4. The Matrix Exponential

Background for the Fundamental Matrix

We seek a solution of a homogeneous first order linear system of differential equations. For illustration purposes we consider the 2×2 case:

$$x' = a\,x[t] + b\,y[t]$$
$$y' = c\,x[t] + d\,y[t]$$

First, write the system in vector and matrix form $\vec{X}'[t] = A\,\vec{X}[t]$

$$\vec{X}'[t] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{X}[t].$$

Then, find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, denote the eigenpairs of $\mathbf{A}$ by

$$\lambda_1,\ \vec{v}_1 = \begin{pmatrix} v_{1,1} \\ v_{2,1} \end{pmatrix} \quad \text{and} \quad \lambda_2,\ \vec{v}_2 = \begin{pmatrix} v_{1,2} \\ v_{2,2} \end{pmatrix}.$$

Assumption. Assume that there are two linearly independent eigenvectors $\vec{v}_1$ and $\vec{v}_2$, which correspond to the eigenvalues λ_1 and λ_2, respectively. Then two linearly independent solution to $\vec{X}'[t] = A\,\vec{X}[t]$ are

$$\vec{x}_1[t] = \vec{v}_1 e^{\lambda_1 t} = \begin{pmatrix} v_{1,1} \\ v_{2,1} \end{pmatrix} e^{\lambda_1 t}, \quad \text{and}$$
$$\vec{x}_2[t] = \vec{v}_2 e^{\lambda_2 t} = \begin{pmatrix} v_{1,2} \\ v_{2,2} \end{pmatrix} e^{\lambda_2 t}.$$

Definition (Fundamental Matrix Solution) The fundamental matrix solution $\vec{\Phi}[t]$, is formed by using the two column vectors $\vec{v}_1 e^{\lambda_1 t}$ and $\vec{v}_2 e^{\lambda_2 t}$.

(1) $$\vec{\Phi}[t] = [\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}] = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

The general solution to $\vec{X}'[t] = A\,\vec{X}[t]$ is the linear combination

(2) $$\vec{X}[t] = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

It can be written in matrix form using the fundamental matrix solution $\vec{\Phi}[t]$ as follows

$$\vec{X}[t] = \vec{\Phi}[t] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Notation. When we introduce the notation

$$\mathbf{p} = [\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2] = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix},$$

and

$$\mathbf{e}[t] = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

The fundamental matrix solution $\vec{\Phi}[t]$ can be written as

(3) $$\vec{\Phi}[t] = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

or

(4) $$\vec{\Phi}[t] = \mathbf{p}\ \mathbf{e}[t].$$

The initial condition $\vec{X}[0]$

If we desire to have the initial condition $\vec{X}[0] = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, then this produces the equation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \vec{\Phi}[0] \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The vector of constant $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ can be solved as follows

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{\Phi}^{-1}[0] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

The solution with the prescribed initial conditions is

$$\vec{X}[t] = \vec{\Phi}[t]\,\vec{\Phi}^{-1}[0] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Observe that $\vec{\Phi}[0]\,\vec{\Phi}^{-1}[0] = \mathbf{I}_{2\times 2}$ where $\mathbf{I}_{2\times 2}$ is the identity matrix. This leads us to make the following important definition

Definition (Matrix Exponential) If $\vec{\Phi}[t]$ is a fundamental matrix solution to $\vec{X}'[t] = \mathbf{A}\vec{X}[t]$, then the matrix exponential is defined to be

$$\vec{\Phi}[t] = \vec{\Phi}[t]\,\vec{\Phi}^{-1}[0].$$

Notation. This can be written as

(5) $$\vec{\Phi}[t] = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}^{-1},$$

or

(6) $$\vec{\Phi}[t] = \mathbf{p}\ \mathbf{e}[t]\ \mathbf{p}^{-1}.$$

Fact. For a 2×2 system, the initial condition is

$$\vec{\Phi}[0] = \vec{\Psi}[0]\,\vec{\Psi}^{-1}[0] = \mathbf{I}_{2\times2},$$

and the solution with the initial condition $\vec{X}[0] = \begin{pmatrix} \vec{x}_1[0] \\ \vec{x}_1[0] \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is

$$\vec{X}[t] = \vec{\Phi}[t] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

or

$$\vec{X}[t] = \vec{\Phi}[t]\,\vec{X}[0].$$

Theorem (Matrix Diagonalization) The eigen decomposition of a 2×2 square matrix **A** is

$$\mathbf{A} = \mathbf{p}\ \mathbf{d}\ \mathbf{p}^{-1},$$

which exists when **A** has a full set of eigenpairs $\lambda_i, \vec{v}_i$ for $i = 1, 2, \ldots, m$, and **d** is the diagonal matrix

$$\mathbf{d} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and

$$\mathbf{p} = [\vec{v}_1, \vec{v}_2]$$

is the augmented matrix whose columns are the eigenvectors of **A**.

$$\mathbf{p} = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}.$$

Matrix power $\mathbf{A}^n$

How do you compute the higher powers of a matrix ? For example, given $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$\mathbf{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + b c & a b + b d \\ a c + c d & b c + d^2 \end{pmatrix},$$

and

$$\mathbf{A}^3 = \begin{pmatrix} a^3 + 2 a b c + b c d & a^2 b + b^2 c + a b d + b d^2 \\ a^2 c + b c^2 + a c d + c d^2 & a b c + 2 b c d + d^3 \end{pmatrix},$$ etc.

The higher powers seem to be intractable! But if we have an eigen decomposition, then we are permitted to write

$$A^2 = p\, d\, p^{-1} p\, d\, p^{-1} = p\, d^2\, p^{-1}$$

and

$$A^3 = p\, d^2\, p^{-1} p\, d\, p^{-1} = p\, d^3\, p^{-1}$$

in general

$$A^n = p\, d^n\, p^{-1}$$

Fact. For a 2×2 matrix this is

$$A^n = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}^{-1}$$

which can be simplified

$$A^n = \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix}^{-1}$$

Theorem (Series Representation for the Matrix Exponential) The solution to $\vec{X}'[t] = A\,\vec{X}[t]$ is given by the series

$$\vec{\Phi}[t] = e^{At} = \sum_{n=0}^{\infty} \frac{1}{n!} (A t)^n \,,$$ which becomes

$$\vec{\Phi}[t] = e^{At} = p \left(\sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \lambda_1^n t^n & 0 \\ 0 & \lambda_2^n t^n \end{pmatrix}^n \right) p^{-1}$$

and has the simplified form

$$\vec{\Phi}[t] = e^{At} = p \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} p^{-1},$$

or

$$\vec{\Phi}[t] = e^{At} = p\, e[t]\, p^{-1}.$$

Example 1. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

1 (a) Find $\Phi = e^{A}$.

