# 7. Frobenius Series Solution of a Differential Equation

## Background.

Consider the second order linear differential equation

(1) $$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y(x) = 0.$$

Rewrite this equation in the form $y''(x) + \frac{Q(x)}{P(x)}y'(x) + \frac{R(x)}{P(x)}y(x) = 0$, then use the substitutions $f_1(x) = \frac{Q(x)}{P(x)}$ and $f_2(x) = \frac{R(x)}{P(x)}$ and rewrite the differential equation (1) in the form

(2) $$y''(x) + f_1(x)\,y'(x) + f_2(x)\,y(x) = 0.$$

**Definition (Analytic).** The functions $f_1(x)$ and $f_2(x)$ are analytic at $x = x_0$ if they have Taylor series expansions with radius of convergence $r_1 > 0$ and $r_2 > 0$, respectively. That is

$$f_1(x) = \frac{Q(x)}{P(x)} = \sum_{n=0}^{\infty} p_n (x - x_0)^n$$ which converges for $|x - x_0| < r_1$

and

$$f_2(x) = \frac{R(x)}{P(x)} = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$ which converges for $|x - x_0| < r_2$

**Definition (Ordinary Point).** If the functions $f_1(x) = \frac{Q(x)}{P(x)}$ and $f_2(x) = \frac{R(x)}{P(x)}$ are analytic at $x = x_0$, then the point $x = x_0$ is called an ordinary point of the differential equation

$$y''(x) + f_1(x)\,y'(x) + f_2(x)\,y(x) = 0.$$

Otherwise, the point $x = x_0$ is called a singular point.

**Definition (Regular Singular Point).** Assume that $x_0 = 0$ is a singular point of (1) and that $\phi_1(x) = x\frac{Q(x)}{P(x)}$ and $\phi_2(x) = x^2\frac{R(x)}{P(x)}$ are analytic at $x = 0$.

They will have Maclaurin series expansions with radius of convergence $r_1 > 0$ and $r_2 > 0$, respectively. That is

$$\phi_1(x) = x\frac{Q(x)}{P(x)} = \sum_{n=0}^{\infty} p_n x^n$$ which converges for $|x| < r_1$

and

$$\phi_2(x) = x^2 \frac{R(x)}{P(x)} = \sum_{n=0}^{\infty} q_n x^n$$ which converges for $|x| < r_2$

Then the point $x_0 = 0$ is called a regular singular point of the differential equation (1).

## Method of Frobenius.

This method is attributed to the german mathemematican Ferdinand Georg Frobenius (1849-1917 ). Assume that $x_0 = 0$ is regular singular point of the differential equation

$$P(x)\,y''(x) + Q(x)\,y'(x) + R(x)\,y(x) = 0.$$

A Frobenius series (generalized Laurent series) of the form

$$y(x) = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

can be used to solve the differential equation. The parameter $r$ must be chosen so that when the series is substituted into the D.E. the coefficient of the smallest power of $x$ is zero. This is called the indicial equation. Next, a recursive equation for the coefficients is obtained by setting the coefficient of $x^{n+r}$ equal to zero. Caveat: There are some instances when only one Frobenius solution can be constructed.

**Definition (Indicial Equation).** The parameter $r$ in the Frobenius series is a root of the indicial equation

$$r(r-1) + p_0 r + q_0 = 0.$$

Assuming that the singular point is $x_0$, we can calculate $p_0$ and $q_0$ as follows:

$$p_0 = \lim_{x \to x_0} x \frac{Q(x)}{P(x)}$$

and

$$q_0 = \lim_{x \to x_0} x^2 \frac{R(x)}{P(x)}$$

## The Recursive Formulas.

For each root $r$ of the indicial equation, recursive formulas are used to calculate the unknown coefficients $\{c_n\}_{n=1}^{\infty}$. This is custom work because a numerical value for $r$ is easier use.

**Example 1.** Use Frobenius series to solve the D. E.

$$x^2 y''[x] + x y'[x] + \left(x^2 - \left(\frac{1}{2}\right)^2\right) y[x] = 0.$$

**Solution 1.**

**Example 2.** Use Frobenius series to solve the D. E.

$$2 x y''[x] + 3 y'[x] - y[x] = 0.$$

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**Example 1.** Use Frobenius series to solve the D. E.

$$x^2 y''[x] + x y'[x] + \left(x^2 - \left(\frac{1}{2}\right)^2\right) y[x] = 0.$$

**Solution 1.**

Determine the nature of the singularity at $x_0 = 0$.

$$\left(-\frac{1}{4} + x^2\right) y[x] + x y'[x] + x^2 y''[x] == 0$$

$P[x] = x^2$

$P[0] = 0$

The point $x_0 = 0$ is a singular point of the D. E.

Proceed with a Frobenius solution.

**Construct the Indicial Equation.**

$P[x] = x^2$

$Q[x] = x$

$R[x] = -\frac{1}{4} + x^2$

$$\phi_1[x\_] = x\frac{Q[x]}{P[x]} = 1$$

$$p_0 = \lim_{x\to 0} x\frac{Q[x]}{P[x]} = 1$$

$$\phi_2[x\_] = x^2\frac{R[x]}{P[x]} = -\frac{1}{4} + x^2$$

$$q_0 = \lim_{x\to 0} x^2\frac{R[x]}{P[x]} = -\frac{1}{4}$$

The indicial equation is

$$r(r-1) + p_0\, r + q_0 == 0$$

$$-\frac{1}{4} + r + (-1+r)\, r == 0$$

$$-\frac{1}{4} + r^2 == 0$$

**Find the Roots of the Indicial Equation.**

The indicial equation is

$$-\frac{1}{4}+r^2==0$$

The roots are

$$r_1 = \frac{1}{2}$$

$$r_2 = -\frac{1}{2}$$

Form the first Frobenius solution corresponding to the larger root $r_1 = \frac{1}{2}$.

