8. Chebyshev Polynomials

Background for the Chebyshev approximation polynomial.

We now turn our attention to polynomial interpolation for f(x) over [-1,1] based on the nodes $-1 \le x_0 \le x_2 \le \ldots \le x_n \le 1$. Both the Lagrange and Newton polynomials satisfy

$$f(x) = P_n(x) + R_n(x),$$

where the remainder term $R_n(x)$ has the form

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} Q(x).$$

and $Q(x)$ is the polynomial of degree $n+1$ given by

$$Q(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$$

Using the relationship

$$\left| R_n(x) \right| \le \frac{1}{(n+1)!} \max_{-1 \le x \le 1} \{ | f^{(n+1)}(x) | \} * \max_{-1 \le x \le 1} \{ | Q(x) | \}$$

our task is to determine how to select the set of nodes $\{x_k\}_{k=0}^{n}$ that minimizes $\max_{-1 \le x \le 1} \{ | Q(x) | \}$. Research investigating the minimum error in polynomial interpolation is attributed to the Russian mathematician Pafnuty Lvovich Chebyshev (1821-1894).

Table of Chebyshev Polynomials.

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_5(x) = 16x^5 - 20x^3 + 5x$

$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

TABLE Chebyshev polynomials

Recursive Relationship.

The Chebyshev polynomials can be generated recursively in the following way. First, set

$T_0(x) = 1$

$T_1(x) = x$

and use the recurrence relation

$$T_k(x) = 2x\,T_{k-1}(x) - T_{k-2}(x).$$

Exploration 1.

Relation to trigonometric functions.

The signal property of Chebyshev polynomials is the trigonometric representation on [-1,1].

Consider the following expansion.

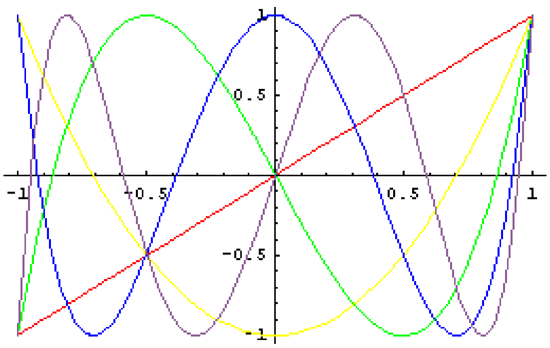
$T_2[x] = \text{Cos}[2\ \text{ArcCos}[x]]$

$T_2[x] = -1 + 2x^2$

Exploration 2.

These celebrated Chebyshev polynomials are readily available in *Mathematica* and called under the reserved name "ChebyshevT[n,x]."

```
Needs["Graphics`Colors`"];
Plot[Evaluate[Table[ChebyshevT[n, x], {n, 1, 5}]], {x, -1, 1}, PlotStyle → {Red, Yellow, Green, Blue, Violet}];
For[n = 1, n ≤ 5, n++,
 Print["\t\t\t", "T"_n, "[x] = ", ChebyshevT[n, x]]; ];
```



$T_1[x] = x$

$T_2[x] = -1 + 2x^2$

$T_3[x] = -3x + 4x^3$

$T_4[x] = 1 - 8x^2 + 8x^4$

$T_5[x] = 5x - 20x^3 + 16x^5$

Roots of the Chebyshev polynomials

The roots of $T_{n+1}(x)$ are $\text{Cos}\left[\frac{(2n+1-2k)\pi}{2n+2}\right]$ for $k = 0, 1, \ldots, n$. These will be the nodes for polynomial approximation of degree n.

Exploration 3.

The Minimax Problem

An upper bound for the absolute value of the remainder term, $|R_n(x)|$, is the product of $\frac{1}{(n+1)!}\max_{-1\le x\le 1}\{|f^{(n+1)}(x)|\}$ and $\max_{-1\le x\le 1}\{|Q(x)|\}$. To minimize the factor $\max_{-1\le x\le 1}\{|Q(x)|\}$, Chebyshev discovered that $x_0, x_2, \ldots, x_n$ must be chosen so that $Q(x) = \frac{1}{2^n}T_{n+1}(x)$, which is stated in the following result.

Theorem (Minimax Property). Assume that n is fixed. Among all possible choices for Q(x) and thus among all possible choices for the distinct nodes $\{x_k\}_{k=0}^{n}$ in [-1,1],

the polynomial $T(x) = \frac{1}{2^n}T_{n+1}(x)$ is the unique choice which has the property

$$\max_{-1\le x\le 1}\{|T(x)|\} \le \max_{-1\le x\le 1}\{|Q(x)|\}$$

Moreover,

$$\max_{-1 \le x \le 1} \{ | T(x) | \} = \frac{1}{2^n}.$$

Exploration for the theorem. Construct Q(x) of degree n using the n+1 Chebyshev nodes and compare it to $T_{n+1}(x)$.
Exploration 4.

Rule of Thumb.

The "best a priori choice" of interpolation nodes for the interval [-1,1] are the n+1 nodes that are zeros of the Chebyshev polynomial $T_{n+1}(x)$.

Here is a visual analysis of equally spaced nodes verses Chebyshev nodes on [-1,1], and their affect on the magnitude of Q(x) in the remainder term $R_n(x)$.
Exploration 5.

Transforming the Interval for Interpolation

Sometimes it is necessary to take a problem stated on an interval [a,b] and reformulate the problem on the interval [c,d] where the solution is known. If the approximation $P_n(x)$ to $f(x)$ is to be obtained on the interval [a,b], then we change the variable so that the problem is reformulated on [-1,1]:

$$x = \frac{b-a}{2} t + \frac{a+b}{2} \quad \text{or} \quad t = 2\frac{x-a}{b-a} - 1,$$

where $a \le x \le b$ and $-1 \le t \le 1$. The required Chebyshev nodes of $T_{n+1}(x)$ on [-1,1] are

$$t_k = \cos\left(\frac{2n+1-2k}{2n+2}\pi\right) \text{ for } k = 0, 1, \ldots, n,$$

and the interpolating nodes $\{x_k\}_{k=0}^{n}$ on [a,b] are obtained using the change of variable $x_k = \frac{b-a}{2} t_k + \frac{a+b}{2}$ for $k = 0, 1, \ldots, n$.

Theorem (Lagrange-Chebyshev Approximation). Assume that $P_n(x)$ is the Lagrange polynomial that is based on the Chebyshev interpolating nodes $\{x_k\}_{k=0}^{n}$ on [a,b] mentioned above.

If $f(x) \in C^{n+1}[a, b]$ then

$$\left| f(x) - P_n(x) \right| \le \frac{2(b-a)^{n+1}}{4^{n+1}(n+1)!} \max_{a \le x \le b} \{ | f^{(n+1)}(x) | \}$$

holds for all $x \in [a, b]$.

Algorithm (Lagrange-Chebyshev Approximation). The Chebyshev approximation polynomial $P_n(x)$ of degree $\le n$ for f(x) over [-1,1] can be written as a sum of $\{T_j(x)\}$:

$$f(x) \approx P_n(x) = \sum_{j=0}^{n} c_j T_j(x).$$

The coefficients $\{c_j\}$ are computed with the formulas

$$c_0 = \frac{1}{n+1} \sum_{k=0}^{n} f(x_k) T_0(x_k) = \frac{1}{n+1} \sum_{k=0}^{n} f(x_k),$$

and

$$c_j = \frac{2}{n+1} \sum_{k=0}^{n} f(x_k) T_j(x_k) = \frac{2}{n+1} \sum_{k=0}^{n} f(x_k) \cos\left(j \frac{2k+1}{2n+2}\pi\right),$$

for $j = 1, 2, \ldots, n$ where $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$ for $k = 0, 1, 2, \ldots, n$.

Example 1. Find the Chebyshev polynomial approximation for $f[x] = \frac{1}{\sqrt{1-x}}$, on the interval $[-1.5, 0.95]$.
Solution 1.

Example 2. Find the Chebyshev polynomial approximation for $f[x] = \frac{1}{1 + 10x^2}$, on the interval $[-1, 1]$.
Solution 2.

Example 3. Find the Chebyshev polynomial approximation for $f[x] = \text{Log}[x]$, on the interval $[0.02, 2]$.
Solution 3.

