4. Adams-Bashforth-Moulton Method for O.D.E.'s

The methods of Euler, Taylor and Runge-Kutta are called single-step methods because they use only the information from one previous point to compute the successive point, that is, only the initial point (t_0, y_0) is used to compute (t_1, y_1) and in general y_k is needed to compute y_{k+1}. After several points have been found it is feasible to use several prior points in the calculation. The Adams-Bashforth-Moulton method uses $y_{k-3}, y_{k-2}, y_{k-1}$ and y_k in the calculation of y_{k+1}. This method is not self-starting; four initial points (t_0, y_0), (t_1, y_1), (t_2, y_2), and (t_3, y_3) must be given in advance in order to generate the points $\{(t_k, y_k)\}_{k=4}^{m}$.

A desirable feature of a multistep method is that the local truncation error (L. T. E.) can be determined and a correction term can be included, which improves the accuracy of the answer at each step. Also, it is possible to determine if the step size is small enough to obtain an accurate value for y_{k+1}, yet large enough so that unnecessary and time-consuming calculations are eliminated. If the code for the subroutine is fine-tuned, then the combination of a predictor and corrector requires only two function evaluations of $f(t,y)$ per step.

Theorem (Adams-Bashforth-MoultonMethod) Assume that $f(t,y)$ is continuous and satisfies a Lipschits condition in the variable y, and consider the I. V. P. (initial value problem)

$$y' = f(t, y) \text{ with } y(a) = t_0 = \alpha, \text{ over the interval } a \le t \le b.$$

The Adams-Bashforth-Moulton method uses the formulas $t_{k+1} = t_k + h$, and

the predictor $$p_{k+1} = y_k + \frac{h}{24}(-9 f_{k-3} + 37 f_{k-2} - 59 f_{k-1} + 55 f_k), \text{ and}$$

the corrector $$y_{k+1} = y_k + \frac{h}{24}(f_{k-2} - 5 f_{k-1} + 19 f_k + 9 f[t_{k+1}, p_{k+1}]) \text{ for } k = 3, 4, \ldots, m-1$$

as an approximate solution to the differential equation using the discrete set of points $\{(t_k, y_k)\}_{k=0}^{m}$.

Remark. The Adams-Bashforth-Moulton method is not a self-starting method. Three additional starting values $y_1, y_2,$ and y_3 must be given. They are usually computed using the Runge-Kutta method.

Theorem (Precision of Adams-Bashforth-MoultonMethod) Assume that $y = y(t)$ is the solution to the I.V.P. $y' = f(t, y)$ with $y(t_0) = y_0$. If $y(t) \in C^5[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{m}$ is the sequence of approximations generated by Adams-Bashforth-Moulton method, then at each step, the local truncation error is of the order $O(h^5)$, and the overall global truncation error e_k is of the order

$$|e_k| = |y(t_k) - y_k| = O(h^4), \text{ for } k = 1, 2, \ldots, m.$$

The error at the right end of the interval is called the final global error

$$E(y(b), h) = |y(b) - y_m| = O(h^4).$$

Example 1. Solve the I.V.P. $y' = 1 - t y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

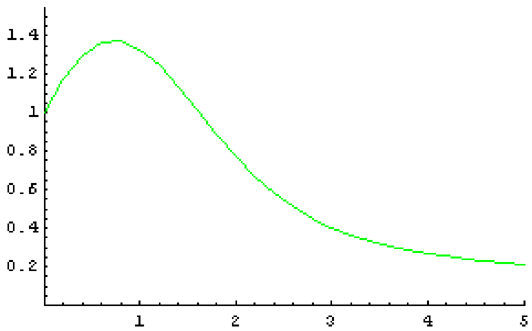
Example 2. Solve $y' = 30 - 5y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 2.

Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

Example 1. Solve the I.V.P. $y' = 1 - t\,y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

Compute the Adams-Bashforth-Moulton solution based on 25 subintervals and plot the results.

