3. The Newton Polynomial

Background.

We have seen how to expand a function $f(x)$ in a Maclaurin polynomial about $x_0 = 0$ involving the powers x^k and a Taylor polynomial about $x_0 \neq 0$ involving the powers $(x - x_0)^k$. These polynomials have a single "center" x_0. Polynomial interpolation can be used to construct the polynomial of degree $\leq n$ that passes through the n+1 points $(x_k, y_k) = (x_k, f(x_k))$, for $k = 0, 1, \cdots, n$. If multiple "centers" $x_0, x_1, \ldots, x_n$ are used, then the result is the so called Newton polynomial. We attribute much of the founding theory to Sir Isaac Newton (1643-1727).

Theorem (Newton Polynomial). Assume that $f \in C^{n+1}[a, b]$ and $x_k \in [a, b]$ for $k = 0, 1, \cdots, n$ are distinct values. Then

$$f(x) = P_n(x) + R_n(x),$$

where $P_n(x)$ is a polynomial that can be used to approximate $f(x)$,

$$\begin{aligned} P_n(x) = {} & a_0 + a_1 (x - x_0) + a_2 (x - x_0)(x - x_1) + a_3 (x - x_0)(x - x_1)(x - x_2) \\ & + \ldots \\ & + a_n (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) \end{aligned}$$

and we write

$$f(x) \approx P_n(x).$$

The Newton polynomial goes through the $n + 1$ points $\{(x_k, y_k)\}_{k=0}^{n}$, i.e.

$$P_n(x_k) = f(x_k) \quad \text{for} \quad k = 0, 1, \ldots, n.$$

The remainder term $R_n(x)$ has the form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1})(x - x_n),$$

for some value $c = c(x)$ that lies in the interval $[a, b]$. The coefficients a_i are constructed using divided differences.

Definition. Divided Differences.

The divided differences for a function $f[x]$ are defined as follows:

$$f[x_{i-1}, x_i] = \frac{f[x_i] - f[x_{i-1}]}{x_i - x_{i-1}}$$

$$f[x_{i-2}, x_{i-1}, x_i] = \frac{f[x_{i-1}, x_i] - f[x_{i-2}, x_{i-1}]}{x_i - x_{i-2}}$$

$$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i] = \frac{f[x_{i-2}, x_{i-1}, x_i] - f[x_{i-3}, x_{i-2}, x_{i-1}]}{x_i - x_{i-3}}$$

$$f[x_{i-j}, x_{i-j+1}, \ldots, x_i] = \frac{f[x_{i-j+1}, \ldots, x_i] - f[x_{i-j}, \ldots, x_{i-1}]}{x_i - x_{i-j}}$$

The divided difference formulae are used to construct the divided difference table:

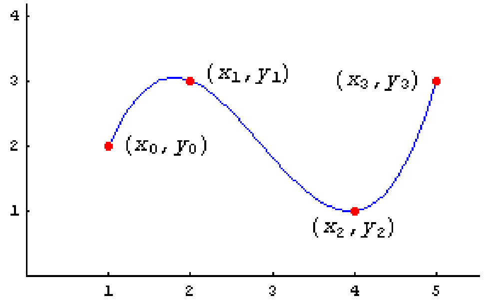
x_i	$f[x_i]$	$f[x_{i-1}, x_i]$	$f[x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-3}, x_{i-2}, x_{i-1}, x_i]$	$f[x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}, x_i]$
x_0	$f[x_0]$				
x_1	$f[x_1]$	$f[x_0, x_1]$			
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	
x_4	$f[x_4]$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4]$

The coefficient a_i of the Newton polynomial $P_n(x)$ is $a_i = f[x_0, x_1, \ldots, x_i]$ and it is the top element in the column of the i-th divided differences.

