6. Determinants and Conic Section Curves

Background.

Five points in the plane uniquely determine an equation for a conic section. The implicit formula for a conic section is often mentioned in textbooks, and the special cases for an ellipse, hyperbola, parabola, circle are obtained by either setting some coefficients equal to zero or making them the same value.

Implicit Equation for a Line.

The equation $c_1 x + c_2 y + c_3 = 0$ of the line through the two points (x_1, y_1), (x_2, y_2) can be computed with the determinant

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Example 1. Use the determinant method to find the line through the points (1,4) and {5,3).
Solution 1.

Implicit Equation for a Circle.

The equation of the circle $c_1 \left(x^2 + y^2\right) + c_2 x + c_3 y + c_4 = 0$ through the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) can be computed with the determinant

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example 2. Use the determinant method to find the circle through the points (6,1), (2,2) and (1,4).
Remark. In Exercises 3 and 4 the same points are used to find the standard parabola and ellipse.
Solution 2.

Implicit Equation for a Parabola.

The equation of the parabola $c_1 x^2 + c_2 x + c_3 y + c_4 = 0$ through the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) can be computed with the determinant

$$\begin{vmatrix} x^2 & x & y & 1 \\ x_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Alternate Equation of a Parabola.

The equation of the parabola $c_1 y^2 + c_2 x + c_3 y + c_4 = 0$ through the three points (x_1, y_1), (x_2, y_2), (x_3, y_3) can be computed with the determinant

$$\begin{vmatrix} y^2 & x & y & 1 \\ y_1^2 & x_1 & y_1 & 1 \\ y_2^2 & x_2 & y_2 & 1 \\ y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example 3. Use the determinant method to find the alternate equation of a parabola through the points (6,1), (2,2) and (1,4).
Solution 3.

Implicit Equation for a Standard Ellipse.

The equation of the ellipse $c_1 x^2 + c_2 y^2 + c_3 x + c_4 y + c_5 = 0$ through the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) can be computed with the determinant

$$\begin{vmatrix} x^2 & y^2 & x & y & 1 \\ x_1^2 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0$$

Example 4. Use the determinant method to find the standard ellipse through the points (6,1), (2,2), (1,4), (9,2).
Solution 4.

The Implicit Equation for a 5 Point Conic.

The equation of the conic through the five points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) can be computed with the determinant

$$\begin{vmatrix} x^2 & x\,y & y^2 & x & y & 1 \\ x_1^2 & x_1\,y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2\,y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3\,y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4\,y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5\,y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0$$

Example 5. Determine the conic that passes through the five points
$\left(\frac{14}{5}, \frac{22}{5}\right), (2, 2), \left(\frac{22}{5}, \frac{14}{5}\right), (7, 5), (10, 10)$.
Solution 5.

Example 6. Determine the conic that passes through the five points
$\left(\frac{7}{4}, \frac{13}{4}\right), (2, 2), \left(\frac{31}{12}, \frac{7}{4}\right), \left(\frac{13}{4}, \frac{7}{4}\right), \left(\frac{38}{5}, \frac{14}{5}\right)$.
Solution 6.

Example 7. Determine the conic that passes through the five points
$(1, 1), (3, 5), (5, 4), (7, 3), (9, 7)$.
Solution 7.

Example 1. Use the determinant method to find the line through the points (1,4) and {5,3).
Solution 1.

The matrix **A** for the system of equations is:

$$A = \begin{pmatrix} x & y & 1 \\ x1 & y1 & 1 \\ x2 & y2 & 1 \end{pmatrix};$$

Enter the points (1,4) and (5,3) and put them into rows 2 and 3 of the matrix **A**:

$$\begin{pmatrix} x & y & 1 \\ 1 & 4 & 1 \\ 5 & 3 & 1 \end{pmatrix}$$

Form the equation for the line by setting the determinant of **A** equal to zero.

$$-17 + x + 4y == 0$$

Then graph the line.

