

EE301 Homework-3

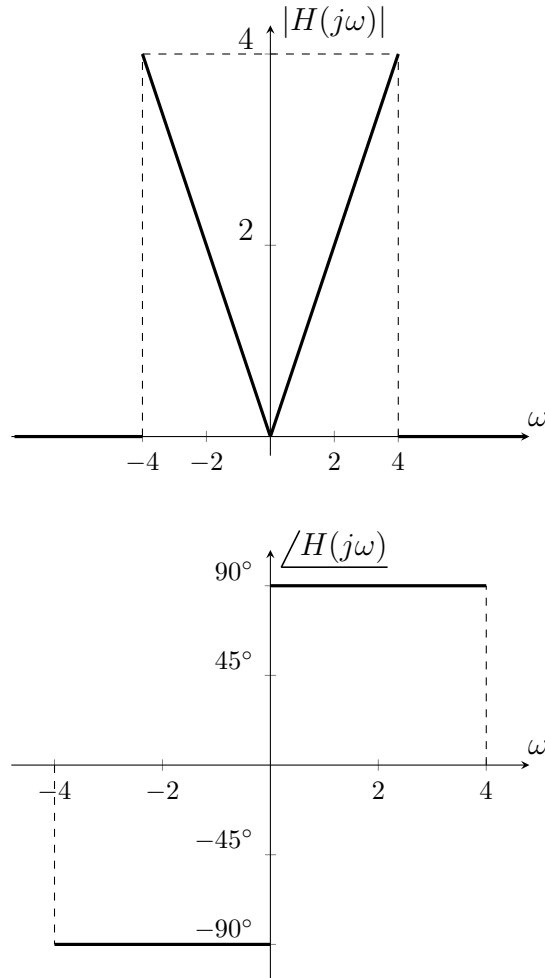
İsmail Enes Bülbül, Eren Meydanlı, Ahmet Caner Akar

December 14, 2022

Question 1

a)

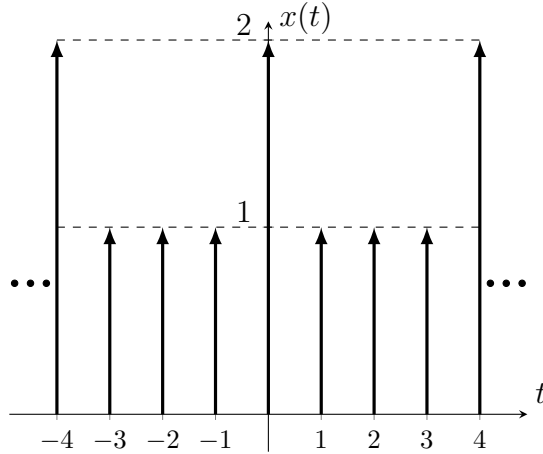
The magnitude and phase responses of $H(j\omega)$ can be seen below:



By differentiation property, we know that CTFT of a signal is multiplied by $j\omega$ if the signal is differentiated in time domain. Therefore, this system acts like an differentiator and since it has nonzero values only between $\omega = -4$ and $\omega = 4$, it is named as ideal band-limited differentiator.

b)

$x(t)$ can be plotted as:



$x(t)$ is a periodic signal with fundamental period $T_0 = 4$. Also, $x(t)$ can be written in CTFS representation: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, where $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$.

Let's first calculate the FS coefficients: $a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt = \frac{1}{4} \int_{0^-}^{4^-} x(t) e^{-jk\frac{\pi}{2}t} dt$
 $\Rightarrow a_0 = \frac{5}{4}, a_1 = \frac{1}{4}, a_{-1} = \frac{1}{4}, a_2 = \frac{1}{4}, a_{-2} = \frac{1}{4}$

Also, recall that:

$$e^{jk\omega_0 t} \longrightarrow \boxed{\text{LTI system}} \longrightarrow H(jk\omega_0) e^{jk\omega_0 t}$$

By linearity:

$$\underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}_{x(t)} \longrightarrow \boxed{\text{LTI system}} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}}_{y(t)}$$

