1. Numerical Differentiation, Part I

Background.

Numerical differentiation formulas can be derived by first constructing the Lagrange interpolating polynomial $P_2(x)$ through three points, differentiating the Lagrange polynomial, and finally evaluating $P_2'(x)$ at the desired point. In this module the truncation error will be investigated, but round off error from computer arithmetic using computer numbers will be studied in another module.

Theorem (Three point rule for $f'[x]$). The central difference formula for the first derivative, based on three points is

$$f'[x] \approx D1[x, h] = \frac{f[x+h] - f[x-h]}{2h},$$

and the remainder term is

$$R1[x, h] = \frac{-f^{(3)}[c]}{6} h^2.$$

Together they make the equation $f'[x] = D1[x, h] + R1[x, h]$, and the truncation error bound is

$$EB1[h] = \left| \frac{-f^{(3)}[c]}{6} h^2 \right| \leq \frac{M_3}{6} h^2$$

where $M_3 = \max_{a \leq x \leq b} |f^{(3)}[x]|$. This gives rise to the Big "O" notation for the error term for $f'[x]$:

$$f'[x] = \frac{f[x+h] - f[x-h]}{2h} + O(h^2).$$

Theorem (Three point rule for $f''[x]$). The central difference formula for the second derivative, based on three points is

$$f''[x] \approx D2[x, h] = \frac{f[x-h] - 2f[x] + f[x+h]}{h^2},$$

and the remainder term is

$$R2[x, h] = \frac{-f^{(4)}[x]}{12} h^2.$$

Together they make the equation $f''[x] = D2[x, h] + R2[x, h]$, and the truncation error bound is

$$EB2[h] = \left| \frac{-f^{(4)}[x]}{12} h^2 \right| \leq \frac{M_4}{12} h^2$$

where $M_4 = max_{a \leq x \leq b} \; | f^{(4)}[x] |$. This gives rise to the Big "O" notation for the error term for $f''[x]$:

$$f''[x] = \frac{f[x-h] - 2 f[x] + f[x+h]}{h^2} + O(h^2).$$

Example 1. Compute numerical approximations for the derivative $f'[0]$, using step sizes $h = 0.1, 0.01, 0.001$, include the details. Plot the numerical approximation $D1[x, 0.01]$ over the interval $[0, \pi]$. Compare it with the graph of $f'[x]$ over the interval $[0, \pi]$.
Solution 1.

Example 2. Consider the function $f[x] = e^{-x} Sin[x]$. Find the formula for the fourth derivative $f^{(4)}[x]$, it will be used in our explorations for the remainder term and the truncation error bound. Graph $f^{(4)}[x]$. Find the bound $M_4 = max_{0 \leq x \leq \pi} \; | f^{(4)}[x] |$. Look at it's graph and estimate the value M_4, be sure to take the absolute value if necessary.
Solution 2.

Various Scenarios for Numerical Differentiation.

Example 3. Given $f[x] = e^{-x} Sin[x]$, find numerical approximations to the derivative $f'[1.0]$, using two points and the forward difference formula.
Solution 3.

Example 4. Given $f[x] = e^{-x} Sin[x]$, find numerical approximations to the derivative $f'[1.0]$, using two points and the backward difference formula.
Solution 4.

Example 5. Given $f[x] = e^{-x} Sin[x]$, find numerical approximations to the derivative $f'[1.0]$, using two points and the central difference formula.
Solution 5.

Example 6. Given $f[x] = e^{-x} Sin[x]$, find numerical approximations to the derivative $f'[1.0]$, using three points and the forward difference formula.
Solution 6.

Example 7. Given $f[x] = e^{-x} Sin[x]$, find numerical approximations to the derivative $f'[1.0]$, using three points and the backward difference formula.
Solution 7.

Example 8. Given $f[x] = e^{-x} \text{Sin}[x]$, find numerical approximations to the derivative $f'[1.0]$, using three points and the central difference formula.
Solution 8.

Example 9. Given $f[x] = e^{-x} \text{Sin}[x]$, find numerical approximations to the second derivative $f''[1.0]$, using three points and the forward difference formula.
Solution 9.

Example 10. Given $f[x] = e^{-x} \text{Sin}[x]$, find numerical approximations to the second derivative $f''[1.0]$, using three points and the backward difference formula.
Solution 10.

Example 11. Given $f[x] = e^{-x} \text{Sin}[x]$, find numerical approximations to the second derivative $f''[1.0]$, using three points and the central difference formula.
Solution 11.

Example 1. Compute numerical approximations for the derivative f'[0], using step sizes h = 0.1, 0.01, 0.001, include the details.Plot the numerical approximation D1[x, 0.01] over the interval [0, π]. Compare it with the graph of f'[x] over the interval [0, π].

Solution 1.

f[x] = e^{-x} Sin[x]
Using h = 0.1
f'[1.] ≈ ((0.296657) - (0.318477))/(0.2)
f'[1.] ≈ (-0.0218198)/(0.2)
f'[1.] ≈ -0.109099

f[x] = e^{-x} Sin[x]
Using h = 0.01
f'[1.] ≈ ((0.308432) - (0.310648))/(0.02)
f'[1.] ≈ (-0.00221554)/(0.02)
f'[1.] ≈ -0.110777

f[x] = e^{-x} Sin[x]
Using h = 0.001
f'[1.] ≈ ((0.309449) - (0.30967))/(0.002)
f'[1.] ≈ (-0.000221587)/(0.002)
f'[1.] ≈ -0.110794

f[x] = e^{-x} Sin[x]
The true value is:
$$f'[1] = \frac{\text{Cos}[1]}{e} - \frac{\text{Sin}[1]}{e}$$
f'[1] = -0.110794

Aside. Which step size h = 0.1, h = 0.01, or h = 0.001 gave the best approximation. Do you think that it will be better with $h = 1.0 * 10^{-12}$?

f[x] = e^{-x} Sin[x]
Using h = 0.001
f'[1.] ≈ -0.110793595865

f[x] = e^{-x} Sin[x]
Using $h = 1. \times 10^{-12}$
f'[1.] ≈ -0.110800257858

f[x] = e^{-x} Sin[x]

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The true value is:
f'[1] = -0.110793765307
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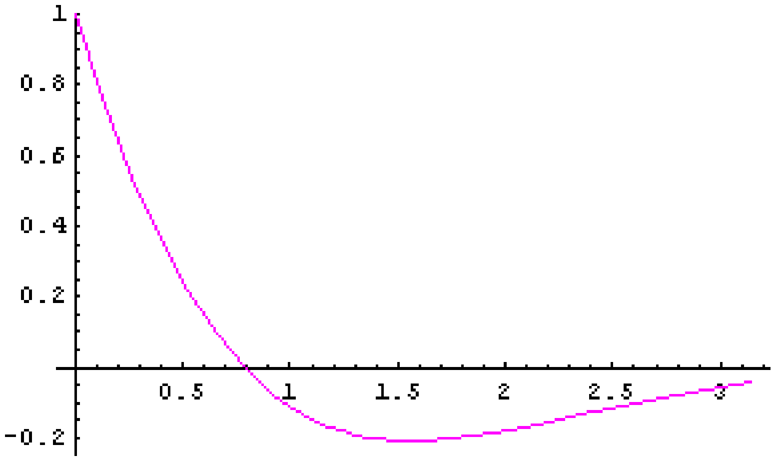
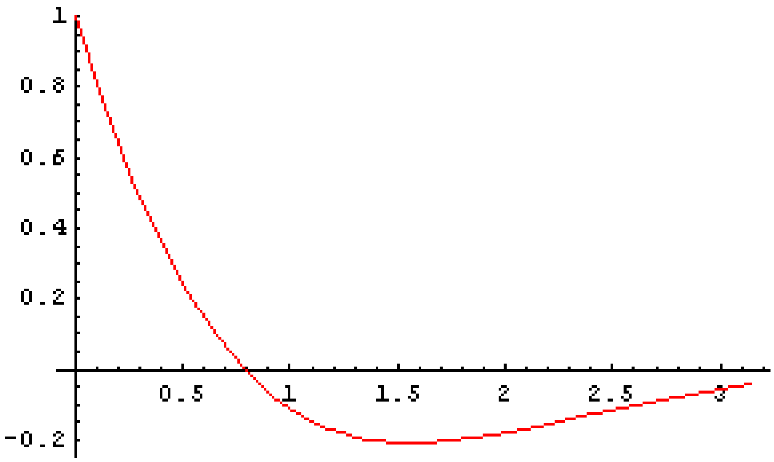
Let us plot `D1[x, 0.01]` over the interval $[0, \pi]$.



D1[x,0.01] = (f[x+0.01] - f[x-0.01])/0.02

D1[x,0.01] = $50.\,(e^{0.01-x}\,\mathrm{Sin}[0.01-x]+e^{-0.01-x}\,\mathrm{Sin}[0.01+x])$



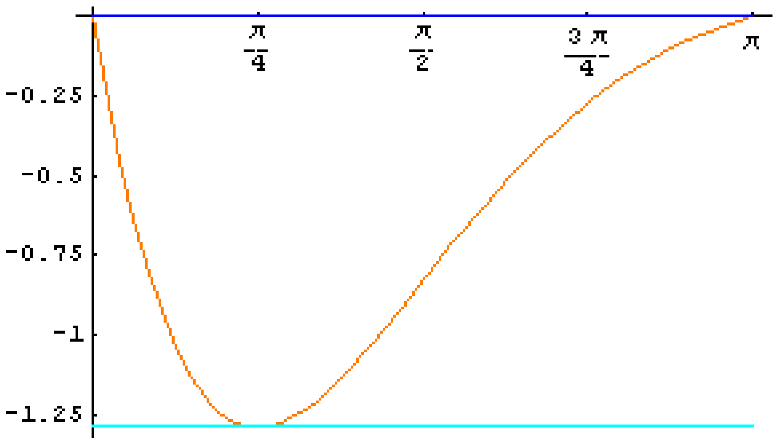
f'[x] = $e^{-x}\,\mathrm{Cos}[x]-e^{-x}\,\mathrm{Sin}[x]$

Example 2. Consider the function $f[x] = e^{-x} \mathrm{Sin}[x]$. Find the formula for the fourth derivative $f^{(4)}[x]$, it will be used in our explorations for the remainder term and the truncation error bound. Graph $f^{(4)}[x]$. Find the bound $M_4 = \max_{0 \le x \le \pi} | f^{(4)}[x] |$. Look at it's graph and estimate the value M_4, be sure to take the absolute value if necessary.

Solution 2.

$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$$
$$f^{(3)}[x] = 2 e^{-x} \mathrm{Cos}[x] + 2 e^{-x} \mathrm{Sin}[x]$$

$$f^{(4)}[x] = -4 e^{-x} \mathrm{Sin}[x]$$



$y = f^{(4)}[x] = -4 e^{-x} \mathrm{Sin}[x]$

The minimum occurs at $x = \frac{\pi}{4}$ and the maximum occurs at $x = 0, \pi$.

