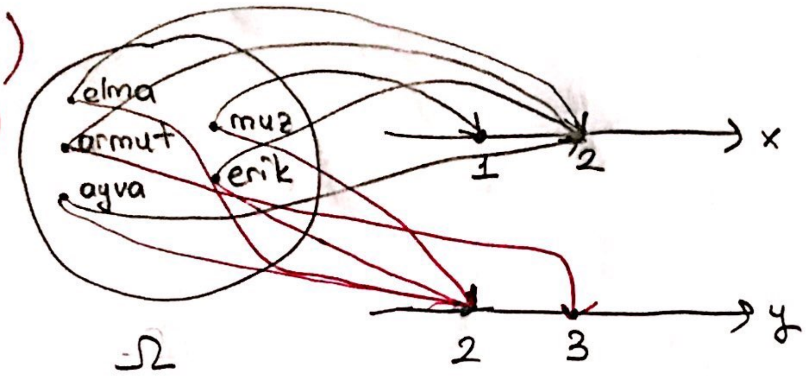
1)

a)



$X = \#$ vowels

$Y = \#$ consonants

$$P_X(1) = P(\{X=1\}) = P(\{\text{muz}\}) = 1/5$$

$$P_X(2) = P(\{X=2\}) = P(\{\text{elma, armut, ayva, erik}\}) = 4/5$$

$$P_X(x) = \begin{cases} 1/5, & \text{if } x=1 \\ 4/5, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

$$P_Y(2) = P(\{Y=2\}) = P(\{\text{elma, ayva, muz, erik}\}) = 4/5$$

$$P_Y(3) = P(\{Y=3\}) = P(\{\text{armut}\}) = 1/5$$

$$P_Y(y) = \begin{cases} 4/5, & \text{if } y=2 \\ 1/5, & \text{if } y=3 \\ 0, & \text{otherwise} \end{cases}$$

$$\Omega = \{\text{elma, armut, ayva, muz, erik}\}$$

$$(X,Y): \quad (2,2) \quad (2,3) \quad (2,2) \quad (1,2) \quad (2,2)$$

$$P_{X,Y}(2,2) = 3/5 \qquad P_{X,Y}(1,2) = 1/5 \qquad P_{X,Y}(2,3) = 1/5$$

$$P_{X,Y}(x,y) = \begin{cases} 3/5, & \text{if } x=2,\ y=2 \\ 1/5, & \text{if } x=1,\ y=2 \\ 1/5, & \text{if } x=2,\ y=3 \\ 0, & \text{otherwise} \end{cases}$$

1) b) Win → W Lose → L Draw → D Game 1 → G_1 Game 2 → G_2

i)

$$\Omega = \{WW, WL, WD, LW, LL, LD, DW, DL, DD\}$$

$$X: \quad 6 \quad 3 \quad 4 \quad 3 \quad 0 \quad 1 \quad 4 \quad 1 \quad 2$$

(total points)

$$P(\{WW\}) = P(\{\text{Win Game 1}\}) \cdot P(\{\text{Win Game 2}\} \mid \{\text{Win Game 1}\})$$

first game second game

$$= 0.5 \cdot 0.5$$

$$= 0.25$$

$P(\{WL\}) = 0.5 \times 0.1 = 0.05$ $P(\{WD\}) = 0.5 \times 0.4 = 0.2$

$P(\{LW\}) = 0.1 \times 0.4 = 0.04$ $P(\{LL\}) = 0.1 \times 0.4 = 0.04$

$P(\{LD\}) = 0.1 \times 0.2 = 0.02$ $P(\{DW\}) = 0.4 \times 0.4 = 0.16$

$P(\{D,L\}) = 0.4 \times 0.4 = 0.16$ $P(\{D,D\}) = 0.4 \times 0.2 = 0.08$

$P_X(6) = P(\{WW\}) = 0.25$

$P_X(3) = P(\{WL, LW\}) = 0.05 + 0.04 = 0.09$

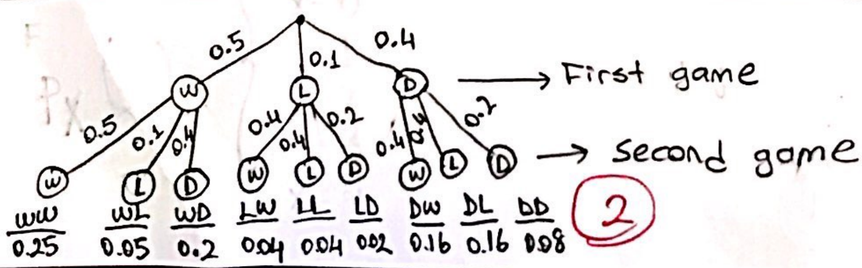
$P_X(4) = P(\{WD, DW\}) = 0.2 + 0.16 = 0.36$

