, solve the non-linear fixed point system

$$x = g_1(x, y)$$
$$y = g_2(x, y)$$

given one initial approximation $\vec{P}_0 = (p_0, q_0)$, and generating a sequence $\{\vec{P}_k\} = \{(p_k, q_k)\}$ which converges to the solution $\vec{P} = (p, q)$, i.e.

$$p = g_1(p, q)$$

$$q = g_2(p, q)$$

Algorithm (Fixed Point Iteration for Non-Linear Systems) In three dimensions, solve the non-linear fixed point system

$$x = g_1(x, y, z)$$

$$y = g_2(x, y, z)$$

$$z = g_3(x, y, z)$$

given one initial approximation $\vec{P}_0 = (p_0, q_0, r_0)$, and generating a sequence $\{\vec{P}_k\} = \{(p_k, q_k, r_k)\}$ which converges to the solution $\vec{P} = (p, q, r)$, i.e.

$$p = g_1(p, q, r)$$

$$q = g_2(p, q, r)$$

$$r = g_3(p, q, r)$$

Algorithm (Fixed Point Iteration for Non-Linear Systems) Solve the non-linear fixed point system

$$\vec{X} = \vec{G}(\vec{X}),$$

given one initial approximation $\vec{P}_0$, and generating a sequence $\{\vec{P}_k\}$ which converges to the solution $\vec{P}$,

$$\vec{G}(\vec{P}) = \vec{P}.$$

Example 1. Use fixed point iteration to find a solution to the nonlinear system

$$x = g_1(x, y) = \frac{2x^2 - 2y^3 + 1}{4}$$

$$y = g_2(x, y) = \frac{-x^4 - 4y^4 + 8y + 4}{12}$$

Solution 1.

Newton's Method for Nonlinear Systems

We now outline the derivation of Newton's method in two dimensions. Newton's method can easily be extended to higher dimensions. Consider the system

$$\begin{aligned} u &= f_1(x, y) \\ v &= f_2(x, y) \end{aligned} \tag{1}$$

which can be considered a transformation from the xy-plane to the uv-plane. We are interested in the behavior of this transformation near the point (x_0, y_0) whose image is the point (u_0, v_0). If the two functions have continuous partial derivatives, then the differential can be used to write a system of linear approximations that is valid near the point (x_0, y_0):

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

then substitute the changes $\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta u = u - u_0$, $\Delta v = v - v_0$ for dx, dy, du, dv, respectively. Then we will have

$$\begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \quad \text{or}$$

(2)

$$\Delta \vec{\mathbf{F}} = \mathbf{J}(x_0, y_0)\, \Delta \vec{\mathbf{X}}.$$

Consider the system (1) with u and v set equal to zero,

(3)

$$\begin{aligned} 0 &= f_1(x, y) \\ 0 &= f_2(x, y) \end{aligned}$$

Suppose we are trying to find the solution (p, q) and we start iteration at the nearby point (p_0, q_0), then we can apply (2) and write

$$\begin{pmatrix} f_1(p, q) - f_1(p_0, q_0) \\ f_2(p, q) - f_2(p_0, q_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix}.$$

Since $f_1(p, q) = 0$, and $f_2(p, q) = 0$, this becomes

$$-\begin{pmatrix} f_1(p_0, q_0) \\ f_2(p_0, q_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix},$$

or

$$-\vec{\mathbf{F}}(p_0, q_0) = \mathbf{J}(p_0, q_0)\, \Delta \vec{\mathbf{P}}.$$

When we solve this latter equation for $\Delta \vec{\mathbf{P}}$ we get $\Delta \vec{\mathbf{P}} = -(\mathbf{J}(p_0, q_0))^{-1} \vec{\mathbf{F}}(p_0, q_0)$ and the next approximation $\vec{\mathbf{P}}_1 = \vec{\mathbf{P}}_0 + \Delta \vec{\mathbf{P}}$ is

$$\vec{P}_1 = \vec{P}_0 - \left(J(\vec{P}_0)\right)^{-1} \vec{F}(\vec{P}_0)$$

Theorem (Newton-Raphson Method for 2-dimensional Systems). To solve the non-linear system

$$\vec{F}(\vec{X}) = \vec{0},$$

given one initial approximation $\vec{P}_0$, and generating a sequence $\{\vec{P}_k\}$ which converges to the solution $\vec{P}$, i.e.

$$\vec{F}(\vec{P}) = \vec{0}.$$

Suppose that $\vec{P}_k$ has been obtained, use the following steps to obtain $\vec{P}_{k+1}$.