Example 4. Error Analysis. Investigate the error for the Chebyshev polynomial approximations of degree $n = 4$ and 5 or the function $f[x] = \text{Cos}[x]$ over the interval $[0, 1]$ using Chebyshev's abscissas.
Solution 4.

Recursive Relationship.

The Chebyshev polynomials can be generated recursively in the following way. First, set

$T_0(x) = 1$
$T_1(x) = x$

and use the recurrence relation

$T_k(x) = 2x\,T_{k-1}(x) - T_{k-2}(x)$.

Exploration 1.

This is a "classic example" of recursion programming. Check it out and see how recursion works.

$T_2[x] = -1 + 2x^2$

$T_3[x] = -x + 2x(-1 + 2x^2)$
$T_3[x] = -3x + 4x^3$

$T_4[x] = 1 - 2x^2 + 2x(-x + 2x(-1 + 2x^2))$
$T_4[x] = 1 - 8x^2 + 8x^4$

$T_5[x] = x - 2x(-1 + 2x^2) + 2x(1 - 2x^2 + 2x(-x + 2x(-1 + 2x^2)))$
$T_5[x] = 5x - 20x^3 + 16x^5$

$T_6[x] = -1 + 2x^2 - 2x(-x + 2x(-1 + 2x^2)) + 2x(x - 2x(-1 + 2x^2) + 2x(1 - 2x^2 + 2x(-x + 2x(-1 + 2x^2))))$
$T_6[x] = -1 + 18x^2 - 48x^4 + 32x^6$

$T_7[x] =$
$-x + 2x(-1 + 2x^2) - 2x(1 - 2x^2 + 2x(-x + 2x(-1 + 2x^2))) + 2x(-1 + 2x^2 - 2x(-x + 2x(-1 + 2x^2)) + 2x(x - 2x(-1 + 2x^2) + 2x(1 - 2x^2 + 2x(-x + 2x(-1 + 2x^2)))))$
$T_7[x] = -7x + 56x^3 - 112x^5 + 64x^7$

Relation to trigonometric functions.

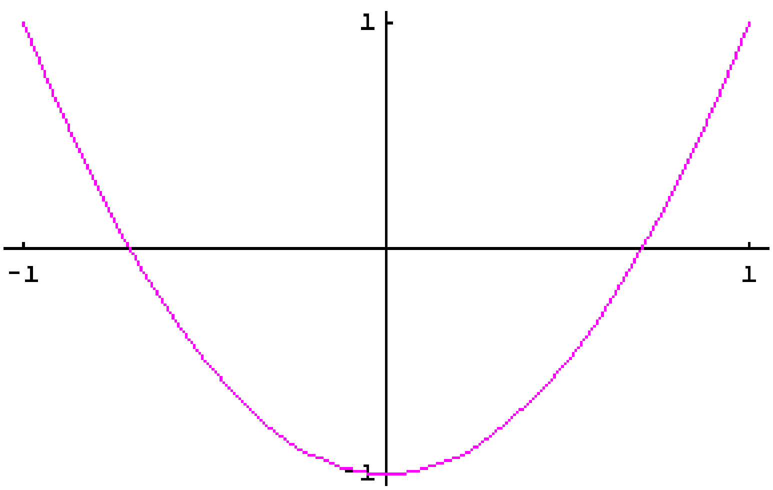
The signal property of Chebyshev polynomials is the trigonometric representation on [-1,1].

$$T_2[x] = \text{Cos}[2\ \text{ArcCos}[x]]$$

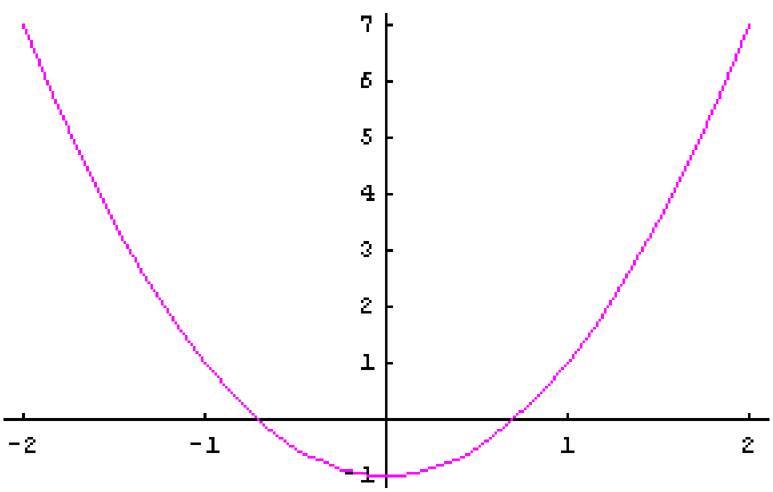
$$T_2[x] = -1 + 2x^2$$

Exploration 2.

We are interested in the polynomial form of Cos[n ArcCos[x]], however we will restrict our analysis to [-1,1].

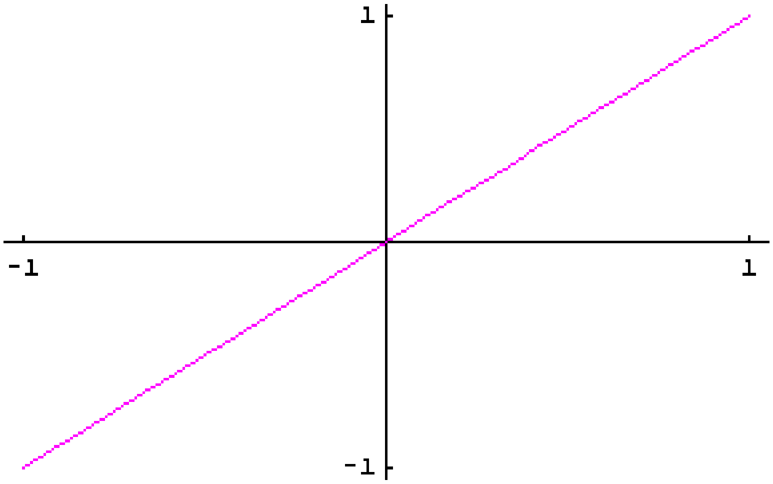


$$y = T_2[x] = \text{Cos}[2\,\text{ArcCos}[x]] = -1 + 2x^2$$



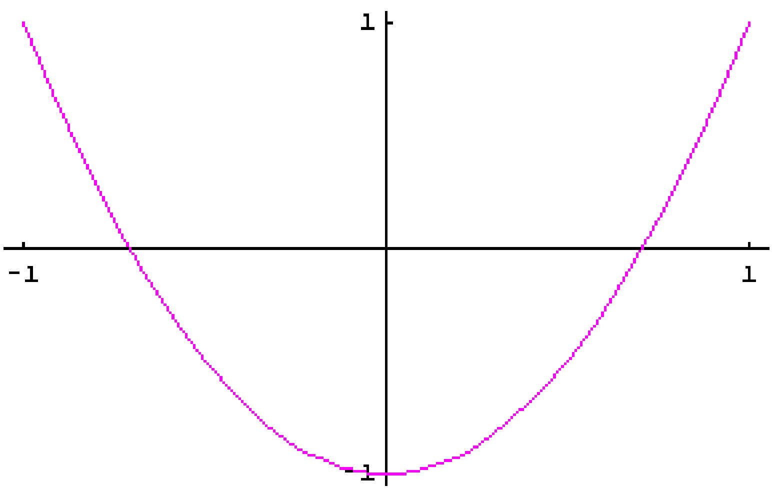
$$y = T_2[x] = \text{Cos}[2\,\text{ArcCos}[x]] = -1 + 2x^2$$

Here is a list of several expansions.



