2. Lagrange Polynomials

Background.

We have seen how to expand a function $f(x)$ in a Maclaurin polynomial about $x_0 = 0$ involving the powers x^k and a Taylor polynomial about $x_0 \neq 0$ involving the powers $(x - x_0)^k$. The Lagrange polynomial of degree n passes through the $n+1$ points (x_k, y_k) for $k = 0, 1, \ldots, n$ and were investigated by the mathematician Joseph-Louis Lagrange (1736-1813).

Theorem (Lagrange Polynomial). Assume that $f \in C^{n+1}[a, b]$ and $x_k \in [a, b]$ for $k = 0, 1, \ldots, n$ are distinct values. Then

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ is a polynomial that can be used to approximate $f(x)$,

$$P_n(x) = \sum_{k=0}^{n} y_k \frac{(x - x_0) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_n)}{(x_k - x_0) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_n)}$$

and we write

$$f(x) \approx P_n(x).$$

The Lagrange polynomial goes through the $n+1$ points $\{(x_k, y_k)\}_{k=0}^{n}$, i.e.

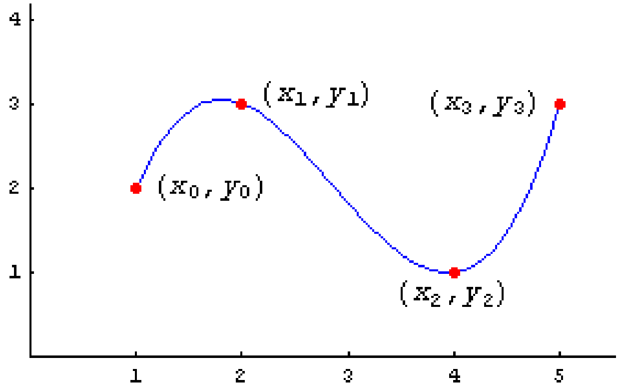
$$P_n(x_k) = f(x_k) \quad \text{for} \quad k = 0, 1, \ldots, n.$$

The remainder term $R_n(x)$ has the form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1})(x - x_n),$$

for some value $c = c(x)$ that lies in the interval $[a, b]$.

The cubic curve in the figure below illustrates a Lagrange polynomial of degree n = 3, which passes through the four points (x_k, y_k) for $k = 0, 1, 2, 3$.



$$p[x] = \frac{(x - x_1)(x - x_2)(x - x_3)\, y_0}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \frac{(x - x_0)(x - x_2)(x - x_3)\, y_1}{(-x_0 + x_1)(x_1 - x_2)(x_1 - x_3)} + \frac{(x - x_0)(x - x_1)(x - x_3)\, y_2}{(-x_0 + x_2)(-x_1 + x_2)(x_2 - x_3)} + \frac{(x - x_0)(x - x_1)(x - x_2)\, y_3}{(-x_0 + x_3)(-x_1 + x_3)(-x_2 + x_3)}$$

$p[x_0] = y_0$

$p[x_1] = y_1$

$p[x_2] = y_2$

$p[x_3] = y_3$

Theorem. (Error Bounds for Lagrange Interpolation, Equally Spaced Nodes) Assume that $f(x)$ defined on $[a, b]$, which contains the equally spaced nodes $x_k = x_0 + kh$. Additionally, assume that $f(x)$ and the derivatives of $f(x)$ up to the order $n+1$ are continuous and bounded on the special subintervals $[x_0, x_1]$, $[x_0, x_2]$, $[x_0, x_3]$, $[x_0, x_4]$, and $[x_0, x_5]$, respectively; that is,

$$|f^{(n+1)}(x)| \le M_{n+1} \quad \text{for } x_0 < x < x_n,$$

for $n = 1, 2, 3, 4, 5$. The error terms corresponding to these three cases have the following useful bounds on their magnitude

(i). $|R_1(x)| \le \frac{M_2}{8} h^2$ is valid for $x \in [x_0, x_1]$,

(ii). $|R_2(x)| \le \frac{M_3}{9\sqrt{3}} h^3$ is valid for $x \in [x_0, x_2]$,

(iii). $|R_3(x)| \le \frac{M_4}{24} h^4$ is valid for $x \in [x_0, x_3]$,

(iv). $|R_4(x)| \le \frac{\sqrt{4750 + 290\sqrt{145}}}{3000} M_5 h^5$ is valid for $x \in [x_0, x_4]$,

(v). $| R_5(x) | \le \frac{10 + 7\sqrt{7}}{1215} M_6 h^6$ is valid for $x \in [x_0, x_5]$.