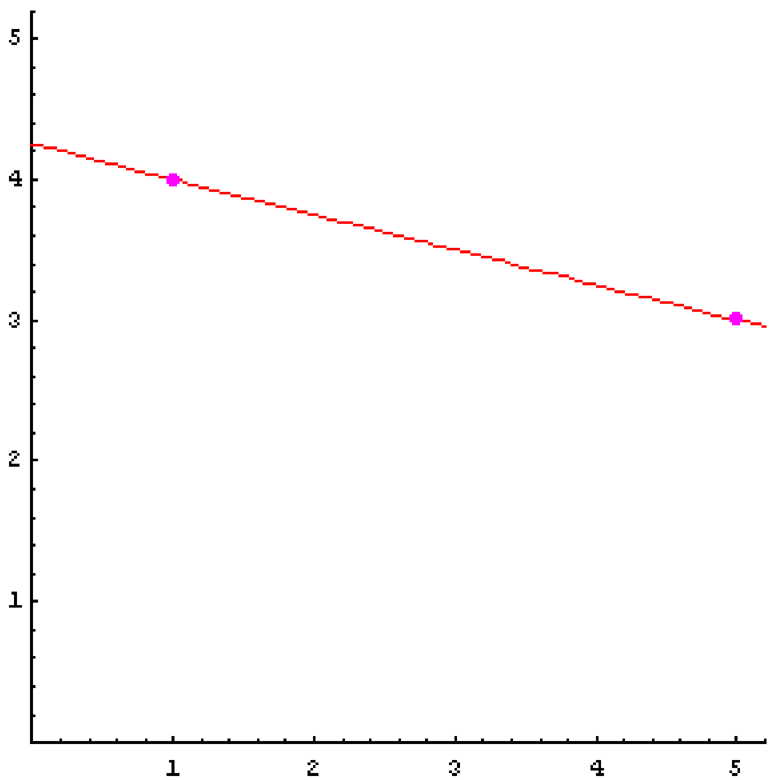

Figure 1. The Line $-17 + x + 4y == 0$

Through the points {{1, 4}, {5, 3}}

Example 2. Use the determinant method to find the circle through the points (6,1), (2,2) and (1,4).
Remark. In Exercises 3 and 4 the same points are used to find the standard parabola and ellipse.
Solution 2.

The matrix **A** for the linear system in (2) is:

$$\begin{pmatrix} x^2+y^2 & x & y & 1 \\ 37 & 6 & 1 & 1 \\ 8 & 2 & 2 & 1 \\ 17 & 1 & 4 & 1 \end{pmatrix}$$

For the given three points, the homogeneous system **AC** = **0** is:

$$\begin{pmatrix} x^2+y^2 & x & y & 1 \\ 37 & 6 & 1 & 1 \\ 8 & 2 & 2 & 1 \\ 17 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0$$

The determinant of this matrix is computed by typing:

```
d = Together[Det[A]]
```

$$-208 + 67\,x - 7\,x^2 + 65\,y - 7\,y^2$$

The desired equation is:

```
eqn = d == 0
```

$$-208 + 67\,x - 7\,x^2 + 65\,y - 7\,y^2 == 0$$

The conic is the circle shown in Figure 4.

