15. Linear Programming - The Simplex Method

Background for Linear Programming

Linear programming is an area of linear algebra in which the goal is to maximize or minimize a linear function $f(\vec{X})$ of n variables $\vec{X} = (x_1, x_2, \ldots, x_n)$ on a region R whose boundary is defined by linear inequalities and equations. In this context, when we speak of a "linear function of n variables" we mean that $f(\vec{X})$ has the form

$$f(\vec{X}) = f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j .$$

The region R is a convex polytope. If all the vertices of R are known, then the maximum of $f(\vec{X})$ will occur at one of these vertices.

The solution can be constructed using the simplex method and is attributed to George Dantzig (1914 -) who was born in Portland, Oregon. The simplex method starts at the origin and follows a path along the edges of the polytope to the vertex where the maximum occurs.

Definition (Convex Polytope). In two dimensions a convex polytope is a region that is the intersection of a finite set of half-planes (the general idea of a convex polygon). In three dimensions a convex polytope is solid region that is the intersection of a finite set of half-spaces (the generalized idea of a convex polyhedron). The generalization in n dimensions is called a polytope.

Example 1. Two students Ann and Carl work x and y hours per week, respectively. Together they can work at most 40 hours per week. According to the rules for part timers Ann can work at most 8 hours more that Carl. But Carl can work at most 6 hours more than Ann. There is an extra constraint $18 \le 2y + x$. Determine the region R for these constraints.
1 (a). If Ann and Carl earn \$15 and \$17 per hour, respectively, then find their maximum combined income per week.
1 (b). If Ann and Carl earn \$17 and \$15 per hour, respectively, then find their maximum combined income per week.
1 (c). If Ann and Carl both earn \$16 per hour, respectively, then find their maximum combined income per week.
Solution 1 (a).
Solution 1 (b).
Solution 1 (c).

Standard Form of the Linear Programming Problem

The standard form of the linear programming problem is to maximize $F(\vec{X})$ of n variables

$\vec{X} = (x_1, x_2, \ldots, x_n)$.

(1) Maximize

$$z = F(x_1, x_2, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

$$z = F(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} c_j x_j \quad \text{where} \quad 0 \le c_j \text{ for } j = 1, 2, \ldots, n.$$

(2) Subject to the m constraints

$$a_{i,1} x_1 + a_{i,2} x_2 + \ldots + a_{i,n} x_n \le b_i \quad \text{where} \quad 0 \le b_i \text{ for } i = 1, 2, \ldots, m.$$

(3) With the primary constraints $0 \le x_j$ for $j = 1, 2, \ldots, n$.

The coefficients $\{c_j\}$ and $\{a_{i,j}\}$ can be any real number. It is often the case that $m > n$, but the cases $m = n$ or $m < n$ can occur.

Setting up the Extended Simplex Tableau

For computational purposes, we construct a tableau. The first m rows consist of the coefficients matrix $(a_{i,j})_{m \times n}$, the identity matrix $I_{m \times m}$ and the column vector $(b_i)_{m,1}$. In the $(m+1)^{st}$-row of the tableau the first n elements are the coefficients $\{-c_j\}$, which are called the coefficients of the augmented objective equation. The remainder of the bottom row is filled in with zeros. An extra column on the right will be used in the decision process in solving for the variables.

Row	Decision variables					Slack variables					Right Side	Ratio's Column
	x_1	x_2	x_3	$\ldots$	x_n	x_{n+1}	x_{n+2}	x_{n+3}	$\ldots$	x_{n+m}		
1	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$\ldots$	$a_{1,n}$	1	0	0	$\ldots$	0	b_1	
2	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$\ldots$	$a_{2,n}$	0	1	0	$\ldots$	0	b_2	
3	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$\ldots$	$a_{3,n}$	0	0	1	$\ldots$	0	b_3	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	
m	$a_{m,1}$	$a_{m,2}$	$a_{m,3}$	$\ldots$	$a_{m,n}$	0	0	0	$\ldots$	1	b_m	
$m+1$	$-c_1$	$-c_2$	$-c_3$	$\ldots$	$-c_n$	0	0	0	$\ldots$	0	0	

The non-negative variable x_{n+i} is called a slack variable and is the amount that the linear combination $a_{i,1} x_1 + a_{i,2} x_2 + \ldots + a_{i,n} x_n$ falls short of the bound b_i. It's purpose is to change an inequality to an equation, i.e. we have

(4) $a_{i,1} x_1 + a_{i,2} x_2 + ... + a_{i,n} x_n + x_{n+i} = b_i$ for rows $i = 1, 2, ..., m$ in the tableau.

The goal of the simplex method is to exchange some of the columns of 1's and 0's of the slack variables into columns of 1's and 0's of the decision variables.

Algorithm (Simplex Method). To find the minimum of $F(\vec{X})$ over the convex polytope R.

(i) Use non-negative slack variables $x_{n+1}, x_{n+2}, ..., x_{n+m}$ and form a system of equations and the initial tableau.
(ii) Determine the exchange variable, the pivot row and pivotal element.
The exchange variable x_e is chosen in the pivot column e where $-c_e$ is the smallest negative coefficient.
The pivot row p is chosen where the minimum ratio $\frac{b_i}{a_{i,e}}$ occurs for all rows with $a_{i,e} > 0$.
The pivot element is $a_{p,e}$.
(iii) Perform row operations to zero out elements in the pivotal column e above and below the pivot row p.
(iv) Repeat steps (ii) and (iii) until there are no negative coefficients $-c_j$ in the bottom row.

Example 2. Consider the region R in the plane defined by the inequalities:

$$2x + y \le 30$$
$$x + 4y \le 64$$
$$5x + 6y \le 110$$
$$x \ge 0$$
$$y \ge 0$$

Find the maximum of $f[x, y] = 10x + 20y$ over the region R.
Solution 2.

Example 3. Consider the region R in the plane defined by the inequalities:

$$-x + 2y \le 36$$
$$x + 6y \le 132$$
$$3x + 5y \le 136$$
$$5x + 3y \le 136$$
$$6x + y \le 132$$
$$2x - y \le 36$$
$$x \ge 0$$
$$y \ge 0$$

Find the maximum of $f[x, y] = 10x + 10y$ over the region R.

Solution 3.

Example 4. Find the maximum of $f[x_1, x_2, x_3, x_4] = 5x_1 + 9x_2 + 8x_3 + 6x_4$ subject to the constraints:

$$2x_1 + 3x_2 + 4x_3 \le 500$$
$$3x_2 + 2x_3 + 5x_4 \le 300$$
$$2x_1 + 4x_2 + 2x_4 \le 400$$
$$4x_1 + 5x_3 + 3x_4 \le 800$$
$$x_1 + x_2 + x_3 + x_4 \le 200$$

Solution 4.

Example 1. Two students Ann and Carl work x and y hours per week, respectively. Together they can work at most 40 hours per week. According to the rules for part timers Ann can work at most 8 hours more that Carl. But Carl can work at most 6 hours more than Ann. There is an extra constraint $18 \le 2y + x$. Determine the region R for these constraints.

1 (a). If Ann and Carl earn $15 and $17 per hour, respectively, then find their maximum combined income per week.

Solution 1 (a).

Enter the linear function and the constraints.

Find the maximum of

$f[\{x,y\}] = 15x + 17y$

Subject to the constraints

$80 \le 4x + 2y$

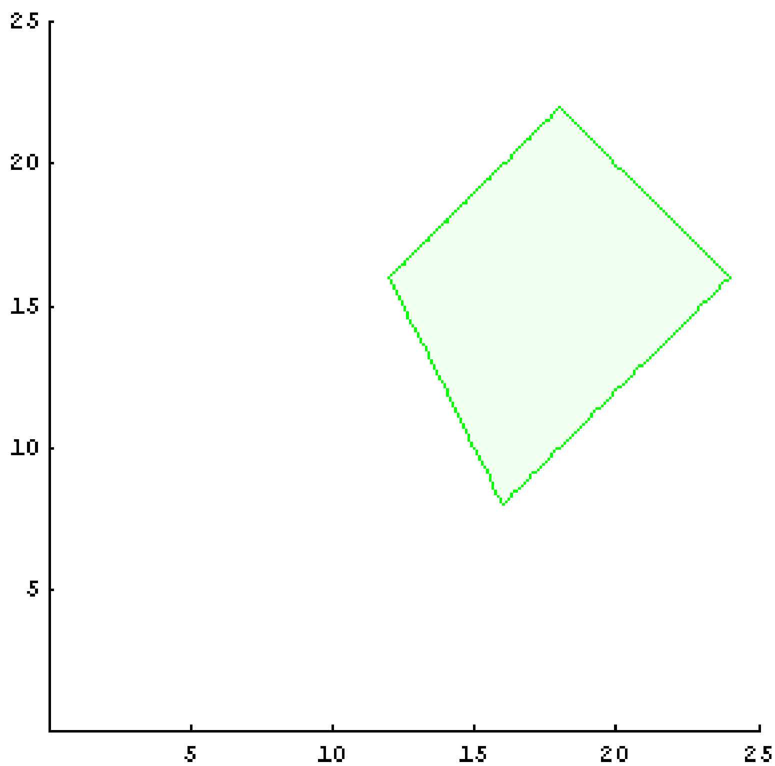
$y \le 4 + x$

$x \le 8 + y$

$x + y \le 40$

$x \ge 0$

$y \ge 0$



The region R defined by

$80 \le 4x + 2y$
$y \le 4 + x$
$x \le 8 + y$
$x + y \le 40$
$x \ge 0$
$y \ge 0$

The solution will occur at one of the vertices of the convex polytope. We now solve for these four points.

