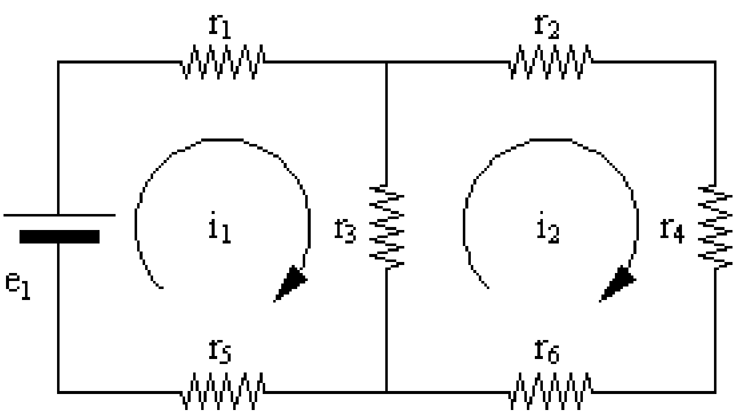
13. Kirchoff's Law

Background

Solution of linear systems can be applied to resistor network circuits. Kirchoff's voltage law says that the sum of the voltage drops around any closed loop in the network must equal zero. A closed loop has the obvious definition: starting at a node, trace a path through the circuit that returns you to the original starting node.

Network #1

Consider the network consisting of six resistors and two battery, shown in the figure below.



There are two closed loops. When Kirchoff's voltage law is applied, we obtain the following linear system of equations.

$$\begin{aligned} (r_1 + r_3 + r_5)\, i_1 \quad - \quad r_3\, i_2 &= e_1 \\ -r_3\, i_1 \quad + \quad (r_2 + r_3 + r_4 + r_6)\, i_2 &= 0 \end{aligned}$$

Example 1. Solve the network #1 for the currents i_1, i_2 given the following value for the resistors and battery:

$r_1 = 10,\ r_2 = 10,\ r_3 = 10,\ r_4 = 20,\ r_5 = 10,\ r_6 = 30$

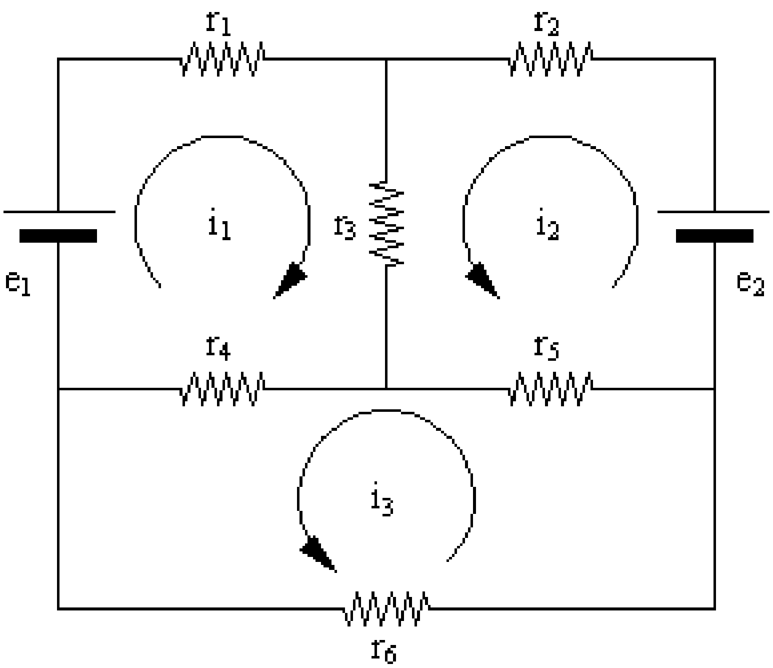
and .

$e_1 = 20$

Solution 1.

Network #2

Consider the network consisting of six resistors and two batteries, shown in the figure below.