Algorithm (Lagrange Polynomial). To construct the Lagrange polynomial

$$P(x) = \sum_{k=0}^{n} y_k L_{n,k}(x)$$

of degree n, based on the $n+1$ points (x_k, y_k) for $k = 0, 1, \ldots, n$. The Lagrange coefficient polynomials $L_{n,k}(x)$ for degree n are:

$$L_{n,k}(x) = \frac{(x - x_0)\ \ldots\ (x - x_{k-1})\ (x - x_{k+1})\ \ldots\ (x - x_n)}{(x_k - x_0)\ \ldots\ (x_k - x_{k-1})\ (x_k - x_{k+1})\ \ldots\ (x_k - x_n)}$$ for $k = 0, 1, \ldots, n$.

Example 1. Find the Lagrange polynomial approximation for $f[x] = \sqrt{x}$, on the interval $[0, 8]$.
Solution 1.

Example 2. Find the Lagrange polynomial approximation for $f[x] = \frac{1}{1 + 10x^2}$, on the interval $[-1, 1]$.

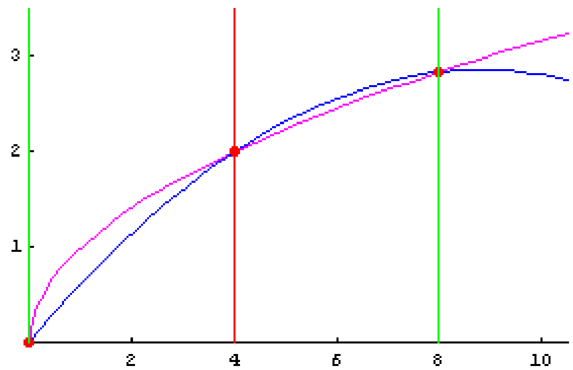
Solution 2.

Example 3. Find the Lagrange polynomial approximation for $f[x] = Log[x]$, on the interval $[0.02, 2]$.
Solution 3.

Example 4. Error Analysis. Investigate the error for the Lagrange polynomial approximations of degree n = 4 and 5 for the function $f[x] = Cos[x]$ over the interval $[0, 1]$.
Solution 4 (a).
Solution 4 (b).

Example 1. Find the Lagrange polynomial approximation for $f[x] = \sqrt{x}$, on the interval $[0, 8]$.

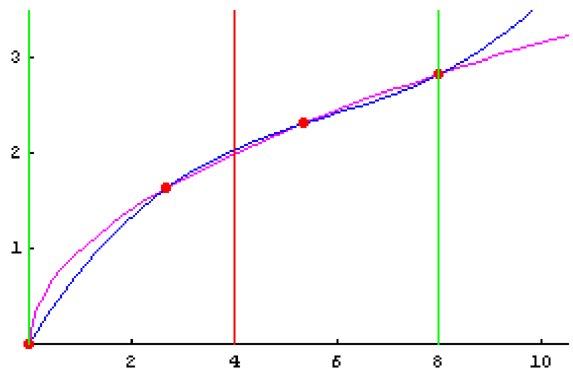
Solution 1.