$$s[x] = c_0\sqrt{x} + c_1 x^{3/2} + c_2 x^{5/2} + c_3 x^{7/2} + c_4 x^{9/2} + c_5 x^{11/2} + c_6 x^{13/2} + c_7 x^{15/2} + c_8 x^{17/2} + c_9 x^{19/2} + O[x]^{21/2}$$

$$s'[x] = \frac{c_0}{2\sqrt{x}} + \frac{3c_1\sqrt{x}}{2} + \frac{5}{2}c_2 x^{3/2} + \frac{7}{2}c_3 x^{5/2} + \frac{9}{2}c_4 x^{7/2} + \frac{11}{2}c_5 x^{9/2} + \frac{13}{2}c_6 x^{11/2} + \frac{15}{2}c_7 x^{13/2} + \frac{17}{2}c_8 x^{15/2} + \frac{19}{2}c_9 x^{17/2} + O[x]^{19/2}$$

$$s''[x] = -\frac{c_0}{4x^{3/2}} + \frac{3c_1}{4\sqrt{x}} + \frac{15c_2\sqrt{x}}{4} + \frac{35}{4}c_3 x^{3/2} + \frac{63}{4}c_4 x^{5/2} + \frac{99}{4}c_5 x^{7/2} + \frac{143}{4}c_6 x^{9/2} + \frac{195}{4}c_7 x^{11/2} + \frac{255}{4}c_8 x^{13/2} + \frac{323}{4}c_9 x^{15/2} + O[x]^{17/2}$$

Substitute into

$$x^2\ s''[x] + x\ s'[x] + (x^2 - (\frac{1}{2})^2)s[x] == 0$$

Get

$$2c_1 x^{3/2} + (c_0+6c_2)x^{5/2} + (c_1+12c_3)x^{7/2} + (c_2+20c_4)x^{9/2} + (c_3+30c_5)x^{11/2} + (c_4+42c_6)x^{13/2} + (c_5+56c_7)x^{15/2} + (c_6+72c_8)x^{17/2} + (c_7+90c_9)x^{19/2} + O[x]^{21/2} == 0$$

Form the set of equations to solve and do it.

$$2c_1 x^{3/2} + (c_0+6c_2)x^{5/2} + (c_1+12c_3)x^{7/2} + (c_2+20c_4)x^{9/2} + (c_3+30c_5)x^{11/2} + (c_4+42c_6)x^{13/2} + (c_5+56c_7)x^{15/2} + (c_6+72c_8)x^{17/2} + (c_7+90c_9)x^{19/2} + O[x]^{21/2} == 0$$

$2c_1 == 0$
$c_0 + 6c_2 == 0$
$c_1 + 12c_3 == 0$
$c_2 + 20c_4 == 0$
$c_3 + 30c_5 == 0$
$c_4 + 42c_6 == 0$
$c_5 + 56c_7 == 0$
$c_6 + 72c_8 == 0$
$c_7 + 90c_9 == 0$

$c_1 \to 0$

$c_2 \to -\frac{c_0}{6}$

$c_3 \to 0$

$c_4 \to \frac{c_0}{120}$

$c_5 \to 0$

$c_6 \to -\frac{c_0}{5040}$

$c_7 \to 0$

$c_8 \to \frac{c_0}{362880}$

$c_9 \to 0$

$$y = c_0\sqrt{x} - \frac{1}{6} c_0 x^{5/2} + \frac{1}{120} c_0 x^{9/2} - \frac{c_0 x^{13/2}}{5040} + \frac{c_0 x^{17/2}}{362880} + 0[x]^{21/2}$$

$$y = \sqrt{x}\, c_0 - \frac{1}{6} x^{5/2} c_0 + \frac{1}{120} x^{9/2} c_0 - \frac{x^{13/2} c_0}{5040} + \frac{x^{17/2} c_0}{362880} + \ldots$$

$$y = c_0\sqrt{x}\,(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{362880} + \ldots)$$

The first Frobenius solution is:

$$f1[x] = \sqrt{x} - \frac{x^{5/2}}{6} + \frac{x^{9/2}}{120} - \frac{x^{13/2}}{5040} + \frac{x^{17/2}}{362880} + 0[x]^{21/2}$$

$$f1[x] = \sqrt{x} - \frac{x^{5/2}}{6} + \frac{x^{9/2}}{120} - \frac{x^{13/2}}{5040} + \frac{x^{17/2}}{362880} + \ldots$$

$$f1[x] = \sqrt{x}\,(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{362880} + \ldots)$$

Form the second Frobenius solution corresponding to the smaller root $r_2 = \frac{-1}{2}$.

$$s[x] = \frac{c_0}{\sqrt{x}} + c_1\sqrt{x} + c_2 x^{3/2} + c_3 x^{5/2} + c_4 x^{7/2} + c_5 x^{9/2} + c_6 x^{11/2} + c_7 x^{13/2} + c_8 x^{15/2} + c_9 x^{17/2} + c_{10} x^{19/2} + c_{11} x^{21/2} + 0[x]^{23/2}$$

$$s'[x] = -\frac{c_0}{2x^{3/2}} + \frac{c_1}{2\sqrt{x}} + \frac{3c_2\sqrt{x}}{2} + \frac{5}{2} c_3 x^{3/2} + \frac{7}{2} c_4 x^{5/2} + \frac{9}{2} c_5 x^{7/2} + \frac{11}{2} c_6 x^{9/2} + \frac{13}{2} c_7 x^{11/2} + \frac{15}{2} c_8 x^{13/2} + \frac{17}{2} c_9 x^{15/2} + \frac{19}{2} c_{10} x^{17/2} + \frac{21}{2} c_{11} x^{19/2} + 0[x]^{21/2}$$

$$s''[x] = \frac{3c_0}{4x^{5/2}} - \frac{c_1}{4x^{3/2}} + \frac{3c_2}{4\sqrt{x}} + \frac{15 c_3\sqrt{x}}{4} + \frac{35}{4} c_4 x^{3/2} + \frac{63}{4} c_5 x^{5/2} + \frac{99}{4} c_6 x^{7/2} + \frac{143}{4} c_7 x^{9/2} + \frac{195}{4} c_8 x^{11/2} + \frac{255}{4} c_9 x^{13/2} + \frac{323}{4} c_{10} x^{15/2} + \frac{399}{4} c_{11} x^{17/2} + 0[x]^{19/2}$$