Therefore, $y(t) = \sum_{k=-2}^2 a_k H(jk\omega_0) e^{jk\omega_0 t} = -\frac{1}{4}\pi j e^{-j\pi t} - \frac{1}{8}\pi j e^{-j\frac{\pi}{2}t} + 0 + \frac{1}{8}\pi j e^{j\frac{\pi}{2}t} + \frac{1}{4}\pi j e^{j\pi t}$

$$y(t) = -\frac{\pi}{2} \left(\frac{1}{2j} e^{j\pi t} - \frac{1}{2j} e^{-j\pi t} \right) - \frac{\pi}{4} \left(\frac{1}{2j} e^{j\frac{\pi}{2}t} - \frac{1}{2j} e^{-j\frac{\pi}{2}t} \right) = -\frac{\pi}{2} \sin(\pi t) - \frac{\pi}{4} \sin\left(\frac{\pi}{2}t\right)$$

Question 2

a)

We know that $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$. Then

$$\begin{aligned}\frac{d}{d\omega}X(j\omega) &= \frac{d}{d\omega} \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \frac{d}{d\omega}(x(t)e^{-j\omega t}) dt \\ &= \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t} dt \\ &= \mathcal{F}\{x(t)(-jt)\}\end{aligned}$$

Therefore, $\mathcal{F}^{-1}\{\frac{d}{d\omega}X(j\omega)\} = x(t)(-jt)$

b)

Let $g_1(t) = \frac{\sin(\pi t)}{\pi t}$. Then we know that $\mathcal{F}\{g_1(t)\} = G_1(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{else} \end{cases}$

Since $g(t) = g_1(t)g_1(t)$, by multiplication property, we know that $G(j\omega) = \frac{1}{2\pi}G_1(j\omega)*G_1(j\omega)$

$$G(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\theta)G(j(\omega - \theta)) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(j(\omega - \theta)) d\theta$$

Let $\theta' = \omega - \theta$. Then $d\theta' = -d\theta$ and

$$G(j\omega) = \frac{1}{2\pi} \int_{\omega+\pi}^{\omega-\pi} G_1(j\theta') d(-\theta') = \frac{1}{2\pi} \int_{\omega-\pi}^{\omega+\pi} G_1(j\theta') d\theta'$$

For $\omega < -2\pi$ or $\omega > 2\pi$, $G(j\omega) = 0$.

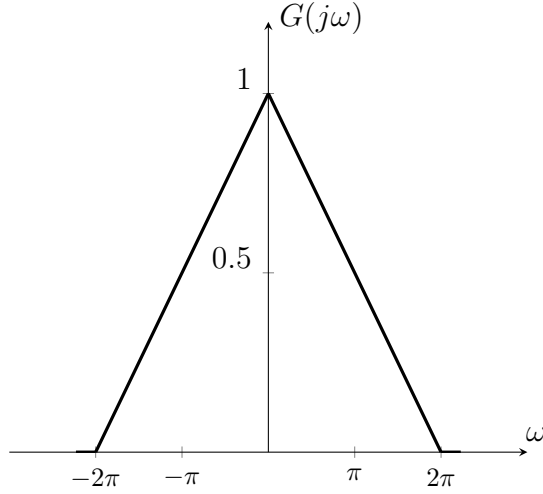
For $-2\pi < \omega < 0$

$$G(j\omega) = \frac{1}{2\pi} \int_{-\pi}^{\omega+\pi} d\theta' = \frac{\omega+2\pi}{2\pi}$$

For $0 < \omega < 2\pi$

$$G(j\omega) = \frac{1}{2\pi} \int_{\omega-\pi}^{\pi} d\theta' = \frac{2\pi-\omega}{2\pi}$$

Therefore, $G(j\omega) = \begin{cases} \frac{\omega+2\pi}{2\pi}, & -2\pi < \omega < 0 \\ \frac{2\pi-\omega}{2\pi}, & 0 < \omega < 2\pi \\ 0, & \text{else} \end{cases}$, and $G(j\omega)$ is plotted below



c)

i)

$$H(j\omega) = \int_{-\infty}^{\infty} f^*(-t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} (f(-t)e^{j\omega t})^* dt$$

Let $t' = -t$. Then $dt' = -dt$ and

$$H(j\omega) = \int_{\infty}^{-\infty} (f(t')e^{-j\omega t'})^* d(-t') = (\int_{-\infty}^{\infty} f(t')e^{-j\omega t'} d(t'))^* = F^*(j\omega)$$

ii)

Let $\mathcal{F}\{y(t)\} = Y(j\omega)$. Then $Y(j\omega) = F^*(j\omega)F(j\omega) = |F(j\omega)|^2$

$Y^*(j\omega) = |F(j\omega)|$, therefore we can say that $Y(j\omega)$ is even and $y(t)$ is real valued.