The Newton polynomial of degree $\leq n$ that passes through the $n+1$ points $(x_k, y_k) = (x_k, f(x_k))$, for $k = 0, 1, \cdots, n$ is

$$\begin{aligned} P_n(x) = {} & a_0 + a_1 (x - x_0) + a_2 (x - x_0)(x - x_1) + a_3 (x - x_0)(x - x_1)(x - x_2) \\ & + \ldots \\ & + a_n (x - x_0)(x - x_1)(x - x_2) \ldots (x - x_{n-1}) \end{aligned}$$

The cubic curve in the figure below illustrates a Newton polynomial of degree n = 3.



$$p[x] = f[x_0] + \frac{(-f[x_0] + f[x_1])(x - x_0)}{-x_0 + x_1} + \frac{(x - x_0)(x - x_1)\left(-\frac{-f[x_0]+f[x_1]}{-x_0+x_1} + \frac{-f[x_1]+f[x_2]}{-x_1+x_2}\right)}{-x_0 + x_2} + \frac{(x - x_0)(x - x_1)(x - x_2)\left(-\frac{-\frac{-f[x_0]+f[x_1]}{-x_0+x_1}+\frac{-f[x_1]+f[x_2]}{-x_1+x_2}}{-x_0+x_2} + \frac{-\frac{-f[x_1]+f[x_2]}{-x_1+x_2}+\frac{-f[x_2]+f[x_3]}{-x_2+x_3}}{-x_1+x_3}\right)}{-x_0 + x_3}$$

$$p[x_0] = f[x_0]$$

$$p[x_1] = f[x_1]$$

$$p[x_2] = f[x_0] + \frac{(-f[x_0] + f[x_1])(-x_0 + x_2)}{-x_0 + x_1} + (-x_1 + x_2)\left(-\frac{-f[x_0] + f[x_1]}{-x_0 + x_1} + \frac{-f[x_1] + f[x_2]}{-x_1 + x_2}\right)$$

$$p[x_2] = f[x_2]$$

$$p[x_3] = f[x_0] + \frac{(-f[x_0] + f[x_1])\,(-x_0 + x_3)}{-x_0 + x_1} + \frac{\left(-\frac{-f[x_0]+f[x_1]}{-x_0+x_1} + \frac{-f[x_1]+f[x_2]}{-x_1+x_2}\right)(-x_0 + x_3)\,(-x_1 + x_3)}{-x_0 + x_2} + (-x_1 + x_3)\,(-x_2 + x_3)\left(-\frac{-\frac{-f[x_0]+f[x_1]}{-x_0+x_1} + \frac{-f[x_1]+f[x_2]}{-x_1+x_2}}{-x_0 + x_2} + \frac{-\frac{-f[x_1]+f[x_2]}{-x_1+x_2} + \frac{-f[x_2]+f[x_3]}{-x_2+x_3}}{-x_1 + x_3}\right)$$

$$p[x_3] = f[x_3]$$

Theorem. (Error Bounds for Newton Interpolation, Equally Spaced Nodes) Assume that $f(x)$ defined on $[a, b]$, which contains the equally spaced nodes $x_k = x_0 + kh$. Additionally, assume that $f(x)$ and the derivatives of $f(x)$ up to the order $n+1$ are continuous and bounded on the special subintervals $[x_0, x_1]$, $[x_0, x_2]$, $[x_0, x_3]$, $[x_0, x_4]$, and $[x_0, x_5]$, respectively; that is,

$$|f^{(n+1)}(x)| \le M_{n+1} \text{ for } x_0 < x < x_n,$$

for $n = 1, 2, 3, 4, 5$. The error terms corresponding to these three cases have the following useful bounds on their magnitude

(i). $|R_1(x)| \le \frac{M_2}{8} h^2$ is valid for $x \in [x_0, x_1]$,

(ii). $|R_2(x)| \le \frac{M_3}{9\sqrt{3}} h^3$ is valid for $x \in [x_0, x_2]$,

(iii). $|R_3(x)| \le \frac{M_4}{24} h^4$ is valid for $x \in [x_0, x_3]$,

(iv). $|R_4(x)| \le \frac{\sqrt{4750 + 290\sqrt{145}}}{3000} M_5 h^5$ is valid for $x \in [x_0, x_4]$,

(v). $|R_5(x)| \le \frac{10 + 7\sqrt{7}}{1215} M_6 h^6$ is valid for $x \in [x_0, x_5]$.

