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1)

a) Conditions for all N cases $(N=0, N=1, N=2)$

$\underline{N=0}$

$$\left.\begin{aligned}\frac{\ell}{2}|\sin\theta| &< y\\ \frac{\ell}{2}|\sin\theta| &< d-y\end{aligned}\right\} \longrightarrow \boxed{\frac{\ell}{2}|\sin\theta| < y < d-\frac{\ell}{2}|\sin\theta|}$$

$+$

$$|\sin\theta| < \frac{d}{\ell} \longrightarrow \boxed{-\sin^{-1}\left(\frac{d}{\ell}\right) < \theta < \sin^{-1}\left(\frac{d}{\ell}\right)}$$

$\underline{N=2}$

$$\left.\begin{aligned}\frac{\ell}{2}|\sin\theta| &> y\\ \frac{\ell}{2}|\sin\theta| &> d-y\end{aligned}\right\} \longrightarrow d-\frac{\ell}{2}|\sin\theta| < y < \frac{\ell}{2}|\sin\theta|$$

$+$

$$|\sin\theta| > \frac{d}{\ell} \Bigg\} \longrightarrow \begin{aligned}&\text{if } \theta>0, \quad \sin^{-1}\left(\frac{d}{\ell}\right) < \theta < \frac{\pi}{2}\\ &\text{if } \theta<0, \quad \frac{-\pi}{2} < \theta < -\sin^{-1}\left(\frac{d}{\ell}\right)\end{aligned}$$

$N=1$

$0<y<\frac{d}{2}$

$$\begin{cases} \frac{\ell}{2}|\sin\theta| > y \\ \frac{\ell}{2}|\sin\theta| < d-y \end{cases}$$

$\rightarrow y<\frac{\ell}{2}|\sin\theta|<d-y$

$\downarrow$

$$\frac{2y}{\ell}<|\sin\theta|<\frac{2(d-y)}{\ell}$$

if $\theta>0$, $\sin^{-1}\left(\frac{2y}{\ell}\right)<\theta<\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)$

if $\theta<0$, $-\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)<\theta<-\sin^{-1}\left(\frac{2y}{\ell}\right)$

$\frac{d}{2}<y<d$

$$\frac{\ell}{2}|\sin\theta|>d-y$$

$$\frac{\ell}{2}|\sin\theta|<y$$

$\downarrow$

$$\frac{2(d-y)}{\ell}<|\sin\theta|<\frac{2y}{\ell}$$

if $\theta>0$, $\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)<\theta<\sin^{-1}\left(\frac{2y}{\ell}\right)$

if $\theta<0$, $-\sin^{-1}\left(\frac{2y}{\ell}\right)<\theta<-\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)$

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* Taking integrals of each cases

$N=0$

$$\int_{-\sin^{-1}(\frac{d}{\ell})}^{\sin^{-1}(\frac{d}{\ell})}\int_{\frac{\ell}{2}|\sin\theta|}^{d-\frac{\ell}{2}|\sin\theta|}\frac{\cos\theta}{2d}\,dy\,d\theta = \frac{1}{2d}\int_{-\sin^{-1}(\frac{d}{\ell})}^{\sin^{-1}(\frac{d}{\ell})}\cos\theta(d-\ell|\sin\theta|)\,d\theta$$

$$=\frac{1}{2\cancel{d}}\cancel{d}\int_{-\sin^{-1}(\frac{d}{\ell})}^{\sin^{-1}(\frac{d}{\ell})}\cos\theta\,d\theta-\frac{\cancel{2}\ell}{\cancel{2}d}\int_{0}^{\sin^{-1}(\frac{d}{\ell})}\sin\theta\cos\theta\,d\theta=\boxed{\frac{d}{2\ell}}=P(N=0)$$

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$N=1$

$0 < y < \frac{d}{2}$

$$2 \cdot \frac{1}{2d} \int_0^{d/2} \int_{\sin^{-1}\left(\frac{2y}{\ell}\right)}^{\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)} \cos\theta \, d\theta \, dy = \frac{1}{d} \int_0^{d/2} \frac{2d-4y}{\ell} dy$$

since $\theta > 0$ and $\theta < 0$ give same results

$$\frac{2}{d\ell} \int_0^{d/2} d-2y \, dy = \frac{2}{d\ell} \left[dy - y^2\right] \Big|_0^{d/2} = \frac{d}{2\ell}$$