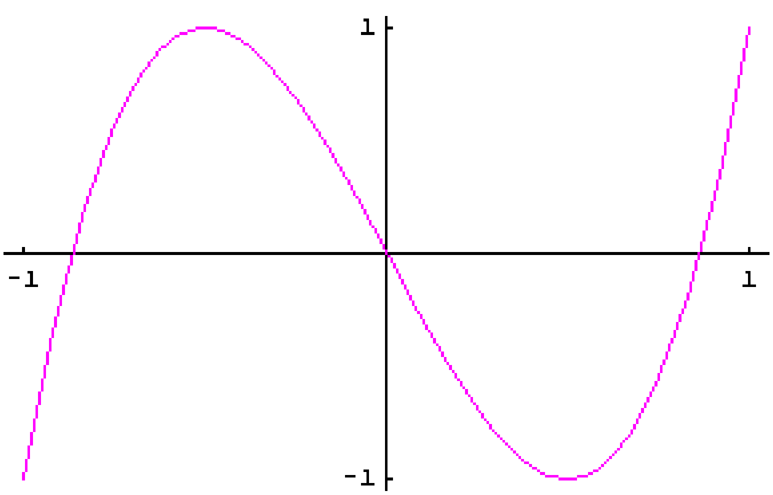
$T_1[x] = x$

$T_1[x] = x$



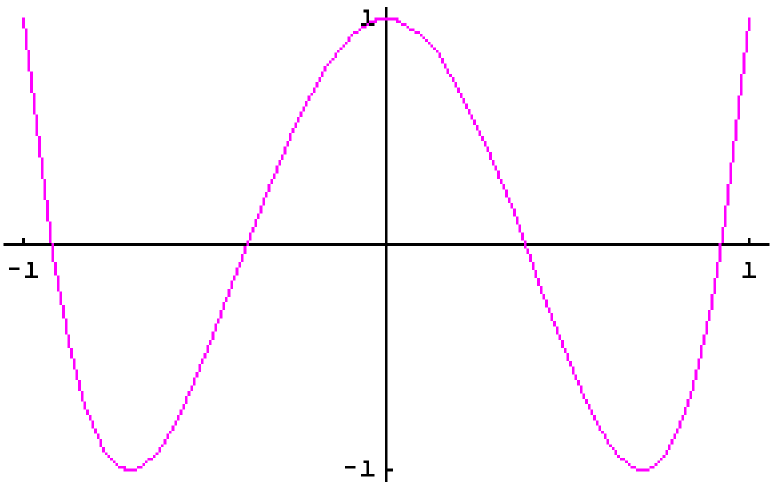
$T_2[x] = \mathrm{Cos}[2\,\mathrm{ArcCos}[x]]$

$T_2[x] = -1 + 2x^2$

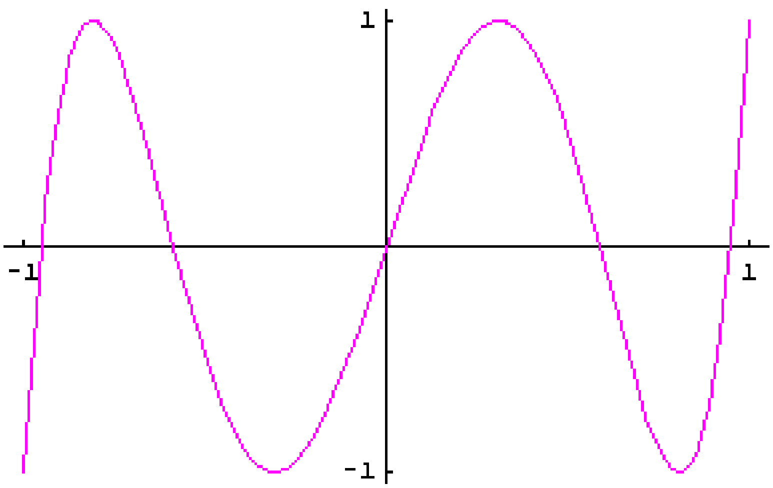


$T_3[x] = \mathrm{Cos}[3\,\mathrm{ArcCos}[x]]$

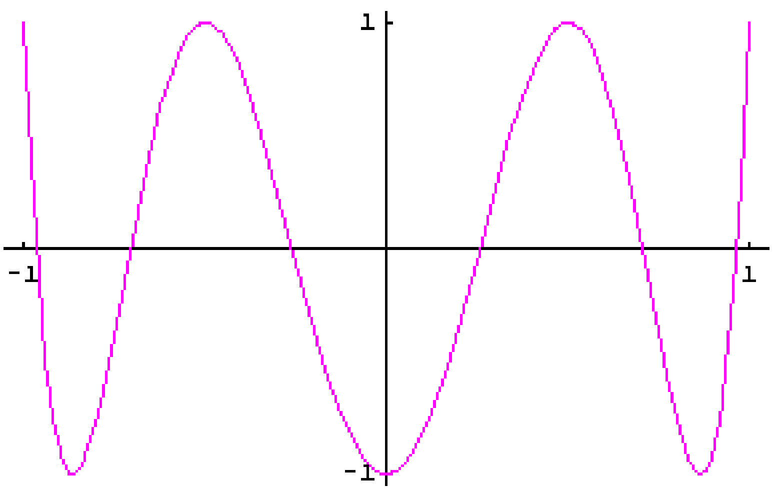
$$T_3[x] = -3x + 4x^3$$



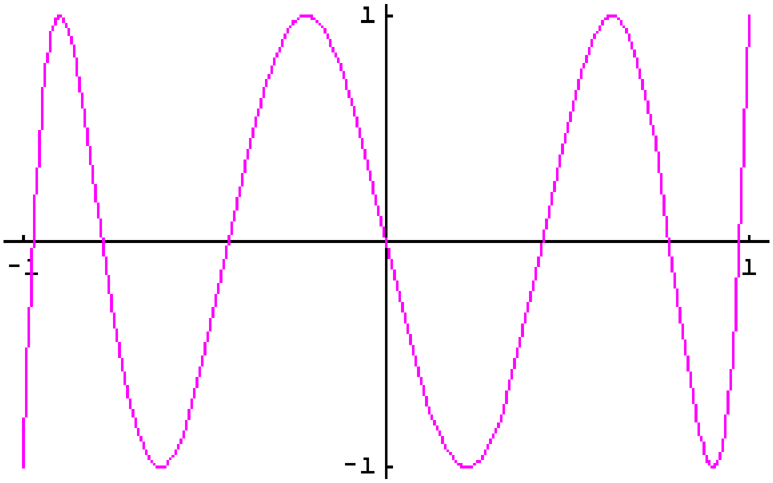
$$T_4[x] = \mathrm{Cos}[4\,\mathrm{ArcCos}[x]]$$
$$T_4[x] = 1 - 8x^2 + 8x^4$$



$$T_5[x] = \mathrm{Cos}[5\,\mathrm{ArcCos}[x]]$$
$$T_5[x] = 5x - 20x^3 + 16x^5$$



$T_6[x] = \text{Cos}[6\,\text{ArcCos}[x]]$
$T_6[x] = -1 + 18x^2 - 48x^4 + 32x^6$



$T_7[x] = \text{Cos}[7\,\text{ArcCos}[x]]$
$T_7[x] = -7x + 56x^3 - 112x^5 + 64x^7$

Roots of the Chebyshev polynomials

The roots of $T_{n+1}(x)$ are $\text{Cos}\left[\frac{(2n+1-2k)\pi}{2n+2}\right]$ for $k = 0, 1, \ldots, n$. These will be the nodes for polynomial approximation of degree n.

Exploration 3.

$T_2[x] = -1+2x^2 = 0$

$\left\{\left\{x \to -\frac{1}{\sqrt{2}}\right\}, \left\{x \to \frac{1}{\sqrt{2}}\right\}\right\}$

$\{\{x \to -0.707107\}, \{x \to 0.707107\}\}$

$\text{Cos}[\frac{1}{4}(3-2k)\pi]$ for k=0,...,1

$\left\{-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$

$\{-0.707107, 0.707107\}$

$T_3[x] = -3x+4x^3 = 0$

$\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{3}}{2}\right\}, \left\{x \to \frac{\sqrt{3}}{2}\right\}\right\}$

$\{\{x \to 0.\}, \{x \to -0.866025\}, \{x \to 0.866025\}\}$

$\text{Cos}[\frac{1}{6}(5-2k)\pi]$ for k=0,...,2

$\left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$

$\{-0.866025, 0., 0.866025\}$

$T_4[x] = 1-8x^2+8x^4 = 0$

$\left\{\left\{x \to -\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}\right\}, \left\{x \to \sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}\right\}, \left\{x \to -\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}\right\}, \left\{x \to \sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}\right\}\right\}$

$\{\{x \to -0.382683\}, \{x \to 0.382683\}, \{x \to -0.92388\}, \{x \to 0.92388\}\}$

$\text{Cos}[\frac{1}{8}(7-2k)\pi]$ for k=0,...,3

$\left\{\text{Cos}\left[\frac{7\pi}{8}\right], \text{Cos}\left[\frac{5\pi}{8}\right], \text{Cos}\left[\frac{3\pi}{8}\right], \text{Cos}\left[\frac{\pi}{8}\right]\right\}$