$P_X(0) = P(\{LL\}) = 0.04$

$P_X(1) = P(\{LD, DL\}) = 0.02 + 0.16 = 0.18$

$P_X(2) = P(\{D, D\}) = 0.08$

$$\Rightarrow \boxed{P_X(x) = \begin{cases} 0.04, & \text{if } x = 0 \\ 0.18, & \text{if } x = 1 \\ 0.08, & \text{if } x = 2 \\ 0.09, & \text{if } x = 3 \\ 0.36, & \text{if } x = 4 \\ 0.25, & \text{if } x = 6 \\ 0, & \text{o.w} \end{cases}}$$



2

1) b) ii) $A = \{WW, WD, DW, DD\}$ ⇒ event that they don't lose both games

$$P_{X|A}(6) = \frac{P(\{WW\} \cap A)}{P(A)} = \frac{P(\{WW\})}{P(A)} = \frac{0.25}{\underbrace{0.25+0.2+0.16+0.08}_{0.69}}$$

$$= 0.3623$$

$$P_{X|A}(3) = \frac{P(\{WL, LW\} \cap A)}{P(A)} = \frac{P(\emptyset)}{P(A)} = 0$$

$$P_{X|A}(4) = \frac{P(\{WD, DW\})}{P(A)} = \frac{0.36}{0.69} = 0.5217$$

$P_{X|A}(0) = 0$ $\quad P_{X|A}(1) = 0$ $\quad P_{X|A}(2) = \frac{0.08}{0.69} = 0.116$

$$\boxed{P_{X|A}(x) = \begin{cases} 0.116, & \text{if } x=2 \\ 0.5217, & \text{if } x=4 \\ 0.3623, & \text{if } x=6 \\ 0, & \text{o.w.} \end{cases}}$$

2) $\sum_k p_X(k) = 1 = p_X(0) + p_X(-3) + p_X(-1) + p_X(1) + p_X(2)$

a)

$$= 2c + c + c + c + c$$

$$= 6c = 1 \Rightarrow \boxed{c = 1/6}$$

$$E[X] = \sum_k k\,P(X=k) = 0.2c - 3.c - 1c + 1.c + 2.c$$

$$= -1c = \boxed{-1/6}$$

$$E[(X - \underbrace{E[X]}_{-1/6})^2] = \sum_k (k + 1/6)^2 P(X=k)$$

$$= \frac{1}{36}.2c + c\left[\left(\frac{17}{6}\right)^2 + \left(\frac{5}{6}\right)^2 + \left(\frac{7}{6}\right)^2 + \left(\frac{13}{6}\right)^2\right]$$

$$= \boxed{2.4722}$$

b)

X	Y
0	1
-3	4
-1	2
1	2
2	3

$P_Y(1) = P_X(0) = 1/3$

$P_Y(4) = P_X(-3) = 1/6$

$P_Y(2) = P_X(1) + P_X(-1) = 1/3$

$P_Y(3) = P_X(2) = 1/6$

$$P_Y(y) = \begin{cases} 1/3, & \text{if } y=1 \\ 1/3, & \text{if } y=2 \\ 1/6, & \text{if } y=3 \\ 1/6, & \text{if } y=4 \\ 0, & \text{o.w} \end{cases}$$

$$E[Y] = \frac{1}{3}.1 + \frac{1}{6}.4 + \frac{1}{3}.2 + \frac{1}{6}.3 = \boxed{13/6}$$

$$\text{var}(Y) = \frac{1}{3}(1 - 13/6)^2 + \frac{1}{6}(4 - 13/6)^2 + \frac{1}{3}(2 - 13/6)^2 + \frac{1}{6}(3 - 13/6)^2$$

