1. The Golden Ratio Search for a Minimum

Bracketing Search Methods

An approach for finding the minimum of $f(x)$ in a given interval is to evaluate the function many times and search for a local minimum. To reduce the number of function evaluations it is important to have a good strategy for determining where $f(x)$ is to be evaluated. Two efficient bracketing methods are the golden ratio and Fibonacci searches. To use either bracketing method for finding the minimum of $f(x)$, a special condition must be met to ensure that there is a proper minimum in the given interval.

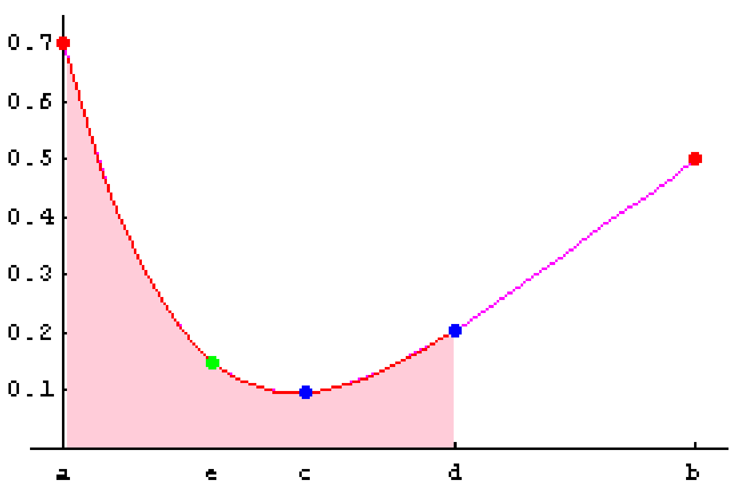
Definition (Unimodal Function) The function $f(x)$ is unimodal on $I = [a, b]$, if there exists a unique number $p \in I$ such that

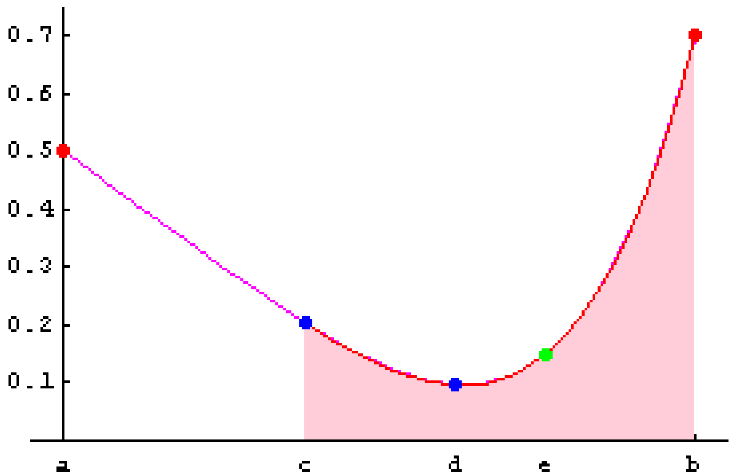
$f(x)$ is decreasing on $[a, p]$,
and
$f(x)$ is increasing on $[p, b]$.

Golden Ratio Search

If $f(x)$ is known to be unimodal on $[a, b]$, then it is possible to replace the interval with a subinterval on which $f(x)$ takes on its minimum value. One approach is to select two interior points $c < d$. This results in $a < c < d < b$. The condition that $f(x)$ is unimodal guarantees that the function values $f(c)$ and $f(d)$ are less than $\max\{f(a), f(b)\}$.

If $f(c) \le f(d)$, then the minimum must occur in the subinterval $[a, d]$, and we replace b with d and continue the search in the new subinterval $[a, d]$. If $f(d) < f(c)$, then the minimum must occur in the subinterval $[c, b]$, and we replace a with c and continue the search in the new subinterval $[c, b]$. These choices are shown in Figure 1 below.





If $f(c) \le f(d)$, then squeeze from the right and use the new interval $[a, d]$ and the four points $\{a, e, c, d\}$.

