**Homework 2**
**Due: April 24, 2022**

1. (a) A supermarket sells the following fruits : elma, armut, ayva, muz, erik. You go to the supermarket and buy one fruit and you are equally likely to buy any fruit. Let $X$ be the number of vowels and $Y$ be the number of consonants in the name of the fruit you buy. Define a sample space for this random experiment, and by showing the mapping between the sample space and the values the random variables take, find
   i. the PMF of $X$
   ii. the PMF of $Y$
   iii. the joint PMF of $X$ and $Y$.

   (b) Your favorite soccer team will play two games next week. For each game, they get 3 points if they win, 1 point if they draw and no points if they lose. For the first game, they have 0.5 probability of winning and 0.1 probability of losing. If they win the first game, they have the same probabilities of winning and losing in the second game; otherwise the probabilities of winning and losing are 0.4 each in the second game. Let $X$ be the total points they get next week.
   i. Find the PMF of $X$.
   ii. Find the PMF of $X$ given they don't lose both games.

2. The PMF of random variable $X$ is defined as follows.

$$p_X(k) = \begin{cases} 2c, & \text{if } k = 0, \\ c, & \text{if } k \in \{-3, -1, 1, 2\}, \\ 0. & \text{otherwise} \end{cases}$$

   (a) i. Determine $c$ and plot $p_X(k)$.
   ii. Compute the mean and variance of $X$.

   (b) Let $Y = |X| + 1$. Compute the mean and variance of $Y$. Find and plot PMF of $Y$.

   (c) Random variable $W$ is defined as follows.

$$W = \begin{cases} -1, & X \leq 0, \\ X^2, & otherwise. \end{cases}$$

   i. Compute $E[W^2]$ without using/finding PMF of $W$.
   ii. Compute $var(W)$ without using/finding PMF of $W$.
   iii. Determine and plot the PMF $p_W(k)$ of $W$.
   iv. Compute $E[X \cdot W]$.

   (d) (Not to be graded) Random variable $U$ is uniformly distributed in $\{0, 1, 2\}$ and is independent of $X$. Compute $P(U - X < 1)$.

3. A probabilistic experiment has a sample space with 9 equally likely outcomes:

$$\Omega = \{bb, hb, hbh, hbb, hh, bbh, ah, aha, bah\}$$

   Let $X$ be the number of b's in the outcome and $Y$ be the number of h's.

   (a) By clearly showing the mapping between the outcomes in the sample space and the random variables, find the joint pmf of X and Y and tabulate the joint PMF.

(b) Find the marginal PMF $p_X(x)$ by marginalization.

(c) Find the conditional PMF $p_{X|Y}(x|y)$.

(d) Define a new random variable $Z$ which is 1 if $XY > 0$ and zero otherwise. Find the probability of the event $P(\{Z = 1\} \cap \{Y = 1\})$.

(e) Determine the $E[Z]$, $E[XY]$, $E[X|Y = 2]$.

4. (a) Consider a sequence of six independent rolls of a tetrahedral unfair die with faces numbered 1, 2, 3 and 4. let $X_i$ be the random variable corresponding to the result of the $i^{th}$ roll. The PMF of $X_i$ for all $i$ is given as follows:

$$p_{X_i}(x) = \begin{cases} 2/5, & if \text{ x} = 1, \\ 1/5, & if \text{ x} = 2, \\ 1/5, & if \text{ x} = 3, \\ 1/5, & if \text{ x} = 4, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

i. What is the probability that exactly three of the rolls have result equal to 1?

ii. Given that exactly three of the rolls resulted in 1 and exactly three resulted in 3, determine the probability that the sequence of rolls is 111333.

iii. Given that at least one roll resulted in 1, determine the conditional PMF of the number of 1's.

(b) (Not to be graded) Suppose that $X$ and $Y$ are independent, identically distributed, geometric random variables with parameter $p$. Show that

$$P(X = m|X + Y = n) = \frac{1}{n-1} \text{ for all } m = 1, 2, ..., n-1.$$

5. In this problem, you will use matlab to plot several PMF's and compute their means and variances. You will also write matlab functions to generate Bernoulli and Binomial random variables and will also obtain histograms using these functions.

(a) First, from the *Help* menu of matlab window, select Product Help, and in the new window that opens, type *pdf* to search and read the documentation of the *pdf()* command. You can see that the *pdf()* command can provide PMF values for many random variables including the ones we learned in class.

(b) Run the code in the provided matlab script *HW2_Binomial.m* (provided below) to plot the PMF of a Binomial random variable and to compute its mean and variance. Provide the plot. Now run the same code for $p = 0.2$ and provide the plot you obtain with this parameter as well.

(c) Modify the provided script so that you plot the PMF of a discrete uniform random variable defined on the integers in $[2, 6]$ and compute its mean and variance. Provide the plot.

(d) From the *Help* menu of Matlab, read the documentation of *rand*() command. Using the *rand*() command, write a matlab function to generate a Bernoulli random variable with parameter $p$. The function should have one input, $p$, and one output, the Bernoulli random variable. ($ber\_rv = get\_bernoulli\_rv(p)$) Check in your function so that $0 \leq p \leq 1$ is satisfied.

(e) Write a matlab function to get a Binomial random variable which calls your $get\_bernoulli\_rv(p)$ function. The function should have 3 inputs, $p, k, n$, and one output, the Binomial random variable. ($binom\_rv = get\_binomial\_rv(p, k, n)$) Check in your function so that $0 \leq k \leq n$ is satisfied.

(f) A histogram[1] can be seen as an estimate of the probability distribution of a random variable that is obtained from many independent trials. In other words, a random experiment is performed

[1] read histogram on Wikipedia

independently many times, and the number of times each outcome occurs is recorded as a bar graph, which can be seen (after scaling by the total number of trials) as an estimate of the probability mass function of a random variable.

i. In this part of the question, you will obtain a histogram for the Bernoulli random variable. Write a matlab function $plot\_bernoulli\_histogram(p, numberoftimes)$ which runs the previously written matlab function $get\_bernoulli\_rv(p)$ $numberoftimes$ times to generate and plot a histogram. (Normalize the histogram values by $numberoftimes$ so that histogram plot adds up to 1.) Run your function with $p = 0.7$ three times for $numberoftimes = 10, 100, 1000$. Finally, plot the Bernoulli PMF using the *pdf()* command and comment on your histogram plots.

ii. Repeat i for the Binomial random variable. (Use $p = 0.7$, $n = 20$ and $numberoftimes = 10, 100, 1000$.)

## MATLAB CODE

```
% Determine parameters of Binomial PMF
p = 0.35 ;
n = 10 ;
k = 0:10 ;

% Use pdf() function in Matlab to get PMF values of Binomial PMF at required
% points
pXk = pdf('bino',k ,n, p) ;

% Compute Expected Value
ExpVal = sum( k .* pXk ) ;

% Compute Variance
Var = sum( (k-ExpVal).^2 .* pXk ) ;

% Plot PDF and label axes
figure, bar(k,pXk); grid ; xlabel('k'); ylabel('p_X(k)')

% Put a title on plot
title(sprintf('PMF of Binomial r.v. with n=%d, p=%2.2f, E[X]=%3.2f, var(X)
    =%3.2f',n,p,ExpVal,Var));
```