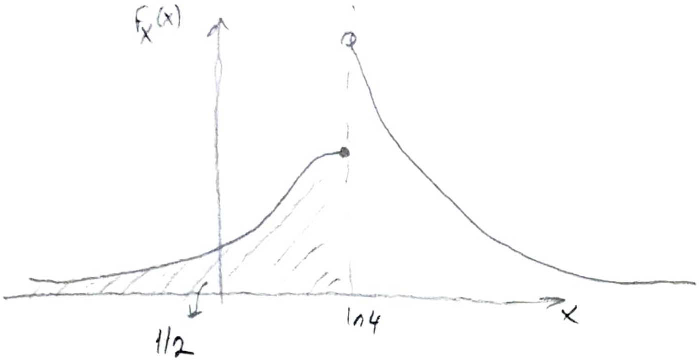
EE230 HW3

Abdullah Emir Göğüşdere - 2443075

İsmail Enes Bülbül - 2442630

Group 13

1a)



$$\int_{-\infty}^{\ln 4} f_X(x)\,dx = \frac{1}{2}$$

$$\Downarrow$$

$$\int_{\ln 4}^{\infty} f_X(x)\,dx = \frac{1}{2}$$

$$\frac{1}{2} = \int_{\ln 4}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \left(-e^{-\lambda x}\right)\Big|_{\ln 4}^{\infty} = e^{-\lambda \ln 4} = 4^{-\lambda} = \frac{1}{2}$$

$$\boxed{\lambda = \frac{1}{2}}$$

1b)

$$\int_{\ln 4}^{\infty} x \lambda e^{-\lambda x}\,dx = \qquad x = u + \ln 4, \quad dx = du$$

$$= \int_{0}^{\infty} (u + \ln 4)\lambda e^{-\lambda(u+\ln 4)}\,du$$

$$= \underbrace{e^{-\lambda \ln 4}}_{\frac{1}{2}} \underbrace{\int_{0}^{\infty} u \lambda e^{-\lambda u}\,du}_{\frac{1}{\lambda}=2} + \underbrace{e^{-\lambda \ln 4}}_{\frac{1}{2}} \cdot \ln 4 \cdot \underbrace{\int_{0}^{\infty} \lambda e^{-\lambda u}\,du}_{1}$$

$$= \ln 2 + 1$$

$$E[X] = \underbrace{\int_{-\infty}^{\ln 4} \frac{x}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}}}_{\ln 2 - 1} + \underbrace{\int_{\ln 4}^{\infty} x \lambda e^{-\lambda x}\,dx}_{\ln 2 + 1} = \ln 4$$

$$\ln 2 - 1 = \int_{-\infty}^{\ln 4} \frac{x}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}}\,dx$$

$$= \int_{-\infty}^{\ln 4} \frac{x - \ln 4}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}}\,dx + \ln 4 \underbrace{\int_{-\infty}^{\ln 4} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}}\,dx}_{\frac{1}{2}}$$

$$-1 = \int_{-\infty}^{\ln 4} \frac{x - \ln 4}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\ln 4)^2}{2\sigma^2}}\,dx \qquad (x-\ln 4)^2 = u, \quad 2(x - \ln 4)\,dx = du$$

$$= \frac{1}{2} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{u}{2\sigma^2}}\,du$$

$$= \frac{1}{2\sqrt{2\pi}\,\sigma} \left(-2\sigma^2 e^{-\frac{u}{2\sigma^2}}\right)\Big|_{\infty}^{0}$$

$$+1 = \frac{+\cancel{2}\sigma^{\cancel{2}}}{\cancel{2}\sqrt{2\pi}\,\cancel{\sigma}}$$

$$\boxed{\sigma = \sqrt{2\pi}}$$

1c)

