2. Taylor's Method for solving O.D.E.'s

The Taylor series method is of general applicability and it is the standard to which we can compare the accuracy of the various other numerical methods for solving an I. V. P. It can be devised to have any specified degree of accuracy.

Theorem (Taylor Series Method of Order n) Assume that $f(t,y)$ is continuous and satisfies a Lipschits condition in the variable y, and consider the I. V. P. (initial value problem)

$$y' = f(t, y) \text{ with } y(a) = t_0 = \alpha, \text{ over the interval } a \le t \le b.$$

The Taylor series method uses the formulas $t_{k+1} = t_k + h$, and

$$y_{j+1} = y_j + d_1 h + \frac{d_2}{2!} h^2 + \frac{d_3}{3!} h^3 + \ldots + \frac{d_n}{n!} h^n \quad \text{for } k = 0, 1, 2, \ldots, m-1$$

where d_k is $y^{(k)}[t]$ evaluated at t_k, as an approximate solution to the differential equation using the discrete set of points $\{(t_k, y_k)\}_{k=0}^{m}$.

Theorem (Precision of Taylor Series Method of Order n) Assume that $y = y(t)$ is the solution to the I.V.P. $y' = f(t, y)$ with $y(t_0) = y_0$. If $y(t) \in C^{n+1}[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{m}$ is the sequence of approximations generated by the Taylor series method of order n, then at each step, the local truncation error is of the order $O(h^{n+1})$, and the overall global truncation error e_k is of the order

$$|e_k| = |y(t_k) - y_k| = O(h^n), \text{ for } k = 1, 2, \ldots, m.$$

The error at the right end of the interval is called the final global error

$$E(y(b), h) = |y(b) - y_m| = O(h^n).$$

Example 1. Solve the I.V.P. $y' = 1 - t\,y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

Example 2. Solve $y' = 30 - 5\,y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 2.

Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

Example 1. Solve the I.V.P. $y' = 1 - t\,y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

Compute the Taylor series solution based on 25 subintervals and plot the results.

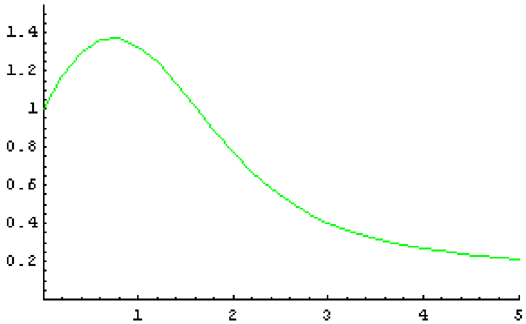
Find numerical solutions to the D.E.
$y' = 1 - t\,y$

First, enter the function $y'[t]$ and create explicit formulas for $D1, D2, D3, D4$ for y', y'', y''', y'''', respectively, which will involve t and $y[t]$.

$y'[t] = 1 - t\,y[t]$
$y''[t] = -t - y[t] + t^2\,y[t]$
$y'''[t] = -2 + t^2 + 3\,t\,y[t] - t^3\,y[t]$
$y''''[t] = 5\,t - t^3 + 3\,y[t] - 6\,t^2\,y[t] + t^4\,y[t]$

Second, replace $y'[t]$ with y and construct the implicit formulas $d_1[t, y], d_2[t, y], d_3[t, y], d_4[t, y]$ for y', y'', y''', y'''', respectively.

$y'[t] = 1 - t\,y$
$y''[t] = -t - y + t^2\,y$
$y'''[t] = -2 + t^2 + 3\,t\,y - t^3\,y$
$y''''[t] = 5\,t - t^3 + 3\,y - 6\,t^2\,y + t^4\,y$



