7. Module for Newton-Cotes Integration

Introduction to Quadrature

We now approach the subject of numerical integration. The goal is to approximate the definite integral of $f(x)$ over the interval $[a,b]$ by evaluating $f(x)$ at a finite number of sample points.

Definition (Quadrature Formula) Suppose that $a = x_0 < x_1 < \ldots < x_m = b$. A formula of the form

(1) $$Q[f] = \sum_{k=0}^{m} w_k f(x_k) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + \ldots + w_m f(x_m)$$

with the property that

(2) $$\int_a^b f(x)\,dx = Q[f] + E[f]$$

is called a numerical integration or quadrature formula. The term $E[f]$ is called the truncation error for integration. The values $\{x_k\}_{k=0}^{m}$ are called the quadrature nodes and $\{w_k\}_{k=0}^{m}$ are called the weights.

Depending on the application, the nodes $\{x_k\}_{k=0}^{m}$ are chosen in various ways. For the Trapezoidal Rule, Simpson's Rule, and Boole's Rule, the nodes are chosen to be equally spaced. For Gauss-Legendre quadrature, the nodes are chosen to be zeros of certain Legendre polynomials. When the integration formula is used to develop a predictor formula for differential equations, all the nodes are chosen less than b. For all applications, it is necessary to know something about the accuracy of the numerical solution. This leads us to the next definition.

Definition (Degree of Precision) The degree of precision of a quadrature formula is the positive integer n such that $E[P_i] = 0$ for all polynomials $P_i(x)$ of degree $i \le n$, but for which $E[P_{i+1}] \ne 0$ for some polynomial $P_{i+1}(x)$ of degree $n+1$. That is

$$\int_a^b P_i(x)\,dx = Q[P_i] \quad \text{when degree } i \le n,$$

and

$$\int_a^b P_{i+1}(x)\,dx \ne Q[P_{i+1}] \quad \text{when degree } i = n+1.$$

The form of $E[P_i]$ can be anticipated by studying what happens when $f(x)$ is a polynomial. Consider the arbitrary polynomial

$$P_i(x) = a_i x^i + a_{i-1} x^{i-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

of degree i. If $i \le n$, then $P_i^{(n+1)}(x) \equiv 0$ for all x, and $P_{n+1}^{(n+1)}(x) \equiv (n+1)!\,a_{n+1}$ for all x. Thus it is not surprising that the general form for the truncation error term is

(3) $$E(f) = K f^{(n+1)}(c),$$

where K is a suitably chosen constant and n is the degree of precision. The proof of this general result can be found in advanced books on numerical integration. The derivation of quadrature formulas is sometimes based on polynomial interpolation. Recall that there exists a unique polynomial $P_m(x)$ of degree $\le m$, passing through the $m+1$ equally spaced points $\{(x_k, f(x_k))\}_{k=0}^{m}$. When this polynomial is used to approximate $f(x)$ over $[a,b]$, and then the integral of $f(x)$ is approximated by the integral of $P_m(x)$, the resulting formula is called a Newton-Cotes quadrature formula. When the sample points $x_0 = a$ and $x_m = b$ are used, it is called a closed Newton-Cotes formula. The next result gives the formulas when approximating polynomials of degree $m = 1, 2, 3, 4$ are used.

Theorem (Closed Newton-Cotes Quadrature Formula) Assume that $x_k = x_0 + kh$ are equally spaced nodes and $f_k = f(x_k)$. The first four closed Newton-Cotes quadrature formulas:

(4) Trapezoidal Rule $$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{h}{2}(f(x_0) + f(x_1))$$

(5) Simpson's Rule $$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$$

(6) Simpson 3/8 Rule $$\int_{x_0}^{x_3} f(x)\,dx \approx \frac{3h}{8}(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3))$$

(7) Boole's Rule $$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{2h}{45}(7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4))$$

Corollary (Newton-Cotes Precision) Assume that $f(x)$ is sufficiently differentiable; then $E[f]$ for Newton-Cotes quadrature involves an appropriate higher derivative.

(8) The trapezoidal rule has degree of precision $n=1$. If $f \in C^2[a,b]$, then

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(f_0 + f_1) - \frac{1}{12} f^{(2)}(c)\,h^3.$$

(9) Simpson's rule has degree of precision $n=3$. If $f \in C^4[a,b]$, then

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2) - \frac{1}{90} f^{(4)}(c)\,h^5.$$