If $f(d) < f(c)$, then squeeze from the left and use the new interval $[c, b]$ and the four points $\{c, d, e, b\}$.

Figure 1. The decision process for the golden ratio search.

The interior points c and d of the original interval $[a, b]$, must be constructed so that the resulting intervals $[a, c]$, and $[d, b]$ are symmetrical in $[a, b]$. This requires that $b - d = c - a$, and produces the two equations

(1) $$c = a + (1 - r)(b - a) = r a + (1 - r) b,$$

and

(2) $$d = b - (1 - r)(b - a) = (1 - r) a + r b,$$

where $\frac{1}{2} < r < 1$ (to preserve the ordering $c < d$).

We want the value of r to remain constant on each subinterval. If r is chosen judicially then only one new point e (shown in green in Figure 1) needs to be constructed for the next iteration. Additionally, one of the old interior points (either c or d) will be used as an interior point of the next subinterval, while the other interior point (d or c) will become an endpoint of the next subinterval in the iteration process. Thus, for each iteration only one new point e will have to be constructed and only one new function evaluation $f(e)$, will have to be made. As a consequence, this means that the value r must be chosen carefully to split the interval of $[a, b]$ into subintervals which have the following ratios:

(3) $$c - a = (1 - r)(b - a) \quad \text{and} \quad b - c = r(b - a),$$

and

(4) $$b - d = (1 - r)(b - a) \quad \text{and} \quad d - a = r(b - a).$$

If $f(c) \le f(d)$ and only one new function evaluation is to be made in the interval $[a, d]$, then we must have

$$\frac{d-a}{b-a} = \frac{c-a}{d-a}.$$

Use the facts in (3) and (4) to rewrite this equation and then simplify.

$$\frac{r(b-a)}{b-a} = \frac{(1-r)(b-a)}{r(b-a)},$$

$$r = \frac{1-r}{r},$$

$$r^2 = 1 - r,$$

$$r^2 + r - 1 = 0.$$

Now the quadratic equation can be applied and we get

$$r = \frac{1}{2}\left(-1 \pm \sqrt{5}\right).$$

The value we seek is $r = \frac{\sqrt{5}-1}{2}$ and it is often referred to as the "golden ratio." Similarly, if $f(d) < f(c)$, then it can be shown that $r = \frac{\sqrt{5}-1}{2}$.

Example 1. Find the minimum of the unimodal function $f(x) = x^2 - \sin(x)$ on the interval $[0, 1]$.
Solution 1.

Example 2. Find the local maximum of the function $f(x) = e^x - \sin(2x)$ in the interval $[-3, 1]$.
Solution 2.

Exercise 3. The voltage input to an electrical component is $f(x) = 140\, e^{-x/9} \sin(x)$ for $0 \le x$. The manufacturer states that the voltage is not to exceed ± 120 or else the component will "burn out." Can we expect that the component will be o.k. or will it "burn out."
Solution 3.

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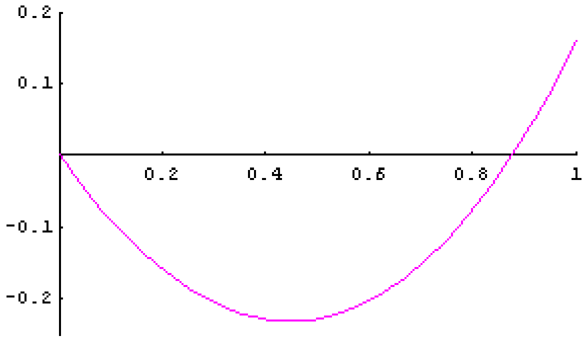
Solution 1.



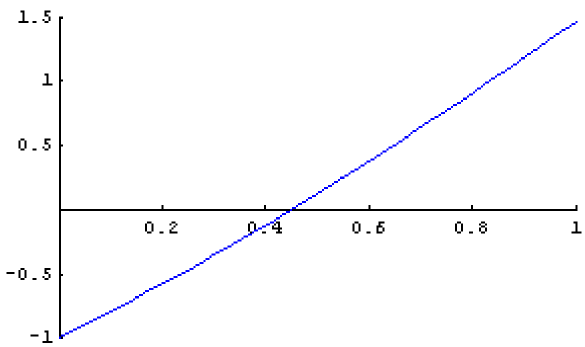
$y = f[x] = x^2 - \mathrm{Sin}[x]$

$g[x] = f'[x] = 2x - \mathrm{Cos}[x]$

Solution by solving $f'(x) = 0$

A root-finding method can be used to determine where the derivative $f'(x) = 2x - \cos(x)$ is zero.

