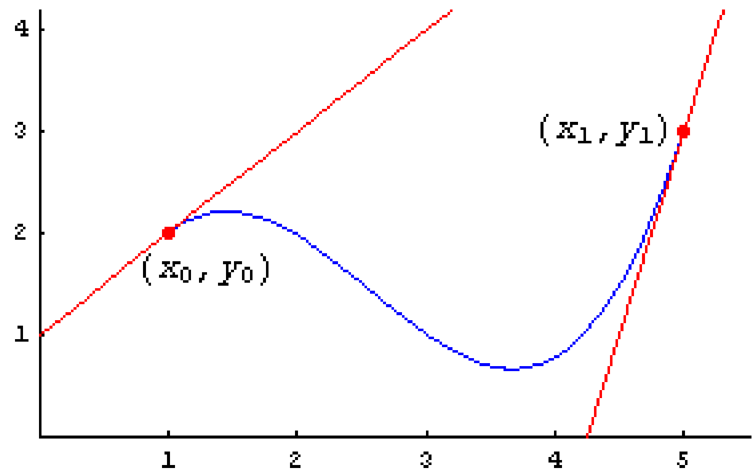
4. Hermite Polynomial Interpolation

Background for the Hermite Polynomial

The cubic Hermite polynomial $p(x)$ has the interpolative properties $p(x_0) = y_0$, $p(x_1) = y_1$, $p'(x_0) = d_0$, and $p'(x_1) = d_1$, both the function values and their derivatives are known at the endpoints of the interval $[x_0, x_1]$. Hermite polynomials were studied by the French Mathematician Charles Hermite (1822-1901), and are referred to as a "clamped cubic," where "clamped" refers to the slope at the endpoints being fixed. This situation is illustrated in the figure below.



Theorem (Cubic Hermite Polynomial). If $f[x]$ is continuous on the interval $[x_0, x_1]$, there exists a unique cubic polynomial $p[x] = a x^3 + b x^2 + c x + d$ such that

$$p[x_0] = f[x_0],$$
$$p[x_1] = f[x_1],$$
$$p'[x_0] = f'[x_0],$$
$$p'[x_2] = f'[x_2].$$

Remark. The cubic Hermite polynomial is a generalization of both the Taylor polynomial and Lagrange polynomial, and it is referred to as an "osculating polynomial." Hermite polynomials can be generalized to higher degrees by requiring that the use of more nodes $\{x_0, x_1, \ldots, x_n\}$ and the extension to agreement at higher derivatives $p^{(k)}[x_i] = f^{(k)}[x_i]$ for $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m_i$. The details are found in advanced texts on numerical analysis.

Example 1. Find the cubic Hermite polynomial or "clamped cubic" that satisfies

$p(1) = 2$

$p'(1) = 1$

$p(3) = 1$

$p'(3) = 2$

Solution 1.

More Background. The Clamped Cubic Spline

A clamped cubic spline is obtained by forming a piecewise cubic function which passes through the given set of knots $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with the condition the function values, their derivatives and second derivatives of adjacent cubics agree at the interior nodes. The endpoint conditions are $S'(x_0) = d_0$ and $S'(x_n) = d_n$, where d_0 and d_n are given.

Example 2. Find the "clamped cubic spline" that satisfies

$S_1(1) = 2$

$S_1'(1) = 1$

$S_1(2) = 1$

$S_1(2) = S_2(2)$

$S_1'(2) = S_2'(2)$

$S_1''(2) = S_2''(2)$

$S_2(3) = 1$

$S_2'(3) = 2$

Solution 2.

More Background. The Natural Cubic Spline

A natural cubic spline is obtained by forming a piecewise cubic function which passes through the given set of knots $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ with the condition the function values, their derivatives and second derivatives of adjacent cubics agree at the interior nodes. The endpoint conditions are $S''(x_0) = 0$ and $S''(x_n) = 0$. The natural cubic spline is said to be "a relaxed curve."