$$= 1.1389$$

2) c) $X \xrightarrow{g(x)=w} w$

X	w
0	-1
-3	-1
-1	-1
1	1
2	4

$P_w(-1) = P_X(0) + P_X(-3) + P_X(-1) = 2/3$

$P_w(1) = P_X(1) = 1/6$

$P_w(4) = P_X(2) = 1/6$

$$P_w(w) = \begin{cases} 2/3, & \text{if } w=-1 \\ 1/6, & \text{if } w=1 \\ 1/6, & \text{if } w=4 \\ 0, & \text{o.w} \end{cases}$$

$$E[w] = \sum_{x} g(x) \cdot P_X(x) = \sum_{x=0,-3,-1} -1 \cdot P_X(x) + \sum_{x=1,2} x^2 P_X(x) = \boxed{\frac{1}{6}}$$

$$E[w^2] = \sum_{x} (g(x)^2) P_X(x) = \sum_{x=0,-3,-1} P_X(x) + \sum_{x=1,2} x^4 P_X(x) = \boxed{\frac{21}{6}}$$

$$\text{var}(w) = E[w^2] - (E[w])^2 = \frac{21}{6} - \frac{1}{36} = \boxed{\frac{125}{36}}$$

$$P_{X,w}(x,w) = \begin{cases} 1/3, & \text{if } x=0, w=-1 \\ 1/6, & \text{if } x=-3, w=-1 \\ 1/6, & \text{if } x=-1, w=-1 \\ 1/6, & \text{if } x=1, w=1 \\ 1/6, & \text{if } x=2, w=4 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X.w] = \sum_{x,w} x.w \, P_{X,w}(x,w) = \frac{1}{3} \cdot 0 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 8$$

$$= \boxed{13/6}$$

2) d) $P(u-x<1) = \frac{1}{3}P(u-x<1 \mid u=0) + \frac{1}{3}P(u-x<1 \mid u=1)$

$$+\frac{1}{3}P(u-x<1 \mid u=2)$$

$$= \frac{1}{3}P(X+1>0) + \frac{1}{3}P(X+1>1) + \frac{1}{3}P(X+1>2)$$

$$= \frac{1}{3}\left(\frac{1}{3}+\frac{1}{6}+\frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{6}+\frac{1}{6}\right) + \frac{1}{3}\left(\frac{1}{6}\right) = \frac{1}{3}\cdot\frac{7}{6} = \boxed{\frac{7}{18}}$$

3) a) $\Omega = \{bb, hb, hbh, hbb, hh, bbh, ah, aha, bah\}$

(X,Y): $(2,0)$, $(1,1)$, $(1,2)$, $(2,1)$, $(0,2)$, $(2,1)$, $(0,1)$, $(0,1)$, $(1,1)$

$$P_{X,Y}(x,y) = \begin{cases} 1/9, & \text{if } (x,y)=(2,0) \\ 2/9, & \text{if } (x,y)=(1,1) \\ 1/9, & \text{if } (x,y)=(1,2) \\ 2/9, & \text{if } (x,y)=(2,1) \\ 1/9, & \text{if } (x,y)=(0,2) \\ 2/9, & \text{if } (x,y)=(0,1) \\ 0, & \text{otherwise} \end{cases}$$

X

$P_{X,Y}(x,y)$	0	1	2
Y 0	0	0	1/9
1	2/9	2/9	2/9
2	1/9	1/9	0

PMF tabular form

b) $P_X(x) = \sum_y P_{X,Y}(x,y) = P_{X,Y}(x,0) + P_{X,Y}(x,1) + P_{X,Y}(x,2)$

(summation of rows in PMF tabular form)

$$P_X(x) = \begin{cases} 3/9, & \text{if } x=0 \\ 3/9, & \text{if } x=1 \\ 3/9, & \text{if } x=2 \\ 0, & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} 1/9, & \text{if } y=0 \\ 6/9, & \text{if } y=1 \\ 2/9, & \text{if } y=2 \\ 0, & \text{otherwise} \end{cases}$$

(summation of columns)

3) c) $P_{X|Y}(x|y) = \dfrac{P_{X,Y}(x,y)}{P_Y(y)}$

$P_{X|Y}(x|0) = \begin{cases} 1, & \text{if } x=2 \\ 0, & \text{o.w} \end{cases}$ $\quad P_{X|Y}(x|1) = \begin{cases} 1/3, & \text{if } x=0,1,2 \\ 0, & \text{o.w.} \end{cases}$