Since $f'(0) = -1$ and $f'(1) = 1.4596977$ have opposite signs, by the intermediate value theorem a root of $f'(x)$ lies in the interval $[0, 1]$. The results obtained by using the secant method with the initial values $p_0 = 0$ and $p_1 = 1$ are given by the following computations.



$g[x] = f'[x] = 2x - \mathrm{Cos}[x]$

$p_0 = 0.0000000000000000,\quad g[p_0] = -1.$
$p_1 = 1.0000000000000000,\quad g[p_1] = 1.45969769413186$
$p_2 = 0.4065540258811950,\quad g[p_2] = -0.1053809204842636$
$p_3 = 0.4465123272893702,\quad g[p_3] = -0.00893398567074244$
$p_4 = 0.4502137087059809,\quad g[p_4] = 0.0000732915423633517 8$
$p_5 = 0.4501835908416719,\quad g[p_5] = -4.98062224796314\times 10^{-8}$
$p_6 = 0.4501836112947598,\quad g[p_6] = -2.77111666946439 1\times 10^{-13}$
$p_7 = 0.4501836112948736,\quad g[p_7] = 0.$

$p = 0.4501836112948736$
$\Delta p = \pm 1.13798\times 10^{-13}$
$g[p] = 0.$

The conclusion from applying the secant method is that

$$f'(0.45018361129475976) = -2.7711166694643907 * 10^{-13} \approx 0$$

The second derivative is $f''(x) = 2 + \sin(x)$ and we compute

$$f''(0.45018361129475976) = 2.435130859036607 > 0$$

Hence, by the second derivative test, the local minimum of $f(x)$ on the interval $[0, 1]$ is

$$f(0.45018361129475976) = -0.23246557515821564$$

Solution using the golden ratio search for the minimum of $f(x)$

Let $a_0 = 0$ and $b_0 = 1$, and start with the initial interval $[a_0, b_0] = [0, 1]$. Formulas (1) and (2) yield

$$c_0 = a_0 + \left(1 - \frac{-1+\sqrt{5}}{2}\right)(b_0 - a_0)$$
$$c_0 = 0. + \left(1 + \frac{1}{2}\left(1 - \sqrt{5}\right)\right)(1.)$$
$$c_0 = 0. + (0.381966)(1.)$$
$$c_0 = 0.381966$$

$$d_0 = b_0 - \left(1 - \frac{-1+\sqrt{5}}{2}\right)(b_0 - a_0)$$
$$d_0 = 1. - \left(1 + \frac{1}{2}\left(1 - \sqrt{5}\right)\right)(1.)$$
$$d_0 = 1. - (0.381966)(1.)$$
$$d_0 = 0.618034$$

The function values are

$f[c_0] = -0.226847$

$f[d_0] = -0.197468$

Is $f[c_0] < f[d_0]$? True

Since $f[c_0] < f[d_0]$, the new subinterval is $[a_0, d_0] = [0.000000, 0.618034]$.
To continue the iteration we set $a_1 = a_0$, $b_1 = d_0$, $d_1 = c_0$, and compute $c_1 = a_1 + (1 - r)(b_1 - a_1)$.