$\frac{d}{2} < y < d$

When we follow similar steps as above, we'll again find the same result.

$$2 \cdot \frac{1}{2d} \int_{d/2}^{d} \int_{\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)}^{\sin^{-1}\left(\frac{2y}{\ell}\right)} \cos\theta \, d\theta \, dy = \frac{d}{2\ell}$$

Then, we sum the results up.

$$P(N=1) = \frac{d}{2\ell} + \frac{d}{2\ell} = \boxed{\frac{d}{\ell}}$$

$\underline{N=2}$

$$2\cdot\frac{1}{2d}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\int_{d-\frac{\ell}{2}|\sin\theta|}^{\frac{\ell}{2}|\sin\theta|}\cos\theta\, dy\, d\theta = \frac{\cancel{2}}{\cancel{2}d}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\cos\theta\left[\ell|\sin\theta|-d\right]d\theta$$

since $\theta>0$ and $\theta<0$ give same results

$$P(N=2)=\frac{\ell}{d}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\cos\theta\sin\theta\, d\theta-\frac{\cancel{d}}{\cancel{d}}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\cos\theta\, d\theta=\boxed{\frac{\ell}{2d}-\frac{d}{2\ell}-1+\frac{d}{\ell}}$$

$$P(N=0)+P(N=1)+P(N=2)=1$$

$$=\cancel{\frac{d}{2\ell}}+\frac{d}{\ell}+\frac{\ell}{2d}-\cancel{\frac{d}{2\ell}}+\frac{d}{\ell}-1=1$$

$$=\frac{2d}{\ell}+\frac{\ell}{2d}=2 \Rightarrow \ell^2-4d\ell+4d^2=0=(\ell-2d)^2$$

$$\Rightarrow \ell=2d$$

$$\Rightarrow P(N=0)=\frac{d}{2\ell}=\frac{1}{4}$$

$$P(N=1)=\frac{d}{\ell}=\frac{1}{2}$$

$$P(N=2)=\frac{1}{4}$$

$$\boxed{P_N(n)=\begin{cases}1/4 & \text{if } n=0\\ 1/2 & \text{if } n=1\\ 1/4 & \text{if } n=2\end{cases}}$$

b) $$f_{Y|\{N=2\}}(y) = \frac{f_Y(y)\, P(N=2|Y=y)}{\int f_Y(t)\, P(N=2|Y=t)\,dt}$$

$$f_Y(y) = \frac{1}{2d}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{1}{d} \Rightarrow \begin{cases} \frac{1}{d}, & 0 \le y \le d \\ 0, & \text{o.w.} \end{cases}$$

$$f_{Y|\{N=2\}}(y) = \frac{\frac{1}{d}\,\frac{1}{4}}{\int_0^d \frac{1}{d}\,\frac{1}{4}\,dt} = \boxed{\frac{1}{d}} \text{ for } 0 < y < d$$

c) $$f_{\Theta|N}(\theta|0) = \frac{f_\Theta(\theta)\, P(N=0|\Theta=\theta)}{\int f_\Theta(t)\, P(N=0|\Theta=t)\,dt}$$

$$f_\Theta(\theta) = \int_0^d \frac{\cos\theta}{2d}\, dy = \frac{\cos\theta}{2} = \begin{cases} \frac{\cos\theta}{2} & \frac{-\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 & \text{o.w.} \end{cases}$$

$$f_{\Theta|\{N=0\}}(\theta) = \frac{\frac{\cos\theta}{\cancel{2}}\,\cancel{\frac{1}{4}}}{\int_{-\sin^{-1}(\frac{1}{2})}^{\sin^{-1}(\frac{1}{2})} \frac{\cos t}{\cancel{2}}\,\cancel{\frac{1}{4}}\,dt} = \frac{\cos\theta}{\frac{1}{2}+\frac{1}{2}} = \boxed{\cos\theta}$$

for $-\sin^{-1}(\frac{1}{2}) < \theta < \sin^{-1}(\frac{1}{2})$

2)

a)

$$Y = g(X) = \begin{cases} 1, & -2 \le X < -1 \Rightarrow y = 1 \\ -X, & -1 \le X < 0 \Rightarrow 0 \le y \le 1 \\ X, & 0 \le X < 1 \Rightarrow 0 \le y \le 1 \\ (X-2)^2, & 1 \le X \le 2 \Rightarrow 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_X(x) = \begin{Bmatrix} 1, & 0 \le X \le 1 \\ 0, & \text{o.w.} \end{Bmatrix}, \quad Y = X \text{ for } 0 \le X < 1$$

$$f_Y(y) = \frac{1}{|1|} f_X\left(\frac{y}{1}\right) = f_X(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

b) $$f_X(x) = \begin{Bmatrix} \frac{1}{3}, & -1 \le X \le 2 \\ 0, & \text{o.w.} \end{Bmatrix},$$

