1. Euler's Method for O.D.E.'s

The first method we shall study for solving differential equations is called Euler's method, it serves to illustrate the concepts involved in the advanced methods. It has limited use because of the larger error that is accumulated with each successive step. However, it is important to study Euler's method because the remainder term and error analysis is easier to understand.

Theorem (Euler's Method) Assume that $f(t,y)$ is continuous and satisfies a Lipschits condition in the variable y, and consider the I. V. P. (initial value problem)

$y' = f(t, y)$ with $y(a) = t_0 = \alpha$, over the interval $a \le t \le b$.

Euler's method uses the formulas $t_{k+1} = t_k + h$, and

$y_{k+1} = y_k + h f(t_k, y_k)$ for $k = 0, 1, 2, \ldots, m-1$

as an approximate solution to the differential equation using the discrete set of points $\{(t_k, y_k)\}_{k=0}^{m}$.

Error analysis for Euler's Method

When we obtained the formula $y_{k+1} = y_k + h f(t_k, y_k)$ for Euler's method, the neglected term for each step has the form $\frac{y^{(2)}(c_k)}{2} h^2$. If this was the only error at each step, then at the end of the interval $[a, b]$, after m steps have been made, the accumulated error would be

$$\sum_{k=1}^{m} \frac{y^{(2)}(c_k)}{2} h^2 = m \frac{y^{(2)}(c)}{2} h^2 = \frac{h\, m\, y^{(2)}(c)}{2} h = \frac{(b-a)\, y^{(2)}(c)}{2} h = O(h^1).$$

The error is more complicated, but this estimate predominates.

Theorem (Precision of Euler's Method) Assume that $y = y(t)$ is the solution to the I.V.P. $y' = f(t, y)$ with $y(t_0) = y_0$. If $y(t) \in C^2[t_0, b]$ and $\{(t_k, y_k)\}_{k=0}^{m}$ is the sequence of approximations generated by Euler's method, then at each step, the local trunctaion error is of the order $O(h^2)$, and the overall global truncation error e_k is of the order

$|e_k| = |y(t_k) - y_k| = O(h^1)$, for $k = 1, 2, \ldots, m$.

The error at the right end of the interval is called the final global error

$E(y(b), h) = |y(b) - y_m| = O(h^1)$.

Remark. The global truncation error $E(y(b), h)$ is used to study the behavior of the error for various step sizes. It can be used to give us an idea of how much computing effort must be done to obtain an accurate approximation.

Algorithm (Modified Euler's Method). To approximate the solution of the initial value problem $y' = f(t, y)$ with $y(a) = y_0 = \alpha$ over $[t_0, b] = [a, b]$ at a discrete set of points using the formulas

$t_{k+1} = t_k + h$, and $y_{k+1} = y_k + h f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f(t_k, y_k)\right)$ for $k = 0, 1, 2, \ldots, m-1$

Example 1. Solve the I.V.P. $y' = 1 - t y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

Example 2. Solve $y' = 30 - 5y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 2.

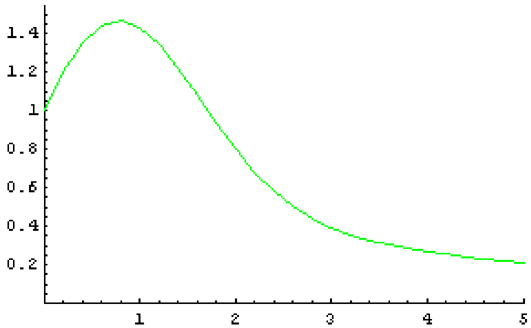
Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

Example 1. Solve the I.V.P. $y' = 1 - t\,y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 1.

Compute the Euler and modified Euler solutions based on 25 subintervals and plot the results.

Find numerical solutions to the D.E.
$y' = 1 - t\,y$

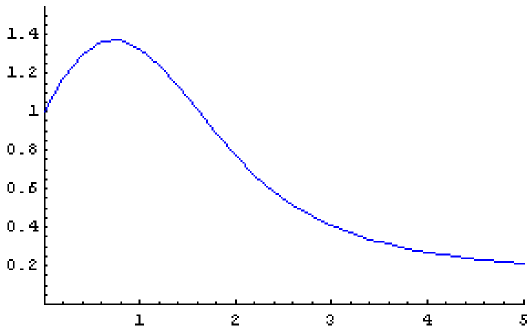
First, find Euler's solution.



