Math 630 Notes

Survival Models

- Furure Life Time T_x and Its Distribution
  - (x): a life aged x
  - T_x: the future life time of (x)
    Example: Consider a life (55). If $T_{55} = 30$, then (55) dies at age $55 + T_{55} = 85$.
  - The CDF of T_x: $F_x(t) = P(T_x \le t)$, the probability that (x) dies within t years
  - Survival function of (x): $S_x(t) = 1 - F_x(t) = P(T_x > t)$, the probability that (x) survives for at least t years
  - Notation: ${}_tp_x = S_x(t) = P(T_x > t)$, ${}_tq_x = F_x(t) = P(T_x \le t)$. For $t = 1$, we drop the front subscript and use $p_x = {}_1p_x$ and $q_x = {}_1q_x$
  - **An important formula** $P(T_x \le t) = P(T_0 \le x + t | T_0 > x)$.
  - Similarly, $P(T_x > t) = P(T_0 > x + t \mid T_0 > x)$
  - **Exercise** Show that

    1) $F_x(t) = \dfrac{F_0(x+t) - F_0(x)}{S_0(x)}$

    2) $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)}$ or $S_0(x+t) = S_0(x)S_x(t)$

    3) $S_x(t+u) = S_x(t)S_{x+t}(u)$

  - Conditions on $S_x(t)$: $S_x(0) = 1$, $\lim_{t\to\infty} S_x(t) = 0$, $S_x(t)$ is non-increasing.
  - Assumptions: $S_x(t)$ is smooth, $\lim_{t\to\infty} t^2 S_x(t) = 0$ ($\Rightarrow \lim_{t\to\infty} tS_x(t) = 0$.)
  - Example (Exercise 2.3) Given the survival function $S_0(x) = \frac{1}{10}\sqrt{100 - x}$ for $0 \le x \le 100$, find the probability that (0) will die between ages 19 and 36.
  - Example. You are given the following survival data of a group of 100 people, where l_x is the number of people in the group who survive to age x.

x	0	1	2	$\cdots$	50	51	52	53
l_x	100	97	94	$\cdots$	32	28	26	23

    Find $F_0(53)$, $S_0(51)$, $f_0(51)$, $S_2(50)$, and the probability that someone age 2 will die between 51 and 53.

$$P(51 < T_0 \le 53 \mid T_0 > 2) = \frac{S_0(51) - S_0(53)}{S_0(2)} = P(49 < T_2 \le 51)$$

    or just count how many among the 94 survived to age 2 die between 51 and 53.

- The Force of Mortality
  - $\mu_x = \lim_{\Delta x \to 0} \dfrac{1}{\Delta x} P(T_x \le \Delta x) \left(= \lim_{\Delta x \to 0} \dfrac{1}{\Delta x} P(T_0 \le x + \Delta x \mid T_0 > x) = \lim_{\Delta x \to 0} \dfrac{1}{\Delta x} F_x(\Delta x) \right)$
  - $\mu_x \Delta x \approx P(T_x \le \Delta x)$
  - $F_x(\Delta x) = 1 - S_x(\Delta x) = 1 - \dfrac{S_0(x + \Delta x)}{S_0(x)} = \dfrac{S_0(x) - S_0(x+\Delta x)}{S_0(x)}$ implies that $\mu_x = \lim_{\Delta x \to 0} \dfrac{1}{\Delta x} \dfrac{S_0(x) - S_0(x+\Delta x)}{S_0(x)} = \dfrac{-1}{S_0(x)} \dfrac{dS_0(x)}{dx} = \dfrac{-S_0'(x)}{S_0(x)}$
  - $\mu_x = \dfrac{-S_0'(x)}{S_0(x)} = -\dfrac{d}{dx} \ln S_0(x)$
  - Let $f_0(x)$ be the density function of T_0. Then $-S_0'(x) = F_0'(x) = f_0(x)$, and so
  - $\mu_x = \dfrac{f_0(x)}{S_0(x)}$
  - Let x be fixed and t be variable. Then $\mu_{x+t} = \cdots = \dfrac{-1}{S_x(t)} \dfrac{d}{dt} S_x(t) = \dfrac{-S_x'(t)}{S_x(t)}$
  - $\mu_{x+t} = \dfrac{-S_x'(t)}{S_x(t)} = \dfrac{f_x(t)}{S_x(t)} = -\dfrac{d}{dt} \ln S_x(t)$
  - Integrating $\mu_{x+s} = -\frac{d}{ds} \ln S_x(s)$ from $s = 0$ to $s = t$, we get

$$S_x(t) = \exp\left(- \int_0^t \mu_{x+s}\, ds \right) = \exp\left(- \int_x^{x+t} \mu_r\, dr \right).$$