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} (\int_{-\infty}^{\infty} |F(j\omega)|^2 \cos(\omega t) d\omega + j \int_{-\infty}^{\infty} |F(j\omega)|^2 \sin(\omega t) d\omega)$$

Since $y(t)$ is real valued, $y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \cos(\omega t) d\omega$.

$$y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 d\omega$$

$$|y(t)| < \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 |\cos(\omega t)| d\omega < y(0)$$

$$|y(t)| < |y(0)| \text{ for all } t.$$

Question 3

a)

i)

$$x(t) = \frac{\sin(4\pi t)}{\pi t} \cos(2\pi t) = \frac{4\sin(4\pi t)}{4\pi t} \cos(2\pi t) = x_1(t)x_2(t) \left(\text{where } x_1(t) = \frac{4\sin(4\pi t)}{4\pi t}, x_2(t) = \cos(2\pi t) \right)$$

$$\text{Recall that: } \mathcal{F}\{rect(\theta)\} = \frac{\sin(\omega/2)}{\omega/2}$$

$$\text{By duality property of CTFT: } \frac{\sin(t/2)}{t/2} \longleftrightarrow 2\pi rect(-\omega) = 2\pi rect(\omega)$$

$$\text{By scaling property of CTFT: } \frac{\sin(4\pi t)}{4\pi t} \longleftrightarrow \frac{rect(\frac{\omega}{8\pi})}{4}$$

$$\text{By linearity: } \frac{4\sin(4\pi t)}{4\pi t} \longleftrightarrow rect(\frac{\omega}{8\pi}) = X_1(j\omega)$$

$$x_2(t) = \cos(2\pi t) = \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$$

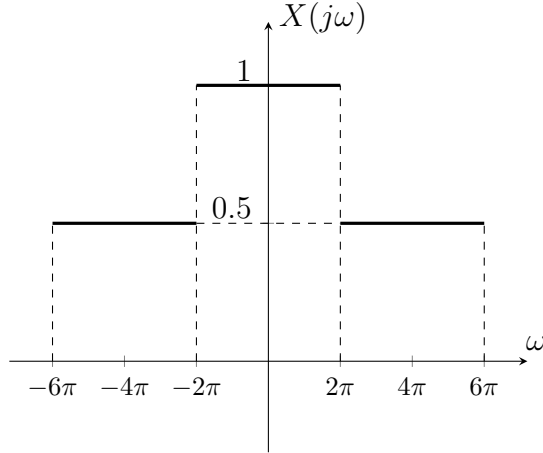
$$\mathcal{F}\{\frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}\} \longleftrightarrow \pi[\delta(\omega + 2\pi) + \delta(\omega - 2\pi)] = X_2(j\omega)$$

By modulation property of CTFT:

$$x(t) = x_1(t)x_2(t) \longleftrightarrow X(j\omega) = \frac{1}{2}X_1(j\omega) * X_2(j\omega)$$

$$X(j\omega) = \frac{1}{2\pi} rect(\frac{\omega}{8\pi}) * \pi[\delta(\omega + 2\pi) + \delta(\omega - 2\pi)] = \frac{1}{2}[rect(\frac{\omega}{8\pi}) * \delta(\omega + 2\pi) + rect(\frac{\omega}{8\pi}) * \delta(\omega - 2\pi)]$$

$$X(j\omega) = \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi})$$



ii)

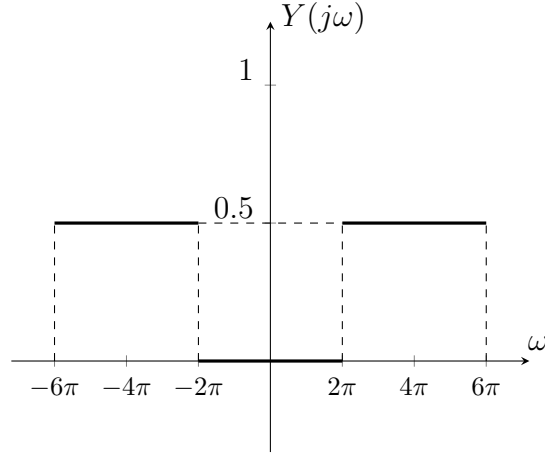
$$y(t) = h(t) * x(t)$$

$$\text{By convolution property of CTFT: } y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(j\omega) = (1 - rect(\frac{\omega}{4\pi})) (\frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}))$$

