Whole B

AES311 2013/14 SESSION TEST

(4)

$$f(x,y) = \begin{cases} 24xy & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$Z = x + y$ and $w = x$

from $Z = x + y$

$Z = w + y$

Then $y = Z - w$ and $x = w$

since

$f(w, z) = f(x(w, z)\, y(w, z))\,|J|$

where $|J| = \begin{vmatrix} \frac{dx}{dw} & \frac{dx}{dz} \\ \frac{dy}{dw} & \frac{dy}{dz} \end{vmatrix}$

deriving Jacobians

$\frac{dx}{dw} = 1$ $\qquad \frac{dx}{dz} = 0$

$\frac{dy}{dw} = -1$ $\qquad \frac{dy}{dz} = 1$

$J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 - (-0) = 1$

$\therefore |J| = 1$

$f(x(w,z)\,y(w,z)) = 24(w)(z-w)$

$f(w,z) = 24w(z-w) \quad 0<w<1 \quad 0<z<2$

(b) (a) Stating the moment generating function

i Poisson distribution

$X \sim P_o(\lambda)$

$$f(x) = \begin{cases} \dfrac{e^{-\lambda}\lambda^x}{x!} & x = 0,1,2 \ldots \infty \\ 0 & \text{elsewhere} \end{cases}$$

$E(x) = \lambda$

$Var(x) = \lambda$

$M_x(t) = e^{\lambda(e^t-1)}$

ii Binomial distribution

$X \sim B(n,\theta)$

$$f(x) = \begin{cases} {}^nC_x\,\theta^x(1-\theta)^{n-x} & x = 0,1,2 \ldots n \\ 0 & \text{elsewhere} \end{cases}$$

$E(x) = n\theta \qquad Var(x) = n\theta(1-\theta)$

$M_x(t) = [(1-\theta) + \theta e^t]^n$

(iii) Exponential distribution

$X \sim exp(\lambda)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$E(x) = \dfrac{1}{\lambda} \qquad Var(x) = \dfrac{1}{\lambda^2}$

$M_x(t) = \dfrac{\lambda}{\lambda - t}$

(iv) Gamma distribution

$X \sim G(\lambda,\alpha)$

$$f(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$E(x) = \dfrac{\alpha}{\lambda} \qquad Var(x) = \dfrac{\alpha}{\lambda^2}$

$M_x(t) = \left(\dfrac{\lambda}{\lambda - t}\right)^{\alpha}$

(Q) Deriving the means and variances

(i) Poisson distribution

since $M_x(t) = e^{\lambda(e^t-1)}$

$$E(x) = \frac{d}{dt} M_x(t)\Big|_{t=0} = \frac{d}{dt} e^{\lambda e^t - \lambda}$$

$$= e^{\lambda e^t - \lambda} \times \lambda e^t \Big|_{t=0}$$

$$= e^{\lambda(e^0-1)} \cdot \lambda e^0 = e^{\lambda(1-1)} \cdot \lambda$$

$$= e^{\lambda(0)} \cdot \lambda = e^0 \cdot \lambda$$

$$\underline{E(x) = \lambda}$$

$$Var(x) = E(x^2) - (E(x))^2$$

but $E(x^2) = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0}$

$$E(x^2) = \frac{d}{dt}\left(\frac{d}{dt} M_x(t)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(e^{\lambda(e^t-1)} \cdot \lambda e^t\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\lambda e^{\lambda(e^t-1)+t}\right)\Big|_{t=0}$$

$$E(x^2) = \lambda e^{\lambda(e^t-1)+t} \times (\lambda e^t + 1)\Big|_{t=0}$$

$$= \lambda e^{0+0} \times (\lambda e^0 + 1) = \lambda(\lambda +$$

$$E(x^2) = \lambda^2 + \lambda$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$Var(x) = \underline{\lambda}$$

(ii) Binomial distribution

Since $M_x(t) = [(1-\theta) + \theta e^t]^n$

$$E(x) = \frac{d}{dt} M_x(t)\Big|_{t=0}$$

$$= \frac{d}{dt}[(1-\theta) + \theta e^t]^n\Big|_{t=0}$$

$$= n[(1-\theta) + \theta e^t]^{n-1} \times \theta e^t\Big|_{t=0}$$

$$= n((1-\theta) + \theta e^0)^{n-1} \times \theta e^0$$

$$= n(1-\theta+\theta)^{n-1} \times \theta = n(1)^{n-}$$

$$E(x) = \underline{n\theta}$$

$\boxed{\begin{array}{l}\theta(n^2-n)+\cancel{n\theta}-n^2\theta^2 \\ n^2\theta-n\theta+n\theta-n^2\theta^2\end{array}}$

$u\frac{dv}{dx}+v\frac{du}{dx}$

$n^2(\theta-\theta^2)$

$n^2\theta(1-\theta)$

$Var(x) = E(x^2)-(E(x))^2$

but $E(x^2) = \frac{d}{dt}\left[\frac{d}{dt}M_x(t)\right]\Big|_{t=0}$

$= \frac{d}{dt}\left(n\left((1-\theta)+\theta e^t\right)^{n-1}\times\theta e^t\right)$

$E(x^2) = \theta e^t\left[(n-1)n\left[(1-\theta)+\theta e^t\right]^{n-2}\theta e^t + n\left((1-\theta)+\theta e^t\right)^{n-1}\theta e^t\right]$

$= \theta e^0\left[(n-1)n\left((1-\theta)+\theta e^0\right)^{n-2}+n\left((1-\theta)+\theta e^0\right)^{n-1}\right]\theta e^0$

$= \theta^2(n-1)n(1-\theta+\theta)^{n-2}+n\left((1-\theta)+\theta\right)^{n-1}\cdot\theta$

$= \theta^2(n^2-n)+n\theta$

$Var(X) = \cancel{\theta n^2}-\theta n+n\theta-\cancel{n^2\theta^2}$

$= \cancel{\theta n^2 - n\theta} = n\theta-\theta^2 n$

$= \cancel{n\theta(1-\theta)} = n\theta(1-\theta)$

$Var(x) = n\theta(1-\theta)$

(iii) Exponential distribution

Since $M_x(t) = \frac{\lambda}{\lambda-t}$

$E(x) = \frac{d}{dt}M_x(t)\Big|_{t=0}$

$E(x) = \frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\Big|_{t=0} = \frac{d}{dt}\lambda(\lambda-t)^{-1}$

$= -\lambda(\lambda-t)^{-2}\times-1$

$= \lambda(\lambda-t)^{-2}\Big|_{t=0}$

$= \frac{\lambda}{(\lambda-0)^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

$E(x^2) = \frac{d}{dt}\left(\frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\right)\Big|_{t=0}$