Step 1. Evaluate the function $\vec{F}(\vec{P}_k) = \begin{pmatrix} f_1(p_k, q_k) \\ f_2(p_k, q_k) \end{pmatrix}$.

Step 2. Evaluate the Jacobian $J(\vec{P}_k) = \begin{pmatrix} \frac{\partial}{\partial x} f_1(p_k, q_k) & \frac{\partial}{\partial y} f_1(p_k, q_k) \\ \frac{\partial}{\partial x} f_2(p_k, q_k) & \frac{\partial}{\partial y} f_2(p_k, q_k) \end{pmatrix}$.

Step 3. Solve the linear system $J(\vec{P}_k)\,\Delta\vec{P} = -\vec{F}(\vec{P}_k)$ for $\Delta\vec{P}$.

Step 4. Compute the next approximation $\vec{P}_{k+1} = \vec{P}_k + \Delta\vec{P}$.

Example 2. Use Newton's method to solve the nonlinear system

$$0 = f_1(x, y) = 1 - 4x + 2x^2 - 2y^3$$
$$0 = f_2(x, y) = -4 + x^4 + 4y + 4y^4$$

Solution 2.

Exercise 3. Observe that the subroutine **NewtonSystem** involves vector functions and is not dependent on the dimension.
Use the subroutine NewtonSystem to solve the nonlinear system in 3D space:

$$0 = 9x^2 + 36y^2 + 4z^2 - 36$$
$$0 = x^2 - 2y^2 - 20z$$
$$0 = x^2 - y^2 + z^2$$

Hint. There are four solutions. Good starting vectors are $\mathbf{P0} = \{\pm 1, \pm 1, 0\}$.
Solution 3.

Example 1. Use fixed point iteration to find a solution to the nonlinear system

$$x = g_1(x, y) = \frac{2x^2 - 2y^3 + 1}{4}$$

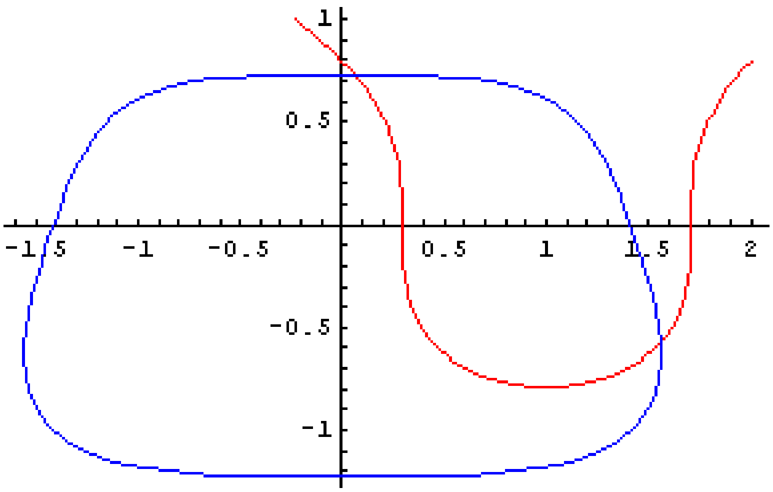
$$y = g_2(x, y) = \frac{-x^4 - 4y^4 + 8y + 4}{12}$$

Solution 1.

First, enter the coordinate functions $g_1(x, y)$ and $g_2(x, y)$, and vector function $\mathbf{G}$.

$$\vec{G}[\{x,y\}] = \left\{\frac{1}{4}\left(1 + 2x^2 - 2y^3\right), \frac{1}{12}\left(4 - x^4 + 8y - 4y^4\right)\right\}$$

Second, graph the curves $g_1(x, y) = x$ and $g_2(x, y) = y$.
The points of intersection are the solutions we seek.