$$var(x) = E[x^2] - (E[x])^2$$

$$= \int_{-\infty}^{\ln 4} \frac{x^2}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\ln 4)^2}{2\sigma^2}} dx + \int_{\ln 4}^{\infty} x^2 \lambda e^{-\lambda x} dx - (\ln 4)^2$$

$$\int_{\ln 4}^{\infty} x^2 \lambda e^{-\lambda x} dx \qquad \begin{aligned} x &= u + \ln 4 \\ dx &= du \end{aligned}$$

$$= \int_0^{\infty} (u^2 + 2u\ln 4 + (\ln 4)^2)\, \lambda e^{-\lambda u} \underbrace{e^{-\lambda \ln 4}}_{1/2} du$$

$$= \frac{1}{2}\left[\underbrace{\int_0^{\infty} u^2 \lambda e^{-\lambda u} du}_{\substack{E[U^2] = var(U) + (E[U])^2 \\ = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2}} + 2\ln 4 \underbrace{\int_0^{\infty} u \lambda e^{-\lambda u} du}_{\frac{1}{\lambda}} + (\ln 4)^2 \underbrace{\int_0^{\infty} \lambda e^{-\lambda u} du}_{1} \right]$$

$$= \frac{1}{2}\left(4 + 4 + 2\ln 4 \cdot 2 + (\ln 4)^2\right) = 4 + 2\ln 4 + \frac{(\ln 4)^2}{2}$$

$$\int_{-\infty}^{\ln 4} \frac{x^2}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\ln 4)^2}{2\sigma^2}} dx = \int_{-\infty}^{0} \frac{(u+\ln 4)^2}{\sqrt{2\pi}\,\sigma} e^{\frac{-u^2}{2\sigma^2}}$$

$$= \underbrace{\int_{-\infty}^{0} \frac{u^2}{\sqrt{2\pi}\,\sigma} e^{\frac{-u^2}{2\sigma^2}} du}_{\frac{E[U^2]}{2}} + 2\ln 4 \underbrace{\int_{-\infty}^{0} \frac{u}{\sqrt{2\pi}\,\sigma} e^{\frac{-u^2}{2\sigma^2}} du}_{-1} + (\ln 4)^2 \underbrace{\int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-u^2}{2\sigma^2}} du}_{\frac{1}{2}}$$

(Since the integrand is even function)

(From part 1b)

$$\frac{var(U) + E[U]}{2} = \frac{\sigma^2}{2}$$

$$= \pi - 2\ln 4 + \frac{(\ln 4)^2}{2}$$

$$var(x) = \pi - 2\ln 4 + \frac{(\ln 4)^2}{2} + 4 + 2\ln 4 + \frac{(\ln 4)^2}{2} - (\ln 4)^2 \qquad \boxed{var(x) = \pi + 4}$$

1d)

For $x \leq \ln 4$

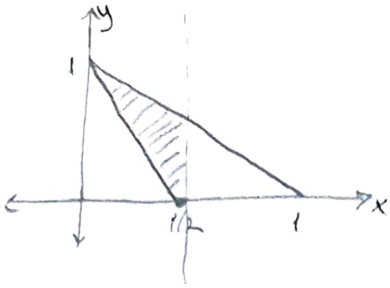
$$CDF = \Phi\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\ln 4}{\sqrt{2\pi}}\right)$$

For $x > \ln 4$

$$CDF = \frac{1}{2} + \int_{\ln 4}^{x} \lambda e^{-\lambda x} dx = \frac{1}{2} + \left(-e^{-\lambda x}\right)\Big|_{\ln 4}^{\infty} = 1 - e^{-\lambda x}$$

$$\boxed{F_X(x) = \begin{cases} \Phi\left(\frac{x-\ln 4}{\sqrt{2\pi}}\right) &, x \leq \ln 4 \\ 1 - e^{-x/2} &, x > \ln 4 \end{cases}}$$

