part II. PROVE THAT THE BOX-MULLER ALGORITHM BELOW GENERATES A NORMALLY DISTRIBUTED RANDOM NUMBER, FOLLOWING THE LECTURE SLIDES ON "MONTE CARLO BASICS".

$$p(\zeta) = \sqrt{\frac{1}{2\pi}} \exp\left(-\zeta^2/2\right)$$

(a) Generate uniform random numbers r_1, r_2 in the range $(0,1)$

(b) Calculate $\zeta_1 = (-2\ln(r_1))^{1/2} \cos(2\pi r_2)$

$\zeta_2 = (-2\ln(r_1))^{1/2} \sin(2\pi r_2)$

Let r_1 and r_2 be uniform random numbers in the range between [0,1]
then it follows that there are a total Number of samples:

$$\underbrace{NP(r_1,r_2)}_{1}\underbrace{dr_1 dr_2}_{\text{rectangle}} = N \times \underbrace{P'(\zeta_1,\zeta_2)}_{\text{density}}\underbrace{S(r_1,r_2,dr_1,dr_2)}_{\text{area}} \qquad (1)$$

in a small rectangle of size $dr_1 dr_2$ around the point r_1 and r_2
there are a total number of samples

$$S = \left|\frac{\partial(\zeta_1,\zeta_2)}{\partial(r_1,r_2)}\right| dr_1 dr_2 \quad \text{(proven in lecture notes)} \qquad (2)$$

Now, let us transform each point at each of these end point
from r_1 and r_2 coordinate to this ζ_1 and ζ_2 coordinate

Now, lets substitute eqn (2) into (1) N and $dr_1 dr_2$ cancels; so we end up with:

$$P(r_1 r_2) = P'(\zeta_1,\zeta_2)\left|\frac{\partial(\zeta_1,\zeta_2)}{\partial(r_1,r_2)}\right| \quad \text{then solving for } P'(\zeta_1,\zeta_2) \text{ we get:}$$

$$\therefore \; P'(\zeta_1,\zeta_2) = \left|\frac{\partial(\zeta_1,\zeta_2)}{\partial(r_1,r_2)}\right|^{-1} P(r_1,r_2) = \left|\frac{\partial(r_1,r_2)}{\partial(\zeta_1,\zeta_2)}\right| \quad \text{(see lecture notes for proof)} \qquad (3)$$

$$\left|\frac{\partial(r_1,r_2)}{\partial(\zeta_1,\zeta_2)}\right| = \det\begin{vmatrix} \frac{\partial r_1}{\partial \zeta_1} & \frac{\partial r_2}{\partial \zeta_1} \\ \frac{\partial r_1}{\partial \zeta_2} & \frac{\partial r_2}{\partial \zeta_2} \end{vmatrix}$$

To take the derivatives we need to express r_1 and r_2 in terms of ζ_1 and ζ_2:

by squaring ζ_1 and ζ_2 we find that:

$$\zeta_1^2 + \zeta_2^2 = \left(\sqrt{-2\ln(r_1)}\cos(2\pi r_2)\right)^2 + \left(\sqrt{-2\ln(r_1)}\sin(2\pi r_1)\right)^2 = 2\ln(r_1)\left(\cos^2(2\pi r_2) + \sin^2(2\pi r_2)\right)$$

$$= -2\ln(r_1) \Rightarrow \quad r_1 = e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)} \qquad (4)$$

so that $\frac{\partial r_1}{\partial \zeta_1}$

And to find r_2 in terms of ζ_1 and ζ_2, notice that the ratio:

$$\frac{\zeta_2}{\zeta_1} = \frac{-2\ln(r_1)\sin(2\pi r_2)}{\cancel{-2\ln(r_1)}\cos(2\pi r_2)} = \tan(2\pi r_2)$$ From which we can we get $\frac{1}{\cos^2(2\pi r_2)} = 1+\tan^2(2\pi r_2) = 1+\left(\frac{\zeta_2}{\zeta_1}\right)^2$

then we can get $d\zeta_1$ and $d\zeta_2$ by:

$$\boxed{-\frac{\zeta_2}{\zeta_1} d\zeta_1 = \frac{2\pi dr_2}{\cos^2(2\pi r_2)}} \quad \text{and} \quad \boxed{\frac{d\zeta_2}{\zeta_1} = \frac{2\pi dr_2}{\cos^2(2\pi r_2)}} \qquad (5)$$