$= \frac{d}{dt}\left(\frac{\lambda}{(\lambda-t)^2}\right)\Big|_{t=0} = \frac{d}{dt}\lambda(\lambda-t)^{-2}$

$= -2\lambda(\lambda-t)^{-3}\times-1 = \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0}$

$= \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$

$Var(x) = \frac{2}{\lambda^2}-\left(\frac{1}{\lambda}\right)^2 = \frac{2}{\lambda^2}-\frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

(iv) Gamma distribution

$$E(x) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)^{\alpha} \Big|_{t=0}$$

$$= \alpha\left(\frac{\lambda}{\lambda-t}\right)^{\alpha-1} \times \frac{\lambda}{(\lambda-t)^2}\Big|_{t=0} = \alpha\left(\frac{\lambda}{\lambda}\right)^{\alpha-1} \times \frac{\lambda}{\lambda^2}$$

$$= \alpha(1)^{\alpha-1} \times \frac{\lambda}{\lambda^2} = \alpha \times \frac{1}{\lambda}$$

$$E(x) = \frac{\alpha}{\lambda}$$

$$E(x^2) = \frac{d}{dt}\left(\frac{d}{dt} M_x(t)\right)$$

$$= \frac{d}{dt}\left(\alpha\left(\frac{\lambda}{\lambda-t}\right)^{\alpha-1} \times \frac{\lambda}{(\lambda-t)^2}\right)\Big|_{t=0}$$

$$= \alpha(\alpha-1)\left(\frac{\lambda}{\lambda-t}\right)^{\alpha-2} \times \frac{\lambda}{(\lambda-t)^2} + \alpha\left(\frac{\lambda}{\lambda-t}\right)^{\alpha-1} \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=}$$

$$E(x^2) = \alpha(\alpha-1)\left(\frac{\lambda}{\lambda}\right)^{\alpha-2} \frac{\lambda}{\lambda^2} + \alpha\left(\frac{\lambda}{\lambda}\right)^{\alpha-1} \times \frac{2\lambda}{\lambda^3}$$

$$= (\alpha^2-\alpha)\frac{1}{\lambda \times \lambda} + \frac{2\alpha}{\lambda^2}$$

$$E(x^2) = \frac{\alpha^2}{\lambda^2} - \frac{\alpha}{\lambda^2} + \frac{2\alpha}{\lambda^2}$$

$$Var(x) = \cancel{\frac{\alpha^2}{\lambda^2}} - \frac{\alpha}{\lambda^2} + \frac{2\alpha}{\lambda^2} - \cancel{\frac{\alpha^2}{\lambda^2}}$$

$$= \frac{2\alpha}{\lambda^2} - \frac{\alpha}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

$y \sim \exp(x)$

(3) $y \sim \exp(4)$

$$f(y) = \begin{cases} \frac{1}{4} e^{-y/4} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$u = 3y + 1$

pdf of u, $= f(u) = f(y(u))|J|$

from $u = 3y + 1$

$u - 1 = 3y \qquad y = \frac{u-1}{3}$

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$$\frac{dy}{du} = \frac{1}{3}$$

$$f(y(u)) = \frac{1}{4} e^{-\frac{\left(\frac{u-1}{3}\right)}{4}} = \frac{1}{4} e^{-\left(\frac{u-1}{12}\right)}$$

$$f(u) = \left(\frac{1}{3}\right)\frac{1}{4} e^{-\left(\frac{u-1}{12}\right)} = \frac{1}{12} e^{-\left(\frac{u-1}{12}\right)}$$

$$f(u) = \begin{cases} \frac{1}{12} e^{-\left(\frac{u-1}{12}\right)} & u \geq 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$E(u) = \int_1^{\infty} u f(u)\, du = \int_1^{\infty} u \times \frac{1}{12} e^{-\left(\frac{u-1}{12}\right)} du$$

$$= \frac{1}{12} \int_1^{\infty} u e^{-\left(\frac{u-1}{12}\right)} du$$

$$= \frac{1}{12} \left\{ \frac{u e^{-\left(\frac{u-1}{12}\right)}}{-\frac{1}{12}} \Bigg|_1^{\infty} + \int_1^{\infty} 12 e^{-\left(\frac{u-1}{12}\right)} du \right\}$$

$$= \frac{1}{12} \left(-12u e^{-\left(\frac{u-1}{12}\right)} \Big|_1^{\infty} + \left(\frac{12 e^{-\left(\frac{u-1}{12}\right)}}{-\frac{1}{12}} \Bigg|_1^{\infty} \right) \right)$$

$$E(u) = \frac{1}{12} \left(+12 e^{0} + \left(-144 e^{-\left(\frac{u-1}{12}\right)} \Big|_1^{\infty} \right) \right)$$

$$= \frac{1}{12} \left(12 e^{0} + 144 e^{0} \right) = \frac{1}{12} (12 + 144)$$

$$= \frac{1}{\cancel{12}} \times \cancel{12} (1 + 12) = 1 + 12 = 13$$

$$E(u) = 13$$

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University of Lagos
Faculty of Business Administration
Department of Actuarial Science and Insurance
B.Sc Degree Examination, 2017
ACS 311 (Probability & Statistics)

Instruction: Correct answers to <u>ANY FOUR</u> questions bear full marks **Time: 2hrs.**

1) Explain briefly with examples, the following properties of an estimator
   - (i) Unbiasedness
   - (ii) Efficiency
   - (iii) Sufficiency
   - (iv) Consistency

$-\frac{n}{2}\cdot\frac{1}{\sigma^2} \quad -\frac{n}{2\sigma^2} - \frac{\sum(y_i-\mu)^2}{2\sigma^4}$

2) (a) State the Cramer-Rao Inequality

(b) Let $Y_1,\ Y_2,\ .\ .\ .\ ,\ Y_n$ denote a random sample from a probability density function $f(y)$.

i. Suppose $f(y)$ is the normal density with mean μ and variance σ^2. Find a efficient estimator of μ.

ii. The inequality also holds for discrete probability function $p(y)$. Suppose that $p(y)$ is the Poisson probability function with mean λ. Show that $\overline{Y}$ is an efficient estimator of λ.

3) (a) Given the following joint pdf

$$f(x,y) = \begin{cases} 24xy, & 0 < x < 1, 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

Find the joint pdf of $Z = x + y$ and $W = x$

(b) The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean of 4 tons. The cost of the flour is proportional to $U = 3Y + 1$

(i) Find the probability density function for U

(ii) Use the answer in (i) to find $E(U)$

(b) Let $Y_1,\ Y_2,\ .\ .\ .\ ,\ Y_n$ be a random sample in which Y_i possesses the probability density function $f(y_i/\theta) = (1/\theta)e^{-y_i/\theta},\ \ for\ \ 0 \le y_i \le \infty$ and $i = 1,\ 2, .\ .\ .\ \ ,\ n$. Show that $\overline{Y}$ is a sufficient statistic for the estimation of θ.