  - $\mu_x \Longleftrightarrow S_x$
  - Example. (Exercise 2.1 (a)–(d))
    Let $F_0(t) = 1 - (1 - t/105)^{1/5}$ for $0 \le t \le 105$. Calculate
    (a) the probability that a newborn dies before age 60.
    (b) the probability that a life aged 30 survives to at least age 70.
    (b) the probability that a life aged 20 dies between ages 90 and 100.
    (a) the force of mortality at age 50.
  - Example. (Exercise 2.5 (a) and (b))
    Let $F_0(t) = 1 - e^{-\lambda t}$, where $\lambda > 0$.
    (a) Show that $S_x(t) = e^{-\lambda t}$.
    (b) Show that $\mu_x = \lambda$.
  - Special Mortality Laws
    * Constant mortality: $\mu_x = c$
    * Gompertz' Law: $\mu_x = Bc^x$ (See Example 2.3)
    * Makeham's Law: $\mu_x = A + Bc^x$
    * De Moivre's Law: $\mu_x = \frac{1}{\omega - x}$ for $0 \le x < \omega$

- Example

  Given that μ_x is constant μ and that the probability that a life aged 60 survives to age 80 is 0.1, find μ_x.

  $${}_{20}p_{60} = 0.1 \;\Rightarrow\; \exp\left(-\int_0^{20} \mu_{x+t}\,dt\right) = 0.1 \;\Rightarrow\; e^{-20\mu} = 0.1 \;\Rightarrow\; \mu = 0.11513.$$

- Example. DML with $\omega = 100$.

  $\mu_x = \dfrac{1}{100-x}$ for $0 \le x < 100 \;\Rightarrow\;$ for $0 \le t < 100 - x$,

  $${}_tp_x = \exp\left(-\int_0^t \frac{1}{100-(x+s)}\,ds\right) = \frac{100-(x+t)}{100-x}, \quad {}_tq_x = \frac{t}{100-x}.$$

  (Use a time line to express these expressions.)

- More on Notations
  - ${}_tp_x = S_x(t) = P(T_x > t), \quad {}_tq_x = F_x(t) = P(T_x \le t),$
    ${}_{u|t}q_x = P(u < T_x \le u+t) = S_x(u) - S_x(u+t) = F_x(u+t) - F_x(u)$
  - ${}_{u|t}q_x = {}_up_{xt} \cdot q_{x+u} = {}_up_x - {}_{u+t}p_x = {}_{u+t}q_x - {}_uq_x, \quad {}_{u+t}p_x = {}_up_x \cdot {}_tp_{x+u}$
  - $\mu_x = -\dfrac{1}{{}_xp_0}\dfrac{d}{dx}\,({}_xp_0), \quad \mu_{x+t} = -\dfrac{1}{{}_tp_x}\dfrac{d}{dt}\,({}_tp_x)$
  - $f_x(t) = \dfrac{d}{dt}F_x(t) = {}_tp_x\mu_{x+t}, \quad {}_tq_x = \displaystyle\int_0^t {}_sp_x\mu_{x+s}\,ds$

- Mean and Variance of T_x
  - $\mathring{e}_x = \mathbb{E}(T_x) = \displaystyle\int_0^\infty t f_x(t)\,dt = \int_0^\infty t\,{}_tp_x\mu_{x+t}\,dt = \cdots = \int_0^\infty {}_tp_x\,dt$
  - $\mathbb{E}(T_x^2) = \displaystyle\int_0^\infty t^2 f_x(t)\,dt = \cdots = 2\int_0^\infty t \cdot {}_tp_x\,dt$

    $\mathbb{V}(T_x) = \mathbb{E}(T_x^2) - [\mathbb{E}(T_x)]^2 = \mathbb{E}(T_x^2) - \left(\mathring{e}_x\right)^2$
  - Example

    For constant force of mortality $\mu_x = 0.3$, ${}_tp_x = e^{-0.3t} \;\Rightarrow$

    $$\mathring{e}_x = \int_0^\infty e^{-0.3t}\,dt = \frac{1}{0.3} = \frac{1}{\mu_x}.$$

    $\mathbb{E}(T_x^2) = 2\displaystyle\int_0^\infty t \cdot e^{-0.3t}\,dt = \cdots = \frac{2}{0.3^2} \;\Rightarrow$

    $$\mathbb{V}(T_x) = \frac{2}{0.3^2} - \left(\frac{1}{0.3}\right)^2 = \frac{1}{0,3^2}.$$

