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HW4

1)

a) Conditions for all N cases ($N=0$, $N=1$, $N=2$)

$N=0$

$$\left.\begin{aligned} \frac{\ell}{2}|\sin\theta| &< y \\ \frac{\ell}{2}|\sin\theta| &< d-y \end{aligned}\right\} \longrightarrow \boxed{\frac{\ell}{2}|\sin\theta| < y < d-\frac{\ell}{2}|\sin\theta|}$$

$+$

$$|\sin\theta| < \frac{d}{\ell} \longrightarrow \boxed{-\sin^{-1}\left(\frac{d}{\ell}\right) < \theta < \sin^{-1}\left(\frac{d}{\ell}\right)}$$

$N=2$

$$\left.\begin{aligned} \frac{\ell}{2}|\sin\theta| &> y \\ \frac{\ell}{2}|\sin\theta| &> d-y \end{aligned}\right\} \longrightarrow d-\frac{\ell}{2}|\sin\theta| < y < \frac{\ell}{2}|\sin\theta|$$

$+$

$$|\sin\theta| > \frac{d}{\ell} \longrightarrow \begin{cases} \text{if } \theta > 0, & \sin^{-1}\left(\frac{d}{\ell}\right) < \theta < \frac{\pi}{2} \\ \text{if } \theta < 0, & \frac{-\pi}{2} < \theta < -\sin^{-1}\left(\frac{d}{\ell}\right) \end{cases}$$

$\underline{N=1}$

$$0<y<\frac{d}{2}$$

$$\begin{cases} \frac{\ell}{2}|\sin\theta| > y \\ \frac{\ell}{2}|\sin\theta| < d-y \end{cases}$$

$$\rightarrow y < \frac{\ell}{2}|\sin\theta| < d-y$$

$$\downarrow$$

$$\frac{2y}{\ell} < |\sin\theta| < \frac{2(d-y)}{\ell}$$

$$\frac{d}{2}<y<d$$

$$\frac{\ell}{2}|\sin\theta| > d-y$$

$$\frac{\ell}{2}|\sin\theta| < y$$

$$\downarrow$$

$$\frac{2(d-y)}{\ell} < |\sin\theta| < \frac{2y}{\ell}$$

if $\theta>0$, $\sin^{-1}\left(\frac{2y}{\ell}\right) < \theta < \sin^{-1}\left(\frac{2(d-y)}{\ell}\right)$ if $\theta>0$, $\sin^{-1}\left(\frac{2(d-y)}{\ell}\right) < \theta < \sin^{-1}\left(\frac{2y}{\ell}\right)$

if $\theta<0$, $-\sin^{-1}\left(\frac{2(d-y)}{\ell}\right) < \theta < -\sin^{-1}\left(\frac{2y}{\ell}\right)$ if $\theta<0$, $-\sin^{-1}\left(\frac{2y}{\ell}\right) < \theta < -\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)$

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* Taking integrals of each cases

$\underline{N=0}$

$$\int_{-\sin^{-1}(\frac{d}{\ell})}^{\sin^{-1}(\frac{d}{\ell})} \int_{\frac{\ell}{2}|\sin\theta|}^{d-\frac{\ell}{2}|\sin\theta|} \frac{\cos\theta}{2d}\, dy\, d\theta = \frac{1}{2d}\int_{-\sin^{-1}(\frac{d}{\ell})}^{\sin^{-1}(\frac{d}{\ell})} \cos\theta\left(d-\ell|\sin\theta|\right) d\theta$$

$$= \frac{1}{2\cancel{d}}\cancel{d}\int_{-\sin^{-1}(\frac{d}{\ell})}^{\sin^{-1}(\frac{d}{\ell})} \cos\theta\, d\theta - \frac{\cancel{2}\ell}{\cancel{2}d}\int_{0}^{\sin^{-1}(\frac{d}{\ell})} \sin\theta\cos\theta\, d\theta = \boxed{\frac{d}{2\ell}} = P(N=0)$$

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$N = 1$

$0 < y < \frac{d}{2}$

$$2 \cdot \frac{1}{2d} \int_0^{d/2} \int_{\sin^{-1}\left(\frac{2y}{\ell}\right)}^{\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)} \cos\theta \, d\theta \, dy = \frac{1}{d} \int_0^{d/2} \frac{2d-4y}{\ell} dy$$

since
$\theta > 0$ and $\theta < 0$
give same results

$$\frac{2}{d\ell} \int_0^{d/2} d-2y \, dy = \frac{2}{d\ell} \left[dy - y^2 \right] \Big|_0^{d/2} = \frac{d}{2\ell}$$