1 (b) Find $\Phi(t) = e^{At}$.

Solution 1 (a).

Solution 1 (b).

Matrix Exponential 2D examples

The following examples illustrate the situation when there is a full set of eigenvectors.

Example 2. Use the matrix exponential to find the general solution for the system of D. E.'s

$$x'[t] = -2x[t] + y[t]$$
$$y'[t] = x[t] - 2y[t]$$

Solution 2.

Example 3. Use the matrix exponential to find the general solution for the system of D. E.'s

$$x'[t] = -x[t] - 2y[t]$$
$$y'[t] = 2x[t] - y[t]$$

Solution 3.

Matrix Exponential 3D examples

The following examples illustrate the situation when there is a full set of eigenvectors.

Example 4. Use the matrix exponential to find the general solution for the system of D.E.'s

$\vec{\mathbf{X}}'[t] = \mathbf{A}\vec{\mathbf{X}}[t]$, where

$$\mathbf{A} = \begin{pmatrix} -1 & -2 & 2 \\ 1 & -4 & 2 \\ -1 & -3 & 1 \end{pmatrix}.$$

Solution 4.

Example 1. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

1 (a) Find $\Phi = e^{A}$.

Solution 1 (a).

We want to find

$$e^{A} = I + A + \frac{1}{2!} A^{2} + \frac{1}{3!} A^{3} + \frac{1}{4!} A^{4} + \ldots + \frac{1}{n!} A^{n} + \ldots$$

First look at some powers A^{i}

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$A^{2} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$A^{3} = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix}$$

$$A^{4} = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$$

$$A^{5} = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}$$

Now use the calculation $A^{i} = p.d^{i}.p^{-1}$

```
d = {{1, 0}, {0, 3}};
p = {{-1, 1}, {1, 1}};
pi = Inverse[p];
For[i = 1, i ≤ 5, i++,
 Print["p", "d"^i, "p^-1", " = ", MatrixForm[p], MatrixForm[MatrixPower[d, i]], MatrixForm[pi], " = ",
  MatrixForm[p.MatrixPower[d, i].Inverse[p]]]];
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$$pdp^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$pd^{2}p^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$pd^{3}p^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix}$$

$$pd^{4}p^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 81 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$$

$$pd^{5}p^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 243 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}$$

Find the expression for the general term $A^{n} = p.d^{n}.p^{-1}$

$$pd^{n}p^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^{n} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{3^{n}}{2} & -\frac{1}{2} + \frac{3^{n}}{2} \\ -\frac{1}{2} + \frac{3^{n}}{2} & \frac{1}{2} + \frac{3^{n}}{2} \end{pmatrix}$$

Find matrix exponential $\phi = e^{A}$ will be the sum of the infinite series

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}+\frac{3}{2} & -\frac{1}{2}+\frac{3}{2} \\ -\frac{1}{2}+\frac{3}{2} & \frac{1}{2}+\frac{3}{2} \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \frac{1}{2}+\frac{3^2}{2} & -\frac{1}{2}+\frac{3^2}{2} \\ -\frac{1}{2}+\frac{3^2}{2} & \frac{1}{2}+\frac{3^2}{2} \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \frac{1}{2}+\frac{3^3}{2} & -\frac{1}{2}+\frac{3^3}{2} \\ -\frac{1}{2}+\frac{3^3}{2} & \frac{1}{2}+\frac{3^3}{2} \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} \frac{1}{2}+\frac{3^4}{2} & -\frac{1}{2}+\frac{3^4}{2} \\ -\frac{1}{2}+\frac{3^4}{2} & \frac{1}{2}+\frac{3^4}{2} \end{pmatrix} + \ldots$$

The sum of the first few terms are:

Clear[n];

$$s = \sum_{n=0}^{4} \frac{1}{n!} \begin{pmatrix} \frac{1}{2}+\frac{3^n}{2} & -\frac{1}{2}+\frac{3^n}{2} \\ -\frac{1}{2}+\frac{3^n}{2} & \frac{1}{2}+\frac{3^n}{2} \end{pmatrix};$$

Print[MatrixForm[s], " = ", MatrixForm[N[s]]];

$$\begin{pmatrix} \frac{229}{24} & \frac{41}{6} \\ \frac{41}{6} & \frac{229}{24} \end{pmatrix} = \begin{pmatrix} 9.54167 & 6.83333 \\ 6.83333 & 9.54167 \end{pmatrix}$$

$$s = \sum_{n=0}^{9} \frac{1}{n!} \begin{pmatrix} \frac{1}{2}+\frac{3^n}{2} & -\frac{1}{2}+\frac{3^n}{2} \\ -\frac{1}{2}+\frac{3^n}{2} & \frac{1}{2}+\frac{3^n}{2} \end{pmatrix};$$

Print[MatrixForm[s], " = ", MatrixForm[N[s]]];

$$\begin{pmatrix} \frac{590501}{51840} & \frac{3147097}{362880} \\ \frac{3147097}{362880} & \frac{590501}{51840} \end{pmatrix} = \begin{pmatrix} 11.3908 & 8.67256 \\ 8.67256 & 11.3908 \end{pmatrix}$$

$$s = \sum_{n=0}^{99} \frac{1}{n!} \begin{pmatrix} \frac{1}{2}+\frac{3^n}{2} & -\frac{1}{2}+\frac{3^n}{2} \\ -\frac{1}{2}+\frac{3^n}{2} & \frac{1}{2}+\frac{3^n}{2} \end{pmatrix};$$

Print[MatrixForm[N[s]]];

$$\begin{pmatrix} 11.4019 & 8.68363 \\ 8.68363 & 11.4019 \end{pmatrix}$$

$$s = \sum_{n=0}^{199} \frac{1}{n!} \begin{pmatrix} \frac{1}{2}+\frac{3^n}{2} & -\frac{1}{2}+\frac{3^n}{2} \\ -\frac{1}{2}+\frac{3^n}{2} & \frac{1}{2}+\frac{3^n}{2} \end{pmatrix};$$

Print[MatrixForm[N[s]]];

$$\begin{pmatrix} 11.4019 & 8.68363 \\ 8.68363 & 11.4019 \end{pmatrix}$$

Each element in ϕ can be calculated by the sum of an infinite series and *Mathematica* can assist us in these computations.

ϕ = Table[0, {2}, {2}];

$$\phi_{[[1,1]]} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2}\left(e + e^3\right)$$

$$\phi_{[[1,2]]} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2} \left(-e + e^3\right)$$

$$\phi_{[[2,1]]} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2} \left(-e + e^3\right)$$

$$\phi_{[[2,2]]} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2} \left(e + e^3\right)$$

Therefore, the matrix exponential $\phi = e^A$ is

$$\phi = \begin{pmatrix} \frac{1}{2} \left(e + e^3\right) & \frac{1}{2} \left(-e + e^3\right) \\ \frac{1}{2} \left(-e + e^3\right) & \frac{1}{2} \left(e + e^3\right) \end{pmatrix}$$

Example 1. Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

1 (a) Find $\Phi = e^{A}$.

1 (b) Find $\Phi(t) = e^{At}$.

Solution 1 (b).

We want to find

$$e^{At} = I + A\,t + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \frac{1}{4!} A^4 t^4 + \ldots + \frac{1}{n!} A^n t^n + \ldots$$

First look at some powers $A^i t^i$

$$A = \begin{pmatrix} 2t & t \\ t & 2t \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 5t^2 & 4t^2 \\ 4t^2 & 5t^2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 14t^3 & 13t^3 \\ 13t^3 & 14t^3 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 41t^4 & 40t^4 \\ 40t^4 & 41t^4 \end{pmatrix}$$

$$A^5 = \begin{pmatrix} 122t^5 & 121t^5 \\ 121t^5 & 122t^5 \end{pmatrix}$$

Now use the calculation $A^i t^i = p.d^i.p^{-1} t^i$