Example 3. Find the "natural cubic spline" that satisfies

$S_1(1) = 2$

$S_1''(1) = 0$

$S_1(2) = 1$

$S_1(2) = S_2(2)$

$S_1'(2) = S_2'(2)$

$S_1''(2) = S_2''(2)$

$S_2(3) = 1$

$S_2''(3) = 0$

Solution 3.

Example 1. Find the cubic Hermite polynomial or "clamped cubic" that satisfies

$$p(1) = 2$$
$$p'(1) = 1$$
$$p(3) = 1$$
$$p'(3) = 2$$

Solution 1.

Enter the formula for a general cubic equation.

$$p[x] = d + c\,x + b\,x^2 + a\,x^3$$

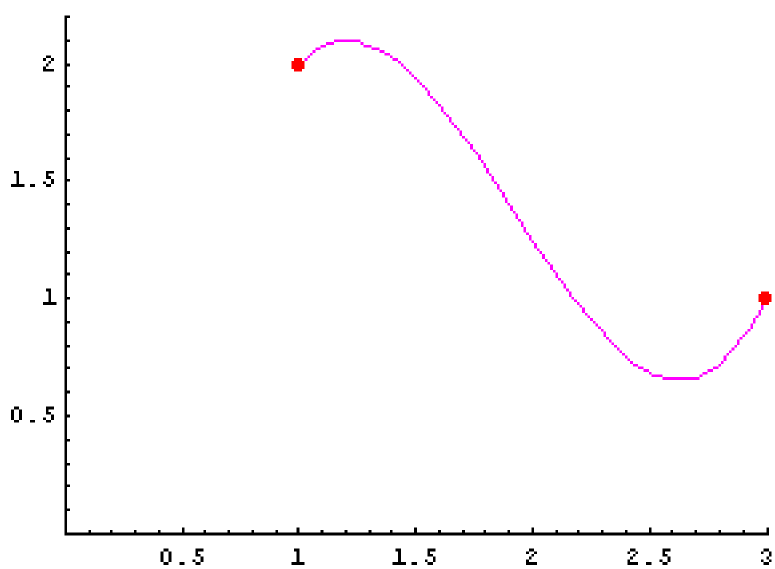
$$p'[x] = c + 2\,b\,x + 3\,a\,x^2$$

Set up four equations using the prescribed endpoint conditions. Then find the solution set to this linear system.

$$a + b + c + d == 2$$
$$3\,a + 2\,b + c == 1$$
$$27\,a + 9\,b + 3\,c + d == 1$$
$$27\,a + 6\,b + c == 2$$

$$\left\{\left\{a \to 1,\ b \to -\frac{23}{4},\ c \to \frac{19}{2},\ d \to -\frac{11}{4}\right\}\right\}$$

Use the solution given above for the coefficients and form the cubic function.



$$y = f[x] = -\frac{11}{4} + \frac{19x}{2} - \frac{23x^2}{4} + x^3$$

$$f[1] = 2, \qquad f[3] = 1$$

$$f'[1] = 1, \qquad f'[3] = 2$$

Example 2. Find the "clamped cubic spline" that satisfies

$$S_1(1) = 2$$
$$S_1'(1) = 1$$
$$S_1(2) = 1$$
$$S_1(2) = S_2(2)$$
$$S_1'(2) = S_2'(2)$$
$$S_1''(2) = S_2''(2)$$
$$S_2(3) = 1$$
$$S_2'(3) = 2$$

Solution 2.

Set up the formulas for the two cubic polynomials and form the equations to solve. Subscripted variables are used if you prefer you can use ordinary variables $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$.

$$S1[x\_] = x^3 a_1 + x^2 b_1 + x c_1 + d_1$$
$$S2[x\_] = x^3 a_2 + x^2 b_2 + x c_2 + d_2$$

Set up eight equations using the prescribed endpoint conditions. Then find the solution set to this linear system.