$\{x == 8 + y, x + y == 40\}$

$\vec{X}_1 = \{24, 16\}$

$\{y == 4 + x, x + y == 40\}$

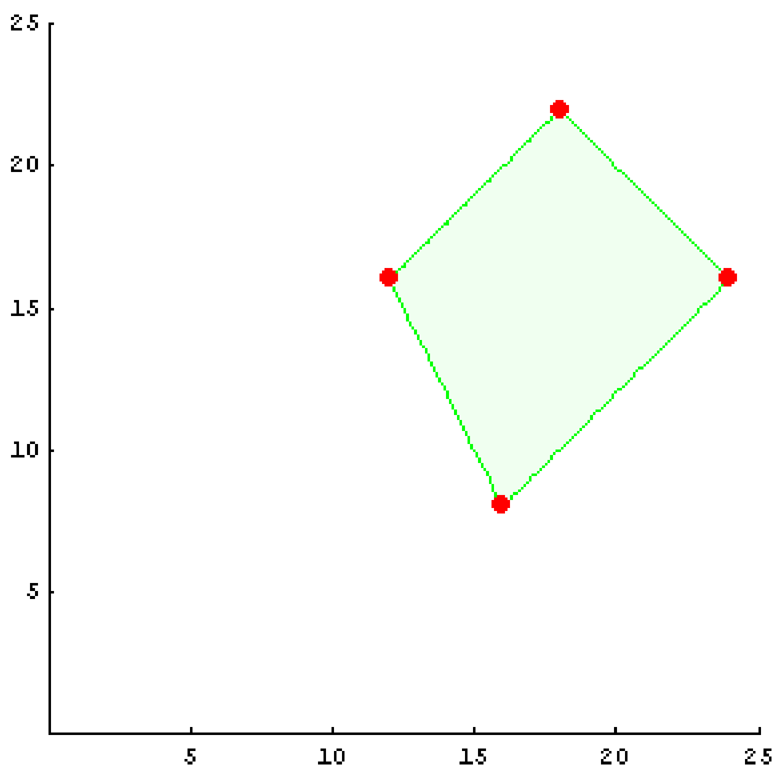
$\vec{X}_2 = \{18, 22\}$

$\{80 == 4x + 2y, x == 8 + y\}$

$\vec{X}_3 = \{16, 8\}$

$\{80 == 4x + 2y, y == 4 + x\}$

$\vec{X}_4 = \{12, 16\}$



$f[\{24, 16\}] = 632$

$f[\{18, 22\}] = 644$

$f[\{16, 8\}] = 376$

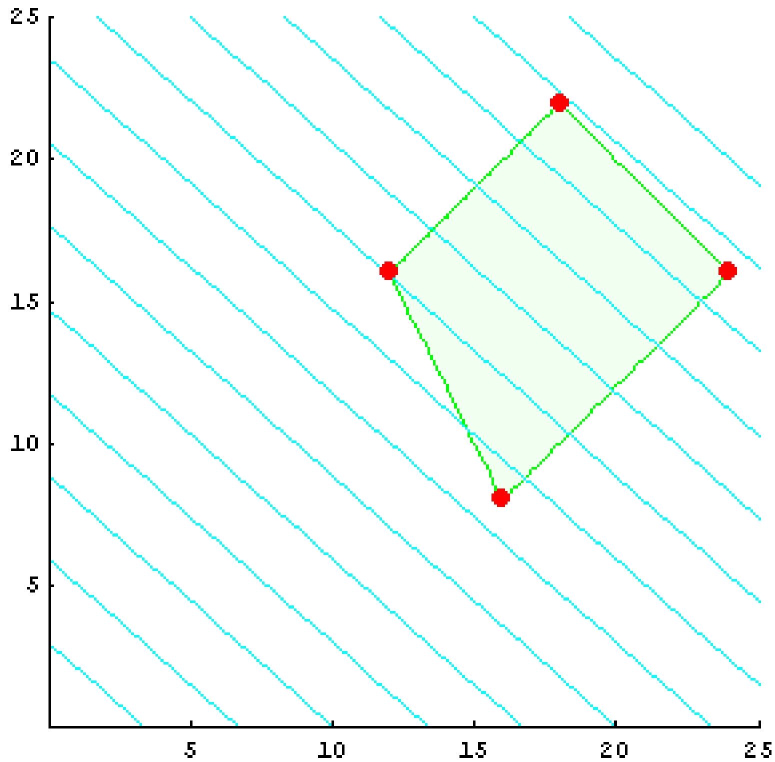
$f[\{12, 16\}] = 452$

The maximum over the region R is

$\text{Max}[f[\vec{X}_1], f[\vec{X}_2], f[\vec{X}_3], f[\vec{X}_4]] = \text{Max}[\{632, 644, 376, 452\}]$

$\text{Max}[f[\vec{X}_1], f[\vec{X}_2], f[\vec{X}_3], f[\vec{X}_4]] = 644$

Graph the level curves of the objective function.



The level curves of the objective function

$f[\{x, y\}] = 15x + 17y = c$

$c = \{50, 100, 150, 200, 250, 300, 350, 400, 450, 500, 550, 600, 650, 700\}$

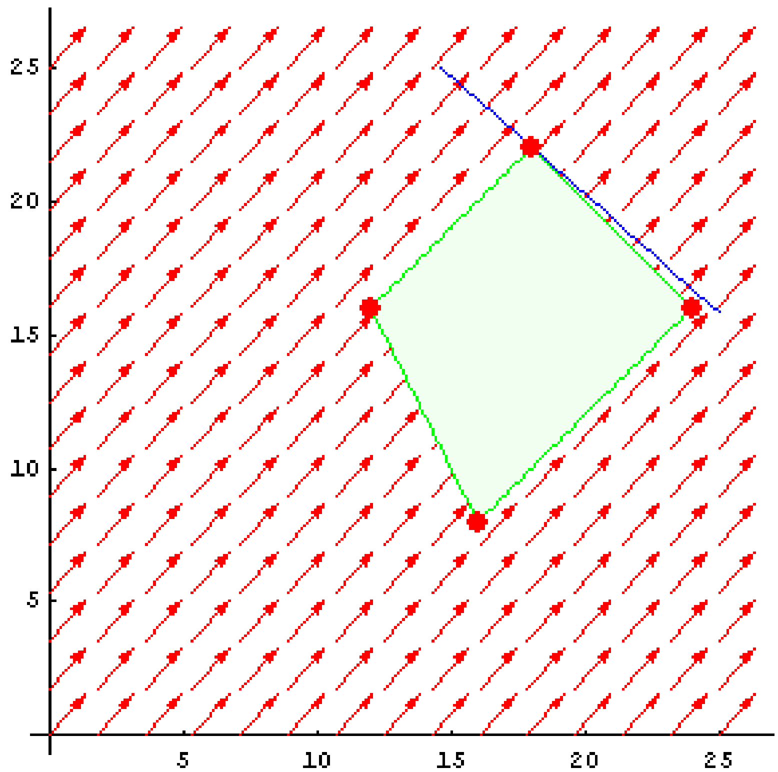
The maximum occurs at $\vec{X} = \{18, 22\}$

$f[\{18, 22\}] = 644$

The solution point for the maximum is the furthest point in the region in the direction of the gradient $\nabla f[\vec{X}]$.

Find the gradient vector $\nabla f[\vec{X}]$.

$$f[\vec{X}] = 15x + 17y$$
$$\partial_x f[\vec{X}] = \partial_x(15x + 17y) = 15$$
$$\partial_y f[\vec{X}] = \partial_y(15x + 17y) = 17$$
$$\nabla f[\vec{X}] = \{15, 17\}$$



We have found the maximum over the region R.

