EE 230 Spring 2021-2022
HW 1

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1)

a)

$\Omega = \{x|x \in N, x > 0\}$
The sample space consists of natural numbers greater than 0. The number represents number of total flipping before someone wins. For example, 2 represents that Ayşe get tail in the first trial, and Bora get head in his first trial. Any number represents that Ayşe wins if the remainder from the division of the number by 3 is 1, and represents Bora wins if the remainder is 2, and represents Ceyda wins otherwise.

b)

$A = \{x|x \in \Omega, x\%3 = 1\}$
$B = \{x|x \in \Omega, x\%3 = 2\}$
$A \cup B = \{x|x \in \Omega, x\%3 \neq 0\}$
$(A \cup B)^c = \{x|x \in \Omega, x\%3 = 0\}$

c)

Let the events E_k be defined as follows:
E_k: the event for the case that first k-1 trial is tail and kth trial is head.
Then we can write events A and B in terms of E_k as follows:
$A = E_1 \cup E_4 \cup E_7 \cup ...$
$B = E_2 \cup E_5 \cup E_8 \cup ...$
Since these events are disjoint, we can calculate the probabilities as follows:
$P(A) = P(E_1) + P(E_4) + P(E_7) + ...$
$P(B) = P(E_2) + P(E_5) + P(E_8) + ...$
The probability of E_k can be calculated as:
$P(E_k) = (1 - p)^{k-1} \cdot p$

Then we can calculate $P(A)$ and $P(B)$ as:
$P(A) = \sum_{k=1}^{\infty}(1-p)^{3\cdot(k-1)} \cdot p$
$= p \cdot (((1-p)^3)^0 + ((1-p)^3)^1 + ((1-p)^3)^2 + ...) = \frac{p}{1-(1-p)^3} = \frac{1}{p^2-3p+3}$

$P(B) = \sum_{k=1}^{\infty}(1-p)^{3\cdot(k-1)+1} \cdot p$
$= p \cdot (1-p) \cdot (((1-p)^3)^0 + ((1-p)^3)^1 + ((1-p)^3)^2 + ...) = \frac{p\cdot(1-p)}{1-(1-p)^3} = \frac{1-p}{p^2-3p+3}$

d)

Since events A and B are disjoint, by axiom 2, we have
$P(A \cup B) = P(A) + P(B) = \frac{1}{p^2-3p+3} + \frac{1-p}{p^2-3p+3} = \frac{2-p}{p^2-3p+3}$

Since $(A \cup B)$ and$(A \cup B)^c$ are disjoint, and $\Omega = (A \cup B) + (A \cup B)^c$ we have
$P((A \cup B)^c) = P(\Omega) - P(A \cup B)$
By axiom 3, $P(\Omega) = 1$. Therefore $P((A \cup B)^c) = \frac{p^2-2p+1}{p^2-3p+3}$

e)

For A and B:
$P(A) + P(B) - 1 = \frac{2-p}{p^2-3p+3} - 1 = \frac{-p^2+2p-1}{p^2-3p+3} = \frac{-(p-1)^2}{(p-3/2)^2+3/4} \leq 0 \ \forall p \ 0 \leq p \leq 1$
$P(A \cap B) = 0$ (events are disjoint)
$0 \geq \frac{-(p-1)^2}{(p-3/2)^2+3/4}$ (equation holds)

For arbitrary two events C and D:
$P(C \cap D) \geq P(C) + P(D) - 1$
$P(C \cap D) \geq P(C \backslash D) + P(D \backslash C) + 2P(C \cap D) - 1$
$1 \geq P(C \backslash D) + P(D \backslash C) + P(C \cap D)$
$1 \geq P(C \cup D)$ (equation holds)

2)

a)

$B : A_3 \cap A_5 \ given \ A_2$

b)

