3. Runge Kutta Method for O.D.E.'s

Theorem (Runge-Kutta Method of order 4) Assume that $f(t,y)$ is continuous and satisfies a Lipschits condition in the variable y, and consider the I. V. P. (initial value problem)

$y' = f(t, y)$ with $y(a) = t_0 = \alpha$, over the interval $a \le t \le b$.

The Runge-Kutta method uses the formulas $t_{k+1} = t_k + h$, and

$$y_{j+1} = y_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{for } k = 0, 1, 2, \ldots, m-1$$

where

$$k_1 = h f(t_j, y_j)$$
$$k_2 = h f\left(t_j + \frac{h}{2}, y_j + \frac{k_1}{2}\right)$$
$$k_3 = h f\left(t_j + \frac{h}{2}, y_j + \frac{k_2}{2}\right)$$
$$k_4 = h f(t_j + h, y_j + k_3)$$

as an approximate solution to the differential equation using the discrete set of points $\{(t_k, y_k)\}_{k=0}^{m}$.

Theorem (Precision of the Runge-Kutta Method of Order 4) Assume that $y = y(t)$ is the solution to the I.V.P. $y' = f(t, y)$ with $y(t_0) = y_0$. If $y(t) \in C^2[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{m}$ is the sequence of approximations generated by the Runge-Kutta method of order 4, then at each step, the local truncation error is of the order $O(h^5)$, and the overall global truncation error e_k is of the order

$$|e_k| = |y(t_k) - y_k| = O(h^4), \text{ for } k = 1, 2, \ldots, m.$$

The error at the right end of the interval is called the final global error

$$E(y(b), h) = |y(b) - y_m| = O(h^4).$$

Example 1. Solve the I.V.P. $y' = 1 - t y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

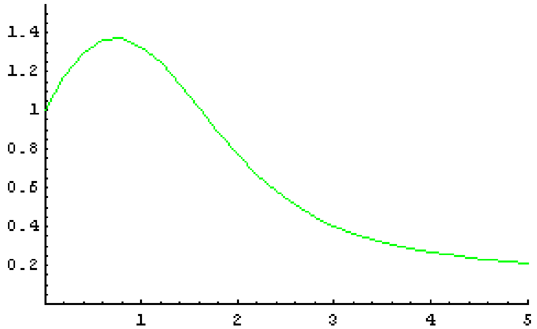
Example 2. Solve $y' = 30 - 5y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 2.

Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

Example 1. Solve the I.V.P. $y' = 1 - t\,y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

Compute the Runge-Kutta solution based on 25 subintervals and plot the results.

Find numerical solutions to the D.E.
$y' = 1 - t\,y$

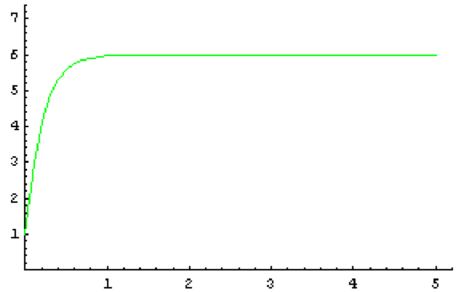


The Runge-Kutta solution for $y' = 1 - t\,y$
Using $n = 26$ points.
{{0., 1.}, {0.2, 1.17755}, {0.4, 1.30245}, {0.6, 1.36819}, {0.8, 1.37546}, {1., 1.3313}, {1.2, 1.24732}, {1.4, 1.13727}, {1.6, 1.01473},
{1.8, 0.89124}, {2., 0.77533}, {2.2, 0.672211}, {2.4, 0.584154}, {2.6, 0.511195}, {2.8, 0.451948}, {3., 0.404325}, {3.2, 0.366085}, {3.4, 0.335157},
{3.6, 0.309811}, {3.8, 0.288687}, {4., 0.270764}, {4.2, 0.255297}, {4.4, 0.241752}, {4.6, 0.229741}, {4.8, 0.218982}, {5., 0.209267}}

The final value is $y(5) = y_{26} = 0.209267$

Example 2. Solve $y' = 30 - 5y$ with $y(0) = 1$ over $0 \le t \le 5$.

Solution 2.



The Runge-Kutta solution for $y' = 30 - 5y$

Using $n = 51$ points.