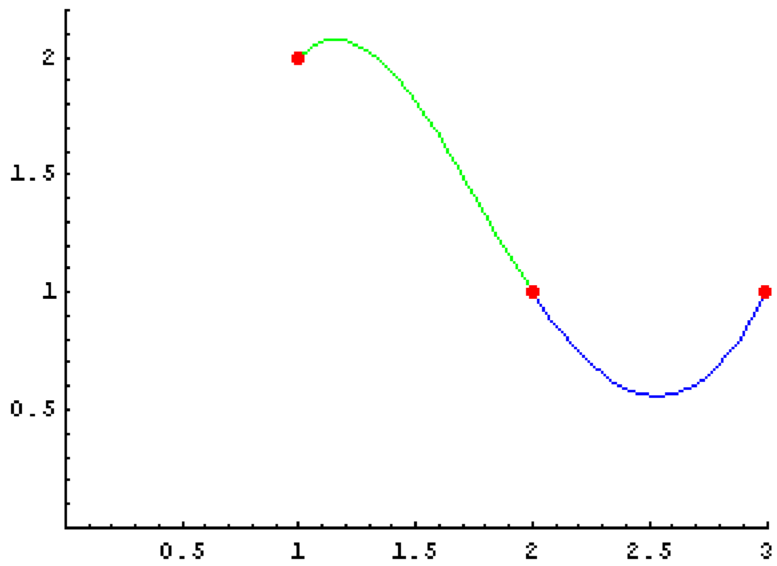
$$a_1 + b_1 + c_1 + d_1 == 2$$
$$3 a_1 + 2 b_1 + c_1 == 1$$
$$8 a_1 + 4 b_1 + 2 c_1 + d_1 == 1$$
$$8 a_1 + 4 b_1 + 2 c_1 + d_1 == 8 a_2 + 4 b_2 + 2 c_2 + d_2$$
$$12 a_1 + 4 b_1 + c_1 == 12 a_2 + 4 b_2 + c_2$$
$$12 a_1 + 2 b_1 == 12 a_2 + 2 b_2$$
$$27 a_2 + 9 b_2 + 3 c_2 + d_2 == 1$$
$$27 a_2 + 6 b_2 + c_2 == 2$$

$$\left\{\left\{a_1 \to \frac{3}{2}, b_1 \to -8, c_1 \to \frac{25}{2}, d_1 \to -4, a_2 \to \frac{1}{2}, b_2 \to -2, c_2 \to \frac{1}{2}, d_2 \to 4\right\}\right\}$$

Use the solution given above for the coefficients and form the cubic functions.

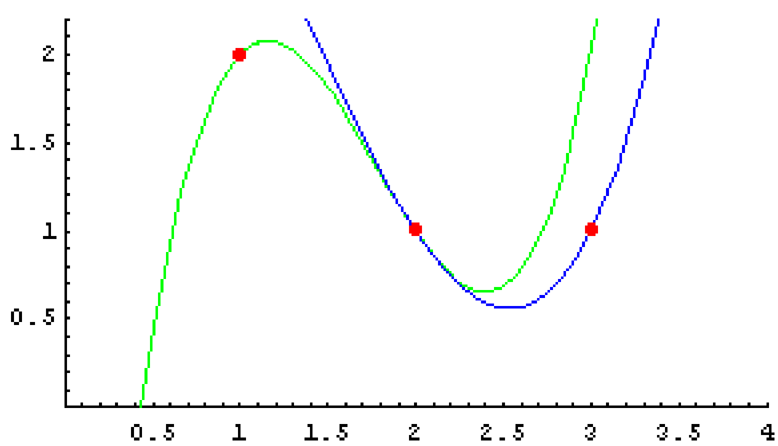
$$f_1[x] = -4 + \frac{25 x}{2} - 8 x^2 + \frac{3 x^3}{2}$$
$$f_2[x] = 4 + \frac{x}{2} - 2 x^2 + \frac{x^3}{2}$$

Now graph the portion of each cubic in the interval over which it is to be used. Then combine the two piecewise cubic graphs to form the spline.



$$f_1[x] = -4 + \frac{25x}{2} - 8x^2 + \frac{3x^3}{2}$$
$$f_2[x] = 4 + \frac{x}{2} - 2x^2 + \frac{x^3}{2}$$

