3. The Compartment Model

The "compartment" model is used to describe the concentration of a dissolved substance in several compartments in a system. For example, the "three-stage system" consists of three tanks containing $V_1, V_2,$ and V_3 gallons of solute. Pure water flows at the rate r into the first tank and is mixed, then flows at the rate r into the second tank and is mixed, then flows at the rate r into the third tank and is mixed, and finally flows out of the third tank at the rate r. We define the constants $k_1 = \frac{r}{V_1}$, $k_2 = \frac{r}{V_2}$, and $k_3 = \frac{r}{V_3}$. Then the differential equation for the system is

$$\frac{dx_1}{dt} = -k_1 x_1$$
$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2$$
$$\frac{dx_3}{dt} = k_2 x_2 - k_3 x_3$$

with the initial conditions :

$$x_1(0) = a_1$$
$$x_2(0) = a_2$$
$$x_3(0) = a_3$$

Example 1. Find the solution to the "three-stage system" where

$$V_1 = 80,\ V_2 = 100,\ V_3 = 120,\ r = 20$$

with the initial conditions

$$x_1(0) = 10,\ x_2(0) = 0,\ x_3(0) = 0.$$

Plot the solution curves for $x_1(t), x_2(t), x_3(t)$ for $0 \le t \le 40$.

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Solution 1.

For this enter the following volumes and rate of flow.

```
Clear[c, k, r, v];
v1 = 80;
v2 = 100;
v3 = 120;
r = 20;
```

$$k_1 = \frac{r}{v_1};$$
$$k_2 = \frac{r}{v_2};$$
$$k_3 = \frac{r}{v_3};$$
$$A = \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \end{pmatrix};$$

```
Print["", MatrixForm[A]];
```

$$\begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ \frac{1}{4} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{6} \end{pmatrix}$$

Find the eigenvalues and eigenvectors of the matrix **A**.

$$\{\lambda_i\} = \left\{-\frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}\right\}$$
$$\{V_i\} = \left\{\left\{\frac{1}{12}, -\frac{5}{12}, 1\right\}, \left\{0, -\frac{1}{6}, 1\right\}, \{0, 0, 1\}\right\}$$

Construct the three vector eigenfunctions and the general solution.

```
X1[t_] = V[[1]] e^(λ[[1]] t);
X2[t_] = V[[2]] e^(λ[[2]] t);
X3[t_] = V[[3]] e^(λ[[3]] t);
X[t_] = c1 X1[t] + c2 X2[t] + c3 X3[t];
Print["X1[t] = ", MatrixForm[X1[t]] ];
Print["X2[t] = ", MatrixForm[X2[t]] ];
Print["X3[t] = ", MatrixForm[X3[t]] ];
Print[""];
Print["X[t] = ", MatrixForm[X[t]] ];
```

$$X_1[t] = \begin{pmatrix} \frac{e^{-t/4}}{12} \\ -\frac{5}{12} e^{-t/4} \\ e^{-t/4} \end{pmatrix}$$

$$X_2[t] = \begin{pmatrix} 0 \\ -\frac{1}{6} e^{-t/5} \\ e^{-t/5} \end{pmatrix}$$

$$X_3[t] = \begin{pmatrix} 0 \\ 0 \\ e^{-t/6} \end{pmatrix}$$

$$X[t] = \begin{pmatrix} \frac{1}{12} e^{-t/4} c_1 \\ -\frac{5}{12} e^{-t/4} c_1 - \frac{1}{6} e^{-t/5} c_2 \\ e^{-t/4} c_1 + e^{-t/5} c_2 + e^{-t/6} c_3 \end{pmatrix}$$

The following calculation will verify that $X[t]$ is the general solution.

$$X'[t]] = A\ X[t]$$

$$\begin{pmatrix} -\frac{1}{48} e^{-t/4} c_1 \\ \frac{5}{48} e^{-t/4} c_1 + \frac{1}{30} e^{-t/5} c_2 \\ -\frac{1}{4} e^{-t/4} c_1 - \frac{1}{5} e^{-t/5} c_2 - \frac{1}{6} e^{-t/6} c_3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ \frac{1}{4} & -\frac{1}{5} & 0 \\ 0 & \frac{1}{5} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} \frac{1}{12} e^{-t/4} c_1 \\ -\frac{5}{12} e^{-t/4} c_1 - \frac{1}{6} e^{-t/5} c_2 \\ e^{-t/4} c_1 + e^{-t/5} c_2 + e^{-t/6} c_3 \end{pmatrix}$$