$\frac{d}{2} < y < d$

When we follow similar steps as above, we'll again find the same result.

$$2 \cdot \frac{1}{2d} \int_{d/2}^{d} \int_{\sin^{-1}\left(\frac{2(d-y)}{\ell}\right)}^{\sin^{-1}\left(\frac{2y}{\ell}\right)} \cos\theta \, d\theta \, dy = \frac{d}{2\ell}$$

Then, we sum the results up.

$$P(N=1) = \frac{d}{2\ell} + \frac{d}{2\ell} = \boxed{\frac{d}{\ell}}$$

$\underline{N=2}$

$$2\cdot\frac{1}{2d}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\int_{d-\frac{\ell}{2}|\sin\theta|}^{\frac{\ell}{2}|\sin\theta|}\cos\theta\, dy\, d\theta = \frac{\cancel{2}}{\cancel{2}d}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\cos\theta\left[\ell|\sin\theta|-d\right]d\theta$$

since $\theta>0$ and $\theta<0$ give same results

$$P(N=2) = \frac{\ell}{d}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\cos\theta\sin\theta\, d\theta - \frac{\cancel{d}}{\cancel{d}}\int_{\sin^{-1}(\frac{d}{\ell})}^{\pi/2}\cos\theta\, d\theta = \boxed{\frac{\ell}{2d}-\frac{d}{2\ell}-1+\frac{d}{\ell}}$$

$$P(N=0)+P(N=1)+P(N=2)=1$$

$$=\cancel{\frac{d}{2\ell}}+\frac{d}{\ell}+\frac{\ell}{2d}-\cancel{\frac{d}{2\ell}}+\frac{d}{\ell}-1=1$$

$$=\frac{2d}{\ell}+\frac{\ell}{2d}=2 \Rightarrow \ell^2-4d\ell+4d^2=0=(\ell-2d)^2$$

$$\Rightarrow \ell=2d$$

$$\left.\begin{aligned}\Rightarrow P(N=0)&=\frac{d}{2\ell}=\frac{1}{4}\\ P(N=1)&=\frac{d}{\ell}=\frac{1}{2}\\ P(N=2)&=\frac{1}{4}\end{aligned}\right\} \boxed{P_N(n)=\begin{cases}1/4 & \text{if } n=0\\ 1/2 & \text{if } n=1\\ 1/4 & \text{if } n=2\end{cases}}$$

b) $$f_{Y|\{N=2\}}(y) = \frac{f_Y(y)\, P(N=2|Y=y)}{\int f_Y(t)\, P(N=2|Y=t)\,dt}$$

$$f_Y(y) = \frac{1}{2d}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta = \frac{1}{d} \Rightarrow \begin{cases} \frac{1}{d}, & 0 \le y \le d \\ 0, & o.w. \end{cases}$$

$$f_{Y|\{N=2\}}(y) = \frac{\frac{1}{d}\,\frac{1}{4}}{\int_0^d \frac{1}{d}\,\frac{1}{4}\,dt} = \boxed{\frac{1}{d}} \text{ for } 0 < y < d$$

c) $$f_{\Theta|N}(\theta|0) = \frac{f_\Theta(\theta)\, P(N=0|\Theta=\theta)}{\int f_\Theta(t)\, P(N=0|\Theta=t)\,dt}$$

$$f_\Theta(\theta) = \int_0^d \frac{\cos\theta}{2d}\,dy = \frac{\cos\theta}{2} = \begin{cases} \frac{\cos\theta}{2} & \frac{-\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 & o.w. \end{cases}$$

$$f_{\Theta|\{N=0\}}(\theta) = \frac{\frac{\cos\theta}{\cancel{2}}\,\cancel{\frac{1}{4}}}{\int_{-\sin^{-1}(\frac{1}{2})}^{\sin^{-1}(\frac{1}{2})} \frac{\cos t}{\cancel{2}}\,\cancel{\frac{1}{4}}\,dt} = \frac{\cos\theta}{\frac{1}{2}+\frac{1}{2}} = \boxed{\cos\theta}$$

for $-\sin^{-1}(\frac{1}{2}) < \theta < \sin^{-1}(\frac{1}{2})$

2)

a)