Substitute into

$$x^2\, s''[x] + x\, s'[x] + (x^2 - (\frac{1}{2})^2) s[x] == 0$$

Get

$$(c_0 + 2c_2)\, x^{3/2} + (c_1 + 6c_3)\, x^{5/2} + (c_2 + 12c_4)\, x^{7/2} + (c_3 + 20c_5)\, x^{9/2} + (c_4 + 30c_6)\, x^{11/2} + (c_5 + 42c_7)\, x^{13/2} + (c_6 + 56c_8)\, x^{15/2} + (c_7 + 72c_9)\, x^{17/2} + (c_8 + 90c_{10})\, x^{19/2} + (c_9 + 110c_{11})\, x^{21/2} + 0[x]^{23/2} == 0$$

Form the set of equations to solve and do it.

$$(c_0+2c_2)x^{3/2}+(c_1+6c_3)x^{5/2}+(c_2+12c_4)x^{7/2}+(c_3+20c_5)x^{9/2}+(c_4+30c_6)x^{11/2}+(c_5+42c_7)x^{13/2}+(c_6+56c_8)x^{15/2}+(c_7+72c_9)x^{17/2}+(c_8+90c_{10})x^{19/2}+(c_9+110c_{11})x^{21/2}+0[x]^{23/2}==0$$

$c_0+2c_2==0$
$c_1+6c_3==0$
$c_2+12c_4==0$
$c_3+20c_5==0$
$c_4+30c_6==0$
$c_5+42c_7==0$
$c_6+56c_8==0$
$c_7+72c_9==0$
$c_8+90c_{10}==0$
$c_9+110c_{11}==0$

$c_2\to-\frac{c_0}{2}$
$c_3\to-\frac{c_1}{6}$
$c_4\to\frac{c_0}{24}$
$c_5\to\frac{c_1}{120}$
$c_6\to-\frac{c_0}{720}$
$c_7\to-\frac{c_1}{5040}$
$c_8\to\frac{c_0}{40320}$
$c_9\to\frac{c_1}{362880}$
$c_{10}\to-\frac{c_0}{3628800}$
$c_{11}\to-\frac{c_1}{39916800}$

$$y=\frac{c_0}{\sqrt{x}}+c_1\sqrt{x}-\frac{1}{2}c_0x^{3/2}-\frac{1}{6}c_1x^{5/2}+\frac{1}{24}c_0x^{7/2}+\frac{1}{120}c_1x^{9/2}-\frac{1}{720}c_0x^{11/2}-\frac{c_1x^{13/2}}{5040}+\frac{c_0x^{15/2}}{40320}+\frac{c_1x^{17/2}}{362880}-\frac{c_0x^{19/2}}{3628800}-\frac{c_1x^{21/2}}{39916800}+0[x]^{23/2}$$

$$y=\frac{c_0}{\sqrt{x}}-\frac{1}{2}x^{3/2}c_0+\frac{1}{24}x^{7/2}c_0-\frac{1}{720}x^{11/2}c_0+\frac{x^{15/2}c_0}{40320}-\frac{x^{19/2}c_0}{3628800}+\sqrt{x}\,c_1-\frac{1}{6}x^{5/2}c_1+\frac{1}{120}x^{9/2}c_1-\frac{x^{13/2}c_1}{5040}+\frac{x^{17/2}c_1}{362880}-\frac{x^{21/2}c_1}{39916800}+\dots$$

$$y=c_0\frac{1}{\sqrt{x}}\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\frac{x^{10}}{3628800}+\frac{x\,c_1}{c_0}-\frac{x^3c_1}{6c_0}+\frac{x^5c_1}{120c_0}-\frac{x^7c_1}{5040c_0}+\frac{x^9c_1}{362880c_0}-\frac{x^{11}c_1}{39916800c_0}+\dots\right)$$

The second Frobenius solution is:

$$f2[x]=\frac{1}{\sqrt{x}}+c_1\sqrt{x}-\frac{x^{3/2}}{2}-\frac{1}{6}c_1x^{5/2}+\frac{x^{7/2}}{24}+\frac{1}{120}c_1x^{9/2}-\frac{x^{11/2}}{720}-\frac{c_1x^{13/2}}{5040}+\frac{x^{15/2}}{40320}+\frac{c_1x^{17/2}}{362880}-\frac{x^{19/2}}{3628800}-\frac{c_1x^{21/2}}{39916800}+0[x]^{23/2}$$

$$f2[x]=\frac{1}{\sqrt{x}}-\frac{x^{3/2}}{2}+\frac{x^{7/2}}{24}-\frac{x^{11/2}}{720}+\frac{x^{15/2}}{40320}-\frac{x^{19/2}}{3628800}+\sqrt{x}\,c_1-\frac{1}{6}x^{5/2}c_1+\frac{1}{120}x^{9/2}c_1-\frac{x^{13/2}c_1}{5040}+\frac{x^{17/2}c_1}{362880}-\frac{x^{21/2}c_1}{39916800}+\dots$$

Observe that the coefficients that involve $c_1$ are that multiple of the first Frobenius solution. Hence we can set $c_1=0$.

$$f2[x] = \frac{1}{\sqrt{x}} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24} - \frac{x^{11/2}}{720} + \frac{x^{15/2}}{40320} - \frac{x^{19/2}}{3628800} + O[x]^{23/2}$$

$$f2[x] = \frac{1}{\sqrt{x}} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24} - \frac{x^{11/2}}{720} + \frac{x^{15/2}}{40320} - \frac{x^{19/2}}{3628800} + \ldots$$

$$y = \frac{1}{\sqrt{x}} \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \ldots\right)$$