```
d = (1  0
     0  3);
p = (-1  1
      1  1);
pi = Inverse[p];
For[i = 1, i ≤ 5, i++,
 Print["p", "d"^i, "p^-1", "t"^i, " = ", MatrixForm[p], MatrixForm[MatrixPower[d, i]], MatrixForm[pi], "t"^i, " = ",
  MatrixForm[p.MatrixPower[d, i].Inverse[p] t^i]]];
```

$$\text{pdp}^{-1}\text{t} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} t = \begin{pmatrix} 2\,t & t \\ t & 2\,t \end{pmatrix}$$

$$\text{pd}^2\text{p}^{-1}t^2 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} t^2 = \begin{pmatrix} 5\,t^2 & 4\,t^2 \\ 4\,t^2 & 5\,t^2 \end{pmatrix}$$

$$\text{pd}^3\text{p}^{-1}t^3 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 27 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} t^3 = \begin{pmatrix} 14\,t^3 & 13\,t^3 \\ 13\,t^3 & 14\,t^3 \end{pmatrix}$$

$$\text{pd}^4\text{p}^{-1}t^4 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 81 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} t^4 = \begin{pmatrix} 41\,t^4 & 40\,t^4 \\ 40\,t^4 & 41\,t^4 \end{pmatrix}$$

$$\text{pd}^5\text{p}^{-1}t^5 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 243 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} t^5 = \begin{pmatrix} 122\,t^5 & 121\,t^5 \\ 121\,t^5 & 122\,t^5 \end{pmatrix}$$

Find the expression for the general term $A^n t^n = p.d^n.p^{-1}\, t^n$

$$\text{pd}^n\text{p}^{-1}t^n = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} t^n = t^n \begin{pmatrix} \frac{1}{2} + \frac{3^n}{2} & -\frac{1}{2} + \frac{3^n}{2} \\ -\frac{1}{2} + \frac{3^n}{2} & \frac{1}{2} + \frac{3^n}{2} \end{pmatrix}$$

Find matrix exponential $\phi(t) = e^{At}$ will be the sum of the infinite series

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} + \frac{3}{2} & -\frac{1}{2} + \frac{3}{2} \\ -\frac{1}{2} + \frac{3}{2} & \frac{1}{2} + \frac{3}{2} \end{pmatrix} + \frac{t^2}{2!} \begin{pmatrix} \frac{1}{2} + \frac{3^2}{2} & -\frac{1}{2} + \frac{3^2}{2} \\ -\frac{1}{2} + \frac{3^2}{2} & \frac{1}{2} + \frac{3^2}{2} \end{pmatrix} + \frac{t^3}{3!} \begin{pmatrix} \frac{1}{2} + \frac{3^3}{2} & -\frac{1}{2} + \frac{3^3}{2} \\ -\frac{1}{2} + \frac{3^3}{2} & \frac{1}{2} + \frac{3^3}{2} \end{pmatrix} + \frac{t^4}{4!} \begin{pmatrix} \frac{1}{2} + \frac{3^4}{2} & -\frac{1}{2} + \frac{3^4}{2} \\ -\frac{1}{2} + \frac{3^4}{2} & \frac{1}{2} + \frac{3^4}{2} \end{pmatrix} + \ldots$$

The sum of the first five terms is

Clear[n];

$$\mathbf{s} = \sum_{n=0}^{4} \frac{\mathbf{t^n}}{\mathbf{n!}} \begin{pmatrix} \frac{1}{2} + \frac{3^n}{2} & -\frac{1}{2} + \frac{3^n}{2} \\ -\frac{1}{2} + \frac{3^n}{2} & \frac{1}{2} + \frac{3^n}{2} \end{pmatrix};$$

Print[MatrixForm[s]];

$$\begin{pmatrix} 1 + 2t + \frac{5t^2}{2} + \frac{7t^3}{3} + \frac{41t^4}{24} & t + 2t^2 + \frac{13t^3}{6} + \frac{5t^4}{3} \\ t + 2t^2 + \frac{13t^3}{6} + \frac{5t^4}{3} & 1 + 2t + \frac{5t^2}{2} + \frac{7t^3}{3} + \frac{41t^4}{24} \end{pmatrix}$$

The sum of the first ten terms is

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Clear[n];
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$$\mathbf{s} = \sum_{n=0}^{9} \frac{t^n}{n!} \begin{pmatrix} \frac{1}{2} + \frac{3^n}{2} & -\frac{1}{2} + \frac{3^n}{2} \\ -\frac{1}{2} + \frac{3^n}{2} & \frac{1}{2} + \frac{3^n}{2} \end{pmatrix};$$