Algorithm (Newton Interpolation Polynomial). To construct and evaluate the Newton polynomial of degree $\le n$ that passes through the n+1 points $(x_i, y_i) = (x_i, f(x_i))$, for $i = 0, 1, \cdots, n$

$$\begin{aligned} P_n(x) = {} & d_{0,0} + d_{1,1}(x - x_0) + d_{2,2}(x - x_0)(x - x_1) \\ & + d_{3,3}(x - x_0)(x - x_1)(x - x_2) + \ldots \\ & + d_{n,n}(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1}) \end{aligned}$$

where

$d_{i,0} = y_i$ for $i = 0, 1, \ldots, n$

and

$$d_{i,j} = \frac{d_{i,j-1} - d_{i-1,j-1}}{x_i - x_{i-j}} \text{ for } i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, i$$

Remark 1. Newton polynomials are created "recursively."

$$P_n(x) = P_{n-1}(x) + d_{n,n}(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_{n-1}).$$

Example 1. Find the Newton polynomial approximation for $f[x] = \sqrt{x}$, on the interval $[0, 8]$.
Solution 1.

Example 2. Find the Newton polynomial approximation for $f[x] = \frac{1}{1 + 10x^2}$, on the interval $[-1, 1]$.

Solution 2.

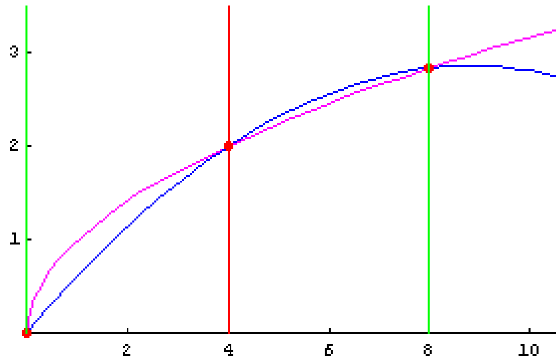
Example 3. Find the Newton polynomial approximation for $f[x] = \mathrm{Log}[x]$, on the interval $[0.02, 2]$.
Solution 3.

Example 4. Application to number theory.
4 (a). Find the formula for the sum of the first n integers.
4 (b). Find the formula for the sum of the squares of the first n integers.
Solution 4.

Example 5. Error Analysis. Investigate the error for the Newton polynomial approximations of degree $n = 3$ and 4 for the function `f[x] = Cos[x]` over the interval $[x_0, x_n]$.
Solution 5 (a).
Solution 5 (b).

Example 1. Find the Newton polynomial approximation for $f[x] = \sqrt{x}$, on the interval $[0, 8]$.

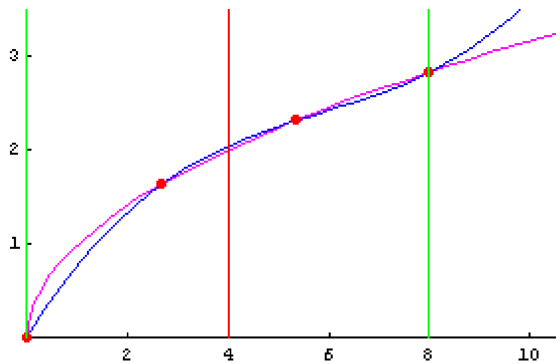
Solution 1.



$f[x] = \sqrt{x}$

$P[x] = 0. + 0.5 (0. + x) - 0.0366117 (-4. + x) (0. + x)$

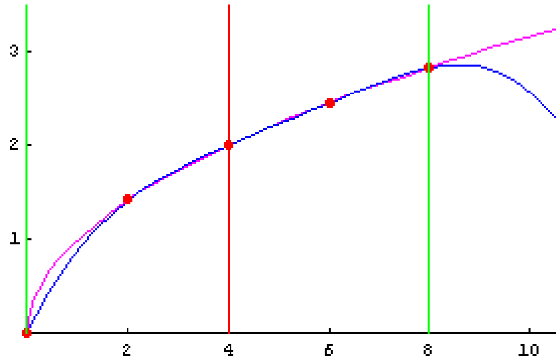
$P[x] = 0.646447 x - 0.0366117 x^2$



$f[x] = \sqrt{x}$

$P[x] = 0. + 0.612372 (0. + x) - 0.0672599 (-2.66667 + x) (0. + x) + 0.00702425 (-5.33333 + x) (-2.66667 + x) (0. + x)$

