

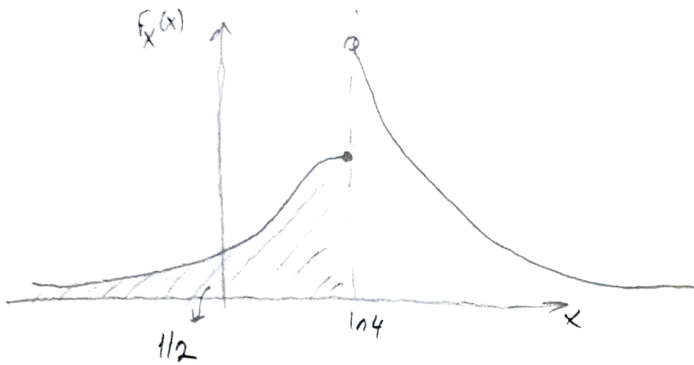
# EE230 HW3

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Group 13

(a)



$$\int_{-\infty}^{\ln 4} f_X(x) dx = \frac{1}{2}$$

↓

$$\int_{\ln 4}^{\infty} f_X(x) dx = \frac{1}{2}$$

$$\frac{1}{2} = \int_{\ln 4}^{\infty} \lambda e^{-\lambda x} dx$$

$$= (-e^{-\lambda x}) \Big|_{\ln 4}^{\infty} = e^{-\lambda \ln 4} = 4^{-\lambda} = \frac{1}{2}$$

$$\boxed{\lambda = \frac{1}{2}}$$

(b)

$$\int_{\ln 4}^{\infty} x \lambda e^{-\lambda x} dx = \quad \begin{matrix} x = u + \ln 4 \\ dx = du \end{matrix}$$

$$= \int_0^{\infty} (u + \ln 4) \lambda e^{-\lambda(u + \ln 4)} du$$

$$= e^{-\lambda \ln 4} \int_0^{\infty} u \lambda e^{-\lambda u} du + e^{-\lambda \ln 4} \cdot \ln 4 \cdot \int_0^{\infty} \lambda e^{-\lambda u} du$$

$$= \ln 2 + 1$$

$$\rightarrow E[X] = \frac{\int_{-\infty}^{\ln 4} \frac{x}{\sqrt{2\pi}} \sigma e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} dx}{\ln 2 - 1} + \frac{\int_{\ln 4}^{\infty} x \lambda e^{-\lambda x} dx}{\ln 2 + 1} = \ln 4$$

$$\ln 2 - 1 = \int_{-\infty}^{\ln 4} \frac{x}{\sqrt{2\pi}} \sigma e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\ln 4} \frac{x - \ln 4}{\sqrt{2\pi}} \sigma e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} dx + \ln 4 \int_{-\infty}^{\ln 4} \frac{1}{\sqrt{2\pi}} \sigma e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} dx$$

$$-1 = \int_{-\infty}^{\ln 4} \frac{x - \ln 4}{\sqrt{2\pi}} \sigma e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} dx \quad \begin{matrix} (x - \ln 4)^2 = u \\ 2(x - \ln 4) dx = du \end{matrix}$$

$$= \frac{1}{2} \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} \sigma e^{-\frac{u}{2\sigma^2}} du$$

$$= \frac{1}{2\sqrt{2\pi}} \sigma \left( -2\sigma^2 e^{-\frac{u}{2\sigma^2}} \right) \Big|_{-\infty}^0$$

$$\sigma = \sqrt{2 - \ln 2}$$

1c)

$$\text{var}(X) = E[X^2] - (E[X])^2$$

$$= \int_{-\infty}^{\ln 4} \frac{x^2}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} dx + \int_{\ln 4}^{\infty} x^2 \lambda e^{-\lambda x} dx - (\ln 4)^2$$

$$\int_{\ln 4}^{\infty} x^2 \lambda e^{-\lambda x} dx \quad \begin{array}{l} x = u + \ln 4 \\ dx = du \end{array}$$

$$= \int_0^{\infty} (u^2 + 2u \ln 4 + (\ln 4)^2) \lambda e^{-\lambda u} \frac{1}{\lambda} du$$

$$= \frac{1}{2} \left[ \underbrace{\int_0^{\infty} u^2 \lambda e^{-\lambda u} du}_{E[U^2] = \text{var}(U) + (E[U])^2} + 2 \ln 4 \underbrace{\int_0^{\infty} u \lambda e^{-\lambda u} du}_{\frac{1}{\lambda}} + (\ln 4)^2 \underbrace{\int_0^{\infty} \lambda e^{-\lambda u} du}_1 \right]$$

$$= \frac{1}{2} + \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{2} \left( 4 + 4 + 2 \ln 4 \cdot 2 + (\ln 4)^2 \right) = 4 + 2 \ln 4 + \frac{(\ln 4)^2}{2}$$

$$\int_{-\infty}^{\ln 4} \frac{x^2}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}} dx = \int_{-\infty}^0 \frac{(u+\ln 4)^2}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}} du$$

