2. Gauss-Jordan Elimination

In this module we develop a algorithm for solving a general linear system of equations $AX = B$ consisting of n equations and n unknowns where it is assumed that the system has a unique solution. The method is attributed Johann Carl Friedrich Gauss (1777-1855) and Wilhelm Jordan (1842 to 1899). The following theorem states the sufficient conditions for the existence and uniqueness of solutions of a linear system $AX = B$.

Theorem (Unique Solutions) Assume that A is an $n \times n$ matrix. The following statements are equivalent.

(i) Given any $n \times 1$ matrix B, the linear system $AX = B$ has a unique solution.

(ii) The matrix A is nonsingular (i.e., A^{-1} exists).

(iii) The system of equations $AX = 0$ has the unique solution $X = 0$.

(iv) The determinant of A is nonzero, i.e. $\det(A) \neq 0$.

It is convenient to store all the coefficients of the linear system $AX = B$ in one array of dimension $n \times n + 1$. The coefficients of B are stored in column $n + 1$ of the array (i.e. $a_{i,n+1} = b_i$). Row k contains all the coefficients necessary to represent the i^{th} equation in the linear system. The augmented matrix is denoted $M = [A \mid B]$ and the linear system is represented as follows:

$$M = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} & b_2 \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} & b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n,1} & a_{n,2} & a_{n,3} & \cdots & a_{n,n} & b_n \end{pmatrix}$$

The system $AX = B$, with augmented matrix M, can be solved by performing row operations on M. The variables are placeholders for the coefficients and cam be omitted until the end of the computation.

Theorem (Elementary Row Operations). The following operations applied to the augmented matrix M yield an equivalent linear system.

(i) Interchanges: The order of two rows can be interchanged.

(ii) Scaling: Multiplying a row by a nonzero constant.

(iii) Replacement: Row r can be replaced by the sum of that row and a nonzero multiple of any other row;

that is: $\text{row}_r = \text{row}_r - c\,\text{row}_p$.

It is common practice to implement (iii) by replacing a row with the difference of that row and a multiple of another row.

Definition (Pivot Element). The number $a_{p,p}$ in the coefficient matrix $\mathbf{A}$ that is used to eliminate $a_{i,p}$ where $i = p+1,\ p+2,\ \ldots,\ n$, is called the p^{th} pivot element, and the p^{th} row is called the pivot row.

Theorem (Gaussian Elimination with Back Substitution). Assume that $\mathbf{A}$ is an $n \times n$ nonsingular matrix. There exists a unique system $\mathbf{UX} = \mathbf{Y}$ that is equivalent to the given system $\mathbf{AX} = \mathbf{B}$, where $\mathbf{U}$ is an upper-triangular matrix with $u_{i,i} \neq 0$ for $i = 1,\ 2,\ \ldots,\ n$. After $\mathbf{U}$ and $\mathbf{Y}$ are constructed, back substitution can be used to solve $\mathbf{UX} = \mathbf{Y}$ for $\mathbf{X}$.

Example 1. Use the Gauss-Jordan elimination method to solve the linear system

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & 5 \\ 2 & 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}.$$

Solution 1.

Example 2. Use the Gauss-Jordan subroutine to solve $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ -3 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$

Solution 2.

Example 1. Use the Gauss-Jordan elimination method to solve the linear system

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & 5 \\ 2 & 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}.$$

Solution 1. First form the augmented matrix $\mathbf{M} = [\mathbf{A}, \mathbf{B}]$.

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ -3 & 1 & 5 & -2 \\ 2 & 4 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 7 & 14 & 7 \\ 0 & 0 & -7 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -7 & -7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Example 2. Use the Gauss-Jordan subroutine to solve $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ -3 & 1 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$.

Solution 2.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ -3 & 1 & 5 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$$

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & -1 & -1 \\ -3 & 1 & 5 & -2 \end{pmatrix}$$

Then try to perform Gauss-Jordan elimination.

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 2 & 4 & -1 & -1 \\ -3 & 1 & 5 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 3 \\ 0 & 0 & -7 & -7 \\ 0 & 7 & 14 & 7 \end{pmatrix}$$

.....Indeterminate expression 0 encountered.

Gauss-Jordan elimination does not work without row interchanges.