2. The Bisection Method

Background. The bisection method is one of the bracketing methods for finding roots of equations.
Implementation. Given a function f(x) and an interval which might contain a root, perform a predetermined number of iterations using the bisection method.

Theorem (Bisection Theorem). Assume that $f \in C[a, b]$ and that there exists a number $r \in [a, b]$ such that $f(r) = 0$.
If $f(a)$ and $f(b)$ have opposite signs, and $\{c_n\}$ represents the sequence of midpoints generated by the bisection process, then

$$\left| r - c_n \right| \le \frac{b - a}{2^{n+1}} \quad \text{for} \quad n = 0, 1, \ldots,$$

and the sequence $\{c_n\}$ converges to the zero $x = r$.

That is, $\lim_{k \to \infty} c_n = r$.

Example 1. Find all the real solutions to the cubic equation $x^3 + 4x^2 - 10 = 0$.
Solution 1.

Concise Program for the Bisection Method

```
Bisection[a0_, b0_, m_] :=
 Module[{a = N[a0], b = N[b0]},
  c = (a + b)/2;
  k = 0;
  While[k < m,
   If[Sign[f[b]] == Sign[f[c]],
    b = c, a = c; ];
   c = (a + b)/2;
   k = k + 1; ];
  Print["   c   = ", NumberForm[c, 16]];
  Print[" Δc   = ±", (b - a)/2];
  Print["f[c]  = ", NumberForm[f[c], 16]]; ];
```

Example 2. **Convergence** Find the solution to the cubic equation $x^3 + 4x^2 - 10 = 0$. Use the starting

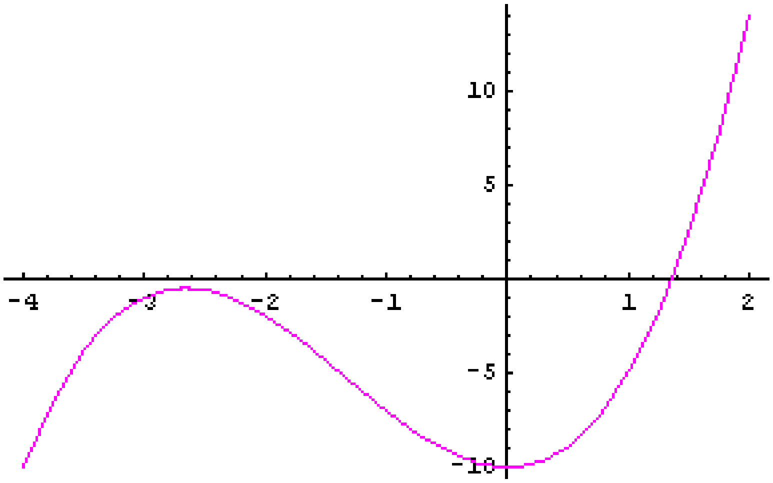
interval `[a, b] = [-1, 2]`.
Solution 2.

Example 3. **Not a root located** Find the solution to the equation `Tan[x] = 0`. Use the starting interval `[a, b] = [0, 2]`.
Solution 3.

Example 1. Find all the real solutions to the cubic equation $x^3 + 4x^2 - 10 = 0$.

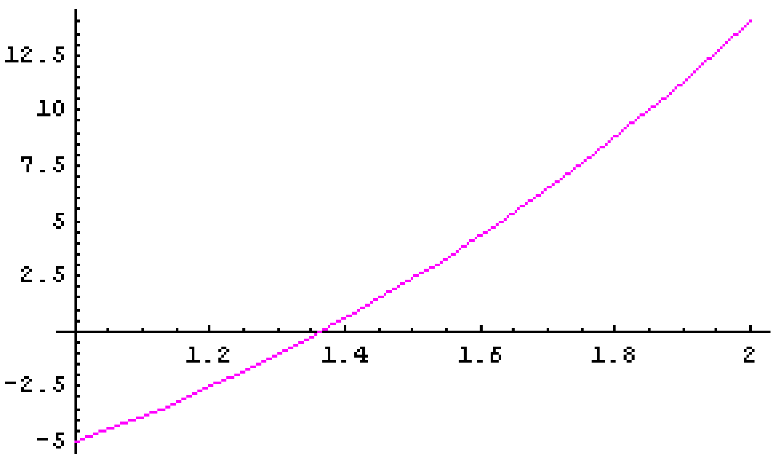
Solution 1.

Plot the function.



$y = f[x] = -10 + 4x^2 + x^3$

There appears to be only one real root which lies in the interval [1,2].