$Y_1 = -X$ for $-1 \le X < 0$

$Y_2 = X$ for $0 \le X < 1$

$Y_3 = (X-2)^2$ for $1 \le X \le 2$

$Y = Y_1 + Y_2 + Y_3$

$$F_{Y_1}(y) = \int_{(x \mid -x \le y,\ -1 \le x < 0)} f_X(x)\,dx \rightarrow -y \le x,\ -1 \le x < 0$$

$$= \int_{-y}^{0} \frac{1}{3}\,dx = \frac{y}{3} \text{ for } 0 \le y \le 1$$

$$\Rightarrow f_{Y_1}(y) = \frac{\partial}{\partial y} F_{Y_1}(y) = \frac{1}{3} = \begin{cases} \frac{1}{3}, & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$F_{Y_2}(y) = \int\limits_{(x \mid x \le y,\ 0 \le x < 1)} f_X(x)\,dx = \int_0^y \frac{1}{3}\,dx = \frac{y}{3} \quad \text{for } 0 \le y \le 1$$

$$f_{Y_2}(y) = \begin{cases} \frac{1}{3}, & 0 \le y \le 1 \\ 0, & \text{o.w} \end{cases}$$

$$F_{Y_3}(y) = \int\limits_{(x \mid (x-2)^2 \le y,\ 1 \le x \le 2)} f_X(x)\,dx \rightarrow -\sqrt{y} \le x-2 \le \sqrt{y} \quad \text{also}$$

$$-\sqrt{y}+2 \le x \le \sqrt{y}+2$$

$$1 \le x \le 2$$

if $1 \le x \le 2$, $0 \le y \le 1$ (it is given at the beginning)

$$= \int_{-\sqrt{y}+2}^{2} \frac{1}{3}\,dx = \frac{\sqrt{y}}{3} \quad \text{for } 0 \le y \le 1$$

$$f_{Y_3}(y) = \begin{cases} \frac{1}{6\sqrt{y}}, & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_Y(y) = f_{Y_1}(y) + f_{Y_2}(y) + f_{Y_3}(y) = \begin{cases} \frac{2}{3} + \frac{1}{6\sqrt{y}}, & 0 \le y \le 1 \\ 0, & \text{o.w} \end{cases}$$

c) $f_X(x) = \left\{ \begin{array}{ll} \frac{1}{4}, & -2 \le x \le 2 \\ 0, & \text{o.w.} \end{array} \right\}$

$Y_1 = 1$ for $-2 \le x < -1$

$Y_2 = -x$ for $-1 \le x < 0$

$Y_3 = x$ for $0 \le x < 1$

$Y_4 = (x-2)^2$ for $1 \le x \le 2$

$$Y = Y_1 + Y_2 + Y_3 + Y_4$$

$$F_{Y_1}(y) = \int\limits_{(x \mid 1 \le y,\ -2 < x < -1)} f_X(x)\,dx = \int_{-2}^{-1} \frac{1}{4}\,dx = \frac{1}{4} \quad \text{for } y = 1$$

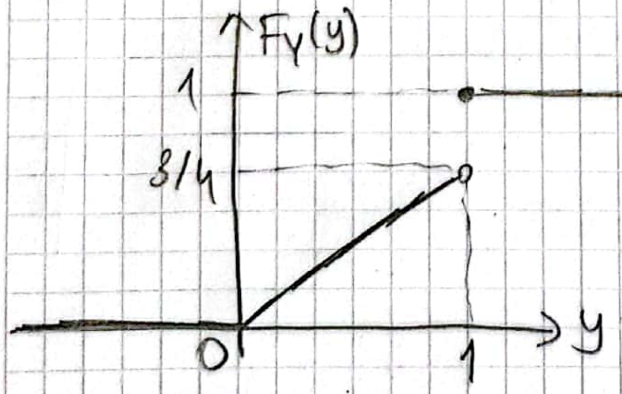
$$F_{Y_2}(y) = \int\limits_{(x \mid -x \le y,\ -1 \le x < 0)} f_X(x)\,dx = \int_{-y}^{0} \frac{1}{4}\,dx = \frac{y}{4} \quad \text{for } 0 \le y \le 1$$