$\{-0.92388, -0.382683, 0.382683, 0.92388\}$

$T_5[x] = 5x - 20x^3 + 16x^5 = 0$

$$\left\{\{x \to 0\}, \left\{x \to -\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}\right\}, \left\{x \to \sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}\right\}, \left\{x \to -\sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}\right\}, \left\{x \to \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}}\right\}\right\}$$

$\{\{x \to 0.\}, \{x \to -0.587785\}, \{x \to 0.587785\}, \{x \to -0.951057\}, \{x \to 0.951057\}\}$

$\mathrm{Cos}[\frac{1}{10}(9 - 2k)\pi]$ for k=0,...,4

$$\left\{-\frac{1}{2}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}, -\frac{1}{2}\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}, 0, \frac{1}{2}\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)}, \frac{1}{2}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}\right\}$$

$\{-0.951057, -0.587785, 0., 0.587785, 0.951057\}$

$T_6[x] = -1 + 18x^2 - 48x^4 + 32x^6 = 0$

$$\left\{\left\{x \to -\frac{1}{\sqrt{2}}\right\}, \left\{x \to \frac{1}{\sqrt{2}}\right\}, \left\{x \to -\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}\right\}, \left\{x \to \sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}\right\}, \left\{x \to -\sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}\right\}, \left\{x \to \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}\right\}\right\}$$

$\{\{x \to -0.707107\}, \{x \to 0.707107\}, \{x \to -0.258819\}, \{x \to 0.258819\}, \{x \to -0.965926\}, \{x \to 0.965926\}\}$

$\mathrm{Cos}[\frac{1}{12}(11 - 2k)\pi]$ for k=0,...,5

$$\left\{-\frac{1+\sqrt{3}}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{-1+\sqrt{3}}{2\sqrt{2}}, \frac{-1+\sqrt{3}}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1+\sqrt{3}}{2\sqrt{2}}\right\}$$

$\{-0.965926, -0.707107, -0.258819, 0.258819, 0.707107, 0.965926\}$

$T_7[x] = -7x + 56x^3 - 112x^5 + 64x^7 = 0$

$$\left\{\{x \to 0\}, \left\{x \to -\sqrt{\frac{7}{12} + \frac{7^{2/3}}{6\, 2^{2/3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} + \frac{1}{12} \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3}}\right\}, \left\{x \to \sqrt{\frac{7}{12} + \frac{7^{2/3}}{6\, 2^{2/3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} + \frac{1}{12} \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3}}\right\},\right.$$

$$\left\{x \to -\sqrt{\frac{7}{12} - \frac{7^{2/3}}{12\, 2^{2/3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} + \frac{i\, 7^{2/3}}{4\, 2^{2/3} \sqrt{3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} - \frac{1}{24} \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3} - \frac{i \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3}}{8 \sqrt{3}}}\right\},$$

$$\left\{x \to \sqrt{\frac{7}{12} - \frac{7^{2/3}}{12\, 2^{2/3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} + \frac{i\, 7^{2/3}}{4\, 2^{2/3} \sqrt{3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} - \frac{1}{24} \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3} - \frac{i \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3}}{8 \sqrt{3}}}\right\},$$

$$\left\{x \to -\sqrt{\frac{7}{12} - \frac{7^{2/3}}{12\, 2^{2/3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} - \frac{i\, 7^{2/3}}{4\, 2^{2/3} \sqrt{3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} - \frac{1}{24} \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3} + \frac{i \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3}}{8 \sqrt{3}}}\right\},$$

$$\left.\left\{x \to \sqrt{\frac{7}{12} - \frac{7^{2/3}}{12\, 2^{2/3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} - \frac{i\, 7^{2/3}}{4\, 2^{2/3} \sqrt{3} \left(-1 + 3 i \sqrt{3}\right)^{1/3}} - \frac{1}{24} \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3} + \frac{i \left(\frac{7}{2} \left(-1 + 3 i \sqrt{3}\right)\right)^{1/3}}{8 \sqrt{3}}}\right\}\right\}$$

$\{\{x \to 0.\}, \{x \to -0.974928 + 2.84694 \times 10^{-17}\, i\}, \{x \to 0.974928 - 2.84694 \times 10^{-17}\, i\}, \{x \to -0.781831 - 8.87518 \times 10^{-17}\, i\},$
$\{x \to 0.781831 + 8.87518 \times 10^{-17}\, i\}, \{x \to -0.433884 + 9.59551 \times 10^{-17}\, i\}, \{x \to 0.433884 - 9.59551 \times 10^{-17}\, i\}\}$

$\mathrm{Cos}[\frac{1}{14}(13 - 2k)\pi]$ for k=0,...,6

$\left\{\mathrm{Cos}\left[\frac{13\pi}{14}\right], \mathrm{Cos}\left[\frac{11\pi}{14}\right], \mathrm{Cos}\left[\frac{9\pi}{14}\right], 0, \mathrm{Cos}\left[\frac{5\pi}{14}\right], \mathrm{Cos}\left[\frac{3\pi}{14}\right], \mathrm{Cos}\left[\frac{\pi}{14}\right]\right\}$

{-0.974928, -0.781831, -0.433884, 0., 0.433884, 0.781831, 0.974928}

Theorem (Minimax Property). Assume that n is fixed. Among all possible choices for Q(x) and thus among all possible choices for the distinct nodes $\{x_k\}_{k=0}^{n}$ in [-1,1], the polynomial $T(x) = \frac{1}{2^n} T_{n+1}(x)$ is the unique choice which has the property

$$\max_{-1 \le x \le 1} \{ |T(x)| \} \le \max_{-1 \le x \le 1} \{ |Q(x)| \}$$

Moreover,

$$\max_{-1 \le x \le 1} \{ |T(x)| \} = \frac{1}{2^n}.$$

Exploration for the theorem. Construct Q(x) of degree n using the n+1 Chebyshev nodes and compare it to $T_{n+1}(x)$.

Exploration 4.

Construct Q(x) of degree n using the n+1 Chebyshev nodes and compare it to $T_{n+1}(x)$.

Case (i). Using 2 nodes.

$$\mathbf{Q2} = \left(x - \mathrm{Cos}\left[\frac{\pi}{4}\right]\right)\left(x - \mathrm{Cos}\left[\frac{3\pi}{4}\right]\right)$$

$$\left(-\frac{1}{\sqrt{2}} + x\right)\left(\frac{1}{\sqrt{2}} + x\right)$$

$$-\frac{1}{2} + x^2$$

$$\frac{1}{2}\,\mathbf{ChebyshevT[2, x]}$$

$$\frac{1}{2}\left(-1 + 2x^2\right)$$

$$\mathbf{Expand}\left[\frac{1}{2}\,\mathbf{ChebyshevT[2, x]}\right]$$

$$-\frac{1}{2} + x^2$$

Case (ii). Using 3 nodes.

$$\mathbf{Q3} = \left(x - \mathrm{Cos}\left[\frac{\pi}{6}\right]\right)\left(x - \mathrm{Cos}\left[\frac{\pi}{2}\right]\right)\left(x - \mathrm{Cos}\left[\frac{5\pi}{6}\right]\right)$$

$$x\left(-\frac{\sqrt{3}}{2} + x\right)\left(\frac{\sqrt{3}}{2} + x\right)$$

Expand[Q3]

$$-\frac{3x}{4} + x^3$$

$$\frac{1}{2^2}\ \text{ChebyshevT}[3,\ x]$$

$$\frac{1}{4}\ (-3\,x+4\,x^3)$$

$$\text{Expand}\left[\frac{1}{2^2}\ \text{ChebyshevT}[3,\ x]\right]$$

$$-\frac{3\,x}{4}+x^3$$

Case (iii). Using 4 nodes.