The Taylor series solution for $y' = 1 - t\,y$
Using $n = 26$ points.
{{0., 1.}, {0.2, 1.17753}, {0.4, 1.30242}, {0.6, 1.36818}, {0.8, 1.37546}, {1., 1.33132}, {1.2, 1.24737}, {1.4, 1.13733}, {1.6, 1.01478}, {1.8, 0.891278}, {2., 0.775345}, {2.2, 0.672202}, {2.4, 0.584123}, {2.6, 0.511148}, {2.8, 0.451892}, {3., 0.404267}, {3.2, 0.366029}, {3.4, 0.335107}, {3.6, 0.309768}, {3.8, 0.28865}, {4., 0.270732}, {4.2, 0.255271}, {4.4, 0.241729}, {4.6, 0.22972}, {4.8, 0.218964}, {5., 0.20925}}

The final value is $y(5) = y_{26} = 0.20925$

Example 2. Solve $y' = 30 - 5y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 2.

Find numerical solutions to the D.E.
$y' = 30 - 5y$

First, enter the function y'[t] and create explicit formulas for D1, D2, D3, D4 for y', y'', y''', y'''', respectively, which will involve t and y[t].

```
y'[t] = -5 (-6 + y[t])
y''[t] = 25 (-6 + y[t])
y'''[t] = -125 (-6 + y[t])
y''''[t] = 625 (-6 + y[t])
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Second, replace y'[t] with y and construct the implicit formulas d_1[t, y], d_2[t, y], d_3[t, y], d_4[t, y] for y', y'', y''', y'''', respectively.

```
y'[t] = -5 (-6 + y)
y''[t] = 25 (-6 + y)
y'''[t] = -125 (-6 + y)
y''''[t] = 625 (-6 + y)
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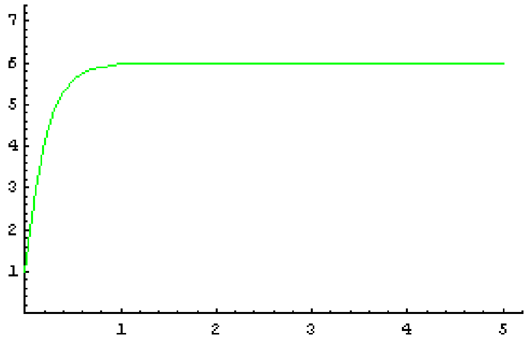


The Taylor series solution for $y' = 30 - 5y$
Using n = 51 points.
{{0., 1}, {0.1, 2.96615}, {0.2, 4.15915}, {0.3, 4.88302}, {0.4, 5.32225}, {0.5, 5.58876}, {0.6, 5.75047}, {0.7, 5.84859},
{0.8, 5.90813}, {0.9, 5.94426}, {1., 5.96618}, {1.1, 5.97948}, {1.2, 5.98755}, {1.3, 5.99244}, {1.4, 5.99542}, {1.5, 5.99722}, {1.6, 5.99831},
{1.7, 5.99898}, {1.8, 5.99938}, {1.9, 5.99962}, {2., 5.99977}, {2.1, 5.99986}, {2.2, 5.99992}, {2.3, 5.99995}, {2.4, 5.99997}, {2.5, 5.99998},
{2.6, 5.99999}, {2.7, 5.99999}, {2.8, 6.}, {2.9, 6.}, {3., 6.}, {3.1, 6.}, {3.2, 6.}, {3.3, 6.}, {3.4, 6.}, {3.5, 6.}, {3.6, 6.}, {3.7, 6.},
{3.8, 6.}, {3.9, 6.}, {4., 6.}, {4.1, 6.}, {4.2, 6.}, {4.3, 6.}, {4.4, 6.}, {4.5, 6.}, {4.6, 6.}, {4.7, 6.}, {4.8, 6.}, {4.9, 6.}, {5., 6.}}

The final value is $y(5) = y_{51} = 6.$

Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

Compute the Taylor series solution based on 50 subintervals and plot the results.