```
Print[MatrixForm[s]];
```

$$\begin{pmatrix} 1 + 2t + \frac{5t^2}{2} + \frac{7t^3}{3} + \frac{41t^4}{24} + \frac{61t^5}{60} + \frac{73t^6}{144} + \frac{547t^7}{2520} + \frac{3281t^8}{40320} + \frac{703t^9}{25920} & t + 2t^2 + \frac{13t^3}{6} + \frac{5t^4}{3} + \frac{121t^5}{120} + \frac{91t^6}{180} + \frac{1093t^7}{5040} + \frac{41t^8}{504} + \frac{9841t^9}{362880} \\ t + 2t^2 + \frac{13t^3}{6} + \frac{5t^4}{3} + \frac{121t^5}{120} + \frac{91t^6}{180} + \frac{1093t^7}{5040} + \frac{41t^8}{504} + \frac{9841t^9}{362880} & 1 + 2t + \frac{5t^2}{2} + \frac{7t^3}{3} + \frac{41t^4}{24} + \frac{61t^5}{60} + \frac{73t^6}{144} + \frac{547t^7}{2520} + \frac{3281t^8}{40320} + \frac{703t^9}{25920} \end{pmatrix}$$

Each element in $\phi(t)$ can be calculated by the sum of an infinite series and *Mathematica* can assist us in these computations.