$P(B) = P(A_3 \cup A_5 | A_2) = \frac{P(A_2 \cap (A_3 \cup A_5))}{P(A_2)} = \frac{P((A_2 \cap A_3) \cup (A_2 \cap A_5))}{P(A_2)}$
$= \frac{P((A_2 \cap A_3)) + P((A_2 \cap A_5)) - P((A_2 \cap A_3 \cap A_5))}{P(A_2)}$
$A_2 = \{2, 4, 6, ..., 100\}$ has 50 elements
$A_2 \cap A_3 = \{6, 12, 18, ..., 96\}$ has 16 elements
$A_2 \cap A_5 = \{10, 20, 30, ..., 100\}$ has 10 elements
$A_2 \cap A_3 \cap A_5 = \{30, 60, 90\}$ has 3 elements
$\frac{P((A_2 \cap A_3)) + P((A_2 \cap A_5)) - P((A_2 \cap A_3 \cap A_5))}{P(A_2)} = \frac{0.16+0.1-0.03}{0.5} = 0.58$

c)

$P(A_a \cup A_b|A_2) = \frac{A_2 \cap (A_a \cup A_b)}{P(A_2)} = \frac{P(A_2 \cap A_a) + P(A_2 \cap A_b) - P(A_2 \cap A_a \cap A_b)}{P(A_2)}$
If we assume that $a \neq b$, $a \neq 2$, and $b \neq 2$; than we have
$P(A_a \cup A_b|A_2) = \frac{P(A_{2a}) + P(A_{2b}) - P(A_{2ab})}{P(A_2)}$
To obtain a formula for general case (without any assumption like $a \neq 2$) we can divide subscripts in the numerator by the greatest common dividor. Therefore our formula will be
$P(A_a \cup A_b|A_2) = \frac{P(A_{\frac{2a}{\gcd(2,a)}}) + P(A_{\frac{2b}{\gcd(2,b)}}) - P(A_{\frac{2ab}{\gcd(2,a) \cdot \gcd(a,b)}})}{P(A_2)}$

3)

Let the events A_k and B be defined as :
A_k: The event that the family has k children
B: The event that the chosen child is the youngest one
B': The event that the chosen child is the oldest one
By theorem 2:
$P(B|A_k) = P(B'|A_k) = 1/k$
$P(B') = \sum_{k=1}^{4} P(B'|A_k) \cdot P(A_k) = \sum_{k=1}^{4} (1/k) \cdot p_k$
$P(B) = \sum_{k=1}^{4} P(B|A_k) \cdot P(A_k) = \sum_{k=1}^{4} (1/k) \cdot p_k$
$P(B) = P(B') = 0.25 + 0.45/2 + 0.2/3 + 0.1/4 = 0.56$
$P(A_k|B) = \frac{P(B|A_k) \cdot P(A_k)}{P(B)} = \frac{(1/k) \cdot p_k}{0.56}$

a)

$P(A_1|B) = \frac{p_k}{0.56k} = 0.25/0.56 = 0.45$

b)

$P(A_3|B) = \frac{p_k}{0.56k} = 0.2/(3 * 0.56) = 0.12$

c)

$P(B') = 0.56$

4)

a)

$\binom{5}{3} \cdot p^3 \cdot q^2 = 10 \cdot p^3 \cdot q^2$

b)

Let the index of the first outcome ω_q be n, then we should have outcomes ω_p or ω_r for the indexes between 1 and n-1. We can calculate this probability as:
$\sum_{k=2}^{5}(1-q)^{n-1} \cdot q$
But we have to remove the case that all outcomes before nth trial is ω_r, so our formula will be:
$\sum_{k=2}^{5}((1-q)^{n-1} - r^{n-1}) \cdot q = q \cdot \sum_{k=1}^{4}((1-q)^{n} - r^{n})$

c)

We can use the formula with replacing the upper limit of summation with ∞.
$q \cdot \sum_{k=1}^{\infty}((1-q)^{n} - r^{n}) = q \cdot (\frac{1}{1-(1-q)} - \frac{1}{1-r}) = 1 - \frac{q}{1-r} = \frac{1-r-q}{1-r} = \frac{p}{1-r}$