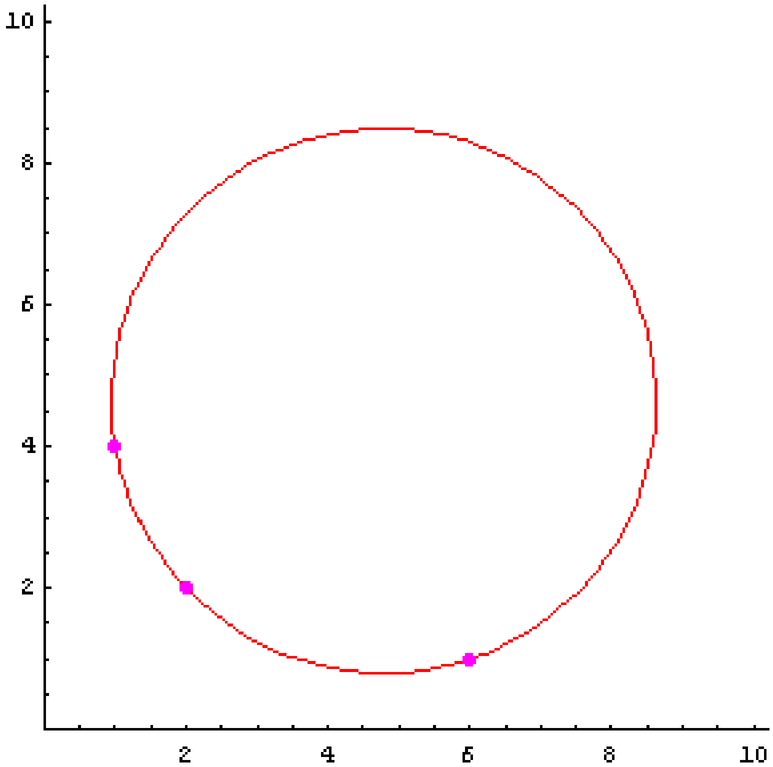

Figure 4. The Circle $-208 + 67\,x - 7\,x^2 + 65\,y - 7\,y^2 == 0$

Through the points {{6, 1}, {2, 2}, {1, 4}}

Example 3. Use the determinant method to find the alternate equation of a parabola through the points (6,1), (2,2) and (1,4).
Solution 3.

The matrix **A** for the linear system in (3) is:

$$\begin{pmatrix} y^2 & x & y & 1 \\ 1 & 6 & 1 & 1 \\ 4 & 2 & 2 & 1 \\ 16 & 1 & 4 & 1 \end{pmatrix}$$

For the given three points, the homogeneous system **AC** = **0** is:

$$\begin{pmatrix} y^2 & x & y & 1 \\ 1 & 6 & 1 & 1 \\ 4 & 2 & 2 & 1 \\ 16 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0$$

The determinant of this matrix is computed by typing:

```
d = Together[Det[A]]
```

$$-74 + 6x + 45y - 7y^2$$

The desired equation is:

```
eqn = d == 0
```

$$-74 + 6x + 45y - 7y^2 == 0$$

The conic is the standard parabola shown in Figure 6.

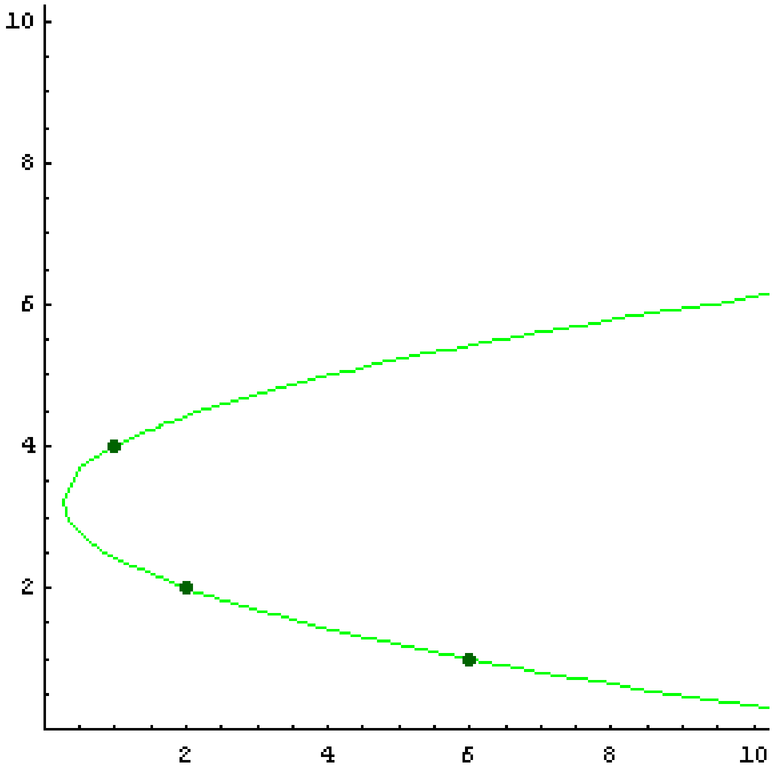

Figure 6. The Parabola $-74 + 6\,x + 45\,y - 7\,y^2 == 0$

Through the points {{6, 1}, {2, 2}, {1, 4}}

Example 4. Use the determinant method to find the standard ellipse through the points (6,1), (2,2), (1,4), (9,2).