$$= \int_{-\infty}^0 \frac{u^2}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}} du + 2 \ln 4 \int_{-\infty}^0 \frac{u}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}} du + (\ln 4)^2 \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}} du$$

$$\frac{E[U^2]}{2} \quad \left( \begin{array}{l} \text{Since the} \\ \text{integrand is} \\ \text{even function} \end{array} \right)$$

$$\downarrow$$

$$\frac{\text{var}(U) + E[U]^2}{2} = \frac{\sigma^2}{2}$$

$$-1 \quad (\text{From part 1b})$$

$$\frac{1}{2}$$

$$= \pi - 2 \ln 4 + \frac{(\ln 4)^2}{2}$$

$$\text{var}(X) = \pi - 2 \ln 4 + \frac{(\ln 4)^2}{2} + 4 + 2 \ln 4 + \frac{(\ln 4)^2}{2} - (\ln 4)^2$$

$$\boxed{\text{var}(X) = \pi + \ln 4}$$

1d) For  $x \leq \ln 4$ 

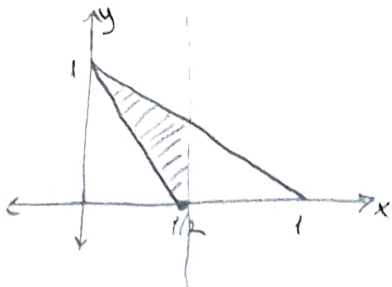
$$\text{CDF} = \Phi\left(\frac{x-u}{\sigma}\right) = \Phi\left(\frac{x-\ln 4}{\sqrt{2\pi}}\right)$$

For  $x \geq 4$ 

$$\text{CDF} = \frac{1}{2} + \int_{\ln 4}^x \lambda e^{-\lambda x} dx = \frac{1}{2} + (-e^{-\lambda x}) \Big|_{\ln 4}^{\infty} = 1 - e^{-\lambda x}$$

$$F_X(x) = \begin{cases} \Phi\left(\frac{x-\ln 4}{\sqrt{2\pi}}\right), & x \leq \ln 4 \\ 1 - e^{-x/2}, & x > \ln 4 \end{cases}$$

2a)



$$\begin{aligned}
 P\left(x \leq \frac{1}{2}\right) &= \int_0^{1/2} \int_{1-x}^{1-x} 12y \, dy \, dx \\
 &= \int_0^{1/2} (6y^2) \Big|_{1-x}^{1-x} dx \\
 &= \int_0^{1/2} 6(1-2x+x^2 - 1+4x-4x^2) dx \\
 &= 6 \int_0^{1/2} (2x-3x^2) dx \\
 &= 6(x^2-x^3) \Big|_0^{1/2} = 6\left(\frac{1}{4}-\frac{1}{8}\right) = \frac{6}{8} = \boxed{\frac{3}{4}}
 \end{aligned}$$

2b)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y) \, dx$$

$$\begin{aligned}
 &= \int_{\frac{1-y}{2}}^{1-y} 12y \, dx = 12y \left(1-y - \frac{1-y}{2}\right) \\
 &= 6y(1-y)
 \end{aligned}$$

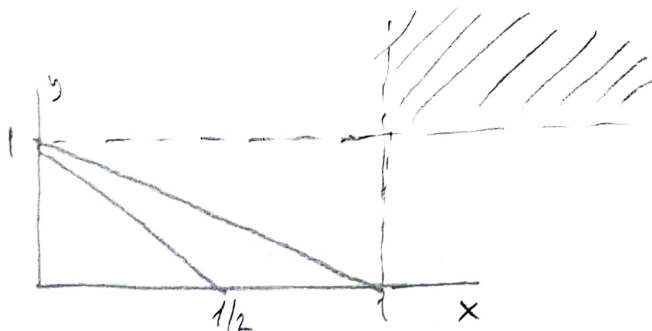
$$f_Y(y) = \begin{cases} 6y(1-y) & , 0 < y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

2c)



$$\begin{aligned}
 F_{X,Y}\left(\frac{1}{2}, \frac{1}{2}\right) &= \int_0^{1/2} \int_{\frac{1-y}{2}}^{1-x} 12y \, dx \, dy \\
 &= \int_0^{1/2} 6y \left(\frac{1-x+y}{2}\right) dy \\
 &= \int_0^{1/2} 6y^2 \, dy = (2y^3) \Big|_0^{1/2} = \boxed{\frac{1}{4}}
 \end{aligned}$$