Extrema of $f^{(4)}[x]$: $\{0, -2\sqrt{2}\, e^{-\pi/4}, 0\}$

$$|f^{(4)}[c]| \le 2\sqrt{2}\, e^{-\pi/4} = 1.28959 \quad \text{for } 0 \le c \le \pi$$

The absolute value of the remainder term:

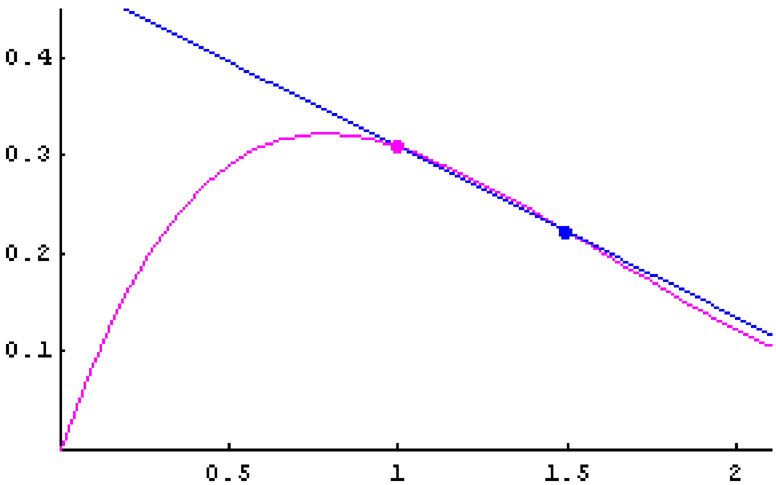
$$|R[x,h]| = \left| \frac{-f^{(4)}[c]}{12} h^2 \right| \le \frac{e^{-\pi/4} h^2}{3\sqrt{2}}$$

The error bound is:

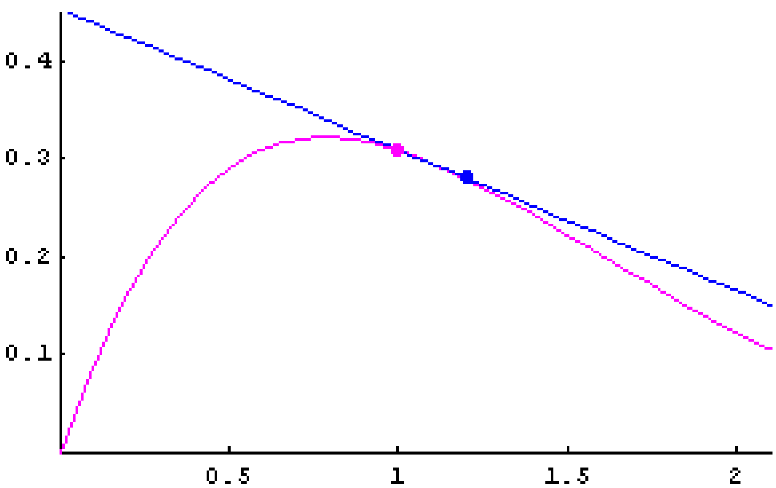
$$EB2[h] = \frac{e^{-\pi/4} h^2}{3\sqrt{2}} = 0.107466\, h^2$$

Example 3. Given $f[x] = e^{-x} \mathrm{Sin}[x]$, find numerical approximations to the derivative $f'[1.0]$, using two points and the forward difference formula.

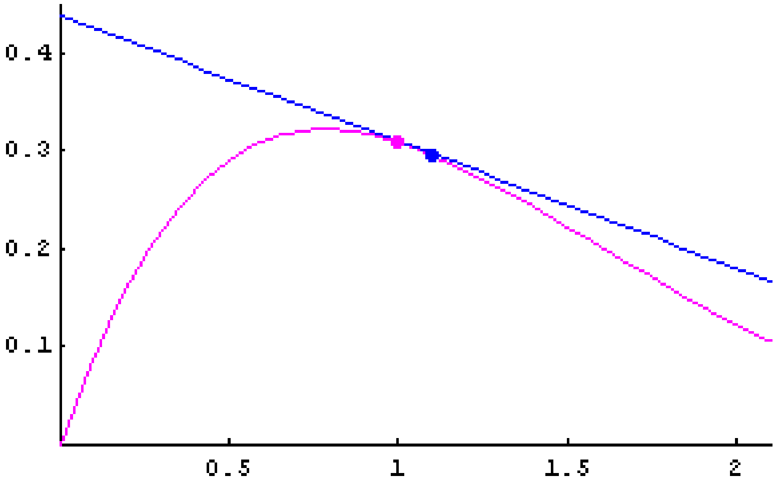
Solution 3.



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.5$$
$$f'[1.] \approx (f[1.+0.5] - f[1.])/(0.5) = -0.173977$$



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.2$$
$$f'[1.] \approx (f[1.+0.2] - f[1.])/(0.2) = -0.144175$$



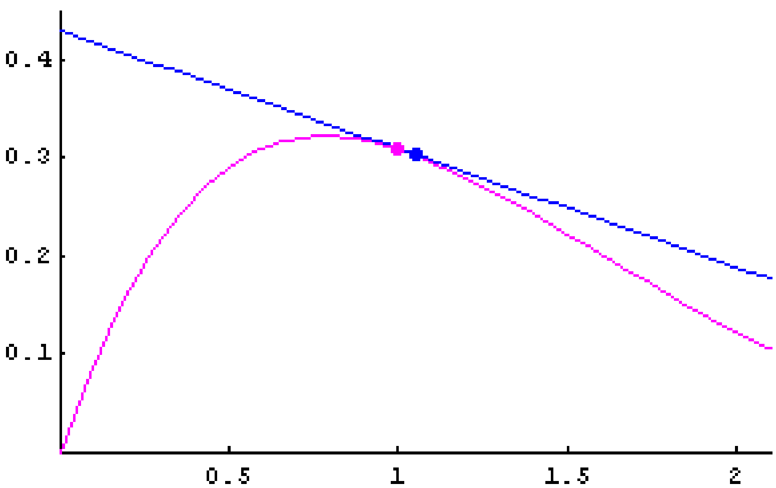
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.1$$

$$f'[1.] \approx (f[1.+0.1] - f[1.])/(0.1) = -0.129027$$



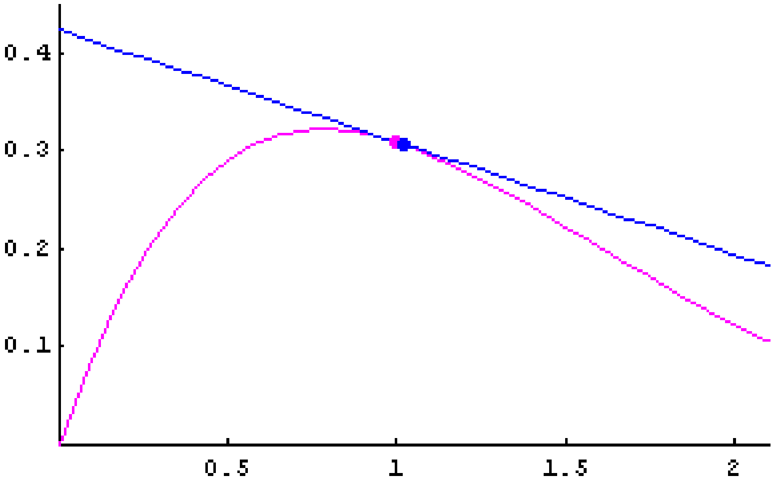
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.05$$

$$f'[1.] \approx (f[1.+0.05] - f[1.])/(0.05) = -0.120315$$



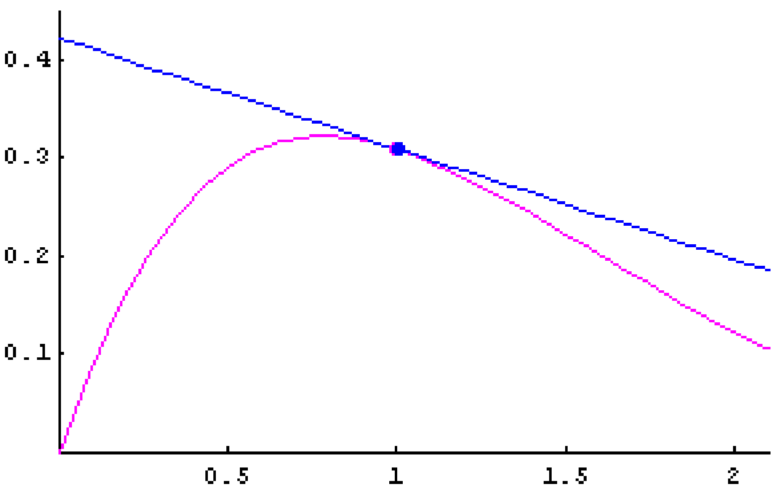
$f[x] = e^{-x}\,\mathrm{Sin}[x]$

$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.02$

$f'[1.] \approx (f[1.+0.02] - f[1.])/(0.02) = -0.114702$



$f[x] = e^{-x}\,\mathrm{Sin}[x]$

$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$

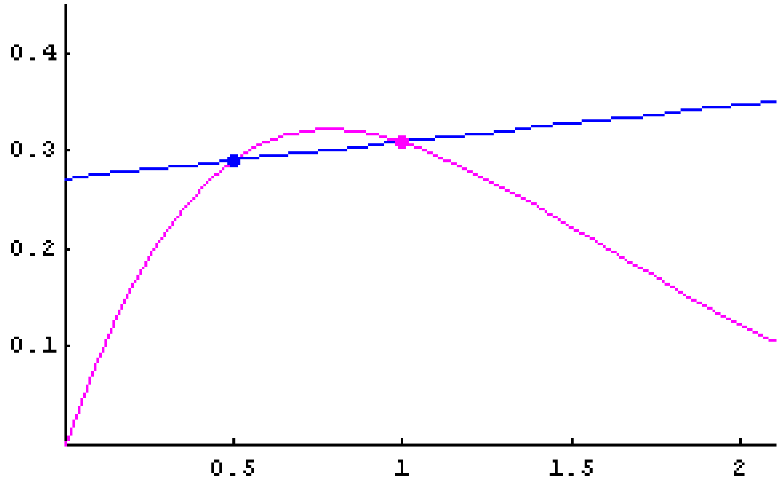
$f'[1.] = -0.110794$

$h = 0.01$

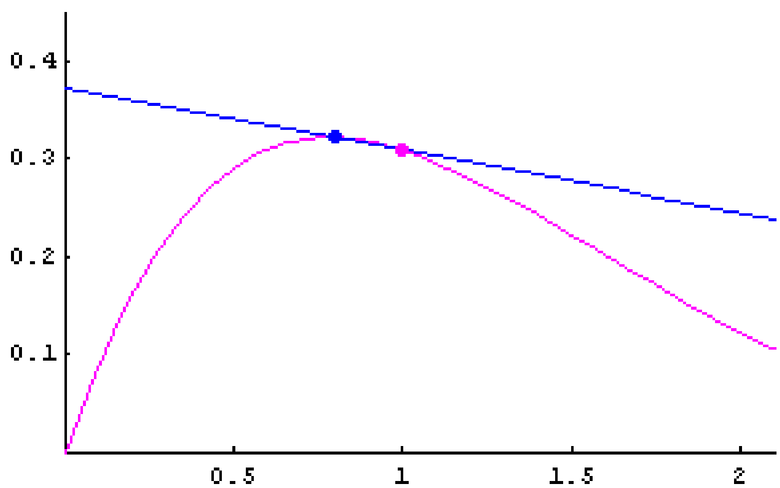
$f'[1.] \approx (f[1.+0.01] - f[1.])/(0.01) = -0.112765$

Example 4. Given $f[x] = e^{-x} \sin[x]$, find numerical approximations to the derivative $f'[1.0]$, using two points and the backward difference formula.