After you are done, use *Mathematica*'s DSolve subroutine to get the answer and check out its series expansion.

$$\left(-\frac{1}{4} + x^2\right) y[x] + x\, y'[x] + x^2\, y''[x] == 0$$

$$\left\{\left\{y[x] \to \frac{e^{-i x} C[1]}{\sqrt{x}} - \frac{i\, e^{i x} C[2]}{2\sqrt{x}}\right\}\right\}$$

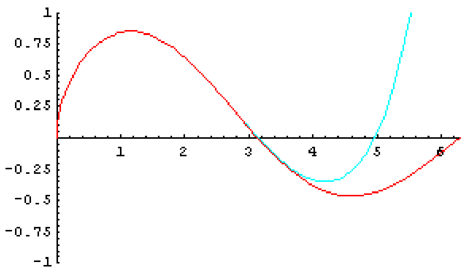
$$y[x] = \frac{Cos[x]\, c_1}{\sqrt{x}} + \frac{Sin[x]\, c_2}{\sqrt{x}}$$

Two linearly independent solutions are

$$g1[x] = \frac{Sin[x]}{\sqrt{x}}$$

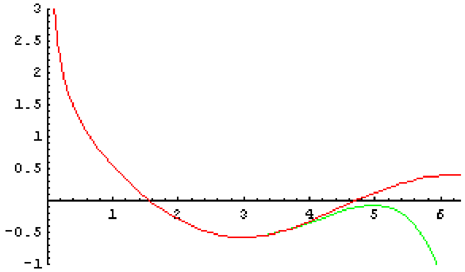
$$g2[x] = \frac{Cos[x]}{\sqrt{x}}$$

Now we plot the series approximations and the analytic solutions.




$$S1[x] = \sqrt{x} - \frac{x^{5/2}}{6} + \frac{x^{9/2}}{120} - \frac{x^{13/2}}{5040} + \frac{x^{17/2}}{362880}$$

$$g1[x] = \frac{Sin[x]}{\sqrt{x}}$$




$$S2[x] = \frac{1}{\sqrt{x}} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24} - \frac{x^{11/2}}{720} + \frac{x^{15/2}}{40320} - \frac{x^{19/2}}{3628800}$$

$$g2[x] = \frac{\mathrm{Cos}[x]}{\sqrt{x}}$$

**The recursive formula for the coefficients.**

If we look at the series in more depth we will be able to obtain the analytic solutions as infinite sums. First find the recursive formula for the coefficients of $x^{r+n}$. If you try this be sure to use the " := " replacement delayed structure to avoid an infinite recursion. For this example the trial term $t[x] = \sum_{k=n-2}^{n} c_k x^{r+k}$ works with the replacements $\{x^{n+r} \to 1,\ x^{-1+n+r} \to 0,\ x^{-2+n+r} \to 0,\ x^{1+n+r} \to 0,\ x^{2+n+r} \to 0\}$.

The general term in the sum is

$$\frac{15}{4} x^{-2+n+r} c_{-2+n} - 4 n x^{-2+n+r} c_{-2+n} + n^2 x^{-2+n+r} c_{-2+n} - 4 r x^{-2+n+r} c_{-2+n} + 2 n r x^{-2+n+r} c_{-2+n} + r^2 x^{-2+n+r} c_{-2+n} + x^{n+r} c_{-2+n} + \frac{3}{4} x^{-1+n+r} c_{-1+n} - 2 n x^{-1+n+r} c_{-1+n} + n^2 x^{-1+n+r} c_{-1+n} - 2 r x^{-1+n+r} c_{-1+n} + 2 n r x^{-1+n+r} c_{-1+n} + r^2 x^{-1+n+r} c_{-1+n} + x^{1+n+r} c_{-1+n} - \frac{1}{4} x^{n+r} c_n + n^2 x^{n+r} c_n + 2 n r x^{n+r} c_n + r^2 x^{n+r} c_n + x^{2+n+r} c_n == 0$$

Simplify it.

$$c_{-2+n} - \frac{c_n}{4} + n^2 c_n + 2 n r c_n + r^2 c_n == 0$$

Solve it for the recursive formula.

$$c_n \to -\frac{c_{-2+n}}{-\frac{1}{4} + n^2 + 2 n r + r^2}$$

Now look at each series individually. The first Frobenius series corresponds to $r_1 = \frac{1}{2}$.

$c_0 = 1$
$c_1 = 0$
$c_2 = -\frac{1}{6}$
$c_3 = 0$
$c_4 = \frac{1}{120}$
$c_5 = 0$
$c_6 = -\frac{1}{5040}$
$c_7 = 0$
$c_8 = \frac{1}{362880}$
$c_9 = 0$
$c_{10} = -\frac{1}{39916800}$

$$c_n = -\frac{c_{n-2}}{n\,(1+n)}$$

$$s1[x] = \sqrt{x} - \frac{x^{5/2}}{6} + \frac{x^{9/2}}{120} - \frac{x^{13/2}}{5040} + \frac{x^{17/2}}{362880} - \frac{x^{21/2}}{39916800}$$

Now look at the second Frobenius series which corresponds to $r_2 = \frac{-1}{2}$.