So, from (4) we have $\frac{\partial r_1}{\partial \zeta_1} = -\zeta_1 e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)}$ and $\frac{\partial r_1}{\partial \zeta_2} = -\zeta_2 e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)}$ (6)

From (5) we get $$\frac{\partial r_2}{\partial \zeta_1} = -\frac{\zeta_2}{\zeta_1}\cos^2(2\pi r_2) = -\frac{\zeta_2}{\zeta_1}\left(1+\tan^2(2\pi r_2)\right)^{-1} = -\frac{\zeta_2}{\zeta_1}\frac{1}{1+\left(\frac{\zeta_2}{\zeta_1}\right)^2} \qquad (7)$$

Finally, $$\frac{\partial r_2}{\partial \zeta_2} = \frac{1}{2\pi\zeta_1}\cos^2(2\pi r_2) = \frac{1}{2\pi\zeta_1}\frac{1}{1+(\zeta_2/\zeta_1)^2} \qquad (8)$$

Now we can compute the jacobian in equation (3) using (6), (7) and (8):

$$P'(\zeta_1,\zeta_2) = \left|\frac{\partial r_1}{\partial \zeta_1}\frac{\partial r_2}{\partial \zeta_2} - \frac{\partial r_1}{\partial \zeta_2}\frac{\partial r_2}{\partial \zeta_1}\right| = \left|\left(-\zeta_1 e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)}\right)\left(\frac{1}{2\pi\zeta_1}\frac{1}{1+\left(\frac{\zeta_2}{\zeta_1}\right)^2}\right) - \left(-\zeta_2 e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)}\right)\left(-\frac{\zeta_2}{\zeta_1}\frac{1}{\left(1+\left(\frac{\zeta_2}{\zeta_1}\right)^2\right)}\right)\right|$$

$$= \left|-e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)}\left(\frac{1}{2\pi\left(1+\left(\frac{\zeta_2}{\zeta_1}\right)^2\right)} + \frac{\zeta_2}{\zeta_1\left(1+\frac{\zeta_2^2}{\zeta_1^2}\right)}\right)\frac{\zeta_1^2}{\zeta_1^2}\right| = \frac{e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)}}{2\pi}\left(\frac{\zeta_1^2}{(\zeta_1^2+\zeta_2^2)} + \frac{\zeta_2\zeta_1}{(\zeta_1^2+\zeta_2^2)}\right)$$

$$\boxed{= \frac{e^{-\frac{1}{2}(\zeta_1^2+\zeta_2^2)}}{2\pi}}$$ Which is a normal distribution

Which can be further expanded into a product of factors, each depending only on one variable :

$$P'(\xi_1, \xi_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(\xi_1^2 + \xi_2^2)} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi_2^2} = P'(\xi_1) P'(\xi_2) \qquad (10)$$

Which means that the expectation value of the two variable functon is the same as product of the two independently computed expectation values for the two single variable functions i.e :

$$\langle A B(\cdot) \rangle = \int d\xi_1 \int d\xi_2 \, P'(\xi_1, \xi_2) \, A(\xi_1) B(\xi_2) = \int d\xi_1 \, P'(\xi_1) A(\xi_1) \int d\xi_2 \, P'(\xi_2) B(\xi_2)$$

$$= \langle A \rangle \langle B \rangle$$

here, they are uncorribted where each of them follow a normal distribution, independenty :

$$P'(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}$$