There are three loops. When Kirchoff's voltage law is applied, we obtain the following linear system of equations.

$$\begin{aligned}
(r_1 + r_3 + r_4)\, i_1 &+ r_3\, i_2 &+ r_4\, i_3 &= e_1 \\
r_3\, i_1 &+ (r_2 + r_3 + r_5)\, i_2 &- r_5\, i_3 &= e_2 \\
r_4\, i_1 &- r_5\, i_2 &+ (r_4 + r_5 + r_6)\, i_3 &= 0
\end{aligned}$$

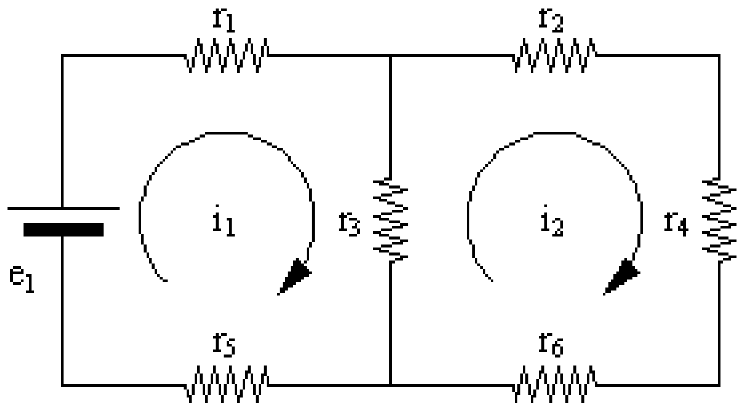
Example 2. Solve the network #2 for the currents i_1, i_2, i_3 given the following value for the resistors and batteries:

$r_1 = 1,\ r_2 = 2,\ r_3 = 4,\ r_4 = 3,\ r_5 = 1,\ r_6 = 5$

and .

$e_1 = 41,\ e_2 = 38$

Solution 2.



Example 1. Solve the network #1 for the currents i_1, i_2 given the following value for the resistors and battery:

$r_1 = 10,\ r_2 = 10,\ r_3 = 10,\ r_4 = 20,\ r_5 = 10,\ r_6 = 30$

and .

$e_1 = 20$

Solution 1.

Solve the linear system

$$\begin{pmatrix} r_1 + r_3 + r_5 & -r_3 \\ -r_3 & r_2 + r_3 + r_4 + r_6 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ 0 \end{pmatrix}$$

Solve the linear system

$$\begin{pmatrix} 30 & -10 \\ -10 & 70 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 0 \end{pmatrix}$$

The solution is

$$\begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} \frac{7}{10} \\ \frac{1}{10} \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$$

Aside. The solution could be found using Cramer's rule.

$$A = \begin{pmatrix} 30 & -10 \\ -10 & 70 \end{pmatrix}$$

$$B = \begin{pmatrix} 20 \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 20 & -10 \\ 0 & 70 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 30 & 20 \\ -10 & 0 \end{pmatrix}$$

$$i_1 = \text{Det}\left[\begin{pmatrix} 20 & -10 \\ 0 & 70 \end{pmatrix}\right] \;/\; \text{Det}\left[\begin{pmatrix} 30 & -10 \\ -10 & 70 \end{pmatrix}\right]$$