```
φ = Table[0, {2}, {2}];
```

$$\phi[[1,1]] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2} e^t (1 + e^{2t})$$

$$\phi[[1,2]] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(-\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2} e^t (-1 + e^{2t})$$

$$\phi[[2,1]] = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(-\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2} e^t (-1 + e^{2t})$$

$$\phi_{[[2,2]]} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{1}{2} + \frac{3^n}{2}\right)$$

$$\frac{1}{2} e^t \left(1 + e^{2t}\right)$$

Therefore, the matrix exponential $\phi(t) = e^{At}$ is

$$\phi = \begin{pmatrix} \frac{1}{2} e^t \left(1 + e^{2t}\right) & \frac{1}{2} e^t \left(-1 + e^{2t}\right) \\ \frac{1}{2} e^t \left(-1 + e^{2t}\right) & \frac{1}{2} e^t \left(1 + e^{2t}\right) \end{pmatrix}$$

Example 2. Use the matrix exponential to find the general solution for the system of D. E.'s

$x'[t] = -2x[t] + y[t]$

$y'[t] = x[t] - 2y[t]$

Solution 2.

First, write the system in vector and matrix form $\vec{X}'[t] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{X}[t]$.

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$p = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$d = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} 3 & -1 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

A fundamental matrix solution is $\vec{\Phi}[t] = \mathbf{p} \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$.

$$\vec{\Phi}[t] = \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

$$A.\vec{\Phi}[t] = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}$$

$$A.\vec{\Phi}[t] = \begin{pmatrix} 3e^{-3t} & -e^{-t} \\ -3e^{-3t} & -e^{-t} \end{pmatrix}$$

$$\vec{\Phi}'[t] = \begin{pmatrix} 3e^{-3t} & -e^{-t} \\ -3e^{-3t} & -e^{-t} \end{pmatrix}$$

Does $\vec{\Phi}'[t] = A.\vec{\Phi}[t]$?

True

The matrix exponential is $\vec{\Phi}[t] = \vec{\Phi}[t]\,\vec{\Phi}^{-1}[0]$.

$$\vec{\Psi}[t] = \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}$$

$$\vec{\Psi}[0] = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\vec{\Phi}[t] = \vec{\Psi}[t].\text{Inverse}[\vec{\Psi}[0]]$$

$$\vec{\Phi}[t] = \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix} \text{Inverse}\left[\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\right]$$

$$\vec{\Phi}[t] = \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\vec{\Phi}[t] = \begin{pmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & -\frac{1}{2} e^{-3t} + \frac{e^{-t}}{2} \\ -\frac{1}{2} e^{-3t} + \frac{e^{-t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{pmatrix}$$

$$\vec{\Phi}[0] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The solution to the D. E. with initial conditions $\vec{X}[0] = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, is $\vec{F}[t, x_0, y_0] = \vec{\Phi}[t] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

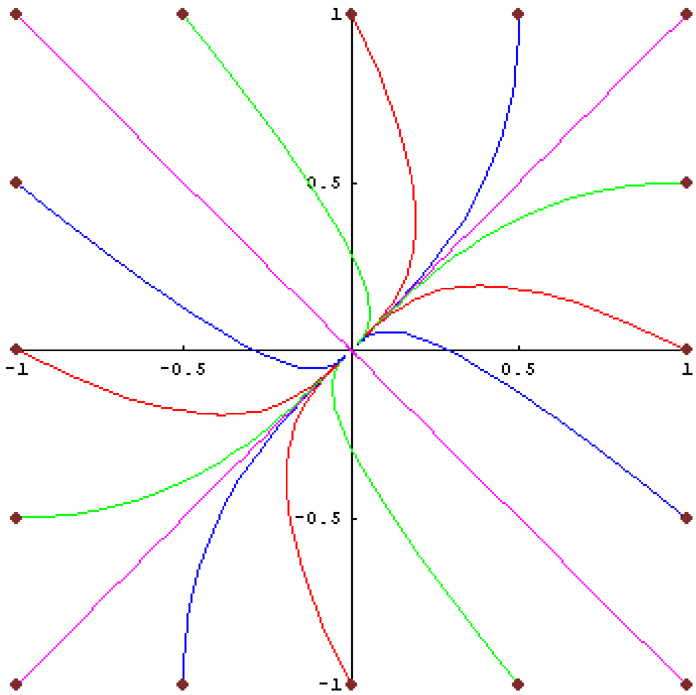
$$\vec{\Phi}[t] = \begin{pmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & -\frac{1}{2} e^{-3t} + \frac{e^{-t}}{2} \\ -\frac{1}{2} e^{-3t} + \frac{e^{-t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{pmatrix}$$

$$\vec{F}[t] = \vec{\Phi}[t].\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\vec{F}[t] = \begin{pmatrix} \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} & -\frac{1}{2} e^{-3t} + \frac{e^{-t}}{2} \\ -\frac{1}{2} e^{-3t} + \frac{e^{-t}}{2} & \frac{e^{-3t}}{2} + \frac{e^{-t}}{2} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\vec{F}[t] = \begin{pmatrix} \frac{1}{2} e^{-3t} x_0 + \frac{1}{2} e^{-t} x_0 - \frac{1}{2} e^{-3t} y_0 + \frac{1}{2} e^{-t} y_0 \\ -\frac{1}{2} e^{-3t} x_0 + \frac{1}{2} e^{-t} x_0 + \frac{1}{2} e^{-3t} y_0 + \frac{1}{2} e^{-t} y_0 \end{pmatrix}$$

$$\vec{F}[0] = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



$$\vec{F}[t] = \begin{pmatrix} \frac{1}{2} e^{-3t} x_0 + \frac{1}{2} e^{-t} x_0 - \frac{1}{2} e^{-3t} y_0 + \frac{1}{2} e^{-t} y_0 \\ -\frac{1}{2} e^{-3t} x_0 + \frac{1}{2} e^{-t} x_0 + \frac{1}{2} e^{-3t} y_0 + \frac{1}{2} e^{-t} y_0 \end{pmatrix}$$

For t in the interval $0 \le t \le 2.5$

I.C.'s $\{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\}$ =

$$\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -0.5 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0.5 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}\right\}$$