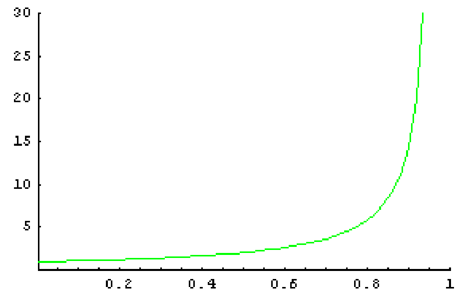
{{0., 1}, {0.1, 2.96615}, {0.2, 4.15915}, {0.3, 4.88302}, {0.4, 5.32225}, {0.5, 5.58876}, {0.6, 5.75047}, {0.7, 5.84859}, {0.8, 5.90813}, {0.9, 5.94426}, {1., 5.96618},
{1.1, 5.97948}, {1.2, 5.98755}, {1.3, 5.99244}, {1.4, 5.99542}, {1.5, 5.99722}, {1.6, 5.99831}, {1.7, 5.99898}, {1.8, 5.99938}, {1.9, 5.99962}, {2., 5.99977}, {2.1, 5.99986},
{2.2, 5.99992}, {2.3, 5.99995}, {2.4, 5.99997}, {2.5, 5.99998}, {2.6, 5.99999}, {2.7, 5.99999}, {2.8, 6.}, {2.9, 6.}, {3., 6.}, {3.1, 6.}, {3.2, 6.}, {3.3, 6.}, {3.4, 6.},
{3.5, 6.}, {3.6, 6.}, {3.7, 6.}, {3.8, 6.}, {3.9, 6.}, {4., 6.}, {4.1, 6.}, {4.2, 6.}, {4.3, 6.}, {4.4, 6.}, {4.5, 6.}, {4.6, 6.}, {4.7, 6.}, {4.8, 6.}, {4.9, 6.}, {5., 6.}}

The final value is $y(5) = y_{51} = 6.$

Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

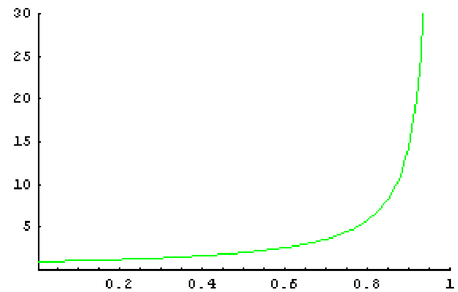
Compute the Runge-Kutta solution based on 50 subintervals and plot the results.

Find numerical solutions to the D.E.
$y' = t^2 + y^2$



The Runge-Kutta solution for $y' = t^2 + y^2$
Using n = 51 points.
{{0., 1.}, {0.02, 1.02041}, {0.04, 1.04169}, {0.06, 1.0639}, {0.08, 1.08713}, {0.1, 1.11146}, {0.12, 1.13698}, {0.14, 1.16378}, {0.16, 1.19198}, {0.18, 1.22168}, {0.2, 1.25302}, {0.22, 1.28613}, {0.24, 1.32117}, {0.26, 1.35831}, {0.28, 1.39774}, {0.3, 1.43967}, {0.32, 1.48433}, {0.34, 1.53199}, {0.36, 1.58294}, {0.38, 1.63752}, {0.4, 1.69611}, {0.42, 1.75915}, {0.44, 1.82713}, {0.46, 1.90063}, {0.48, 1.98033}, {0.5, 2.067}, {0.52, 2.16156}, {0.54, 2.2651}, {0.56, 2.37893}, {0.58, 2.50459}, {0.6, 2.644}, {0.62, 2.79948}, {0.64, 2.97393}, {0.66, 3.171}, {0.68, 3.39531}, {0.7, 3.6529}, {0.72, 3.9517}, {0.74, 4.30241}, {0.76, 4.71982}, {0.78, 5.22491}, {0.8, 5.8486}, {0.82, 6.63827}, {0.84, 7.67048}, {0.86, 9.07759}, {0.88, 11.1098}, {0.9, 14.3037}, {0.92, 20.0556}, {0.94, 33.4719}, {0.96, 96.9561}, {0.98, 65146.}, {1., 1.4206×10^{47}}}
The final value is $y(5) = y_{51} = 1.4206 \times 10^{47}$

Compute the Runge-Kutta solution based on 100 subintervals and plot the results.
Observe that one fewer subinterval is computed for this case.

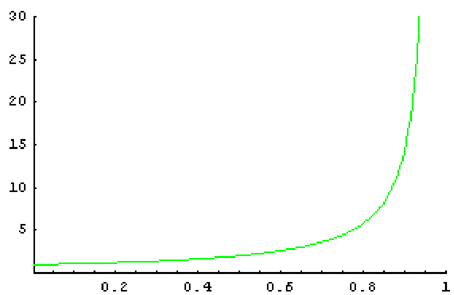


The Runge-Kutta solution for $y' = t^2 + y^2$
Using n = 100 points.