$f[\{x,y\}] = 15x + 17y$

$\nabla f[\vec{X}] = \{15, 17\}$

A level curves of the objective function

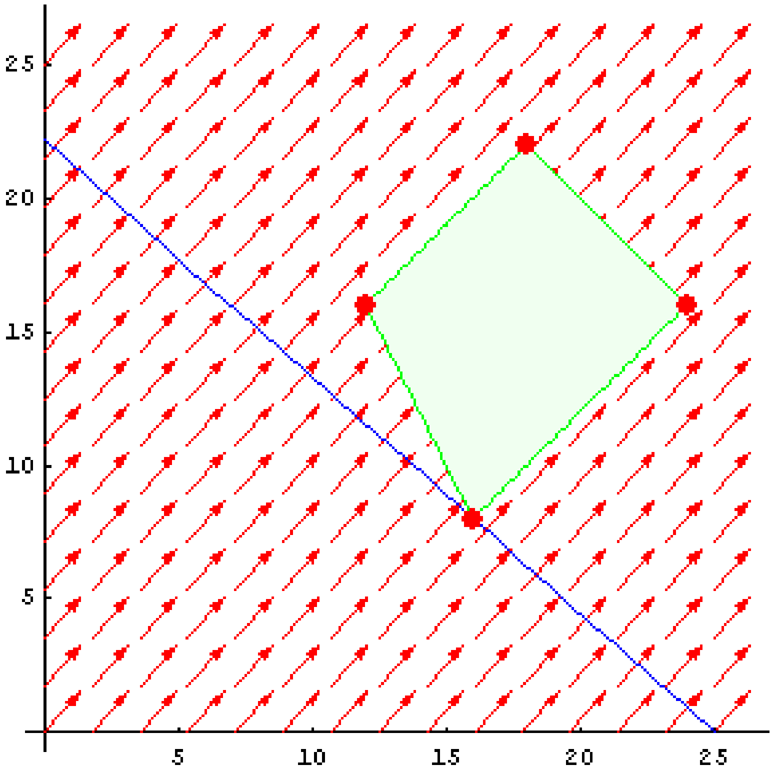
$f[\{x,y\}] = 15x + 17y = 644$

The maximum occurs at $\vec{X} = \{18, 22\}$

$f[\{18, 22\}] = 644$

It it the furthest point in the region in the direction $\nabla f[\vec{X}]$

Aside. The related problem of finding the minimum is solved in a similar fashion. All we need to do is check out the function values at the vertices.



Suppose that we want to find the minimum over the region R.

$f[\{x,y\}] = 15x + 17y$

$\nabla f[\vec{X}] = \{15, 17\}$

A level curves of the objective function

$f[\{x,y\}] = 15x + 17y = 376$

The minimum occurs at $\vec{X} = \{16, 8\}$

$f[\{16, 8\}] = 376$

Example 1. Two students Ann and Carl work x and y hours per week, respectively. Together they can work at most 40 hours per week. According to the rules for part timers Ann can work at most 8 hours more that Carl. But Carl can work at most 6 hours more than Ann. There is an extra constraint $18 \le 2y + x$. Determine the region R for these constraints.
1 (b). If Ann and Carl earn $17 and $15 per hour, respectively, then find their maximum combined income per week.
Solution 1 (b).

Enter the linear function and the constraints.

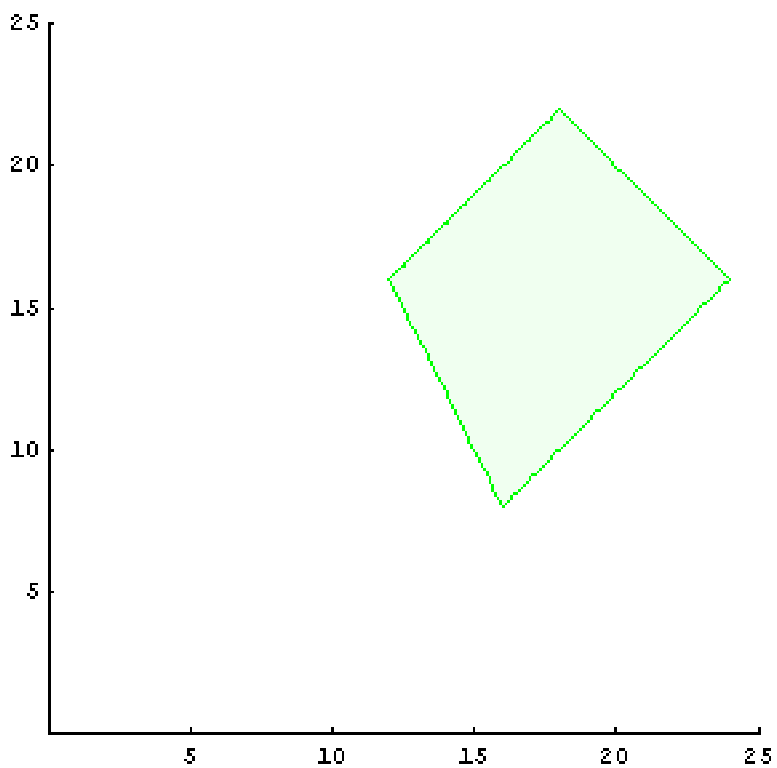
Find the maximum of

$f[\{x,y\}] = 17x + 15y$

Subject to the constraints

$80 \le 4x + 2y$
$y \le 4 + x$
$x \le 8 + y$
$x + y \le 40$
$x \ge 0$
$y \ge 0$

Graph the region R defined by the constraints.



The region R defined by

$80 \leq 4x + 2y$
$y \leq 4 + x$
$x \leq 8 + y$
$x + y \leq 40$
$x \geq 0$
$y \geq 0$

The solution will occur at one of the vertices of the convex polytope. We now solve for these four points.

$\{x == 8 + y, x + y == 40\}$

$\vec{X}_1 = \{24, 16\}$

$\{y == 4 + x, x + y == 40\}$

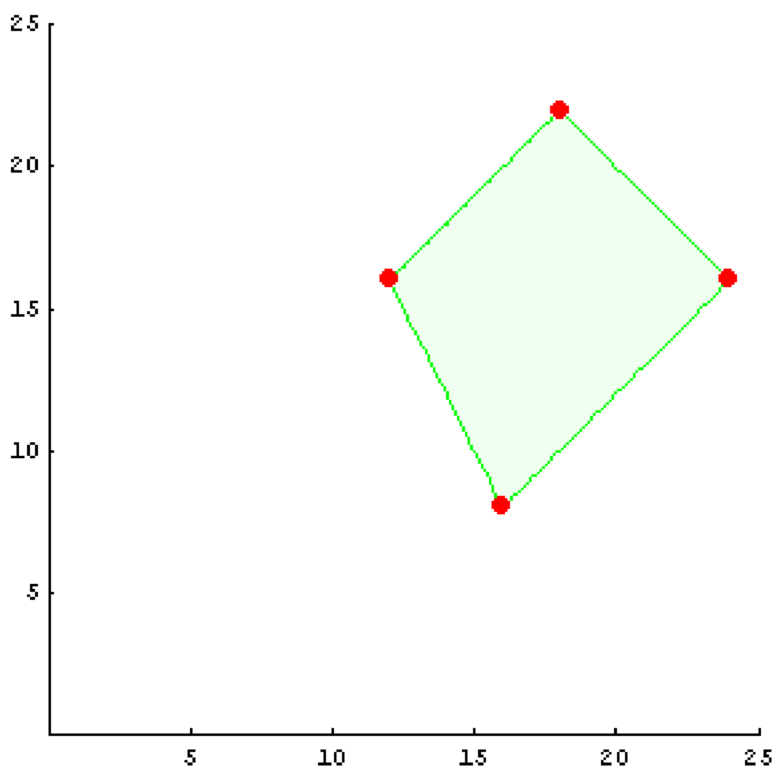
$\vec{X}_2 = \{18, 22\}$

$\{80 == 4x + 2y, x == 8 + y\}$

$\vec{X}_3 = \{16, 8\}$

$\{80 == 4x + 2y, y == 4 + x\}$

$\vec{X}_4 = \{12, 16\}$



$f[\{24, 16\}] = 648$

$f[\{18, 22\}] = 636$

$f[\{16, 8\}] = 392$

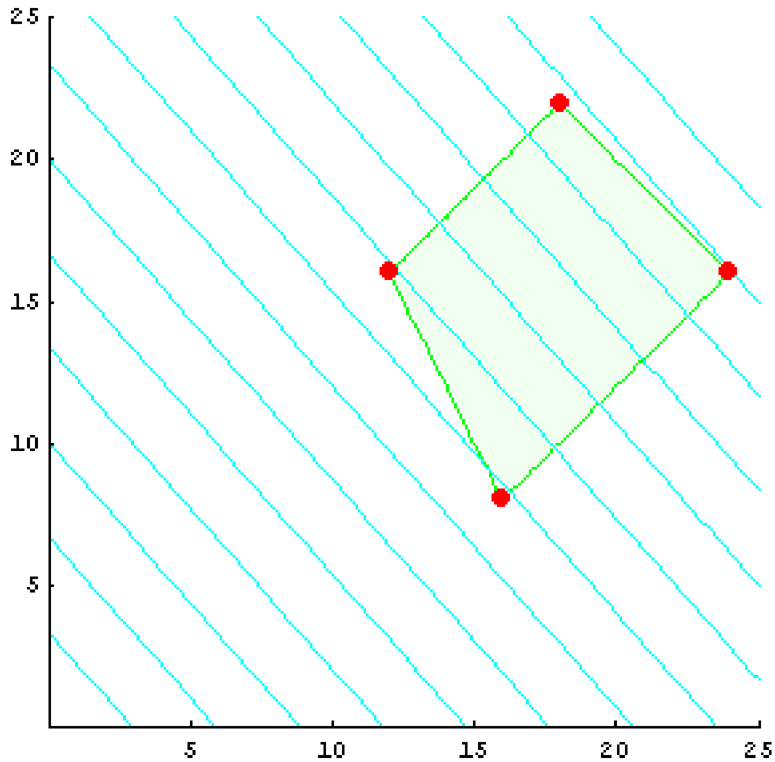
$f[\{12, 16\}] = 444$

The maximum over the region R is

$\text{Max}[f[\vec{X}_1], f[\vec{X}_2], f[\vec{X}_3], f[\vec{X}_4]] = \text{Max}[\{648, 636, 392, 444\}]$