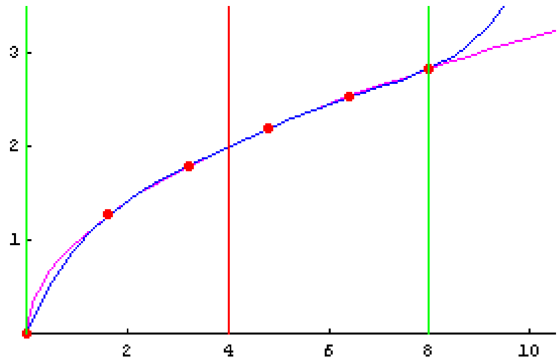
$P[x] = 0.891633 x - 0.123454 x^2 + 0.00702425 x^3$



f[x] = $\sqrt{x}$

P[x] = 0. + 0.707107 (0. + x) - 0.103553 (-2. + x) (0. + x) + 0.0144194 (-4. + x) (-2. + x) (0. + x) - 0.00163121 (-6. + x) (-4. + x) (-2. + x) (0. + x)

P[x] = 1.10787 x - 0.261843 x^2 + 0.0339939 x^3 - 0.00163121 x^4



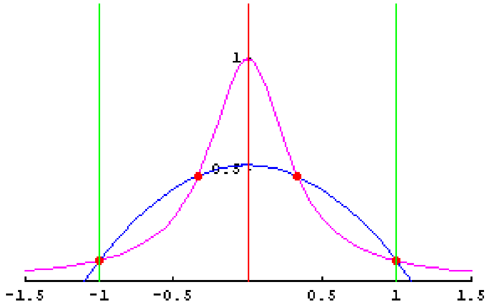
f[x] = $\sqrt{x}$

P[x] = 0. + 0.790569 (0. + x) - 0.14472 (-1.6 + x) (0. + x) + 0.0251896 (-3.2 + x) (-1.6 + x) (0. + x) -
0.00356202 (-4.8 + x) (-3.2 + x) (-1.6 + x) (0. + x) + 0.000416621 (-6.4 + x) (-4.8 + x) (-3.2 + x) (-1.6 + x) (0. + x)

P[x] = 1.30416 x - 0.451261 x^2 + 0.0967142 x^3 - 0.0102279 x^4 + 0.000416621 x^5

Example 2. Find the Newton polynomial approximation for $f[x] = \frac{1}{1 + 10x^2}$, on the interval $[-1, 1]$.

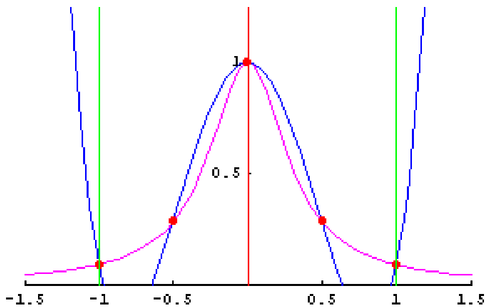
Solution 2.



$$f[x] = \frac{1}{1 + 10x^2}$$

$$P[x] = 0.0909091 + 0.574163\,(1. + x) - 0.430622\,(0.333333 + x)\,(1. + x) - 2.77556 \times 10^{-16}\,(-0.333333 + x)\,(0.333333 + x)\,(1. + x)$$