Find numerical solutions to the D.E.
$y' = 1 - t\,y$

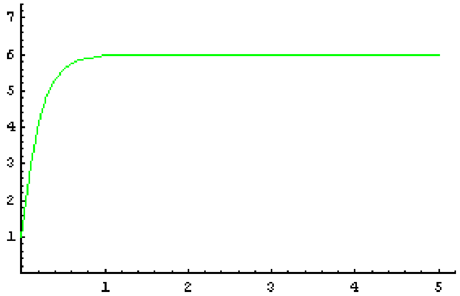


The Adams-Bashforth-Moulton solution for $y' = 1 - t\,y$
Using $n = 26$ points.
{{0., 1.}, {0.2, 1.17755}, {0.4, 1.30245}, {0.6, 1.36819}, {0.8, 1.37543}, {1., 1.33119}, {1.2, 1.2471}, {1.4, 1.13696}, {1.6, 1.01436}, {1.8, 0.890889}, {2., 0.775047}, {2.2, 0.67203}, {2.4, 0.584077}, {2.6, 0.511202}, {2.8, 0.452002}, {3., 0.404394}, {3.2, 0.366141}, {3.4, 0.335189}, {3.6, 0.309816}, {3.8, 0.288671}, {4., 0.270734}, {4.2, 0.255262}, {4.4, 0.241716}, {4.6, 0.229708}, {4.8, 0.218953}, {5., 0.209241}}

The final value is $y(5) = y_{26} = 0.209241$

Example 2. Solve $y' = 30 - 5y$ with $y(0) = 1$ over $0 \le t \le 5$.

Solution 2.



The Adams-Bashforth-Moulton solution for $y' = 30 - 5y$

Using $n = 51$ points.

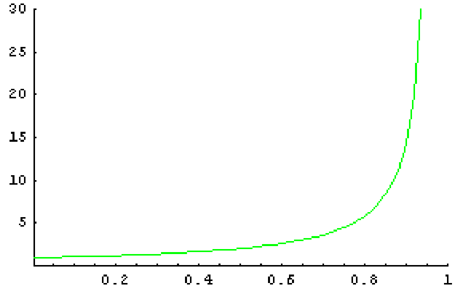
{{0., 1}, {0.1, 2.96615}, {0.2, 4.15915}, {0.3, 4.88302}, {0.4, 5.32762}, {0.5, 5.59541}, {0.6, 5.75597}, {0.7, 5.85309}, {0.8, 5.91164}, {0.9, 5.94679}, {1., 5.96797}, {1.1, 5.98072}, {1.2, 5.9884}, {1.3, 5.99302}, {1.4, 5.9958}, {1.5, 5.99747}, {1.6, 5.99848}, {1.7, 5.99908}, {1.8, 5.99945}, {1.9, 5.99967}, {2., 5.9998}, {2.1, 5.99988}, {2.2, 5.99993}, {2.3, 5.99996}, {2.4, 5.99997}, {2.5, 5.99998}, {2.6, 5.99999}, {2.7, 5.99999}, {2.8, 6.}, {2.9, 6.}, {3., 6.}, {3.1, 6.}, {3.2, 6.}, {3.3, 6.}, {3.4, 6.}, {3.5, 6.}, {3.6, 6.}, {3.7, 6.}, {3.8, 6.}, {3.9, 6.}, {4., 6.}, {4.1, 6.}, {4.2, 6.}, {4.3, 6.}, {4.4, 6.}, {4.5, 6.}, {4.6, 6.}, {4.7, 6.}, {4.8, 6.}, {4.9, 6.}, {5., 6.}}

The final value is $y(5) = y_{51} = 6.$

Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

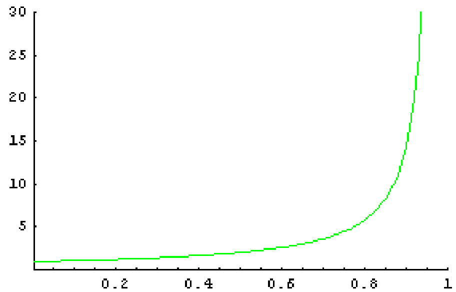
Compute the Adams-Bashforth-Moulton solution based on 50 subintervals and plot the results.