$$Y(j\omega) = \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) - \frac{1}{2}(rect(\frac{\omega}{4\pi})rect(\frac{\omega-2\pi}{8\pi}) + rect(\frac{\omega}{4\pi})rect(\frac{\omega+2\pi}{8\pi}))$$

$$Y(j\omega) = \frac{1}{2} \left(\text{rect}\left(\frac{\omega+4\pi}{4\pi}\right) + \text{rect}\left(\frac{\omega-4\pi}{4\pi}\right) \right)$$



b)

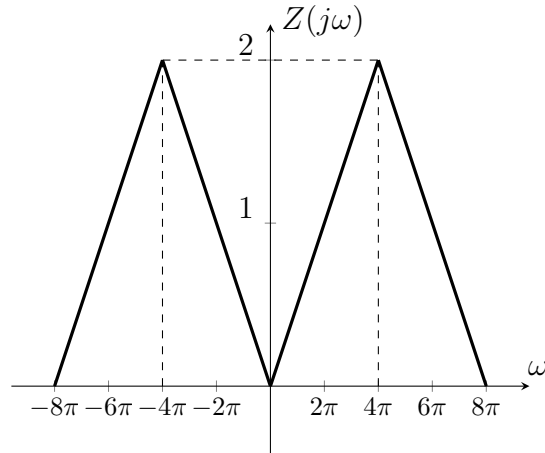
i)

By modulation property of CTFT:

$$z(t) \longleftrightarrow Z(j\omega) = \frac{1}{2\pi} Y(j\omega) * \mathcal{F}\left\{\frac{\sin(2\pi t)}{\pi t}\right\} = \frac{1}{2\pi} Y(j\omega) * \text{rect}\left(\frac{\omega}{4\pi}\right)$$

$$\Rightarrow Z(j\omega) = \frac{1}{2\pi} \left(\text{rect}\left(\frac{\omega+4\pi}{4\pi}\right) + \text{rect}\left(\frac{\omega-4\pi}{4\pi}\right) \right) * \text{rect}\left(\frac{\omega}{4\pi}\right)$$

$$\Rightarrow Z(j\omega) = \begin{cases} \frac{8\pi-\omega}{2\pi}, & 4\pi < \omega < 8\pi \\ \frac{\omega}{2\pi}, & 0 < \omega \leq 4\pi \\ \frac{-\omega}{2\pi}, & -4\pi < \omega < 0 \\ \frac{8\pi+\omega}{2\pi}, & -8\pi < \omega \leq -4\pi \end{cases}$$



ii)

$$y(t) \longleftrightarrow Y(j\omega) = \text{rect}\left(\frac{\omega+4\pi}{4\pi}\right) + \text{rect}\left(\frac{\omega-4\pi}{4\pi}\right)$$

$$Y(j\omega) = \frac{1}{\pi} \text{rect}\left(\frac{\omega}{4\pi}\right) * \pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$$

By modulation property of CTFT:

$$\mathcal{F}^{-1}\{Y(j\omega)\} = y(t) = \frac{2\sin(2\pi t)}{\pi t} \cos(4\pi t)$$

Question 4

a)

i)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n]e^{-j\Omega n}}_{\delta[n]e^{-j\Omega 0}} = \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

ii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} (2\delta[n-3] - \delta[n-10])e^{-j\Omega n} \stackrel{(\text{by linearity})}{=} 2 \sum_{n=-\infty}^{\infty} \delta[n-3]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} \delta[n-10]e^{-j\Omega n}$$