How many points of intersection are there ?
Use fixed point iteration to find a numerical approximation to the solution near $(0.1, 0.7)$.

$$\vec{\mathbf{P}}_0 = \{0.1, 0.7\}$$

$$\vec{\mathbf{P}}_1 = \{0.0835, 0.719958\}$$

$$\vec{\mathbf{P}}_2 = \{0.0668945, 0.723743\}$$

$$\vec{\mathbf{P}}_3 = \{0.0626879, 0.72437\}$$

$$\vec{\mathbf{P}}_4 = \{0.061922, 0.724471\}$$

$$\vec{\mathbf{P}}_5 = \{0.0617947, 0.724487\}$$

$\vec{\mathbf{P}}_6 = \{0.0617741, 0.72449\}$

$\vec{\mathbf{P}}_7 = \{0.0617708, 0.72449\}$

$\vec{\mathbf{P}}_8 = \{0.0617702, 0.724491\}$

$\vec{\mathbf{P}}_9 = \{0.0617701, 0.724491\}$

$\vec{\mathbf{P}}_{10} = \{0.0617701, 0.724491\}$

$\vec{\mathbf{P}}_{11} = \{0.0617701, 0.724491\}$

A fixed point satisfies the equation $\vec{\mathbf{P}} = \vec{\mathbf{G}}(\vec{\mathbf{P}})$. Our last approximation is stored in $\vec{\mathbf{P}}$, check it out.

$\vec{P} = \{0.0617701, 0.724491\}$

$\vec{G}[\vec{P}] = \{0.0617701, 0.724491\}$

Do you think that iteration produced the solution ? Why ?

Accuracy is determined by the tolerance and number of iterations. How accurate was the solution "really"?

$\vec{P} = \{0.0617701267578, 0.724490515295\}$

$\vec{G}[\vec{P}] = \{0.0617701264058, 0.724490515339\}$

Since our tolerance was only 10^{-8}, the accuracy is what we should expect.

Use fixed point iteration to attempt finding a numerical approximation to the solution near $(1.5, -0.5)$.
But this time attempt only 10 iterations !

$\vec{\mathbf{P}}_0 = \{1.5, -0.5\}$

$\vec{\mathbf{P}}_1 = \{1.4375, -0.442708\}$

$\vec{\mathbf{P}}_2 = \{1.32659, -0.330446\}$

$\vec{\mathbf{P}}_3 = \{1.14796, -0.149022\}$

$\vec{\mathbf{P}}_4$ = {0.910558, 0.089103}

$\vec{\mathbf{P}}_5$ = {0.664204, 0.335428}

$\vec{\mathbf{P}}_6$ = {0.451714, 0.536514}

$\vec{\mathbf{P}}_7$ = {0.274806, 0.659921}

$\vec{\mathbf{P}}_8$ = {0.144063, 0.709587}

$\vec{\mathbf{P}}_9$ = {0.081734, 0.721847}

$\vec{\mathbf{P}}_{10}$ = {0.0652766, 0.724059}

$\vec{\mathbf{P}}_{11}$ = {0.0623327, 0.724421}

A fixed point satisfies the equation $\vec{\mathbf{P}} = \vec{\mathbf{G}}(\vec{\mathbf{P}})$. Our last approximation is stored in $\vec{\mathbf{P}}$, check it out.

$\vec{P}$ = {0.0623327, 0.724421}

$\vec{G}[\vec{P}]$ = {0.0618597, 0.724479}

Did iteration find the desired solution near $(1.5, -0.5)$? Why ?

Use *Mathematica* to find that solution near $(1.5, -0.5)$.