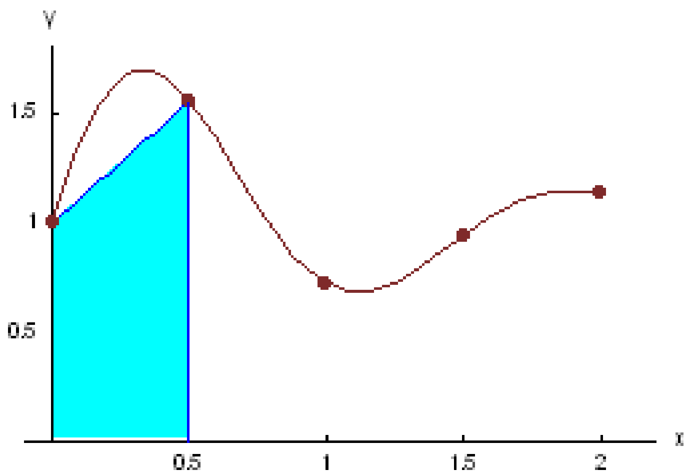
(10) Simpson's $\frac{3}{8}$ rule has degree of precision $n=3$. If $f \in C^4[a, b]$, then

$$\int_{x_0}^{x_3} f(x)\,dx = \frac{3h}{8}(f_0 + 3f_1 + 3f_2 + f_3) - \frac{3}{80} f^{(4)}(c)\,h^5.$$

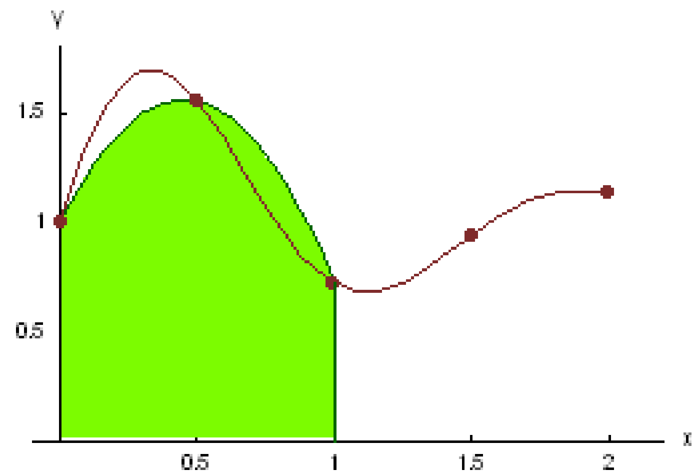
(11) Boole's rule has degree of precision $n=5$. If $f \in C^6[a, b]$, then

$$\int_{x_0}^{x_4} f(x)\,dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) - \frac{8}{945} f^{(6)}(c)\,h^7.$$

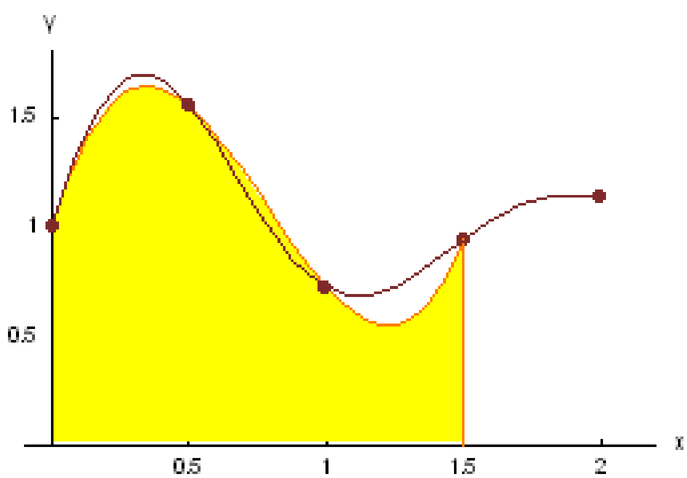
Example 1. Consider the function $f(x) = 1 + e^{-x}\sin(4x)$, the equally spaced quadrature nodes $x_0 = 0.0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, and $x_4 = 2.0$, and the corresponding function values $f_0 = 1.00000$, $f_1 = 1.55152$, $f_2 = 0.72159$, $f_3 = 0.93765$, and $f_4 = 1.13390$. Apply the various quadrature formulas (4) through (7).



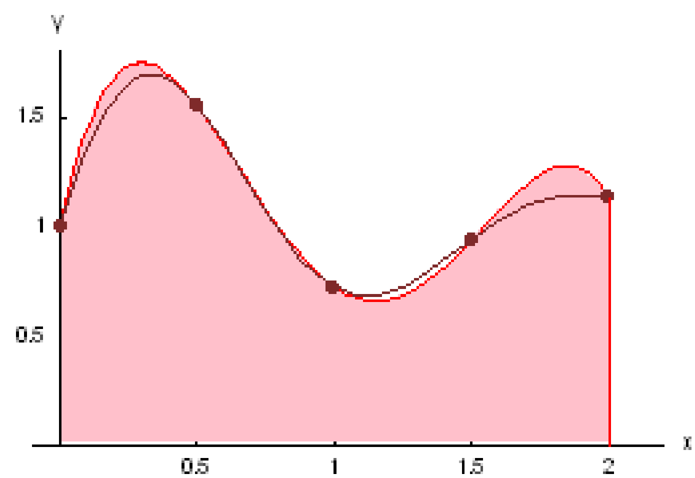
Trapezoidal Rule



Simpson's Rule



Simpson's 3/8 Rule



Boole's Rule

Solution 1.