$c_0 = 1$
$c_1 = 0$
$c_2 = -\frac{1}{2}$
$c_3 = 0$
$c_4 = \frac{1}{24}$
$c_5 = 0$
$c_6 = -\frac{1}{720}$
$c_7 = 0$
$c_8 = \frac{1}{40320}$
$c_9 = 0$
$c_{10} = -\frac{1}{3628800}$

$$c_n = \frac{c_{n-2}}{n - n^2}$$

$$s2[x] = \frac{1}{\sqrt{x}} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24} - \frac{x^{11/2}}{720} + \frac{x^{15/2}}{40320} - \frac{x^{19/2}}{3628800}$$

When the explicit formulas for the coefficients of the first Frobenius are used we get:

$a_0 = 1$

$a_1 = -\frac{1}{6}$

$a_2 = \frac{1}{120}$

$a_3 = -\frac{1}{5040}$

$a_4 = \frac{1}{362880}$

$a_5 = -\frac{1}{39916800}$

$a_6 = \frac{1}{6227020800}$

$a_7 = -\frac{1}{1307674368000}$

$a_8 = \frac{1}{355687428096000}$

$a_9 = -\frac{1}{121645100408832000}$

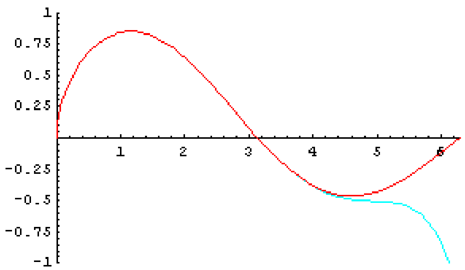
$a_{10} = \frac{1}{51090942171709440000}$

$$a_k = \frac{(-1)^k}{(1+2k)!}$$

$$f1[x] = \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + \frac{x^8}{362880} - \frac{x^{10}}{39916800} + \frac{x^{12}}{6227020800} - \frac{x^{14}}{1307674368000} + \frac{x^{16}}{355687428096000} - \frac{x^{18}}{121645100408832000} + \frac{x^{20}}{51090942171709440000}\right) + \ldots$$

$$f1[x] = \frac{\sqrt{x^2}\,\mathrm{Sin}[\sqrt{x^2}]}{x^{3/2}}$$

$$f1[x] = \frac{\mathrm{Sin}[x]}{\sqrt{x}}$$




$$s1[x] = \sqrt{x} - \frac{x^{5/2}}{6} + \frac{x^{9/2}}{120} - \frac{x^{13/2}}{5040} + \frac{x^{17/2}}{362880} - \frac{x^{21/2}}{39916800}$$

$$f1[x] = \frac{\mathrm{Sin}[x]}{\sqrt{x}}$$

When the explicit formulas for the coefficients of the second Frobenius are used we get:

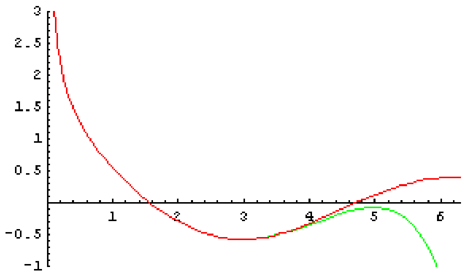
$$b_0 = 1$$
$$b_1 = -\frac{1}{2}$$
$$b_2 = \frac{1}{24}$$
$$b_3 = -\frac{1}{720}$$
$$b_4 = \frac{1}{40320}$$
$$b_5 = -\frac{1}{3628800}$$
$$b_6 = \frac{1}{479001600}$$
$$b_7 = -\frac{1}{87178291200}$$
$$b_8 = \frac{1}{20922789888000}$$
$$b_9 = -\frac{1}{6402373705728000}$$
$$b_{10} = \frac{1}{2432902008176640000}$$

$$b_k = \frac{(-1)^k}{(2k)!}$$

$$f2[x] = \frac{1}{\sqrt{x}}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} - \frac{x^{14}}{87178291200} + \frac{x^{16}}{20922789888000} - \frac{x^{18}}{6402373705728000} + \frac{x^{20}}{2432902008176640000} + \ldots\right)$$

$$f2[x] = \frac{\mathrm{Cos}\left[\sqrt{x^2}\right]}{\sqrt{x}}$$

$$f2[x] = \frac{\mathrm{Cos}[x]}{\sqrt{x}}$$




$$s2[x] = \frac{1}{\sqrt{x}} - \frac{x^{3/2}}{2} + \frac{x^{7/2}}{24} - \frac{x^{11/2}}{720} + \frac{x^{15/2}}{40320} - \frac{x^{19/2}}{3628800}$$

$$f2[x] = \frac{\mathrm{Cos}[x]}{\sqrt{x}}$$

**Example 2.** Use Frobenius series to solve the D. E.

$2 x y''[x] + 3 y'[x] - y[x] = 0.$

**Solution 2.**

Determine the nature of the singularity at $x_0 = 0$.

$-y[x] + 3 y'[x] + 2 x y''[x] == 0$

$P[x] = 2x$

$P[0] = 0$

The point $x_0 = 0$ is a singular point.

Proceed with a Frobenius solution.

**Construct the Indicial Equation.**

$P[x] = 2x$

$Q[x] = 3$

$R[x] = -1$

$$\phi_1[x\_] = x\frac{Q[x]}{P[x]} = \frac{3}{2}$$

$$p_0 = \lim_{x\to 0} x\frac{Q[x]}{P[x]} = \frac{3}{2}$$

$$\phi_2[x\_] = x^2\frac{R[x]}{P[x]} = -\frac{x}{2}$$

$$q_0 = \lim_{x\to 0} x^2\frac{R[x]}{P[x]} = 0$$

The indicial equation is

$r(r-1) + p_0\, r + q_0 == 0$

$$\frac{3r}{2} + (-1+r)\ r == 0$$

$$\frac{r}{2} + r^2 == 0$$

**Find the Roots of the Indicial Equation.**

The indicial equation is

$$\frac{r}{2} + r^2 == 0$$

The roots are

$$r_1 = -\frac{1}{2}$$

$r_2 = 0$

Form the first Frobenius solution corresponding to the root $r_1 = \frac{-1}{2}$ .