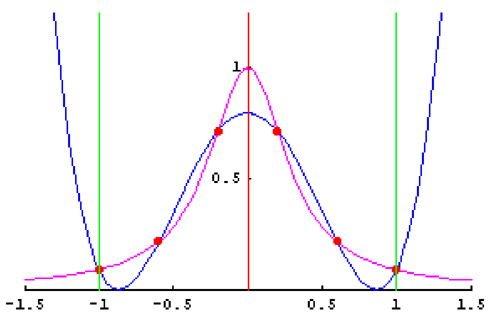
$$P[x] = 0.521531 - 0.430622\,x^2$$



$$f[x] = \frac{1}{1 + 10x^2}$$

$$P[x] = 0.0909091 + 0.38961\,(1. + x) + 1.03896\,(0.5 + x)\,(1. + x) - 2.5974\,(0. + x)\,(0.5 + x)\,(1. + x) + 2.5974\,(-0.5 + x)\,(0. + x)\,(0.5 + x)\,(1. + x)$$

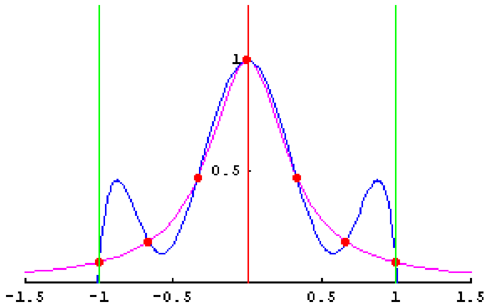
$$P[x] = 1. - 3.50649\,x^2 + 2.5974\,x^4$$



$$f[x] = \frac{1}{1 + 10 x^2}$$

$$P[x] = 0.0909091 + 0.316206\,(1. + x) + 1.15754\,(0.6 + x)\,(1. + x) - 2.25861\,(0.2 + x)\,(0.6 + x)\,(1. + x) + 1.41163\,(-0.2 + x)\,(0.2 + x)\,(0.6 + x)\,(1. + x) - 2.55351 \times 10^{-15}\,(-0.6 + x)\,(-0.2 + x)\,(0.2 + x)\,(0.6 + x)\,(1. + x)$$

$$P[x] = 0.796725 - 2.11745\,x^2 + 1.41163\,x^4$$

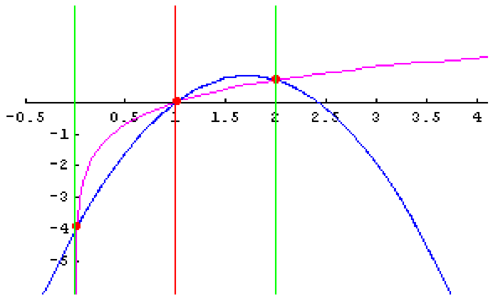


$$f[x] = \frac{1}{1 + 10 x^2}$$

$$P[x] = 0.0909091 + 0.278293\,(1. + x) + 0.887609\,(0.666667 + x)\,(1. + x) + 0.175764\,(0.333333 + x)\,(0.666667 + x)\,(1. + x) - 4.48198\,(0. + x)\,(0.333333 + x)\,(0.666667 + x)\,(1. + x) + 7.90938\,(-0.333333 + x)\,(0. + x)\,(0.333333 + x)\,(0.666667 + x)\,(1. + x) - 7.90938\,(-0.666667 + x)\,(-0.333333 + x)\,(0. + x)\,(0.333333 + x)\,(0.666667 + x)\,(1. + x)$$

$$P[x] = 1. - 6.09413\,x^2 + 13.0944\,x^4 - 7.90938\,x^6$$

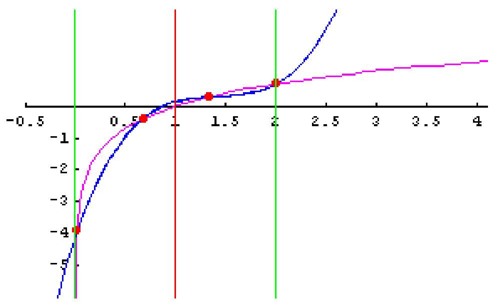
Example 3. Find the Newton polynomial approximation for `f[x] = Log[x]`, on the interval `[0.02, 2]`.
Solution 3.