$P_{X|Y}(x|2) = \begin{cases} 1/2, & \text{if } x=0,1 \\ 0, & \text{o.w} \end{cases}$

d)

	(x,y)	z
1/9 →	(2,0)	0
2/9 →	(1,1)	1
1/9 →	(1,2)	1
2/9 →	(2,1)	1
1/9 →	(0,2)	0
2/9 →	(0,1)	0

→ $P(\{Z=1\} \cap \{Y=1\}) = 4/9$

e) $P_Z(z) = \begin{cases} 5/9, & \text{if } z=1 \\ 4/9, & \text{if } z=0 \\ 0, & \text{o.w} \end{cases}$ $\Rightarrow \sum_z z\, P_Z(z) = \boxed{E[Z] = 5/9}$

$E[XY] = \sum_{x,y} x \cdot y \; P_{X,Y}(x,y) = \frac{1}{9}\cdot 0 + \frac{2}{9}\cdot 1 + \frac{1}{9}\cdot 2 + \frac{2}{9}\cdot 2 + \frac{1}{9}\cdot 0 + \frac{2}{9}\cdot 0$

$= \boxed{8/9}$

$E[X|Y=2] = \sum_x x \cdot P_{X|Y}(x|2) = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \boxed{1/2}$

from part c

4) a) i) $\boxed{\binom{6}{3}\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^3} = p_B(3) = P(B=3) = P(3\ 1\text{'s}, 3\ \text{not } 1\text{'s})$

(Binomial r.v with $n=6$ and $p=2/5$)

$p = P(X_i = 1),\ i=1,\dots,6$

ii) $\boxed{\dfrac{1}{\binom{6}{3}}}$ Only the allocation is important, since the rolls are given.

iii) A: any roll don't resulted in 1.

$P(A) = (3/5)^6$

B: at least one roll resulted in 1.

$P(B) = 1-(3/5)^6$

C: x rolls resulted in 1. X = number of ones

$P(C) = \binom{6}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{6-x}$

$$p_{X|B}(x) = \begin{cases} \dfrac{\binom{6}{x}\left(\frac{2}{5}\right)^x\left(\frac{3}{5}\right)^{6-x}}{1-(3/5)^6}, & \text{if } x=1,2,\dots,5 \\ 0, & \text{o.w} \end{cases}$$

b) $\underbrace{p_X(x)}_{P(X=x)} = (1-p)^{x-1}p,\ x=1,2,\dots$

$$P(X=m \mid X+Y=n) = \frac{P(\{X=m\}\cap\{X+Y=n\})}{P(\{X+Y=n\})} = \frac{P(\{X=m\}\cap\{Y=n-m\})}{\sum_{k=1}^{n} P(\{X=k\}\cap\{Y=n-k\})}$$

$$= \frac{P(X=m)\,P(Y=n-m)}{\sum_{k=1}^{n-1} P(X=k)\,P(Y=n-k)} = \frac{(1-p)^{m-1}p\,(1-p)^{n-m-1}p}{\sum_{k=1}^{n-1}(1-p)^{k-1}p\,(1-p)^{n-k-1}p} = \frac{p^2(1-p)^{n-2}}{p^2\sum_{k=1}^{n-1}(1-p)^{n-2}}$$