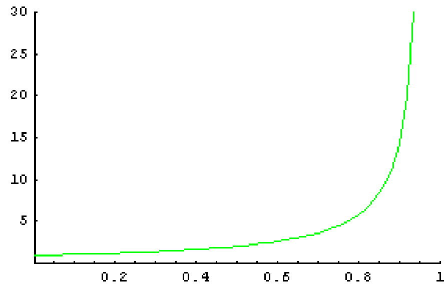
Find numerical solutions to the D.E.
$y' = t^2 + y^2$

First, enter the function $y'[t]$ and create explicit formulas for $D1, D2, D3, D4$ for y', y'', y''', y'''', respectively, which will involve t and $y[t]$.

$$y'[t] = t^2 + y[t]^2$$
$$y''[t] = 2\,(t + t^2\,y[t] + y[t]^3)$$
$$y'''[t] = 2\,(1 + t^4 + 2\,t\,y[t] + 4\,t^2\,y[t]^2 + 3\,y[t]^4)$$
$$y''''[t] = 4\,(3\,t^3 + y[t] + 4\,t^4\,y[t] + 5\,t\,y[t]^2 + 10\,t^2\,y[t]^3 + 6\,y[t]^5)$$

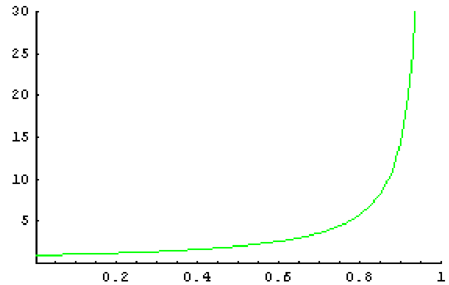
Second, replace $y'[t]$ with y and construct the implicit formulas $d_1[t, y]$, $d_2[t, y]$, $d_3[t, y]$, $d_4[t, y]$ for y', y'', y''', y'''', respectively.

$$y'[t] = t^2 + y^2$$
$$y''[t] = 2\,(t + t^2\,y + y^3)$$
$$y'''[t] = 2\,(1 + t^4 + 2\,t\,y + 4\,t^2\,y^2 + 3\,y^4)$$
$$y''''[t] = 4\,(3\,t^3 + y + 4\,t^4\,y + 5\,t\,y^2 + 10\,t^2\,y^3 + 6\,y^5)$$



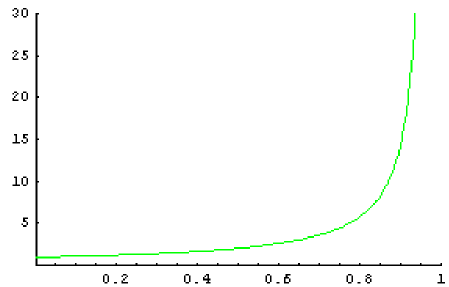
The Taylor series solution for $y' = t^2 + y^2$
Using $n = 51$ points.
{{0., 1.}, {0.02, 1.02041}, {0.04, 1.04169}, {0.06, 1.0639}, {0.08, 1.08713}, {0.1, 1.11146}, {0.12, 1.13698}, {0.14, 1.16378}, {0.16, 1.19198},
{0.18, 1.22168}, {0.2, 1.25302}, {0.22, 1.28613}, {0.24, 1.32117}, {0.26, 1.35831}, {0.28, 1.39774}, {0.3, 1.43967}, {0.32, 1.48433},
{0.34, 1.53199}, {0.36, 1.58294}, {0.38, 1.63752}, {0.4, 1.69611}, {0.42, 1.75915}, {0.44, 1.82713}, {0.46, 1.90063}, {0.48, 1.98033},
{0.5, 2.067}, {0.52, 2.16156}, {0.54, 2.2651}, {0.56, 2.37892}, {0.58, 2.50459}, {0.6, 2.64399}, {0.62, 2.79947}, {0.64, 2.97392}, {0.66, 3.17098},
{0.68, 3.39529}, {0.7, 3.65287}, {0.72, 3.95166}, {0.74, 4.30235}, {0.76, 4.71972}, {0.78, 5.22475}, {0.8, 5.84833}, {0.82, 6.63777}, {0.84, 7.66952},
{0.86, 9.07551}, {0.88, 11.1046}, {0.9, 14.2879}, {0.92, 19.9899}, {0.94, 32.9947}, {0.96, 84.922}, {0.98, 1597.15}, {1., 1.71659×10^9}}
The final value is $y(5) = y_{51} = 1.71659 \times 10^9$

Compute the Taylor series solution based on 100 subintervals and plot the results.
Observe that one fewer subinterval is computed for this case.