$$\text{X'[t]]} = \text{A X[t]}$$

$$\begin{pmatrix} -\frac{1}{48} e^{-t/4} c_1 \\ \frac{5}{48} e^{-t/4} c_1 + \frac{1}{30} e^{-t/5} c_2 \\ -\frac{1}{4} e^{-t/4} c_1 - \frac{1}{5} e^{-t/5} c_2 - \frac{1}{6} e^{-t/6} c_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{48} e^{-t/4} c_1 \\ \frac{1}{48} e^{-t/4} c_1 + \frac{1}{5}\left(\frac{5}{12} e^{-t/4} c_1 + \frac{1}{6} e^{-t/5} c_2\right) \\ \frac{1}{5}\left(-\frac{5}{12} e^{-t/4} c_1 - \frac{1}{6} e^{-t/5} c_2\right) + \frac{1}{6}\left(-e^{-t/4} c_1 - e^{-t/5} c_2 - e^{-t/6} c_3\right) \end{pmatrix}$$

$$\text{X'[t]]} = \text{A X[t]}$$

$$\begin{pmatrix} -\frac{1}{48} e^{-t/4} c_1 \\ \frac{5}{48} e^{-t/4} c_1 + \frac{1}{30} e^{-t/5} c_2 \\ -\frac{1}{4} e^{-t/4} c_1 - \frac{1}{5} e^{-t/5} c_2 - \frac{1}{6} e^{-t/6} c_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{48} e^{-t/4} c_1 \\ \frac{5}{48} e^{-t/4} c_1 + \frac{1}{30} e^{-t/5} c_2 \\ -\frac{1}{4} e^{-t/4} c_1 - \frac{1}{5} e^{-t/5} c_2 - \frac{1}{6} e^{-t/6} c_3 \end{pmatrix}$$

Does X'[t]] = A X[t] ?

True

Now solve for the constants using the initial conditions, and replace these values for the constants in **X**[t] and call the solution **Y**[t].
Then check out the initial condition **Y**[0].

$$\text{Solve} \quad \begin{pmatrix} \frac{c_1}{12} \\ -\frac{5 c_1}{12} - \frac{c_2}{6} \\ c_1 + c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

Get

$$c_1 \to 120$$
$$c_2 \to -300$$
$$c_3 \to 180$$

Form the solution

$$\text{Y[t]} = \begin{pmatrix} 10 e^{-t/4} \\ -50 e^{-t/4} + 50 e^{-t/5} \\ 120 e^{-t/4} - 300 e^{-t/5} + 180 e^{-t/6} \end{pmatrix}$$

$$Y[0] = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$$

Extract the three coordinate functions from **Y**[t] and call them y_1[t], y_2[t], and y_3[t] .

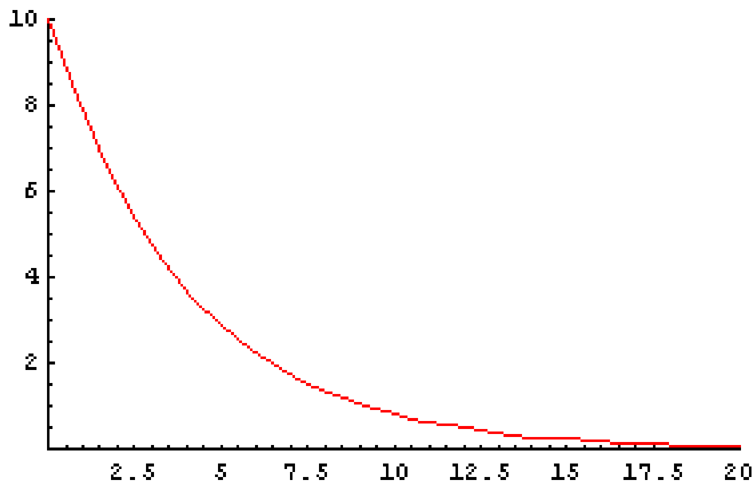
```
y1[t_] = Y[t][[1]];
y2[t_] = Y[t][[2]];
y3[t_] = Y[t][[3]];
y[t_] = y1[t] + y2[t] + y3[t];
Print["y1[t] = ", y1[t] ];
Print["y2[t] = ", y2[t] ];
Print["y3[t] = ", y3[t] ];
```

