3. Nonlinear Curve Fitting

Data Linearization Method for Exponential Curve Fitting.

Fit the curve $y = c\,e^{a x}$ to the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Taking the logarithm of both sides we obtain $\ln(y) = \ln(c\,e^{a x}) =$
$\ln(c) + \ln(e^{a x}) = \ln(c) + a x$, thus

$$\ln(y) = a x + \ln(c).$$

Introduce the change of variable $X = x$ and $Y = \ln(y)$. Then the previous equation becomes

$$Y = a X + \ln(c)$$

which is a linear equation in the variables X and Y.

Use the change of variables $X = x$ and $Y = \ln(y)$ on all the data points and obtain

$$X_k = x_k \text{ and } Y_k = \ln(y_k) \quad \text{for } k = 1, 2, \ldots, n.$$

Fit the points $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ with a "least squares line" of the form $Y = A X + B$.

Comparing the equations $Y = A X + B$ and $Y = a X + \ln(c)$ we see that $A = a$ and $B = \ln(c)$. Thus

$$a = A \text{ and } c = e^B$$

are used to construct the coefficients which are then used to "fit the curve"

$$y = c\,e^{a x}$$

to the given data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane.

Example 1. Fit the curve $y = c\,e^{a x}$ to the data points
$(0.0, 1.5), (1.0, 2.5), (2.0, 3.5), (3.0, 5.0), (4.0, 7.5)$.
Solution 1.

Data Linearization Method for a Power Function Curve Fitting.

Fit the curve $y = c\,x^a$ to the data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Taking the logarithm of both sides we obtain $\ln(y) = \ln(c\,x^a) =$
$\ln(c) + \ln(x^a) = \ln(c) + a \ln(x)$, thus

$$\ln(y) = a\ln(x) + \ln(c).$$

Introduce the change of variable $X = \ln(x)$ and $Y = \ln(y)$. Then the previous equation becomes

$$Y = aX + \ln(c)$$

which is a linear equation in the variables X and Y.

Use the change of variables $X = \ln(x)$ and $Y = \ln(y)$ on all the data points and obtain

$$X_k = \ln(x_k) \text{ and } Y_k = \ln(y_k) \text{ for } k = 1, 2, \ldots, n.$$

Fit the points $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ with a "least squares line" of the form $Y = AX + B$.

Comparing the equations $Y = AX + B$ and $Y = aX + \ln(c)$ we see that $A = a$ and $B = \ln(c)$. Thus

$$a = A \text{ and } c = e^B$$

are used to construct the coefficients which are then used to "fit the curve"

$$y = cx^a$$

to the given data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane.

Example 2. Fit the curve $y = cx^a$ to the data points
$(1.0,\ 1.1), (2.0,\ 2.8), (3.0,\ 5.2), (4.0,\ 8.0), (5.0,\ 11.1)$.
Solution 2.

Example 1. Fit the curve $y = c\,e^{a x}$ to the data points
(0.0, 1.5), (1.0, 2.5), (2.0, 3.5), (3.0, 5.0), (4.0, 7.5) .

Solution 1.

$\{x_k\} = \{0., 1., 2., 3., 4.\}$

$\{y_k\} = \{1.5, 2.5, 3.5, 5., 7.5\}$

$X_k = x_k$
$Y_k = \text{Log}[x_k]$

$\{X_k\} = \{0., 1., 2., 3., 4.\}$

$\{Y_k\} = \{0.405465, 0.916291, 1.25276, 1.60944, 2.0149\}$

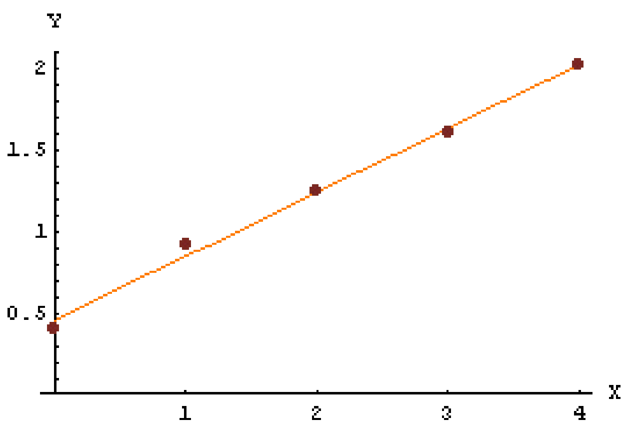
Now "glue together" the transformed parts to form the pairs $\{(X_k, Y_k)\}$.

$\{(X_k, Y_k)\} = \{\{0., 0.405465\}, \{1., 0.916291\}, \{2., 1.25276\}, \{3., 1.60944\}, \{4., 2.0149\}\}$

In the XY - plane

$Y = g[X] = 0.457367 + 0.391202\,X$

Now plot the "least squares line" $Y = g[X] = A\,X + B$ in the XY-plane.



$\{(X_k, Y_k)\} = \{\{0., 0.405465\}, \{1., 0.916291\}, \{2., 1.25276\}, \{3., 1.60944\}, \{4., 2.0149\}\}$

$Y = g[X] = 0.457367 + 0.391202\,X$

So the coefficients A is located at [[2, 1]] and B is located at [[1]] .

$A = 0.391202$
$B = 0.457367$

We use $c = e^{B}$ and $a = A$ to get the coefficients of $y = f_1[x] = c\,e^{a x}$ back in the original xy-plane.