The system to solve is

$$\left\{\frac{1}{4}\left(1+2x^2-2y^3\right)==x,\ \frac{1}{12}\left(4-x^4+8y-4y^4\right)==y\right\}$$

The solution is

$x \to -3.51817 + 2.19862\,i$	$y \to -0.627399 + 2.85361\,i$
$x \to -3.51817 - 2.19862\,i$	$y \to -0.627399 - 2.85361\,i$
$x \to 0.287034 - 2.98083\,i$	$y \to 1.33491 + 1.67649\,i$
$x \to 0.287034 + 2.98083\,i$	$y \to 1.33491 - 1.67649\,i$
$x \to 0.705424 - 1.33921\,i$	$y \to -1.31959 + 0.151703\,i$
$x \to 0.705424 + 1.33921\,i$	$y \to -1.31959 - 0.151703\,i$
$x \to 0.377463 + 0.754727\,i$	$y \to 0.218369 + 1.02812\,i$
$x \to 0.377463 - 0.754727\,i$	$y \to 0.218369 - 1.02812\,i$
$x \to 1.33932 - 0.155119\,i$	$y \to 0.319288 + 0.678892\,i$
$x \to 1.33932 + 0.155119\,i$	$y \to 0.319288 - 0.678892\,i$
$x \to 0.0617701$	$y \to 0.724491$
$x \to 1.5561$	$y \to -0.575651$

Example 2. Use Newton's method to solve the nonlinear system

$$0 = f_1(x, y) = 1 - 4x + 2x^2 - 2y^3$$
$$0 = f_2(x, y) = -4 + x^4 + 4y + 4y^4$$

Solution 2.

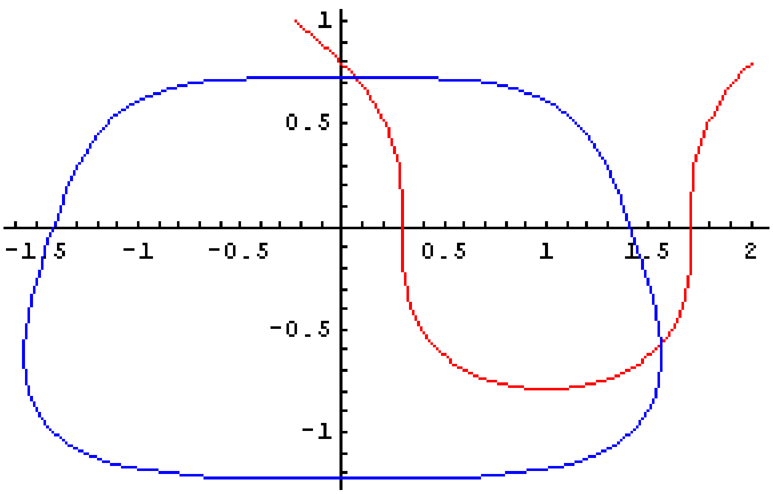
First, enter the coordinate functions $f_1[\{x, y\}]$ and $f_2[\{x, y\}]$ and construct the vector function $\vec{F}(\vec{X})$, and then find the Jacobian matrix $J(x, y)$.

$$0 = f_1(x,y) = 1 - 4x + 2x^2 - 2y^3$$

$$0 = f_2(x,y) = -4 + x^4 + 4y + 4y^4$$

$$J(x,y) = \begin{pmatrix} -4 + 4x & -6y^2 \\ 4x^3 & 4 + 16y^3 \end{pmatrix}$$

Second, graph the curves $f_1(x, y) = 0$ and $f_2(x, y) = 0$. The points of intersection are the solutions we seek.



Use the Newton-Raphson method to find a numerical approximation to the solution near $(0.1, 0.7)$.

$$\vec{P}_0 = \{0.1, 0.7\}$$

$$\vec{P}_1 = \{0.0610386, 0.725259\}$$

$$\vec{P}_2 = \{0.0617699, 0.724491\}$$

$$\vec{P}_3 = \{0.0617701, 0.724491\}$$

$$\vec{P}_4 = \{0.0617701, 0.724491\}$$

A solution to the system satisfies $\vec{\mathbf{F}}\,(\vec{\mathbf{P}}) = \vec{\mathbf{0}}$. Our last approximation is stored in $\vec{\mathbf{P}}$, check it out.

$\vec{P}$ = {0.0617701, 0.724491}

$\vec{F}[\vec{P}]$ = {0., 0.}

Accuracy is determined by the tolerance and number of iterations. How accurate was the solution "really"?