When the trapezoidal rule is applied on the four subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, and $[x_3, x_4]$ it is called a composite trapezoidal rule:

$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{h}{2}(f_0 + f_1) + \frac{h}{2}(f_1 + f_2) + \frac{h}{2}(f_2 + f_3) + \frac{h}{2}(f_3 + f_4)$$

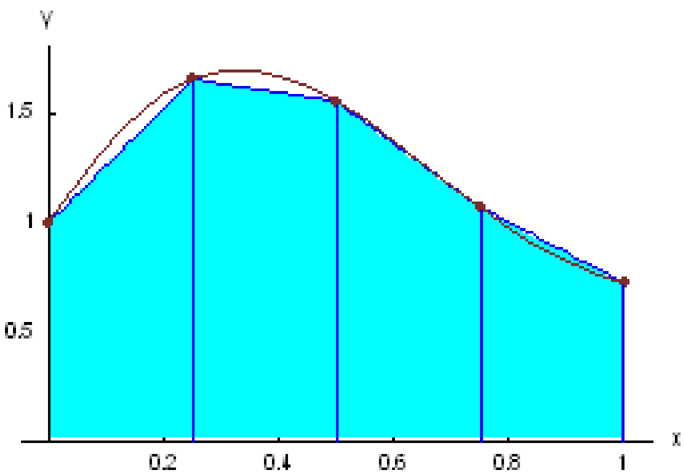
(12) $$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4)$$

When Simpson's rule is applied on the two subintervals $[x_0, x_2]$ and $[x_2, x_4]$ it is called a composite Simpson's rule:

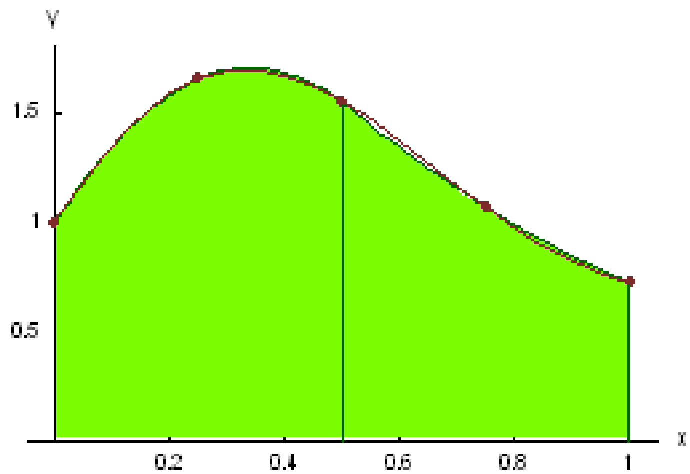
$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2) + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

(13) $$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4)$$

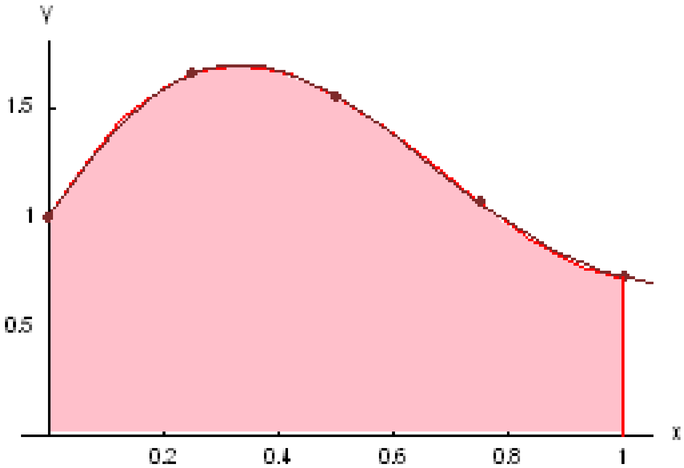
Example 2. Consider the integration of the function $f(x) = 1 + e^{-x}\sin(4x)$ over $[a, b] = [0, 1]$. Use exactly five function evaluations and compare the results from the composite trapezoidal rule, composite Simpson rule, and Boole's rule. Use the uniform step size $h = \frac{1}{4} = 0.25$.



Composite Trapezoidal Rule



Composite Simpson's Rule



Boole's Rule

Solution 2.

Degree of Precision of the Quadrature Rules

We can use formula (3) to determine the degree of precision the trapezoidal rule, composite Simpson rule, and Boole's rule. Assume that

$$E(f) = K f^{(n+1)}(c),$$

where K is a suitably chosen constant and n is the degree of precision. It will suffice to use $f_i(x) = x^i$ and find the largest power n for which the quadrature formula is exact, i. e.

$$E[f] = \int_a^b f_n(x)\,dx - \sum_{k=0}^{m} w_k f_n(x_k) = 0.$$

The constant K is determined by solving

$$E[f] = \int_a^b f_{n+1}(x)\,dx - \sum_{j=0}^{m} w_j f_{n+1}(x_j) = K f_{n+1}^{(n+1)}(x).$$

Since this involves $f_{n+1}(x) = x^{n+1}$, and $f_{n+1}^{(n+1)}(x) = (n+1)!$ is will be easy to solve for K.