The Euler solution for $y' = 1 - t\,y$
Using $n = 26$ points.
{{0., 1.}, {0.2, 1.2}, {0.4, 1.352}, {0.6, 1.44384}, {0.8, 1.47058}, {1., 1.43529}, {1.2, 1.34823}, {1.4, 1.22465}, {1.6, 1.08175}, {1.8, 0.935591}, {2., 0.798778}, {2.2, 0.679267}, {2.4, 0.580389}, {2.6, 0.501803}, {2.8, 0.440865}, {3., 0.393981}, {3.2, 0.357592}, {3.4, 0.328733}, {3.6, 0.305195}, {3.8, 0.285454}, {4., 0.268509}, {4.2, 0.253702}, {4.4, 0.240592}, {4.6, 0.228871}, {4.8, 0.21831}, {5., 0.208732}}

The final value is $y(5) = y_{26} = 0.208732$

Second, find the modified Euler solution.

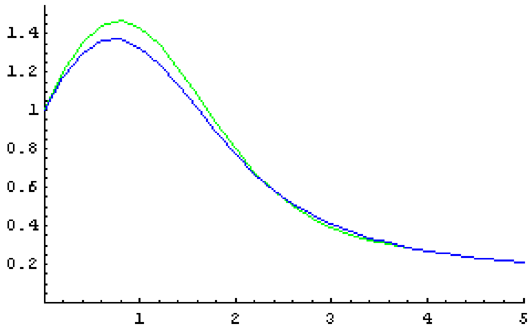


The modified Euler solution for $y' = 1 - t\,y$
Using $n = 26$ points.

{{0., 1.}, {0.2, 1.178}, {0.4, 1.30273}, {0.6, 1.36767}, {0.8, 1.37369}, {1., 1.3282}, {1.2, 1.24322}, {1.4, 1.13277}, {1.6, 1.01052}, {1.8, 0.887912}, {2., 0.773239}, {2.2, 0.671431}, {2.4, 0.584521}, {2.6, 0.512403}, {2.8, 0.453647}, {3., 0.406204}, {3.2, 0.367911}, {3.4, 0.336793}, {3.6, 0.311195}, {3.8, 0.289813}, {4., 0.271659}, {4.2, 0.256003}, {4.4, 0.242309}, {4.6, 0.230185}, {4.8, 0.219343}, {5., 0.209566}}

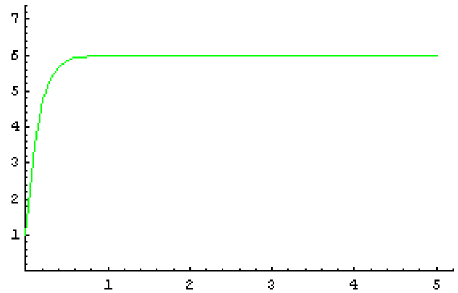
The final value is $y(5) = y_{26} = 0.209566$

Just for fun, plot both the Euler solution and the modified Euler solution. Notice that there is a difference.

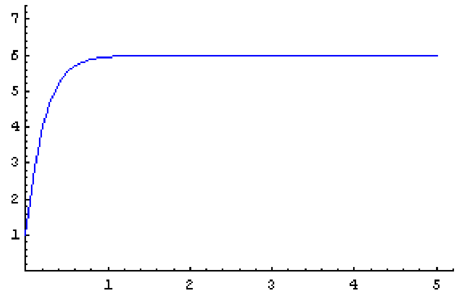


The Euler and modified Euler solutions for $y' = 1 - t y$

Example 2. Solve $y' = 30 - 5y$ with $y(0) = 1$ over $0 \le t \le 5$.
Solution 2.