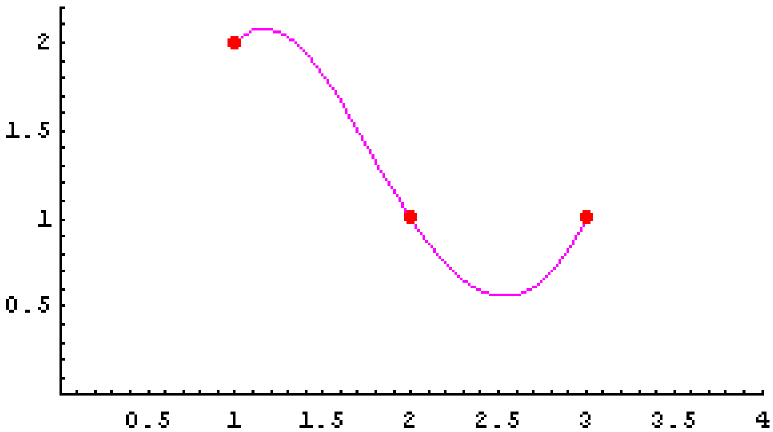
Remark. Note that the individual cubics were graphed over different intervals and then the graphs were combined. If they were plotted together over a common interval it would look different.



$$f_1[x] = -4 + \frac{25x}{2} - 8x^2 + \frac{3x^3}{2}$$
$$f_2[x] = 4 + \frac{x}{2} - 2x^2 + \frac{x^3}{2}$$

Remark. It would be nice to have one piecewise cubic function S[z] that is used. The following formulas for S[x] uses the condition syntax `/;` for making a piecewise function. Under the help

menu we can find the following information about Condition.
patt /; test is a pattern which matches only if the evaluation of test yields True



$$S[x\_/;1\le x\le 2] = -4 + \frac{25\,x}{2} - 8\,x^2 + \frac{3\,x^3}{2}$$

$$S[x\_/;2\le x\le 3] = 4 + \frac{x}{2} - 2\,x^2 + \frac{x^3}{2}$$

The clamped cubic spline forces the slope at the endpoints to be prescribed values.

Example 3. Find the "natural cubic spline" that satisfies

$$S_1(1) = 2$$
$$S_1''(1) = 0$$
$$S_1(2) = 1$$
$$S_1(2) = S_2(2)$$
$$S_1'(2) = S_2'(2)$$
$$S_1''(2) = S_2''(2)$$
$$S_2(3) = 1$$
$$S_2''(3) = 0$$

Solution 3.

Set up the formulas for the two cubic polynomials and form the equations to solve. Subscripted variables are used if you prefer you can use ordinary variables $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2$.

$$S1[x\_] = x^3 a_1 + x^2 b_1 + x c_1 + d_1$$
$$S2[x\_] = x^3 a_2 + x^2 b_2 + x c_2 + d_2$$

Set up eight equations using the prescribed endpoint conditions. Then find the solution set to this linear system.

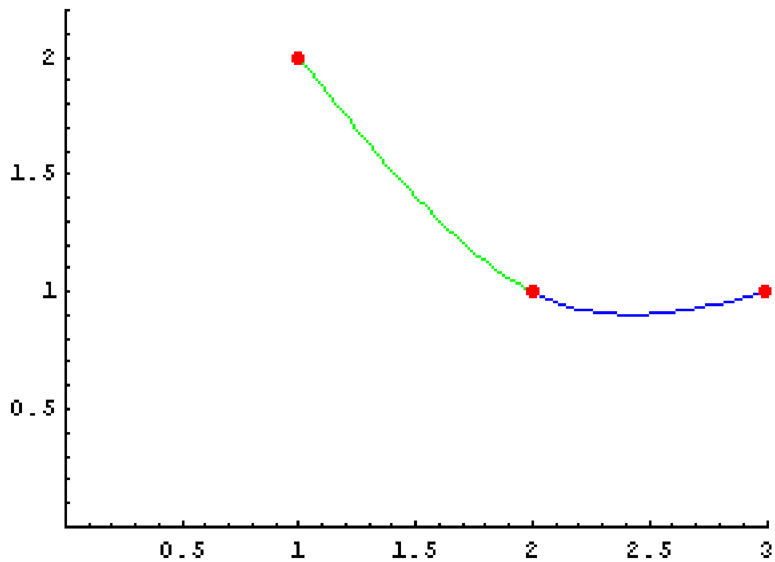
$$a_1 + b_1 + c_1 + d_1 == 2$$
$$6 a_1 + 2 b_1 == 0$$
$$8 a_1 + 4 b_1 + 2 c_1 + d_1 == 1$$
$$8 a_1 + 4 b_1 + 2 c_1 + d_1 == 8 a_2 + 4 b_2 + 2 c_2 + d_2$$
$$12 a_1 + 4 b_1 + c_1 == 12 a_2 + 4 b_2 + c_2$$
$$12 a_1 + 2 b_1 == 12 a_2 + 2 b_2$$
$$27 a_2 + 9 b_2 + 3 c_2 + d_2 == 1$$
$$18 a_2 + 2 b_2 == 0$$