Example 3. Show that the degree of precision of the Trapezoidal Rule is $E[f] = -\frac{1}{12} f^{(2)}[c]\,h^2$.

Solution 3.

Example 4. Show that the degree of precision of Simpson's Rule is $E[f] = -\frac{1}{90} f^{(4)}[c]\,h^4$.

Solution 4.

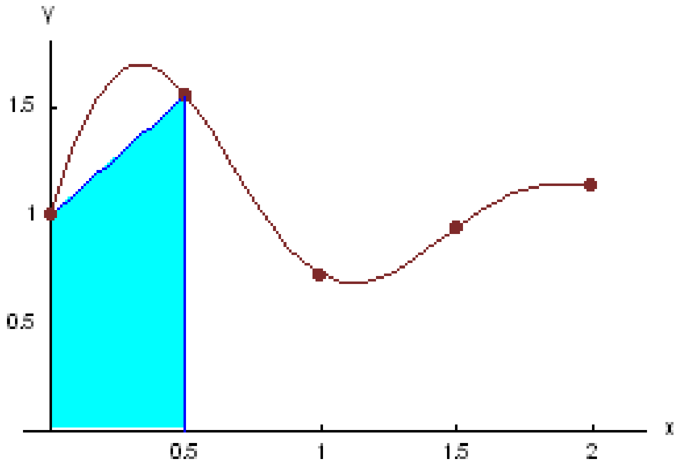
Example 5. Show that the degree of precision of Simpson's $\frac{3}{8}$ Rule is $E[f] = -\frac{3}{80} f^{(4)}[c] h^4$.

Solution 5.

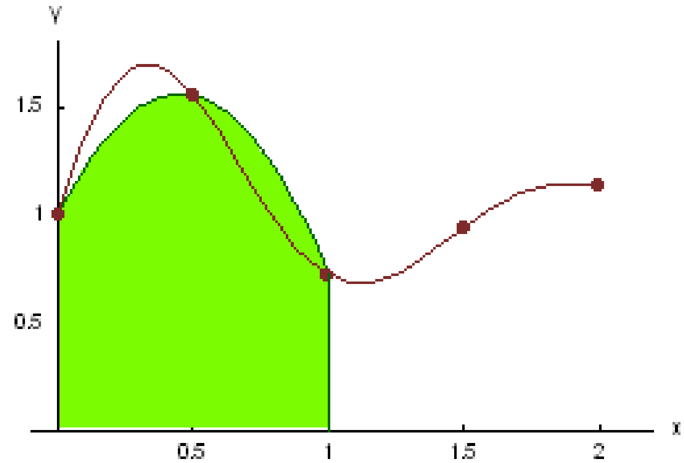
Example 6. Show that the degree of precision of Boole's Rule is $E[f] = -\frac{8}{945} f^{(6)}[c] h^6$.

Solution 6.

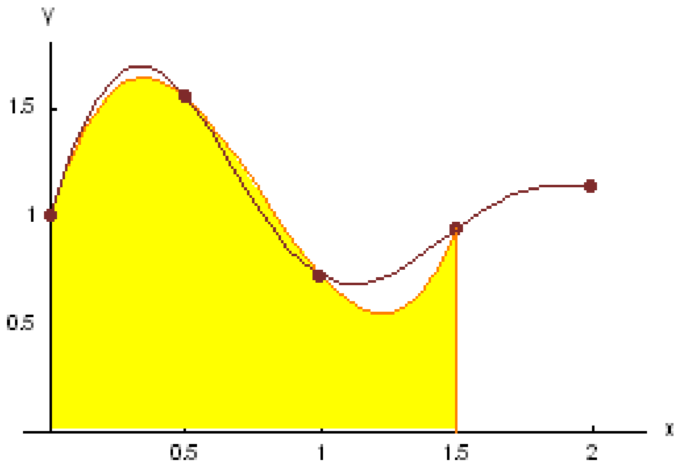
Example 1. Consider the function $f(x) = 1 + e^{-x} \sin(4x)$, the equally spaced quadrature nodes $x_0 = 0.0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, and $x_4 = 2.0$, and the corresponding function values $f_0 = 1.00000$, $f_1 = 1.55152$, $f_2 = 0.72159$, $f_3 = 0.93765$, and $f_4 = 1.13390$. Apply the various quadrature formulas (4) through (7).