$\text{Max}[f[\vec{X}_1], f[\vec{X}_2], f[\vec{X}_3], f[\vec{X}_4]] = 648$

Graph the level curves of the objective function.



The level curves of the objective function

$f[\{x, y\}] = 17x + 15y = c$

$c = \{50, 100, 150, 200, 250, 300, 350, 400, 450, 500, 550, 600, 650, 700\}$

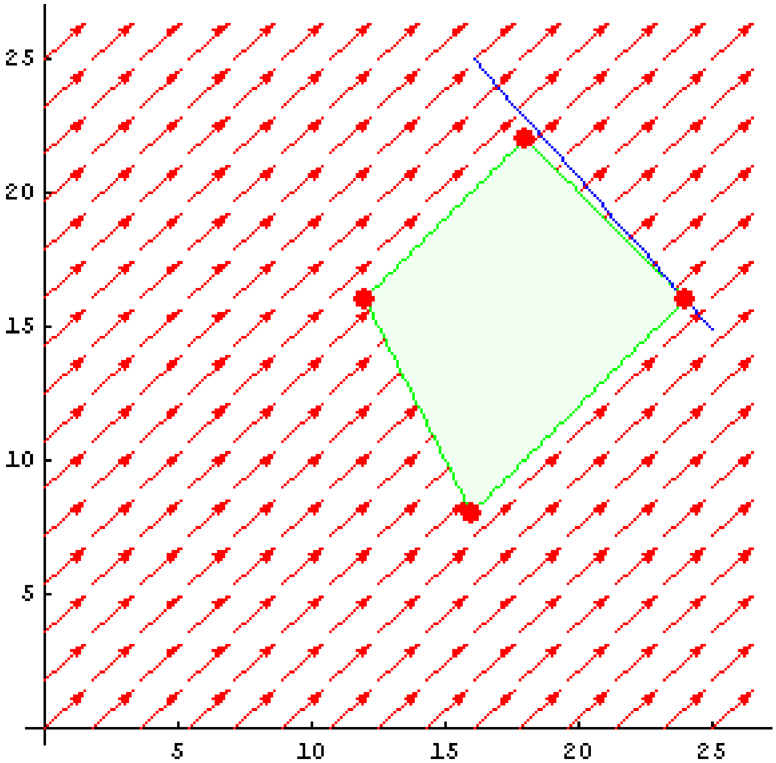
The maximum occurs at $\vec{X} = \{24, 16\}$

$f[\{24, 16\}] = 648$

The solution point for the maximum is the furthest point in the region in the direction of the gradient $\nabla f[\vec{X}]$.

Find the gradient vector $\nabla f[\vec{X}]$.

$$f[\vec{X}] = 17x + 15y$$
$$\partial_x f[\vec{X}] = \partial_x(17x + 15y) = 17$$
$$\partial_y f[\vec{X}] = \partial_y(17x + 15y) = 15$$
$$\nabla f[\vec{X}] = \{17, 15\}$$



$f[\{x,y\}] = 17x + 15y$

$\nabla f[\vec{X}] = \{17, 15\}$

A level curves of the objective function

$f[\{x,y\}] = 17x + 15y = 648$

The maximum occurs at $\vec{X} = \{24, 16\}$

$f[\{24, 16\}] = 648$

It it the furthest point in the region in the direction $\nabla f[\vec{X}]$

Example 1. Two students Ann and Carl work x and y hours per week, respectively. Together they can work at most 40 hours per week. According to the rules for part timers Ann can work at most 8 hours more that Carl. But Carl can work at most 6 hours more than Ann. There is an extra constraint $18 \leq 2y + x$. Determine the region R for these constraints.

1 (c). If Ann and Carl both earn $16 per hour, respectively, then find their maximum combined income per week.

Solution 1 (c).

Enter the linear function and the constraints.

Find the maximum of

$f[\{x,y\}] = 16x + 16y$

Subject to the constraints

$80 \leq 4x + 2y$

$y \leq 4 + x$

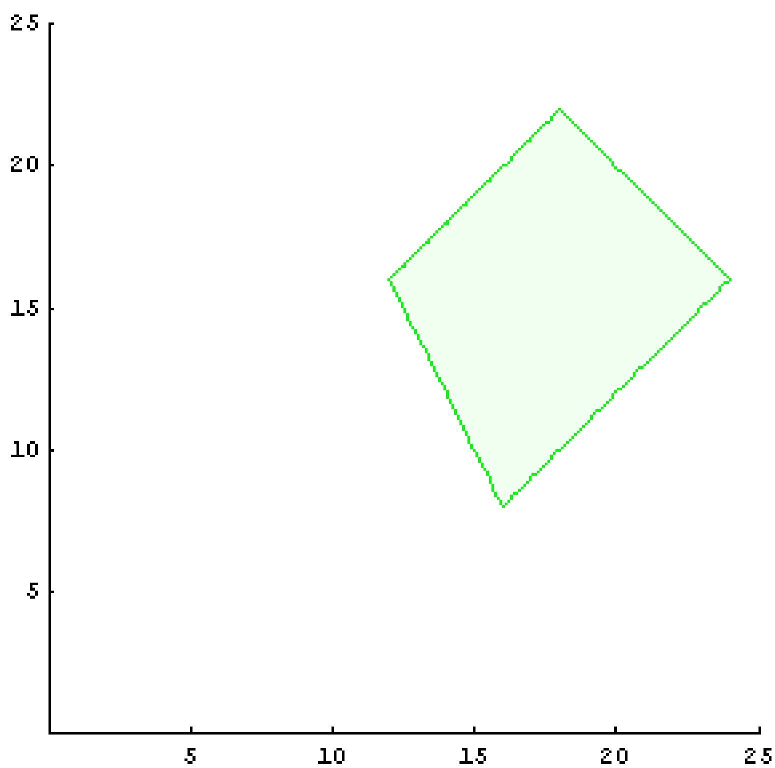
$x \leq 8 + y$

$x + y \leq 40$

$x \geq 0$

$y \geq 0$

Graph the region R defined by the constraints.



The region R defined by

$80 \le 4x + 2y$
$y \le 4 + x$
$x \le 8 + y$
$x + y \le 40$
$x \ge 0$
$y \ge 0$

The solution will occur at one of the vertices of the convex polytope. We now solve for these four points.