$$s[x] = \frac{c_0}{\sqrt{x}} + c_1\sqrt{x} + c_2 x^{3/2} + c_3 x^{5/2} + c_4 x^{7/2} + c_5 x^{9/2} + c_6 x^{11/2} + c_7 x^{13/2} + O[x]^{15/2}$$

$$s'[x] = -\frac{c_0}{2x^{3/2}} + \frac{c_1}{2\sqrt{x}} + \frac{3c_2\sqrt{x}}{2} + \frac{5}{2}c_3 x^{3/2} + \frac{7}{2}c_4 x^{5/2} + \frac{9}{2}c_5 x^{7/2} + \frac{11}{2}c_6 x^{9/2} + \frac{13}{2}c_7 x^{11/2} + O[x]^{13/2}$$

$$s''[x] = \frac{3c_0}{4x^{5/2}} - \frac{c_1}{4x^{3/2}} + \frac{3c_2}{4\sqrt{x}} + \frac{15c_3\sqrt{x}}{4} + \frac{35}{4}c_4 x^{3/2} + \frac{63}{4}c_5 x^{5/2} + \frac{99}{4}c_6 x^{7/2} + \frac{143}{4}c_7 x^{9/2} + O[x]^{11/2}$$

Substitute into

$$2\,x\,s''[x] + 3\,s'[x] - s[x] == 0$$

Get

$$\frac{-c_0 + c_1}{\sqrt{x}} + (-c_1 + 6c_2)\sqrt{x} + (-c_2 + 15c_3)x^{3/2} + (-c_3 + 28c_4)x^{5/2} + (-c_4 + 45c_5)x^{7/2} + (-c_5 + 66c_6)x^{9/2} + (-c_6 + 91c_7)x^{11/2} + O[x]^{13/2} == 0$$

Form the set of equations to solve and do it.

$$\frac{-c_0 + c_1}{\sqrt{x}} + (-c_1 + 6c_2)\sqrt{x} + (-c_2 + 15c_3)x^{3/2} + (-c_3 + 28c_4)x^{5/2} + (-c_4 + 45c_5)x^{7/2} + (-c_5 + 66c_6)x^{9/2} + (-c_6 + 91c_7)x^{11/2} + O[x]^{13/2} == 0$$

$-c_0 + c_1 == 0$
$-c_1 + 6c_2 == 0$
$-c_2 + 15c_3 == 0$
$-c_3 + 28c_4 == 0$
$-c_4 + 45c_5 == 0$
$-c_5 + 66c_6 == 0$
$-c_6 + 91c_7 == 0$

$c_1 \to c_0$
$c_2 \to \frac{c_0}{6}$
$c_3 \to \frac{c_0}{90}$
$c_4 \to \frac{c_0}{2520}$
$c_5 \to \frac{c_0}{113400}$
$c_6 \to \frac{c_0}{7484400}$
$c_7 \to \frac{c_0}{681080400}$

$$y = \frac{c_0}{\sqrt{x}} + c_0\sqrt{x} + \frac{1}{6}c_0 x^{3/2} + \frac{1}{90}c_0 x^{5/2} + \frac{c_0 x^{7/2}}{2520} + \frac{c_0 x^{9/2}}{113400} + \frac{c_0 x^{11/2}}{7484400} + \frac{c_0 x^{13/2}}{681080400} + O[x]^{15/2}$$

$$y = \frac{c_0}{\sqrt{x}} + \sqrt{x}\,c_0 + \frac{1}{6}x^{3/2}c_0 + \frac{1}{90}x^{5/2}c_0 + \frac{x^{7/2}c_0}{2520} + \frac{x^{9/2}c_0}{113400} + \frac{x^{11/2}c_0}{7484400} + \frac{x^{13/2}c_0}{681080400} + \ldots$$

The first Frobenius solution is:

$$f1[x] = \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{6} + \frac{x^{5/2}}{90} + \frac{x^{7/2}}{2520} + \frac{x^{9/2}}{113400} + \frac{x^{11/2}}{7484400} + \frac{x^{13/2}}{681080400} + O[x]^{15/2}$$

$$f1[x] = \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{6} + \frac{x^{5/2}}{90} + \frac{x^{7/2}}{2520} + \frac{x^{9/2}}{113400} + \frac{x^{11/2}}{7484400} + \frac{x^{13/2}}{681080400} + \ldots$$

$$f1[x] = \frac{1}{\sqrt{x}} \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + \frac{x^6}{7484400} + \frac{x^7}{681080400} + \ldots\right)$$

Form the second Frobenius solution corresponding to the root $r_2 = 0$.

$$s[x] = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + O[x]^8$$

$$s'[x] = c_1 + 2 c_2 x + 3 c_3 x^2 + 4 c_4 x^3 + 5 c_5 x^4 + 6 c_6 x^5 + 7 c_7 x^6 + O[x]^7$$

$$s''[x] = 2 c_2 + 6 c_3 x + 12 c_4 x^2 + 20 c_5 x^3 + 30 c_6 x^4 + 42 c_7 x^5 + O[x]^6$$

Substitute into

2 x s''[x] + 3 s'[x] - s[x] == 0

Get

$$(-c_0 + 3 c_1) + (-c_1 + 10 c_2) x + (-c_2 + 21 c_3) x^2 + (-c_3 + 36 c_4) x^3 + (-c_4 + 55 c_5) x^4 + (-c_5 + 78 c_6) x^5 + (-c_6 + 105 c_7) x^6 + O[x]^7 == 0$$

Form the set of equations to solve and do it.

$$(-c_0 + 3 c_1) + (-c_1 + 10 c_2) x + (-c_2 + 21 c_3) x^2 + (-c_3 + 36 c_4) x^3 + (-c_4 + 55 c_5) x^4 + (-c_5 + 78 c_6) x^5 + (-c_6 + 105 c_7) x^6 + O[x]^7 == 0$$

$-c_0 + 3 c_1 == 0$
$-c_1 + 10 c_2 == 0$
$-c_2 + 21 c_3 == 0$
$-c_3 + 36 c_4 == 0$
$-c_4 + 55 c_5 == 0$
$-c_5 + 78 c_6 == 0$
$-c_6 + 105 c_7 == 0$