$$\text{Q4}=\left(x-\text{Cos}\left[\frac{\pi}{8}\right]\right)\left(x-\text{Cos}\left[\frac{3\,\pi}{8}\right]\right)\left(x-\text{Cos}\left[\frac{5\,\pi}{8}\right]\right)\left(x-\text{Cos}\left[\frac{7\,\pi}{8}\right]\right)$$

$$\left(x-\text{Cos}\left[\frac{\pi}{8}\right]\right)\left(x-\text{Cos}\left[\frac{3\,\pi}{8}\right]\right)\left(x-\text{Cos}\left[\frac{5\,\pi}{8}\right]\right)\left(x-\text{Cos}\left[\frac{7\,\pi}{8}\right]\right)$$

$$\text{Expand}[\text{Q4}]$$

$$x^4-x^3\,\text{Cos}\left[\frac{\pi}{8}\right]-x^3\,\text{Cos}\left[\frac{3\,\pi}{8}\right]+x^2\,\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]-x^3\,\text{Cos}\left[\frac{5\,\pi}{8}\right]+x^2\,\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]+x^2\,\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]-$$
$$x\,\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]-x^3\,\text{Cos}\left[\frac{7\,\pi}{8}\right]+x^2\,\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]+x^2\,\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]-x\,\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]+$$
$$x^2\,\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]-x\,\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]-x\,\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]+\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]$$

The symbolic manipulation required to simplify the above polynomial is overwhelming. However we can simplify the list of coefficients.

$$\text{CoefficientList}[\text{Expand}[\text{Q4}],\ x]$$

$$\Big\{\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right],\ -\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]-\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]-\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]-\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right],$$
$$\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{3\,\pi}{8}\right]+\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]+\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{5\,\pi}{8}\right]+\text{Cos}\left[\frac{\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]+\text{Cos}\left[\frac{3\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right]+\text{Cos}\left[\frac{5\,\pi}{8}\right]\text{Cos}\left[\frac{7\,\pi}{8}\right],$$
$$-\text{Cos}\left[\frac{\pi}{8}\right]-\text{Cos}\left[\frac{3\,\pi}{8}\right]-\text{Cos}\left[\frac{5\,\pi}{8}\right]-\text{Cos}\left[\frac{7\,\pi}{8}\right],\ 1\Big\}$$

$$\text{c}=\text{MapAll}[\text{FullSimplify},\ \text{CoefficientList}[\text{Expand}[\text{Q4}],\ x]]$$

$$\left\{\frac{1}{8},\ 0,\ -1,\ 0,\ 1\right\}$$

$$\text{c}.\{1,\ x,\ x^2,\ x^3,\ x^4\}$$

$$\frac{1}{8}-x^2+x^4$$

$$\frac{1}{2^3}\ \text{ChebyshevT}[4,\ x]$$

$$\frac{1}{8}\ (1-8\,x^2+8\,x^4)$$

$$\text{Expand}\left[\frac{1}{2^3}\ \text{ChebyshevT}[4,\ x]\right]$$

$$\frac{1}{8} - x^2 + x^4$$

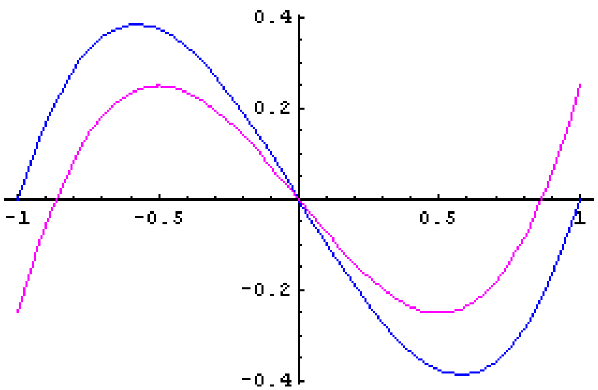
Rule of Thumb.

The "best a priori choice" of interpolation nodes for the interval [-1,1] are the n+1 nodes that are zeros of the Chebyshev polynomial $T_{n+1}(x)$.

Here is a visual analysis of equally spaced nodes verses Chebyshev nodes on [-1,1], and their affect on the magnitude of Q(x) in the remainder term $R_n(x)$.

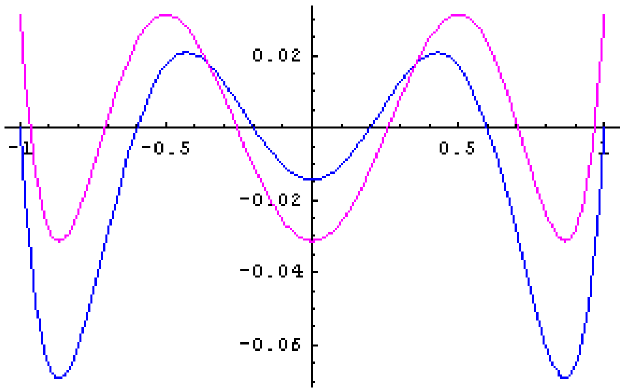
Exploration 5.

j = 3



q[3,x] = (-1 + x) x (1 + x)
Q[3,x] = (-0.866025 + x) x (0.866025 + x)

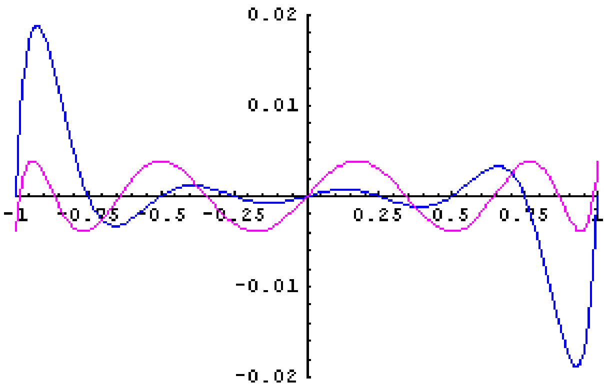
j = 6



$$q[6,x] = (-1+x)\left(-\frac{3}{5}+x\right)\left(-\frac{1}{5}+x\right)\left(\frac{1}{5}+x\right)\left(\frac{3}{5}+x\right)(1+x)$$

$$Q[6,x] = (-0.965926+x)(-0.707107+x)(-0.258819+x)(0.258819+x)(0.707107+x)(0.965926+x)$$

j = 9

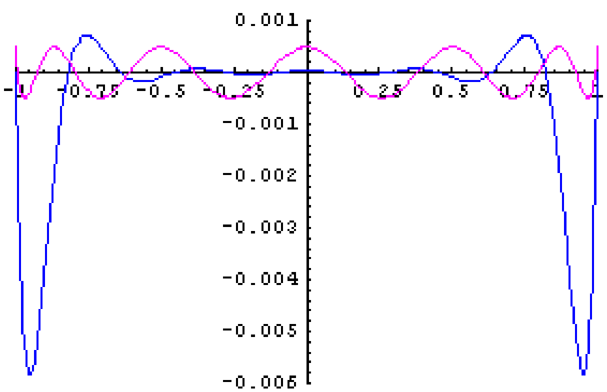


$$q[9,x] = (-1+x)\left(-\frac{3}{4}+x\right)\left(-\frac{1}{2}+x\right)\left(-\frac{1}{4}+x\right)x\left(\frac{1}{4}+x\right)\left(\frac{1}{2}+x\right)\left(\frac{3}{4}+x\right)(1+x)$$

$$Q[9,x] = (-0.984808+x)(-0.866025+x)(-0.642788+x)(-0.34202+x)\,x\,(0.34202+x)(0.642788+x)(0.866025+x)(0.984808+x)$$

Observation. The magnitude of Q(x) is less when the Chebyshev nodes are used and larger when equally spaced notes are used. This becomes more pronounced when the degree is larger.