Find numerical solutions to the D.E.
$y' = t^2 + y^2$



The Adams-Bashforth-Moulton solution for $y' = t^2 + y^2$
Using n = 51 points.
{{0., 1.}, {0.02, 1.02041}, {0.04, 1.04169}, {0.06, 1.0639}, {0.08, 1.08713}, {0.1, 1.11146}, {0.12, 1.13698}, {0.14, 1.16378}, {0.16, 1.19198}, {0.18, 1.22168}, {0.2, 1.25302}, {0.22, 1.28613}, {0.24, 1.32117}, {0.26, 1.35831}, {0.28, 1.39774}, {0.3, 1.43967}, {0.32, 1.48433}, {0.34, 1.53199}, {0.36, 1.58294}, {0.38, 1.63752}, {0.4, 1.69611}, {0.42, 1.75915}, {0.44, 1.82713}, {0.46, 1.90064}, {0.48, 1.98033}, {0.5, 2.067}, {0.52, 2.16157}, {0.54, 2.26511}, {0.56, 2.37893}, {0.58, 2.5046}, {0.6, 2.64401}, {0.62, 2.7995}, {0.64, 2.97395}, {0.66, 3.17102}, {0.68, 3.39535}, {0.7, 3.65295}, {0.72, 3.95177}, {0.74, 4.30251}, {0.76, 4.71995}, {0.78, 5.22512}, {0.8, 5.84891}, {0.82, 6.63874}, {0.84, 7.67122}, {0.86, 9.07873}, {0.88, 11.1113}, {0.9, 14.3035}, {0.92, 20.0323}, {0.94, 33.1297}, {0.96, 84.8408}, {0.98, 1230.47}, {1., 3.70921×10^7}}
The final value is $y(5) = y_{51} = 3.70921 \times 10^7$

Compute the Adams-Bashforth-Moulton solution based on 100 subintervals and plot the results.
Observe that one fewer subinterval is computed for this case.

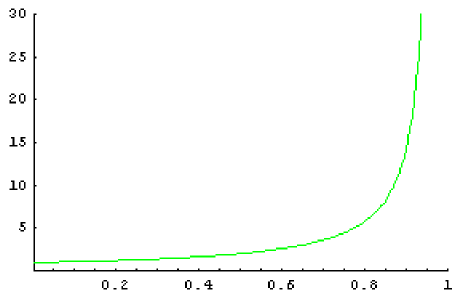


The Adams-Bashforth-Moulton solution for $y' = t^2 + y^2$
Using n = 100 points.