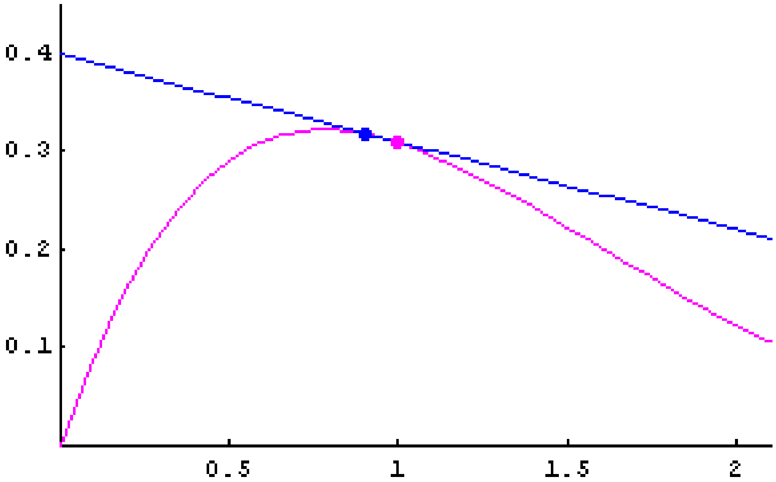
Solution 4.



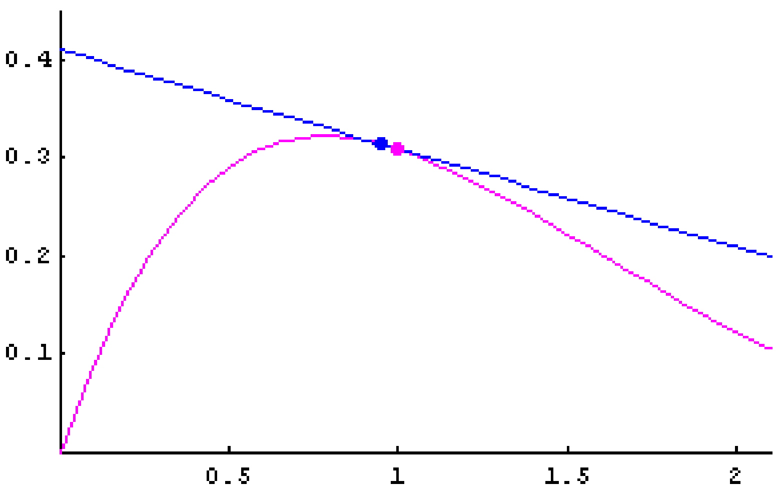
$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.5$$
$$f'[1.] \approx (f[1.] - f[1.-0.5])/(0.5) = 0.0375472$$



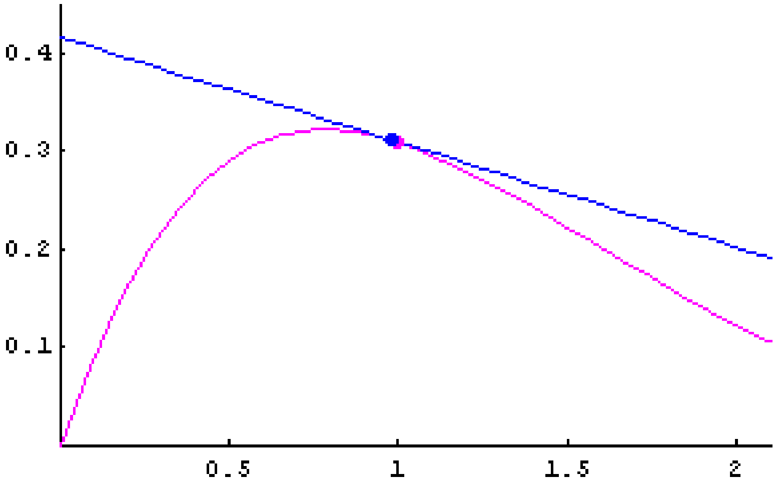
$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.2$$
$$f'[1.] \approx (f[1.] - f[1.-0.2])/(0.2) = -0.063845$$



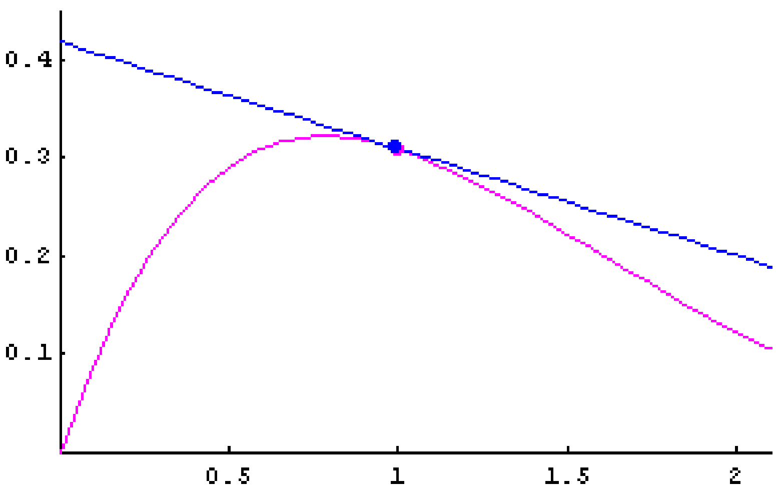
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.1$$
$$f'[1.] \approx (f[1.] - f[1.-0.1])/(0.1) = -0.0891708$$



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.05$$
$$f'[1.] \approx (f[1.] - f[1.-0.05])/(0.05) = -0.100425$$



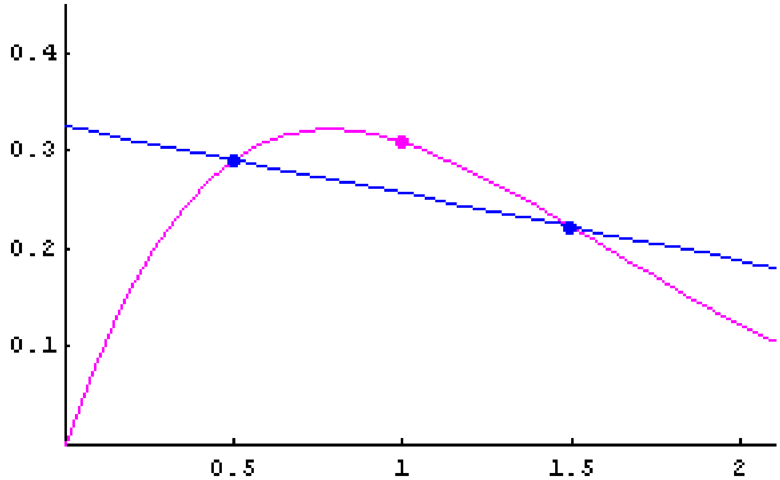
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.02$$
$$f'[1.] \approx (f[1.] - f[1.-0.02])/(0.02) = -0.10675$$



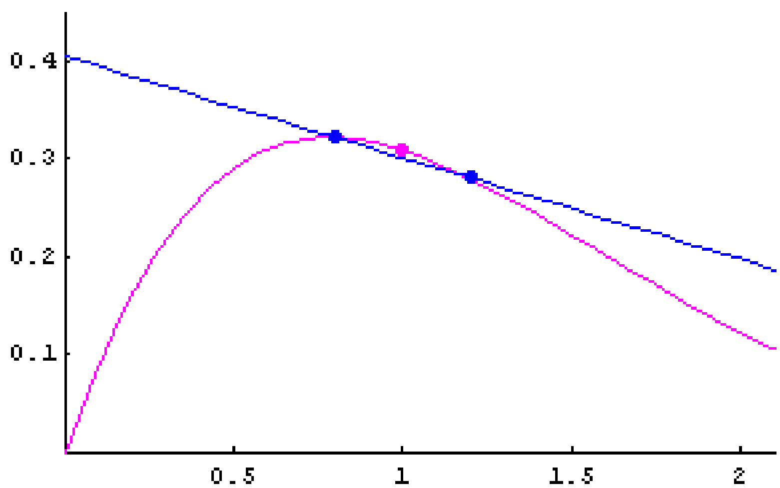
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.01$$
$$f'[1.] \approx (f[1.] - f[1.-0.01])/(0.01) = -0.108789$$

Example 5. Given $f[x] = e^{-x} \mathrm{Sin}[x]$, find numerical approximations to the derivative $f'[1.0]$, using two points and the central difference formula.

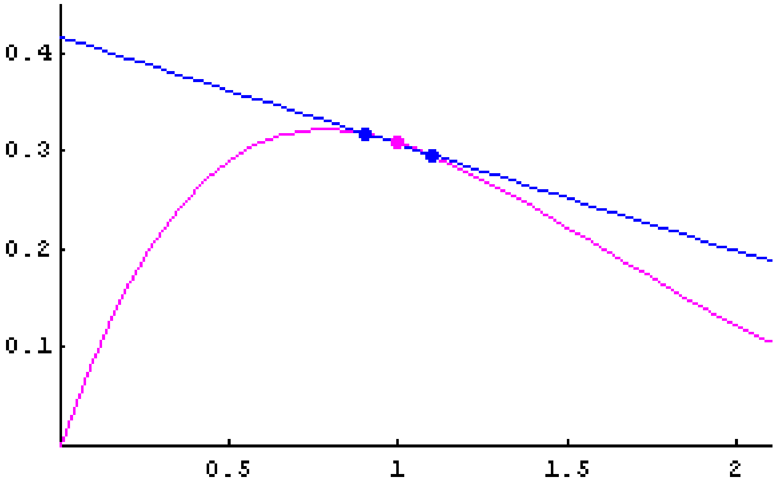
Solution 5.



$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.5$$
$$f'[1.] \approx (f[1.+0.5] - f[1.-0.5])/(1.) = -0.0682151$$



$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f'[1.] = -0.110794$$
$$h = 0.2$$
$$f'[1.] \approx (f[1.+0.2] - f[1.-0.2])/(0.4) = -0.10401$$



