5. Cubic Splines

Cubic Spline Interpolant

Definition (Cubic Spline). Suppose that $\{(x_k, y_k)\}_{k=0}^{n}$ are n+1 points, where $a = x_0 < x_0 < ... < x_n = b$. The function $S(x)$ is called a cubic spline if there exists n cubic polynomials $S_k(x)$ with coefficients $s_{k,0}$, $s_{k,1}$, $s_{k,2}$, and $s_{k,3}$ that satisfy the properties:

I. $S(x) = S_k(x) = s_{k,0} + s_{k,1}(x - x_k) + s_{k,2}(x - x_k)^2 + s_{k,3}(x - x_k)^3$
for $x \in [x_k, x_{k+1}]$ and $k = 0, 1, ..., n-1$.

II. $S(x_k) = y_k$ for $k = 0, 1, ..., n$.
The spline passes through each data point.

III. $S_k(x_{k+1}) = S_{k+1}(x_{k+1})$ for $k = 0, 1, ..., n-2$.
The spline forms a continuous function over [a,b].

IV. $S'_k(x_{k+1}) = S'_{k+1}(x_{k+1})$ for $k = 0, 1, ..., n-2$.
The spline forms a smooth function.

IV. $S''_k(x_{k+1}) = S''_{k+1}(x_{k+1})$ for $k = 0, 1, ..., n-2$.
The second derivative is continuous.

Lemma (Natural Spline). There exists a unique cubic spline with the free boundary conditions $S''(a) = 0$ and $S''(b) = 0$.

Remark. The natural spline is the curve obtained by forcing a flexible elastic rod through the points but letting the slope at the ends be free to equilibrate to the position that minimizes the oscillatory behavior of the curve. It is useful for fitting a curve to experimental data that is significant to several significant digits.

Example 1. Construct the natural cubic spline for the points $(0, 1), (1, 0), (2, 0), (3, 1), (4, 2), (5, 2), (6, 1)$ that has the endpoint constraints $S''(0) = 0$ and $S''(6) = 0$.
Solution 1.

Remark. There are five popular types of splines: natural spline, clamped spline, extrapolated spline, parabolically terminated spline, endpoint curvature adjusted spline.

Lemma (Clamped Spline). There exists a unique cubic spline with the first derivative boundary conditions $S'(a) = d_0$ and $S'(b) = d_n$.

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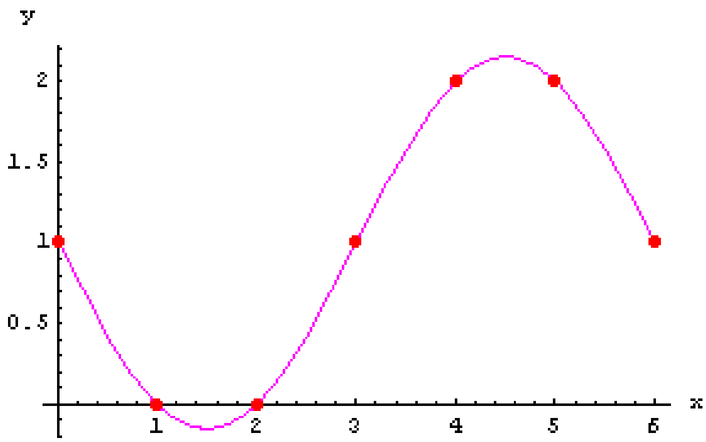
S''(0) = 0 and S''(6) = 0.

Solution 1.



Spline $y = S_i[x]$

data points = {{0., 1.}, {1., 0.}, {2., 0.}, {3., 1.}, {4., 2.}, {5., 2.}, {6., 1.}}

The spline coefficients are

$$\begin{pmatrix} 1. & -1.2 & 0 & 0.2 \\ 0 & -0.6 & 0.6 & 0 \\ 0 & 0.6 & 0.6 & -0.2 \\ 1. & 1.2 & 0 & -0.2 \\ 2. & 0.6 & -0.6 & 0 \\ 2. & -0.6 & -0.6 & 0.2 \end{pmatrix}$$