{{0., 1.}, {0.01, 1.0101}, {0.02, 1.02041}, {0.03, 1.03094}, {0.04, 1.04169}, {0.05, 1.05267}, {0.06, 1.0639}, {0.07, 1.07539}, {0.08, 1.08713}, {0.09, 1.09916},
{0.1, 1.11146}, {0.11, 1.12407}, {0.12, 1.13698}, {0.13, 1.15021}, {0.14, 1.16378}, {0.15, 1.1777}, {0.16, 1.19198}, {0.17, 1.20663}, {0.18, 1.22168},
{0.19, 1.23714}, {0.2, 1.25302}, {0.21, 1.26934}, {0.22, 1.28613}, {0.23, 1.3034}, {0.24, 1.32117}, {0.25, 1.33947}, {0.26, 1.35831}, {0.27, 1.37773},
{0.28, 1.39774}, {0.29, 1.41838}, {0.3, 1.43967}, {0.31, 1.46164}, {0.32, 1.48433}, {0.33, 1.50777}, {0.34, 1.53199}, {0.35, 1.55703}, {0.36, 1.58294},
{0.37, 1.60975}, {0.38, 1.63752}, {0.39, 1.66629}, {0.4, 1.69611}, {0.41, 1.72704}, {0.42, 1.75915}, {0.43, 1.79249}, {0.44, 1.82713}, {0.45, 1.86315},
{0.46, 1.90063}, {0.47, 1.93966}, {0.48, 1.98033}, {0.49, 2.02274}, {0.5, 2.067}, {0.51, 2.11323}, {0.52, 2.16156}, {0.53, 2.21214}, {0.54, 2.2651},
{0.55, 2.32064}, {0.56, 2.37893}, {0.57, 2.44017}, {0.58, 2.50459}, {0.59, 2.57244}, {0.6, 2.644}, {0.61, 2.71957}, {0.62, 2.79948}, {0.63, 2.88413},
{0.64, 2.97394}, {0.65, 3.06938}, {0.66, 3.171}, {0.67, 3.27941}, {0.68, 3.39532}, {0.69, 3.51951}, {0.7, 3.65291}, {0.71, 3.79656}, {0.72, 3.95171},
{0.73, 4.11977}, {0.74, 4.30242}, {0.75, 4.50166}, {0.76, 4.71983}, {0.77, 4.95979}, {0.78, 5.22495}, {0.79, 5.51951}, {0.8, 5.84865}, {0.81, 6.21887},
{0.82, 6.63836}, {0.83, 7.11768}, {0.84, 7.67065}, {0.85, 8.31569}, {0.86, 9.07795}, {0.87, 9.99262}, {0.88, 11.1106}, {0.89, 12.5083}, {0.9, 14.306},
{0.91, 16.7041}, {0.92, 20.0639}, {0.93, 25.1084}, {0.94, 33.5197}, {0.95, 50.2591}, {0.96, 97.7203}, {0.97, 440.75}, {0.98, 84534.4}, {0.99, 1.00668×10^{14}}}

The final value is $y(0.99) = y_{100} = 1.00668 \times 10^{14}$

Compute the Adams-Bashforth-Moulton solution based on 200 subintervals and plot the results.
Observe that four fewer subintervals are computed for this case.



The Adams-Bashforth-Moulton solution for $y' = t^2 + y^2$

Using n = 197 points.