Solution 4.

The matrix **A** for the linear system in (4) is:

$$\begin{pmatrix} x^2 & y^2 & x & y & 1 \\ 36 & 1 & 6 & 1 & 1 \\ 4 & 4 & 2 & 2 & 1 \\ 1 & 16 & 1 & 4 & 1 \\ 81 & 4 & 9 & 2 & 1 \end{pmatrix}$$

For the given three points, the homogeneous system **AC** = **0** is:

$$\begin{pmatrix} x^2 & y^2 & x & y & 1 \\ 36 & 1 & 6 & 1 & 1 \\ 4 & 4 & 2 & 2 & 1 \\ 1 & 16 & 1 & 4 & 1 \\ 81 & 4 & 9 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} = 0$$

The determinant of this matrix is computed by typing:

```
d = Together[Det[A]]
```

$$-14\,(142 - 33\,x + 3\,x^2 - 60\,y + 8\,y^2)$$

This quantity is multiplied by $\frac{-1}{14}$ to get the desired equation:

```
eqn = -d/14 == 0
```

$$142 - 33\,x + 3\,x^2 - 60\,y + 8\,y^2 == 0$$

The conic is the circle shown in Figure 7.

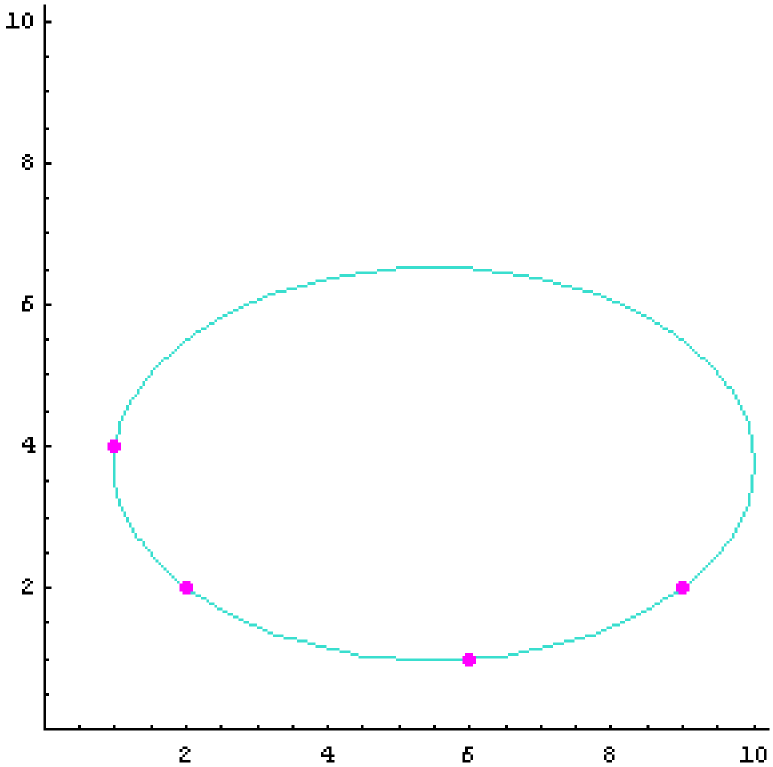

Figure 7. The Ellipse $142 - 33x + 3x^2 - 60y + 8y^2 == 0$

Through the points {{6, 1}, {2, 2}, {1, 4}, {9, 2}}

Example 5. Determine the conic that passes through the five points
$\left(\frac{14}{5}, \frac{22}{5}\right)$, $(2, 2)$, $\left(\frac{22}{5}, \frac{14}{5}\right)$, $(7, 5)$, $(10, 10)$.
Solution 5.