$$Y = g(X) = \begin{cases} 1, & -2 \le X < -1 \Rightarrow y = 1 \\ -X, & -1 \le X < 0 \Rightarrow 0 \le y \le 1 \\ X, & 0 \le X < 1 \Rightarrow 0 \le y \le 1 \\ (X-2)^2, & 1 \le X \le 2 \Rightarrow 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_X(x) = \begin{Bmatrix} 1, & 0 \le X \le 1 \\ 0, & \text{o.w.} \end{Bmatrix}, \quad Y = X \text{ for } 0 \le X < 1$$

$$f_Y(y) = \frac{1}{|1|} f_X\left(\frac{y}{1}\right) = f_X(y) = \boxed{\begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}}$$

b) $$f_X(x) = \begin{Bmatrix} \frac{1}{3}, & -1 \le x \le 2 \\ 0, & \text{o.w.} \end{Bmatrix},$$

$Y_1 = -X$ for $-1 \le X < 0$

$Y_2 = X$ for $0 \le X < 1$

$Y_3 = (X-2)^2$ for $1 \le X \le 2$

$Y = Y_1 + Y_2 + Y_3$

$$F_{Y_1}(y) = \int_{(x \mid -x \le y,\ -1 \le x < 0)} f_X(x)dx \rightarrow -y \le x, \ -1 \le x < 0$$

$$= \int_{-y}^{0} \frac{1}{3} dx = \frac{y}{3} \text{ for } 0 \le y \le 1$$

$$\Rightarrow f_{Y_1}(y) = \frac{\partial}{\partial y} F_{Y_1}(y) = \frac{1}{3} = \begin{cases} \frac{1}{3}, & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$$F_{Y_2}(y) = \int\limits_{(x \mid x \leq y,\ 0 \leq x < 1)} f_X(x)\,dx = \int_0^y \frac{1}{3}\,dx = \frac{y}{3} \quad \text{for } 0 \leq y \leq 1$$

$$f_{Y_2}(y) = \begin{cases} \frac{1}{3}, & 0 \leq y \leq 1 \\ 0, & \text{o.w} \end{cases}$$

$$F_{Y_3}(y) = \int\limits_{(x \mid (x-2)^2 \leq y,\ 1 \leq x \leq 2)} f_X(x)\,dx$$

$(x \mid (x-2)^2 \leq y,\ 1 \leq x \leq 2) \rightarrow -\sqrt{y} \leq x-2 \leq \sqrt{y}$ also if $1 \leq x \leq 2$, $0 \leq y \leq 1$ (it is given at the beginning)

$-\sqrt{y}+2 \leq x \leq \sqrt{y}+2$

$1 \leq x \leq 2$

$$= \int_{-\sqrt{y}+2}^{2} \frac{1}{3}\,dx = \frac{\sqrt{y}}{3} \quad \text{for } 0 \leq y \leq 1$$

$$f_{Y_3}(y) = \begin{cases} \frac{1}{6\sqrt{y}}, & 0 \leq y \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

$$f_Y(y) = f_{Y_1}(y) + f_{Y_2}(y) + f_{Y_3}(y) = \begin{cases} \frac{2}{3} + \frac{1}{6\sqrt{y}}, & 0 \leq y \leq 1 \\ 0, & \text{o.w} \end{cases}$$

c) $f_X(x) = \left\{ \begin{array}{ll} \frac{1}{4}, & -2 \leq x \leq 2 \\ 0, & \text{o.w.} \end{array} \right\}$

$Y_1 = 1$ for $-2 \leq X < -1$

$Y_2 = -X$ for $-1 \leq X < 0$

$Y_3 = X$ for $0 \leq X < 1$

$Y_4 = (X-2)^2$ for $1 \leq X \leq 2$

$$Y = Y_1 + Y_2 + Y_3 + Y_4$$

$$F_{Y_1}(y) = \int_{(x \mid 1 \le y,\ -2 < x < -1)} f_X(x)\,dx = \int_{-2}^{-1} \frac{1}{4}\,dx = \frac{1}{4} \quad \text{for } y = 1$$