$c_1 \to \frac{c_0}{3}$
$c_2 \to \frac{c_0}{30}$
$c_3 \to \frac{c_0}{630}$
$c_4 \to \frac{c_0}{22680}$
$c_5 \to \frac{c_0}{1247400}$
$c_6 \to \frac{c_0}{97297200}$
$c_7 \to \frac{c_0}{10216206000}$

$$y = c_0 + \frac{c_0 x}{3} + \frac{c_0 x^2}{30} + \frac{c_0 x^3}{630} + \frac{c_0 x^4}{22680} + \frac{c_0 x^5}{1247400} + \frac{c_0 x^6}{97297200} + \frac{c_0 x^7}{10216206000} + O[x]^8$$

$$y = c_0 + \frac{x c_0}{3} + \frac{x^2 c_0}{30} + \frac{x^3 c_0}{630} + \frac{x^4 c_0}{22680} + \frac{x^5 c_0}{1247400} + \frac{x^6 c_0}{97297200} + \frac{x^7 c_0}{10216206000} + \ldots$$

The second Frobenius solution is:

$$f2[x] = 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + \frac{x^6}{97297200} + \frac{x^7}{10216206000} + O[x]^8$$

$$f2[x] = 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + \frac{x^6}{97297200} + \frac{x^7}{10216206000} + \ldots$$

After you are done, use *Mathematica*'s DSolve subroutine to get the answer and check out its series expansion.
This will require slight fussing around with the appropriate multiple of "`Sinh`".

$$-y[x] + 3\,y'[x] + 2\,x\,y''[x] == 0$$

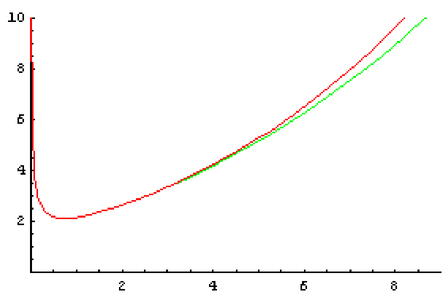
$$\left\{\left\{y[x] \to \frac{e^{\sqrt{2}\sqrt{x}}\,C[1]}{\sqrt{x}} - \frac{e^{-\sqrt{2}\sqrt{x}}\,C[2]}{\sqrt{2}\,\sqrt{x}}\right\}\right\}$$

Two linearly independent solutions are

$$g1[x] = \frac{\mathrm{Cosh}\left[\sqrt{2}\,\sqrt{x}\right]}{\sqrt{x}}$$

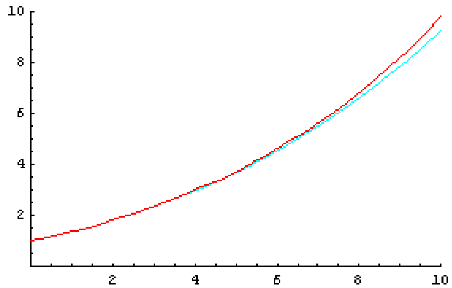
$$g2[x] = \frac{\mathrm{Sinh}\left[\sqrt{2}\,\sqrt{x}\right]}{\sqrt{2}\,\sqrt{x}}$$

At this time we could plot the series approximations and the analytic solutions. To see the difference in the graphs we will reduce the number of terms in the series.




$$S1[x] = \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{6} + \frac{x^{5/2}}{90}$$

$$g1[x] = \frac{\mathrm{Cosh}\left[\sqrt{2}\,\sqrt{x}\right]}{\sqrt{x}}$$




$$S2[x] = 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630}$$

$$g2[x] = \frac{\mathrm{Sinh}\left[\sqrt{2}\ \sqrt{x}\right]}{\sqrt{2}\ \sqrt{x}}$$

**The recursive formula for the coefficients.**

If we look at the series in more depth we will be able to obtain the analytic solutions as infinite sums. First find the recursive formula for the coefficients of $x^{r+n}$. If you try this be sure to use the " := " replacement delayed structure to avoid an infinite recursion. For this example the trial term $t[x] = \sum_{k=n-1}^{n} c_k x^{r+k}$ works with the replacements $\{x^{n+r} \to 0,\ x^{-1+n+r} \to 1,\ x^{-2+n+r} \to 0,\ x^{1+n+r} \to 0,\ x^{2+n+r} \to 0\}$.

The general term in the sum is

$$x^{-2+n+r} c_{-1+n} - 3 n x^{-2+n+r} c_{-1+n} + 2 n^2 x^{-2+n+r} c_{-1+n} - 3 r x^{-2+n+r} c_{-1+n} + 4 n r x^{-2+n+r} c_{-1+n} + 2 r^2 x^{-2+n+r} c_{-1+n} - x^{-1+n+r} c_{-1+n} + n x^{-1+n+r} c_n + 2 n^2 x^{-1+n+r} c_n + r x^{-1+n+r} c_n + 4 n r x^{-1+n+r} c_n + 2 r^2 x^{-1+n+r} c_n - x^{n+r} c_n == 0$$

Simplify it.

$$-c_{-1+n} + n c_n + 2 n^2 c_n + r c_n + 4 n r c_n + 2 r^2 c_n == 0$$

Solve it for the recursive formula.

$$c_n \to \frac{c_{-1+n}}{n + 2 n^2 + r + 4 n r + 2 r^2}$$

Now look at each series individually. The first Frobenius corresponds to $r_1 = \frac{-1}{2}$.