The matrix **A** for the linear system in (5) is:

$$\begin{pmatrix} x^2 & x\,y & y^2 & x & y & 1 \\ \frac{196}{25} & \frac{308}{25} & \frac{484}{25} & \frac{14}{5} & \frac{22}{5} & 1 \\ 4 & 4 & 4 & 2 & 2 & 1 \\ \frac{484}{25} & \frac{308}{25} & \frac{196}{25} & \frac{22}{5} & \frac{14}{5} & 1 \\ 49 & 35 & 25 & 7 & 5 & 1 \\ 100 & 100 & 100 & 10 & 10 & 1 \end{pmatrix}$$

For the given five points, the homogeneous system **AC** = **0** is:

$$\begin{pmatrix} x^2 & x\,y & y^2 & x & y & 1 \\ \frac{196}{25} & \frac{308}{25} & \frac{484}{25} & \frac{14}{5} & \frac{22}{5} & 1 \\ 4 & 4 & 4 & 2 & 2 & 1 \\ \frac{484}{25} & \frac{308}{25} & \frac{196}{25} & \frac{22}{5} & \frac{14}{5} & 1 \\ 49 & 35 & 25 & 7 & 5 & 1 \\ 100 & 100 & 100 & 10 & 10 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = 0$$

The determinant of this matrix is computed by typing:

```
d = Together[Det[A]]
```

$$\frac{49152}{625}\left(80 - 24\,x + 17\,x^2 - 24\,y - 30\,x\,y + 17\,y^2\right)$$

This quantity is multiplied by $\frac{625}{49152}$ to get the desired equation:

$$\mathbf{eqn} = \frac{\mathbf{625\,d}}{\mathbf{49152}} == \mathbf{0}$$

$$80 - 24\,x + 17\,x^2 - 24\,y - 30\,x\,y + 17\,y^2 == 0$$

The conic is the ellipse shown in Figure 9.

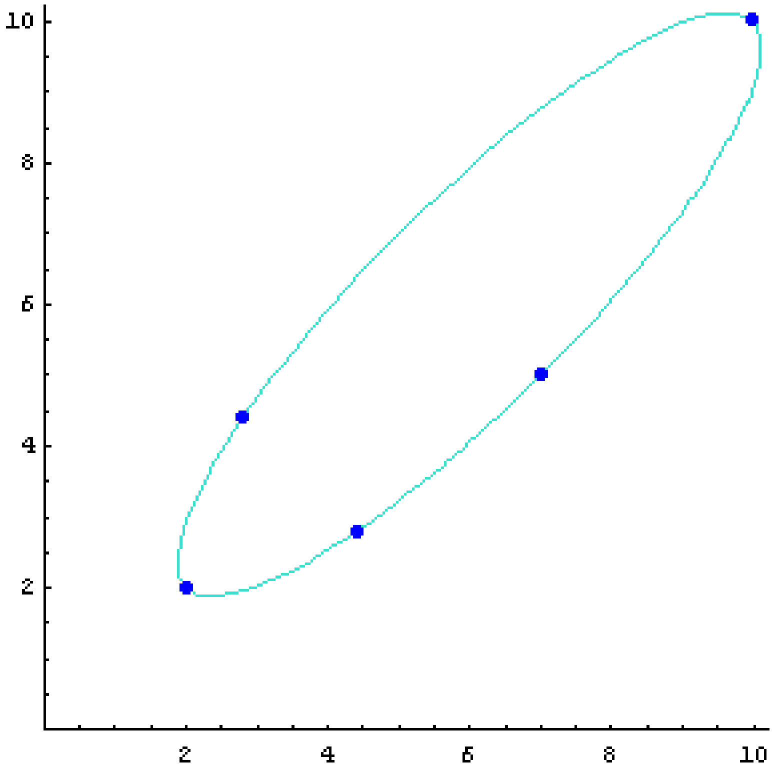

Figure 9. The Ellipse $80 - 24x + 17x^2 - 24y - 30xy + 17y^2 == 0$

Through the points $\left\{\left\{\frac{14}{5}, \frac{22}{5}\right\}, \{2, 2\}, \left\{\frac{22}{5}, \frac{14}{5}\right\}, \{7, 5\}, \{10, 10\}\right\}$

Example 6. Determine the conic that passes through the five points
$\left(\frac{7}{4}, \frac{13}{4}\right)$, $(2, 2)$, $\left(\frac{31}{12}, \frac{7}{4}\right)$, $\left(\frac{13}{4}, \frac{7}{4}\right)$, $\left(\frac{38}{5}, \frac{14}{5}\right)$.

Solution 6.