$y = f[x] = -10 + 4x^2 + x^3$

Call the Bisection subroutine on the interval [1,2] using 10 iterations

k	a_k	c_k	b_k	$f[c_k]$
0	1.	1.5	2.	2.375
1	1.	1.25	1.5	-1.796875
2	1.25	1.375	1.5	0.162109375

3	1.25	1.3125	1.375	-0.848388671875
4	1.3125	1.34375	1.375	-0.350982666015625
5	1.34375	1.359375	1.375	-0.0964088439941406
6	1.359375	1.3671875	1.375	0.0323557853698730 5
7	1.359375	1.36328125	1.3671875	-0.0321499705314636 2
8	1.36328125	1.365234375	1.3671875	0.000072024762630462 65
9	1.36328125	1.3642578125	1.365234375	-0.0160466907545924 2
10	1.3642578125	1.36474609375	1.365234375	-0.00798926281277090 3

$$c = 1.36474609375$$
$$\Delta c = \pm 0.000488281$$
$$f[c] = -0.00798926281277090 3$$

After 10 iterations, the interval has been reduced to [a,b] where

$$a = 1.3642578125$$
$$b = 1.365234375$$

$$[a, b] = [1.36426, 1.36523]$$

The root lies somewhere in the interval [a,b] the width of which is

$$b-a = 0.0009765625$$

The reported root is alleged to be

$$c = 1.36474609375$$

The accuracy we can guarantee is one half of the interval width.

$$\frac{b - a}{2} = 0.00048828125$$

Is this the desired accuracy you want ? If not, more iterations are required.

Remember. The bisection method can only be used to find a real root in an interval [a,b] in which f[x] changes sign.

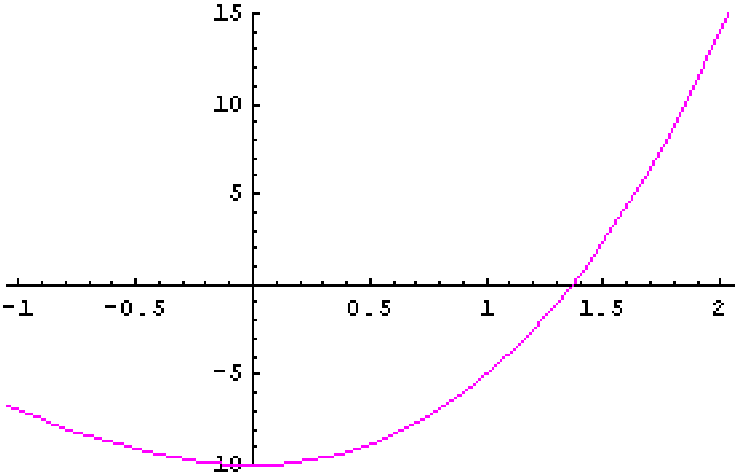
Example 2. Convergence Find the solution to the cubic equation $x^3 + 4x^2 - 10 = 0$. Use the starting interval $[a, b] = [-1, 2]$.

Solution 2.

k	a_k	c_k	b_k	$f[c_k]$
0	-1.	0.5	2.	-8.875
1	0.5	1.25	2.	-1.796875
2	1.25	1.625	2.	4.853515625
3	1.25	1.4375	1.625	1.236083984375
4	1.25	1.34375	1.4375	-0.350982666015625
5	1.34375	1.390625	1.4375	0.4245948791503906
6	1.34375	1.3671875	1.390625	0.03235578536987305
7	1.34375	1.35546875	1.3671875	-0.1604211926460266
8	1.35546875	1.361328125	1.3671875	-0.06431024521589279
9	1.361328125	1.3642578125	1.3671875	-0.01604669075459242
10	1.3642578125	1.36572265625	1.3671875	0.00813717267010361
11	1.3642578125	1.364990234375	1.36572265625	-0.003959101522923447
12	1.364990234375	1.3653564453125	1.36572265625	0.002087949806082179
13	1.364990234375	1.36517333984375	1.3653564453125	-0.0009358472818803342
14	1.36517333984375	1.365264892578125	1.3653564453125	0.0005759834039338333
15	1.36517333984375	1.365219116210937	1.365264892578125	-0.0001799489032272561
16	1.365219116210937	1.365242004394531	1.365264892578125	0.0001980130092538168
17	1.365219116210937	1.365230560302734	1.365242004394531	$9.03099274296437 \times 10^{-6}$
18	1.365219116210937	1.365224838256836	1.365230560302734	-0.0000854592203092253
19	1.365224838256836	1.365227699279785	1.365230560302734	-0.00003821418005012234
20	1.365227699279785	1.36522912979126	1.365230560302734	-0.00001459161022010491
21	1.36522912979126	1.365229845046997	1.365230560302734	$-2.780312880368285 \times 10^{-6}$
22	1.365229845046997	1.365230202674866	1.365230560302734	$3.125338895682006 \times 10^{-6}$
23	1.365229845046997	1.365230023860931	1.365230202674866	$1.72512748974895 7\times 10^{-7}$
24	1.365229845046997	1.365229934453964	1.365230023860931	$-1.30390013053372 \times 10^{-6}$
25	1.365229934453964	1.365229979157448	1.365230023860931	$-5.65693706988668 \times 10^{-7}$
26	1.365229979157448	1.36523000150919	1.365230023860931	$-1.965904834477783 \times 10^{-7}$
27	1.36523000150919	1.36523001268506	1.365230023860931	$-1.203886768053053 \times 10^{-8}$
28	1.36523001268506	1.365230018272996	1.365230023860931	$8.02369402030934 \times 10^{-8}$