{{0., 1.}, {0.005, 1.00503}, {0.01, 1.0101}, {0.015, 1.01523}, {0.02, 1.02041}, {0.025, 1.02565}, {0.03, 1.03094}, {0.035, 1.03628}, {0.04, 1.04169},
{0.045, 1.04715}, {0.05, 1.05267}, {0.055, 1.05826}, {0.06, 1.0639}, {0.065, 1.06961}, {0.07, 1.07539}, {0.075, 1.08123}, {0.08, 1.08713}, {0.085, 1.09311},
{0.09, 1.09916}, {0.095, 1.10527}, {0.1, 1.11146}, {0.105, 1.11773}, {0.11, 1.12407}, {0.115, 1.13048}, {0.12, 1.13698}, {0.125, 1.14356}, {0.13, 1.15021},
{0.135, 1.15696}, {0.14, 1.16378}, {0.145, 1.1707}, {0.15, 1.1777}, {0.155, 1.18479}, {0.16, 1.19198}, {0.165, 1.19926}, {0.17, 1.20663}, {0.175, 1.21411},
{0.18, 1.22168}, {0.185, 1.22935}, {0.19, 1.23714}, {0.195, 1.24502}, {0.2, 1.25302}, {0.205, 1.26112}, {0.21, 1.26934}, {0.215, 1.27768}, {0.22, 1.28613},
{0.225, 1.2947}, {0.23, 1.3034}, {0.235, 1.31222}, {0.24, 1.32117}, {0.245, 1.33025}, {0.25, 1.33947}, {0.255, 1.34882}, {0.26, 1.35831}, {0.265, 1.36795},
{0.27, 1.37773}, {0.275, 1.38766}, {0.28, 1.39774}, {0.285, 1.40798}, {0.29, 1.41838}, {0.295, 1.42894}, {0.3, 1.43967}, {0.305, 1.45057}, {0.31, 1.46164},
{0.315, 1.4729}, {0.32, 1.48433}, {0.325, 1.49595}, {0.33, 1.50777}, {0.335, 1.51978}, {0.34, 1.53199}, {0.345, 1.54441}, {0.35, 1.55703}, {0.355, 1.56988},
{0.36, 1.58294}, {0.365, 1.59623}, {0.37, 1.60975}, {0.375, 1.62352}, {0.38, 1.63752}, {0.385, 1.65178}, {0.39, 1.66629}, {0.395, 1.68106}, {0.4, 1.69611},
{0.405, 1.71143}, {0.41, 1.72704}, {0.415, 1.74295}, {0.42, 1.75915}, {0.425, 1.77566}, {0.43, 1.79249}, {0.435, 1.80964}, {0.44, 1.82713}, {0.445, 1.84496},
{0.45, 1.86315}, {0.455, 1.88171}, {0.46, 1.90063}, {0.465, 1.91995}, {0.47, 1.93966}, {0.475, 1.95978}, {0.48, 1.98033}, {0.485, 2.00131}, {0.49, 2.02274},
{0.495, 2.04463}, {0.5, 2.067}, {0.505, 2.08986}, {0.51, 2.11323}, {0.515, 2.13713}, {0.52, 2.16156}, {0.525, 2.18656}, {0.53, 2.21213}, {0.535, 2.23831},
{0.54, 2.2651}, {0.545, 2.29254}, {0.55, 2.32064}, {0.555, 2.34943}, {0.56, 2.37893}, {0.565, 2.40916}, {0.57, 2.44017}, {0.575, 2.47197}, {0.58, 2.50459},
{0.585, 2.53807}, {0.59, 2.57244}, {0.595, 2.60774}, {0.6, 2.644}, {0.605, 2.68126}, {0.61, 2.71957}, {0.615, 2.75896}, {0.62, 2.79948}, {0.625, 2.84119},
{0.63, 2.88413}, {0.635, 2.92836}, {0.64, 2.97393}, {0.645, 3.02092}, {0.65, 3.06938}, {0.655, 3.11938}, {0.66, 3.171}, {0.665, 3.22431}, {0.67, 3.27941},
{0.675, 3.33638}, {0.68, 3.39532}, {0.685, 3.45632}, {0.69, 3.51951}, {0.695, 3.58499}, {0.7, 3.6529}, {0.705, 3.72338}, {0.71, 3.79656}, {0.715, 3.87261},
{0.72, 3.9517}, {0.725, 4.03402}, {0.73, 4.11976}, {0.735, 4.20915}, {0.74, 4.30242}, {0.745, 4.39982}, {0.75, 4.50165}, {0.755, 4.6082}, {0.76, 4.71982},
{0.765, 4.83687}, {0.77, 4.95977}, {0.775, 5.08895}, {0.78, 5.22492}, {0.785, 5.36823}, {0.79, 5.51948}, {0.795, 5.67936}, {0.8, 5.84862}, {0.805, 6.02812},
{0.81, 6.21882}, {0.815, 6.42181}, {0.82, 6.6383}, {0.825, 6.8697}, {0.83, 7.1176}, {0.835, 7.38385}, {0.84, 7.67055}, {0.845, 7.98016}, {0.85, 8.31555},
{0.855, 8.68009}, {0.86, 9.07774}, {0.865, 9.51326}, {0.87, 9.99233}, {0.875, 10.5218}, {0.88, 11.1102}, {0.885, 11.7678}, {0.89, 12.5077}, {0.895, 13.3464},
{0.9, 14.305}, {0.905, 15.4115}, {0.91, 16.7027}, {0.915, 18.2293}, {0.92, 20.0622}, {0.925, 22.304}, {0.93, 25.1086}, {0.935, 28.7188}, {0.94, 33.5395},
{0.945, 40.3024}, {0.95, 50.4741}, {0.955, 67.4796}, {0.96, 101.48}, {0.965, 198.857}, {0.97, 924.64}, {0.975, 203414.}, {0.98, 4.21811×10^{14}}}

The final value is $y(0.98) = y_{197} = 4.21811 \times 10^{14}$