2a)



$$P\left(X\leq \tfrac{1}{2}\right) = \int_0^{1/2}\int_{1-2x}^{1-x} 12y\,dy\,dx$$

$$= \int_0^{1/2} (6y^2)\Big|_{1-2x}^{1-x}\,dx$$

$$= \int_0^{1/2} 6\left(\cancel{1}-2x+x^2-\cancel{1}+4x-4x^2\right)dx$$

$$= 6\int_0^{1/2}\left(2x-3x^2\right)dx$$

$$= 6\left(x^2-x^3\right)\Big|_0^{1/2} = 6\left(\tfrac{1}{4}-\tfrac{1}{8}\right) = \tfrac{6}{8} = \boxed{\tfrac{3}{4}}$$

2b)

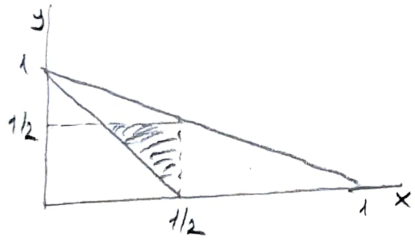
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_{\frac{1-y}{2}}^{1-y} 12y\,dx = 12y\left(1-y-\tfrac{1-y}{2}\right)$$

$$= 6y(1-y)$$

$$f_Y(y) = \begin{cases} 6y(1-y) & , 0<y<1 \\ 0 & , \text{otherwise} \end{cases}$$

2c)

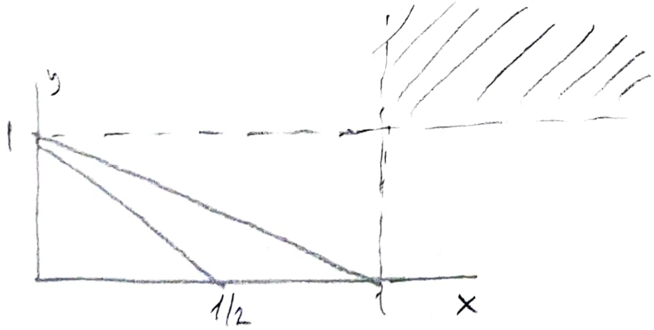


$$F_{X,Y}\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^{1/2}\int_{\frac{1-y}{2}}^{1/2} 12y\,dx\,dy$$

$$= \int_0^{1/2} \cancel{12}6y\left(\frac{\cancel{1}-\cancel{1}+y}{\cancel{2}}\right)dy$$

$$= \int_0^{1/2} 6y^2\,dy = \left(2y^3\right)\Big|_0^{1/2} = \boxed{\tfrac{1}{4}}$$

2d)



$$\boxed{\left\{(x,y)\ \middle|\ x\geq 1,\ y\geq 1\right\}}$$

3a)

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2xy\rho}{\sigma_x\sigma_y}\right)}\, dy$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx = 1$$

$$\int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx = \sqrt{2\pi}\,\sigma$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{y}{\sigma_y}-\frac{\rho x}{\sigma_x}\right)^2+\frac{x^2}{\sigma_x^2}(1-\rho^2)\right)}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{\frac{-x^2}{2\sigma_x^2}} \cdot e^{\frac{-1}{2(1-\rho^2)}\left(\frac{y}{\sigma_y}-\frac{\rho x}{\sigma_x}\right)^2}\, dy$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{\frac{-x^2}{2\sigma_x^2}} \underbrace{\int_{-\infty}^{\infty} e^{\frac{-1}{2(1-\rho^2)\sigma_y^2}\left(y-\frac{\rho x\sigma_y}{\sigma_x}\right)^2}\, dy}_{\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma_y}$$

$$= \frac{1}{2\pi\sigma_x\cancel{\sigma_y}\cancel{\sqrt{1-\rho^2}}}\, e^{\frac{-x^2}{2\sigma_x^2}} \cdot \sqrt{2\pi}\cancel{\sqrt{1-\rho^2}}\,\cancel{\sigma_y}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x}\, e^{\frac{-x^2}{2\sigma_x^2}}$$