$$F_{Y_3}(y) = \int\limits_{(x \mid x \le y,\ 0 \le x < 1)} f_X(x)\,dx = \int_{0}^{y} \frac{1}{4}\,dx = \frac{y}{4} \quad \text{for } 0 \le y \le 1$$

$$F_{Y_4}(y) = \int\limits_{(x \mid (x-2)^2 \le y,\ 1 \le x \le 2)} f_X(x)\,dx = \int_{-\sqrt{y}+2}^{2} \frac{1}{4}\,dx = \frac{\sqrt{y}}{4} \quad \text{for } 0 \le y \le 1$$

$$F_Y(y) = F_{Y_1}(y) + F_{Y_2}(y) + F_{Y_3}(y) + F_{Y_4}(y)$$

$$= F_Y(y) = \begin{cases} 0, & y < 0 \\ \dfrac{\sqrt{y}}{4} + \dfrac{y}{2}, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$



Y is a mixed random variable.

d) $f_Z(z) = \begin{cases} 1, & 0 \le z \le 1 \\ 0, & \text{o.w.} \end{cases}$ $\quad f_W(w) = \begin{cases} 1, & 0 \le w \le 1 \\ 0, & \text{o.w.} \end{cases}$

$$F_V(v) = P\left(\frac{Z}{W+1} \le v\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{vw+v} \underbrace{f_{Z,W}(z,w)}_{f_Z(z)\cdot f_W(w) \text{ since they are independent}} dz\, dw$$

$\Downarrow$

$z \le vw+v$

$$= \int_{-\infty}^{\infty} f_W(w) \underbrace{\int_{-\infty}^{vw+v} f_Z(z)dz}_{F_Z(vw+v)}\, dw = F_V(v)$$

$$f_V(v) = \frac{\partial}{\partial v} F_V(v) = \frac{\partial}{\partial v} \int_{-\infty}^{\infty} f_W(w) F_Z(vw+v) dw$$

$$= \int_{-\infty}^{\infty} f_W(w) \left[\frac{d}{dv} F_Z(vw+v)\right] dw = \int_{-\infty}^{\infty} f_W(w) f_Z(vw+v)(w+1) dw$$

$*$ $f_W(w)$ and $f_Z(vw+1)$ are not zero when $0 \le w \le 1$

$0 \le vw+v \le 1$

$\Downarrow$

$-v \le vw \le 1-v$

If $v>0$, $\cancel{\frac{-v}{v}}\ -1 \le w \le \underbrace{\frac{1}{v} - 1}$ → condition for this to be positive or zero

$0 < v \le 1$

then $0 \le w \le 1$, $-1 \le w \le \frac{1}{v} - 1$ } for $0 < v \le 1$, $\frac{1}{v} - 1 \le 1$ $(\frac{1}{2} \le v)$

the condition for w becomes $0 \le w \le \frac{1}{v} - 1$

if $\frac{1}{2} \le v \le 1$

$0 \le w \le 1$ } for $0 < v < \frac{1}{2}$

the condition for w becomes $0 \le w \le 1$

if $0 < v < \frac{1}{2}$

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* for $\frac{1}{2} \le v \le 1, \quad 0 \le w \le \frac{1}{v} - 1$

$$f_{V_1}(v) = \int_0^{\frac{1}{v}-1} (w+1)\,dw = \frac{1}{2v^2} - \frac{1}{2} \quad \text{for } \frac{1}{2} \le v \le 1$$

* for $0 < v < \frac{1}{2}, \quad 0 \le w \le 1$

$$f_{V_2}(v) = \int_0^{1} (w+1)\,dw = \frac{3}{2} \quad \text{for } 0 < v < \frac{1}{2}$$

$$f_V(v) = f_{V_1}(v) + f_{V_2}(v) = \begin{cases} \frac{3}{2}, & 0 < v < \frac{1}{2} \\ \frac{1}{2v^2} - \frac{1}{2}, & \frac{1}{2} \le v \le 1 \\ 0, & \text{o.w.} \end{cases}$$

3a)

For given different Y values, the range of X changes for the figures with circle and ellipses. Therefore, we have independent random variables only for the first figure, and the covariance for the figure is zero. For the second and third figures, one variable has no tendency to increase or decrease as other variable increases or decreases, hence they are uncorrelated. For the last figure, one variable has tendency to increase as other variable increases and vice versa, therefore, they are positively correlated.