$a_1 = 0.$

$b_1 = 0.618034$

$c_1 = 0.236068$

$d_1 = 0.381966$

The function values are

$f[c_1] = -0.178153$

$f[d_1] = -0.226847$

Now compute and compare $f(c_1)$ and $f(d_1)$ to determine the new subinterval and continue the iteration process. The list of computations are obtained by using the `GoldenSearch` subroutine are:

```
f[{ 0.000000,  0.381966,  0.618034,  1.000000}] = { 0.00000000, -0.22684748, -0.19746793,  0.15852902}
f[{ 0.000000,  0.236068,  0.381966,  0.618034}] = { 0.00000000, -0.17815339, -0.22684748, -0.19746793}
f[{ 0.236068,  0.381966,  0.472136,  0.618034}] = {-0.17815339, -0.22684748, -0.23187724, -0.19746793}
f[{ 0.381966,  0.472136,  0.527864,  0.618034}] = {-0.22684748, -0.23187724, -0.22504882, -0.19746793}
f[{ 0.381966,  0.437694,  0.472136,  0.527864}] = {-0.22684748, -0.23227594, -0.23187724, -0.22504882}
f[{ 0.381966,  0.416408,  0.437694,  0.472136}] = {-0.22684748, -0.23108238, -0.23227594, -0.23187724}
f[{ 0.416408,  0.437694,  0.450850,  0.472136}] = {-0.23108238, -0.23227594, -0.23246503, -0.23187724}
f[{ 0.437694,  0.450850,  0.458980,  0.472136}] = {-0.23227594, -0.23246503, -0.23237125, -0.23187724}
f[{ 0.437694,  0.445825,  0.450850,  0.458980}] = {-0.23227594, -0.23244245, -0.23246503, -0.23237125}
f[{ 0.445825,  0.450850,  0.453955,  0.458980}] = {-0.23244245, -0.23246503, -0.23244825, -0.23237125}
f[{ 0.445825,  0.448930,  0.450850,  0.453955}] = {-0.23244245, -0.23246366, -0.23246503, -0.23244825}
f[{ 0.448930,  0.450850,  0.452036,  0.453955}] = {-0.23246366, -0.23246503, -0.23246140, -0.23244825}
f[{ 0.448930,  0.450117,  0.450850,  0.452036}] = {-0.23246366, -0.23246557, -0.23246503, -0.23246140}
f[{ 0.448930,  0.449663,  0.450117,  0.450850}] = {-0.23246366, -0.23246525, -0.23246557, -0.23246503}
f[{ 0.449663,  0.450117,  0.450397,  0.450850}] = {-0.23246525, -0.23246557, -0.23246552, -0.23246503}
f[{ 0.449663,  0.449944,  0.450117,  0.450397}] = {-0.23246525, -0.23246550, -0.23246557, -0.23246552}
f[{ 0.449944,  0.450117,  0.450224,  0.450397}] = {-0.23246550, -0.23246557, -0.23246557, -0.23246552}
f[{ 0.450117,  0.450224,  0.450290,  0.450397}] = {-0.23246557, -0.23246557, -0.23246556, -0.23246552}
f[{ 0.450117,  0.450183,  0.450224,  0.450290}] = {-0.23246557, -0.23246558, -0.23246557, -0.23246556}
f[{ 0.450117,  0.450157,  0.450183,  0.450224}] = {-0.23246557, -0.23246557, -0.23246558, -0.23246557}
f[{ 0.450157,  0.450183,  0.450198,  0.450224}] = {-0.23246557, -0.23246558, -0.23246557, -0.23246557}
f[{ 0.450157,  0.450173,  0.450183,  0.450198}] = {-0.23246557, -0.23246558, -0.23246558, -0.23246557}
f[{ 0.450173,  0.450183,  0.450189,  0.450198}] = {-0.23246558, -0.23246558, -0.23246558, -0.23246557}
f[{ 0.450173,  0.450179,  0.450183,  0.450189}] = {-0.23246558, -0.23246558, -0.23246558, -0.23246558}
f[{ 0.450179,  0.450183,  0.450185,  0.450189}] = {-0.23246558, -0.23246558, -0.23246558, -0.23246558}
f[{ 0.450179,  0.450181,  0.450183,  0.450185}] = {-0.23246558, -0.23246558, -0.23246558, -0.23246558}

After m = 25 iterations
{  a,   b,   c,   d}  = { 0.450179,  0.450181,  0.450183,  0.450185}
{f[a],f[b],f[c],f[d]} = {-0.232465575132, -0.232465575152, -0.232465575157, -0.232465575156}
f[x] = x^2 - Sin[x]
```

At the twenty-fifth iteration the interval has been narrowed down to

$[a_{25}, b_{25}] = [0.450179, 0.450185]$.