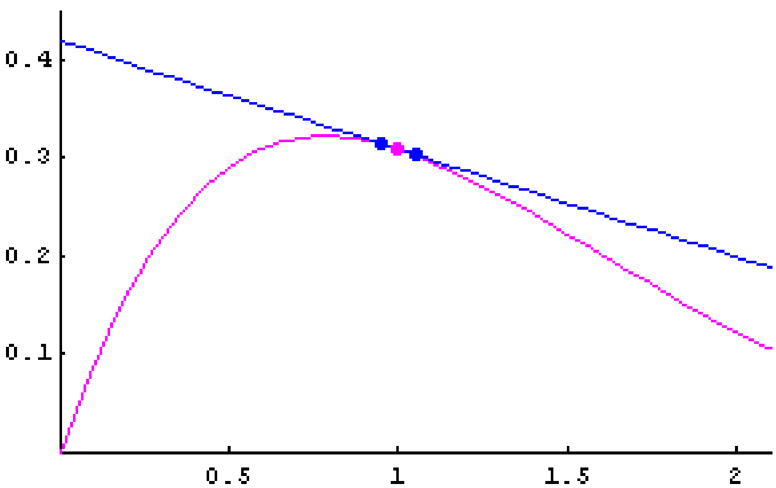
$f[x] = e^{-x} \mathrm{Sin}[x]$

$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.1$

$f'[1.] \approx (f[1.+0.1] - f[1.-0.1])/(0.2) = -0.109099$



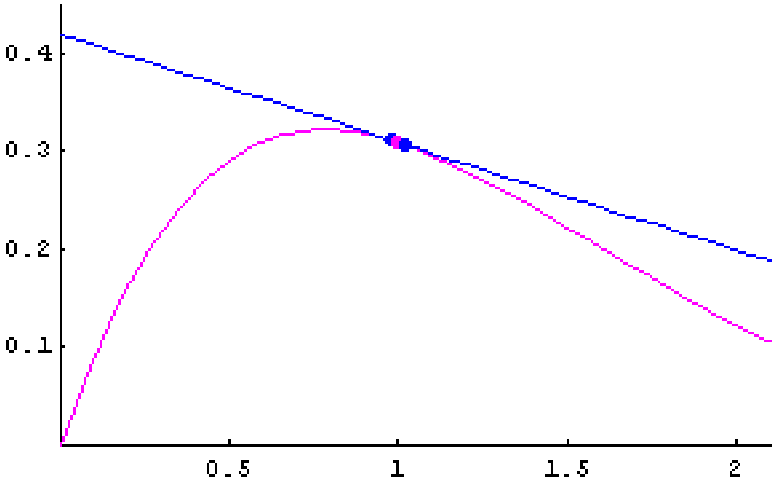
$f[x] = e^{-x} \mathrm{Sin}[x]$

$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.05$

$f'[1.] \approx (f[1.+0.05] - f[1.-0.05])/(0.1) = -0.11037$



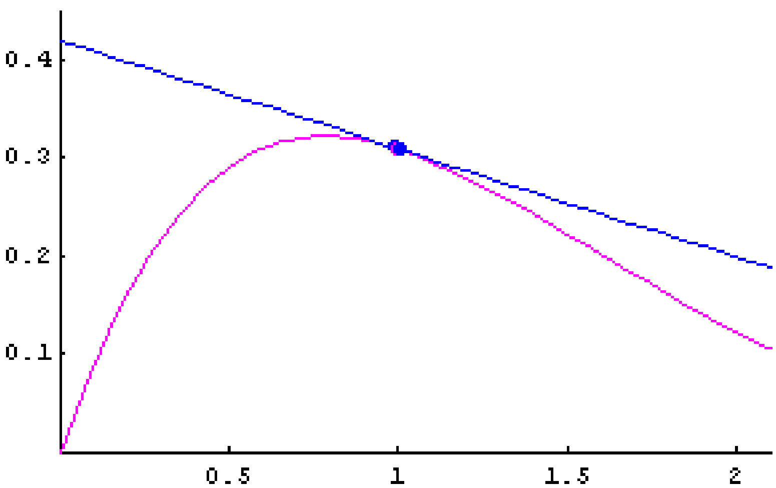
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.02$$

$$f'[1.] \approx (f[1.+0.02] - f[1.-0.02])/(0.04) = -0.110726$$



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

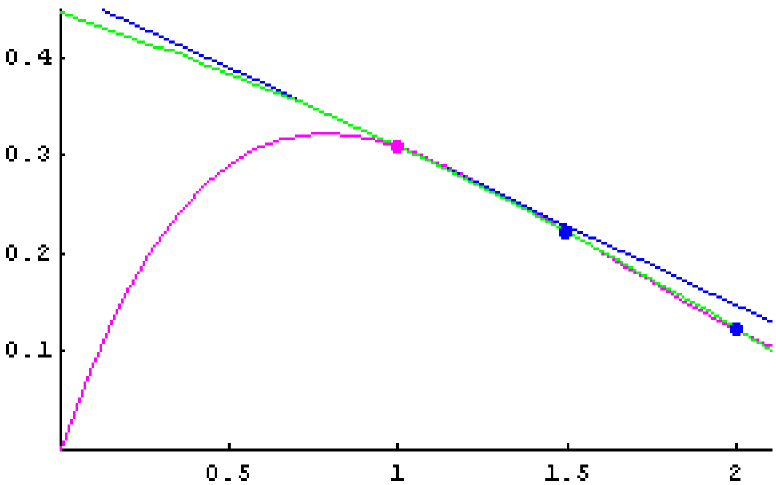
$$f'[1.] = -0.110794$$

$$h = 0.01$$

$$f'[1.] \approx (f[1.+0.01] - f[1.-0.01])/(0.02) = -0.110777$$

Example 6. Given $f[x] = e^{-x} \, Sin[x]$, find numerical approximations to the derivative $f'[1.0]$, using three points and the forward difference formula.

Solution 6.



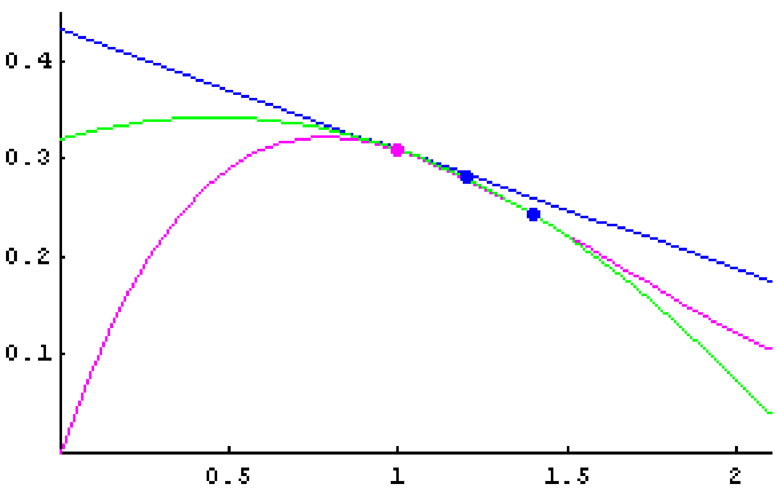
$$f[x] = e^{-x} Sin[x]$$

$$f'[x] = e^{-x} Cos[x] - e^{-x} Sin[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.5$$

$$f'[1.] \approx (-3f[1.] + 4f[1.+0.5] - f[1.+1.])/(1.) = -0.161455$$



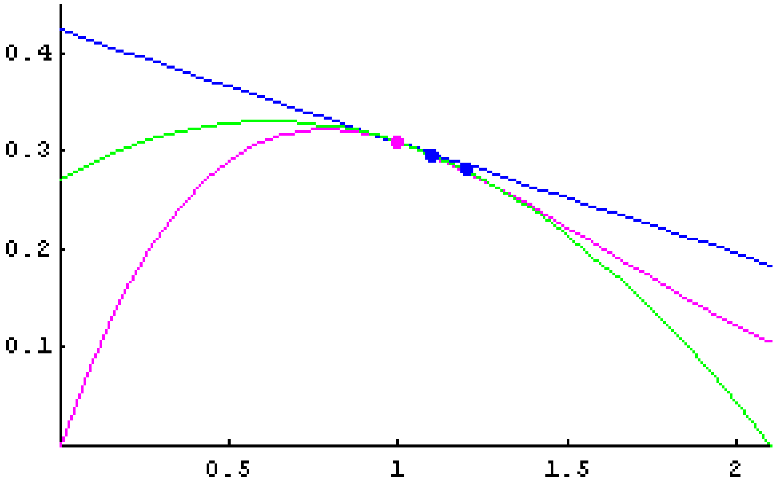
$$f[x] = e^{-x} Sin[x]$$

$$f'[x] = e^{-x} Cos[x] - e^{-x} Sin[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.2$$

$$f'[1.] \approx (-3f[1.] + 4f[1.+0.2] - f[1.+0.4])/(0.4) = -0.121974$$



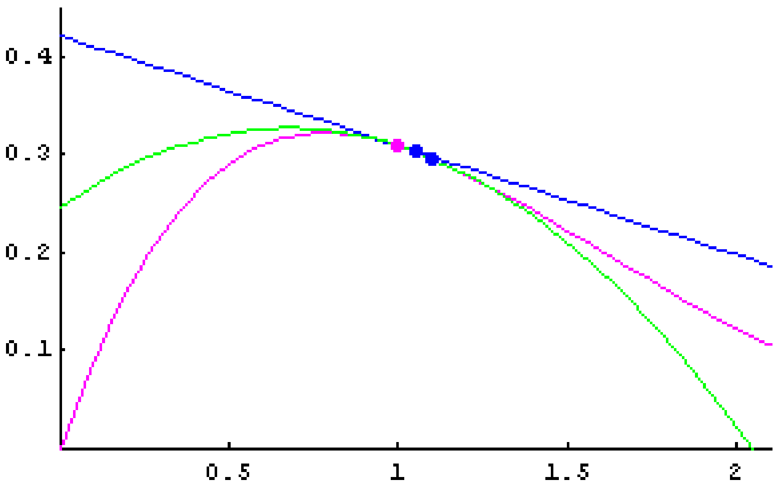
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.1$$

$$f'[1.] \approx (-3f[1.] + 4f[1.+0.1] - f[1.+0.2])/(0.2) = -0.113879$$



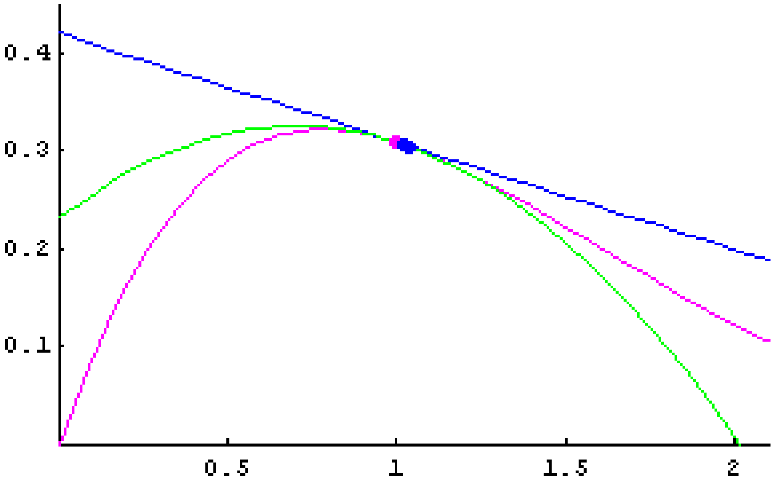
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.05$$