(c) Let $Y_1,\ Y_2,\ .\ .\ .\ ,\ Y_n$ denote a random sample from a Poisson distribution with parameter λ. Show that $\sum_i^n Y_i$ is sufficient for λ.

4) (a) Explain briefly the method of maximum likelihood estimation

(b) Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the normal distribution with mean λ and known variance θ^2.

i. Find the maximum-likelihood estimator $\hat{\lambda}$ for λ.

ii. Find the expected value and variance of $\hat{\lambda}$

iii. Show that the estimator of (i) is consistent for λ.

iv. hence or otherwise estimate the value of λ given the following data; 2.5, 5.1, 3.6, 4.8, 2.3, 5.1, 2.3, 2.6, 5.4, 6.0, 2.5

5) (a) The time t, in minutes to serve a customer has exponential distribution with density:

$$f(t/\theta) = \theta e^{-t\theta}$$

where θ > 0 is an unknown parameter. The prior distribution for θ has gamma distribution with mean 0.16 and standard deviation 4. The average time taken to serve a random sample of 10 customers is 7.6 minutes. Calculate the Bayes estimate of θ.

(b) Twenty identically and independently distributed observations from a Poisson (λ) distribution gave 3, 4, 3, 1, 5, 5, 2, 3, 3, 2, 3, 4, 3, 1, 5, 5, 2, 3, 3, 2. Assuming an Exp(0.8) prior distribution for λ, find the Bayesian estimator of λ.

6) (a) If X has the binomial distribution with the parameters n *and* θ, show that the sample proportion, $\frac{X}{n}$, is unbiased estimator of θ .

(b) If $X_1, X_2, \ldots, X_n$ constitute a random sample from the population given by $f(x) = \begin{cases} e^{-(x-\delta)} & for\ x > \delta \\ 0 & elsewhere \end{cases}$

Show that $\overline{X}$ is a biased estimator of δ.

(c) If X has the exponential distribution with parameter θ. Find the probability density of the random variable

i. $Y = \sqrt{X}$

ii. $U = X^2$

(d) If X has the exponential distribution with parameter θ. Find the sufficient statistic for estimating θ.

University of Lagos
Faculty of Business Administration
Department of Actuarial Science and Insurance
Test Questions, 2015/2016 Session
ACS 311 (Probability & Statistics)

Instruction: Answer ALL questions

Time: 45mins.

1) If X is the number of heads obtained in four tosses of a balanced coin, find the probability distribution of
   i. $Y = \frac{1}{1+X}$
   ii. $Z = (X-2)^2$

2) If X has a binomial distribution with $n = 3, \theta = \frac{1}{3}$. Find the probability distributions of
   i. $Y = \frac{X}{1+X}$
   ii. $U = (X-1)^4$

3) If X has the exponential distribution with parameter θ. Find the probability density of the random variable
   i. $Y = \sqrt{X}$
   ii. $U = X^2$

4) If the joint probability density of X_1, X_2 is by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & for\ x_1 > 0, x_2 > 0 \\ 0 & elsewhere \end{cases}$$

Find the probability density of
   i. $Y = \frac{X_1}{X_1+X_2}$
   ii. The joint density of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$
   iii. The marginal density of Y_2

University of Lagos
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Department of Actuarial Science and Insurance
Test Questions, 2016/2017 Session
ACS 311 (Probability & Statistics)

Instruction: Answer ALL questions **Time: 45mins.**

✓ 1) If X has the binomial distribution with the parameters n and θ, show that the sample proportion, $\frac{X}{n}$, is unbiased estimator of θ .

2) If $X_1, X_2, \dots, X_n$ constitute a random sample from the population given by

$$f(x) = \begin{cases} e^{-(x-\delta)} & for\ x > \delta \\ 0 & elsewhere \end{cases}$$

Show that $\overline{X}$ is a biased estimator of δ.

f(y)

3) If X has the exponential distribution with parameter θ. Find the probability density of the random variable

i. $Y = \sqrt{X}$

ii. $U = X^2$

✓ 4) If X has the exponential distribution with parameter θ. Find the sufficient statistic for estimating θ.

University of Lagos
Faculty of Business Administration
Department of Actuarial Science and Insurance
B.Sc. Degree Examination, 2013/2014 Session
ACS 311 (Probability & Statistics)

Instruction: Correct answers to ANY FOUR questions bear full marks   Time: 2hrs.

1) Explain briefly with examples, the following properties of an estimator
   (i) Unbiasedness
   (ii) Efficiency
   (iii) Sufficiency
   (iv) Consistency

2) (a) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample in which Y_i possesses the probability density function $f(y_i/\theta) = (1/\theta)e^{-y_i/\theta}$, $for\ 0 \le y_i \le \infty$ and $i = 1, 2, \ldots, n$. Find the maximum-likelihood estimate estimator of θ.

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson distribution with parameter λ.

i. Find the maximum-likelihood estimator $\hat{\lambda}$ for λ.

ii. Find the expected value and variance of $\hat{\lambda}$

iii. Show that the estimator of (i) is consistent for λ.

3) (a) The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean of 4 tons. The cost of the flour is proportional to $U = 3Y + 1$
(i) Find the probability density function for U
(ii) Use the answer in (i) to find $E(U)$

(b). The waiting time Y until delivery of a new component for an industrial operation is uniformly distributed over the interval from 1 to 5 days. The cost of this delay is given by $U = 2Y^2 + 3$. Find the probability density function for U.

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4) (a) State the Cramer-Rao Inequality

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a probability density function $f(y)$.

i. Suppose $f(y)$ is the normal density with mean μ and variance σ^2. Show that $\overline{Y}$ is an effidient estimator of μ.

ii. The inequality also holds for discrete probability function $p(y)$. Suppose that $p(y)$ is the Poisson probability function with mean λ. Show that $\overline{Y}$ is an efficient estimator of λ.

5) The opening prices per share Y_1 and Y_2 of two stocks are independent random variables, each with density function given by

$$f(x) = \begin{cases} 0.5e^{-0.5(y-4)} & y \geq 4 \\ 0 & elsewhere \end{cases}$$

On a morning, an investor is going to buy shares of whichever stock is less expensive.

(i) Find the probability density function for the price per share that the investor will pay.

(ii) Find the expected cost per share that the investor will pay.

6) Let $X_1, X_2, \ldots, X_n$ be a random sample from the Bernoulli distribution with parameter θ. Suppose that the prior density of θ is uniform distribution over (0, 1). Find the posterior distribution of θ.

6(b). Let X be a random variable with probability function

$$f(x) = \begin{cases} \frac{2x}{12}, & x = 1, 2, 3 \\ 0 & elsewhere \end{cases}$$

Calculate (a). E(X) (b) Var(X)

(c) Determine the value of k for which the function given by

$$f(x, y) = \begin{cases} kxy; & for\ x = 1,2,3,4; \quad y = 1,2,3,4 \\ 0, & otherwise \end{cases}$$

can serve as a joint probability distribution.