29	1.36523001268506	1.365230015479028	1.365230018272996	$3.409903603923681 \times 10^{-8}$
30	1.36523001268506	1.365230014082044	1.365230015479028	$1.103008440139774 \times 10^{-8}$

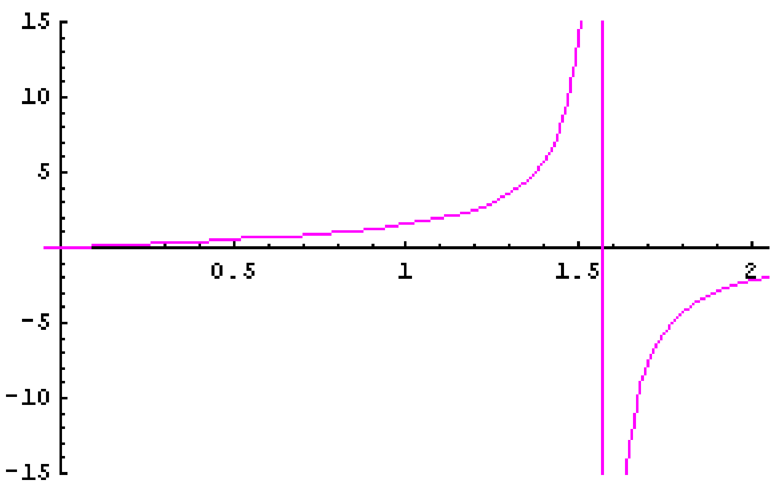
$$c = 1.365230014082044$$
$$\Delta c = \pm 1.39698 \times 10^{-9}$$
$$f[c] = 1.103008440139774 \times 10^{-8}$$



Example 3. **Not a root located** Find the solution to the equation `Tan[x] = 0`. Use the starting interval `[a, b] = [0, 2]`.
Solution 3.

k	a_k	c_k	b_k	$f[c_k]$
0	0.	1.	2.	1.557407724654902
1	1.	1.5	2.	14.10141994717172
2	1.5	1.75	2.	-5.52037992250933
3	1.5	1.625	1.75	-18.43086276236962
4	1.5	1.5625	1.625	120.5325057225426
5	1.5625	1.59375	1.625	-43.55836040673973
6	1.5625	1.578125	1.59375	-136.4479038428448
7	1.5625	1.5703125	1.578125	2066.855189746604
8	1.5703125	1.57421875	1.578125	-292.1894914044914
9	1.5703125	1.572265625	1.57421875	-680.5965439242681
10	1.5703125	1.5712890625	1.572265625	-2029.48539899437
11	1.5703125	1.57080078125	1.5712890625	-224494.3493165802
12	1.5703125	1.570556640625	1.57080078125	4172.12215991226
13	1.570556640625	1.5706787109375	1.57080078125	8502.2548619161
14	1.5706787109375	1.57073974609375	1.57080078125	17673.87074864176
15	1.57073974609375	1.570770263671875	1.57080078125	38368.38735496419

c = 1.570770263671875
Δc = ±0.0000305176
f[c] = 38368.38735496419



Note. The bisection method located a pole of `f[x] = Tan[x]`. This is where the graph has a vertical asymptote.

$$\left\{\left\{x \to \frac{\pi}{2}\right\}\right\}$$