3b)

$X \sim \text{Uniform}(0,5)$

$f_X(x) = \begin{cases} 1/5, & 0<x<5 \\ 0, & 0 \end{cases}$

$Y = g(X) = \begin{cases} 3x+1, & x<2 \\ x, & \text{o.w.} \end{cases}$

$E[Y] = E[g(x)] = \int_0^2 (3x+1)\frac{1}{5}dx + \int_2^5 x\frac{1}{5}dx$

$= \frac{1}{5}\left(\left(\frac{3x^2}{2}+x\right)\Big|_0^2 + \left(\frac{x^2}{2}\right)\Big|_2^5\right)$

$= \frac{1}{5}\left(6+\cancel{2}+\frac{25}{2}-\cancel{2}\right)$

$\boxed{E[Y] = \frac{37}{10}}$

$E[XY] = \int_0^2 x(3x+1)\frac{1}{5}dx + \int_2^5 x^2\frac{1}{5}dx$

$= \frac{1}{5}\left(\left(x^3+\frac{x^2}{2}\right)\Big|_0^2 + \left(\frac{x^3}{3}\right)\Big|_2^5\right)$

$= \frac{1}{5}\left(8+2+\frac{125-8}{3}\right)$

$\boxed{E[XY] = \frac{147}{15}}$

$E[X] = \int_0^5 x\frac{1}{5}dx = \left(\frac{x^2}{10}\right)\Big|_0^5 = \frac{5}{2}$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$= \frac{147}{15} - \frac{5}{2}\cdot\frac{37}{10}$

$= \frac{588-555}{60}$

$\boxed{cov(X,Y) = \frac{11}{20}}$

They have nonzero covariance, so they are dependent.

3c)

i)

$E[g(x)+Y \mid X=2] = E[g(x) \mid X=2] + E[Y \mid X=2]$

$= g(2) + E[Y \mid X=2]$

ii) Let $E[g(X)Y \mid X] = h_1(X)$ and $g(X)E[Y \mid X] = h_2(X)$

For an arbitrary $X = x_0$

$h_1(x_0) = E[g(X)Y \mid X=x_0]$

$= E[g(x_0)Y \mid X=x_0]$

$= g(x_0)E[Y \mid X=x_0] \rightarrow$

$h_2(x_0) = g(x_0)E[Y \mid X=x_0]$

They have same value for any $X = x_0$, hence

$h_1(X) = h_2(X) \Rightarrow E[g(X)Y \mid X] = g(X)E[Y \mid X]$

iii) $f_X(x) = \begin{cases} 1/2, & 0<x<2 \\ 0, & \text{o.w.} \end{cases}$

$E[X^2] = \int_0^2 x^2\frac{1}{2}dx = \left(\frac{x^3}{6}\right)\Big|_0^2 = 4/3$

$E[X^3] = \int_0^2 x^3\frac{1}{2}dx = \left(\frac{x^4}{8}\right)\Big|_0^2 = 2$

$E[Y \mid X] = X^3$

$E[E[Y \mid X]] = E[X^3]$

$E[Y] = E[X^3] = 2$

$E[X^2+Y] = E[X^2] + E[Y]$

$= 4/3 + 2.$

$\boxed{E[X^2+Y] = \frac{10}{3}}$

4) $P_N(k) = e^{-a}\frac{a^k}{k!}$ $\quad M_N(s) = E[e^{sN}] = \sum_{k=0}^{\infty} e^{sk} e^{-a}\frac{a^k}{k!} = e^{-a}\sum_{k=0}^{\infty}\frac{(e^s a)^k}{k!}$

$$= e^{-a} e^{(ae^s)}$$

$$\boxed{M_N(s) = e^{a(e^s-1)}}$$

$f_{X_i}(x) = \lambda e^{-\lambda x}$

$M_{X_i}(s) = \frac{\lambda}{\lambda - s}$

$$M_Y(s) = M_N(\log M_X(s))$$

$$= e^{a(e^{\log M_X(s)}-1)}$$

$$= e^{a(M_X(s)-1)}$$

$$\boxed{M_Y(s) = e^{\frac{as}{\lambda-s}}}$$

$P_{X_i}(x) = \begin{cases} p, & x=1 \\ 1-p, & x=0 \\ 0, & \text{o.w.} \end{cases}$

