Homework 3
Due 23:59 Sunday May 29, 2022

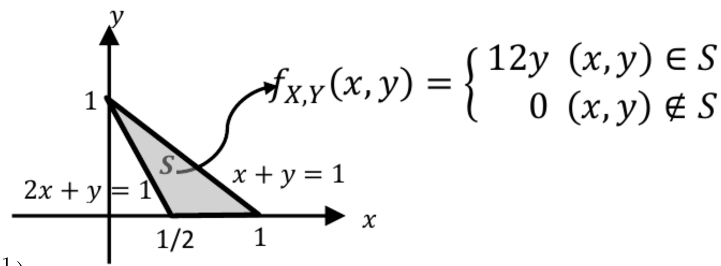
1. Let X be a random variable with the following probability density function

$$f_X(x) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} & x \le \ln 4 \\ \lambda e^{-\lambda x} & x > \ln 4 \end{cases}$$

where σ and λ are some positive constants and $\mathbf{E}[\mathsf{X}] = \ln 4$.

   (a) Determine the value of λ?
   (b) Determine the value of σ?
   (c) Determine variance of the random variable X?
   (d) Determine the CDF of the random variable X in terms of elementary functions and the CDF of a standard normal random variable?

2. *(A Question from Second Midterm Exam of EE230, 2019 Spring)* Assume X and Y are jointly continuous random variables whose joint PDF is given in the figure.



   (a) Find $P(X \le \frac{1}{2})$.
   (b) Find $f_Y(y)$.
   (c) Find $F_{X,Y}(\frac{1}{2}, \frac{1}{2})$.
   (d) Determine the region in xy-plane, where $F_{X,Y}(x,y) = 1$.

3. Consider jointly Gaussian zero mean random variables X and Y that have the following joint PDF:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \rho\frac{2xy}{\sigma_x\sigma_y}\right)\right],$$

which is known as *bivariate normal distribution* with parameters σ_x^2, σ_y^2 and ρ. Here, $\sigma_x, \sigma_y > 0$, and $\rho \in (-1, 1)$ are all constants.

   (a) Prove that X and Y are zero mean Gaussian random variables with variances σ_x^2 and σ_y^2 respectively by deriving their marginal PDFs $f_X(x)$ and $f_Y(y)$.
   (b) Find $E_{X,Y}[XY]$ in terms of σ_x^2, σ_y^2 and ρ.
   (c) Prove that the conditional density $f_{X|Y}(x|y)$ corresponds to another Gaussian random variable. Then, find its mean $\mu_{x|y} = E_{X|Y}[X|Y=y]$.
   (d) The minimum mean square error (MMSE) estimate of X (given Y), denoted by $\hat{X}$, can be obtained by just finding the conditional expectation of X as $\hat{X} = \mu_{x|y}$.
      (i) The estimator is in the form of $\hat{X} = \alpha Y$. Determine α in terms of σ_x^2, σ_y^2 and ρ.
      (ii) Calculate the mean square error (MSE) of this estimate, namely $\text{MSE} = E\left[\left(X - \hat{X}\right)^2\right]$ in terms of σ_x^2, σ_y^2 and ρ.

4. (Matlab Question) We validate the analytical result in Q3-(d) via Monte Carlo (MC) Simulation. MC method is a convenient way of approximating an expectation by getting the sample mean of a function of simulated random variables [E. Anderson, Lecture Notes on Monte Carlo Methods and Importance Sampling, Oct. 1999, Available online]. This method method invokes "Weak Laws of Large Numbers (WLLN)" to approximate expectations. Here, we can obtain the MSE = $E\left[\left(X-\hat{X}\right)^2\right]$ in Q3-(d) with the help of MC expectation over the sequence of samples $\{X_k, Y_k\}$, where X_k and Y_k are zero mean jointly Gaussian random variables. That is to say, we can calculate the MSE by getting the time average of the sequence $\left\{\left(X_k-\hat{X}_k\right)^2\right\},\ k=0,\dots,N-1$.
For $\sigma_x^2=\sigma_y^2=1$, obtain MSE both analytically and numerically (via MC Simulation) for $\rho=-1:0.1:1$. Then, plot these MSE vs ρ curves (theoretical and simulation). Compare your results and comment.
PS: You can use the attached Matlab script to generate sample sequence $\{X_k, Y_k\},\ k=0,\dots,N-1$.