University of Lagos
Faculty of Business Administration
Department of Actuarial Science and Insurance
Exam Questions, 2015/2016 Session
ACS 311 (Probability & Statistics)

Instruction: Answer ANY FOUR questions

Time: 2 hrs.

1) (a) The time t, in minutes to serve a customer has exponential distribution with density:

$$f(t/\theta) = \theta e^{-t\theta}$$

where θ > 0 is an unknown parameter. The prior distribution for θ has gamma distribution with mean 0.8 and standard deviation 2. The average time taken to serve a random sample of 5 customers is 3.8 minutes. Calculate the Bayes estimate of θ.

~~(b)~~ Ten identically and independently distributed observations from a Poisson (λ) distribution gave 3, 4, 3, 1, 5, 5, 2, 3, 3, 2. Assuming an Exp(0.2) prior distribution for λ, find the Bayesian estimator of λ.

2) (a) Explain the meaning of sufficient estimator

(b) If $X_1, X_2, X_3, \ldots, X_n$ constitute a random sample of size n from a Bernoulli population, show that

$$\widehat{\Theta} = \frac{X_1 + X_2 + X_3 + \ldots + X_n}{n}$$

is a sufficient estimator of the parameter θ.

(c) Show that $\bar{X}$ is a sufficient estimator of the mean μ of a normal population with the known variance σ^2.

3) (a) Explain the meaning of likelihood function and list the steps involved in maximum likelihood estimation

(b) If $x_1, x_2, x_3, \ldots, x_n$ are the values of a random sample from an exponential population, find the maximum likelihood estimator of its parameter θ.

(c) If $X_1, X_2, X_3, \ldots, X_n$ constitute a random sample of size n from a normal population with mean μ and variance σ^2, find the joint maximum likelihood estimates of these two parameters.

4) (a) If X has a binomial distribution with $n = 3, \theta = \frac{1}{3}$. Find the probability distributions of

i. $Y = \frac{X}{1+X}$

ii. $U = (X-1)^4$

(b) If X has the exponential distribution with parameter θ. Find the probability density of the random variable

i. $Y = \sqrt{X}$

ii. $U = X^2$

5) (a) If the joint probability density of X_1, X_2 is by

$$f(x_1, x_2) = \begin{cases} e^{-(x_1+x_2)} & for\ x_1 > 0, x_2 > 0 \\ 0 & elsewhere \end{cases}$$

Find the probability density of

i. $Y = \frac{X_1}{X_1+X_2}$ ✓

ii. The joint density of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$ ✓

i. The marginal density of Y_2

(b) If X is the number of heads obtained in four tosses of a balanced coin, find the probability distribution of

ii. $Y = \frac{1}{1+X}$ ✓

iii. $Z = (X-2)^2$

6) (a) State the Cramer-Rao Inequality

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a probability density function $f(y)$.

i. Suppose $f(y)$ is the normal density with mean μ and variance σ^2. Show that $\overline{Y}$ is an efficient estimator of μ.

ii. The inequality also holds for discrete probability function $p(y)$. Suppose that $p(y)$ is the Poisson probability function with mean λ. Show that $\overline{Y}$ is an efficient estimator of λ.

University of Lagos
Faculty of Business Administration
Department of Actuarial Science and Insurance
B.Sc. Degree Examination, 2012/2013 Session
ACS 311 (Probability & Statistics)

Instruction: Correct answers to ANY FOUR questions bear full marks Time: 2hrs.

1) (a) Given the following joint pdf

$$f(x,y) = \begin{cases} 24xy, & 0 < x < 1, 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

Find the joint pdf of $Z = x + y$ and $W = x$

(b) The amount of flour used per day by a bakery is a random variable Y that has an exponential distribution with mean of 4 tons. The cost of the flour is proportional to $U = 3Y + 1$

(i) Find the probability density function for U

(ii) Use the answer in (i) to find $E(U)$

2) (a) State the moment generating function of the following distributions

i. Poisson distribution $- e^{-\lambda + e^t \lambda}$

ii. Binomial distribution $(e^t p + q)^n$

iii. Exponential distribution $= \frac{\lambda}{\lambda - t}$

iv. Gamma distribution $= \frac{\lambda}{\lambda - t}$

(b) For each of the distributions above, derive the means and variances.

3) (a) State the Cramer-Rao Inequality

(b) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a probability density function $f(y)$.

i. Suppose $f(y)$ is the normal density with mean μ and variance σ^2. Show that $\overline{Y}$ is an efficient estimator of μ.

ii. The inequality also holds for discrete probability function $p(y)$. Suppose that $p(y)$ is the Poisson probability function with mean λ. Show that $\overline{Y}$ is an efficient estimator of λ.

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4) Explain briefly with examples, the following properties of an estimator

(i) Unbiasedness

(ii) Efficiency

(iii) Sufficiency

(iv) Consistency

(b) Let $Y_1, Y_2, \ldots, Y_n$ be a random sample in which Y_i possesses the probability density function $f(y_i/\theta) = (1/\theta)e^{-y_i/\theta}$; $for\ 0 \le y_i \le \infty$ and $i = 1, 2, \ldots, n$. Show that $\overline{Y}$ is a sufficient statistic for the estimation of θ.

(c) Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson distribution with parameter λ. Show that $\sum_i^n Y_i$ is sufficient for λ.

5) (a) Explain briefly the method of maximum likelihood estimation

(b) Suppose that $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the Poisson distribution with mean λ

i. Find the maximum-likelihood estimator $\hat{\lambda}$ for λ.

ii. Find the expected value and variance of $\hat{\lambda}$

iii. Show that the estimator of (ii) is consistent for λ.

6(a) Let X be a random variable with probability function

$$f(x) = \begin{cases} \frac{2x}{12}, & x = 1, 2, 3 \\ 0 & elsewhere \end{cases}$$

$E\left(\frac{\partial^2 \ln f(x)}{\partial \theta^2}\right)$

Calculate (a) E(X) (b) Var(X)

(b) Determine the value of k for which the function given by

$$f(x, y) = \begin{cases} kxy, & for\ x = 1, 2, 3, 4; \quad y = 1, 2, 3, 4 \\ 0, & otherwise \end{cases}$$

can serve as a joint probability distribution.