The Taylor series solution for $y' = t^2 + y^2$
Using n = 100 points.
{{0., 1.}, {0.01, 1.0101}, {0.02, 1.02041}, {0.03, 1.03094}, {0.04, 1.04169}, {0.05, 1.05267}, {0.06, 1.0639}, {0.07, 1.07539}, {0.08, 1.08713}, {0.09, 1.09916}, {0.1, 1.11146}, {0.11, 1.12407}, {0.12, 1.13698}, {0.13, 1.15021}, {0.14, 1.16378}, {0.15, 1.1777}, {0.16, 1.19198}, {0.17, 1.20663}, {0.18, 1.22168}, {0.19, 1.23714}, {0.2, 1.25302}, {0.21, 1.26934}, {0.22, 1.28613}, {0.23, 1.3034}, {0.24, 1.32117}, {0.25, 1.33947}, {0.26, 1.35831}, {0.27, 1.37773}, {0.28, 1.39774}, {0.29, 1.41838}, {0.3, 1.43967}, {0.31, 1.46164}, {0.32, 1.48433}, {0.33, 1.50777}, {0.34, 1.53199}, {0.35, 1.55703}, {0.36, 1.58294}, {0.37, 1.60975}, {0.38, 1.63752}, {0.39, 1.66629}, {0.4, 1.69611}, {0.41, 1.72704}, {0.42, 1.75915}, {0.43, 1.79249}, {0.44, 1.82713}, {0.45, 1.86315}, {0.46, 1.90063}, {0.47, 1.93966}, {0.48, 1.98033}, {0.49, 2.02274}, {0.5, 2.067}, {0.51, 2.11323}, {0.52, 2.16156}, {0.53, 2.21213}, {0.54, 2.2651}, {0.55, 2.32064}, {0.56, 2.37893}, {0.57, 2.44017}, {0.58, 2.50459}, {0.59, 2.57244}, {0.6, 2.644}, {0.61, 2.71957}, {0.62, 2.79948}, {0.63, 2.88413}, {0.64, 2.97393}, {0.65, 3.06938}, {0.66, 3.171}, {0.67, 3.27941}, {0.68, 3.39531}, {0.69, 3.51951}, {0.7, 3.6529}, {0.71, 3.79656}, {0.72, 3.9517}, {0.73, 4.11976}, {0.74, 4.30241}, {0.75, 4.50164}, {0.76, 4.71981}, {0.77, 4.95976}, {0.78, 5.22491}, {0.79, 5.51946}, {0.8, 5.8486}, {0.81, 6.21879}, {0.82, 6.63825}, {0.83, 7.11754}, {0.84, 7.67046}, {0.85, 8.31542}, {0.86, 9.07755}, {0.87, 9.99203}, {0.88, 11.1097}, {0.89, 12.5068}, {0.9, 14.3034}, {0.91, 16.6994}, {0.92, 20.0547}, {0.93, 25.0882}, {0.94, 33.4689}, {0.95, 50.1081}, {0.96, 97.2792}, {0.97, 460.683}, {0.98, 264901.}, {0.99, 1.3049×10^{19}}}
The final value is $y(0.99) = y_{100} = 1.3049 \times 10^{19}$

Compute the Taylor series solution based on 200 subintervals and plot the results.
Observe that four fewer subintervals are computed for this case.



The Taylor series solution for $y' = t^2 + y^2$
Using n = 197 points.