$f[x] = \sqrt{x}$

$P[x] = 0. (-8. + x) (-4. + x) - 0.125 (-8. + x) (0. + x) + 0.0883883 (-4. + x) (0. + x)$

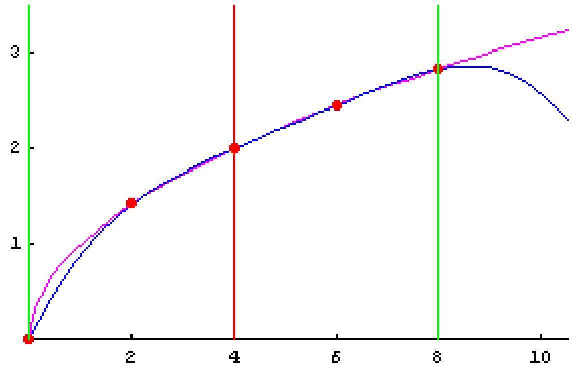
$P[x] = 0.646447 x - 0.0366117 x^2$



$f[x] = \sqrt{x}$

$P[x] = 0. (-8. + x) (-5.33333 + x) (-2.66667 + x) + 0.0430574 (-8. + x) (-5.33333 + x) (0. + x) - 0.0608924 (-8. + x) (-2.66667 + x) (0. + x) + 0.0248592 (-5.33333 + x) (-2.66667 + x) (0. + x)$

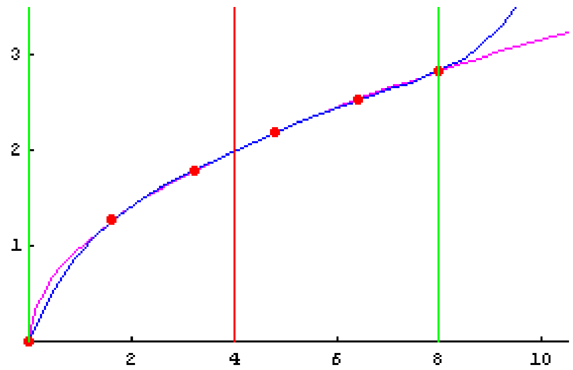
$P[x] = 0.891633 x - 0.123454 x^2 + 0.00702425 x^3$



$f[x] = \sqrt{x}$

$P[x] = 0.\ (-8.+x)\ (-6.+x)\ (-4.+x)\ (-2.+x) - 0.0147314\ (-8.+x)\ (-6.+x)\ (-4.+x)\ (0.+x) + 0.03125\ (-8.+x)\ (-6.+x)\ (-2.+x)\ (0.+x) - 0.0255155\ (-8.+x)\ (-4.+x)\ (-2.+x)\ (0.+x) + 0.0073657\ (-6.+x)\ (-4.+x)\ (-2.+x)\ (0.+x)$

$P[x] = 1.10787\,x - 0.261843\,x^2 + 0.0339939\,x^3 - 0.00163121\,x^4$



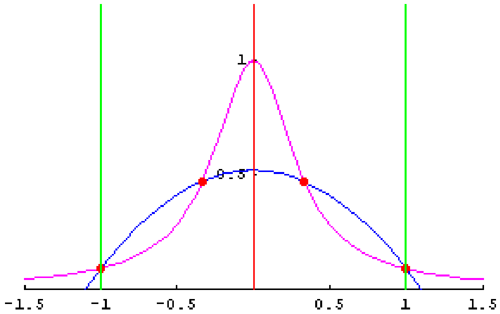
$f[x] = \sqrt{x}$

$P[x] = 0.\ (-8.+x)\ (-6.4+x)\ (-4.8+x)\ (-3.2+x)\ (-1.6+x) + 0.0050263\ (-8.+x)\ (-6.4+x)\ (-4.8+x)\ (-3.2+x)\ (0.+x) - 0.0142165\ (-8.+x)\ (-6.4+x)\ (-4.8+x)\ (-1.6+x)\ (0.+x) + 0.0174116\ (-8.+x)\ (-6.4+x)\ (-3.2+x)\ (-1.6+x)\ (0.+x) - 0.0100526\ (-8.+x)\ (-4.8+x)\ (-3.2+x)\ (-1.6+x)\ (0.+x) + 0.00224783\ (-6.4+x)\ (-4.8+x)\ (-3.2+x)\ (-1.6+x)\ (0.+x)$

$P[x] = 1.30416\,x - 0.451261\,x^2 + 0.0967142\,x^3 - 0.0102279\,x^4 + 0.000416621\,x^5$

Example 2. Find the Lagrange polynomial approximation for $f[x] = \frac{1}{1 + 10x^2}$, on the interval $[-1, 1]$.