UNCLE B

ACS311 (PROBABILITY & STATISTICS)

SECOND SEMESTER EXAMINATION 2014/15 SESSION

QUESTION 1 SOLUTION

(i) UNBIASEDNESS:

An estimator $t(x)$ of θ is said to be unbiased of θ if

$E(t(x)) = \theta$

otherwise, it is a biased estimator

for example: investigate the biasedness of $t(x) = \bar{X}$ Given that $X_i \sim Bern(\theta)$ using a random sample of size n

SOLUTION

$X \sim Ber(\theta)$

$$f(x) = \begin{cases} \theta^x(1-\theta)^{1-x} & x = 0,1 \\ 0 & \text{elsewhere} \end{cases}$$

$E(x) = \theta \qquad Var(X) = \theta(1-\theta)$

$$E(\bar{X}) = E\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right)$$

$$= \frac{1}{n}\left(E(X_1) + E(X_2) + \dots + E(X_n)\right)$$

$$= \frac{\theta + \theta + \dots + \theta}{n}$$

$= \frac{n\theta}{n} = \theta$

Since the $E(\bar{X}) = \theta$

Therefore $T(x) = \bar{X}$ is unbiased for θ

(ii) EFFICIENCY:

If $\bar{X}$ is an unbiased estimator of θ, $\bar{X}$ is said to be efficient parameter for θ. If.

$$Var(\bar{X}) = \frac{1}{n \cdot E\left(\frac{\partial^2 \ln f(x)}{\partial \theta^2}\right)}$$

Then $\bar{X}$ is a minimum variance estimator of θ

for example: Given that a random sample of size n is selected from a normal distribution with parameter μ and σ^2. Show that parameter $\bar{X}$ is an efficient parameter for μ

$x \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty$$

$E(X) = \mu \qquad Var(X) = \sigma^2$

Starting from the left hand side (L.H.S)

$$Var(\bar{X}) = Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right)$$

from the property of variance that

$Var(ax) = a^2 Var(x)$

$$Var(\bar{X}) = \left(\frac{1}{n}\right)^2 \left[Var(X_1) + Var(X_2) + \cdots + Var(X_n)\right]$$

$$= \frac{\sigma^2 + \sigma^2 + \cdots + \sigma^2}{n^2}$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

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then the Right Hand side (R.H.S.)

$$\frac{1}{n \cdot E\left(\frac{\partial^2 \ln f(x)}{\partial \mu^2}\right)} = \frac{1}{n \cdot E\left(\frac{\partial^2 \ln \left[\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right]}{\partial \mu^2}\right)}$$

this function is large so lets start from the origin

$$\ln f(x) = \ln\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\right)$$

from the law of logarithm

$$\ln \frac{w}{uv} = \log_e \frac{w}{uv} = \ln w - \ln u - \ln v$$

$$\ln f(x) = \ln\left[\frac{1}{\sqrt{2\pi}}\right]\left[\frac{1}{\sigma^2}\right]^{1/2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \ln(\sqrt{2\pi})^{-1} + \ln(\sigma^2)^{-1/2} + \ln e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \ln(\sqrt{2\pi})^{-1} - \tfrac{1}{2}\ln \sigma^2 - \frac{(x-\mu)^2}{2\sigma^2}\ln e$$

but $\ln e = \log_e e = 1$

so

$$\ln f(x) = \ln(\sqrt{2\pi})^{-1} - \tfrac{1}{2}\ln \sigma^2 - \frac{(x-\mu)^2}{2\sigma^2}$$

taking the first derivative of $\ln f(x)$ with respect to μ

$$\frac{\partial \ln f(x)}{\partial \mu} = 0 - 0 - \frac{2(x-\mu) \times (-1)}{2\sigma^2}$$

$$= \frac{2(x-\mu)}{2\sigma^2} = \frac{x-\mu}{\sigma^2}$$

Taking the second derivative of $\ln f(x)$ with respect to μ

$$\frac{x}{\sigma} - \frac{u}{\sigma}$$

$$\frac{\partial^2 \ln f(x)}{\partial \mu^2} = -\frac{1}{\sigma^2}$$

this means that we are done with

$$\left[\frac{\partial^2 \ln f(x)}{\partial \mu^2}\right] = -\frac{1}{\sigma^2}$$

so lets complete our original equation

$$-\frac{1}{n \cdot E\left(-\frac{1}{\sigma^2}\right)} = \frac{1}{n \times -\frac{1}{\sigma^2}} = \frac{-1}{\left(-\frac{n}{\sigma^2}\right)}$$

$$-1 \times \frac{\sigma^2}{-n} = \frac{\sigma^2}{n}$$

Since ~~Left~~ Hand side = Right Hand side therefore, $\bar{X}$ is an efficient parameter of μ

(iii) SUFFICIENCY :
The statistic $\bar{X}$ is a sufficient estimator for the parameter θ if and only if for each value of $\bar{X}$ the conditional distribution of the random sample $X_1, X_2, \ldots, x_n$ given $\bar{X} = \bar{x}$ is independent of θ

for example : Given that $X_1, X_2, \ldots, X_n$ are random sample of size n from an exponential distribution with parameter θ show that $\bar{X}$ is a sufficient statistic for θ

SOLUTION

$X_i \sim \exp(\theta)$ / $i = 1, \ldots, n$

$$f(x) = \begin{cases} \theta e^{-\theta x_i} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The joint probability distribution is given

$$\prod_{i=1}^{n} f(x_i/\theta) = \theta e^{-\theta x_1} \times \theta e^{-\theta x_2} \times \ldots \times \theta e^{-\theta x_n}$$

$$= \theta^n e^{-\sum \theta x_i}$$

$$= \underbrace{\theta^n e^{-\theta \sum x_i}}_{g(\bar{x}, \theta)} \times \underbrace{1}_{h(x)}$$

$\Rightarrow \sum x_i$ is a sufficient statistic for θ
$\therefore \bar{X}$ is a sufficient statistic for θ

(iv) CONSISTENCY :
The statistic $\bar{X}$ is a consistent estimator of the parameter θ if and only if for each $c > 0$

$$\lim_{n \to \infty} P(|\bar{X} - \theta| < c) = 1$$

also if $\bar{X}$ is an unbiased estimator of the parameter θ and $Var(\bar{X}) \to 0$ as $n \to \infty$ then $\bar{X}$ is a consistent estimator for θ

Example: Assuming r sample of size n is selected from a normal distribution with μ and σ^2. Determine whether or not $\bar{X}$ is a consistent estimator for μ.