$$f'[1.] \approx (-3f[1.] + 4f[1.+0.05] - f[1.+0.1])/(0.1) = -0.111603$$



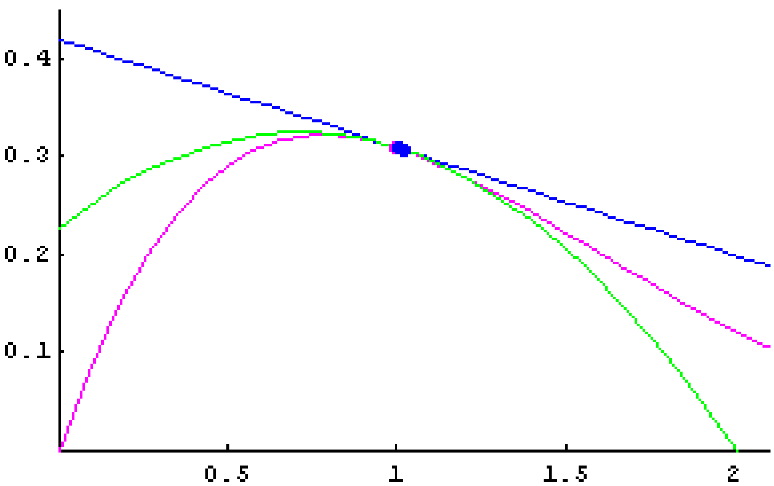
$f[x] = e^{-x} \mathrm{Sin}[x]$

$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.02$

$f'[1.] \approx (-3f[1.] + 4f[1.+0.02] - f[1.+0.04])/(0.04) = -0.110927$



$f[x] = e^{-x} \mathrm{Sin}[x]$

$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

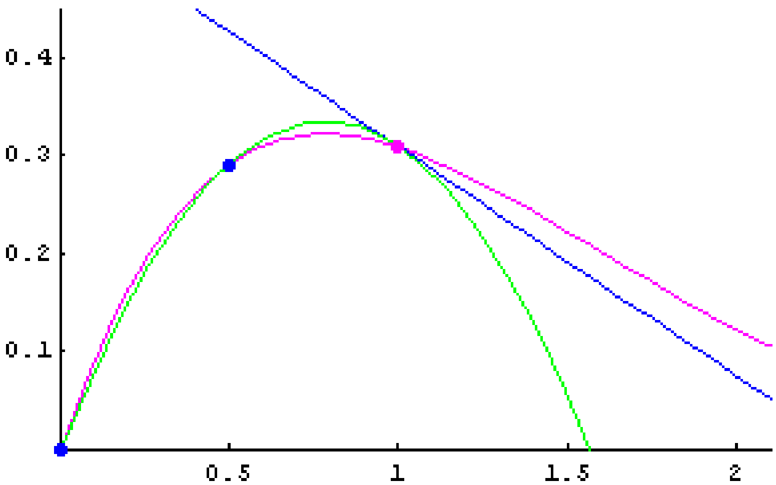
$f'[1.] = -0.110794$

$h = 0.01$

$f'[1.] \approx (-3f[1.] + 4f[1.+0.01] - f[1.+0.02])/(0.02) = -0.110827$

Example 7. Given $f[x] = e^{-x}\,\mathrm{Sin}[x]$, find numerical approximations to the derivative $f'[1.0]$, using three points and the backward difference formula.

Solution 7.



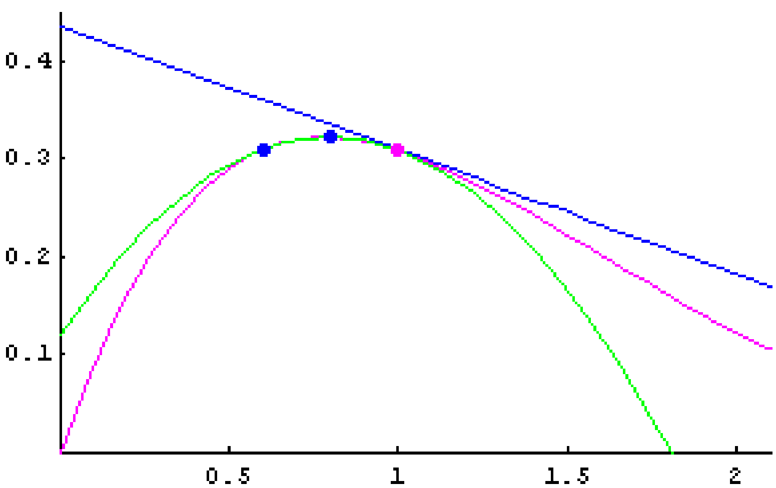
$f[x] = e^{-x}\,\mathrm{Sin}[x]$

$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.5$

$f'[1.] \approx (3f[1.] - 4f[1.-0.5] + f[1.-1.])/(1.) = -0.234466$



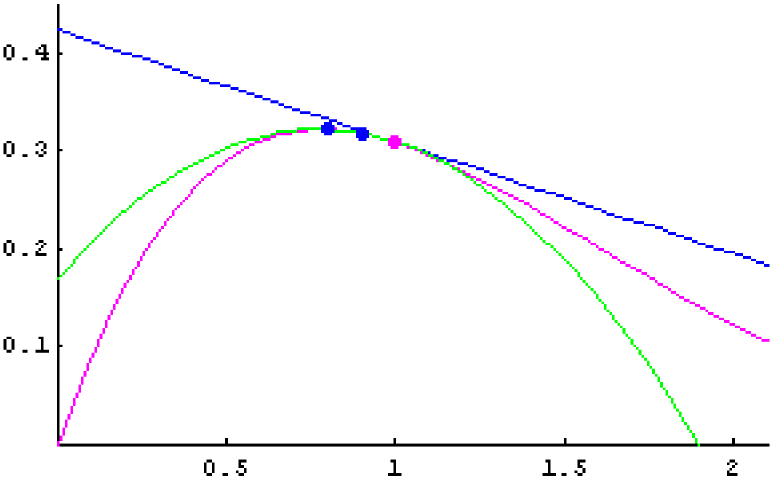
$f[x] = e^{-x}\,\mathrm{Sin}[x]$

$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.2$

$f'[1.] \approx (3f[1.] - 4f[1.-0.2] + f[1.-0.4])/(0.4) = -0.126884$



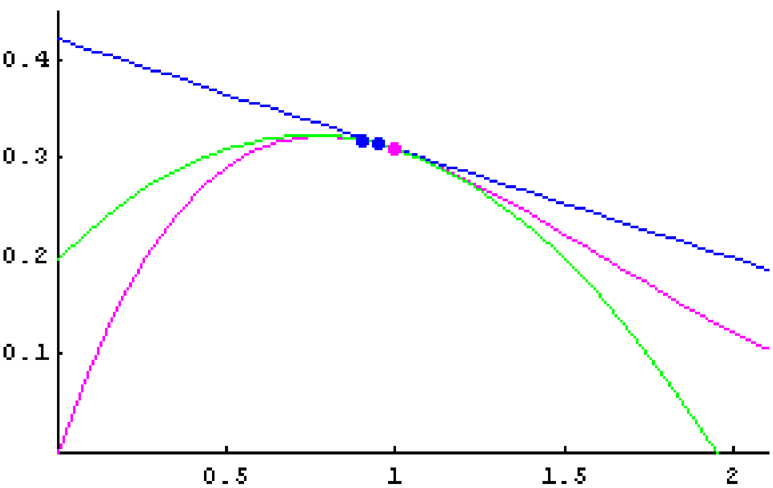
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.1$$

$$f'[1.] \approx (3f[1.] - 4f[1.-0.1] + f[1.-0.2])/(0.2) = -0.114497$$



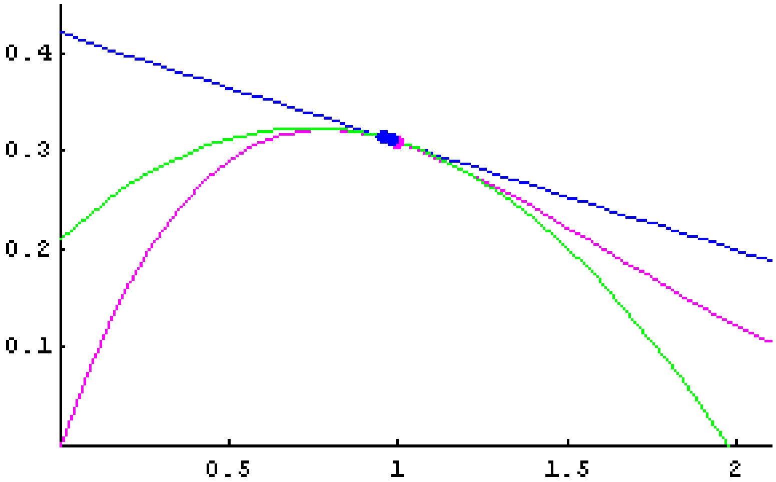
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.05$$

$$f'[1.] \approx (3f[1.] - 4f[1.-0.05] + f[1.-0.1])/(0.1) = -0.11168$$



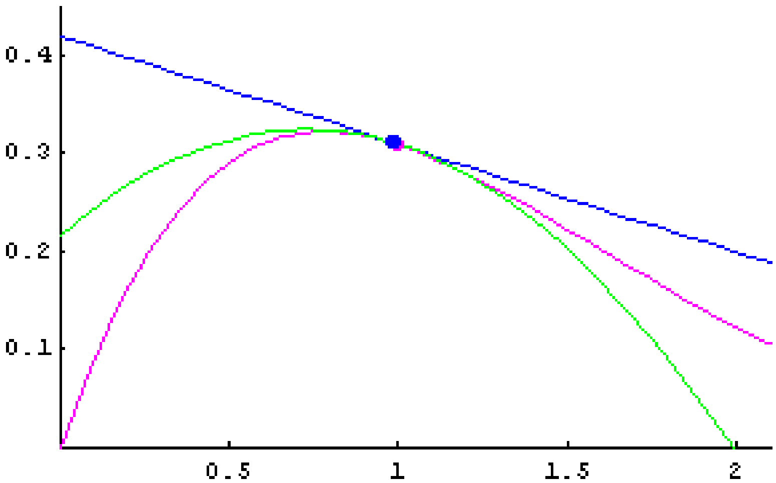
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

$$f'[1.] = -0.110794$$

$$h = 0.02$$

$$f'[1.] \approx (3f[1.] - 4f[1.-0.02] + f[1.-0.04])/(0.04) = -0.110932$$



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$

$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$

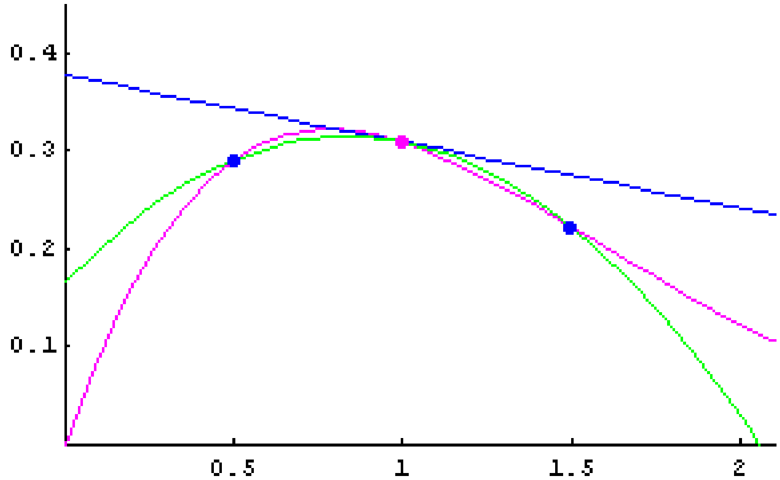
$$f'[1.] = -0.110794$$

$$h = 0.01$$

$$f'[1.] \approx (3f[1.] - 4f[1.-0.01] + f[1.-0.02])/(0.02) = -0.110828$$

Example 8. Given $f[x] = e^{-x} \mathrm{Sin}[x]$, find numerical approximations to the derivative $f'[1.0]$, using three points and the central difference formula.

Solution 8.