independent X and Y

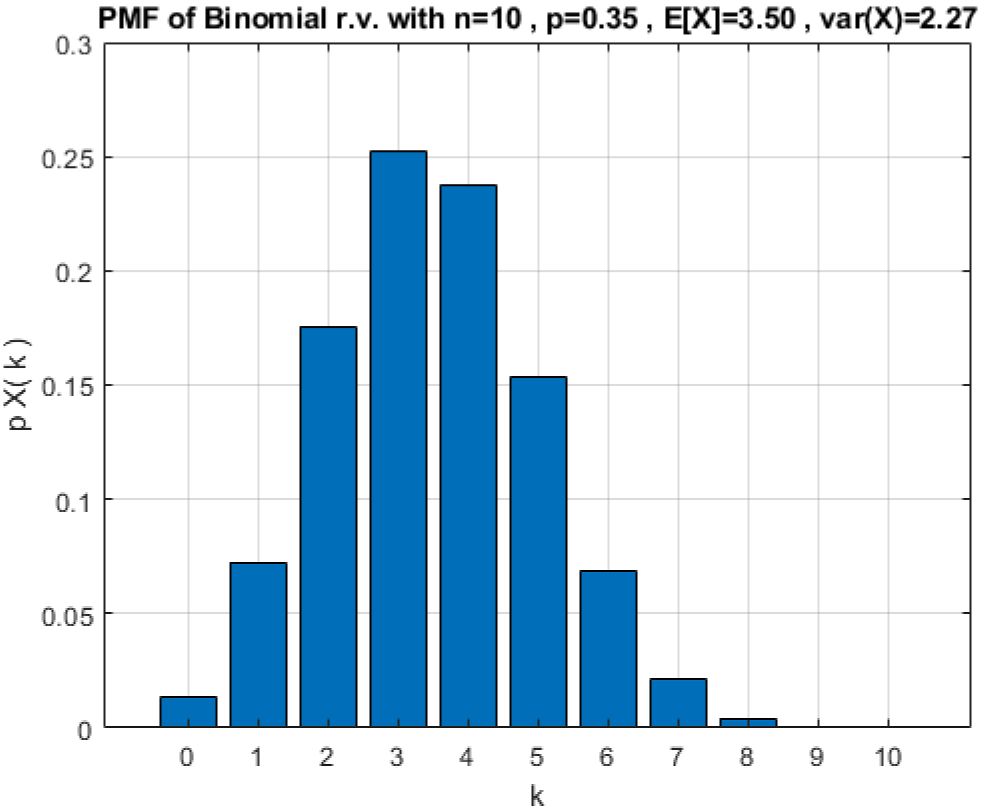
$$= \frac{(1-p)^{n-2}}{(n-1)(1-p)^{n-2}} = 1/(n-1),\ \text{for } 1 < m < n$$

Table of Contents


```
clear
clc
close all
```

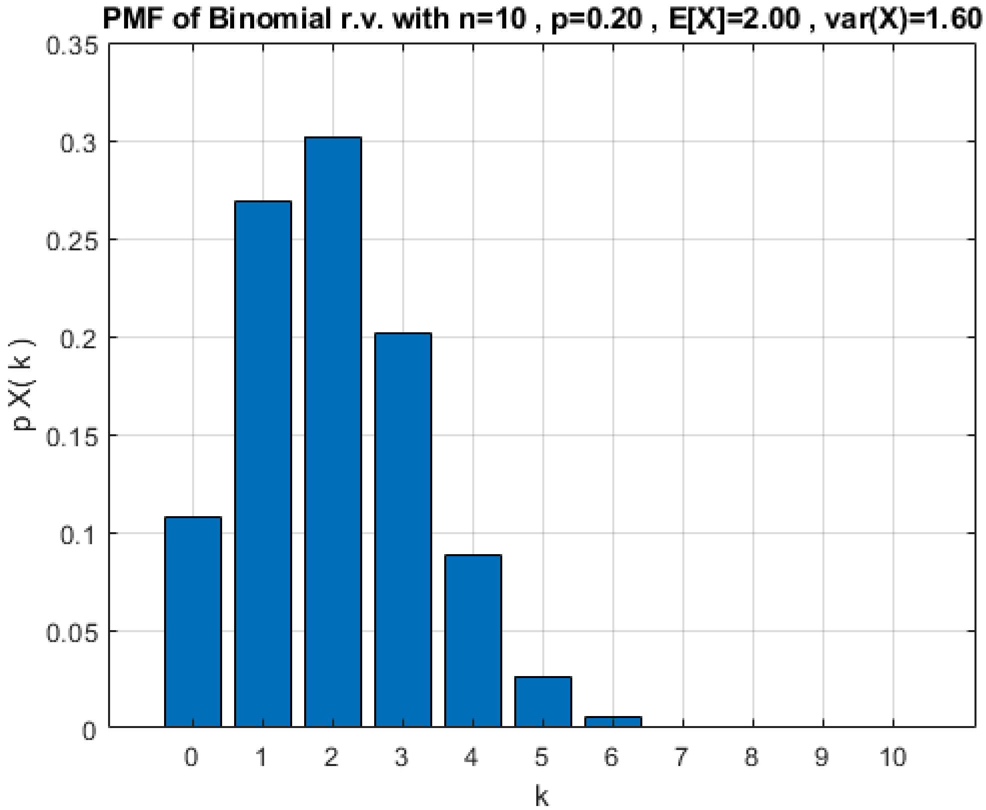
Part b, p=0.35

```
p = 0.35 ;
n = 10 ;
k = 0:10 ;
pXk = pdf('bino' , k ,n , p) ;
ExpVal = sum( k .* pXk ) ;
Var = sum( (k-ExpVal).^2.* pXk ) ;
figure , bar(k,pXk) ; grid ; xlabel('k') ; ylabel('p X( k )')
title(sprintf('PMF of Binomial r.v. with n=%d , p=%2.2f , E[X]=%3.2f , var(X)=
%3.2f' ,n , p , ExpVal , Var) ) ;
```