This interval has width $b_{25} - a_{25} = 0.000006 = 6.0 \times 10^{-6}$.

However, the computed function values at the endpoints agree to ten decimal places

$f(a_{25}) = -0.232465575132$
$f(b_{25}) = -0.232465575156$

hence the algorithm is terminated. A difficulty in using search methods is that the function may be "flat" near the minimum, and this limits the accuracy that can be obtained. The secant method was able to find the more accurate answer

$p_6 = 0.4501836112947597$

and

$f(p_6) = -0.23246557515821564$

Although the golden ratio search is the slower in this example, it has desirable feature that it can be applied in cases where $f(x)$ is not differentiable.

Let us compare these answers with *Mathematica*'s subroutine `FindMinimum`.

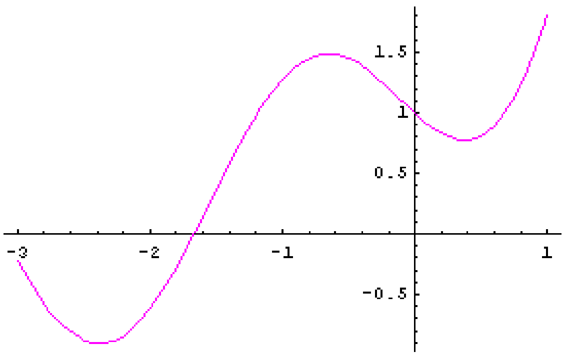
```
f[x]  = x^2 - Sin[x]
{-0.232466, {x → 0.450184}}

  p   = 0.450184
f[p]  = -0.232465575158
```

This is close to the values obtained with the golden ratio search. If more decimal places are needed, we can print them out.

```
  p   = 0.450183607099
f[p]  = -0.2324655751582156
```

However, they are not as accurate as those computed with the secant method applied to solving `f'[x] = 0`.

Example 2. Find the local maximum of the function $f(x) = e^x - \sin(2x)$ in the interval $[-3, 1]$.
Solution 2.



$y = f[x] = e^x - \text{Sin}[2x]$

We see that the local maximum lies in the interval $[-1, 0]$.