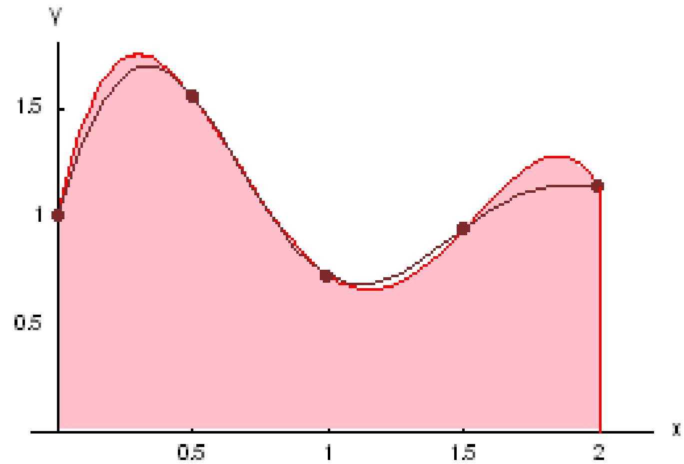
Trapezoidal Rule



Simpson's Rule



Simpson's 3/8 Rule



Boole's Rule

Solution 1.

$$f[x] = 1 + e^{-x} \operatorname{Sin}[4x]$$

Using (8) the Trapezoidal Rule:

$$\int_{0.0}^{0.5} f[x]\,dx \approx \frac{h}{2}(f_0 + f_1)$$

$$= \frac{0.5}{2}(1.00000 + 1.55152)$$

$$= 0.25\,(2.55152)$$

$$= 0.637879$$

Using (9) Simpson's Rule:

$$\int_{0.0}^{1.0} f[x]\,dx \approx \frac{h}{3}(f_0 + 4 f_1 + f_2)$$
$$= \frac{0.5}{3}(1.00000 + 4 * 1.55152 + 0.72159)$$
$$= 0.166667\,(1.00000 + 6.20608 + 0.72159)$$
$$= 0.166667\,(7.92767)$$
$$= 1.32128$$

Using (10) Simpson's $\frac{3}{8}$ Rule:

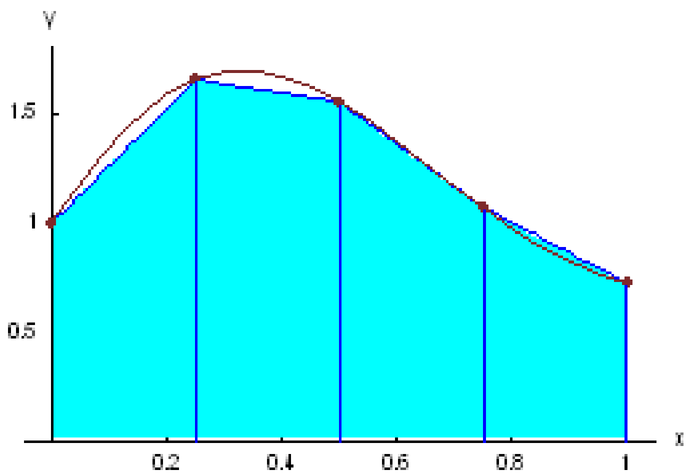
$$\int_{0.0}^{1.5} f[x]\,dx \approx \frac{3h}{8}(f_0 + 3 f_1 + 3 f_2 + f_3)$$
$$= \frac{3 * 0.5}{8}(1.00000 + 3 * 1.55152 + 3 * 0.72159 + 0.93765)$$
$$= 0.1875\,(1.00000 + 4.65456 + 2.16477 + 0.93765)$$
$$= 0.1875\,(8.75698)$$
$$= 1.64193$$

Using (11) Boole's Rule:

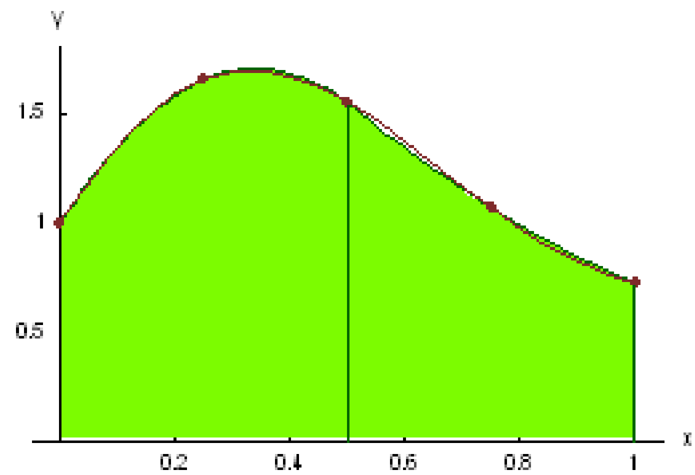
$$\int_{0.0}^{2.0} f[x]\,dx \approx \frac{2h}{45}(7 f_0 + 32 f_1 + 12 f_2 + 32 f_3 + 7 f_4)$$
$$= \frac{2 * 0.5}{45}(7 * 1.00000 + 32 * 1.55152 + 12 * 0.72159 + 32 * 0.93765 + 7 * 1.13390)$$
$$= 0.0222222\,(7.00000 + 49.64864 + 8.65908 + 30.0048 + 7.9373)$$
$$= 0.0222222\,(103.24982)$$
$$= 2.29444$$

It is important to realize that the quadrature formulas (4) through (7) applied in the example above give approximations for definite integrals over different intervals. The graph of the curve $y = f(x)$ and the areas under the Lagrange polynomials $y = P_1(x)$, $y = P_2(x)$, $y = P_3(x)$, and $y = P_4(x)$ are shown in the figures for this example.