SOLUTION

$X_i \sim N(\mu, \sigma^2)$

$$f(x_i) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$E(X) = \mu \qquad Var(X) = \sigma^2$

$$E(\bar{X}) = E\left(\frac{X_1+X_2+\cdots X_n}{n}\right)$$

$$= \frac{E(X_1)+E(X_2)+\cdots+E(X_n)}{n}$$

$$= \frac{\mu+\mu+\cdots+\mu}{n}$$

$$= \frac{n\mu}{n} = \mu$$

Since $\bar{X}$ is unbiased for μ then

$$Var(\bar{X}) = Var\left(\frac{X_1+X_2+\cdots+X_n}{n}\right)$$

$$= \frac{1}{n^2}\left(Var(X_1)+Var(X_2)+\cdots+Var(X_n)\right)$$

$$= \frac{\sigma^2+\sigma^2+\cdots+\sigma^2}{n^2} = \frac{n\sigma^2}{n^2}$$

$$= \frac{\sigma^2}{n}$$

$$\lim_{n\to\infty} Var(\bar{X}) = \frac{\sigma^2}{\infty} = 0$$

$\bar{X}$ is a consistent statistic for parameter μ

QUESTION 2

(a) $Y_i \sim exp(\theta)$

$$f(y_i/\theta) = \frac{1}{\theta}e^{-y_i/\theta} \qquad 0 \le y_i \le \infty$$

The joint probability distribution is given as

$$\prod_{i=1}^{n} f(y_i/\theta) = \frac{1}{\theta}e^{-y_1/\theta} \times \frac{1}{\theta}e^{-y_2/\theta} \times \cdots \times \frac{1}{\theta}e^{-y_n/\theta}$$

$$= \left[\frac{1}{\theta}\right]^n e^{-\sum y_i/\theta}$$

$$= \left[\frac{1}{\theta}\right]^n e^{-\frac{1}{\theta}\sum y_i}$$

$\sum y_i$ is a sufficient statistic for θ

(b) $y_i \sim P_0(\lambda)$ $\quad i = 1, 2 \dots n$

$$f(y_i/\theta) = \begin{cases} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} & y_i = 0, 1, 2 \dots \infty \\ 0 & \text{elsewhere} \end{cases}$$

the joint probability distribution is given as

$$\prod_{i=1}^{n} f(y_i/\theta) = \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \times \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \times \dots \times \frac{e^{-\lambda}\lambda^{y_n}}{y_n!}$$

$$= \frac{e^{-\sum\lambda}\lambda^{\sum y_i}}{\prod_{i=1}^{n} y_i!}$$

$$= \underbrace{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} y_i}}_{g(\bar{y}, \lambda)} \times \underbrace{\frac{1}{\prod_{i=1}^{n} y_i!}}_{h(x)}$$

$\therefore \sum_{i=1}^{n} y_i$ is a sufficient statistic for λ

Question 3

(a) Steps involved in method of maximum likelihood estimation are as follows:

step 1

Derive the likelihood function denoted as $L(\theta)$

step 2

Obtain the log likelihood function denoted as $\ln L(\theta)$

step 3

Derive the partial derivative of log likelihood function with respect to the parameter in the distribution

step 4

equate the partial derivative to zero and solve for the parameter of interest.

The function derived is the maximum likelihood of θ.

(b) (i) $y_i \sim Po(\lambda)$ $\quad i = 1, 2, \ldots, n$

$$f(y_i/\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{y_i}}{y_i!} & y = 0, 1, 2, \ldots \\ 0 & \text{elsewhere} \end{cases}$$

the likelihood function

$$L(\lambda) = \prod_{i=1}^{n} f(y_i|\lambda)$$

$$= \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \times \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \times \cdots \times \frac{e^{-\lambda}\lambda^{y_n}}{y_n!}$$

$$= e^{-n\lambda}\lambda^{\sum_{i=1}^{n} y_i} \times \frac{1}{\prod_{i=1}^{n} y_i!}$$

$$= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}$$

the log likelihood function

$$\ln L(\lambda) = \ln \frac{e^{-\lambda n}\lambda^{\sum_{i=1}^{n} y_i}}{\prod_{i=1}^{n} y_i!}$$

$$= \ln e^{-n\lambda} + \ln \lambda^{\sum_{i=1}^{n} y_i} - \ln \prod_{i=1}^{n} y_i!$$

the derivative of log likelihood function with respect to λ

$$= -n\lambda + \sum_{i=1}^{n} y_i \ln\lambda - \ln \prod_{i=1}^{n} y_i!$$

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} y_i}{\lambda}$$

equating the derivative of log likelihood function with respect to λ to zero

$$-n + \frac{\sum_{i=1}^{n} y_i}{\lambda} = 0$$

$$\frac{-\lambda n + \sum_{i=1}^{n} y_i}{\lambda} = 0$$

$$\sum_{i=1}^{n} y_i = \lambda n$$

$$\lambda = \sum_{i=1}^{n} \frac{y_i}{n}$$

maximum - likelihood estimator

$$\lambda = \sum_{i=1}^{n} \frac{y}{n} \quad \text{or} \quad \bar{y}$$

(ii) $\lambda = \bar{y} = \frac{2.5+5.1+3.6+4.8+2.3+5.1+2.3+2.6+5.4+6.0+2.5}{11}$

$$= \frac{42.2}{11} = 3.8364$$

$$\lambda \approx 3.8$$

Question 4

(a) Cramer - Rao Inequality

$$Var(\hat{\theta}) \geq \frac{1}{n E\left[\left(\frac{\partial \ln f(x)}{\partial \theta}\right)^2\right]}$$

(b) (i) the solution is in Question 1 roman figure (ii)

(ii) $y \sim P_0(\lambda)$

$$f(y) = \begin{cases} \frac{e^{-\lambda}\lambda^y}{y!} & y = 0, 1, 2 \ldots \\ 0 & \text{elsewhere} \end{cases}$$

$E(y) = \lambda \quad Var(y) = \lambda$

Investigating the biasedness of the statistics

$$E(\bar{y}) = E\left(\frac{y_1 + y_2 + \ldots y_n}{n}\right)$$

$$= \frac{E(y_1) + E(y_2) + \ldots E(y_n)}{n}$$

$$= \frac{\lambda + \lambda + \ldots \lambda}{n}$$

$$= \frac{n\lambda}{n} = \lambda$$

Since $\bar{y}$ is unbiased for λ

~~then~~

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$$Var(\bar{y}) = \frac{1}{n \cdot E\left(\frac{\partial^2 \ln f(x)}{\partial \lambda^2}\right)}$$