The matrix **A** for the linear system in (6) is:

$$\begin{pmatrix} x^2 & x\,y & y^2 & x & y & 1 \\ \frac{49}{16} & \frac{91}{16} & \frac{169}{16} & \frac{7}{4} & \frac{13}{4} & 1 \\ 4 & 4 & 4 & 2 & 2 & 1 \\ \frac{961}{144} & \frac{217}{48} & \frac{49}{16} & \frac{31}{12} & \frac{7}{4} & 1 \\ \frac{169}{16} & \frac{91}{16} & \frac{49}{16} & \frac{13}{4} & \frac{7}{4} & 1 \\ \frac{1444}{25} & \frac{532}{25} & \frac{196}{25} & \frac{38}{5} & \frac{14}{5} & 1 \end{pmatrix}$$

For the given five points, the homogeneous system **AC** = **0** is:

$$\begin{pmatrix} x^2 & x\,y & y^2 & x & y & 1 \\ \frac{49}{16} & \frac{91}{16} & \frac{169}{16} & \frac{7}{4} & \frac{13}{4} & 1 \\ 4 & 4 & 4 & 2 & 2 & 1 \\ \frac{961}{144} & \frac{217}{48} & \frac{49}{16} & \frac{31}{12} & \frac{7}{4} & 1 \\ \frac{169}{16} & \frac{91}{16} & \frac{49}{16} & \frac{13}{4} & \frac{7}{4} & 1 \\ \frac{1444}{25} & \frac{532}{25} & \frac{196}{25} & \frac{38}{5} & \frac{14}{5} & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = 0$$

The determinant of this matrix is computed by typing:

```
d = Together[Det[A]]
```

$$\frac{63}{100}\left(16 + 3\,x^2 - 10\,x\,y + 3\,y^2\right)$$

This quantity is multiplied by $\frac{100}{63}$ to get the desired equation:

$$\textbf{eqn} = \frac{\textbf{100 d}}{\textbf{63}} == \textbf{0}$$

$$16 + 3\,x^2 - 10\,x\,y + 3\,y^2 == 0$$

The conic is the hyperbola shown in Figure 10.

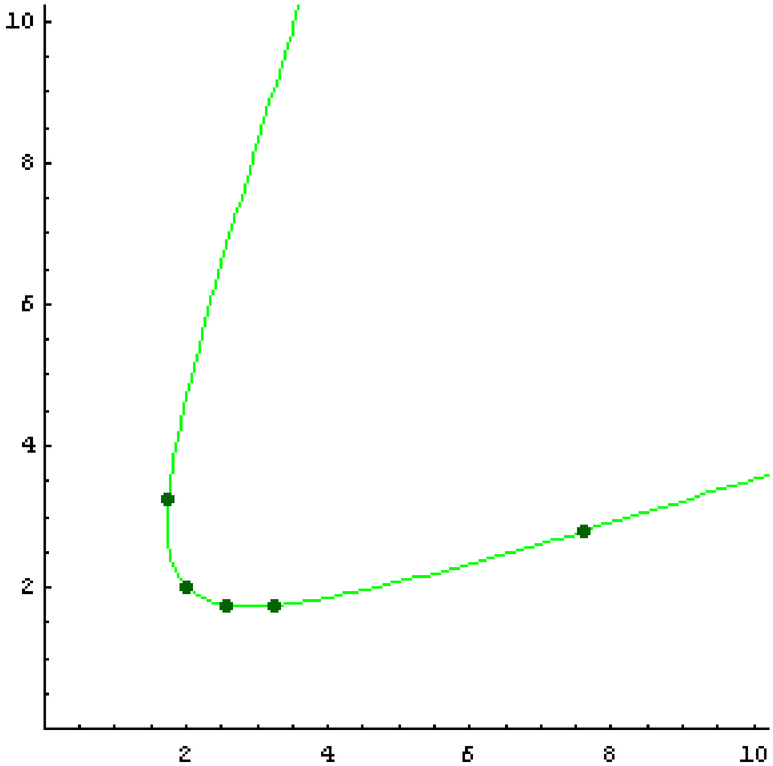

Figure 10. The Hyperbola $16 + 3x^2 - 10xy + 3y^2 == 0$

Through the points $\left\{\left\{\frac{7}{4}, \frac{13}{4}\right\}, \{2, 2\}, \left\{\frac{31}{12}, \frac{7}{4}\right\}, \left\{\frac{13}{4}, \frac{7}{4}\right\}, \left\{\frac{38}{5}, \frac{14}{5}\right\}\right\}$

Example 7. Determine the conic that passes through the five points
$(1, 1), (3, 5), (5, 4), (7, 3), (9, 7)$.