$$\left\{\left\{a_1 \to \frac{1}{4},\ b_1 \to -\frac{3}{4},\ c_1 \to -\frac{1}{2},\ d_1 \to 3,\ a_2 \to -\frac{1}{4},\ b_2 \to \frac{9}{4},\ c_2 \to -\frac{13}{2},\ d_2 \to 7\right\}\right\}$$

Use the solution given above for the coefficients and form the cubic functions.

$$f_1[x] = 3 - \frac{x}{2} - \frac{3 x^2}{4} + \frac{x^3}{4}$$
$$f_2[x] = 7 - \frac{13 x}{2} + \frac{9 x^2}{4} - \frac{x^3}{4}$$

Now graph the portion of each cubic in the interval over which it is to be used. Then combine the two piecewise cubic graphs to form the spline.



$$f_1[x] = 3 - \frac{x}{2} - \frac{3x^2}{4} + \frac{x^3}{4}$$

$$f_2[x] = 7 - \frac{13x}{2} + \frac{9x^2}{4} - \frac{x^3}{4}$$

Remark. It would be nice to have one piecewise cubic function S[z] that is used. The following formulas for S[x] uses the condition syntax **/;** for making a piecewise function. Under the help menu we can find the following information about Condition.

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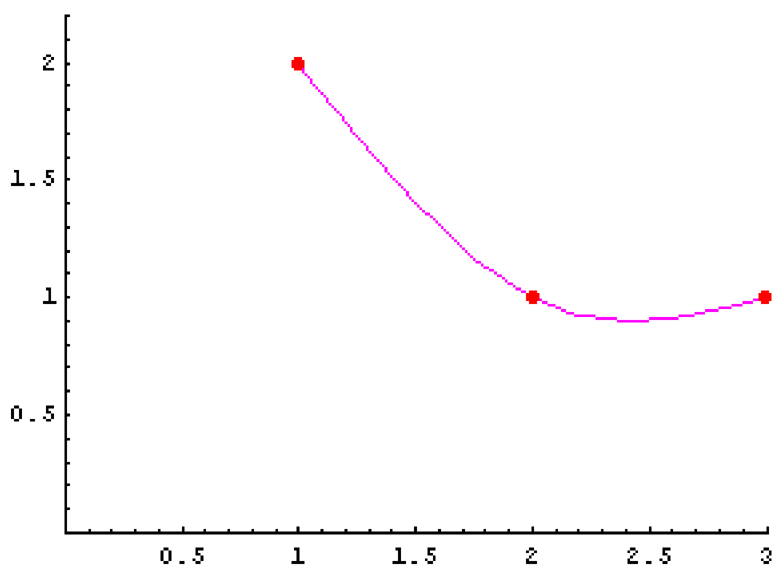

$$S[x\_/;1 \leq x \leq 2] = 3 - \frac{x}{2} - \frac{3x^2}{4} + \frac{x^3}{4}$$

$$S[x\_/;2\leq x\leq 3] = 7 - \frac{13\,x}{2} + \frac{9\,x^2}{4} - \frac{x^3}{4}$$

Notice that the natural cubic spline is different from the clamped cubic spline, it is "a relaxed curve." (and happy too!)