$\{(x_k, y_k)\} = \{\{0., 1.5\}, \{1., 2.5\}, \{2., 3.5\}, \{3., 5.\}, \{4., 7.5\}\}$

In the xy - plane

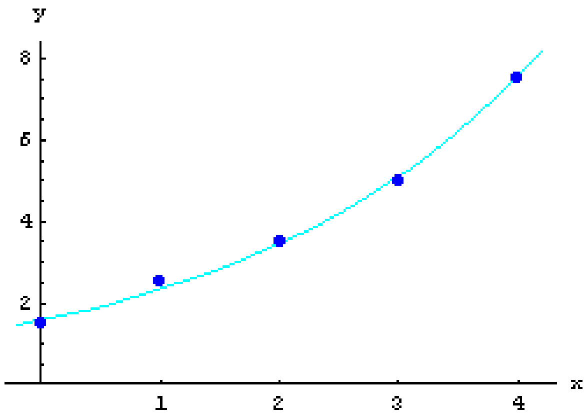
$a = 0.391202$

$c = 1.57991$

$y = f_1[x] = c\, e^{a x}$

$y = f_1[x] = 1.57991\, e^{0.391202 x}$

Now graph the function $y = f_1[x]$ in the xy-plane.



$\{(x_k, y_k)\} = \{\{0., 1.5\}, \{1., 2.5\}, \{2., 3.5\}, \{3., 5.\}, \{4., 7.5\}\}$

$y = f_1[x] = 1.57991\, e^{0.391202 x}$

Example 2. Fit the curve $y = c\,x^a$ to the data points $(1.0,\ 1.1)$, $(2.0,\ 2.8)$, $(3.0,\ 5.2)$, $(4.0,\ 8.0)$, $(5.0,\ 11.1)$.
Solution 2.

$\{x_k\} = \{1.,\ 2.,\ 3.,\ 4.,\ 5.\}$
$\{y_k\} = \{1.1,\ 2.8,\ 5.2,\ 8.,\ 11.1\}$

$X_k = \mathrm{Log}[x_k]$
$Y_k = \mathrm{Log}[x_k]$

$\{X_k\} = \{0.,\ 0.693147,\ 1.09861,\ 1.38629,\ 1.60944\}$
$\{Y_k\} = \{0.0953102,\ 1.02962,\ 1.64866,\ 2.07944,\ 2.40695\}$

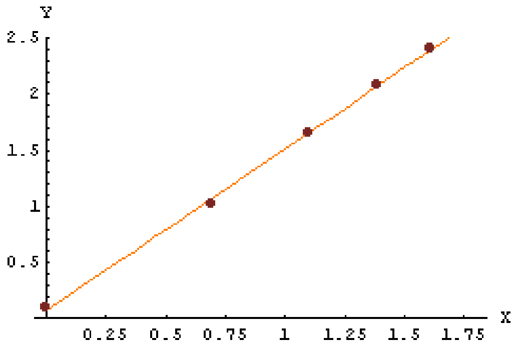
Now "glue together" the transformed parts to form the pairs $\{(X_k,\ Y_k)\}$.

$\{(X_k, Y_k)\} = \{\{0.,\ 0.0953102\},\ \{0.693147,\ 1.02962\},\ \{1.09861,\ 1.64866\},\ \{1.38629,\ 2.07944\},\ \{1.60944,\ 2.40695\}\}$

In the XY - plane

$Y = g[X] = 0.0709788 + 1.44232\,X$

Now plot the "least squares line" $Y = g[X] = A\,X + B$ in the XY-plane.



$\{(X_k, Y_k)\} = \{\{0.,\ 0.0953102\},\ \{0.693147,\ 1.02962\},\ \{1.09861,\ 1.64866\},\ \{1.38629,\ 2.07944\},\ \{1.60944,\ 2.40695\}\}$
$Y = g[X] = 0.0709788 + 1.44232\,X$

So the coefficients A is located at $[\![2,\ 1]\!]$ and B is located at $[\![1]\!]$.

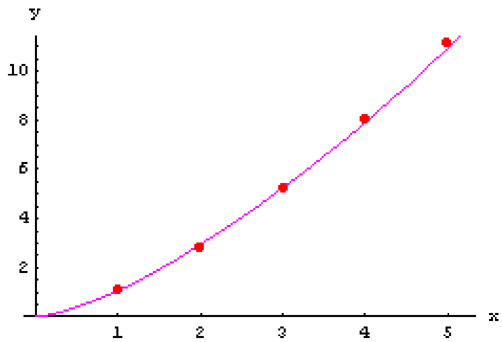
$A = 1.44232$
$B = 0.0709788$

We use $c = e^B$ and a = A to get the coefficients of $y = f_2[x] = c\,x^a$ back in the original xy-plane.

$\{(x_k, y_k)\} = \{\{1.,\ 1.1\},\ \{2.,\ 2.8\},\ \{3.,\ 5.2\},\ \{4.,\ 8.\},\ \{5.,\ 11.1\}\}$
In the xy - plane
$a = 1.44232$
$c = 1.07356$
$y = f_2[x] = c\ x^a$

$y = f_2[x] = 1.07356\,x^{1.44232}$

Now graph the function $y = f_2[x]$ in the xy-plane.



$\{(x_k, y_k)\} = \{\{1., 1.1\}, \{2., 2.8\}, \{3., 5.2\}, \{4., 8.\}, \{5., 11.1\}\}$

$y = f_2[x] = 1.07356\, x^{1.44232}$