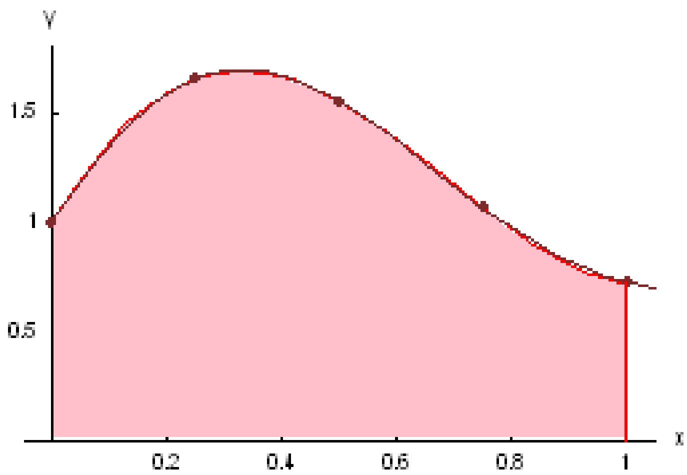
Example 2. Consider the integration of the function $f(x) = 1 + e^{-x}\sin(4x)$ over $[a, b] = [0, 1]$. Use exactly five function evaluations and compare the results from the composite trapezoidal rule, composite Simpson rule, and Boole's rule. Use the uniform step size $h = \frac{1}{4} = 0.25$.



Composite Trapezoidal Rule



Composite Simpson's Rule



Boole's Rule

Solution 3.

$$f[x] = 1 + e^{-x}\,\mathrm{Sin}[4x]$$

Using the Composite Trapezoidal Rule:

$$\begin{aligned}\int_0^1 f[x]\,dx &\approx \frac{h}{2}(f_0 + 2f_1 + 2f_2 + 2f_3 + f_4)\\ &= \frac{0.25}{2}(1.00000 + 2*1.65534 + 2*1.55152 + 2*1.06666 + 0.721588)\\ &= 0.125\,(1.00000 + 3.31068 + 3.10303 + 2.13332 + 0.721588)\end{aligned}$$

$$= 0.125\,(10.2686)$$
$$= 1.28358$$

Using the Composite Simpson's Rule:

$$\int_0^1 f[x]\,dx \approx \frac{h}{3}\,(f_0 + 4 f_1 + 2 f_2 + 4 f_3 + f_4)$$
$$= \frac{0.25}{3}\,(1.00000 + 4*1.65534 + 2*1.55152 + 4*1.06666 + 0.721588)$$
$$= 0.083333\,(1.00000 + 6.62136 + 3.10304 + 4.26664 + 0.721588)$$
$$= 0.083333\,(15.7126)$$
$$= 1.30938$$

Using Boole's Rule:

$$\int_0^1 f[x]\,dx \approx \frac{2h}{45}\,(7 f_0 + 32 f_1 + 12 f_2 + 32 f_3 + 7 f_4)$$
$$= \frac{2*0.25}{45}\,(7*1.00000 + 32*1.65534 + 12*1.55152 + 32*1.06666 + 7*0.721588)$$
$$= 0.0111111\,(7.00000 + 52.9709 + 18.6182 + 34.1331 + 5.05112)$$
$$= 0.0111111\,(117.773)$$
$$= 1.30859$$

The true value of the definite integral is

$$f[x] = 1 + e^{-x}\,\mathrm{Sin}[4x]$$
$$F[x] = \int f[x]dx = x - \frac{4}{17}\,e^{-x}\,\mathrm{Cos}[4x] - \frac{1}{17}\,e^{-x}\,\mathrm{Sin}[4x]$$

$$F[1] = 1 - \frac{4\,\mathrm{Cos}[4]}{17\,e} - \frac{\mathrm{Sin}[4]}{17\,e}$$
$$F[0] = -\frac{4}{17}$$

$$\int_0^1 f[x]dx = \int_0^1 (1 + e^{-x}\,\mathrm{Sin}[4x])dx$$
$$\int_0^1 f[x]dx = F[1] - F[0]$$
$$\int_0^1 f[x]dx = \left(1 - \frac{4\,\mathrm{Cos}[4]}{17\,e} - \frac{\mathrm{Sin}[4]}{17\,e}\right) - \left(-\frac{4}{17}\right)$$
$$\int_0^1 f[x]dx = \frac{21}{17} - \frac{4\,\mathrm{Cos}[4]}{17\,e} - \frac{\mathrm{Sin}[4]}{17\,e}$$

$$\int_0^1 f[x]dx = 1.308250604642669$$

We see that the approximation `1.30938` from Simpson's rule is much better than the value `1.28358` obtained from the trapezoidal rule. Again, the approximation `1.30859` from Boole's rule is closest. Graphs for the areas under the trapezoids and parabolas are shown in the figures for this example.