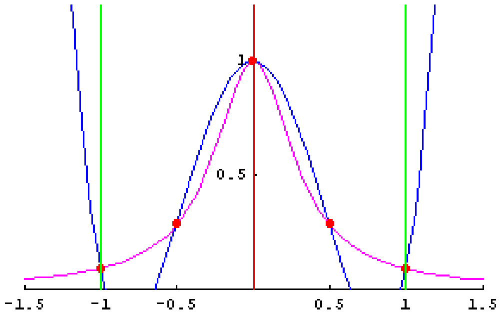
Solution 2.



$$f[x] = \frac{1}{1 + 10x^2}$$

$$P[x] = -0.0511364(-1. + x)(-0.333333 + x)(0.333333 + x) + 0.799342(-1. + x)(-0.333333 + x)(1. + x) - 0.799342(-1. + x)(0.333333 + x)(1. + x) + 0.0511364(-0.333333 + x)(0.333333 + x)(1. + x)$$

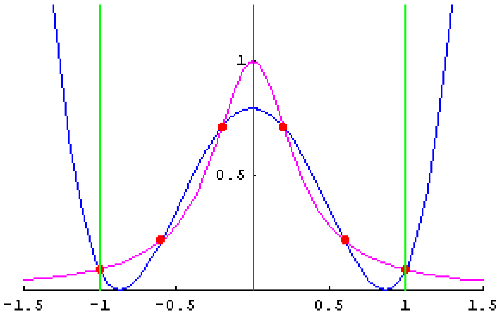
$$P[x] = 0.521531 - 0.430622x^2$$



$$f[x] = \frac{1}{1 + 10x^2}$$

$$P[x] = 0.0606061(-1. + x)(-0.5 + x)(0. + x)(0.5 + x) - 0.761905(-1. + x)(-0.5 + x)(0. + x)(1. + x) + 4.(-1. + x)(-0.5 + x)(0.5 + x)(1. + x) - 0.761905(-1. + x)(0. + x)(0.5 + x)(1. + x) + 0.0606061(-0.5 + x)(0. + x)(0.5 + x)(1. + x)$$

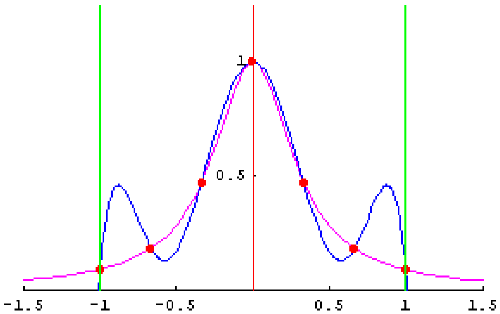
$$P[x] = 1. - 3.50649x^2 + 2.5974x^4$$



$$f[x] = \frac{1}{1 + 10 x^2}$$

$$P[x] = -0.073982 (-1. + x) (-0.6 + x) (-0.2 + x) (0.2 + x) (0.6 + x) + 0.884567 (-1. + x) (-0.6 + x) (-0.2 + x) (0.2 + x) (1. + x) - 5.81287 (-1. + x) (-0.6 + x) (-0.2 + x) (0.6 + x) (1. + x) + 5.81287 (-1. + x) (-0.6 + x) (0.2 + x) (0.6 + x) (1. + x) - 0.884567 (-1. + x) (-0.2 + x) (0.2 + x) (0.6 + x) (1. + x) + 0.073982 (-0.6 + x) (-0.2 + x) (0.2 + x) (0.6 + x) (1. + x)$$