{{0., 1.}, {0.01, 1.0101}, {0.02, 1.02041}, {0.03, 1.03094}, {0.04, 1.04169}, {0.05, 1.05267}, {0.06, 1.0639}, {0.07, 1.07539}, {0.08, 1.08713}, {0.09, 1.09916},
{0.1, 1.11146}, {0.11, 1.12407}, {0.12, 1.13698}, {0.13, 1.15021}, {0.14, 1.16378}, {0.15, 1.1777}, {0.16, 1.19198}, {0.17, 1.20663}, {0.18, 1.22168},
{0.19, 1.23714}, {0.2, 1.25302}, {0.21, 1.26934}, {0.22, 1.28613}, {0.23, 1.3034}, {0.24, 1.32117}, {0.25, 1.33947}, {0.26, 1.35831}, {0.27, 1.37773},
{0.28, 1.39774}, {0.29, 1.41838}, {0.3, 1.43967}, {0.31, 1.46164}, {0.32, 1.48433}, {0.33, 1.50777}, {0.34, 1.53199}, {0.35, 1.55703}, {0.36, 1.58294},
{0.37, 1.60975}, {0.38, 1.63752}, {0.39, 1.66629}, {0.4, 1.69611}, {0.41, 1.72704}, {0.42, 1.75915}, {0.43, 1.79249}, {0.44, 1.82713}, {0.45, 1.86315},
{0.46, 1.90063}, {0.47, 1.93966}, {0.48, 1.98033}, {0.49, 2.02274}, {0.5, 2.067}, {0.51, 2.11323}, {0.52, 2.16156}, {0.53, 2.21213}, {0.54, 2.2651},
{0.55, 2.32064}, {0.56, 2.37893}, {0.57, 2.44017}, {0.58, 2.50459}, {0.59, 2.57244}, {0.6, 2.644}, {0.61, 2.71957}, {0.62, 2.79948}, {0.63, 2.88413},
{0.64, 2.97393}, {0.65, 3.06938}, {0.66, 3.171}, {0.67, 3.27941}, {0.68, 3.39532}, {0.69, 3.51951}, {0.7, 3.6529}, {0.71, 3.79656}, {0.72, 3.9517},
{0.73, 4.11976}, {0.74, 4.30241}, {0.75, 4.50165}, {0.76, 4.71982}, {0.77, 4.95977}, {0.78, 5.22492}, {0.79, 5.51948}, {0.8, 5.84862}, {0.81, 6.21882},
{0.82, 6.63829}, {0.83, 7.11759}, {0.84, 7.67053}, {0.85, 8.31553}, {0.86, 9.07771}, {0.87, 9.99228}, {0.88, 11.1101}, {0.89, 12.5076}, {0.9, 14.3048},
{0.91, 16.7022}, {0.92, 20.0612}, {0.93, 25.1059}, {0.94, 33.531}, {0.95, 50.4364}, {0.96, 101.16}, {0.97, 899.783}, {0.98, 3.8528×10^{13}}, {0.99, 9.59206×10^{182}}}

The final value is $y(0.99) = y_{100} = 9.59206 \times 10^{182}$

Compute the Runge-Kutta solution based on 200 subintervals and plot the results.
Observe that four fewer subintervals are computed for this case.



The Runge-Kutta solution for $y' = t^2 + y^2$
Using n = 197 points.