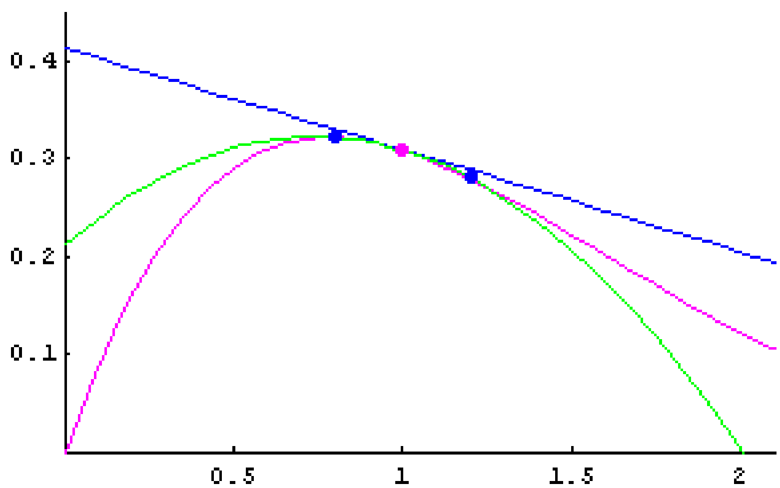
$f[x] = e^{-x} \mathrm{Sin}[x]$

$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.5$

$f'[1.] \approx (f[1.+0.5] - f[1.-0.5])/(1.) = -0.0682151$



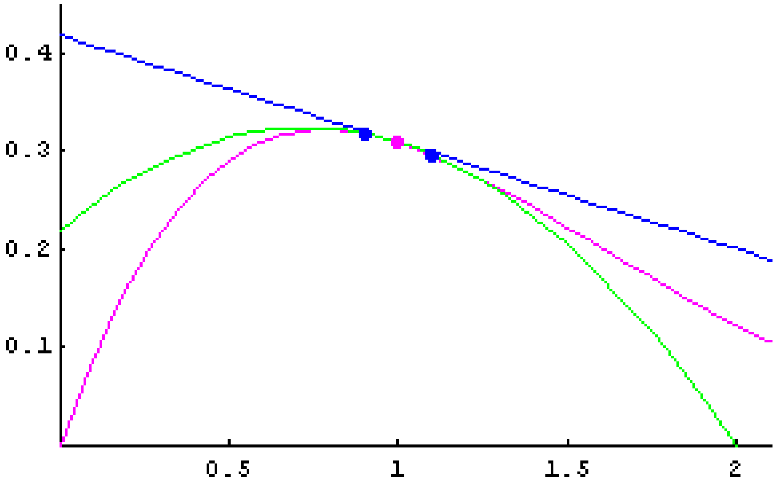
$f[x] = e^{-x} \mathrm{Sin}[x]$

$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.2$

$f'[1.] \approx (f[1.+0.2] - f[1.-0.2])/(0.4) = -0.10401$



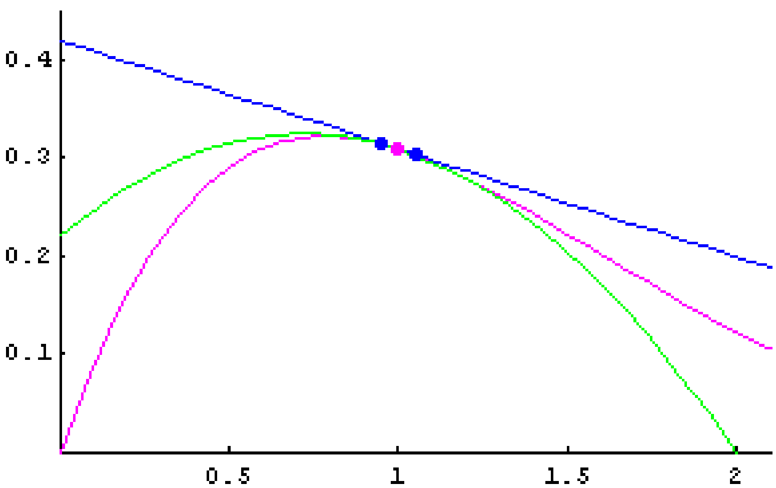
$f[x] = e^{-x} \text{Sin}[x]$

$f'[x] = e^{-x} \text{Cos}[x] - e^{-x} \text{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.1$

$f'[1.] \approx (f[1.+0.1] - f[1.-0.1])/(0.2) = -0.109099$



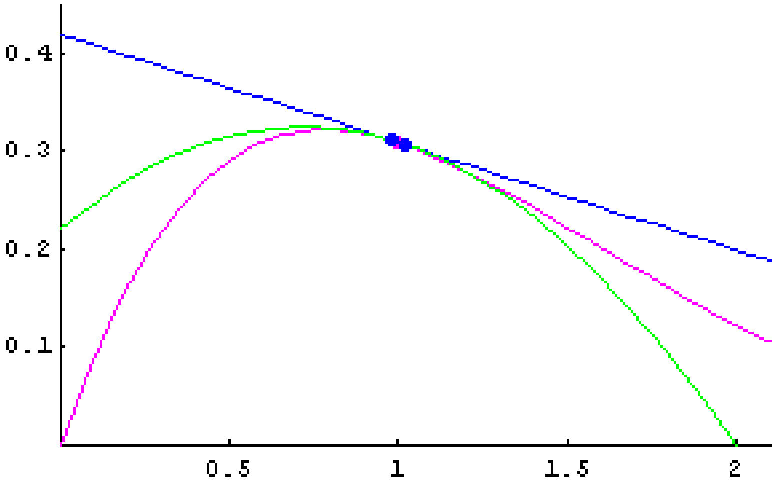
$f[x] = e^{-x} \text{Sin}[x]$

$f'[x] = e^{-x} \text{Cos}[x] - e^{-x} \text{Sin}[x]$

$f'[1.] = -0.110794$

$h = 0.05$

$f'[1.] \approx (f[1.+0.05] - f[1.-0.05])/(0.1) = -0.11037$



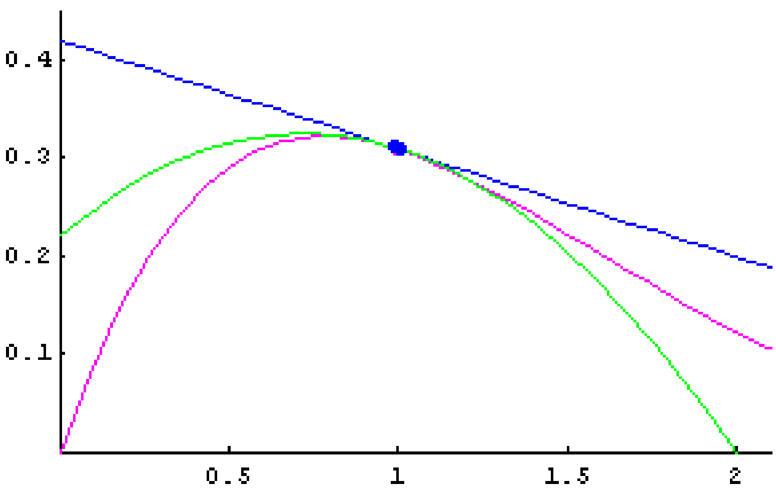
$f[x] = e^{-x}\,Sin[x]$

$f'[x] = e^{-x}\,Cos[x] - e^{-x}\,Sin[x]$

$f'[1.] = -0.110794$

$h = 0.02$

$f'[1.] \approx (f[1.+0.02] - f[1.-0.02])/(0.04) = -0.110726$



$f[x] = e^{-x}\,Sin[x]$

$f'[x] = e^{-x}\,Cos[x] - e^{-x}\,Sin[x]$

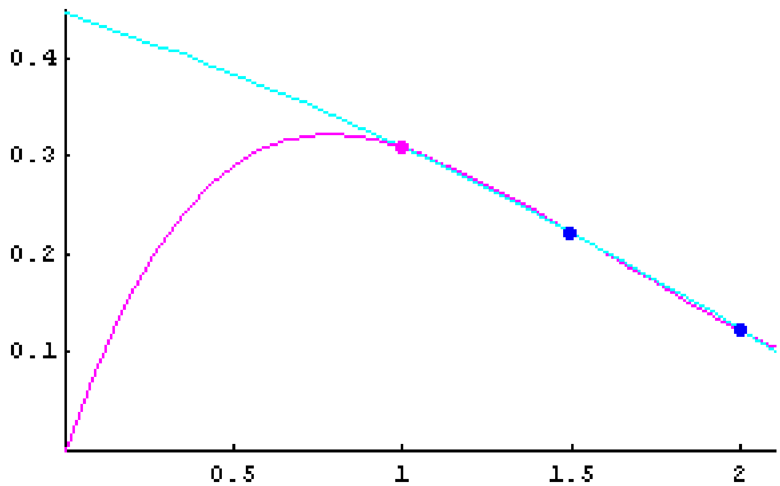
$f'[1.] = -0.110794$

$h = 0.01$

$f'[1.] \approx (f[1.+0.01] - f[1.-0.01])/(0.02) = -0.110777$

Example 9. Given $f[x] = e^{-x} \mathrm{Sin}[x]$, find numerical approximations to the second derivative $f''[1.0]$, using three points and the forward difference formula.

Solution 9.



$f[x] = e^{-x} \mathrm{Sin}[x]$

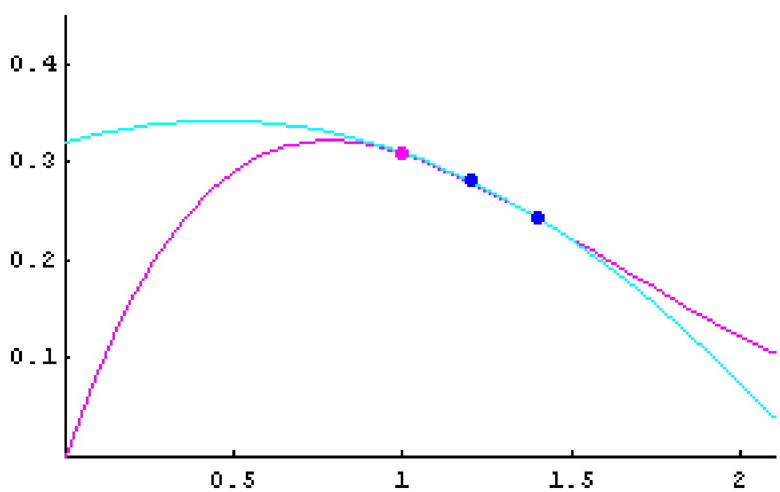
$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$

$f''[1.] = -0.397532$

$h = 0.5$

$f''[1.] \approx (f[1.+1.] - 2f[1.+0.5] + f[1.])/(0.25) = -0.0500901$



$f[x] = e^{-x} \mathrm{Sin}[x]$

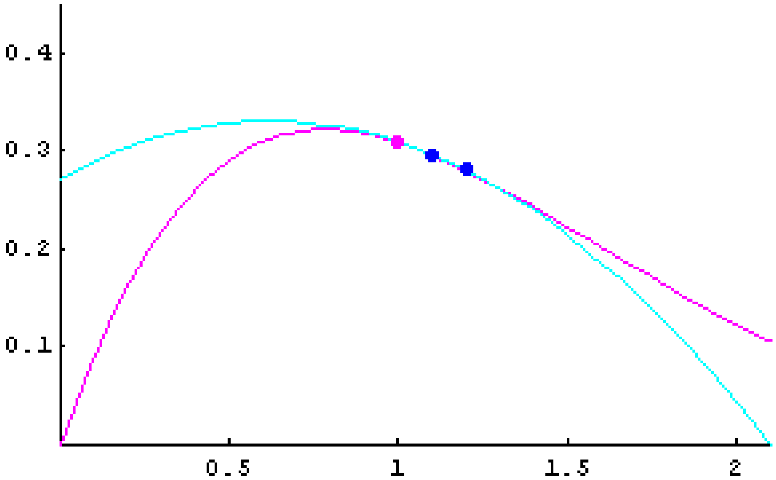
$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$