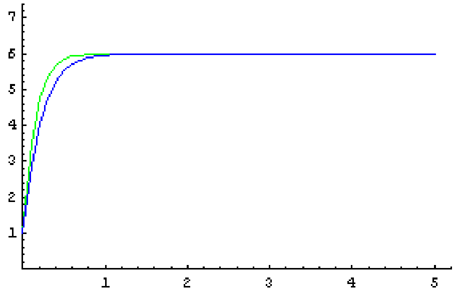


The Euler solution for $y' = 30 - 5y$
Using n = 51 points.
{{0., 1}, {0.1, 3.5}, {0.2, 4.75}, {0.3, 5.375}, {0.4, 5.6875}, {0.5, 5.84375}, {0.6, 5.92188}, {0.7, 5.96094}, {0.8, 5.98047}, {0.9, 5.99023}, {1., 5.99512}, {1.1, 5.99756}, {1.2, 5.99878}, {1.3, 5.99939}, {1.4, 5.99969}, {1.5, 5.99985}, {1.6, 5.99992}, {1.7, 5.99996}, {1.8, 5.99998}, {1.9, 5.99999}, {2., 6.}, {2.1, 6.}, {2.2, 6.}, {2.3, 6.}, {2.4, 6.}, {2.5, 6.}, {2.6, 6.}, {2.7, 6.}, {2.8, 6.}, {2.9, 6.}, {3., 6.}, {3.1, 6.}, {3.2, 6.}, {3.3, 6.}, {3.4, 6.}, {3.5, 6.}, {3.6, 6.}, {3.7, 6.}, {3.8, 6.}, {3.9, 6.}, {4., 6.}, {4.1, 6.}, {4.2, 6.}, {4.3, 6.}, {4.4, 6.}, {4.5, 6.}, {4.6, 6.}, {4.7, 6.}, {4.8, 6.}, {4.9, 6.}, {5., 6.}}
The final value is $y(5) = y_{51} = 6.$



The modified Euler solution for $y' = 30 - 5y$
Using n = 51 points.
{{0., 1}, {0.1, 2.875}, {0.2, 4.04688}, {0.3, 4.7793}, {0.4, 5.23706}, {0.5, 5.52316}, {0.6, 5.70198}, {0.7, 5.81374}, {0.8, 5.88358}, {0.9, 5.92724}, {1., 5.95453}, {1.1, 5.97158}, {1.2, 5.98224}, {1.3, 5.9889}, {1.4, 5.99306}, {1.5, 5.99566}, {1.6, 5.99729}, {1.7, 5.99831}, {1.8, 5.99894}, {1.9, 5.99934}, {2., 5.99959}, {2.1, 5.99974}, {2.2, 5.99984}, {2.3, 5.9999}, {2.4, 5.99994}, {2.5, 5.99996}, {2.6, 5.99998}, {2.7, 5.99998}, {2.8, 5.99999}, {2.9, 5.99999}, {3., 6.}, {3.1, 6.}, {3.2, 6.}, {3.3, 6.}, {3.4, 6.}, {3.5, 6.}, {3.6, 6.}, {3.7, 6.}, {3.8, 6.}, {3.9, 6.}, {4., 6.}, {4.1, 6.}, {4.2, 6.}, {4.3, 6.}, {4.4, 6.}, {4.5, 6.}, {4.6, 6.}, {4.7, 6.}, {4.8, 6.}, {4.9, 6.}, {5., 6.}}
The final value is $y(5) = y_{51} = 6.$

Just for fun, plot both the Euler solution and the modified Euler solution. Notice that there is a difference.



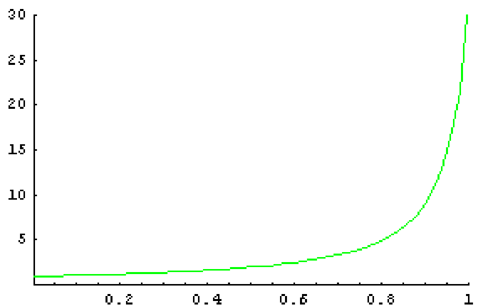
The Euler and modified Euler solutions for $y' = 30 - 5y$

Example 3. Solve the I.V.P. $y' = t^2 + y^2$ with $y(0) = 1$ over $0 \le t \le 1$.
Solution 3.

Compute the Euler and modified Euler solutions based on 50 subintervals and plot the results.
Note, for this example we will extend one subinterval more to the right of t=1.

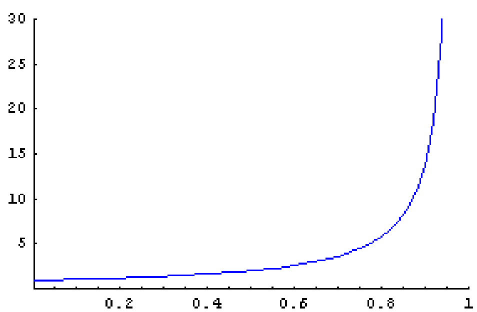
Find numerical solutions to the D.E.
$y' = t^2 + y^2$

First, find Euler's solution.