Working the Left Hand side (L.H.S)

$$Var(\bar{y}) = Var\left(\frac{y_1 + y_2 + y_3 + \cdots + y_n}{n}\right)$$

$$= \frac{1}{n^2}\left(Var(y_1) + Var(y_2) \cdots + Var(y_n)\right)$$

$$= \frac{\lambda + \lambda + \lambda \cdots + \lambda}{n^2}$$

$$= \frac{n\lambda}{n^2} = \frac{\lambda}{n}$$

$$L.H.S = \lambda/n$$

for the Right Hand side (R.H.S)

$$\ln f(x) = \ln \frac{e^{-\lambda}\lambda^{y}}{y!} = \ln e^{-\lambda} + \ln \lambda - \ln y!$$

$$= -\lambda + y\ln\lambda - \ln y!$$

$$\frac{\partial \ln f(x)}{\partial \lambda} = -1 + \frac{y}{\lambda} = -1 + y\lambda^{-1}$$

$$\frac{\partial^2 \ln f(x)}{\partial \lambda^2} = -y\lambda^{-2} = -\frac{y}{\lambda^2}$$

$$n \cdot E\left(\frac{\partial^2 \ln f(x)}{\partial \lambda^2}\right) = n \cdot E\left(-\frac{y}{\lambda^2}\right) = -\frac{n}{\lambda^2}E(y)$$

$$= \frac{-n\lambda}{\lambda^2} = -\frac{n}{\lambda}$$

$$\frac{1}{n E\left(\frac{\partial^2 \ln f(x)}{\partial \lambda^2}\right)} = \frac{1}{\frac{-n}{\lambda}}$$

$$= -1 \times \frac{\lambda}{-n} = \frac{\lambda}{n}$$

Since

$$LHS = RHS = \lambda/n$$

Therefore, $\bar{y}$ is an efficient parameter of λ

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⑤ (a) $f(x,y) = \begin{cases} 24xy & 0<x<1,\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$

$Z = x+y$ $\quad W = x$ ①

since $x = W$

$Z = w + y \Rightarrow y = Z - w$ ②

The best technique to use here is transformation technique because we are given two equations for the joint distribution

$f(w,z) = g(x(w,z)\, y(w,z))\,|J|$

where $|J| = \begin{vmatrix} \frac{dx}{dw} & \frac{dx}{dz} \\ \frac{dy}{dw} & \frac{dy}{dz} \end{vmatrix}$

from $x = w$

$\frac{dx}{dw} = 1 \qquad \frac{dx}{dz} = 0$

from $y = z - w$

$\frac{dy}{dw} = -1 \qquad \frac{dy}{dz} = 1$

$|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1 - 0 = 1$

$g(x(w,z)\, y(w,z)) = 24(w)(z-w)$

$= 24w(z-w)$

$f(w,z) = 24w(z-w) \times 1$

$= 24w(z-w)$

Therefore

$f(w,z) = \begin{cases} 24w(z-w) & 0<w<1,\ 0<z<2 \\ 0 & \text{elsewhere} \end{cases}$

(b)(i) $Y \sim \exp(4)$

$f(y) = \begin{cases} \frac{1}{4}e^{-y/4} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$

$U = 3Y + 1$

We can use either distribution function technique or transformation technique to work the problem. But the transformation technique is the easiest.

So lets make y the subject of the formular $u = 3y + 1$

$u - 1 = 3y \Rightarrow y = \frac{u-1}{3}$

$y = \frac{u}{3} - \frac{1}{3}$

$f(u) = g(y(u))\,|J|$

$\left|\frac{dy}{du}\right| = \left|\frac{1}{3}\right| = \frac{1}{3}$

$g(y(u)) = \frac{1}{4} e^{-\frac{\left(\frac{u-1}{3}\right)}{4}} = \frac{1}{4} e^{-\left(\frac{u-1}{12}\right)}$

$= \frac{1}{4} e^{\frac{1-u}{12}}$

$f(u) = \frac{1}{4} e^{\frac{1-u}{12}} \times \frac{1}{3} = \frac{1}{12} e^{\frac{1-u}{12}}$

$$f(u) = \begin{cases} \frac{1}{12} e^{\frac{1-u}{12}} & u > 1 \\ 0 & \text{elsewhere} \end{cases}$$

(ii) $E(u) = \int_1^\infty u f(u)\, du$

$= \int_1^\infty u \cdot \frac{1}{12} e^{\frac{1-u}{12}}\, du$

$= \frac{1}{12} \int_1^\infty u e^{\frac{1-u}{12}}\, du$

Using integration by part

$\frac{1}{12}\left[u \cdot \frac{e^{\frac{1-u}{12}}}{-\frac{1}{12}} \Big|_1^\infty - \int_1^\infty \frac{e^{\frac{1-u}{12}}}{-\frac{1}{12}}\, du \right]$

$\frac{1}{12}\left[-12u e^{\frac{1-u}{12}} \Big|_1^\infty + \int_1^\infty 12 e^{\frac{1-u}{12}}\, du \right]$

$u = \frac{1-u}{12}$ $\quad \frac{dy}{du} = -\frac{1}{12}$

$12\,dy = -du$

$-12\,dy = du$

$\frac{u^2}{2} e^u + 12 e^{u} \frac{u^2}{2} - 12 e^{\frac{1-u}{12}}$

$\frac{u^2}{2} e^{\frac{-u}{12}} \times -12$

$\frac{1}{3} \int \frac{u^2}{2} e^u$

$$\frac{1}{12}\left[12+\left(\frac{12\cdot e^{\frac{1-y}{12}}}{-\frac{1}{12}}\Big|_{1}^{\infty}\right)\right].$$

$$\frac{1}{12}\left[12+\left(-144\, e^{\frac{1-y}{12}}\Big|_{1}^{\infty}\right)\right]$$

$$\frac{1}{12}\left[12+144\right]=\frac{1}{12}(156)=13$$

$$\therefore E(y)=13$$

(6) step 1
the prior distribution

$$\theta \sim U(0,1)$$

$$f(\theta)=\begin{cases}\frac{1}{1-0} & 0<\theta<1\\ 0 & \text{elsewhere}\end{cases}$$

$$f(\theta)=\begin{cases}1 & 0<\theta<1\\ 0 & \text{elsewhere}\end{cases}$$

step 2
the likelihood function

$$L(x_i/\theta)=\prod_{i=1}^{n}\theta(1-\theta)^{1-x}$$

$$=\theta^{n}(1-\theta)^{\Sigma(1-x)}$$

step 3
find the posterior

$$P(\theta)\propto f(\theta)\cdot L(x_i/\theta)$$

$$\propto 1\times\theta^{n}(1-\theta)^{\Sigma(1-x)}$$

$$\propto \theta^{n}(1-\theta)^{\Sigma(1-x)}$$

$$\propto \theta^{n+1-1}(1-\theta)^{\Sigma(1-x)+1-1}$$

$$\sim Beta\left(n+1,\ \Sigma(1-x)+1\right)$$

ACS811 (PROBABILITY & STATISTICS)
BSc. DEGREE EXAMINATION 2013/14 SESSION

QUESTION 1

Refer to Question NO 1 of 2014/15 session. it has been taken care of.