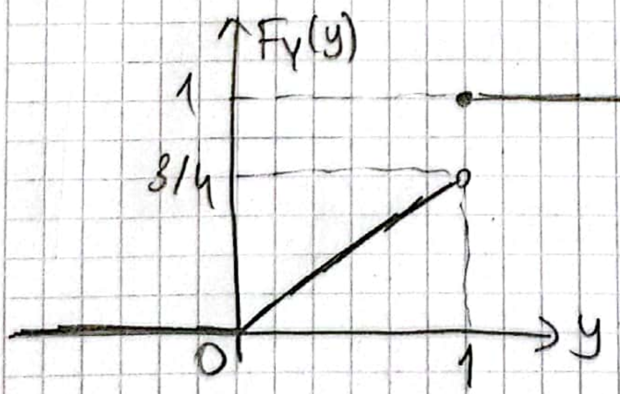
$$F_{Y_2}(y) = \int_{(x \mid -x \le y,\ -1 \le x < 0)} f_X(x)\,dx = \int_{-y}^{0} \frac{1}{4}\,dx = \frac{y}{4} \quad \text{for } 0 \le y \le 1$$

$$F_{Y_3}(y) = \int_{(x \mid x \le y,\ 0 \le x < 1)} f_X(x)\,dx = \int_{0}^{y} \frac{1}{4}\,dx = \frac{y}{4} \quad \text{for } 0 \le y \le 1$$

$$F_{Y_4}(y) = \int_{(x \mid (x-2)^2 \le y,\ 1 \le x \le 2)} f_X(x)\,dx = \int_{-\sqrt{y}+2}^{2} \frac{1}{4}\,dx = \frac{\sqrt{y}}{4} \quad \text{for } 0 \le y \le 1$$

$$F_Y(y) = F_{Y_1}(y) + F_{Y_2}(y) + F_{Y_3}(y) + F_{Y_4}(y)$$

$$= F_Y(y) = \begin{cases} 0, & y < 0 \\ \dfrac{\sqrt{y}}{4} + \dfrac{y}{2}, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases}$$



Y is a mixed random variable.

d) $f_Z(z) = \begin{cases} 1, & 0 \le z \le 1 \\ 0, & o.w. \end{cases}$ $\quad f_W(w) = \begin{cases} 1, & 0 \le w \le 1 \\ 0, & o.w. \end{cases}$

$$F_V(v) = P\left(\frac{Z}{W+1} \le v\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{vw+v} \underbrace{f_{Z,W}(z,w)}_{f_Z(z)\cdot f_W(w) \text{ since they are independent}}\, dz\, dw$$

$\Downarrow$

$z \le vw+v$

$$= \int_{-\infty}^{\infty} f_W(w) \underbrace{\int_{-\infty}^{vw+v} f_Z(z)\, dz}_{F_Z(vw+v)}\, dw = F_V(v)$$

$$f_V(v) = \frac{\partial}{\partial v} F_V(v) = \frac{\partial}{\partial v} \int_{-\infty}^{\infty} f_W(w)\, F_Z(vw+v)\, dw$$

$$= \int_{-\infty}^{\infty} f_W(w) \left[\frac{d}{dv} F_Z(vw+v)\right] dw = \int_{-\infty}^{\infty} f_W(w)\, f_Z(vw+v)(w+1)\, dw$$

* $f_W(w)$ and $f_Z(vw+1)$ are not zero when $0 \le w \le 1$, $0 \le vw+v \le 1$

$\Downarrow$

$-v \le vw \le 1-v$

If $v>0$, $\frac{\cancel{-v}}{\cancel{v}} \le w \le \underbrace{\frac{1}{v}-1}$ → condition for this to be positive or zero $0 < v \le 1$

-1

then $0 \le w \le 1$, $-1 \le w \le \frac{1}{v}-1$ } for $0 < v \le 1$, $\frac{1}{v}-1 \le 1$ $\left(\frac{1}{2} \le v\right)$

the condition for w becomes $0 \le w \le \frac{1}{v}-1$ if $\frac{1}{2} \le v \le 1$

$0 \le w \le 1$ } for $0 < v < \frac{1}{2}$

the condition for w becomes $0 \le w \le 1$ if $0 < v < \frac{1}{2}$

* for $\frac{1}{2} \le v \le 1$, $0 \le w \le \frac{1}{v} - 1$

$$f_{V_1}(v) = \int_0^{\frac{1}{v}-1} (w+1)\,dw = \frac{1}{2v^2} - \frac{1}{2} \quad \text{for } \frac{1}{2} \le v \le 1$$

* for $0 < v < \frac{1}{2}$, $0 \le w \le 1$

$$f_{V_2}(v) = \int_0^1 (w+1)\,dw = \frac{3}{2} \quad \text{for } 0 < v < \frac{1}{2}$$

$$f_V(v) = f_{V_1}(v) + f_{V_2}(v) = \begin{cases} \frac{3}{2}, & 0 < v < \frac{1}{2} \\ \frac{1}{2v^2} - \frac{1}{2}, & \frac{1}{2} \le v \le 1 \\ 0, & \text{o.w.} \end{cases}$$

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