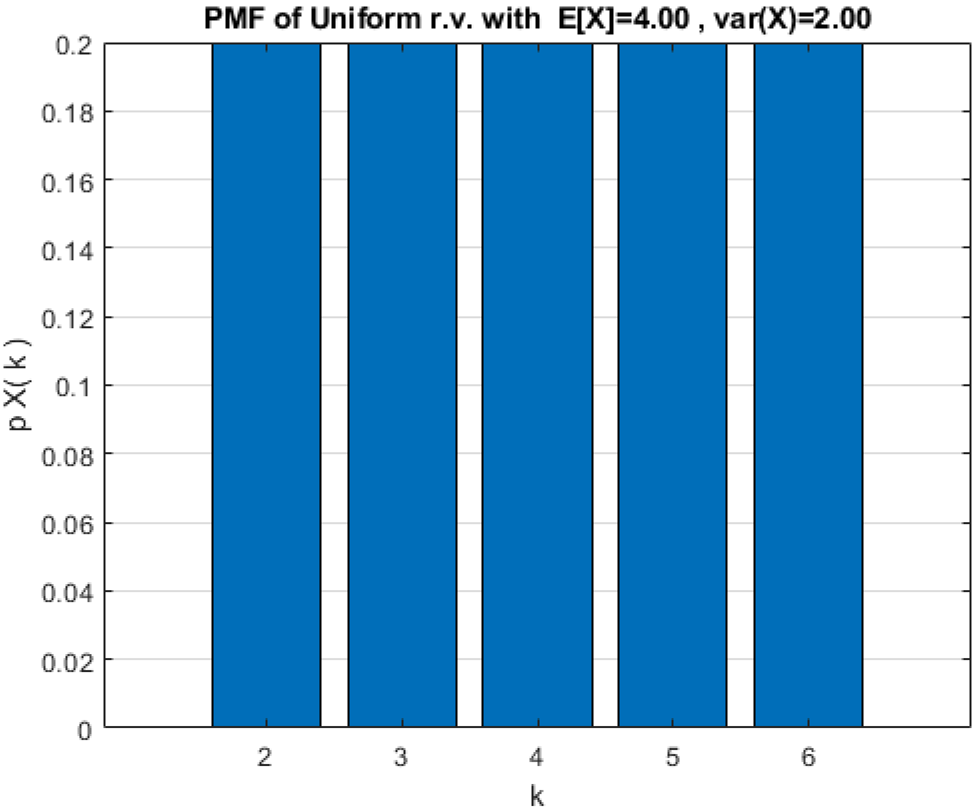
Part b, p=0.2

```
p = 0.2 ;
n = 10 ;
k = 0:10 ;
pXk = pdf('bino' , k ,n , p) ;
ExpVal = sum( k .* pXk ) ;
Var = sum( (k-ExpVal).^2.* pXk ) ;
figure , bar(k,pXk) ; grid ; xlabel('k') ; ylabel('p X( k )')
title(sprintf('PMF of Binomial r.v. with n=%d , p=%2.2f , E[X]=%3.2f , var(X)=
%3.2f' ,n , p , ExpVal , Var) ) ;
```