2d)



$$\{(x,y) \mid x > 1, y > 1\}$$

3a)

$$\begin{aligned}
 f_X(x) &= \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)} \left( \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x \sigma_y} \right)} dy \\
 &= \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)} \left( \left( \frac{y}{\sigma_y} - \frac{\rho x}{\sigma_x} \right)^2 + \frac{x^2}{\sigma_x^2} (1-\rho^2) \right)} dy \\
 &= \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\frac{-x^2}{2\sigma_x^2}} e^{\frac{-1}{2(1-\rho^2)} \left( \frac{y}{\sigma_y} - \frac{\rho x}{\sigma_x} \right)^2} dy \\
 &= \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\frac{-x^2}{2\sigma_x^2}} \underbrace{\int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)} \left( \frac{y}{\sigma_y} - \frac{\rho x}{\sigma_x} \right)^2} dy}_{\sqrt{2\pi} \sqrt{1-\rho^2} \sigma_y}
 \end{aligned}$$

$$= \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\frac{-x^2}{2\sigma_x^2}} \cdot \sqrt{2\pi} \sqrt{1-\rho^2} \sigma_y$$

$$f_X(x) = \frac{1}{\sqrt{2\pi} \sigma_x} e^{\frac{-x^2}{2\sigma_x^2}} \quad \text{Similarly, } f_Y(y) = \frac{1}{\sqrt{2\pi} \sigma_y} e^{\frac{-y^2}{2\sigma_y^2}}$$

3b)

$$\begin{aligned}
 E_{X,Y}[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{xy}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)} \left( \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x \sigma_y} \right)} dy dx \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{xy}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{\frac{-x^2}{2\sigma_x^2}} e^{\frac{-1}{2(1-\rho^2)} \left( \frac{y}{\sigma_y} - \frac{\rho x}{\sigma_x} \right)^2} dy dx \\
 &= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi} \sigma_x} e^{\frac{-x^2}{2\sigma_x^2}} \left( \int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi} \sigma_y \sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)} \left( \frac{y}{\sigma_y} - \frac{\rho x}{\sigma_x} \right)^2} dy \right) dx
 \end{aligned}$$

$$\frac{\rho \sigma_y}{\sigma_x}$$

$$= \int_{-\infty}^{\infty} \frac{x^2 \rho \sigma_y}{\sqrt{2\pi} \sigma_x^2} e^{\frac{-x^2}{2\sigma_x^2}} dx$$

$$= \frac{\rho \sigma_y}{\sigma_x} \underbrace{\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi} \sigma_x} e^{\frac{-x^2}{2\sigma_x^2}} dx}_{E[X^2]}$$

$$= \frac{\rho \sigma_y}{\sigma_x} (var(X) - E[X]^2) = \frac{\rho \sigma_y}{\sigma_x} \sigma_x^2$$

$$\boxed{E_{X,Y}[XY] = \rho \sigma_x \sigma_y}$$

$$\begin{aligned}
 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx &= 1 \\
 \int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx &= \sqrt{2\pi} \sigma
 \end{aligned}$$

3c)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x\sigma_y}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{y^2}{2\sigma_y^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x\sigma_y}\right) + \frac{y^2}{2\sigma_y^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)\sigma_x^2}\left(x - \frac{\rho\sigma_x}{\sigma_y}y\right)^2}$$

Hence, it is a Gaussian r.v.

$$\mu_{X|Y} = E_{X|Y}[X|Y=y] = \frac{\rho\sigma_x}{\sigma_y}$$

$$\begin{aligned} & \left\{ \begin{aligned} & \frac{-x^2}{2(1-\rho^2)\sigma_x^2} - \frac{y^2}{2(1-\rho^2)\sigma_y^2} + \frac{y^2}{2\sigma_y^2} + \frac{2xy\rho}{2(1-\rho^2)\sigma_x\sigma_y} \\ & = \frac{-x^2}{2(1-\rho^2)\sigma_x^2} - \frac{y^2\rho^2}{2(1-\rho^2)\sigma_y^2} + \frac{2xy\rho}{2(1-\rho^2)\sigma_x\sigma_y} \\ & = \frac{-1}{2(1-\rho^2)\sigma_x^2} \left( x^2 + \frac{y^2\rho^2 2(1-\rho^2)\sigma_x^2}{2(1-\rho^2)\sigma_y^2} - \frac{2xy\rho 2(1-\rho^2)\sigma_x^2}{2(1-\rho^2)\sigma_x\sigma_y} \right) \\ & = \frac{-1}{2(1-\rho^2)\sigma_x^2} \left( x - \frac{\rho\sigma_x}{\sigma_y}y \right)^2 \end{aligned} \right. \end{aligned}$$

3d)

$$\hat{X} = \frac{\rho\sigma_x}{\sigma_y} = \alpha Y$$

$$\alpha = \frac{\rho\sigma_x}{\sigma_y}$$

For given  $Y$ ,  $(X - \hat{X})$  is a gaussian random variable with

$$E[X - \hat{X}] = E[X] - \hat{X} = 0 \text{ and}$$

$$\text{var}(X - \hat{X}) = \text{var}(X) = (1 - \rho^2)\sigma_x^2.$$

$$\text{Therefore, } E[(X - \hat{X})^2] = \text{var}(X - \hat{X}) + (E[X - \hat{X}])^2$$

$$\text{1) } \boxed{MSE = (1 - \rho^2)\sigma_x^2}$$