Similarly, $f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y}\, e^{\frac{-y^2}{2\sigma_y^2}}$

3b)

$$E_{X,Y}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{xy}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}-\frac{2xy\rho}{\sigma_x\sigma_y}\right)}\, dy\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{xy}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{\frac{-x^2}{2\sigma_x^2}}\, e^{\frac{-1}{2(1-\rho^2)\sigma_y^2}\left(y-\frac{x\rho\sigma_y}{\sigma_x}\right)^2}\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\,\sigma_x}\, e^{\frac{-x^2}{2\sigma_x^2}} \underbrace{\left(\int_{-\infty}^{\infty} \frac{y}{\sqrt{2\pi}\,\sigma_y\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)\sigma_y^2}\left(y-\frac{x\rho\sigma_y}{\sigma_x}\right)^2}\, dy\right)}_{\frac{x\rho\sigma_y}{\sigma_x}} dx$$

$$= \int_{-\infty}^{\infty} \frac{x^2\rho\sigma_y}{\sqrt{2\pi}\,\sigma_x^2}\, e^{\frac{-x^2}{2\sigma_x^2}}\, dx$$

$$= \frac{\rho\sigma_y}{\sigma_x} \underbrace{\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}\,\sigma_x} \cdot e^{\frac{-x^2}{2\sigma_x^2}}\, dx}_{E[x^2]}$$

$$= \frac{\rho\sigma_y}{\sigma_x}\left(var(x) - E[X]\right) = \frac{\rho\sigma_y}{\cancel{\sigma_x}}\,\sigma_x^{\cancel{2}}$$

$$\boxed{E_{X,Y}[XY] = \rho\,\sigma_x\,\sigma_y}$$

3c)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\frac{1}{\cancel{2\pi}\,\sigma_x \cancel{\sigma_y}\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x\sigma_y}\right)}}{\frac{1}{\cancel{\sqrt{2\pi}}\,\cancel{\sigma_y}}\, e^{\frac{-y^2}{2\sigma_y^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}}\, e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2xy\rho}{\sigma_x\sigma_y}\right) + \frac{y^2}{\sigma_y^2}}$$

$$\frac{-x^2}{2(1-\rho^2)\sigma_x^2} - \frac{y^2}{2(1-\rho^2)\sigma_y^2} + \frac{y^2}{\sigma_y^2} + \frac{2xy\rho}{2(1-\rho^2)\sigma_x\sigma_y}$$

$$= \frac{-x^2}{2(1-\rho^2)\sigma_x^2} - \frac{y^2\rho^2}{2(1-\rho^2)\sigma_y^2} + \frac{2xy\rho}{2(1-\rho^2)\sigma_x\sigma_y}$$

$$= \frac{-1}{2(1-\rho^2)\sigma_x^2}\left(x^2 + \frac{y^2\rho^2\, 2(1-\rho^2)\sigma_x^2}{2(1-\rho^2)\sigma_y^2} - \frac{2xy\rho\, 2(1-\rho^2)\sigma_x^2}{2(1-\rho^2)\sigma_x\sigma_y}\right)$$

$$= \frac{-1}{2(1-\rho^2)\sigma_x^2}\left(x - \frac{y\rho\sigma_x}{\sigma_y}\right)^2$$

$$= \frac{1}{\sqrt{2\pi}\,\underbrace{\sigma_x\sqrt{1-\rho^2}}_{\sigma}}\, e^{\frac{-1}{2\underbrace{(1-\rho^2)\sigma_x^2}_{\sigma^2}}\left(x - \frac{y\rho\sigma_x}{\sigma_y}\right)^2}$$

Hence, it is a Gaussian r.v.

$$\boxed{\mu_{X|Y} = E_{X|Y}[X|Y=y] = \frac{y\rho\sigma_x}{\sigma_y}}$$

3d)

$$\hat{X} = \frac{y\rho\sigma_x}{\sigma_y} = \alpha Y$$

i) $\boxed{\alpha = \frac{\rho\sigma_x}{\sigma_y}}$

For given Y, $(X - \hat{X})$ is a gaussian random variable with

$E[X-\hat{X}] = E[X] - \hat{X} = 0$ and

$var(X-\hat{X}) = var(X) = (1-\rho^2)\sigma_x^2$.