$$P[x] = 0.796725 - 2.11745 x^2 + 1.41163 x^4$$



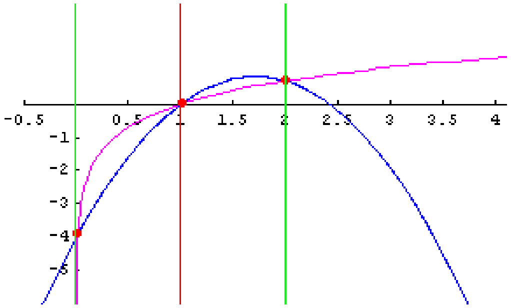
$$f[x] = \frac{1}{1 + 10 x^2}$$

$$P[x] = 0.0920455 (-1. + x) (-0.666667 + x) (-0.333333 + x) (0. + x) (0.333333 + x) (0.666667 + x) - 1.11582 (-1. + x) (-0.666667 + x) (-0.333333 + x) (0. + x) (0.333333 + x) (1. + x) + 7.19408 (-1. + x) (-0.666667 + x) (-0.333333 + x) (0. + x) (0.666667 + x) (1. + x) - 20.25 (-1. + x) (-0.666667 + x) (-0.333333 + x) (0.333333 + x) (0.666667 + x) (1. + x) + 7.19408 (-1. + x) (-0.666667 + x) (0. + x) (0.333333 + x) (0.666667 + x) (1. + x) - 1.11582 (-1. + x) (-0.333333 + x) (0. + x) (0.333333 + x) (0.666667 + x) (1. + x) + 0.0920455 (-0.666667 + x) (-0.333333 + x) (0. + x) (0.333333 + x) (0.666667 + x) (1. + x)$$

$$P[x] = 1. - 6.09413 x^2 + 13.0944 x^4 - 7.90938 x^6$$

Example 3. Find the Lagrange polynomial approximation for `f[x] = Log[x]`, on the interval `[0.02, 2]`.

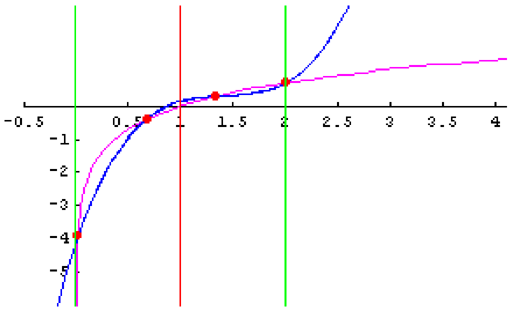
Solution 3.



f[x] = Log[x]

P[x] = -1.99573 (-2. + x) (-1.01 + x) - 0.0101524 (-2. + x) (-0.02 + x) + 0.35361 (-1.01 + x) (-0.02 + x)

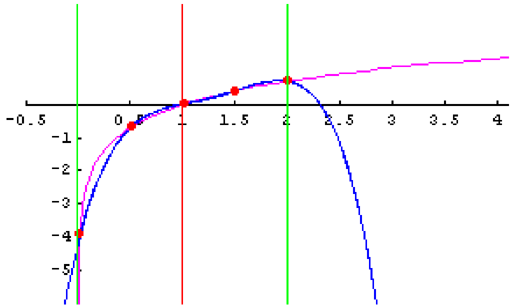
P[x] = $-4.02463 + 5.66343\,x - 1.65227\,x^2$



f[x] = Log[x]

P[x] = 2.26787 (-2. + x) (-1.34 + x) (-0.68 + x) -
0.670727 (-2. + x) (-1.34 + x) (-0.02 + x) - 0.508998 (-2. + x) (-0.68 + x) (-0.02 + x) + 0.40183 (-1.34 + x) (-0.68 + x) (-0.02 + x)

P[x] = $-4.0905 + 9.04919\,x - 6.30864\,x^2 + 1.48998\,x^3$

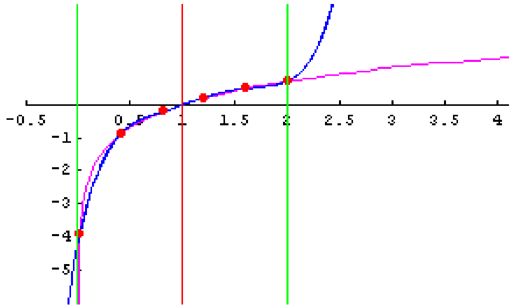


f[x] = Log[x]