Example 3. Use the matrix exponential to find the general solution for the system of D. E.'s

$x'[t] = -x[t] - 2y[t]$

$y'[t] = 2x[t] - y[t]$

Solution 3.

First, write the system in vector and matrix form $\vec{X}'[t] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{X}[t]$.

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$p = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$$

$$d = \begin{pmatrix} -1-2i & 0 \\ 0 & -1+2i \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1-2i & 0 \\ 0 & -1+2i \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} -2+i & -2-i \\ -1-2i & -1+2i \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

A fundamental matrix solution is $\vec{\Phi}[t] = p \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$.

$$\vec{\Phi}[t] = \begin{pmatrix} -i\, e^{(-1-2i)t} & i\, e^{(-1+2i)t} \\ e^{(-1-2i)t} & e^{(-1+2i)t} \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

$$A.\vec{\Phi}[t] = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -i\, e^{(-1-2i)t} & i\, e^{(-1+2i)t} \\ e^{(-1-2i)t} & e^{(-1+2i)t} \end{pmatrix}$$

$$A.\vec{\Phi}[t] = \begin{pmatrix} (-2+i)\, e^{(-1-2i)t} & (-2-i)\, e^{(-1+2i)t} \\ (-1-2i)\, e^{(-1-2i)t} & (-1+2i)\, e^{(-1+2i)t} \end{pmatrix}$$

$$\vec{\Phi}'[t] = \begin{pmatrix} (-2+i)\, e^{(-1-2i)t} & (-2-i)\, e^{(-1+2i)t} \\ (-1-2i)\, e^{(-1-2i)t} & (-1+2i)\, e^{(-1+2i)t} \end{pmatrix}$$

Does $\vec{\Phi}'[t] = A.\vec{\Phi}[t]$?

True

The matrix exponential is $\vec{\Phi}[t] = \vec{\Phi}[t]\, \vec{\Phi}^{-1}[0]$.

$$\vec{\Phi}[t] = \begin{pmatrix} -i\,e^{(-1-2i)t} & i\,e^{(-1+2i)t} \\ e^{(-1-2i)t} & e^{(-1+2i)t} \end{pmatrix}$$

$$\vec{\Phi}[0] = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$$

$$\vec{\Phi}[t] = \vec{\Phi}[t].\text{Inverse}[\vec{\Phi}[0]]$$

$$\vec{\Phi}[t] = \begin{pmatrix} -i\,e^{(-1-2i)t} & i\,e^{(-1+2i)t} \\ e^{(-1-2i)t} & e^{(-1+2i)t} \end{pmatrix} \text{Inverse}\left[\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\right]$$

$$\vec{\Phi}[t] = \begin{pmatrix} -i\,e^{(-1-2i)t} & i\,e^{(-1+2i)t} \\ e^{(-1-2i)t} & e^{(-1+2i)t} \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

$$\vec{\Phi}[t] = \begin{pmatrix} \frac{1}{2} e^{(-1-2i)t} + \frac{1}{2} e^{(-1+2i)t} & -\frac{1}{2} i\,e^{(-1-2i)t} + \frac{1}{2} i\,e^{(-1+2i)t} \\ \frac{1}{2} i\,e^{(-1-2i)t} - \frac{1}{2} i\,e^{(-1+2i)t} & \frac{1}{2} e^{(-1-2i)t} + \frac{1}{2} e^{(-1+2i)t} \end{pmatrix}$$

$$\vec{\Phi}[t] = \begin{pmatrix} e^{-t}\,\text{Cos}[2\,t] & -e^{-t}\,\text{Sin}[2\,t] \\ e^{-t}\,\text{Sin}[2\,t] & e^{-t}\,\text{Cos}[2\,t] \end{pmatrix}$$

$$\vec{\Phi}[0] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The solution to the D. E. with initial conditions $\vec{X}[0] = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, is $\vec{F}[t, x_0, y_0] = \vec{\Phi}[t] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

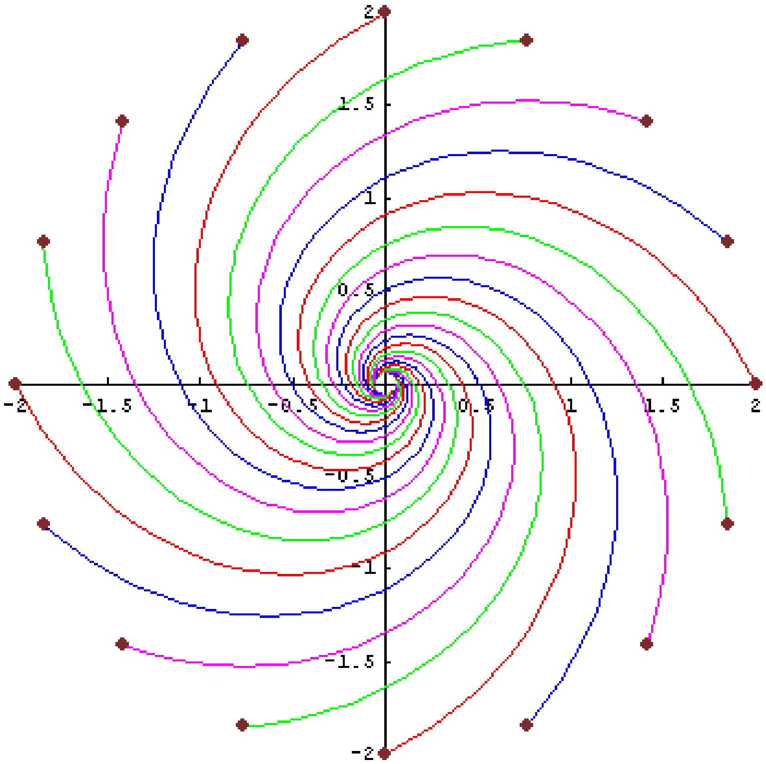
$$\vec{\Phi}[t] = \begin{pmatrix} e^{-t}\,\text{Cos}[2\,t] & -e^{-t}\,\text{Sin}[2\,t] \\ e^{-t}\,\text{Sin}[2\,t] & e^{-t}\,\text{Cos}[2\,t] \end{pmatrix}$$

$$\vec{F}[t] = \vec{\Phi}[t].\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\vec{F}[t] = \begin{pmatrix} e^{-t}\,\text{Cos}[2\,t] & -e^{-t}\,\text{Sin}[2\,t] \\ e^{-t}\,\text{Sin}[2\,t] & e^{-t}\,\text{Cos}[2\,t] \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\vec{F}[t] = \begin{pmatrix} e^{-t}\,\text{Cos}[2\,t]\,x_0 - e^{-t}\,\text{Sin}[2\,t]\,y_0 \\ e^{-t}\,\text{Sin}[2\,t]\,x_0 + e^{-t}\,\text{Cos}[2\,t]\,y_0 \end{pmatrix}$$

$$\vec{F}[0] = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



$$\vec{F}[t] = \begin{pmatrix} e^{-t}\,\mathrm{Cos}[2\,t]\,x_0 - e^{-t}\,\mathrm{Sin}[2\,t]\,y_0 \\ e^{-t}\,\mathrm{Sin}[2\,t]\,x_0 + e^{-t}\,\mathrm{Cos}[2\,t]\,y_0 \end{pmatrix}$$

for t in the interval $0 \le t \le 3.5$