$y_1[t] = 10\,e^{-t/4}$

$y_2[t] = -50\,e^{-t/4} + 50\,e^{-t/5}$

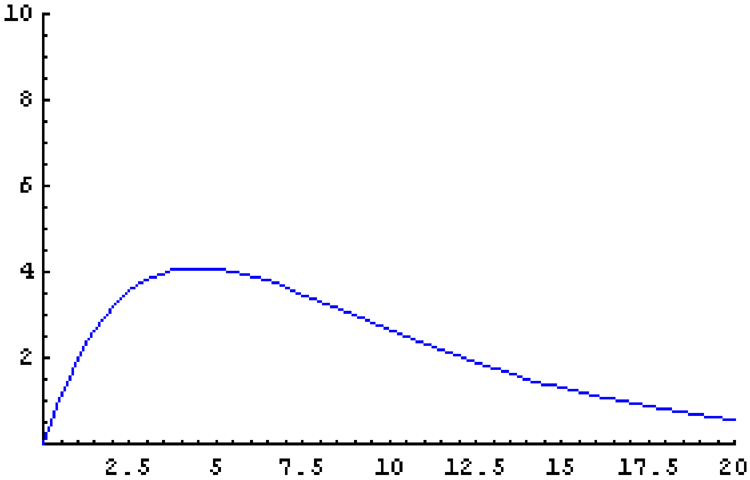
$y_3[t] = 120\,e^{-t/4} - 300\,e^{-t/5} + 180\,e^{-t/6}$

Plot the functions y_1[t], y_2[t], and y_3[t] to see how the solute moves through the system of tanks.

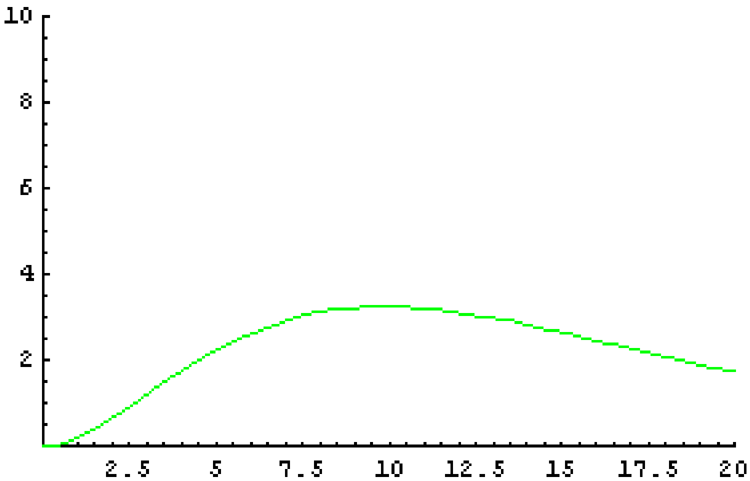


$y_1[t] = 10\,e^{-t/4}$

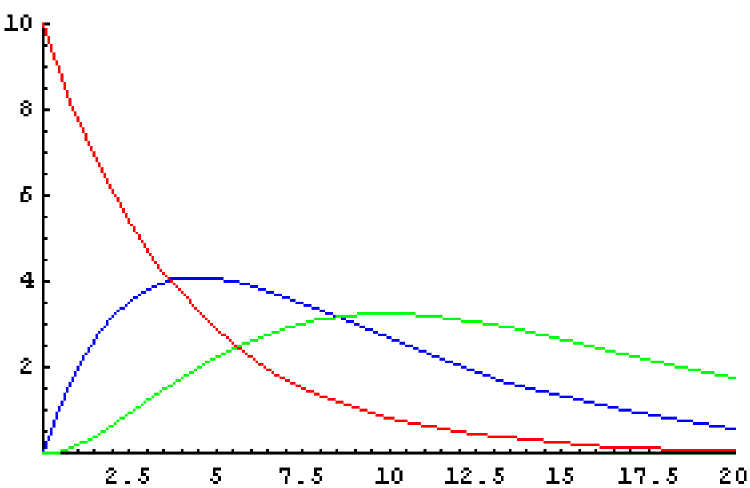
$\lim_{t \to \infty} y_1[t] = 0$



$$y_2[t] = -50\,e^{-t/4} + 50\,e^{-t/5}$$
$$\lim_{t\to\infty} y_2[t] = 0$$



$$y_3[t] = 120\,e^{-t/4} - 300\,e^{-t/5} + 180\,e^{-t/6}$$
$$\lim_{t\to\infty} y_3[t] = 0$$