$f[x] = \text{Log}[x]$

$P[x] = -3.91202 + 3.96159\,(-0.02 + x) - 1.65227\,(-1.01 + x)\,(-0.02 + x)$

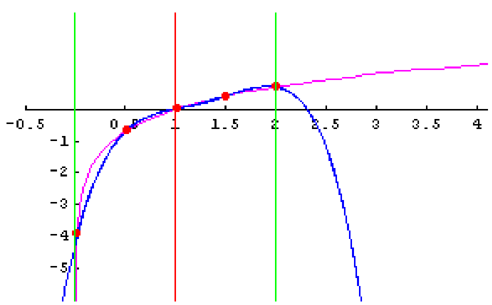
$P[x] = -4.02463 + 5.66343\,x - 1.65227\,x^2$



$f[x] = \text{Log}[x]$

$P[x] = -3.91202 + 5.34297\,(-0.02 + x) - 3.26909\,(-0.68 + x)\,(-0.02 + x) + 1.48998\,(-1.34 + x)\,(-0.68 + x)\,(-0.02 + x)$

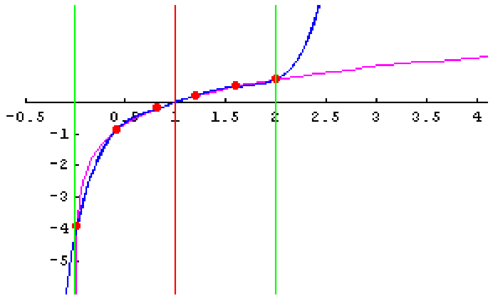
$P[x] = -4.0905 + 9.04919\,x - 6.30864\,x^2 + 1.48998\,x^3$



$f[x] = \text{Log}[x]$

$P[x] = -3.91202 + 6.56249\,(-0.02 + x) - 5.25435\,(-0.515 + x)\,(-0.02 + x) + 3.16081\,(-1.01 + x)\,(-0.515 + x)\,(-0.02 + x) - 1.48518\,(-1.505 + x)\,(-1.01 + x)\,(-0.515 + x)\,(-0.02 + x)$

$P[x] = -4.15353 + 12.3603\,x - 14.409\,x^2 + 7.69062\,x^3 - 1.48518\,x^4$



f[x] = Log[x]

P[x] = -3.91202 + 7.66402 (-0.02 + x) - 7.54431 (-0.416 + x) (-0.02 + x) + 5.62151 (-0.812 + x) (-0.416 + x) (-0.02 + x) -
3.28138 (-1.208 + x) (-0.812 + x) (-0.416 + x) (-0.02 + x) + 1.5656 (-1.604 + x) (-1.208 + x) (-0.812 + x) (-0.416 + x) (-0.02 + x)

P[x] = $-4.21332 + 15.5778\,x - 26.0876\,x^2 + 22.7757\,x^3 - 9.63773\,x^4 + 1.5656\,x^5$

Example 4. Application to number theory.
4 (a). Find the formula for the sum of the first n integers.
4 (b). Find the formula for the sum of the squares of the first n integers.
Solution 4.

4 (a). Find the formula for the sum of the first n integers.

$$\{\{1, 1\}, \{2, 3\}, \{3, 6\}, \{4, 10\}, \{5, 15\}\}$$

$$P[x] = 1 + 2(-1 + x) + \frac{1}{2}(-2 + x)(-1 + x)$$

$$P[x] = \frac{1}{2}(x + x^2)$$

Notice that the divided difference table has zeros for the 2nd and higher order differences.

$$\begin{array}{lllll} 1 & & & & \\ 3 & 2 & & & \\ 6 & 3 & \frac{1}{2} & & \\ 10 & 4 & \frac{1}{2} & 0 & \\ 15 & 5 & \frac{1}{2} & 0 & 0 \end{array}$$

Thus the formula for the sum of the first n integers is:

$$P[n] = \frac{1}{2}(n + n^2)$$

4 (b). Find the formula for the sum of the squares of the first n integers is:

$$\{\{0, 0\}, \{1, 1\}, \{2, 5\}, \{3, 14\}, \{4, 30\}, \{5, 55\}\}$$

$$P[x] = x + \frac{3}{2}(-1 + x)x + \frac{1}{3}(-2 + x)(-1 + x)x$$

$$P[x] = \frac{1}{6}(x + 3x^2 + 2x^3)$$

Notice that the divided difference table has zeros for the 3rd and higher order differences.