Example 3. Show that the degree of precision of the Trapezoidal Rule is $E[f] = -\frac{1}{12} f^{(2)}[c]\,h^2$.

Solution 3.

It will suffice to apply Trapezoidal Rule over the interval $[0,1]$ with the three test functions $f[x] = 1, x, x^2$.

$$f[x] = 1$$

$$F[x] = \int 1\, dx = x$$

$$\int_0^1 1\, dx = 1$$

$$\frac{1}{2}(f[0] + f[1]) = 1$$

$$E[1] = 1 - 1 = 0$$

$$f[x] = x$$

$$F[x] = \int x\, dx = \frac{x^2}{2}$$

$$\int_0^1 x\, dx = \frac{1}{2}$$

$$\frac{1}{2}(f[0] + f[1]) = \frac{1}{2}$$

$$E[x] = \frac{1}{2} - \frac{1}{2} = 0$$

$$f[x] = x^2$$

$$F[x] = \int x^2\, dx = \frac{x^3}{3}$$

$$\int_0^1 x^2\, dx = \frac{1}{3}$$

$$\frac{1}{2}(f[0] + f[1]) = \frac{1}{2}$$

$$E[x^2] = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

- - 3 2 6

The function $f[x] = x^2$ is the lowest power of x for which the trapezoidal rule is not exact.

Solve the equation

$$2K == -\frac{1}{6}$$

For K, and get

$$K \to -\frac{1}{12}$$

The degree of precision for
The Trapezoidal Rule is $n = 1$
And the error term has the form

$$E[f] = -\frac{1}{12} f^{(2)}[c] h^2$$

Example 4. Show that the degree of precision of Simpson's Rule is $E[f] = -\frac{1}{90} f^{(4)}[c]\, h^4$.

Solution 4.

It will suffice to apply Simpson's Rule over the interval $[0,2]$ with the five test functions $f[x] = 1, x, x^2, x^3, x^4$.

$$f[x] = 1$$
$$F[x] = \int 1 \, dx = x$$
$$\int_0^2 1 \, dx = 2$$
$$\frac{1}{3}(f[0] + 4f[1] + f[2]) = 2$$
$$E[1] = 2 - 2 = 0$$

$$f[x] = x$$
$$F[x] = \int x \, dx = \frac{x^2}{2}$$
$$\int_0^2 x \, dx = 2$$
$$\frac{1}{3}(f[0] + 4f[1] + f[2]) = 2$$
$$E[x] = 2 - 2 = 0$$

$$f[x] = x^2$$
$$F[x] = \int x^2 \, dx = \frac{x^3}{3}$$
$$\int_0^2 x^2 \, dx = \frac{8}{3}$$
$$\frac{1}{3}(f[0] + 4f[1] + f[2]) = \frac{8}{3}$$
$$E[x^2] = \frac{8}{3} - \frac{8}{3} = 0$$

$f[x] = x^3$

$F[x] = \int x^3\, dx = \dfrac{x^4}{4}$

$\int_0^2 x^3\, dx = 4$

$\dfrac{1}{3}(f[0] + 4f[1] + f[2]) = 4$

$E[x^3] = 4 - 4 = 0$

$f[x] = x^4$

$F[x] = \int x^4\, dx = \dfrac{x^5}{5}$

$\int_0^2 x^4\, dx = \dfrac{32}{5}$

$\dfrac{1}{3}(f[0] + 4f[1] + f[2]) = \dfrac{20}{3}$

$E[x^4] = \dfrac{32}{5} - \dfrac{20}{3} = -\dfrac{4}{15}$

The function $f[x] = x^4$ is the lowest power of x for which the Simpson's rule is not exact.

Solve the equation

$24 K == -\dfrac{4}{15}$

For K, and get

$K \to -\dfrac{1}{90}$

The degree of precision for
Simpson's Rule is $n = 3$
And the error term has the form

$$E[f] = -\frac{1}{90} f^{(4)}[c] h^4$$

Example 5. Show that the degree of precision of Simpson's $\frac{3}{8}$ Rule is $E[f] = -\frac{3}{80} f^{(4)}[c]\, h^4$.

Solution 5.