$\{x == 8 + y,\ x + y == 40\}$

$\vec{X}_1 = \{24, 16\}$

$\{y == 4 + x,\ x + y == 40\}$

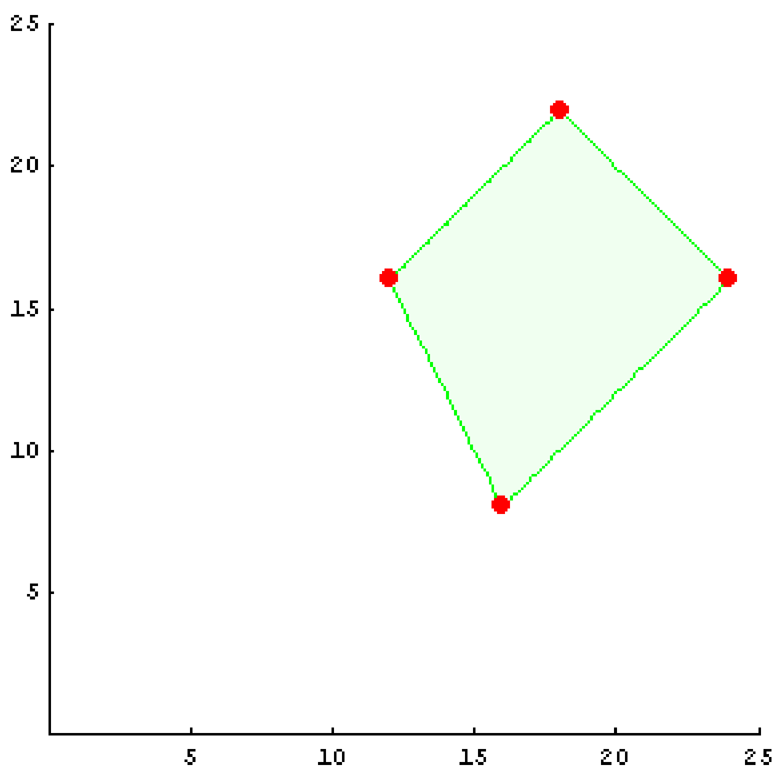
$\vec{X}_2 = \{18, 22\}$

$\{80 == 4x + 2y,\ x == 8 + y\}$

$\vec{X}_3 = \{16, 8\}$

$\{80 == 4x + 2y,\ y == 4 + x\}$

$\vec{X}_4 = \{12, 16\}$



$f[\{24, 16\}] = 640$

$f[\{18, 22\}] = 640$

$f[\{16, 8\}] = 384$

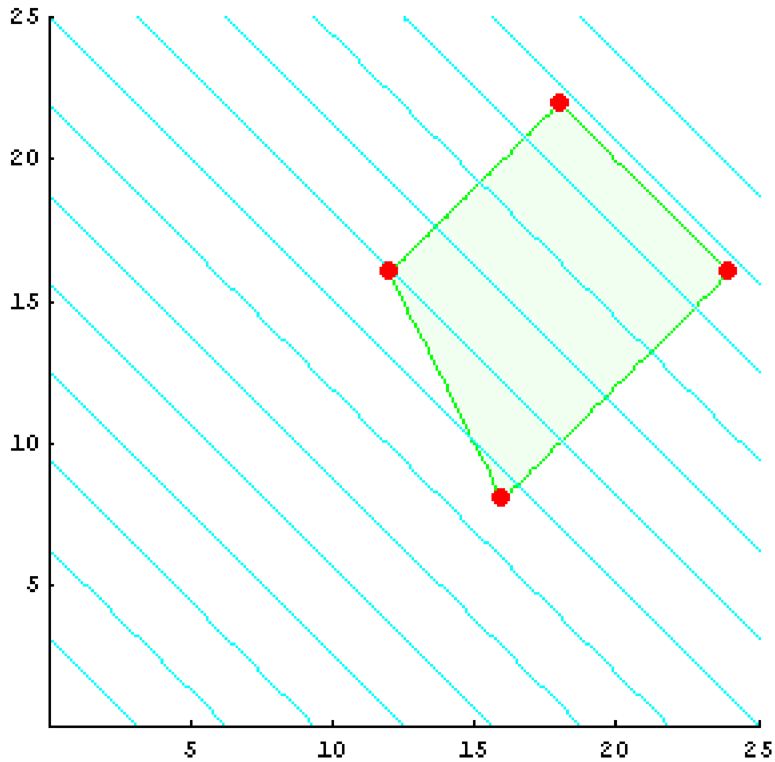
$f[\{12, 16\}] = 448$

The maximum over the region R is

$\text{Max}[f[\vec{X}_1], f[\vec{X}_2], f[\vec{X}_3], f[\vec{X}_4]] = \text{Max}[\{640, 640, 384, 448\}]$

$\text{Max}[f[\vec{X}_1], f[\vec{X}_2], f[\vec{X}_3], f[\vec{X}_4]] = 640$

Graph the level curves of the objective function.



The level curves of the objective function

$f[\{x, y\}] = 16x + 16y = c$

$c = \{50, 100, 150, 200, 250, 300, 350, 400, 450, 500, 550, 600, 650, 700\}$

The maximum occurs at $\vec{X} = \{24, 16\}$

$f[\{24, 16\}] = 640$

And

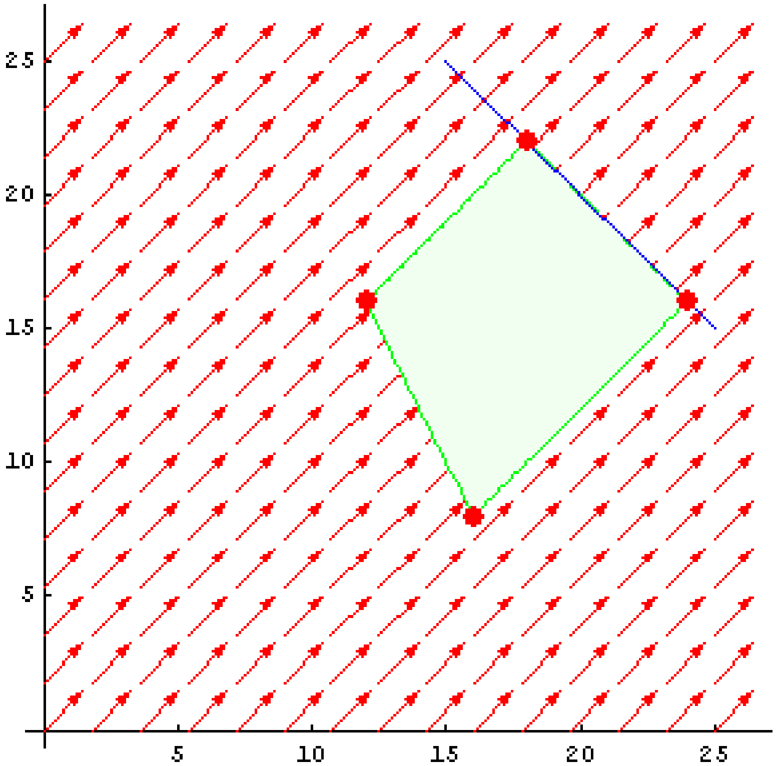
The maximum occurs at $\vec{X} = \{18, 22\}$

$f[\{18, 22\}] = 640$

The solution point for the maximum is the furthest point in the region in the direction of the gradient $\nabla f[\vec{X}]$.

Find the gradient vector $\nabla f[\vec{X}]$.

$$f[\vec{X}] = 16x + 16y$$
$$\partial_x f[\vec{X}] = \partial_x(16x + 16y) = 16$$
$$\partial_y f[\vec{X}] = \partial_y(16x + 16y) = 16$$
$$\nabla f[\vec{X}] = \{16, 16\}$$



$f[\{x,y\}] = 16x + 16y$

$\nabla f[\vec{X}] = \{16, 16\}$

A level curves of the objective function

$f[\{x,y\}] = 16x + 16y = 640$

The maximum occurs at $\vec{X} = \{24, 16\}$

The maximum occurs at $\vec{X} = \{18, 22\}$

The maximum is

$f[\{24, 16\}] = 640$

$f[\{18, 22\}] = 640$

Both points are in the region and furthest in the direction $\nabla f[\vec{X}]$

For this case any point on the line joining them is also a maximum.

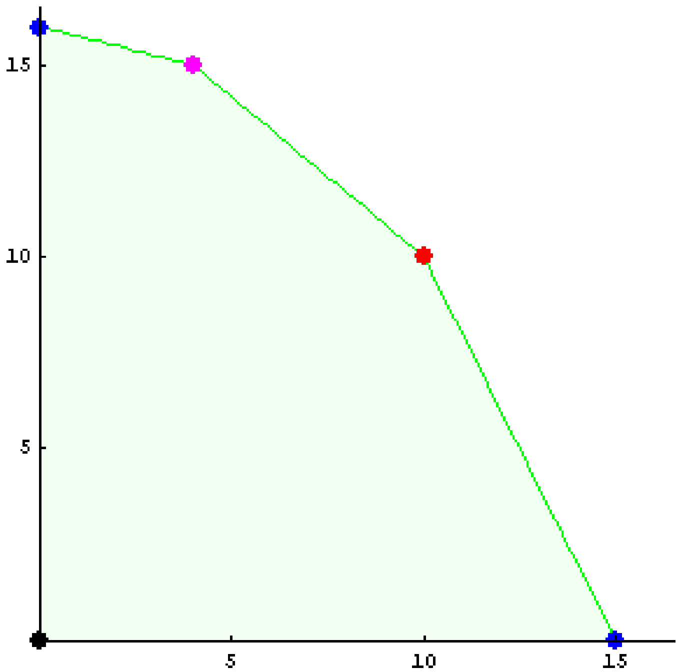
Example 2. Consider the region R in the plane defined by the inequalities:

$$2x + y \le 30$$
$$x + 4y \le 64$$
$$5x + 6y \le 110$$
$$x \ge 0$$
$$y \ge 0$$

Find the maximum of $f[x, y] = 10x + 20y$ over the region R.

Solution 2.

Graph the domain defined by the constraints.



The region defined by

$2x + y \le 30$
$x + 4y \le 64$
$5x + 6y \le 110$
$x \ge 0$
$y \ge 0$

Use the simplex algorithm to find this solution.