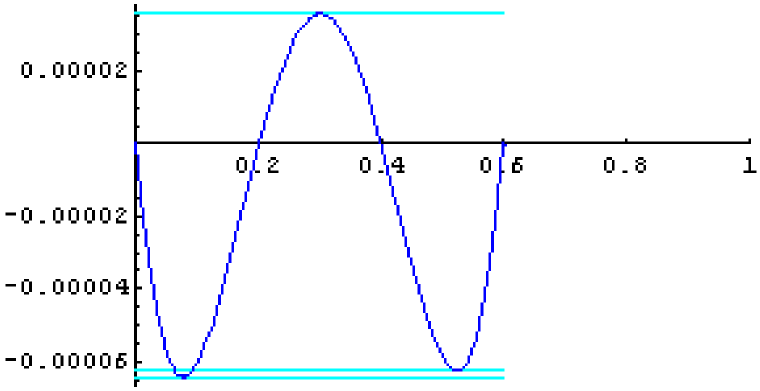
0					
1	1				
5	4	$\frac{3}{2}$			
14	9	$\frac{5}{2}$	$\frac{1}{3}$		
30	16	$\frac{7}{2}$	$\frac{1}{3}$	0	
55	25	$\frac{9}{2}$	$\frac{1}{3}$	0	0

Thus the formula for the sum of the squares of the first n integers is:

$$P[n] = \frac{1}{6}\left(n + 3 n^2 + 2 n^3\right)$$

Example 5. Error Analysis. Investigate the error for the Newton interpolation polynomial $p_3[x]$ of degree n = 3 for the function $f[x] = \mathrm{Cos}[x]$ over the interval $[x_0, x_3] = [0.0, 0.6]$.

Solution 5 (a).



$f[x] = \mathrm{Cos}[x]$

$p_3[x] = 1 - 0.0996671\,x - 0.488402\,(-0.2 + x)\,x + 0.0490076\,(-0.4 + x)\,(-0.2 + x)\,x$

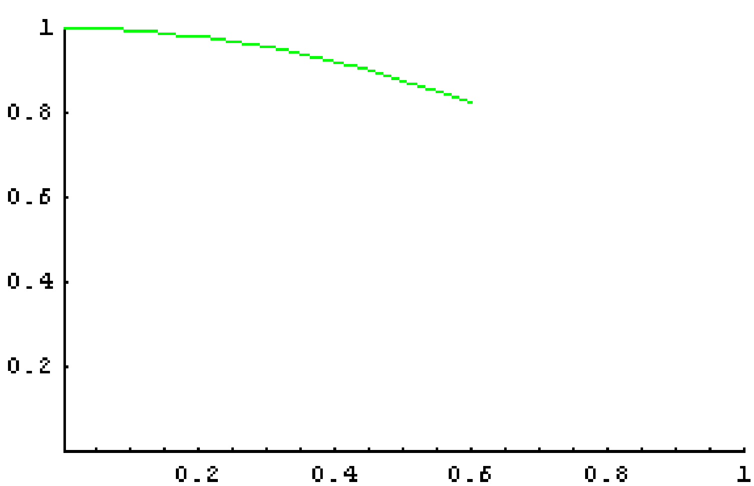
The interval for interpolation is [0.0, 0.6].

Graph of the error $e_3[x] = f[x] - p_3[x]$

Extrema for $e_3[x]$ are {-0.0000642481, 0.0000357062, -0.0000624928}

$|e_3[x]| \le 0.0000642481$

Use formula (iii). $|R_3(x)| \le \dfrac{M_4}{24} h^4$ is valid for $x \in [x_0, x_3] = [0.0, 0.6]$, and find the error bound for this example.