$\vec{P}$ = {0.06177012633860736, 0.7244905153472167}

$\vec{F}[\vec{P}]$ = {0., 0.}

Do you think that iteration produced the solution ? Why ?

Compare with *Mathematica*'s built in routine.

$\{x \to 0.0617701,\ y \to 0.724491\}$

$\vec{P}$ = {0.0617701, 0.724491}

$\vec{F}[\vec{P}] = \{-2.27085 \times 10^{-12},\ 6.86651 \times 10^{-12}\}$

Exercise 3. Observe that the subroutine **NewtonSystem** involves vector functions and is not dependent on the dimension.
Use the subroutine NewtonSystem to solve the nonlinear system in 3D space:

$$0 = 9x^2 + 36y^2 + 4z^2 - 36$$
$$0 = x^2 - 2y^2 - 20z$$
$$0 = x^2 - y^2 + z^2$$

Hint. There are four solutions. Good starting vectors are $\mathbf{P0} = \{\pm 1, \pm 1, 0\}$.
Solution 3.

First, enter the coordinate functions $f_1[\{x, y, z\}]$, $f_2[\{x, y, z\}]$ and $f_2[\{x, y, z\}]$ and construct the vector function $\vec{F}(\vec{X})$, and then find the Jacobian matrix $\mathbf{J}(x, y, z)$.

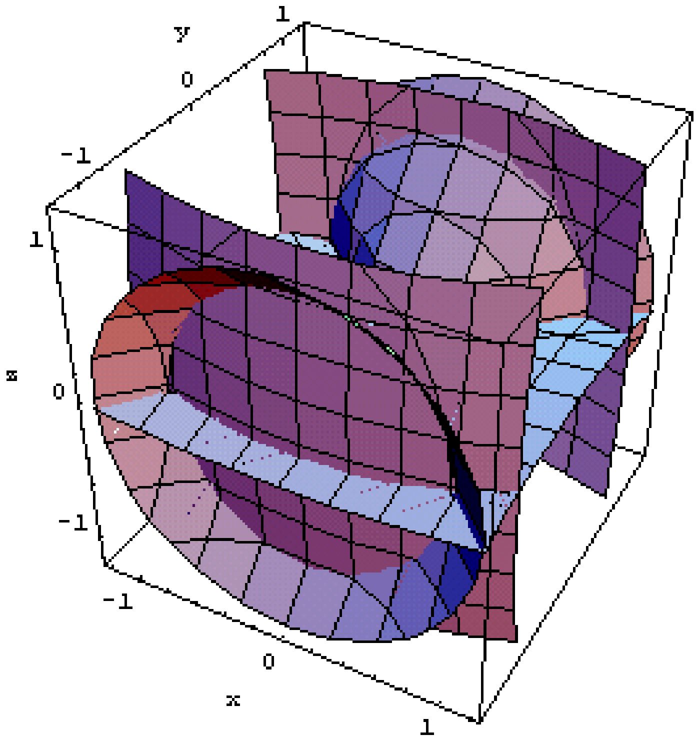
$$0 = f_1(x,y,z) = -36 + 9x^2 + 36y^2 + 4z^2$$

$$0 = f_2(x,y,z) = x^2 - 2y^2 - 20z$$

$$0 = f_3(x,y,z) = x^2 - y^2 + z^2$$

$$J(x,y,z) = \begin{pmatrix} 18x & 72y & 8z \\ 2x & -4y & -20 \\ 2x & -2y & 2z \end{pmatrix}$$

Second, graph the curves $f_1(x, y) = 0$ and $f_2(x, y) = 0$. The points of intersection are the solutions we seek.