Part c

```
k=2:6;
pXk = ones(1,length(k))/length(k);
ExpVal = sum( k .* pXk ) ;
Var = sum( (k-ExpVal).^2.* pXk ) ;
figure , bar(k,pXk) ; grid ; xlabel('k') ; ylabel('p X( k )')
title(sprintf('PMF of Uniform r.v. with  E[X]=%3.2f , var(X)=%3.2f', ExpVal ,
 Var) ) ;
```



Part d

```
p=0.7;
n=1; %generate n number of bernoulli r.v
get_bernoulli_rv=@(p,n) rand(1,n)<p;
ber_rv=get_bernoulli_rv(p,n);
```

Part e

```
p=0.7; n=10;
get_binomial_rv=@(p,n) sum(get_bernoulli_rv(p,n)) ;
binom_rv=get_binomial_rv(p,n); % k is an output
```

Part f (i)

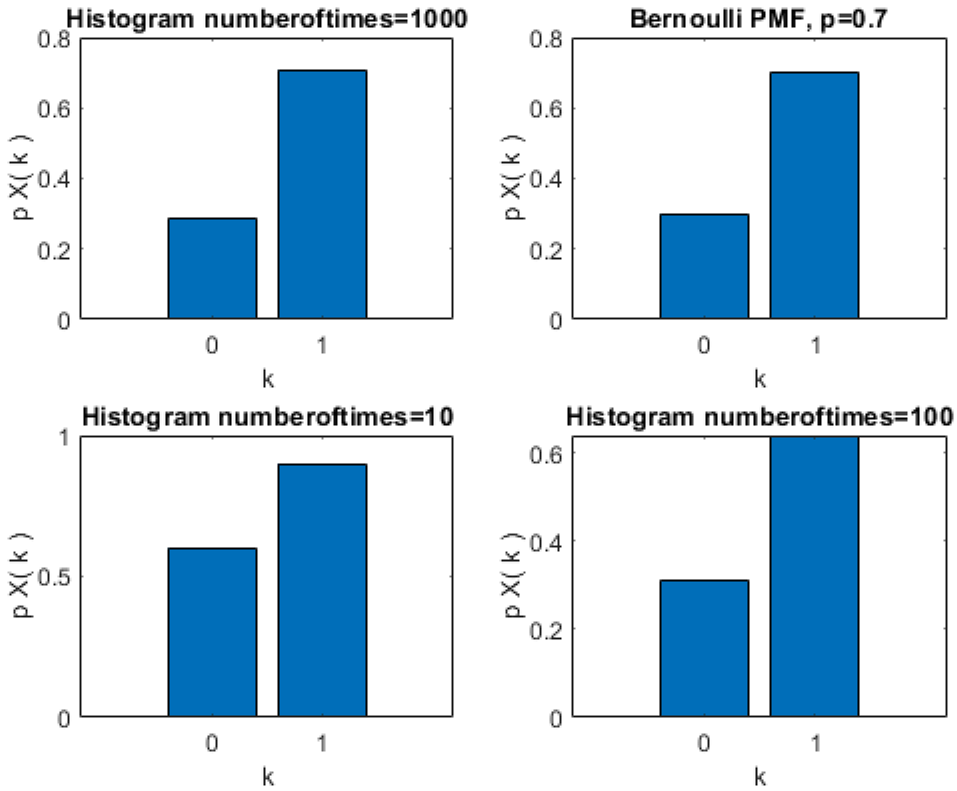
```
p=0.7;
numberoftimes=1000; %generate numberoftimes bernoulli r.v
figure
subplot(2,2,1)
plot_bernoulli_histogram=@(p,numberoftimes) bar(0:1,[1-
sum(get_bernoulli_rv(p,numberoftimes),2)/numberoftimes,
 sum(get_bernoulli_rv(p,numberoftimes),2)/numberoftimes]);
plot_bernoulli_histogram(p,numberoftimes);
xlabel('k') ; ylabel('p X( k )')
```

```
title('Histogram numberoftimes=1000')

subplot(2,2,2)
pXk = pdf('bino' , 0:1 ,1 , p) ;
bar(0:1,pXk)
xlabel('k') ; ylabel('p X( k )')
title('Bernoulli PMF, p=0.7')

subplot(2,2,3)
numberoftimes=10;
plot_bernoulli_histogram=@(p,numberoftimes) bar(0:1,[1-
sum(get_bernoulli_rv(p,numberoftimes),2)/numberoftimes,
 sum(get_bernoulli_rv(p,numberoftimes),2)/numberoftimes]);
plot_bernoulli_histogram(p,numberoftimes);
xlabel('k') ; ylabel('p X( k )')
title('Histogram numberoftimes=10')

subplot(2,2,4)
numberoftimes=100;
plot_bernoulli_histogram=@(p,numberoftimes) bar(0:1,[1-
sum(get_bernoulli_rv(p,numberoftimes),2)/numberoftimes,
 sum(get_bernoulli_rv(p,numberoftimes),2)/numberoftimes]);
plot_bernoulli_histogram(p,numberoftimes);
xlabel('k') ; ylabel('p X( k )')
title('Histogram numberoftimes=100')
```