Enter the coefficients of the decision variables and right side of the tableau.

$$A = M = \begin{pmatrix} 2 & 1 & 30 \\ 1 & 4 & 64 \\ 5 & 6 & 110 \\ -10 & -20 & 0 \end{pmatrix};$$

Fill in the columns for the slack variables and append a column of zeros which will be use to calculate the ratios $\frac{b_i}{a_{i,e}}$ for determining the pivot rows.

x_1	x_2	x_3	x_4	x_5	b_i	$\frac{b_i}{a_{i,e}}$
2	1	1	0	0	30	0
1	4	0	1	0	64	0
5	6	0	0	1	110	0
-10	-20	0	0	0	0	0

$\{x_1, x_2\}$ are the non-basic variables.

$\{x_3, x_4, x_5\}$ are the basic variables.

The objective function is

$f[\{x_1, x_2\}] = 10\,x_1 + 20\,x_2$

The constraints are

$2\,x_1 + x_2 \leq 30$

$x_1 + 4\,x_2 \leq 64$

$5\,x_1 + 6\,x_2 \leq 110$

The bottom line of the tableau corresponds to the augmented objective function

$$F[\{x_1, x_2\}, \{x_3, x_4, x_5\}] = 10\,x_1 + 20\,x_2$$

The first step in the simplex method is to determine an exchange variable.

x_1	x_2	x_3	x_4	x_5	b_i	$\frac{b_i}{a_{i,e}}$
2	1	1	0	0	30	0
1	4	0	1	0	64	0
5	6	0	0	1	110	0
-10	-20	0	0	0	0	0

The first exchange variable $x_e = x_2$ is chosen since the current coefficient $-c_2 = -20$ is negative and is smaller any of the other coefficients $-c_j$.
Eliminate the exchange variable x_2 and use $e = 2$.

x_1	x_2	x_3	x_4	x_5	b_i	$\frac{b_i}{a_{i,e}}$	
2	1	1	0	0	30	30.	
1	4	0	1	0	64	16.	⟵
5	6	0	0	1	110	18.3333	
-10	-20	0	0	0	0		
	↑						pivot
	exchange						row
	x_2						2

To determine the pivot row we need to consider the ratios $\left\{ \frac{b_i}{a_{i,2}} \text{ for all rows with } a_{i,2} > 0 \right\}$. The pivot row p is chosen to be the row where the minimum ratio occurs. Since $\frac{b_i}{a_{i,2}} = 16.0$ is smallest, use simplex Gaussian elimination to eliminate x_2 using the pivot row $p = 2$.

x_1	x_2	x_3	x_4	x_5	b_i	$\frac{b_i}{a_{i,e}}$
$\frac{7}{4}$	0	1	$-\frac{1}{4}$	0	14	
$\frac{1}{4}$	1	0	$\frac{1}{4}$	0	16	
$\frac{7}{2}$	0	0	$-\frac{3}{2}$	1	14	
-5	0	0	5	0	320	

From column 2 we see that $x_2 = 16$. The new point $\{x_1, x_2\} = \{0, 16\}$ is a feasible solution. The simplex method has moved from its previous point $\{x_1, x_2\} = \{0, 0\}$ along the edge $x_1 = 0$ to the point $\{x_1, x_2\} = \{0, 16\}$.

$f[\{0, 16\}] = 320$

The second step in the simplex method is to determine the next exchange variable.

x_1	x_2	x_3	x_4	x_5	b_i	$\frac{b_i}{a_{i,e}}$
$\frac{7}{4}$	0	1	$-\frac{1}{4}$	0	14	
$\frac{1}{4}$	1	0	$\frac{1}{4}$	0	16	
$\frac{7}{2}$	0	0	$-\frac{3}{2}$	1	14	
-5	0	0	5	0	320	

The second exchange variable $x_e = x_1$ is chosen since the current coefficient $-c_1 = -5$ is negative and is smaller any of the other coefficients $-c_j$.
Eliminate the exchange variable x_1 and use $e = 1$.

x_1	x_2	x_3	x_4	x_5	b_i	$\frac{b_i}{a_{i,e}}$	
$\frac{7}{4}$	0	1	$-\frac{1}{4}$	0	14	8.	
$\frac{1}{4}$	1	0	$\frac{1}{4}$	0	16	64.	
$\frac{7}{2}$	0	0	$-\frac{3}{2}$	1	14	4.	⟵
-5	0	0	5	0	320		
↑ exchange x_1							pivot row 3

To determine the pivot row we need to consider the ratios $\left\{\frac{b_i}{a_{i,1}} \text{ for all rows with } a_{i,1} > 0\right\}$. The pivot row p is chosen to be the row where the minimum ratio occurs. Since $\frac{b_i}{a_{i,1}} = 4.0$ is smallest, use simplex Gaussian elimination to eliminate x_1 using the pivot row $p = 3$.

x_1	x_2	x_3	x_4	x_5	b_i	$\frac{b_i}{a_{i,e}}$
0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	7	
0	1	0	$\frac{5}{14}$	$-\frac{1}{14}$	15	
1	0	0	$-\frac{3}{7}$	$\frac{2}{7}$	4	
0	0	0	$\frac{20}{7}$	$\frac{10}{7}$	340	

There are no negative coefficients in the bottom row, so we are done.

From column 1 and row 3 we see that $x_1 = 4$. From column 2 and row 2 we see that $x_2 = 15$. The new point $\{x_1, x_2\} = \{4, 15\}$ is the optimal feasible solution. The simplex method has moved from its previous point $\{x_1, x_2\} = \{0, 16\}$ along the edge $x_1 + 4 x_2 = 64$ to the point $\{x_1, x_2\} = \{4, 15\}$. The value of the objective function at the solution point is located in the bottom row of the column "b_i", i.e. $b_4 = 340.$

$$f[\{x_1, x_2\}] = 10 x_1 + 20 x_2$$
$$f[\{4, 15\}] = 340$$

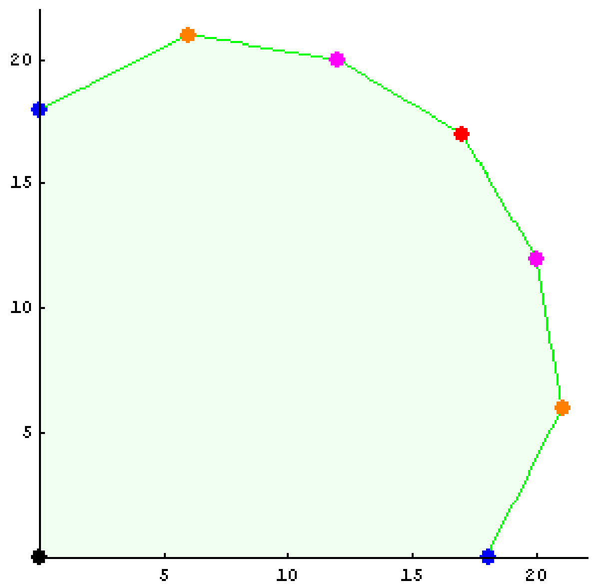
Example 3. Consider the region R in the plane defined by the inequalities:

$$-x + 2y \le 36$$
$$x + 6y \le 132$$
$$3x + 5y \le 136$$
$$5x + 3y \le 136$$
$$6x + y \le 132$$
$$2x - y \le 36$$
$$x \ge 0$$
$$y \ge 0$$

Find the maximum of $f[x, y] = 10x + 10y$ over the region R.

Solution 3.

Graph the domain defined by the constraints.



The region defined by

$$-x + 2y \le 36$$
$$x + 6y \le 132$$
$$3x + 5y \le 136$$
$$5x + 3y \le 136$$
$$6x + y \le 132$$
$$2x - y \le 36$$
$$x \ge 0$$
$$y \ge 0$$

Use the simplex algorithm to find this solution.

Enter the coefficients of the decision variables and right side of the tableau.

$$\mathbf{A} = \mathbf{M} = \begin{pmatrix} -1 & 2 & 36 \\ 1 & 6 & 132 \\ 3 & 5 & 136 \\ 5 & 3 & 136 \\ 6 & 1 & 132 \\ 2 & -1 & 36 \\ -10 & -10 & 0 \end{pmatrix};$$

Fill in the columns for the slack variables and append a column of zeros which will be use to calculate the ratios $\frac{b_i}{a_{i,e}}$ for determining the pivot rows.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$
-1	2	1	0	0	0	0	0	36	0
1	6	0	1	0	0	0	0	132	0
3	5	0	0	1	0	0	0	136	0
5	3	0	0	0	1	0	0	136	0
6	1	0	0	0	0	1	0	132	0
2	-1	0	0	0	0	0	1	36	0
-10	-10	0	0	0	0	0	0	0	0

$\{x_1, x_2\}$ are the non-basic variables.