$$\text{I.C.'s } \left\{\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\right\} = \left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\,\mathrm{Cos}[\frac{\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{\pi}{8}] \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 2\,\mathrm{Cos}[\frac{3\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{3\pi}{8}] \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2\,\mathrm{Cos}[\frac{5\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{5\pi}{8}] \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 2\,\mathrm{Cos}[\frac{7\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{7\pi}{8}] \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2\,\mathrm{Cos}[\frac{9\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{9\pi}{8}] \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 2\,\mathrm{Cos}[\frac{11\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{11\pi}{8}] \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 2\,\mathrm{Cos}[\frac{13\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{13\pi}{8}] \end{pmatrix}, \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} 2\,\mathrm{Cos}[\frac{15\pi}{8}] \\ 2\,\mathrm{Sin}[\frac{15\pi}{8}] \end{pmatrix}\right\}$$

Example 4. Use the matrix exponential to find the general solution for the system of D.E.'s $\vec{\mathbf{X}}'[t] = \mathbf{A}\vec{\mathbf{X}}[t]$, where
$\mathbf{A} = \begin{pmatrix} -1 & -2 & 2 \\ 1 & -4 & 2 \\ -1 & -3 & 1 \end{pmatrix}$.

Solution 4.

$$A = \begin{pmatrix} -1 & -2 & 2 \\ 1 & -4 & 2 \\ -1 & -3 & 1 \end{pmatrix}$$

$$p = \begin{pmatrix} 0 & 1+i & 1-i \\ 1 & 1+i & 1-i \\ 1 & 2 & 2 \end{pmatrix}$$

$$d = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1-2i & 0 \\ 0 & 0 & -1+2i \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} 0 & 1+i & 1-i \\ 1 & 1+i & 1-i \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1-2i & 0 \\ 0 & 0 & -1+2i \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ \frac{1}{4}-\frac{i}{4} & -\frac{1}{4}-\frac{i}{4} & \frac{1}{4}+\frac{i}{4} \\ \frac{1}{4}+\frac{i}{4} & -\frac{1}{4}+\frac{i}{4} & \frac{1}{4}-\frac{i}{4} \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} 0 & 1-3i & 1+3i \\ -2 & 1-3i & 1+3i \\ -2 & -2-4i & -2+4i \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ \frac{1}{4}-\frac{i}{4} & -\frac{1}{4}-\frac{i}{4} & \frac{1}{4}+\frac{i}{4} \\ \frac{1}{4}+\frac{i}{4} & -\frac{1}{4}+\frac{i}{4} & \frac{1}{4}-\frac{i}{4} \end{pmatrix}$$

$$p\ d\ p^{-1} = \begin{pmatrix} -1 & -2 & 2 \\ 1 & -4 & 2 \\ -1 & -3 & 1 \end{pmatrix}$$

A fundamental matrix solution is $\vec{\Psi}[t] = \mathbf{p} \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}$.

$$\vec{\Phi}[t] = \begin{pmatrix} 0 & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & 2\,e^{(-1-2i)t} & 2\,e^{(-1+2i)t} \end{pmatrix}$$

$$A = \begin{pmatrix} -1 & -2 & 2 \\ 1 & -4 & 2 \\ -1 & -3 & 1 \end{pmatrix}$$

$$A.\vec{\Phi}[t] = \begin{pmatrix} -1 & -2 & 2 \\ 1 & -4 & 2 \\ -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & 2\,e^{(-1-2i)t} & 2\,e^{(-1+2i)t} \end{pmatrix}$$

$$A.\vec{\Phi}[t] = \begin{pmatrix} 0 & (1-3i)\,e^{(-1-2i)t} & (1+3i)\,e^{(-1+2i)t} \\ -2\,e^{-2t} & (1-3i)\,e^{(-1-2i)t} & (1+3i)\,e^{(-1+2i)t} \\ -2\,e^{-2t} & (-2-4i)\,e^{(-1-2i)t} & (-2+4i)\,e^{(-1+2i)t} \end{pmatrix}$$

$$\vec{\Phi}'[t] = \begin{pmatrix} 0 & (1-3i)\,e^{(-1-2i)t} & (1+3i)\,e^{(-1+2i)t} \\ -2\,e^{-2t} & (1-3i)\,e^{(-1-2i)t} & (1+3i)\,e^{(-1+2i)t} \\ -2\,e^{-2t} & (-2-4i)\,e^{(-1-2i)t} & (-2+4i)\,e^{(-1+2i)t} \end{pmatrix}$$

Does $\vec{\Phi}'[t] = A.\vec{\Phi}[t]$?

True

The matrix exponential is $\vec{\Phi}[t] = \vec{\Phi}[t]\,\vec{\Phi}^{-1}[0]$.

$$\vec{\Phi}[t] = \begin{pmatrix} 0 & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & 2\,e^{(-1-2i)t} & 2\,e^{(-1+2i)t} \end{pmatrix}$$

$$\vec{\Phi}[0] = \begin{pmatrix} 0 & 1+i & 1-i \\ 1 & 1+i & 1-i \\ 1 & 2 & 2 \end{pmatrix}$$

$$\vec{\Phi}[t] = \vec{\Phi}[t].\text{Inverse}[\vec{\Phi}[0]]$$

$$\vec{\Phi}[t] = \begin{pmatrix} 0 & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & 2\,e^{(-1-2i)t} & 2\,e^{(-1+2i)t} \end{pmatrix} \text{Inverse}\left[\begin{pmatrix} 0 & 1+i & 1-i \\ 1 & 1+i & 1-i \\ 1 & 2 & 2 \end{pmatrix}\right]$$

$$\vec{\Phi}[t] = \begin{pmatrix} 0 & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & (1+i)\,e^{(-1-2i)t} & (1-i)\,e^{(-1+2i)t} \\ e^{-2t} & 2\,e^{(-1-2i)t} & 2\,e^{(-1+2i)t} \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ \frac{1}{4}-\frac{i}{4} & -\frac{1}{4}-\frac{i}{4} & \frac{1}{4}+\frac{i}{4} \\ \frac{1}{4}+\frac{i}{4} & -\frac{1}{4}+\frac{i}{4} & \frac{1}{4}-\frac{i}{4} \end{pmatrix}$$