Comment:

As the number of times increases, the histogram estimate approaches to the true PMF. Histogram estimate becomes more reliable as the number of times increases.

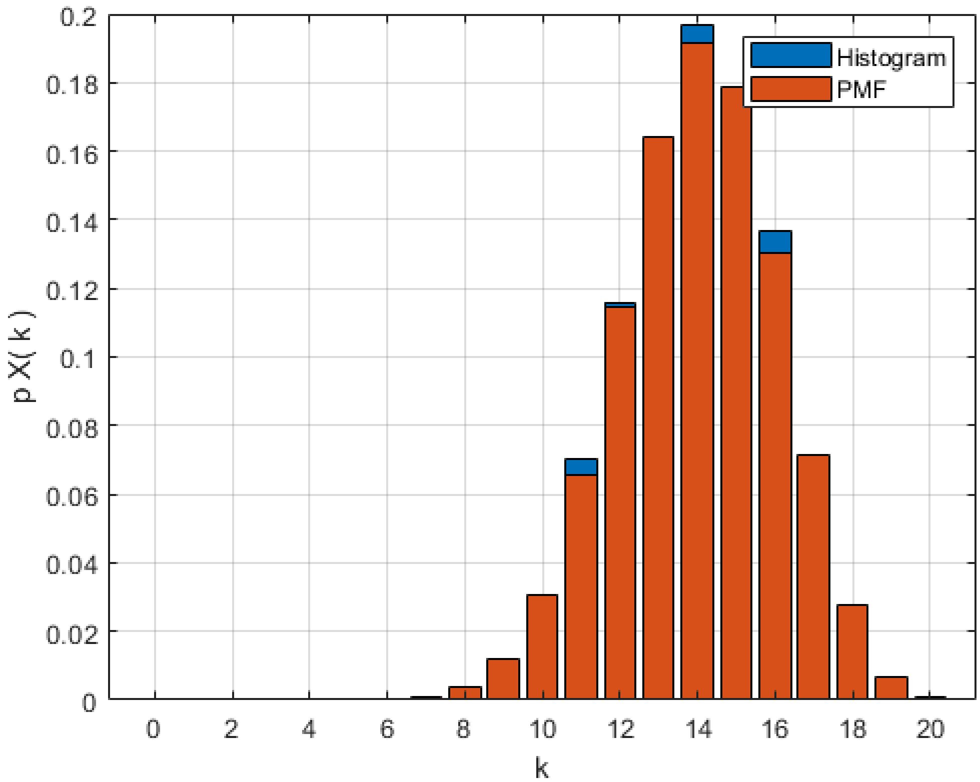
Part f(ii)

```
p=0.7; n=20; numberoftimes=1000;

% Create realizations
binom_rv=zeros(1,numberoftimes);
for i=1:numberoftimes
    binom_rv(i)=get_binomial_rv(p,n); % k is an output
end

% Histogram
hist_est=zeros(1,n);
for k=0:n
    hist_est(k+1)=sum(binom_rv==k)/numberoftimes;
end
figure
bar(0:n, hist_est)
xlabel('k') ; ylabel('p X( k )')
hold on
% PMF binomial
k=0:n;
pXk = pdf('bino' , k ,n , p) ;
bar(0:n, pXk)
xlabel('k') ; ylabel('p X( k )')
grid on

legend('Histogram', 'PMF')
```



Comment:

As the number of times increases, the histogram estimate approaches to the true PMF. Histogram estimate becomes more reliable as the number of times increases.

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