    In general, for constant $\mu_x = \mu$,

    $$\mathring{e}_x = \mathbb{E}(T_x) = \frac{1}{\mu}, \quad \mathbb{V}(T_x) = \frac{1}{\mu^2}.$$

- EXAMPLE. (DML with $\omega = 100$) $\quad {}_tp_x = \frac{100-(x+t)}{100-x} \Rightarrow$

$$\mathring{e}_x = \int_0^{100-x} \frac{100-(x+t)}{100-x}\, dt = \frac{100-x}{2}.$$

It can be computed that

$$\mathbb{V}(T_x) = \frac{(100-x)^2}{12}.$$

In general

$$\mathring{e}_x = \frac{\omega - x}{2}, \quad \mathbb{V}(T_x) = \frac{(\omega-x)^2}{12}.$$

- Curtate Life Time K_x
  - Definition: $K_x = \lfloor T_x \rfloor$, the integer part of T_x.
  - $P(K_x = k) = P(k \le T_x < k+1) = {}_{k|}q_x = {}_kp_x - {}_{k+1}p_x = {}_kp_x \cdot q_{x+k}$
  - $e_x = \mathbb{E}(K_x) = \sum_{k=1}^{\infty} kP(K_x = k) = \sum_{k=1}^{\infty} k({}_kp_x - {}_{k+1}p_x) = \cdots = \sum_{k=1}^{\infty} {}_kp_x$
  - Similarly, $\mathbb{E}(K_x^2) = \cdots = 2\sum_{k=1}^{\infty} k \cdot {}_kp_x - \sum_{k=1}^{\infty} {}_kp_x = 2\sum_{k=1}^{\infty} k \cdot {}_kp_x - e_x$ and so

    $\mathbb{V}(K_x) = 2\sum_{k=1}^{\infty} k \cdot {}_kp_x - e_x - e_x^2$
  - EXAMPLE. Find e_{50} for DML with $\omega = 100$.
  - EXERCISE. Find e_{50} for CFM with $\mu_x = 0.3$.
  - Relation between $\mathring{e}_x$ and e_x: $\mathring{e}_x \approx e_x + \frac{1}{2}$.

- Suggested Exrecises
  - From the text:
    Exercises 2.1–2.3, 2.6–2.7, 2.10–2.11 (Exercises 2.9, 2.14, and 2.15 require more calculus; they are strongly recommended.)
  - Find ${}_{5|}q_{40}$ if $S_0(t) = \left(\frac{100}{100+t}\right)^2$.
  - Given $\mu_x = \frac{2}{100-x}$ for $0 \le x < 100$. Calculate ${}_{10|}q_{65}$.
  - Under DML with $\omega = 100$, calculate the probability that (30) will die in his 70's.
  - Given that $\mu_{70+t} = \begin{cases} 0.01 & \text{if } t \le 5 \\ 0.02 & \text{if } t > 5 \end{cases}$, calculate $\mathring{e}_{70}$.
    (Hint: Find explicit expressions for ${}_tp_{70}$ for $t \le 5$ and for $t > 5$.)

Appendix It is shown in the text that $\overset{\circ}{e}_x = \int_0^\infty {}_tp_x \, dt$ using integration by parts. In order to use integration by parts, it is assumed that $\lim_{t\to\infty} tS_x(t) = 0$. Here is an alternative proof without making the further assumption. First a general result in probability.

Lemma *If X is a non-negative continuous random variable with CDF $F(x)$ and if $\mathbb{E}(X)$ exists, then*

$$\mathbb{E}(X) = \int_0^\infty (1 - F(t))\, dt. \tag{1}$$

Proof. Let $f(x) = F'(x)$, the density function of X. Then, we have

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^\infty s \cdot f(s)\, ds = \int_0^\infty \left(\int_0^s dt \right) \cdot f(s)\, ds \\
&= \int_0^\infty \int_0^s f(s)\, dt\, ds = \int_0^\infty \int_t^\infty f(s)\, ds\, dt \\
&= \int_0^\infty \int_t^\infty f(s)\, ds\, dt = \int_0^\infty P(X > t)\, dt \\
&= \int_0^\infty (1 - F(t))\, dt
\end{aligned}$$

Now for $X = T_x$, $F(t) = P(T_x \le t) = {}_tq_x$, and $1 - F(t) = 1 - {}_tq_x = {}_tp_x$. We have by (1)

$$\overset{\circ}{e}_x = \mathbb{E}[T_x] = \int_0^\infty (1 - F(t))\, dt = \int_0^\infty {}_tp_x\, dt.$$

The formula for $\mathbb{E}(T_x^2)$ and $\mathbb{V}(T_x)$ can be similarly onbtained.