QUESTION 2

(a) The likelihood function

$$L(\theta) = \prod_{i=1}^{n} f(y_i/\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-y_i/\theta}$$

$$= \frac{1}{\theta} e^{-y_1/\theta} \times \frac{1}{\theta} e^{-y_2/\theta} \times \cdots \times \frac{1}{\theta} e^{-\frac{y_n}{\theta}}$$

$$= \left[\frac{1}{\theta}\right]^n e^{-\sum y_i/\theta}$$

$$= \theta^{-n} e^{-\frac{1}{\theta}\sum y}$$

take the log of likelihood function

$$\ln L(\theta) = \ln \theta^{-n} e^{-\frac{1}{\theta}\sum y}$$

$$= \ln \theta^{-n} + \ln e^{-1/\theta \sum y}$$

$$= -n \ln \theta - 1/\theta \sum y \ln e$$

but $\ln e = 1$

$$\ln L(\theta) = -n \ln \theta - \frac{1}{\theta}\sum y$$

taking the derivative of $\ln L(\theta)$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{-n}{\theta} + \frac{\sum y}{\theta^2}$$

equating to zero

$$-\frac{n}{\theta} + \frac{\sum y}{\theta^2} = 0$$

$$\frac{\sum y}{\theta^2} = \frac{n}{\theta} \Rightarrow \frac{\cancel{\theta}\sum y}{\cancel{\theta}} = \frac{\theta^2 n}{\theta}$$

$$\frac{\sum y}{n} = \frac{\theta n}{n} \Rightarrow \theta = \sum y/n$$

$$\theta = \sum y/n = \bar{y}$$

$\bar{y}$ is the maximum likelihood estimator of θ.

(b) (i) $y_i \sim P_o(\lambda)$

$$f(y_i/\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^{y_i}}{y!} & y = 0,1,2,3 \\ 0 & \text{elsewhere} \end{cases}$$

the likelihood function

$$L(\lambda) = \prod_{i=1}^{n} f(y_i/\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y}}{y!}$$

$$= \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \times \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \times \cdots \times \frac{e^{-\lambda}\lambda^{y_n}}{y_n!}$$

$$= \frac{e^{-n\lambda}\lambda^{\sum y}}{\prod_{i=1}^{n} y_i!}$$

taking the log of likelihood function

$$\ln L(\lambda) = \ln\left(\frac{e^{-n\lambda}\lambda^{\sum y}}{\prod_{i=1}^{n} y_i!}\right)$$

$$= \ln e^{-n\lambda} + \ln \lambda^{\sum y} - \ln \prod_{i=1}^{n} y_i!$$

taking the derivative of $\ln L(\lambda)$

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = -n + \frac{\sum y}{\lambda}$$

equating to zero

$$-n + \frac{\sum y}{\lambda} = 0$$

$$\frac{\sum y}{\lambda} = n \Rightarrow \sum y = n\lambda$$

$$\lambda = \sum y/n = \bar{y}$$

$\bar{y}$ is the maximum likelihood estimator of λ

(ii) $E(\hat{\lambda}) = E(\bar{y}) = E\left(\frac{y_1 + y_2 + \cdots y_n}{n}\right)$

$$= \frac{E(y_1) + E(y_2) + \cdots E(y_n)}{n}$$

$E(\bar{y}) = \frac{\lambda + \lambda + \ldots \lambda}{n}$

$= \frac{n\lambda}{n} = \lambda$

$\therefore E(\bar{y}) = \lambda$

$Var(\hat{\lambda}) = Var(\bar{y}) = Var\left(\frac{y_1 + y_2 + \ldots y_n}{n}\right)$

$= \frac{1}{n^2}\left[Var(y_1) + Var(y_2) + \ldots Var(y_n)\right]$

$= \frac{\lambda + \lambda + \ldots \ldots \lambda}{n^2} = \frac{n\lambda}{n^2}$

$Var(\bar{y}) = \frac{\lambda}{n}$

(iii) To test for consistency

$\lim_{n \to \infty} Var(\bar{y}) = 0$

$\lim_{n \to \infty} \frac{\lambda}{n} = 0$

$\bar{y}$ is a consistent statistic for parameter λ

QUESTION 3

(a) Refer to Question 5(b) of 2014/2015 session, it has been taken care of

(b) $y \sim U(1,5)$

$$f(y) = \begin{cases} \frac{1}{4} & 1 < y < 5 \\ 0 & \text{elsewhere} \end{cases}$$

$u = 2y^2 + 3$

Using transformation techniques making y the subject

$u - 3 = 2y^2 \Rightarrow y^2 = \frac{u-3}{2}$

$y = \left(\frac{u-3}{2}\right)^{1/2}$

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$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x-3}{2}\right)^{-\frac{1}{2}} \times \frac{1}{2}$$

$$= \left|\frac{1}{4}\left(\frac{x-3}{2}\right)^{-\frac{1}{2}}\right| = \frac{1}{4\left(\frac{x-3}{2}\right)^{\frac{1}{2}}}$$

$$f(x) = f(g(x))\,|J|$$

$$= \frac{1}{4} \cdot \frac{1}{4\left(\frac{x-3}{2}\right)^{\frac{1}{2}}} = \frac{1}{16\left(\frac{x-3}{2}\right)^{\frac{1}{2}}}$$

$$f(x) = \begin{cases} \dfrac{1}{16\sqrt{\frac{x-3}{2}}} & 5 < x < 53 \\ 0 & \text{elsewhere} \end{cases}$$

Question 4

Refer to Question 4 of 2014/2015 session. this has been ~~taken~~ taken care of

Question 6

(a) this is Question 6 of 2014/2015 session and it has been taken care of

(b)

$$f(x) = \begin{cases} \frac{2x}{12} & x = 1, 2, 3 \\ 0 & \text{elsewhere} \end{cases}$$

(a) $E(x) = \sum_{x} x \cdot P(x)$

$$= \sum_{x=1}^{3} x \cdot \frac{2x}{12}$$

$$= \frac{2}{12} + \frac{8}{12} + \frac{18}{12}$$

$$= \frac{2+8+18}{12} = \frac{28}{12} = \frac{7}{3}$$

$$= 2\tfrac{1}{3}$$

(b) $Var(x) = E(x^2) - (E(x))^2$

$$E(x^2) = \sum_{x} x^2 p(x)$$

$$= \sum_{x=1}^{3} x^2 \cdot \frac{2x}{12}$$

$$= \frac{2}{12} + \frac{16}{12} + \frac{54}{12}$$

$$= \frac{2+16+54}{12} = \frac{72}{12}$$

$$= 6$$

$$Var(x) = 6 - \left(\frac{7}{3}\right)^2$$

$$= 6 - \frac{49}{9} = \frac{54-49}{9}$$

$$= \frac{5}{9}$$

(c) $\sum_{x} \sum_{y} f(x,y) = 1$

$$\sum_{x=1}^{4} \sum_{y=1}^{4} Kxy = 1$$

$$\sum_{x=1}^{4} Kx + 2Kx + 3Kx + 4Kx = 1$$

$$\sum_{x=1}^{4} 10Kx = 1$$

$$10K + 20K + 30K + 40K = 1$$

$$100K = 1$$

$$K = \frac{1}{100}$$