$$\vec{\Phi}[t] = \begin{pmatrix} \frac{1}{2}e^{(-1-2i)t}+\frac{1}{2}e^{(-1+2i)t} & -\frac{1}{2}\,i\,e^{(-1-2i)t}+\frac{1}{2}\,i\,e^{(-1+2i)t} & \frac{1}{2}\,i\,e^{(-1-2i)t}-\frac{1}{2}\,i\,e^{(-1+2i)t} \\ -e^{-2t}+\frac{1}{2}e^{(-1-2i)t}+\frac{1}{2}e^{(-1+2i)t} & e^{-2t}-\frac{1}{2}\,i\,e^{(-1-2i)t}+\frac{1}{2}\,i\,e^{(-1+2i)t} & \frac{1}{2}\,i\,e^{(-1-2i)t}-\frac{1}{2}\,i\,e^{(-1+2i)t} \\ -e^{-2t}+\left(\frac{1}{2}-\frac{i}{2}\right)e^{(-1-2i)t}+\left(\frac{1}{2}+\frac{i}{2}\right)e^{(-1+2i)t} & e^{-2t}-\left(\frac{1}{2}+\frac{i}{2}\right)e^{(-1-2i)t}-\left(\frac{1}{2}-\frac{i}{2}\right)e^{(-1+2i)t} & \left(\frac{1}{2}+\frac{i}{2}\right)e^{(-1-2i)t}+\left(\frac{1}{2}-\frac{i}{2}\right)e^{(-1+2i)t} \end{pmatrix}$$

$$\vec{\Phi}[t] = \begin{pmatrix} e^{-t}\,\text{Cos}[2t] & -e^{-t}\,\text{Sin}[2t] & e^{-t}\,\text{Sin}[2t] \\ -e^{-2t}+e^{-t}\,\text{Cos}[2t] & e^{-2t}-e^{-t}\,\text{Sin}[2t] & e^{-t}\,\text{Sin}[2t] \\ -e^{-2t}+e^{-t}\,\text{Cos}[2t]-e^{-t}\,\text{Sin}[2t] & e^{-2t}-e^{-t}\,\text{Cos}[2t]-e^{-t}\,\text{Sin}[2t] & e^{-t}\,\text{Cos}[2t]+e^{-t}\,\text{Sin}[2t] \end{pmatrix}$$

$$\vec{\Phi}[0] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The solution to the D. E. with initial conditions $\vec{\mathbf{X}}[0] = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, is $\vec{\mathbf{F}}[t, x_0, y_0, z_0] = \vec{\Phi}[t] \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$.

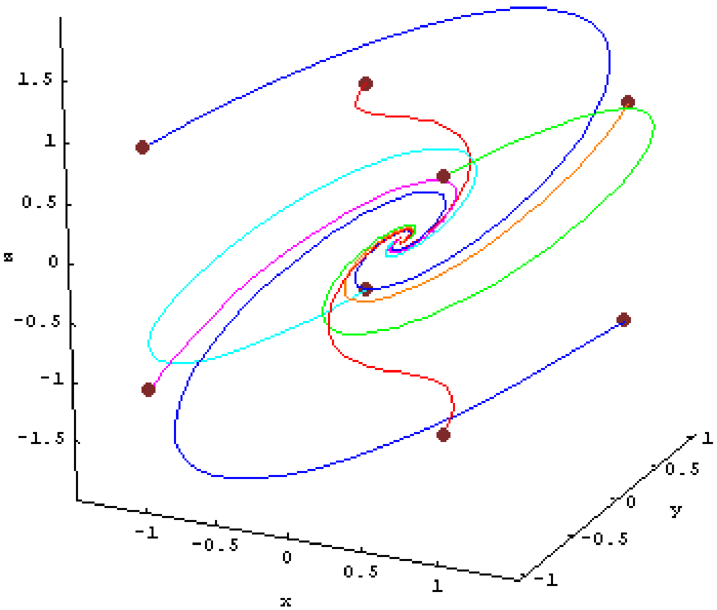
$$\vec{\Phi}[t] = \begin{pmatrix} \frac{1}{2} e^{(-1-2i)t} + \frac{1}{2} e^{(-1+2i)t} & -\frac{1}{2} i e^{(-1-2i)t} + \frac{1}{2} i e^{(-1+2i)t} & \frac{1}{2} i e^{(-1-2i)t} - \frac{1}{2} i e^{(-1+2i)t} \\ -e^{-2t} + \frac{1}{2} e^{(-1-2i)t} + \frac{1}{2} e^{(-1+2i)t} & e^{-2t} - \frac{1}{2} i e^{(-1-2i)t} + \frac{1}{2} i e^{(-1+2i)t} & \frac{1}{2} i e^{(-1-2i)t} - \frac{1}{2} i e^{(-1+2i)t} \\ -e^{-2t} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1-2i)t} + \left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1+2i)t} & e^{-2t} - \left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1-2i)t} - \left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1+2i)t} & \left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1-2i)t} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1+2i)t} \end{pmatrix}$$

$$\vec{F}[t] = \vec{\Phi}[t].\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\vec{F}[t] = \begin{pmatrix} \frac{1}{2} e^{(-1-2i)t} + \frac{1}{2} e^{(-1+2i)t} & -\frac{1}{2} i e^{(-1-2i)t} + \frac{1}{2} i e^{(-1+2i)t} & \frac{1}{2} i e^{(-1-2i)t} - \frac{1}{2} i e^{(-1+2i)t} \\ -e^{-2t} + \frac{1}{2} e^{(-1-2i)t} + \frac{1}{2} e^{(-1+2i)t} & e^{-2t} - \frac{1}{2} i e^{(-1-2i)t} + \frac{1}{2} i e^{(-1+2i)t} & \frac{1}{2} i e^{(-1-2i)t} - \frac{1}{2} i e^{(-1+2i)t} \\ -e^{-2t} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1-2i)t} + \left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1+2i)t} & e^{-2t} - \left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1-2i)t} - \left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1+2i)t} & \left(\frac{1}{2} + \frac{i}{2}\right) e^{(-1-2i)t} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{(-1+2i)t} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\vec{F}[t] = \begin{pmatrix} e^{-t} \operatorname{Cos}[2t]\, x_0 - e^{-t} \operatorname{Sin}[2t]\, y_0 + e^{-t} \operatorname{Sin}[2t]\, z_0 \\ -e^{-2t} x_0 + e^{-t} \operatorname{Cos}[2t]\, x_0 + e^{-2t} y_0 - e^{-t} \operatorname{Sin}[2t]\, y_0 + e^{-t} \operatorname{Sin}[2t]\, z_0 \\ -e^{-2t} x_0 + e^{-t} \operatorname{Cos}[2t]\, x_0 - e^{-t} \operatorname{Sin}[2t]\, x_0 + e^{-2t} y_0 - e^{-t} \operatorname{Cos}[2t]\, y_0 - e^{-t} \operatorname{Sin}[2t]\, y_0 + e^{-t} \operatorname{Cos}[2t]\, z_0 + e^{-t} \operatorname{Sin}[2t]\, z_0 \end{pmatrix}$$

$$\vec{F}[0] = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$



$$\vec{F}[t] = \begin{pmatrix} e^{-t}\,\mathrm{Cos}[2\,t]\,x_0 - e^{-t}\,\mathrm{Sin}[2\,t]\,y_0 + e^{-t}\,\mathrm{Sin}[2\,t]\,z_0 \\ -e^{-2\,t}\,x_0 + e^{-t}\,\mathrm{Cos}[2\,t]\,x_0 + e^{-2\,t}\,y_0 - e^{-t}\,\mathrm{Sin}[2\,t]\,y_0 + e^{-t}\,\mathrm{Sin}[2\,t]\,z_0 \\ -e^{-2\,t}\,x_0 + e^{-t}\,\mathrm{Cos}[2\,t]\,x_0 - e^{-t}\,\mathrm{Sin}[2\,t]\,x_0 + e^{-2\,t}\,y_0 - e^{-t}\,\mathrm{Cos}[2\,t]\,y_0 - e^{-t}\,\mathrm{Sin}[2\,t]\,y_0 + e^{-t}\,\mathrm{Cos}[2\,t]\,z_0 + e^{-t}\,\mathrm{Sin}[2\,t]\,z_0 \end{pmatrix}$$

For t in the interval $0 \le t \le 4$

$$\text{I.C.'s } \{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}\} = \left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}\right\}$$