$f^{(4)}[x] = \mathrm{Cos}[x]$

$|f^{(4)}[x]| \le M_4 = 1 = 1.$

$$h = \frac{1}{5}$$

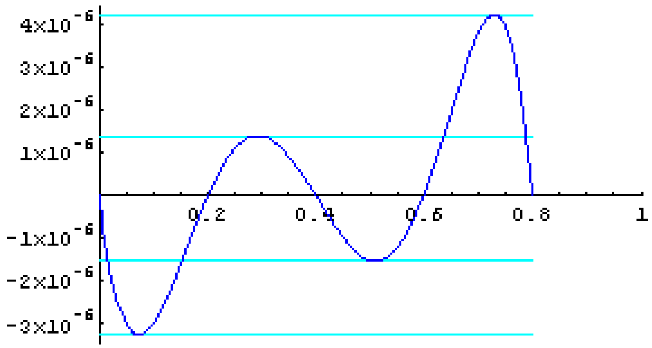
The remainder term $R_3(x)$ has the form

$$|R_3(x)| \le \frac{M_4}{24} h^4 = \frac{1}{15000} = 0.0000666667$$

Thus, $|R_3(x)| \le 0.0000666667$ is valid for $x \in [x_0, x_3] = [0.0, 0.6]$, which is a little bit larger than the maximum error 0.0000642481. After all, it is an error bound.

Example 5. Error Analysis. Investigate the error for the Newton interpolation polynomial $P_4[x]$, of degree n = 4 for the function $f[x] = \mathrm{Cos}[x]$ over the interval $[x_0, x_4] = [0.0, 0.8]$.

Solution 5 (b).



$f[x] = \mathrm{Cos}[x]$

$p_4[x] = 1 - 0.0996671\, x - 0.488402\, (-0.2 + x)\, x + 0.0490076\, (-0.4 + x)\, (-0.2 + x)\, x + 0.0381225\, (-0.6 + x)\, (-0.4 + x)\, (-0.2 + x)\, x$

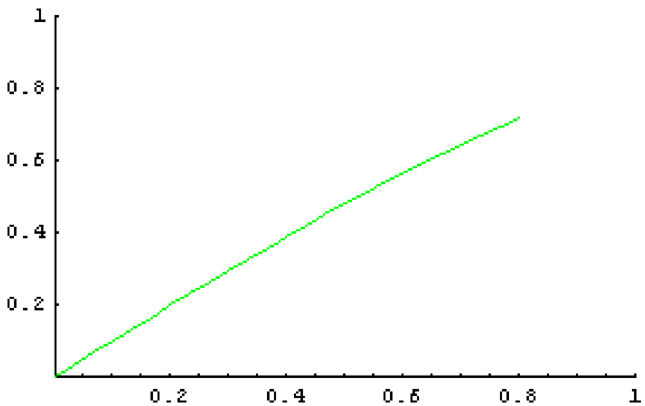
The interval for interpolation is [0.0, 0.8].

Graph of the error $e_4[x] = f[x] - p_4[x]$

Extrema for $e_4[x]$ are $\{-3.25851\times 10^{-6}, 1.40342\times 10^{-6}, -1.52922\times 10^{-6}, 4.23054\times 10^{-6}\}$

$|e_4[x]| \le 4.23054\times 10^{-6}$

Use formula (iv). $|R_4(x)| \le \dfrac{\sqrt{4750 + 290\sqrt{145}}}{3000} M_5 h^5$ is valid for $x \in [x_0, x_4] = [0.0, 0.8]$, and find the error bound for this example.



$|f^{(5)}[x]| \leq M_5 = \text{Sin}\left[\frac{4}{5}\right] = 0.717356$

$h = \frac{1}{5}$

The remainder term $R_4(x)$ has the form

$$|R_4(x)| \leq \frac{\sqrt{4750 + 290\sqrt{145}}}{3000} M_5 h^5 = \frac{\sqrt{4750 + 290\sqrt{145}}\ \text{Sin}\left[\frac{4}{5}\right]}{9375000} = 6.94675 \times 10^{-6}$$

Thus, $|R_4(x)| \leq 6.94675 \times 10^{-6}$ is valid for $x \in [x_0, x_4] = [0.0, 0.8]$, which is a little bit larger than the maximum error 4.23054×10^{-6}. After all, it is an error bound.