The Euler solution for $y' = t^2 + y^2$
Using n = 51 points.
{{0., 1.}, {0.02, 1.02}, {0.04, 1.04082}, {0.06, 1.06251}, {0.08, 1.08516}, {0.1, 1.10884}, {0.12, 1.13364}, {0.14, 1.15963}, {0.16, 1.18691}, {0.18, 1.2156}, {0.2, 1.2458}, {0.22, 1.27764}, {0.24, 1.31126}, {0.26, 1.3468}, {0.28, 1.38443}, {0.3, 1.42433}, {0.32, 1.4667}, {0.34, 1.51177}, {0.36, 1.55979}, {0.38, 1.61105}, {0.4, 1.66584}, {0.42, 1.72454}, {0.44, 1.78755}, {0.46, 1.85533}, {0.48, 1.92841}, {0.5, 2.00739}, {0.52, 2.09298}, {0.54, 2.186}, {0.56, 2.28741}, {0.58, 2.39832}, {0.6, 2.52009}, {0.62, 2.65431}, {0.64, 2.8029}, {0.66, 2.96822}, {0.68, 3.15314}, {0.7, 3.36123}, {0.72, 3.59699}, {0.74, 3.86613}, {0.76, 4.17602}, {0.78, 4.53635}, {0.8, 4.96009}, {0.82, 5.46494}, {0.84, 6.0757}, {0.86, 6.82809}, {0.88, 7.77534}, {0.9, 8.99995}, {0.92, 10.6361}, {0.94, 12.9156}, {0.96, 16.2695}, {0.98, 21.5819}, {1., 30.9167}, {1.02, 50.0536}}
The final value is $y(5) = y_{52} = 50.0536$

Second, find the modified Euler solution.
Note, for this example we will extend one subinterval more to the right of t=1.

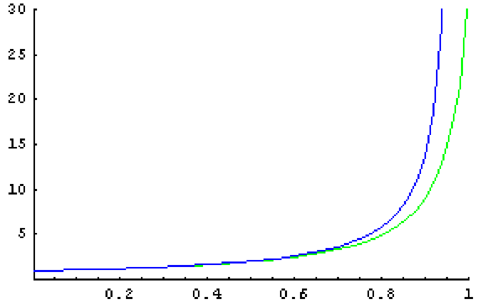


The modified Euler solution for $y' = t^2 + y^2$
Using n = 51 points.

{{0., 1.}, {0.02, 1.0204}, {0.04, 1.04167}, {0.06, 1.06388}, {0.08, 1.0871}, {0.1, 1.11142}, {0.12, 1.13692}, {0.14, 1.16371}, {0.16, 1.19189}, {0.18, 1.22157}, {0.2, 1.25289}, {0.22, 1.28598}, {0.24, 1.32099}, {0.26, 1.3581}, {0.28, 1.39749}, {0.3, 1.43938}, {0.32, 1.48399}, {0.34, 1.53159}, {0.36, 1.58248}, {0.38, 1.63698}, {0.4, 1.69548}, {0.42, 1.75841}, {0.44, 1.82627}, {0.46, 1.89963}, {0.48, 1.97915}, {0.5, 2.06561}, {0.52, 2.15992}, {0.54, 2.26316}, {0.56, 2.3766}, {0.58, 2.50181}, {0.6, 2.64064}, {0.62, 2.79539}, {0.64, 2.96891}, {0.66, 3.16478}, {0.68, 3.38752}, {0.7, 3.64301}, {0.72, 3.93895}, {0.74, 4.2857}, {0.76, 4.69743}, {0.78, 5.19418}, {0.8, 5.80509}, {0.82, 6.57429}, {0.84, 7.57177}, {0.86, 8.91537}, {0.88, 10.8191}, {0.9, 13.7138}, {0.92, 18.5991}, {0.94, 28.3553}, {0.96, 54.8789}, {0.98, 199.415}, {1., 7329.66}, {1.02, 5.93112×10^9}}

The final value is $y(5) = y_{52} = 5.93112\times 10^9$

Just for fun, plot both the Euler solution and the modified Euler solution. Notice that there is a difference.



The Euler and modified Euler solutions for $y' = t^2 + y^2$