$$i_1 = (1400)/(2000)$$

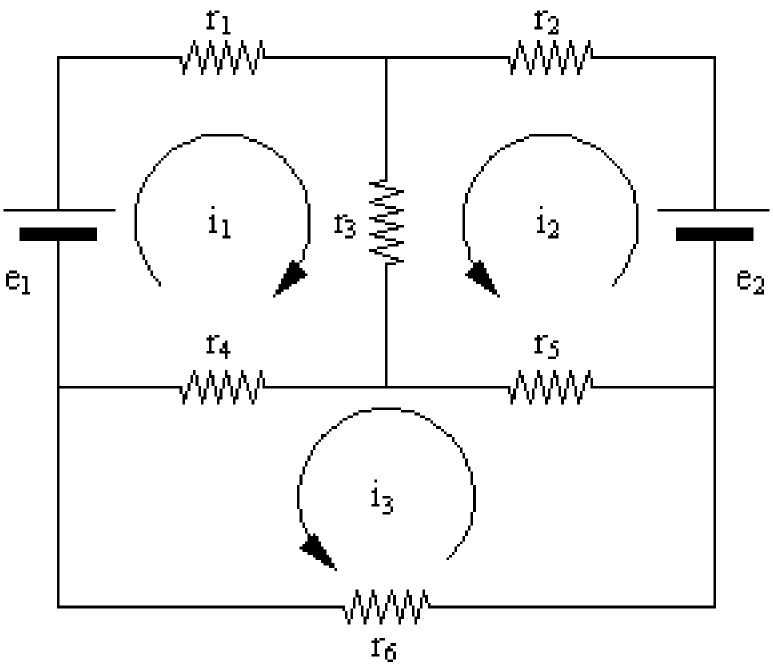
$$i_1 = \frac{7}{10} = 0.7$$

$$i_2 = \text{Det}\left[\begin{pmatrix} 30 & 20 \\ -10 & 0 \end{pmatrix}\right] \;/\; \text{Det}\left[\begin{pmatrix} 30 & -10 \\ -10 & 70 \end{pmatrix}\right]$$

$$i_2 = (200)/(2000)$$

$$i_2 = \frac{1}{10} = 0.1$$

Network #2



Example 2. Solve the network #2 for the currents i_1, i_2, i_3 given the following value for the resistors and batteries:

$r_1 = 1,\ r_2 = 2,\ r_3 = 4,\ r_4 = 3,\ r_5 = 1,\ r_6 = 5$

and .

$e_1 = 41,\ e_2 = 38$

Solution 2.

$$\begin{pmatrix} r_1 + r_3 + r_4 & r_3 & r_4 \\ r_3 & r_2 + r_3 + r_5 & -r_5 \\ r_4 & -r_5 & r_4 + r_5 + r_6 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ 0 \end{pmatrix}$$

Solve the linear system

$$\begin{pmatrix} 8 & 4 & 3 \\ 4 & 7 & -1 \\ 3 & -1 & 9 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 41 \\ 38 \\ 0 \end{pmatrix}$$

The solution is

$$\begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4. \\ 3. \\ -1. \end{pmatrix}$$

Aside. The solution could be found using Cramer's rule.

$$A = \begin{pmatrix} 8 & 4 & 3 \\ 4 & 7 & -1 \\ 3 & -1 & 9 \end{pmatrix}$$

$$B = \begin{pmatrix} 41 \\ 38 \\ 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 41 & 4 & 3 \\ 38 & 7 & -1 \\ 0 & -1 & 9 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 8 & 41 & 3 \\ 4 & 38 & -1 \\ 3 & 0 & 9 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 8 & 4 & 41 \\ 4 & 7 & 38 \\ 3 & -1 & 0 \end{pmatrix}$$

$$i_1 = \mathrm{Det}\left[\begin{pmatrix} 41 & 4 & 3 \\ 38 & 7 & -1 \\ 0 & -1 & 9 \end{pmatrix}\right] / \mathrm{Det}\left[\begin{pmatrix} 8 & 4 & 3 \\ 4 & 7 & -1 \\ 3 & -1 & 9 \end{pmatrix}\right]$$

$$i_1 = (1060)/(265)$$

$$i_1 = 4 = 4.$$

$$i_2 = \mathrm{Det}\left[\begin{pmatrix} 8 & 41 & 3 \\ 4 & 38 & -1 \\ 3 & 0 & 9 \end{pmatrix}\right] / \mathrm{Det}\left[\begin{pmatrix} 8 & 4 & 3 \\ 4 & 7 & -1 \\ 3 & -1 & 9 \end{pmatrix}\right]$$

$$i_2 = (795)/(265)$$

$$i_2 = 3 = 3.$$

$$i_3 = \mathrm{Det}\left[\begin{pmatrix} 8 & 4 & 41 \\ 4 & 7 & 38 \\ 3 & -1 & 0 \end{pmatrix}\right] / \mathrm{Det}\left[\begin{pmatrix} 8 & 4 & 3 \\ 4 & 7 & -1 \\ 3 & -1 & 9 \end{pmatrix}\right]$$

$$i_3 = (-265)/(265)$$

$$i_3 = -1 = -1.$$