$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$

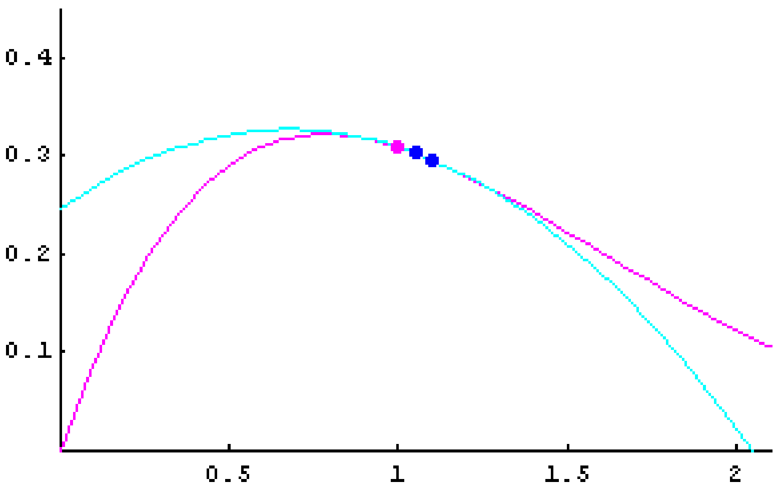
$f''[1.] = -0.397532$

$h = 0.2$

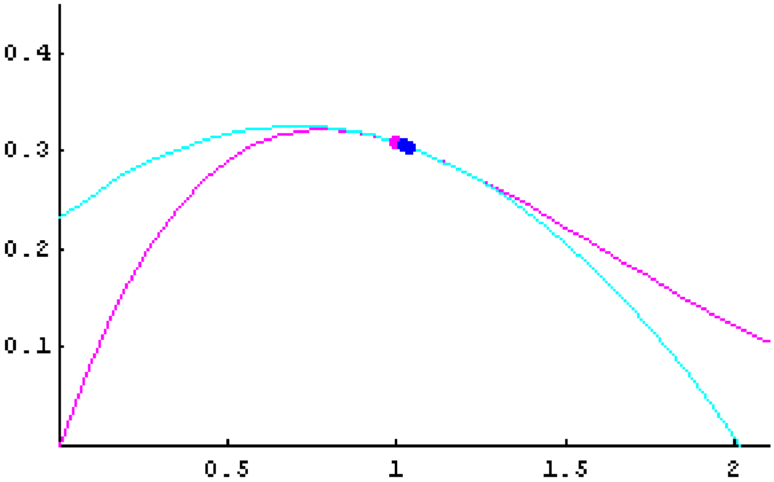
$f''[1.] \approx (f[1.+0.4] - 2f[1.+0.2] + f[1.])/(0.04) = -0.222019$



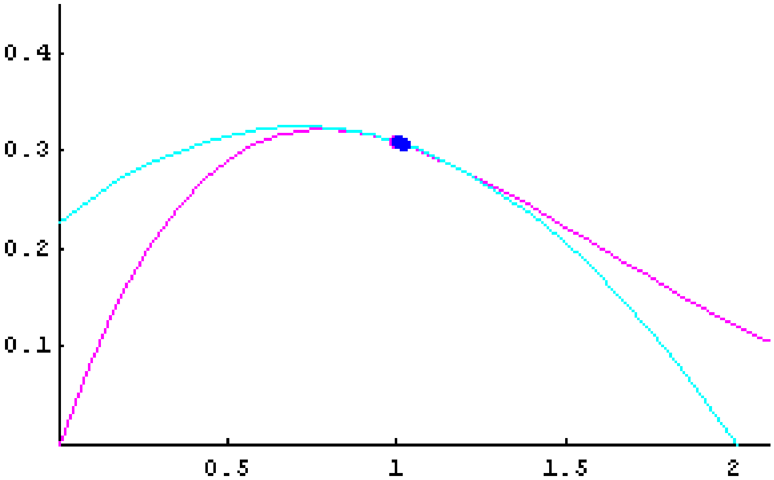
$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.1$$
$$f''[1.] \approx (f[1.+0.2] - 2f[1.+0.1] + f[1.])/(0.01) = -0.302967$$



$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.05$$
$$f''[1.] \approx (f[1.+0.1] - 2f[1.+0.05] + f[1.])/(0.0025) = -0.348491$$



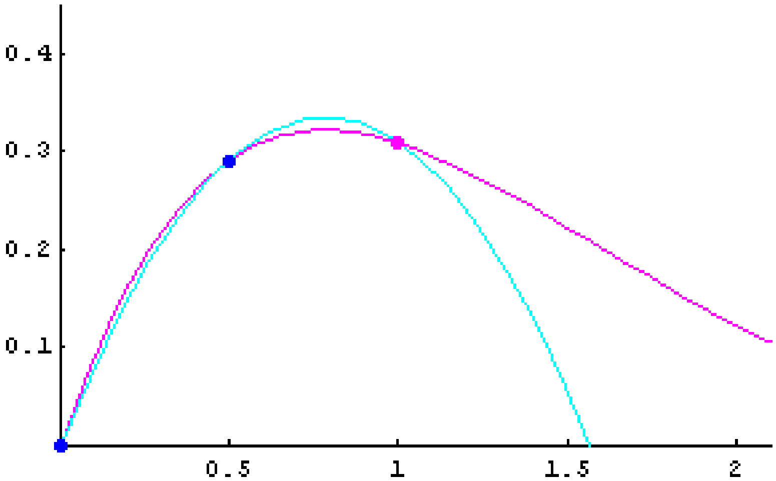
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.02$$
$$f''[1.] \approx (f[1.+0.04] - 2f[1.+0.02] + f[1.])/(0.0004) = -0.377487$$



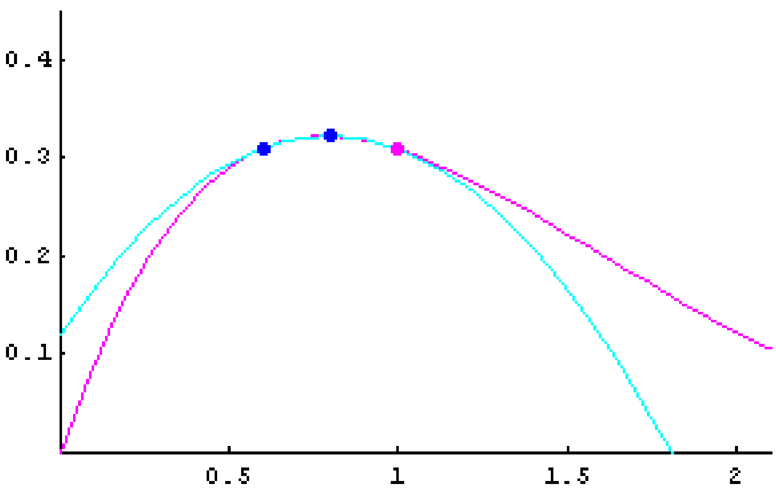
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.01$$
$$f''[1.] \approx (f[1.+0.02] - 2f[1.+0.01] + f[1.])/(0.0001) = -0.387438$$

Example 10. Given $f[x] = e^{-x} \mathrm{Sin}[x]$, find numerical approximations to the second derivative $f''[1.0]$, using three points and the backward difference formula.

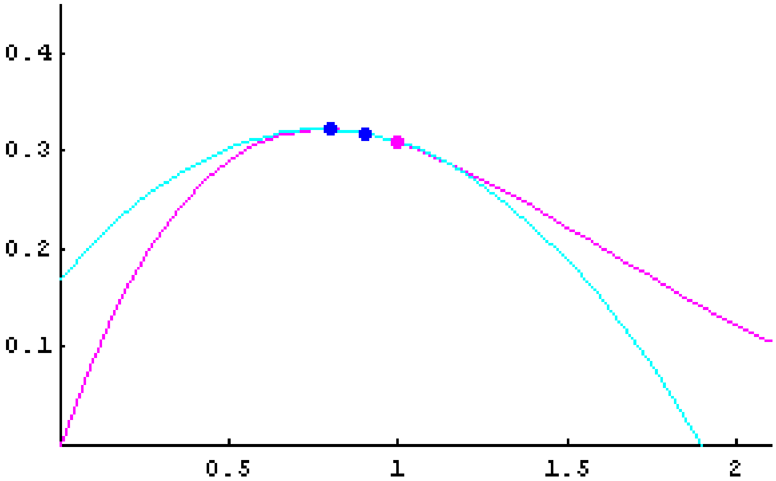
Solution 10.



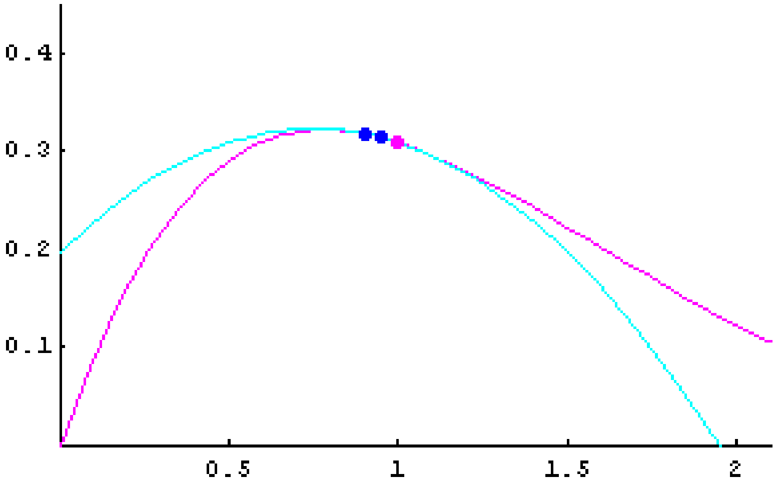
$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.5$$
$$f''[1.] \approx (f[1.-1.] - 2f[1.-0.5] + f[1.])/(0.25) = -1.08805$$



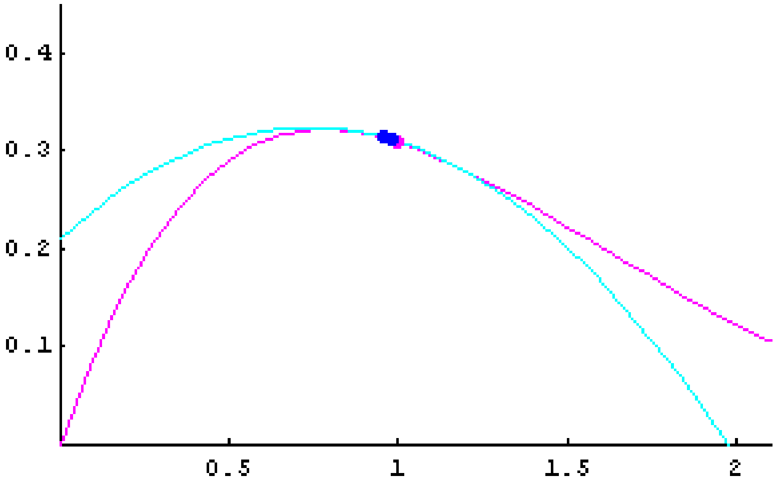
$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.2$$
$$f''[1.] \approx (f[1.-0.4] - 2f[1.-0.2] + f[1.])/(0.04) = -0.630388$$