$c_0 = 1$
$c_1 = 1$
$c_2 = \frac{1}{6}$
$c_3 = \frac{1}{90}$
$c_4 = \frac{1}{2520}$
$c_5 = \frac{1}{113400}$
$c_6 = \frac{1}{7484400}$
$c_7 = \frac{1}{681080400}$
$c_8 = \frac{1}{81729648000}$
$c_9 = \frac{1}{12504636144000}$
$c_{10} = \frac{1}{2375880867360000}$

$$c_n = \frac{c_{n-1}}{-n + 2 n^2}$$

$$s1[x] = \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{6} + \frac{x^{5/2}}{90} + \frac{x^{7/2}}{2520} + \frac{x^{9/2}}{113400} + \frac{x^{11/2}}{7484400} + \frac{x^{13/2}}{681080400} + \frac{x^{15/2}}{81729648000} + \frac{x^{17/2}}{12504636144000} + \frac{x^{19/2}}{2375880867360000} + \frac{x^{21/2}}{548828480360160000}$$

Now look at the second Frobenius series corresponding to $r_2 = 0$.

$c_0 = 1$
$c_1 = \frac{1}{3}$
$c_2 = \frac{1}{30}$
$c_3 = \frac{1}{630}$
$c_4 = \frac{1}{22680}$
$c_5 = \frac{1}{1247400}$
$c_6 = \frac{1}{97297200}$
$c_7 = \frac{1}{10216206000}$
$c_8 = \frac{1}{1389404016000}$
$c_9 = \frac{1}{237588086736000}$
$c_{10} = \frac{1}{49893498214560000}$

$$c_n = \frac{c_{n-1}}{n + 2 n^2}$$

$$s2[x] = 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + \frac{x^6}{97297200} + \frac{x^7}{10216206000} + \frac{x^8}{1389404016000} + \frac{x^9}{237588086736000} + \frac{x^{10}}{49893498214560000} + \frac{x^{11}}{12623055048283680000}$$

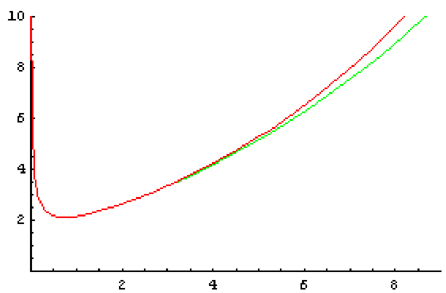
The explicit formulas for the coefficients for the first Frobenius series are:

$$a_0 = 1$$
$$a_1 = 1$$
$$a_2 = \frac{1}{6}$$
$$a_3 = \frac{1}{90}$$
$$a_4 = \frac{1}{2520}$$
$$a_5 = \frac{1}{113400}$$
$$a_6 = \frac{1}{7484400}$$
$$a_7 = \frac{1}{681080400}$$
$$a_8 = \frac{1}{81729648000}$$
$$a_9 = \frac{1}{12504636144000}$$
$$a_{10} = \frac{1}{2375880867360000}$$

$$a_k = \frac{1}{k!\,(-1+2k)!!}$$

$$f1[x] = \frac{1}{\sqrt{x}}\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + \frac{x^6}{7484400} + \frac{x^7}{681080400} + \frac{x^8}{81729648000} + \frac{x^9}{12504636144000} + \frac{x^{10}}{2375880867360000} + \frac{x^{11}}{548828480360160000} + \ldots\right)$$

$$s1[x] = \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{6} + \frac{x^{5/2}}{90} + \frac{x^{7/2}}{2520} + \frac{x^{9/2}}{113400} + \frac{x^{11/2}}{7484400} + \frac{x^{13/2}}{681080400} + \frac{x^{15/2}}{81729648000} + \frac{x^{17/2}}{12504636144000} + \frac{x^{19/2}}{2375880867360000} + \frac{x^{21/2}}{548828480360160000}$$




$$S1[x] = \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{6} + \frac{x^{5/2}}{90}$$

$$f1[x] = \frac{1}{\sqrt{x}} + \sqrt{x} + \frac{x^{3/2}}{6} + \frac{x^{5/2}}{90} + \frac{x^{7/2}}{2520} + \frac{x^{9/2}}{113400} + \frac{x^{11/2}}{7484400} + \frac{x^{13/2}}{681080400} + \frac{x^{15/2}}{81729648000} + \frac{x^{17/2}}{12504636144000} + \frac{x^{19/2}}{2375880867360000} + \frac{x^{21/2}}{548828480360160000}$$

$b_0 = 1$

$b_1 = \frac{1}{3}$

$b_2 = \frac{1}{30}$

$b_3 = \frac{1}{630}$

$b_4 = \frac{1}{22680}$

$b_5 = \frac{1}{1247400}$

$b_6 = \frac{1}{97297200}$

$b_7 = \frac{1}{10216206000}$

$b_8 = \frac{1}{1389404016000}$

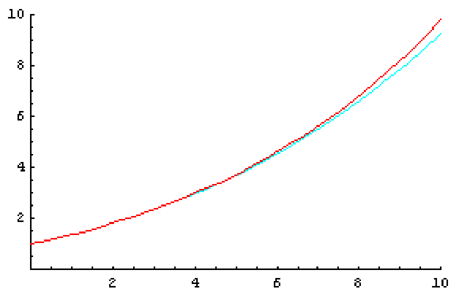
$b_9 = \frac{1}{237588086736000}$

$b_{10} = \frac{1}{49893498214560000}$

$$b_k = \frac{1}{k!\,(1+2k)!!}$$

$$f2[x] = 1(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + \frac{x^6}{97297200} + \frac{x^7}{10216206000} + \frac{x^8}{1389404016000} + \frac{x^9}{237588086736000} + \frac{x^{10}}{49893498214560000} + \frac{x^{11}}{12623055048283680000} + \ldots)$$

$$s2[x] = 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + \frac{x^6}{97297200} + \frac{x^7}{10216206000} + \frac{x^8}{1389404016000} + \frac{x^9}{237588086736000} + \frac{x^{10}}{49893498214560000} + \frac{x^{11}}{12623055048283680000}$$




$$S2[x] = 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630}$$

$$f2[x] = 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + \frac{x^6}{97297200} + \frac{x^7}{10216206000} + \frac{x^8}{1389404016000} + \frac{x^9}{237588086736000} + \frac{x^{10}}{49893498214560000} + \frac{x^{11}}{12623055048283680000}$$