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$$0 = f_3(x,y,z) = x^2 - y^2 + z^2$$

Use the Newton-Raphson method to find a numerical approximation to the solution near $(1.0, 1.0, 0.0)$.

$$\vec{\mathbf{P}}_0 = \{1., 1., 0.\}$$

$$\vec{\mathbf{P}}_1 = \{0.9, 0.9, -0.04\}$$

$$\vec{\mathbf{P}}_2 = \{0.893651, 0.894544, -0.0400893\}$$

$$\vec{\mathbf{P}}_3 = \{0.893628, 0.894527, -0.0400893\}$$

$$\vec{\mathbf{P}}_4 = \{0.893628, 0.894527, -0.0400893\}$$

A solution to the system satisfies $\vec{\mathbf{F}}(\vec{\mathbf{P}}) = \vec{\mathbf{0}}$. Our last approximation is stored in $\vec{\mathbf{P}}$, check it out.

$$\vec{P} = \{0.893628, 0.894527, -0.0400893\}$$

$$\vec{F}[\vec{P}] = \{-5.10269 \times 10^{-15}, 4.44089 \times 10^{-16}, 2.7864 \times 10^{-16}\}$$

Use the Newton-Raphson method to find a numerical approximation to the solution near $(1.0, -1.0, 0.0)$.

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Use the Newton-Raphson method to find a numerical approximation to the solution near $(-1.0, 1.0, 0.0)$.

$\vec{P}_0 = \{-1., 1., 0.\}$

$\vec{P}_1 = \{-0.9, 0.9, -0.04\}$

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A solution to the system satisfies $\vec{F}(\vec{P}) = \vec{0}$. Our last approximation is stored in $\vec{P}$, check it out.

$\vec{P} = \{-0.893628, 0.894527, -0.0400893\}$

$\vec{F}[\vec{P}] = \{-5.10269\times10^{-15}, 4.44089\times10^{-16}, 2.7864\times10^{-16}\}$

Use the Newton-Raphson method to find a numerical approximation to the solution near $(-1.0, -1.0, 0.0)$.

$\vec{P}_0 = \{-1., -1., 0.\}$

$\vec{P}_1 = \{-0.9, -0.9, -0.04\}$

$\vec{P}_2 = \{-0.893651, -0.894544, -0.0400893\}$

$\vec{P}_3 = \{-0.893628, -0.894527, -0.0400893\}$

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Aside. We can have *Mathematica* solve the system analytically. There is a surprise.

Solve the system of equations

$$-36 + 9x^2 + 36y^2 + 4z^2 == 0$$
$$x^2 - 2y^2 - 20z == 0$$
$$x^2 - y^2 + z^2 == 0$$

Get

$z \to \frac{3}{5}\left(-15 - \sqrt{223}\right)$ $\quad x \to -\frac{2}{5}\, i\sqrt{3\left(297 + 20\sqrt{223}\right)}$ $\quad y \to -\sqrt{\frac{468}{25} + \frac{6\sqrt{223}}{5}}$

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$z \to \frac{3}{5}\left(-15 + \sqrt{223}\right)$ $\quad x \to -\frac{2}{5}\sqrt{3\left(-297 + 20\sqrt{223}\right)}$ $\quad y \to -\sqrt{\frac{468}{25} - \frac{6\sqrt{223}}{5}}$

$z \to \frac{3}{5}\left(-15 + \sqrt{223}\right)$ $\quad x \to -\frac{2}{5}\sqrt{3\left(-297 + 20\sqrt{223}\right)}$ $\quad y \to \sqrt{\frac{468}{25} - \frac{6\sqrt{223}}{5}}$

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$z \to -17.9599$	$x \to 0. - 16.9091\, i$	$y \to -6.05308$
$z \to -17.9599$	$x \to 0. - 16.9091\, i$	$y \to 6.05308$
$z \to -17.9599$	$x \to 0. + 16.9091\, i$	$y \to -6.05308$
$z \to -17.9599$	$x \to 0. + 16.9091\, i$	$y \to 6.05308$
$z \to -0.0400893$	$x \to -0.893628$	$y \to -0.894527$
$z \to -0.0400893$	$x \to -0.893628$	$y \to 0.894527$
$z \to -0.0400893$	$x \to 0.893628$	$y \to -0.894527$
$z \to -0.0400893$	$x \to 0.893628$	$y \to 0.894527$

Since *Mathematica* performs its solution using complex number arithmetic, the first four solutions are extraneous.
The solutions that we seek are the latter four solutions where x, y, and z are real numbers.