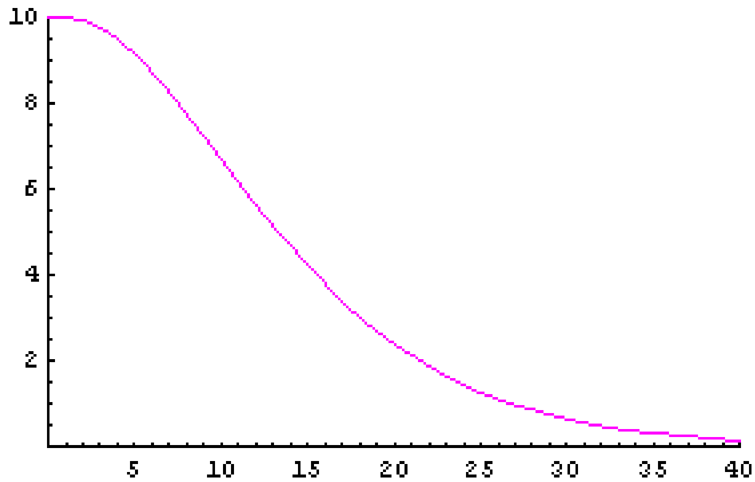
$$y_1[t] = 10\,e^{-t/4}$$

$$y_1[t] = 10\,e$$

$$y_2[t] = -50\,e^{-t/4} + 50\,e^{-t/5}$$

$$y_3[t] = 120\,e^{-t/4} - 300\,e^{-t/5} + 180\,e^{-t/6}$$

From your observations of the graphs, what do you conjecture about the total amount of solute in the system at time t,
i.e. what can you say about the sum $y[t] = y_1[t] + y_2[t] + y_3[t]$.
Plot the graph of the sum over the interval $0 \le t \le 40$.

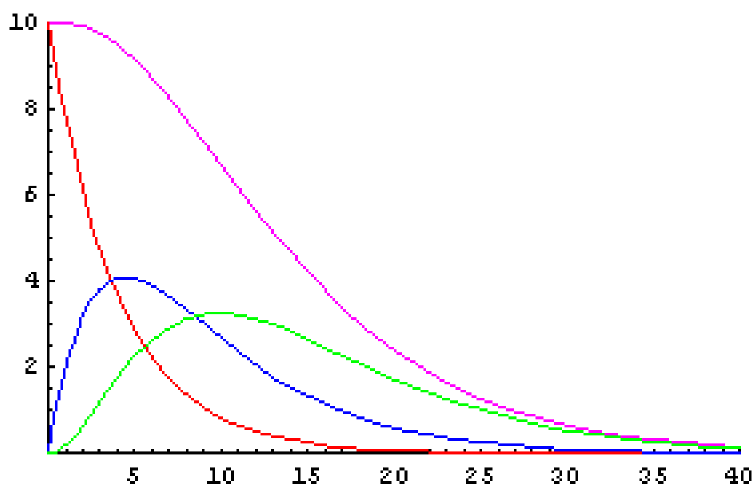


$$y[t] = y_1[t] + y_2[t] + y_3[t]$$

$$y[t] = 80\,e^{-t/4} - 250\,e^{-t/5} + 180\,e^{-t/6}$$

$$\lim_{t\to\infty} y[t] = 0$$

What is the limit of the sum $y[t] = y_1[t] + y_2[t] + y_3[t]$ as $t \to \infty$?
Did you suspect that this was going to happen ?
Why should you expect this to happen ?



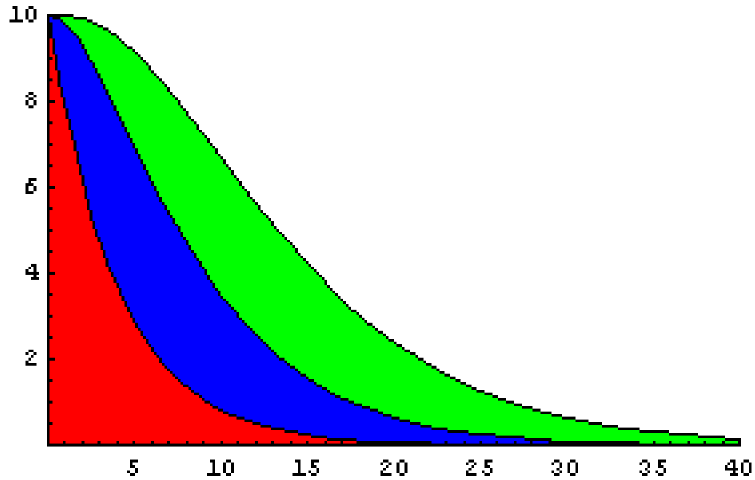
$$y[t] = y_1[t] + y_2[t] + y_3[t]$$

$$y[t] = 80\,e^{-t/4} - 250\,e^{-t/5} + 180\,e^{-t/6}$$

$$y_1[t] = 10\,e^{-t/4}$$
$$y_2[t] = -50\,e^{-t/4} + 50\,e^{-t/5}$$
$$y_3[t] = 120\,e^{-t/4} - 300\,e^{-t/5} + 180\,e^{-t/6}$$

We could look at the portion of the chemical in each compartments over time, and see how it works its way out of the system over time.

This is just for fun !



$$y_1[t] = 10\,e^{-t/4}$$
$$y_1[t]+y_2[t] = -40\,e^{-t/4} + 50\,e^{-t/5}$$
$$y_1[t]+y_2[t]+y_3[t] = 80\,e^{-t/4} - 250\,e^{-t/5} + 180\,e^{-t/6}$$