$\{x_3, x_4, x_5, x_6, x_7, x_8\}$ are the basic variables.

The objective function is

$f[\{x_1, x_2\}] = 10\,x_1 + 10\,x_2$

The constraints are

$-x_1 + 2\,x_2 \le 36$

$x_1 + 6\,x_2 \le 132$

$3\,x_1 + 5\,x_2 \le 136$

$5\,x_1 + 3\,x_2 \le 136$

$6\,x_1 + x_2 \le 132$

$2\,x_1 - x_2 \le 36$

Iteration 1

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$	
-1	2	1	0	0	0	0	0	36	-36.	
1	6	0	1	0	0	0	0	132	132.	
3	5	0	0	1	0	0	0	136	45.3333	
5	3	0	0	0	1	0	0	136	27.2	
6	1	0	0	0	0	1	0	132	22.	
2	-1	0	0	0	0	0	1	36	18.	⟵
-10	-10	0	0	0	0	0	0	0		
↑ exchange x_1										pivot row 6

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$
0	$\frac{3}{2}$	1	0	0	0	0	$\frac{1}{2}$	54	
0	$\frac{13}{2}$	0	1	0	0	0	$-\frac{1}{2}$	114	
0	$\frac{13}{2}$	0	0	1	0	0	$-\frac{3}{2}$	82	
0	$\frac{11}{2}$	0	0	0	1	0	$-\frac{5}{2}$	46	
0	4	0	0	0	0	1	-3	24	
1	$-\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	18	
0	-15	0	0	0	0	0	5	180	

Iteration 2

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$	
0	$\frac{3}{2}$	1	0	0	0	0	$\frac{1}{2}$	54	36.	
0	$\frac{13}{2}$	0	1	0	0	0	$-\frac{1}{2}$	114	17.5385	
0	$\frac{13}{2}$	0	0	1	0	0	$-\frac{3}{2}$	82	12.6154	
0	$\frac{11}{2}$	0	0	0	1	0	$-\frac{5}{2}$	46	8.36364	
0	4	0	0	0	0	1	−3	24	6.	⟵
1	$-\frac{1}{2}$	0	0	0	0	0	$\frac{1}{2}$	18	−36.	
0	−15	0	0	0	0	0	5	180		
	↑ exchange x_2									pivot row 5

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$
0	0	1	0	0	0	$-\frac{3}{8}$	$\frac{13}{8}$	45	
0	0	0	1	0	0	$-\frac{13}{8}$	$\frac{35}{8}$	75	
0	0	0	0	1	0	$-\frac{13}{8}$	$\frac{27}{8}$	43	
0	0	0	0	0	1	$-\frac{11}{8}$	$\frac{13}{8}$	13	
0	1	0	0	0	0	$\frac{1}{4}$	$-\frac{3}{4}$	6	
1	0	0	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	21	
0	0	0	0	0	0	$\frac{15}{4}$	$-\frac{25}{4}$	270	

Iteration 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$	
0	0	1	0	0	0	$-\frac{3}{8}$	$\frac{13}{8}$	45	27.6923	
0	0	0	1	0	0	$-\frac{13}{8}$	$\frac{35}{8}$	75	17.1429	
0	0	0	0	1	0	$-\frac{13}{8}$	$\frac{27}{8}$	43	12.7407	
0	0	0	0	0	1	$-\frac{11}{8}$	$\frac{13}{8}$	13	8.	⟵
0	1	0	0	0	0	$\frac{1}{4}$	$-\frac{3}{4}$	6	-8.	
1	0	0	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	21	168.	
0	0	0	0	0	0	$\frac{15}{4}$	$-\frac{25}{4}$	270		
							↑ exchange x_8			pivot row 4

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$
0	0	1	0	0	-1	1	0	32	
0	0	0	1	0	$-\frac{35}{13}$	$\frac{27}{13}$	0	40	
0	0	0	0	1	$-\frac{27}{13}$	$\frac{16}{13}$	0	16	
0	0	0	0	0	$\frac{8}{13}$	$-\frac{11}{13}$	1	8	
0	1	0	0	0	$\frac{6}{13}$	$-\frac{5}{13}$	0	12	
1	0	0	0	0	$-\frac{1}{13}$	$\frac{3}{13}$	0	20	
0	0	0	0	0	$\frac{50}{13}$	$-\frac{20}{13}$	0	320	

Iteration 4

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$	
0	0	1	0	0	-1	1	0	32	32.	
0	0	0	1	0	$-\frac{35}{13}$	$\frac{27}{13}$	0	40	19.2593	
0	0	0	0	1	$-\frac{27}{13}$	$\frac{16}{13}$	0	16	13.	⟵
0	0	0	0	0	$\frac{8}{13}$	$-\frac{11}{13}$	1	8	-9.45455	
0	1	0	0	0	$\frac{6}{13}$	$-\frac{5}{13}$	0	12	-31.2	
1	0	0	0	0	$-\frac{1}{13}$	$\frac{3}{13}$	0	20	86.6667	
0	0	0	0	0	$\frac{50}{13}$	$-\frac{20}{13}$	0	320		
						↑ exchange x_7				pivot row 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	$\frac{b_i}{a_{i,e}}$
0	0	1	0	$-\frac{13}{16}$	$\frac{11}{16}$	0	0	19	
0	0	0	1	$-\frac{27}{16}$	$\frac{13}{16}$	0	0	13	
0	0	0	0	$\frac{13}{16}$	$-\frac{27}{16}$	1	0	13	
0	0	0	0	$\frac{11}{16}$	$-\frac{13}{16}$	0	1	19	
0	1	0	0	$\frac{5}{16}$	$-\frac{3}{16}$	0	0	17	
1	0	0	0	$-\frac{3}{16}$	$\frac{5}{16}$	0	0	17	
0	0	0	0	$\frac{5}{4}$	$\frac{5}{4}$	0	0	340	

Conclusion

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b_i	
0	0	1	0	$-\frac{13}{16}$	$\frac{11}{16}$	0	0	19	
0	0	0	1	$-\frac{27}{16}$	$\frac{13}{16}$	0	0	13	
0	0	0	0	$\frac{13}{16}$	$-\frac{27}{16}$	1	0	13	
0	0	0	0	$\frac{11}{16}$	$-\frac{13}{16}$	0	1	19	
0	1	0	0	$\frac{5}{16}$	$-\frac{3}{16}$	0	0	17	
1	0	0	0	$-\frac{3}{16}$	$\frac{5}{16}$	0	0	17	done
0	0	0	0	$\frac{5}{4}$	$\frac{5}{4}$	0	0	340	⟵

There are no negative coefficients in the bottom row, so we are done.

From column 1 and row 6 we see that $x_1 = 17$. From column 2 and row 5 we see that $x_2 = 17$. The final point $\{x_1, x_2\} = \{17, 17\}$ is the optimal feasible solution. The simplex method has moved from the origin along the edges to the point $\{x_1, x_2\} = \{17, 17\}$. The value of the objective function at the solution point is located in the bottom row of the column "b_i", i.e. $b_7 = 340$.

$$f[\{x_1, x_2\}] = 10\,x_1 + 10\,x_2$$

$$f[\{17, 17\}] = 340$$

Example 4. Find the maximum of $f[x_1, x_2, x_3, x_4] = 5x_1 + 9x_2 + 8x_3 + 6x_4$ subject to the constraints:

$$2x_1 + 3x_2 + 4x_3 \le 500$$
$$3x_2 + 2x_3 + 5x_4 \le 300$$
$$2x_1 + 4x_2 + 2x_4 \le 400$$
$$4x_1 + 5x_3 + 3x_4 \le 800$$
$$x_1 + x_2 + x_3 + x_4 \le 200$$

Solution 4.

Enter the objective function and constraints.

Find the maximum of

$f[\{x_1,x_2,x_3,x_4\}] = 5x_1 + 9x_2 + 8x_3 + 6x_4$

Subject to the constraints

$$2x_1 + 3x_2 + 4x_3 \le 500$$
$$3x_2 + 2x_3 + 5x_4 \le 300$$
$$2x_1 + 4x_2 + 2x_4 \le 400$$
$$4x_1 + 5x_3 + 3x_4 \le 800$$
$$x_1 + x_2 + x_3 + x_4 \le 200$$

Use the simplex algorithm to find this solution.

Enter the coefficients of the decision variables and right side of the tableau.

$$A = M = \begin{pmatrix} 2 & 3 & 4 & 0 & 500 \\ 0 & 3 & 2 & 5 & 300 \\ 2 & 4 & 0 & 2 & 400 \\ 4 & 0 & 5 & 3 & 800 \\ 1 & 1 & 1 & 1 & 200 \\ -5 & -9 & -8 & -6 & 0 \end{pmatrix};$$

Fill in the columns for the slack variables and append a column of zeros which will be use to calculate the ratios $\frac{b_i}{a_{i,e}}$ for determining the pivot rows.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$
2	3	4	0	1	0	0	0	0	500	0
0	3	2	5	0	1	0	0	0	300	0
2	4	0	2	0	0	1	0	0	400	0
4	0	5	3	0	0	0	1	0	800	0
1	1	1	1	0	0	0	0	1	200	0
-5	-9	-8	-6	0	0	0	0	0	0	0

$\{x_1, x_2, x_3, x_4\}$ are the non-basic variables.