Solution 7.

The matrix **A** for the linear system in (7) is:

$$\begin{pmatrix} x^2 & x\,y & y^2 & x & y & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 9 & 15 & 25 & 3 & 5 & 1 \\ 25 & 20 & 16 & 5 & 4 & 1 \\ 49 & 21 & 9 & 7 & 3 & 1 \\ 81 & 63 & 49 & 9 & 7 & 1 \end{pmatrix}$$

For the given five points, the homogeneous system **AC** = **0** is:

$$\begin{pmatrix} x^2 & x\,y & y^2 & x & y & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 9 & 15 & 25 & 3 & 5 & 1 \\ 25 & 20 & 16 & 5 & 4 & 1 \\ 49 & 21 & 9 & 7 & 3 & 1 \\ 81 & 63 & 49 & 9 & 7 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = 0$$

The determinant of this matrix is computed by typing:

```
d = Together[Det[A]]
```

$$400\,(-13 - 38\,x + 3\,x^2 + 54\,y + 2\,x\,y - 8\,y^2)$$

This quantity is multiplied by $\frac{1}{400}$ to get the desired equation:

$$\mathbf{eqn} = \frac{\mathbf{d}}{\mathbf{400}} == \mathbf{0}$$

$$-13 - 38\,x + 3\,x^2 + 54\,y + 2\,x\,y - 8\,y^2 == 0$$

The conic is the pair of intersecting lines shown in Figure11.

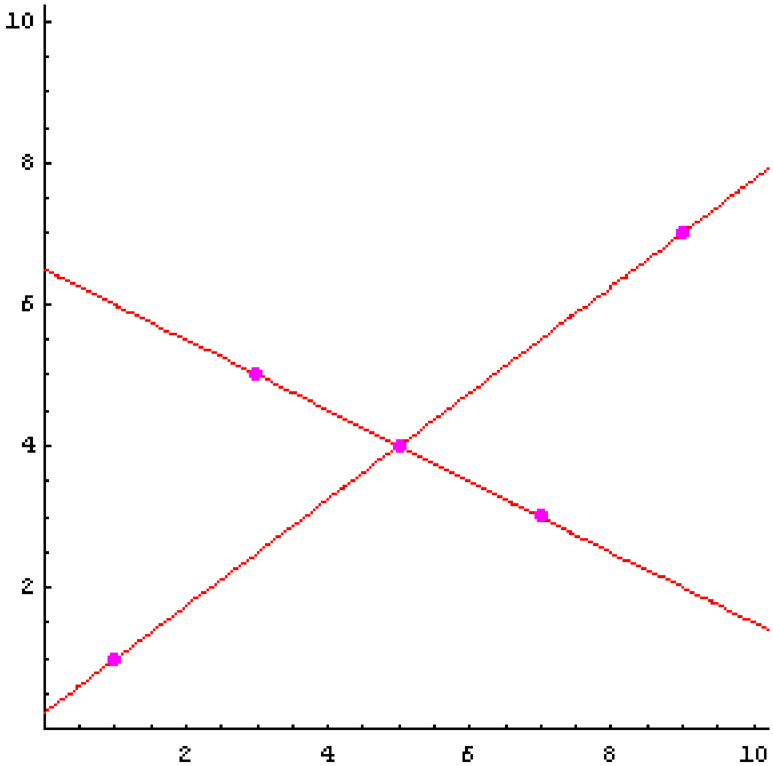

Figure 11. The Lines $-13 - 38x + 3x^2 + 54y + 2xy - 8y^2 == 0$

Through the points {{1, 1}, {3, 5}, {5, 4}, {7, 3}, {9, 7}}