P[x] =

$-2.715\,(-2.+x)\,(-1.505+x)\,(-1.01+x)\,(-0.515+x)+1.84216\,(-2.+x)\,(-1.505+x)\,(-1.01+x)\,(-0.02+x)+0.041434\,(-2.+x)\,(-1.505+x)\,(-0.515+x)\,(-0.02+x)-1.13483\,(-2.+x)\,(-1.01+x)\,(-0.515+x)\,(-0.02+x)+0.481054\,(-1.505+x)\,(-1.01+x)\,(-0.515+x)\,(-0.02+x)$

P[x] = $-4.15353+12.3603\,x-14.409\,x^2+7.69062\,x^3-1.48518\,x^4$



f[x] = Log[x]

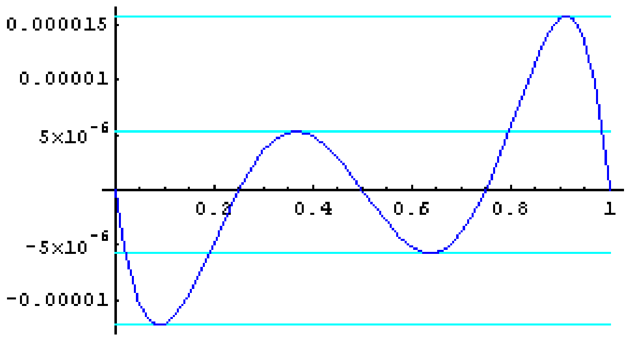
P[x] = $3.34768\,(-2.+x)\,(-1.604+x)\,(-1.208+x)\,(-0.812+x)\,(-0.416+x)-3.75273\,(-2.+x)\,(-1.604+x)\,(-1.208+x)\,(-0.812+x)\,(-0.02+x)+1.78212\,(-2.+x)\,(-1.604+x)\,(-1.208+x)\,(-0.416+x)\,(-0.02+x)+1.61706\,(-2.+x)\,(-1.604+x)\,(-0.812+x)\,(-0.416+x)\,(-0.02+x)-2.02169\,(-2.+x)\,(-1.208+x)\,(-0.812+x)\,(-0.416+x)\,(-0.02+x)+0.593155\,(-1.604+x)\,(-1.208+x)\,(-0.812+x)\,(-0.416+x)\,(-0.02+x)$

P[x] = $-4.21332+15.5778\,x-26.0876\,x^2+22.7757\,x^3-9.63773\,x^4+1.5656\,x^5$

Example 3 (a). Error Analysis. Investigate the error for the Lagrange polynomial approximations of degree n = 4 for the function f[x] = Cos[x] over the interval [0, 1] .

Solution 3 (a).

Investigate the error for the Lagrange interpolation polynomial $p_4[x]$, of degree n = 4.



f[x] = Cos[x]

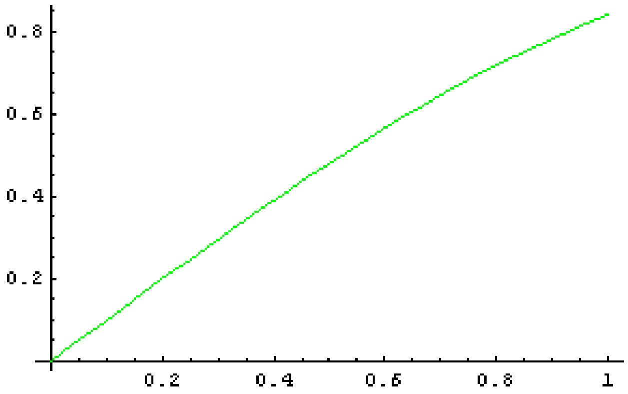
$p_4[x] = 10.6667(-1.+x)(-0.75+x)(-0.5+x)(-0.25+x) - 41.3403(-1.+x)(-0.75+x)(-0.5+x)x + 56.1653(-1.+x)(-0.75+x)(-0.25+x)x - 31.2187(-1.+x)(-0.5+x)(-0.25+x)x + 5.76322(-0.75+x)(-0.5+x)(-0.25+x)x$

The interval for interpolation is [0.0,1.0].