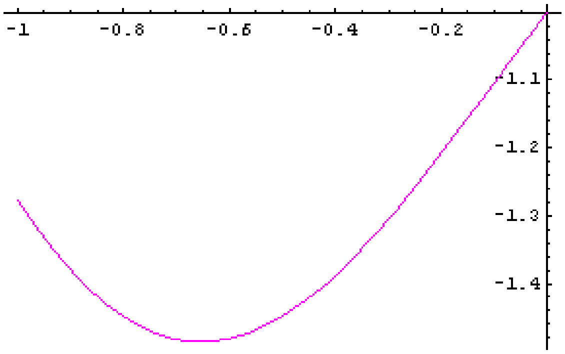
So we will search for the local minimum of $f[x] = -e^x + \text{Sin}[2x]$ in the interval $[-1, 0]$.

```
f[{-1.000000, -0.618034, -0.381966,  0.000000}] = {-1.277176868, -1.483502668, -1.374284434, -1.000000000}
f[{-1.000000, -0.763932, -0.618034, -0.381966}] = {-1.277176868, -1.464909714, -1.483502668, -1.374284434}
f[{-0.763932, -0.618034, -0.527864, -0.381966}] = {-1.464909714, -1.483502668, -1.460122658, -1.374284434}
f[{-0.763932, -0.673762, -0.618034, -0.527864}] = {-1.464909714, -1.484965248, -1.483502668, -1.460122658}
f[{-0.763932, -0.708204, -0.673762, -0.618034}] = {-1.464909714, -1.480633772, -1.484965248, -1.483502668}
f[{-0.708204, -0.673762, -0.652476, -0.618034}] = {-1.480633772, -1.484965248, -1.485625809, -1.483502668}
f[{-0.673762, -0.652476, -0.639320, -0.618034}] = {-1.484965248, -1.485625809, -1.485276157, -1.483502668}
f[{-0.673762, -0.660606, -0.652476, -0.639320}] = {-1.484965248, -1.485553384, -1.485625809, -1.485276157}
f[{-0.660606, -0.652476, -0.647451, -0.639320}] = {-1.485553384, -1.485625809, -1.485560127, -1.485276157}
f[{-0.660606, -0.655581, -0.652476, -0.647451}] = {-1.485553384, -1.485624273, -1.485625809, -1.485560127}
f[{-0.655581, -0.652476, -0.650556, -0.647451}] = {-1.485624273, -1.485625809, -1.485610653, -1.485560127}
f[{-0.655581, -0.653662, -0.652476, -0.650556}] = {-1.485624273, -1.485629028, -1.485625809, -1.485610653}
f[{-0.655581, -0.654395, -0.653662, -0.652476}] = {-1.485624273, -1.485628666, -1.485629028, -1.485625809}
f[{-0.654395, -0.653662, -0.653209, -0.652476}] = {-1.485628666, -1.485629028, -1.485628353, -1.485625809}
f[{-0.654395, -0.653942, -0.653662, -0.653209}] = {-1.485628666, -1.485629101, -1.485629028, -1.485628353}
f[{-0.654395, -0.654115, -0.653942, -0.653662}] = {-1.485628666, -1.485629016, -1.485629101, -1.485629028}
f[{-0.654115, -0.653942, -0.653835, -0.653662}] = {-1.485629016, -1.485629101, -1.485629104, -1.485629028}
f[{-0.653942, -0.653835, -0.653769, -0.653662}] = {-1.485629101, -1.485629104, -1.485629087, -1.485629028}
f[{-0.653942, -0.653876, -0.653835, -0.653769}] = {-1.485629101, -1.485629108, -1.485629104, -1.485629087}
f[{-0.653942, -0.653901, -0.653876, -0.653835}] = {-1.485629101, -1.485629107, -1.485629108, -1.485629104}
f[{-0.653901, -0.653876, -0.653860, -0.653835}] = {-1.485629107, -1.485629108, -1.485629107, -1.485629104}
f[{-0.653901, -0.653886, -0.653876, -0.653860}] = {-1.485629107, -1.485629108, -1.485629108, -1.485629107}
f[{-0.653901, -0.653892, -0.653886, -0.653876}] = {-1.485629107, -1.485629107, -1.485629108, -1.485629108}
f[{-0.653892, -0.653886, -0.653882, -0.653876}] = {-1.485629107, -1.485629108, -1.485629108, -1.485629108}
f[{-0.653886, -0.653882, -0.653880, -0.653876}] = {-1.485629108, -1.485629108, -1.485629108, -1.485629108}
f[{-0.653886, -0.653883, -0.653882, -0.653880}] = {-1.485629108, -1.485629108, -1.485629108, -1.485629108}
f[{-0.653883, -0.653882, -0.653881, -0.653880}] = {-1.485629108, -1.485629108, -1.485629108, -1.485629108}
f[{-0.653882, -0.653881, -0.653881, -0.653880}] = {-1.485629108, -1.485629108, -1.485629108, -1.485629108}
f[{-0.653882, -0.653881, -0.653881, -0.653881}] = {-1.485629108, -1.485629108, -1.485629108, -1.485629108}
f[{-0.653881, -0.653881, -0.653881, -0.653881}] = {-1.485629108, -1.485629108, -1.485629108, -1.485629108}
```

```
{  a,   b,   c,   d}  = {-0.653881, -0.653881, -0.653881, -0.653881}
{f[a],f[b],f[c],f[d]} = {-1.485629107673, -1.485629107673, -1.485629107673, -1.485629107673}
```