It will suffice to apply Simpson's $\frac{3}{8}$ Rule over the interval $[0,3]$ with the five test functions $f[x] = 1, x, x^2, x^3, x^4$.

$$f[x] = 1$$

$$F[x] = \int 1\, dx = x$$

$$\int_0^3 1\, dx = 3$$

$$\frac{3}{8}(f[0] + 3f[1] + 3f[2] + f[3]) = 3$$

$$E[1] = 3 - 3 = 0$$

$$f[x] = x$$

$$F[x] = \int x\, dx = \frac{x^2}{2}$$

$$\int_0^3 x\, dx = \frac{9}{2}$$

$$\frac{3}{8}(f[0] + 3f[1] + 3f[2] + f[3]) = \frac{9}{2}$$

$$E[x] = \frac{9}{2} - \frac{9}{2} = 0$$

$$f[x] = x^2$$

$$F[x] = \int x^2\, dx = \frac{x^3}{3}$$

$$\int_0^3 x^2\, dx = 9$$

$$\frac{3}{8}(f[0] + 3f[1] + 3f[2] + f[3]) = 9$$

$$E[x^2] = 9 - 9 = 0$$

$$f[x] = x^3$$

$$F[x] = \int x^3\,dx = \frac{x^4}{4}$$

$$\int_0^3 x^3\,dx = \frac{81}{4}$$

$$\frac{3}{8}(f[0] + 3f[1] + 3f[2] + f[3]) = \frac{81}{4}$$

$$E[x^3] = \frac{81}{4} - \frac{81}{4} = 0$$

$$f[x] = x^4$$

$$F[x] = \int x^4\,dx = \frac{x^5}{5}$$

$$\int_0^3 x^4\,dx = \frac{243}{5}$$

$$\frac{3}{8}(f[0] + 3f[1] + 3f[2] + f[3]) = \frac{99}{2}$$

$$E[x^4] = \frac{243}{5} - \frac{99}{2} = -\frac{9}{10}$$

The function $f[x] = x^4$ is the lowest power of x for which the Simpson's $\frac{3}{8}$ rule is not exact.

Solve the equation

$$24K == -\frac{9}{10}$$

For K, and get

$$K \to -\frac{3}{80}$$

The degree of precision for

Simpson's $\frac{3}{8}$ Rule is $n = 3$

And the error term has the form

$$E[f] = -\frac{3}{80} f^{(4)}[c] h^4$$

Example 6. Show that the degree of precision of Boole's Rule is $E[f] = -\frac{8}{945} f^{(6)}[c]\, h^6$.

Solution 6.

It will suffice to apply Boole's rule over the interval $[0,4]$ with the seven test functions $f[x] = 1, x, x^2, x^3, x^4, x^5, x^6$.

$$f[x] = 1$$

$$F[x] = \int 1\, dx = x$$

$$\int_0^4 1\, dx = 4$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2] + 32f[3] + 7f[4]) = 4$$

$$E[1] = 4 - 4 = 0$$

$$f[x] = x$$

$$F[x] = \int x\, dx = \frac{x^2}{2}$$

$$\int_0^4 x\, dx = 8$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2] + 32f[3] + 7f[4]) = 8$$

$$E[x] = 8 - 8 = 0$$

$$f[x] = x^2$$

$$F[x] = \int x^2\, dx = \frac{x^3}{3}$$

$$\int_0^4 x^2\, dx = \frac{64}{3}$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2] + 32f[3] + 7f[4]) = \frac{64}{3}$$

$$E[x^2] = \frac{64}{3} - \frac{64}{3} = 0$$

$$f[x] = x^3$$

$$F[x] = \int x^3 \, dx = \frac{x^4}{4}$$

$$\int_0^4 x^3 \, dx = 64$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2] + 32f[3] + 7f[4]) = 64$$

$$E[x^3] = 64 - 64 = 0$$

$$f[x] = x^4$$

$$F[x] = \int x^4 \, dx = \frac{x^5}{5}$$

$$\int_0^4 x^4 \, dx = \frac{1024}{5}$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2] + 32f[3] + 7f[4]) = \frac{1024}{5}$$

$$E[x^4] = \frac{1024}{5} - \frac{1024}{5} = 0$$

$$f[x] = x^5$$

$$F[x] = \int x^5 \, dx = \frac{x^6}{6}$$

$$\int_0^4 x^5 \, dx = \frac{2048}{3}$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2] + 32f[3] + 7f[4]) = \frac{2048}{3}$$

$$E[x^5] = \frac{2048}{3} - \frac{2048}{3} = 0$$

$$f[x] = x^6$$

$$F[x] = \int x^6 \, dx = \frac{x^7}{7}$$

$$\int_0^4 x^6 \, dx = \frac{16384}{7}$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2] + 32f[3] + 7f[4]) = \frac{7040}{3}$$

$$\frac{2}{45}(7f[0] + 32f[1] + 12f[2]+ 32f[3]+ 7f[4]) = \frac{7040}{3}$$

$$E[x^6] = \frac{16384}{7} - \frac{7040}{3} = -\frac{128}{21}$$

The function $f[x] = x^6$ is the lowest power of x for which the Boole's rule is not exact.

Solve the equation

$$720\,K == -\frac{128}{21}$$

For K, and get

$$K \to -\frac{8}{945}$$

The degree of precision for

Boole's Rule is $n = 5$

And the error term has the form

$$E[f] = -\frac{8}{945} f^{(6)}[c]h^6$$