Graph of the error $e_4[x] = f[x] - p_4[x]$

Looking at the above graph we make the following estimate for the error: $|e_4[x]| \le 0.0000157713$

Use formula (iv). $|R_4(x)| \le \frac{\sqrt{4750 + 290\sqrt{145}}}{3000} M_5 h^5$ is valid for $x \in [x_0, x_4]$, and find the error bound for this example.



$|f^{(5)}[x]| = \text{Abs}[\text{Sin}[x]]$

$|f^{(5)}[x]| \leq M_5 = \text{Sin}[1] = 0.841471$

$h = \frac{1}{4}$

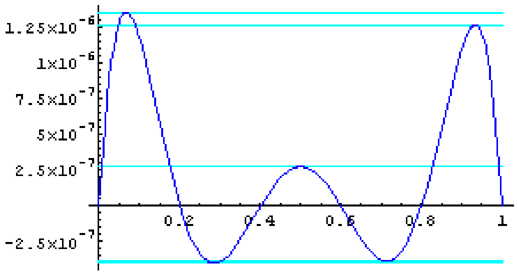
The remainder term $R_4(x)$ has the form

$$|R_4(x)| \leq \frac{\sqrt{4750 + 290\sqrt{145}}}{3000} M_5 h^5 = \frac{\sqrt{4750 + 290\sqrt{145}}\ \text{Sin}[1]}{3072000} = 0.0000248677$$

Thus, $|R_4(x)| \leq 0.0000248677$ is valid for $x \in [0, 1]$, which is a little bit larger than the maximum error 0.0000157713. After all, it is an error bound.

Example 3 (b). Error Analysis. Investigate the error for the Lagrange polynomial approximations of degree n = 5 for the function `f[x] = Cos[x]` over the interval `[0, 1]`
Solution 3 (b).

Investigate the error for the Lagrange interpolation polynomial $p_5[x]$, of degree n = 5.



$f[x] = Cos[x]$

$p_5[x] =$

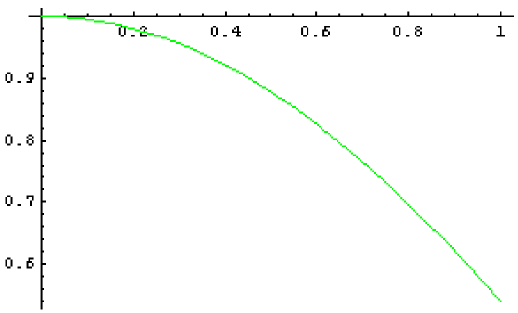
$-26.0417 (-1. + x) (-0.8 + x) (-0.6 + x) (-0.4 + x) (-0.2 + x) + 127.613 (-1. + x) (-0.8 + x) (-0.6 + x) (-0.4 + x) x - 239.86 (-1. + x) (-0.8 + x) (-0.6 + x) (-0.2 + x) x +$
$214.931 (-1. + x) (-0.8 + x) (-0.4 + x) (-0.2 + x) x - 90.717 (-1. + x) (-0.6 + x) (-0.4 + x) (-0.2 + x) x + 14.0704 (-0.8 + x) (-0.6 + x) (-0.4 + x) (-0.2 + x) x$

The interval for interpolation is [0.0,1.0].

Graph of the error $e_5[x] = f[x] - p_5[x]$

Looking at the above graph we make the following estimate for the error: $| e_5[x] | \le 0.00000134999$

Use formula (v). $| R_5 (x) | \le \frac{10 + 7\sqrt{7}}{1215} M_6 h^6$ is valid for $x \in [x_0, x_5]$, and find the error bound for this example.



$|f^{(6)}[x]| = Abs[Cos[x]]$

$|f^{(6)}[x]| \le M_6 = 1 = 1.$

$h = \frac{1}{5}$

The remainder term $R_5(x)$ has the form

$$|R_5(x)| \le \frac{10+7\sqrt{7}}{1215} M_6 h^6 = \frac{10+7\sqrt{7}}{18984375} = 1.5023 \times 10^{-6}$$

Thus, $|R_5(x)| \le 1.5023 \times 10^{-6}$ is valid for $x \in [0, 1]$, which is a little bit larger than the maximum error 1.34999×10^{-6}. After all, it is an error bound.