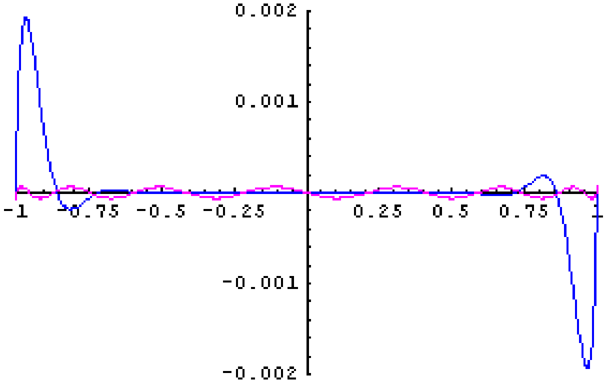
j = 12



$$q[12,x] = (-1+x)\left(-\frac{9}{11}+x\right)\left(-\frac{7}{11}+x\right)\left(-\frac{5}{11}+x\right)\left(-\frac{3}{11}+x\right)\left(-\frac{1}{11}+x\right)\left(\frac{1}{11}+x\right)\left(\frac{3}{11}+x\right)\left(\frac{5}{11}+x\right)\left(\frac{7}{11}+x\right)\left(\frac{9}{11}+x\right)(1+x)$$

Q[12,x] = (-0.991445 + x) (-0.92388 + x) (-0.793353 + x) (-0.608761 + x)
(-0.382683 + x) (-0.130526 + x) (0.130526 + x) (0.382683 + x) (0.608761 + x) (0.793353 + x) (0.92388 + x) (0.991445 + x)

j = 15



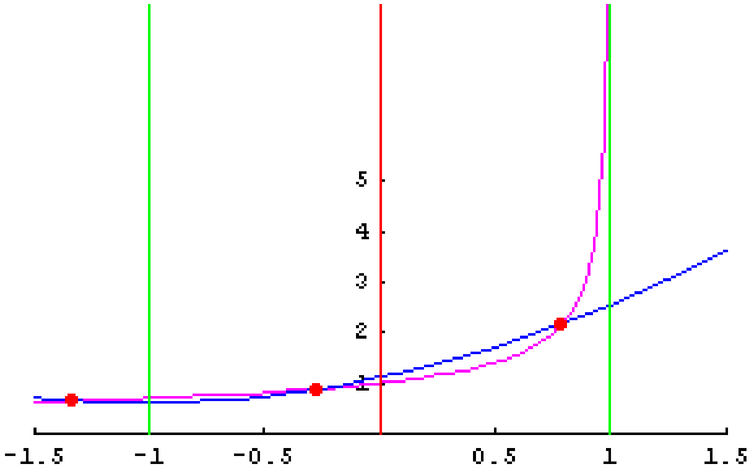
$$q[15,x] = (-1+x)\left(-\frac{6}{7}+x\right)\left(-\frac{5}{7}+x\right)\left(-\frac{4}{7}+x\right)\left(-\frac{3}{7}+x\right)\left(-\frac{2}{7}+x\right)\left(-\frac{1}{7}+x\right)x\left(\frac{1}{7}+x\right)\left(\frac{2}{7}+x\right)\left(\frac{3}{7}+x\right)\left(\frac{4}{7}+x\right)\left(\frac{5}{7}+x\right)\left(\frac{6}{7}+x\right)(1+x)$$

Q[15,x] = (-0.994522 + x) (-0.951057 + x) (-0.866025 + x) (-0.743145 + x) (-0.587785 + x) (-0.406737 + x)
(-0.207912 + x) x (0.207912 + x) (0.406737 + x) (0.587785 + x) (0.743145 + x) (0.866025 + x) (0.951057 + x) (0.994522 + x)

Are you convinced that using the Chebyshev nodes on [-1,1], will decrease the magnitude of the term Q(x) in the remainder term $R_n(x)$?

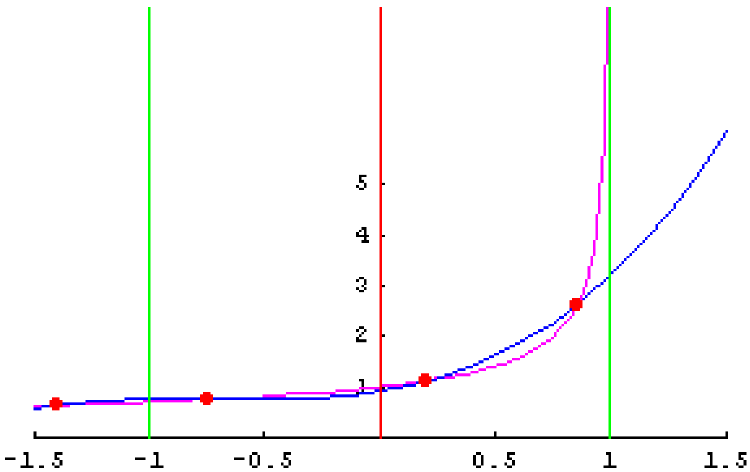
Example 1. Find the Chebyshev polynomial approximation for $f[x] = \frac{1}{\sqrt{1-x}}$, on the interval $[-1.5, 0.95]$.

Solution 1.



