5. PA = LU Factorization with Pivoting

Definition (LU-Factorization). The nonsingular matrix **A** has an LU-factorization if it can be expressed as the product of a lower-triangular matrix **L** and an upper triangular matrix **U**:

$$\mathbf{A} = \mathbf{LU}.$$

When this is possible we say that **A** has an **LU**-decomposition. It turns out that this factorization (when it exists) is not unique. If **L** has 1's on it's diagonal, then it is called a Doolittle factorization. If **U** has 1's on its diagonal, then it is called a Crout factorization. When $\mathbf{L} = \mathbf{U}^{\mathrm{T}}$, it is called a Cholesky decomposition. In this module we will develop an algorithm that produces a Doolittle factorization.

Theorem (LU Factorization with NO pivoting). If row interchanges are not needed to solve the linear system **AX** = **B**, then **A** has the **LU** factorization (illustrated with 4×4 matrices).

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ l_{2,1} & 1 & 0 & 0 \\ l_{3,1} & l_{3,2} & 1 & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & 1 \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} \\ 0 & u_{2,2} & u_{2,3} & u_{2,4} \\ 0 & 0 & u_{3,3} & u_{3,4} \\ 0 & 0 & 0 & u_{4,4} \end{pmatrix}.$$

Remark 1. This is not a linear system.
Remark 2. The easy solution uses row vectors and is a modification of limited Gauss Jordan elimination.
Remark 3. A sufficient condition for the factorization to exist is that all principal minors of **A** are nonsingular.

Example 1. Given $\mathbf{A} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$. Find matrices **L** and **U** so that **LU** = **A**.

Solution 1.

Example 2. Given $\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$. Can **A** be factored **A** = **LU**?

Solution 2.

Example 1. Given $A = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$. Find matrices **L** and **U** so that **LU** = **A**.

Solution 1. Construct matrices **L** and **U** so that **LU** = **A**.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 3 & \frac{7}{2} \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & \frac{6}{5} & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & \frac{6}{5} & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix}$$

Example 2. Given $A = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$. Can **A** be factored **A** = **LU**?

Solution 2.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 6 \\ 0 & 0 & -25 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 6 \\ 0 & 0 & -25 \\ 0 & 7 & 17 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$$

.....Indeterminate expression 0 encountered.