{{0., 1.}, {0.005, 1.00503}, {0.01, 1.0101}, {0.015, 1.01523}, {0.02, 1.02041}, {0.025, 1.02565}, {0.03, 1.03094}, {0.035, 1.03628}, {0.04, 1.04169},
{0.045, 1.04715}, {0.05, 1.05267}, {0.055, 1.05826}, {0.06, 1.0639}, {0.065, 1.06961}, {0.07, 1.07539}, {0.075, 1.08123}, {0.08, 1.08713}, {0.085, 1.09311},
{0.09, 1.09916}, {0.095, 1.10527}, {0.1, 1.11146}, {0.105, 1.11773}, {0.11, 1.12407}, {0.115, 1.13048}, {0.12, 1.13698}, {0.125, 1.14356}, {0.13, 1.15021},
{0.135, 1.15696}, {0.14, 1.16378}, {0.145, 1.1707}, {0.15, 1.1777}, {0.155, 1.18479}, {0.16, 1.19198}, {0.165, 1.19926}, {0.17, 1.20663}, {0.175, 1.21411},
{0.18, 1.22168}, {0.185, 1.22935}, {0.19, 1.23714}, {0.195, 1.24502}, {0.2, 1.25302}, {0.205, 1.26112}, {0.21, 1.26934}, {0.215, 1.27768}, {0.22, 1.28613},
{0.225, 1.2947}, {0.23, 1.3034}, {0.235, 1.31222}, {0.24, 1.32117}, {0.245, 1.33025}, {0.25, 1.33947}, {0.255, 1.34882}, {0.26, 1.35831}, {0.265, 1.36795},
{0.27, 1.37773}, {0.275, 1.38766}, {0.28, 1.39774}, {0.285, 1.40798}, {0.29, 1.41838}, {0.295, 1.42894}, {0.3, 1.43967}, {0.305, 1.45057}, {0.31, 1.46164},
{0.315, 1.4729}, {0.32, 1.48433}, {0.325, 1.49595}, {0.33, 1.50777}, {0.335, 1.51978}, {0.34, 1.53199}, {0.345, 1.54441}, {0.35, 1.55703}, {0.355, 1.56988},
{0.36, 1.58294}, {0.365, 1.59623}, {0.37, 1.60975}, {0.375, 1.62352}, {0.38, 1.63752}, {0.385, 1.65178}, {0.39, 1.66629}, {0.395, 1.68106}, {0.4, 1.69611},
{0.405, 1.71143}, {0.41, 1.72704}, {0.415, 1.74295}, {0.42, 1.75915}, {0.425, 1.77566}, {0.43, 1.79249}, {0.435, 1.80964}, {0.44, 1.82713}, {0.445, 1.84496},
{0.45, 1.86315}, {0.455, 1.88171}, {0.46, 1.90063}, {0.465, 1.91995}, {0.47, 1.93966}, {0.475, 1.95978}, {0.48, 1.98033}, {0.485, 2.00131}, {0.49, 2.02274},
{0.495, 2.04463}, {0.5, 2.067}, {0.505, 2.08986}, {0.51, 2.11323}, {0.515, 2.13713}, {0.52, 2.16156}, {0.525, 2.18656}, {0.53, 2.21213}, {0.535, 2.23831},
{0.54, 2.2651}, {0.545, 2.29254}, {0.55, 2.32064}, {0.555, 2.34943}, {0.56, 2.37893}, {0.565, 2.40916}, {0.57, 2.44017}, {0.575, 2.47197}, {0.58, 2.50459},
{0.585, 2.53807}, {0.59, 2.57244}, {0.595, 2.60774}, {0.6, 2.644}, {0.605, 2.68126}, {0.61, 2.71957}, {0.615, 2.75896}, {0.62, 2.79948}, {0.625, 2.84119},
{0.63, 2.88413}, {0.635, 2.92836}, {0.64, 2.97393}, {0.645, 3.02092}, {0.65, 3.06938}, {0.655, 3.11938}, {0.66, 3.171}, {0.665, 3.22431}, {0.67, 3.27941},
{0.675, 3.33638}, {0.68, 3.39532}, {0.685, 3.45632}, {0.69, 3.51951}, {0.695, 3.58499}, {0.7, 3.6529}, {0.705, 3.72338}, {0.71, 3.79656}, {0.715, 3.87261},
{0.72, 3.9517}, {0.725, 4.03402}, {0.73, 4.11976}, {0.735, 4.20915}, {0.74, 4.30241}, {0.745, 4.39982}, {0.75, 4.50165}, {0.755, 4.6082}, {0.76, 4.71982},
{0.765, 4.83687}, {0.77, 4.95977}, {0.775, 5.08895}, {0.78, 5.22492}, {0.785, 5.36823}, {0.79, 5.51948}, {0.795, 5.67935}, {0.8, 5.84862}, {0.805, 6.02812},
{0.81, 6.21882}, {0.815, 6.4218}, {0.82, 6.63829}, {0.825, 6.86969}, {0.83, 7.11759}, {0.835, 7.38384}, {0.84, 7.67053}, {0.845, 7.98014}, {0.85, 8.31553},
{0.855, 8.68007}, {0.86, 9.07772}, {0.865, 9.51323}, {0.87, 9.99229}, {0.875, 10.5218}, {0.88, 11.1101}, {0.885, 11.7677}, {0.89, 12.5076}, {0.895, 13.3462},
{0.9, 14.3049}, {0.905, 15.4112}, {0.91, 16.7024}, {0.915, 18.2288}, {0.92, 20.0616}, {0.925, 22.303}, {0.93, 25.1071}, {0.935, 28.7163}, {0.94, 33.5358},
{0.945, 40.2969}, {0.95, 50.4692}, {0.955, 67.5037}, {0.96, 101.853}, {0.965, 206.209}, {0.97, 1983.65}, {0.975, 3.17353×10^{14}}, {0.98, 1.31439×10^{193}}}

The final value is $y(0.98) = y_{197} = 1.31439 \times 10^{193}$