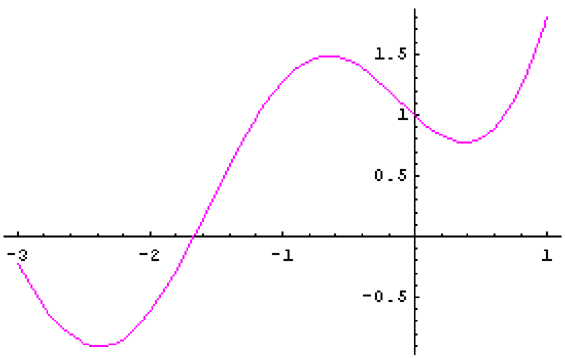
$f[x] = -e^x + \text{Sin}[2x]$

$f[x] = -e^{\wedge} + Sin[2x]$



The function is $f[x] = -e^x + Sin[2x]$
A local minimum of f[x] is:
p = -0.653881
f[p] = -1.48563

Therefore, to find the local maximum of $f[x] = e^x - Sin[2x]$ we take the negative of the local minimum of $f[x] = -e^x + Sin[2x]$.



$y = f[x] = e^x - Sin[2x]$
A local maximum of f[x] is:
p = -0.65388056
f[p] = 1.48562910767

Let us compare this answer with *Mathematica*'s subroutine FindMinimum.

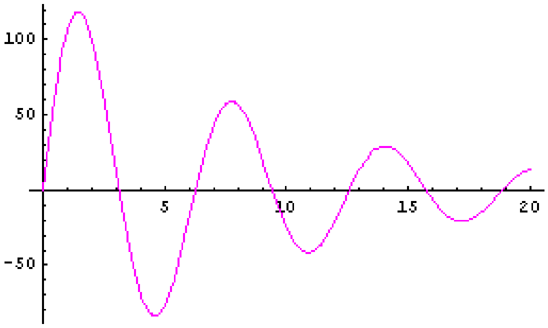
$-f[x] = -e^x + Sin[2x]$
{-1.48563, {x → -0.653881}}

$f[x] = e^x - Sin[2x]$
p = -0.653881
f[p] = 1.48563

Exercise 3. The voltage input to an electrical component is $f(x) = 140\, e^{-x/9} \sin(x)$ for $0 \le x$. The manufacturer states that the voltage is not to exceed ± 120 or else the component will "burn out." Can we expect that the component will be o.k. or will it "burn out."

Solution 3.

We must determine if $|f[x]| < 120$ for $0 \le x$.



$y = f[x] = 140\, e^{-x/9} \mathrm{Sin}[x]$

We see that the local minimum that lies in the interval $[3, 6]$ and this is the minimum for $0 \le x$.