$M_{X_i}(s) = \sum_{k=0}^{1} e^{sk} P_{X_i}(k)$

$= 1-p+pe^s$

$$M_Y(s) = M_N(\log M_X(s))$$

$$= e^{a(e^{\log M_X(s)}-1)}$$

$$= e^{a(M_X(s)-1)}$$

$\boxed{M_Y(s) = e^{ap(-1+e^s)}} \longrightarrow Y$ is a poisson r.v with mean ap

$$\boxed{P_Y(x) = e^{-ap}\frac{(ap)^x}{x!}}$$

5a)

$$Z_n = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} \frac{X_i - 1}{1}\right) = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} X_i - n\right)$$

$$P_{X_i}(k) = e^{-1}\frac{1^k}{k!} = \frac{1}{ek!}$$

$$Z_n\sqrt{n} + n = \sum_{i=1}^{n} X_i$$

$$M_{X_i}(s) = e^{(e^s - 1)}$$

$$E\left[e^{s(X_1 + X_2 \cdots X_n)}\right] = \left(E\left[e^{sX_i}\right]\right)^n = e^{n(e^s - 1)} \leftarrow \text{Transform of poisson with } \mu = n$$

$\sqrt{n}Z_n + n$ is a poisson random variable with mean n.

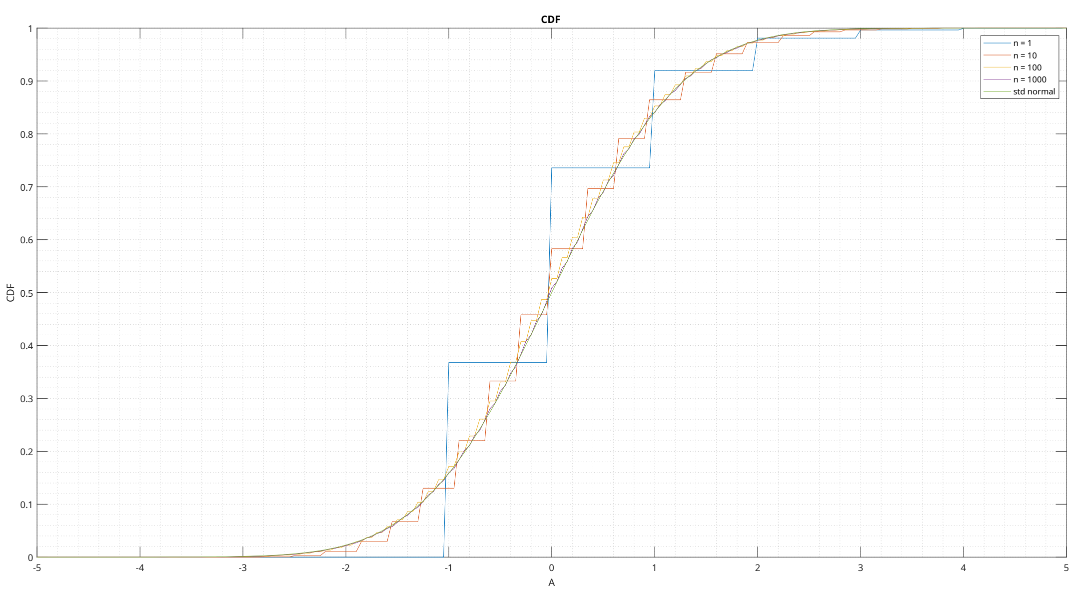

Figure 1: CDF of Zn for n = {1, 10, 100, 1000} and standard normal

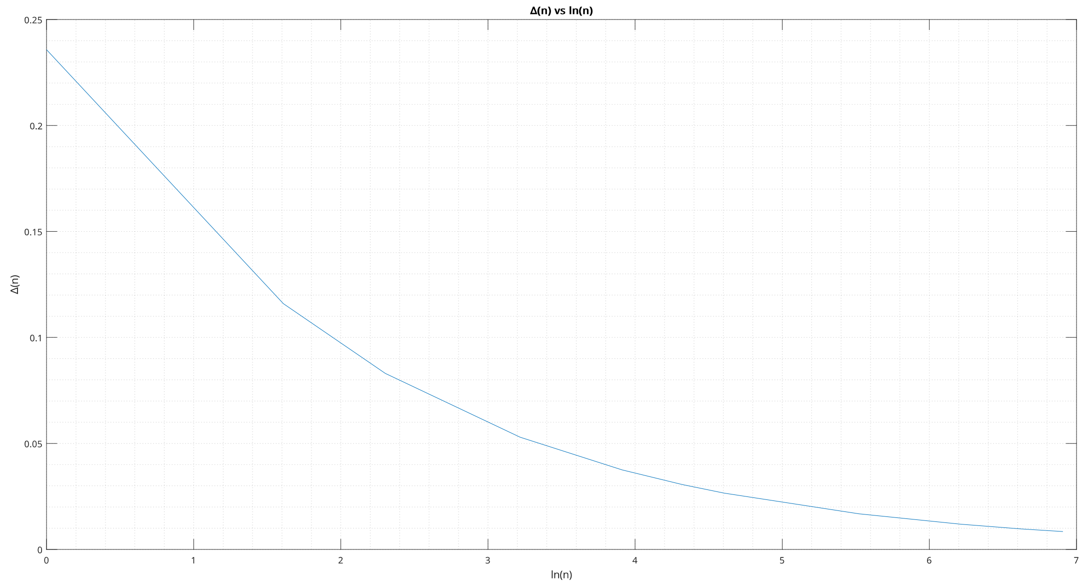

Figure 2: Graph of delta(n) vs ln(n)

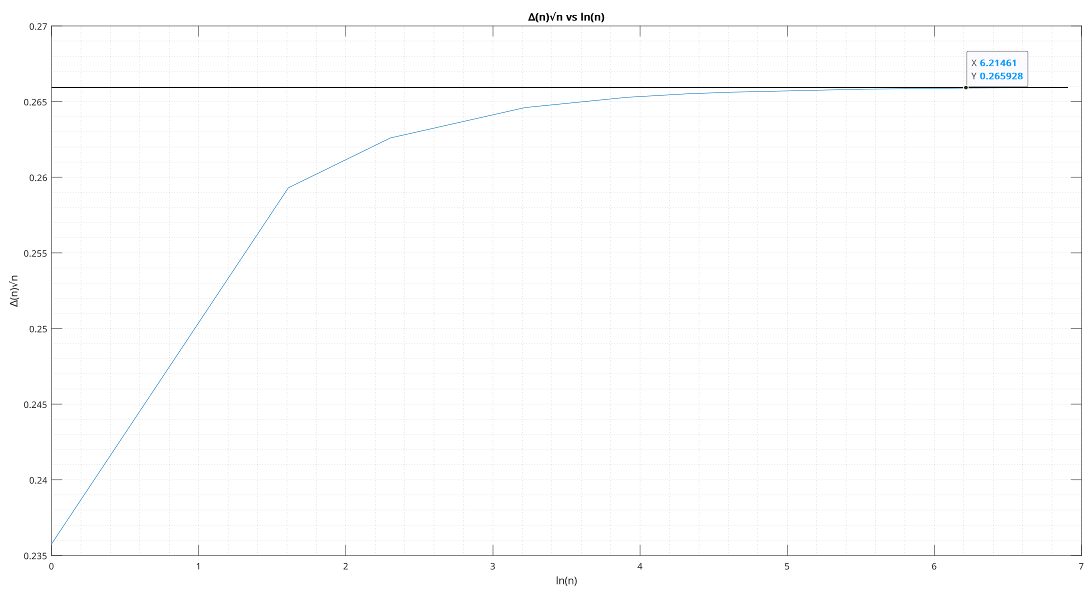

*Figure 3: Graph of delta(n)*sqrt(n) vs ln(n)*

From figure 1, we can see that as n approahes to infinity, cdf of Zn approaches to cdf of standard normal.
From figure 2, we can see that the maximum difference between cdf of Zn and cdf of standard normal for the values of A decreases as n increases.
From figure 3, we can see that one can find a K > 0.27 that satisfies the condition stated in step h for the values of n considered.