{{0., 1.}, {0.005, 1.00503}, {0.01, 1.0101}, {0.015, 1.01523}, {0.02, 1.02041}, {0.025, 1.02565}, {0.03, 1.03094}, {0.035, 1.03628}, {0.04, 1.04169},
{0.045, 1.04715}, {0.05, 1.05267}, {0.055, 1.05826}, {0.06, 1.0639}, {0.065, 1.06961}, {0.07, 1.07539}, {0.075, 1.08123}, {0.08, 1.08713}, {0.085, 1.09311},
{0.09, 1.09916}, {0.095, 1.10527}, {0.1, 1.11146}, {0.105, 1.11773}, {0.11, 1.12407}, {0.115, 1.13048}, {0.12, 1.13698}, {0.125, 1.14356}, {0.13, 1.15021},
{0.135, 1.15696}, {0.14, 1.16378}, {0.145, 1.1707}, {0.15, 1.1777}, {0.155, 1.18479}, {0.16, 1.19198}, {0.165, 1.19926}, {0.17, 1.20663}, {0.175, 1.21411},
{0.18, 1.22168}, {0.185, 1.22935}, {0.19, 1.23714}, {0.195, 1.24502}, {0.2, 1.25302}, {0.205, 1.26112}, {0.21, 1.26934}, {0.215, 1.27768}, {0.22, 1.28613},
{0.225, 1.2947}, {0.23, 1.3034}, {0.235, 1.31222}, {0.24, 1.32117}, {0.245, 1.33025}, {0.25, 1.33947}, {0.255, 1.34882}, {0.26, 1.35831}, {0.265, 1.36795},
{0.27, 1.37773}, {0.275, 1.38766}, {0.28, 1.39774}, {0.285, 1.40798}, {0.29, 1.41838}, {0.295, 1.42894}, {0.3, 1.43967}, {0.305, 1.45057}, {0.31, 1.46164},
{0.315, 1.4729}, {0.32, 1.48433}, {0.325, 1.49595}, {0.33, 1.50777}, {0.335, 1.51978}, {0.34, 1.53199}, {0.345, 1.54441}, {0.35, 1.55703}, {0.355, 1.56988},
{0.36, 1.58294}, {0.365, 1.59623}, {0.37, 1.60975}, {0.375, 1.62352}, {0.38, 1.63752}, {0.385, 1.65178}, {0.39, 1.66629}, {0.395, 1.68106}, {0.4, 1.69611},
{0.405, 1.71143}, {0.41, 1.72704}, {0.415, 1.74295}, {0.42, 1.75915}, {0.425, 1.77566}, {0.43, 1.79249}, {0.435, 1.80964}, {0.44, 1.82713}, {0.445, 1.84496},
{0.45, 1.86315}, {0.455, 1.88171}, {0.46, 1.90063}, {0.465, 1.91995}, {0.47, 1.93966}, {0.475, 1.95978}, {0.48, 1.98033}, {0.485, 2.00131}, {0.49, 2.02274},
{0.495, 2.04463}, {0.5, 2.067}, {0.505, 2.08986}, {0.51, 2.11323}, {0.515, 2.13713}, {0.52, 2.16156}, {0.525, 2.18656}, {0.53, 2.21213}, {0.535, 2.23831},
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{0.81, 6.21882}, {0.815, 6.4218}, {0.82, 6.63829}, {0.825, 6.86969}, {0.83, 7.11759}, {0.835, 7.38383}, {0.84, 7.67053}, {0.845, 7.98014}, {0.85, 8.31553},
{0.855, 8.68006}, {0.86, 9.07771}, {0.865, 9.51322}, {0.87, 9.99228}, {0.875, 10.5218}, {0.88, 11.1101}, {0.885, 11.7677}, {0.89, 12.5076}, {0.895, 13.3462},
{0.9, 14.3048}, {0.905, 15.4111}, {0.91, 16.7021}, {0.915, 18.2285}, {0.92, 20.061}, {0.925, 22.3021}, {0.93, 25.1056}, {0.935, 28.7136}, {0.94, 33.5301},
{0.945, 40.2837}, {0.95, 50.433}, {0.955, 67.3761}, {0.96, 101.171}, {0.965, 197.967}, {0.97, 969.945}, {0.975, 675681.}, {0.98, 8.80476×10^{19}}}

The final value is $y(0.98) = y_{197} = 8.80476\times10^{19}$