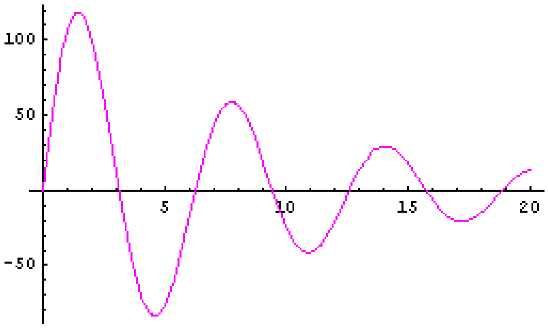
So we will search for the local minimum of $f[x]$ in the interval $[3, 6]$.

```
f[{ 3.000000,  4.145898,  4.854102,  6.000000}] = { 14.156366610, -74.525006510, -80.819856510, -20.083938010}
f[{ 4.145898,  4.854102,  5.291796,  6.000000}] = {-74.525006510, -80.819856510, -65.071058370, -20.083938010}
f[{ 4.145898,  4.583592,  4.854102,  5.291796}] = {-74.525006510, -83.432428110, -80.819856510, -65.071058370}
f[{ 4.145898,  4.416408,  4.583592,  4.854102}] = {-74.525006510, -81.979820640, -83.432428110, -80.819856510}
f[{ 4.416408,  4.583592,  4.686918,  4.854102}] = {-81.979820640, -83.432428110, -83.141947800, -80.819856510}
f[{ 4.416408,  4.519733,  4.583592,  4.686918}] = {-81.979820640, -83.160775510, -83.432428110, -83.141947800}
f[{ 4.519733,  4.583592,  4.623059,  4.686918}] = {-83.160775510, -83.432428110, -83.427163700, -83.141947800}
f[{ 4.519733,  4.559200,  4.583592,  4.623059}] = {-83.160775510, -83.369709230, -83.432428110, -83.427163700}
f[{ 4.559200,  4.583592,  4.598667,  4.623059}] = {-83.369709230, -83.432428110, -83.445947960, -83.427163700}
f[{ 4.583592,  4.598667,  4.607984,  4.623059}] = {-83.432428110, -83.445947960, -83.444694410, -83.427163700}
f[{ 4.583592,  4.592909,  4.598667,  4.607984}] = {-83.432428110, -83.443054750, -83.445947960, -83.444694410}
f[{ 4.592909,  4.598667,  4.602226,  4.607984}] = {-83.443054750, -83.445947960, -83.446334440, -83.444694410}
f[{ 4.598667,  4.602226,  4.604425,  4.607984}] = {-83.445947960, -83.446334440, -83.446038380, -83.444694410}
f[{ 4.598667,  4.600867,  4.602226,  4.604425}] = {-83.445947960, -83.446313130, -83.446334440, -83.446038380}
f[{ 4.600867,  4.602226,  4.603066,  4.604425}] = {-83.446313130, -83.446334440, -83.446269570, -83.446038380}
f[{ 4.600867,  4.601707,  4.602226,  4.603066}] = {-83.446313130, -83.446344730, -83.446334440, -83.446269570}
f[{ 4.600867,  4.601386,  4.601707,  4.602226}] = {-83.446313130, -83.446339700, -83.446344730, -83.446334440}
f[{ 4.601386,  4.601707,  4.601905,  4.602226}] = {-83.446339700, -83.446344730, -83.446343490, -83.446334440}
f[{ 4.601386,  4.601584,  4.601707,  4.601905}] = {-83.446339700, -83.446343830, -83.446344730, -83.446343490}
f[{ 4.601584,  4.601707,  4.601782,  4.601905}] = {-83.446343830, -83.446344730, -83.446344650, -83.446343490}
f[{ 4.601584,  4.601660,  4.601707,  4.601782}] = {-83.446343830, -83.446344530, -83.446344730, -83.446344650}
f[{ 4.601660,  4.601707,  4.601736,  4.601782}] = {-83.446344530, -83.446344730, -83.446344750, -83.446344650}
f[{ 4.601707,  4.601736,  4.601753,  4.601782}] = {-83.446344730, -83.446344750, -83.446344730, -83.446344650}
f[{ 4.601707,  4.601725,  4.601736,  4.601753}] = {-83.446344730, -83.446344750, -83.446344750, -83.446344730}
f[{ 4.601725,  4.601736,  4.601742,  4.601753}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344730}
f[{ 4.601725,  4.601731,  4.601736,  4.601742}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344750}
f[{ 4.601725,  4.601729,  4.601731,  4.601736}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344750}
f[{ 4.601729,  4.601731,  4.601733,  4.601736}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344750}
f[{ 4.601729,  4.601730,  4.601731,  4.601733}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344750}
f[{ 4.601730,  4.601731,  4.601732,  4.601733}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344750}
f[{ 4.601731,  4.601732,  4.601732,  4.601733}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344750}
f[{ 4.601731,  4.601732,  4.601732,  4.601732}] = {-83.446344750, -83.446344750, -83.446344750, -83.446344750}

{ a,  b,  c,  d}  = { 4.601731,  4.601732,  4.601732,  4.601732}
```

{ a, b, c, d} = { 4.601731, 4.601732, 4.601732, 4.601732}

{f[a],f[b],f[c],f[d]} = {-83.446344753440, -83.446344753450, -83.446344753450, -83.446344753430}

$f[x] = 140\, e^{-x/9}\, \mathrm{Sin}[x]$



The function is $f[x] = 140\, e^{-x/9}\, \mathrm{Sin}[x]$

A local minimum of f[x] is:

p = 4.60173

f[p] = -83.4463

Combining this with the result from Exercises 4 we obtain

$-83.4463 \le f(t) \le 118.3054$ for $0 \le t$.

Therefore, the component will be o.k.