$\{x_5, x_6, x_7, x_8, x_9\}$ are the basic variables.

The objective function is

$f[\{x_1, x_2, x_3, x_4\}] = 5x_1 + 9x_2 + 8x_3 + 6x_4$

The constraints are

$2x_1 + 3x_2 + 4x_3 \leq 500$

$3x_2 + 2x_3 + 5x_4 \leq 300$

$2x_1 + 4x_2 + 2x_4 \leq 400$

$4x_1 + 5x_3 + 3x_4 \leq 800$

$x_1 + x_2 + x_3 + x_4 \leq 200$

Iteration 1

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$	
2	3	4	0	1	0	0	0	0	500	166.667	
0	3	2	5	0	1	0	0	0	300	100.	
2	4	0	2	0	0	1	0	0	400	100.	$\longleftarrow$
4	0	5	3	0	0	0	1	0	800	∞	
1	1	1	1	0	0	0	0	1	200	200.	
-5	-9	-8	-6	0	0	0	0	0	0		

$\uparrow$

exchange x_2

pivot row 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$
$\frac{1}{2}$	0	4	$-\frac{3}{2}$	1	0	$-\frac{3}{4}$	0	0	200	
$-\frac{3}{2}$	0	2	$\frac{7}{2}$	0	1	$-\frac{3}{4}$	0	0	0	
$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	100	
4	0	5	3	0	0	0	1	0	800	
$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	1	100	
$-\frac{1}{2}$	0	-8	$-\frac{3}{2}$	0	0	$\frac{9}{4}$	0	0	900	

Iteration 2

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$	
$\frac{1}{2}$	0	4	$-\frac{3}{2}$	1	0	$-\frac{3}{4}$	0	0	200	50.	
$-\frac{3}{2}$	0	2	$\frac{7}{2}$	0	1	$-\frac{3}{4}$	0	0	0	0.	⟵
$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	100	∞	
4	0	5	3	0	0	0	1	0	800	160.	
$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	0	1	100	100.	
$-\frac{1}{2}$	0	-8	$-\frac{3}{2}$	0	0	$\frac{9}{4}$	0	0	900		
		↑ exchange x_3									pivot row 2

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$
$\frac{7}{2}$	0	0	$-\frac{17}{2}$	1	-2	$\frac{3}{4}$	0	0	200	
$-\frac{3}{4}$	0	1	$\frac{7}{4}$	0	$\frac{1}{2}$	$-\frac{3}{8}$	0	0	0	
$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	100	
$\frac{31}{4}$	0	0	$-\frac{23}{4}$	0	$-\frac{5}{2}$	$\frac{15}{8}$	1	0	800	
$\frac{5}{4}$	0	0	$-\frac{5}{4}$	0	$-\frac{1}{2}$	$\frac{1}{8}$	0	1	100	
$-\frac{13}{2}$	0	0	$\frac{25}{2}$	0	4	$-\frac{3}{4}$	0	0	900	

Iteration 3

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$	
$\frac{7}{2}$	0	0	$-\frac{17}{2}$	1	-2	$\frac{3}{4}$	0	0	200	57.1429	⟵
$-\frac{3}{4}$	0	1	$\frac{7}{4}$	0	$\frac{1}{2}$	$-\frac{3}{8}$	0	0	0	0.	
$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	100	200.	
$\frac{31}{4}$	0	0	$-\frac{23}{4}$	0	$-\frac{5}{2}$	$\frac{15}{8}$	1	0	800	103.226	
$\frac{5}{4}$	0	0	$-\frac{5}{4}$	0	$-\frac{1}{2}$	$\frac{1}{8}$	0	1	100	80.	
$-\frac{13}{2}$	0	0	$\frac{25}{2}$	0	4	$-\frac{3}{4}$	0	0	900		
↑ exchange x_1											pivot row 1

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$
1	0	0	$-\frac{17}{7}$	$\frac{2}{7}$	$-\frac{4}{7}$	$\frac{3}{14}$	0	0	$\frac{400}{7}$	
0	0	1	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$	0	0	$\frac{300}{7}$	
0	1	0	$\frac{12}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	0	$\frac{500}{7}$	
0	0	0	$\frac{183}{14}$	$-\frac{31}{14}$	$\frac{27}{14}$	$\frac{3}{14}$	1	0	$\frac{2500}{7}$	
0	0	0	$\frac{25}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$-\frac{1}{7}$	0	1	$\frac{200}{7}$	
0	0	0	$-\frac{23}{7}$	$\frac{13}{7}$	$\frac{2}{7}$	$\frac{9}{14}$	0	0	$\frac{8900}{7}$	

Iteration 4

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$	
1	0	0	$-\frac{17}{7}$	$\frac{2}{7}$	$-\frac{4}{7}$	$\frac{3}{14}$	0	0	$\frac{400}{7}$	-23.5294	
0	0	1	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$	0	0	$\frac{300}{7}$	-600.	
0	1	0	$\frac{12}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	0	0	$\frac{500}{7}$	41.6667	
0	0	0	$\frac{183}{14}$	$-\frac{31}{14}$	$\frac{27}{14}$	$\frac{3}{14}$	1	0	$\frac{2500}{7}$	27.3224	
0	0	0	$\frac{25}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$-\frac{1}{7}$	0	1	$\frac{200}{7}$	16.	⟵
0	0	0	$-\frac{23}{7}$	$\frac{13}{7}$	$\frac{2}{7}$	$\frac{9}{14}$	0	0	$\frac{8900}{7}$		
			↑ exchange x_4								pivot row 5

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	$\frac{b_i}{a_{i,e}}$
1	0	0	0	$-\frac{1}{5}$	$-\frac{7}{25}$	$\frac{1}{50}$	0	$\frac{34}{25}$	96	
0	0	1	0	$\frac{1}{5}$	$\frac{2}{25}$	$-\frac{11}{50}$	0	$\frac{1}{25}$	44	
0	1	0	0	$\frac{1}{5}$	$\frac{2}{25}$	$\frac{7}{25}$	0	$-\frac{24}{25}$	44	
0	0	0	0	$\frac{2}{5}$	$\frac{9}{25}$	$\frac{63}{50}$	1	$-\frac{183}{25}$	148	
0	0	0	1	$-\frac{1}{5}$	$\frac{3}{25}$	$-\frac{2}{25}$	0	$\frac{14}{25}$	16	
0	0	0	0	$\frac{6}{5}$	$\frac{17}{25}$	$\frac{19}{50}$	0	$\frac{46}{25}$	1324	

Conclusion

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	b_i	
1	0	0	0	$-\frac{1}{5}$	$-\frac{7}{25}$	$\frac{1}{50}$	0	$\frac{34}{25}$	96	
0	0	1	0	$\frac{1}{5}$	$\frac{2}{25}$	$-\frac{11}{50}$	0	$\frac{1}{25}$	44	
0	1	0	0	$\frac{1}{5}$	$\frac{2}{25}$	$\frac{7}{25}$	0	$-\frac{24}{25}$	44	
0	0	0	0	$\frac{2}{5}$	$\frac{9}{25}$	$\frac{63}{50}$	1	$-\frac{183}{25}$	148	
0	0	0	1	$-\frac{1}{5}$	$\frac{3}{25}$	$-\frac{2}{25}$	0	$\frac{14}{25}$	16	done
0	0	0	0	$\frac{6}{5}$	$\frac{17}{25}$	$\frac{19}{50}$	0	$\frac{46}{25}$	1324	$\longleftarrow$

There are no negative coefficients in the bottom row, so we are done.

Aside. The bottom line of the tableau corresponds to the augmented objective function

From column 1 and row 1 we see that $x_1 = 96$. From column 2 and row 3 we see that $x_2 = 44$. From column 3 and row 2 we see that $x_3 = 44$. From column 4 and row 4 we see that $x_4 = 16$. The final point $\{x_1, x_2, x_3\} = \left\{\frac{15}{2}, 3, \frac{9}{2}\right\}$ is the optimal feasible solution. The simplex method has moved from the origin along the edges to the point $\{x_1, x_2, x_3\} = \left\{\frac{15}{2}, 3, \frac{9}{2}\right\}$. The value of the objective function at the solution point is located in the bottom row of the column "b_i", i.e. $b_6 = 1324$.

$$f[\{x_1, x_2, x_3, x_4\}] = 5x_1 + 9x_2 + 8x_3 + 6x_4$$
$$f[\{96, 44, 44, 16\}] = 1324$$