Therefore, $E[(X-\hat{X})^2] = var(X-\hat{X}) + (E[X-\hat{X}])^2$

ii) $\boxed{MSE = (1-\rho^2)\sigma_x^2}$

4)
The MSE vs ρ curves obtained both theoretically and in simulation are plotted in the figure below.

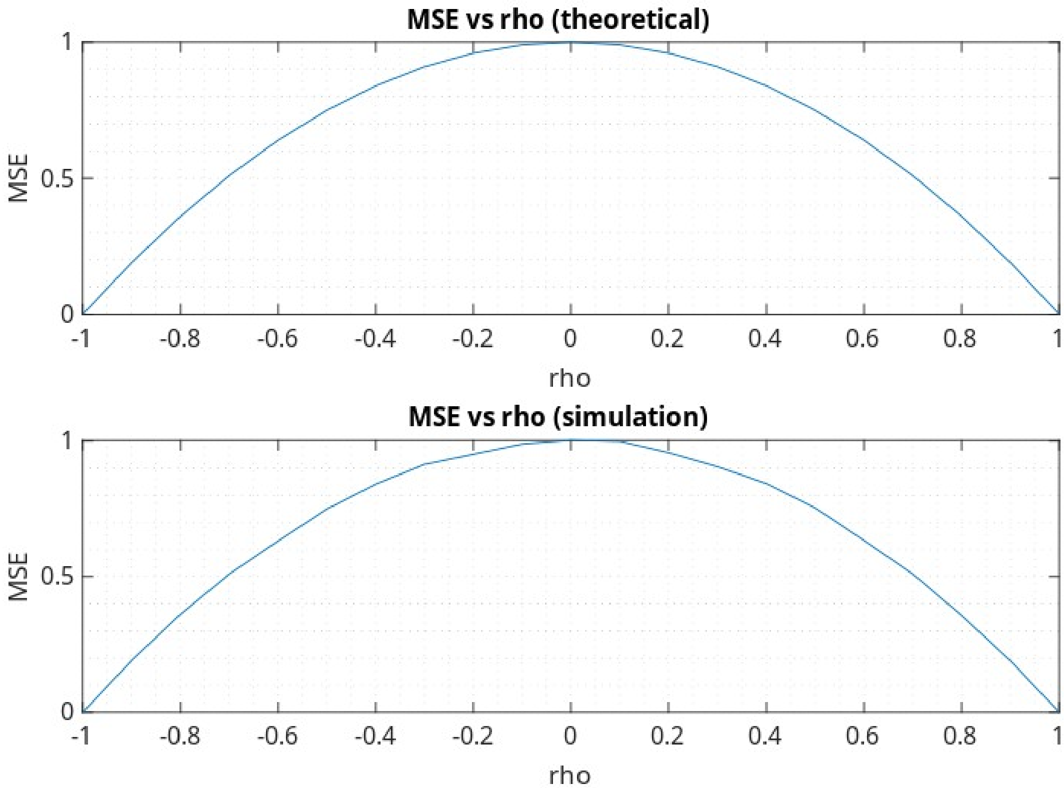


Figure 1: The graphs of MSE vs ρ curves *(theoretical and in simulation)*

We obtained similar graphs for analytical and numerical results. Since we used too many samples (N = 1E5) to calculate MSE numerically, we obtained almost smooth curve. From graph, we can observe that the MSE becomes smaller as ρ becomes greater in magnitude.