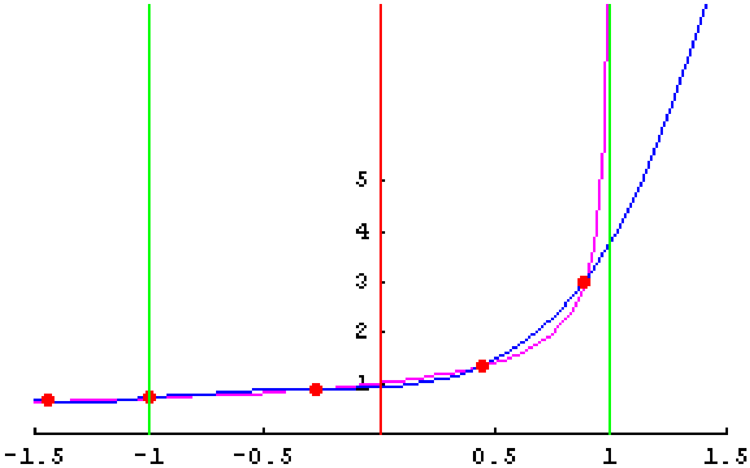
$$f[x] = \frac{1}{\sqrt{1-x}}$$

$$P[x] = 1.11599 + 0.965292\,x + 0.463876\,x^2$$



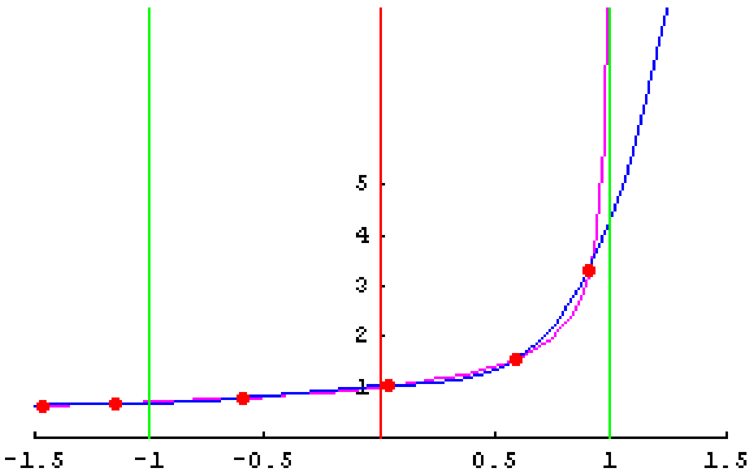
$$f[x] = \frac{1}{\sqrt{1-x}}$$

$$P[x] = 0.925125 + 0.750454\,x + 1.05767\,x^2 + 0.473402\,x^3$$



$$f[x] = \frac{1}{\sqrt{1-x}}$$

$$P[x] = 0.926092 + 0.284974\,x + 0.805809\,x^2 + 1.2476\,x^3 + 0.506725\,x^4$$

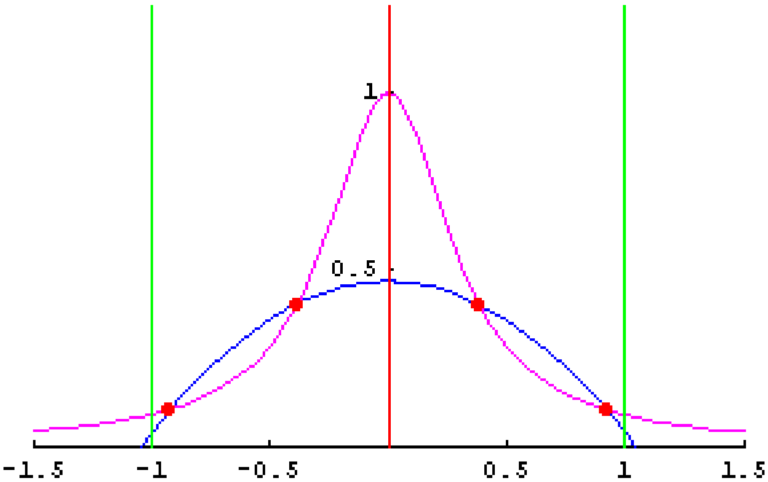


$$f[x] = \frac{1}{\sqrt{1-x}}$$

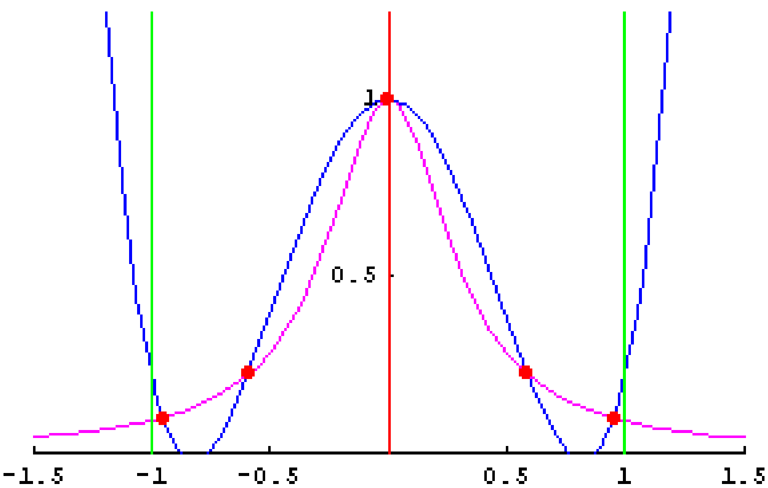
$$P[x] = 1.01176 + 0.236958\,x - 0.0512178\,x^2 + 0.993259\,x^3 + 1.51115\,x^4 + 0.556978\,x^5$$

Example 2. Find the Chebyshev polynomial approximation for $f[x] = \frac{1}{1 + 10x^2}$, on the interval $[-1, 1]$.

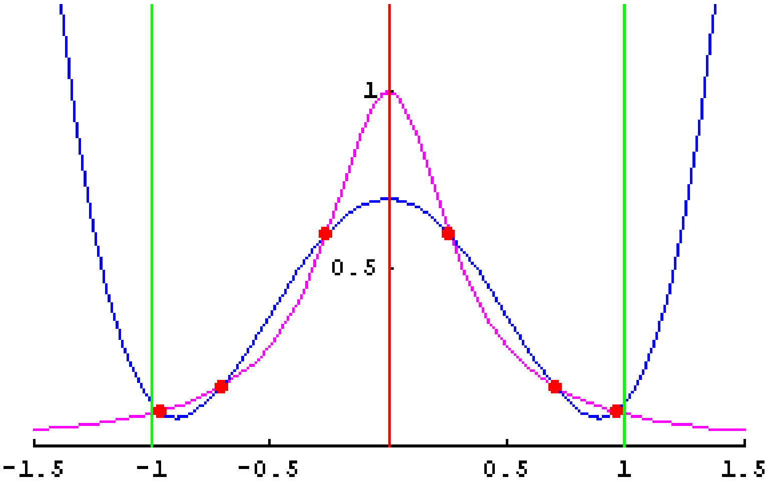
Solution 2.



$$f[x] = \frac{1}{1 + 10x^2}$$
$$P[x] = 0.468085 - 0.425532x^2$$

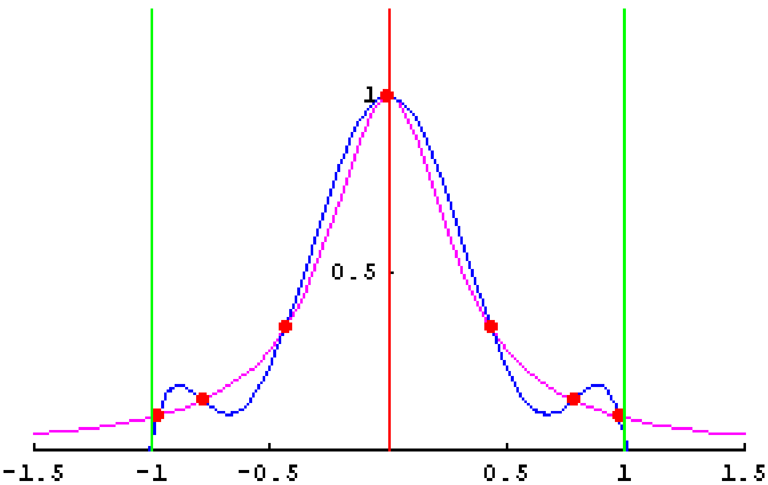


$$f[x] = \frac{1}{1 + 10x^2}$$
$$P[x] = 1. - 3.01676x^2 + 2.23464x^4$$



$$f[x] = \frac{1}{1 + 10x^2}$$

$$P[x] = 0.698068 - 1.54589\,x^2 + 0.966184\,x^4$$

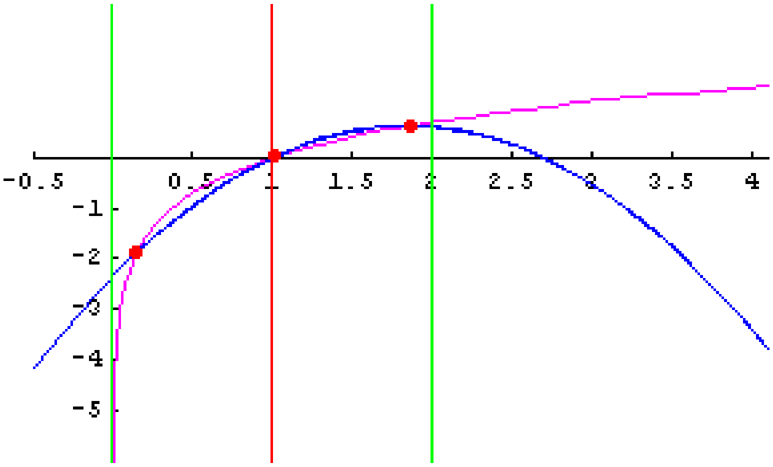


$$f[x] = \frac{1}{1 + 10x^2}$$

$$P[x] = 1. - 4.92165\,x^2 + 8.58967\,x^4 - 4.64306\,x^6$$

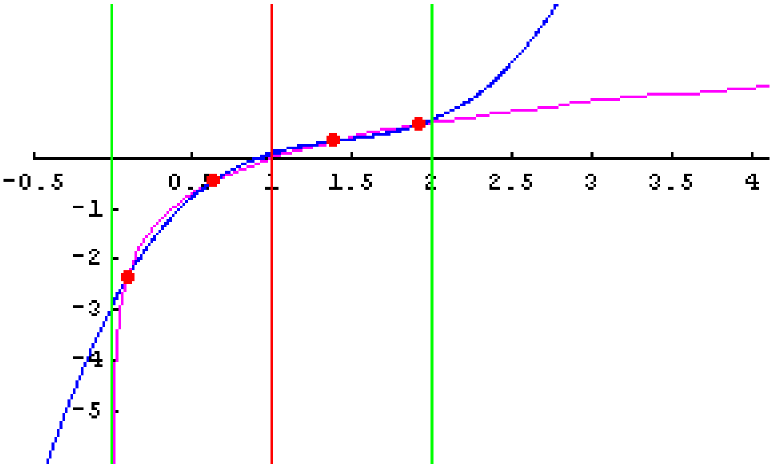
Example 3. Find the Chebyshev polynomial approximation for `f[x] = Log[x]`, on the interval `[0.02, 2]`.

Solution 3.