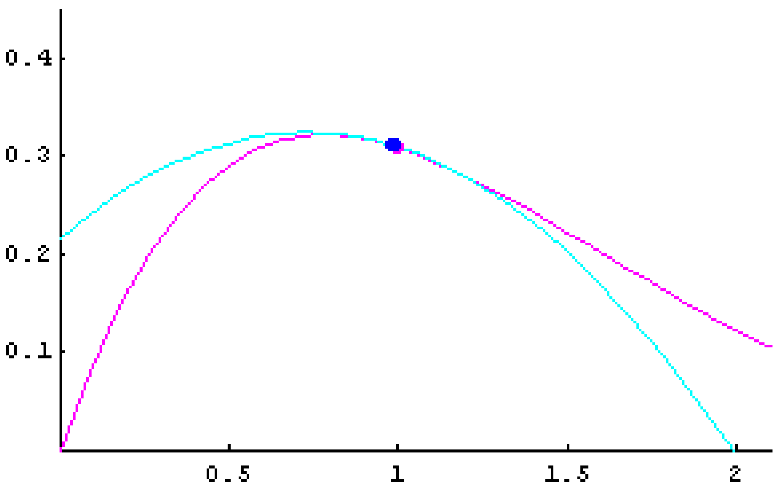
$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.1$$
$$f''[1.] \approx (f[1.-0.2] - 2f[1.-0.1] + f[1.])/(0.01) = -0.506517$$



$$f[x] = e^{-x} \mathrm{Sin}[x]$$
$$f'[x] = e^{-x} \mathrm{Cos}[x] - e^{-x} \mathrm{Sin}[x]$$
$$f''[x] = -2 e^{-x} \mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.05$$
$$f''[1.] \approx (f[1.-0.1] - 2f[1.-0.05] + f[1.])/(0.0025) = -0.450184$$



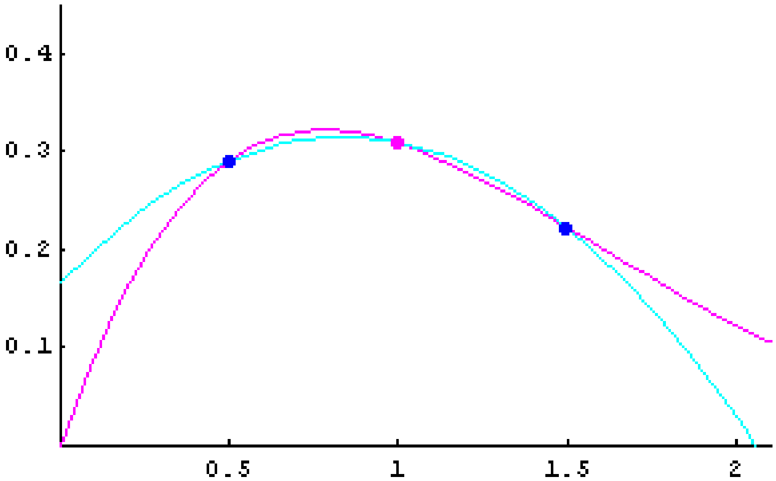
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.02$$
$$f''[1.] \approx (f[1.-0.04] - 2f[1.-0.02] + f[1.])/(0.0004) = -0.418155$$



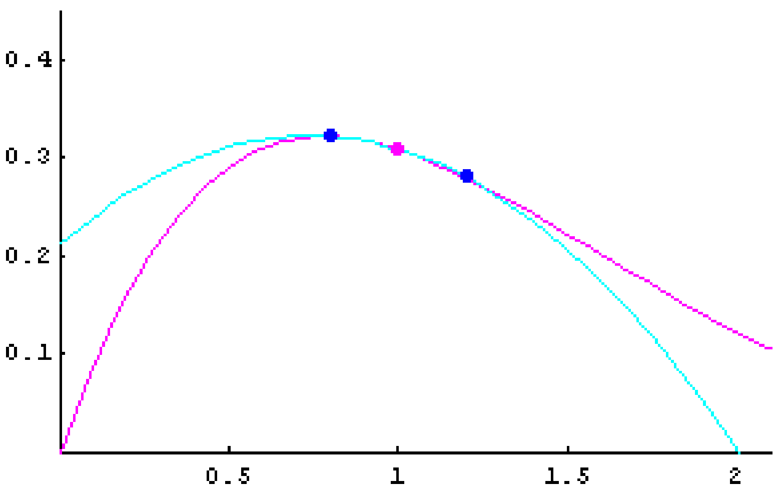
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.01$$
$$f''[1.] \approx (f[1.-0.02] - 2f[1.-0.01] + f[1.])/(0.0001) = -0.407771$$

Example 11. Given $f[x] = e^{-x} \text{Sin}[x]$, find numerical approximations to the second derivative $f''[1.0]$, using three points and the central difference formula.

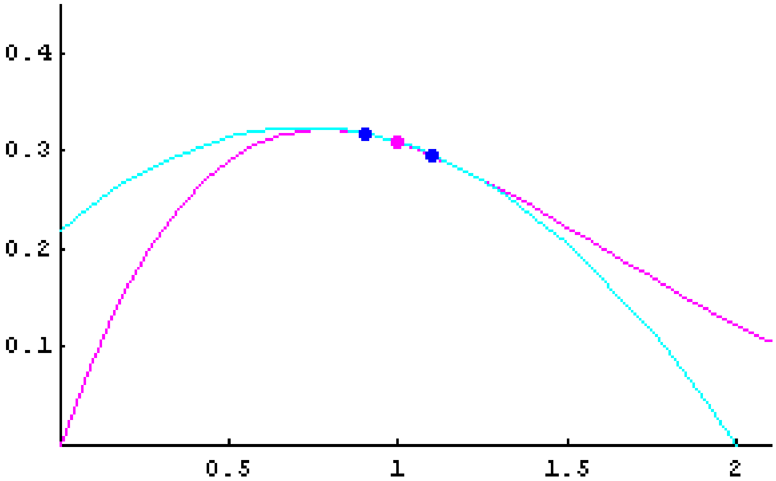
Solution 11.



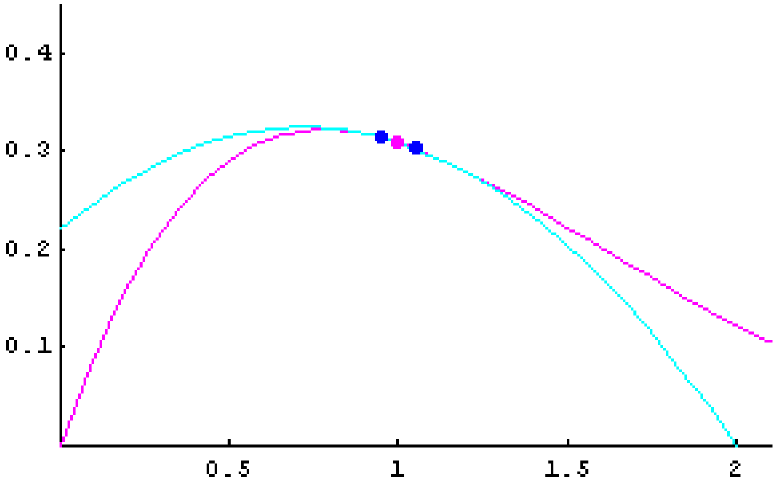
$$f[x] = e^{-x} \text{Sin}[x]$$
$$f'[x] = e^{-x} \text{Cos}[x] - e^{-x} \text{Sin}[x]$$
$$f''[x] = -2 e^{-x} \text{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.5$$
$$f''[1.] \approx (f[1.-0.5] - 2f[1.] + f[1.+0.5])/(0.25) = -0.423049$$



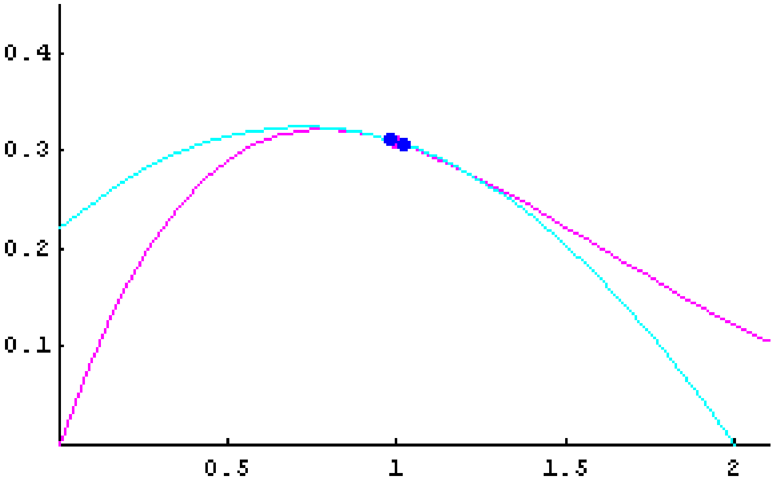
$$f[x] = e^{-x} \text{Sin}[x]$$
$$f'[x] = e^{-x} \text{Cos}[x] - e^{-x} \text{Sin}[x]$$
$$f''[x] = -2 e^{-x} \text{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.2$$
$$f''[1.] \approx (f[1.-0.2] - 2f[1.] + f[1.+0.2])/(0.04) = -0.401653$$



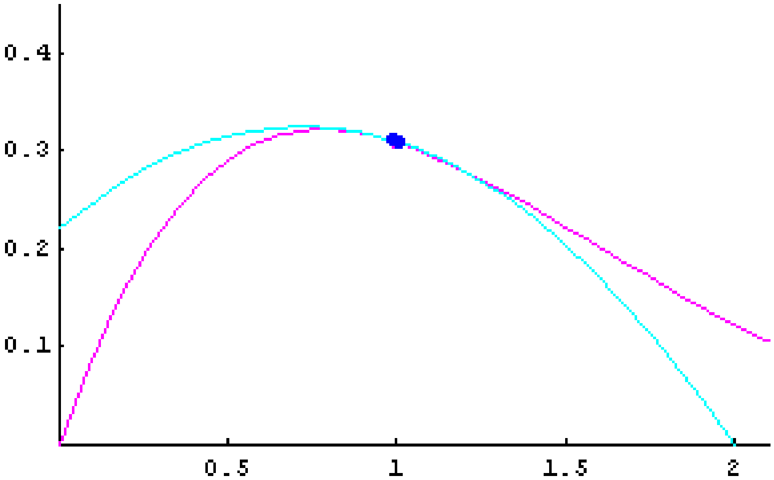
$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.1$$
$$f''[1.] \approx (f[1.-0.1] - 2f[1.] + f[1.+0.1])/(0.01) = -0.398564$$



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.05$$
$$f''[1.] \approx (f[1.-0.05] - 2f[1.] + f[1.+0.05])/(0.0025) = -0.39779$$



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.02$$
$$f''[1.] \approx (f[1.-0.02] - 2f[1.] + f[1.+0.02])/(0.0004) = -0.397573$$



$$f[x] = e^{-x}\,\mathrm{Sin}[x]$$
$$f'[x] = e^{-x}\,\mathrm{Cos}[x] - e^{-x}\,\mathrm{Sin}[x]$$
$$f''[x] = -2\,e^{-x}\,\mathrm{Cos}[x]$$
$$f''[1.] = -0.397532$$
$$h = 0.01$$
$$f''[1.] \approx (f[1.-0.01] - 2f[1.] + f[1.+0.01])/(0.0001) = -0.397543$$