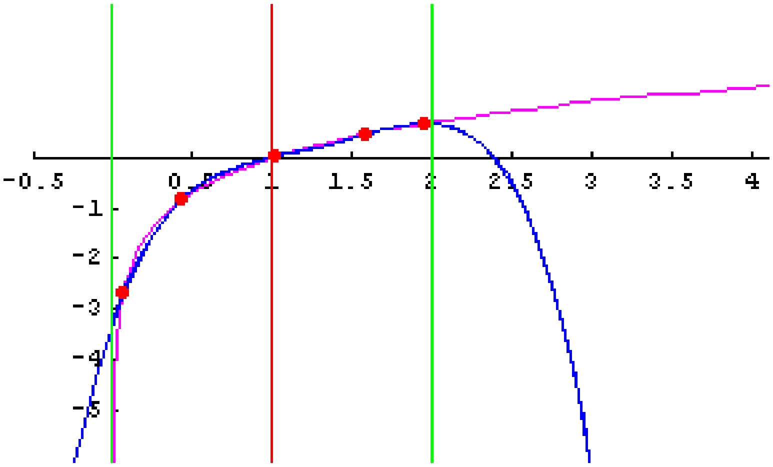
$f[x] = Log[x]$

$P[x] = -2.34982 + 3.2124\,x - 0.867312\,x^2$

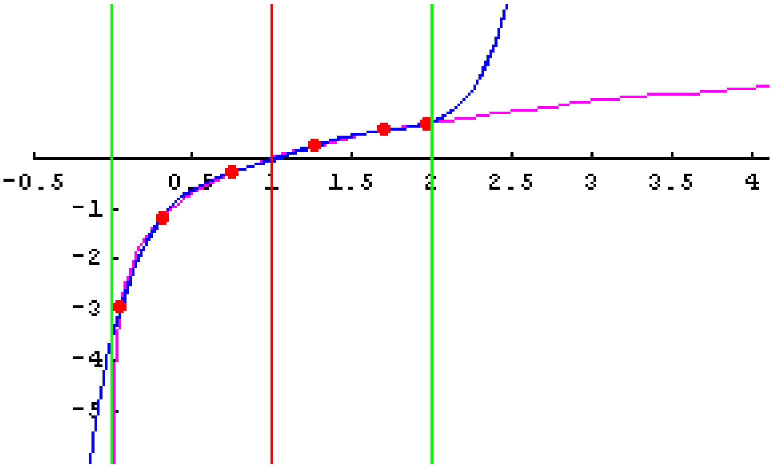


$f[x] = Log[x]$

$P[x] = -2.87473 + 5.85218\,x - 3.75893\,x^2 + 0.868263\,x^3$



$$f[x] = \text{Log}[x]$$
$$P[x] = -3.25381 + 9.01987\,x - 10.0077\,x^2 + 5.24985\,x^3 - 1.00553\,x^4$$

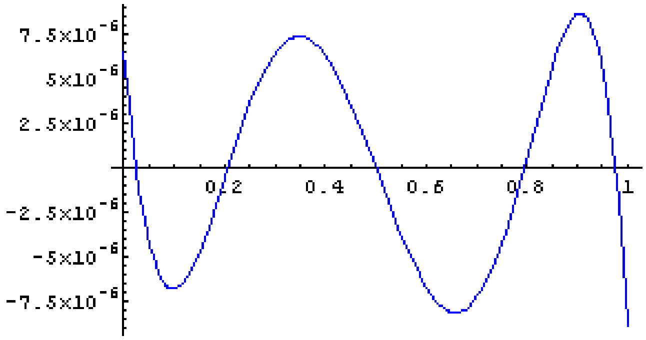


$$f[x] = \text{Log}[x]$$
$$P[x] = -3.54325 + 12.6046\,x - 20.9728\,x^2 + 18.4137\,x^3 - 7.79863\,x^4 + 1.26265\,x^5$$

Example 4. Error Analysis. Investigate the error for the Chebyshev polynomial approximations of degree n = 4 and 5 or the function `f[x] = Cos[x]` over the interval `[0, 1]` using Chebyshev's abscissas.

Solution 4.

4 (a). Investigate the error for the Chebyshev interpolation polynomial $P_4[x]$, of degree n = 4.



f[x] = Cos[x]

$P_4[x] = 0.999993 + 0.000323521\,x - 0.502482\,x^2 + 0.00628968\,x^3 + 0.0361867\,x^4$

The interval for interpolation is [0.0,1.0].

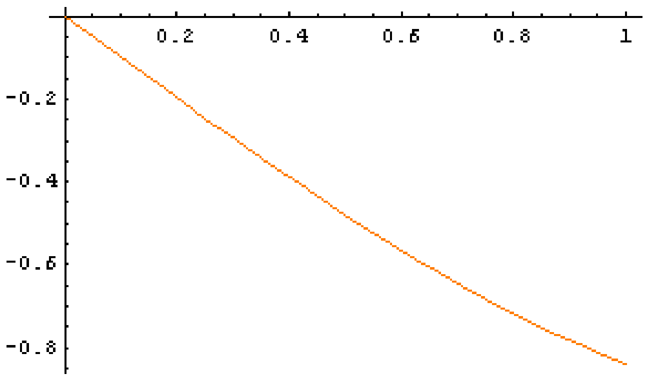
Graph of the error $f[x]-P_4[x]$

Extrema for $e_4[x]$; $\{-6.76182\times10^{-6}, 7.37886\times10^{-6}, -8.10946\times10^{-6}, 8.67066\times10^{-6}\}$

$|e_4[x]| \le 8.67066\times10^{-6}$

Compare the maximum error with the theoretical error bound:

$$\left| f(x) - P_n(x) \right| \le \frac{2\,(1-0)^{n+1}}{4^{n+1}\,(n+1)!} \max_{0\le x\le 1} \{ |f^{(n+1)}(x)| \}$$



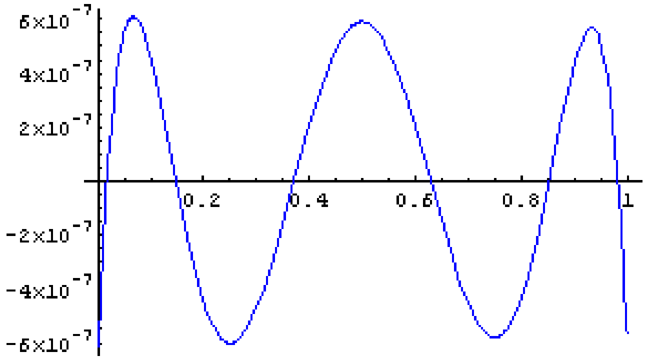
Graph of the derivative $f^{(n+1)}[x] = -\mathrm{Sin}[x]$

$$|e_4[x]| = |f(x) - P_4(x)| \le \frac{2\,(1-0)^{n+1}}{4^{n+1}\,(n+1)!} \mathrm{Abs}[f^{(n+1)}[1]] = 0.0000136958$$

$$|e_4[x]| = |f(x) - P_4(x)| \le 0.0000136958$$

The error bound is about 1.6 times as large as the maximum error. This is to be expected, after all it is an "error bound."

4 (b). Investigate the error for the Chebyshev interpolation polynomial $P_5[x]$, of degree n = 5.



f[x] = Cos[x]

$P_5[x] = 1. - 0.0000440675\,x - 0.499484\,x^2 - 0.00222058\,x^3 + 0.0460104\,x^4 - 0.00395968\,x^5$

The interval for interpolation is [0.0,1.0].

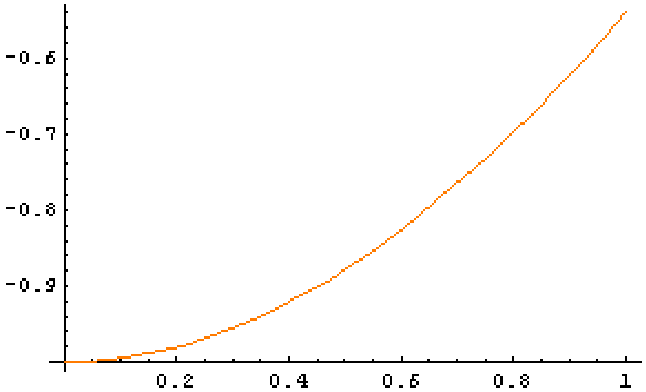
Graph of the error $f[x]-P_5[x]$

Extrema for $e_5[x]$; $\{6.0915\times10^{-7}, -6.02063\times10^{-7}, 5.91188\times10^{-7}, -5.78985\times10^{-7}, 5.69242\times10^{-7}\}$

$|e_5[x]| \le 6.0915\times10^{-7}$

Compare the maximum error with the theoretical error bound:

$$\left| f(x) - P_n(x) \right| \le \frac{2\,(1-0)^{n+1}}{4^{n+1}\,(n+1)!} \max_{0\le x\le 1} \{ |f^{(n+1)}(x)| \}$$



Graph of the derivative $f^{(n+1)}[x] = -\mathrm{Cos}[x]$

$$|e_5[x]| = |f(x) - P_5(x)| \le \frac{2\,(1-0)^{n+1}}{4^{n+1}\,(n+1)!}\mathrm{Abs}[f^{(n+1)}[0]] = 6.78168\times10^{-7}$$

$$|e_5[x]| = |f(x) - P_5(x)| \le 6.78168\times10^{-7}$$

The error bound is about 1.11 times as large as the maximum error. This is to be expected, after all it is an "error bound."