By time-shift property of DTFT:

$$X(e^{j\Omega}) = 2e^{-j3\Omega} - e^{-j10\Omega}$$

iii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=1}^4 \frac{1}{n^2} e^{-j\Omega n} = e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega} + \frac{1}{9}e^{-j3\Omega} + \frac{1}{16}e^{-j4\Omega}$$

iv)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1] \right) e^{-j\Omega n}$$

$$\stackrel{(\text{by linearity})}{=} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} 3^n u[-n-1]e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n - \sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} \quad \left(\text{since } \left| \frac{1}{2}e^{-j\Omega} \right| = \frac{1}{2} < 1, \text{ so the expression is convergent} \right)$$

$$\text{Let } m = -n : \sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n} = \sum_{m=1}^{\infty} 3^{-m} e^{j\Omega m} = \sum_{m=1}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m = \underbrace{\left[\sum_{m=0}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m \right]}_{\frac{1}{1 - \frac{1}{3}e^{j\Omega}}} - 1$$

$$\Rightarrow X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1 - \frac{1}{3}e^{j\Omega}} - 1 \right)$$

v)

Say that, $\hat{x}[n] = \left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]$ and $\mathcal{F}\{\hat{x}[n]\} = \hat{X}(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1 - \frac{1}{3}e^{j\Omega}} - 1 \right)$

By time-shift property of DTFT:

$$x[n] = \hat{x}[n-7] \longleftrightarrow X(e^{j\Omega}) = \hat{X}(e^{j\Omega})e^{-j7\Omega}$$

$$\Rightarrow X(e^{j\Omega}) = \frac{e^{-j7\Omega}}{1 - \frac{1}{2}e^{-j\Omega}} - \left(\frac{e^{-j7\Omega}}{1 - \frac{1}{3}e^{j\Omega}} - e^{-j7\Omega} \right)$$

vi)

Let $x[n]$ be a periodic signal with fundamental period N . Then,

$$\mathcal{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} \sum_{k=k_0}^{k_0+N-1} a_k 2\pi \delta(\Omega - k\frac{2\pi}{N} - 2\pi m)$$

Note that for this signal if we consider the interval $0 \leq \Omega < 2\pi$, DTFT of the signal will be written as:

$$\mathcal{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{k=0}^2 a_k 2\pi \delta(\Omega - k\frac{2\pi}{3})$$

First, find the DTFS coefficients of $x[n]$: $a_k = \frac{1}{3} \sum_{n=0}^2 x[n] e^{-jk\frac{2\pi}{3}n}$

$$\Rightarrow a_0 = \frac{1}{3} \sum_{n=0}^2 x[n] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow a_1 = \frac{1}{3} \sum_{n=0}^2 x[n] e^{-j\frac{2\pi}{3}n} = \frac{1}{3} e^{-j\frac{2\pi}{3}} + \frac{1}{3} e^{-j\frac{4\pi}{3}} = \frac{-1}{3}$$

$$\Rightarrow a_2 = \frac{1}{3} \sum_{n=0}^2 x[n] e^{-j\frac{4\pi}{3}n} = \frac{1}{3} e^{-j\frac{4\pi}{3}} + \frac{1}{3} e^{-j\frac{8\pi}{3}} = \frac{-1}{3}$$

$$X(e^{j\Omega}) = \sum_{k=0}^2 a_k 2\pi \delta(\Omega - k\frac{2\pi}{3}) = a_0 2\pi \delta(\Omega) + a_1 2\pi \delta(\Omega - \frac{2\pi}{3}) + a_2 2\pi \delta(\Omega - \frac{4\pi}{3})$$

$$X(e^{j\Omega}) = \frac{2\pi}{3} \left(2\delta(\Omega) - \delta(\Omega - \frac{2\pi}{3}) - \delta(\Omega - \frac{4\pi}{3}) \right)$$

b)

i)

$$x[n] = \frac{1}{2\pi} \int_{\Omega_0}^{\Omega_0+2\pi} \underbrace{X(e^{j\Omega})}_{=1} e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega n} d\Omega = \frac{1}{2\pi j n} (e^{j\pi n} - e^{-j\pi n})$$

$$x[n] = \frac{1}{\pi n} \frac{1}{2j} (e^{j\pi n} - e^{-j\pi n}) = \frac{\sin(\pi n)}{\pi n}$$

Recall that, $\text{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

Here, n is an integer. So, $\frac{\sin(\pi n)}{\pi n} = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow x[n] = \delta[n]$

ii)

We know that, $\mathcal{F}\{1\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$

By frequency-shift property of DTFT:

$$\mathcal{F}\{e^{j\Omega_0 n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$$

$$\Rightarrow \mathcal{F}\{e^{j\frac{\pi}{3}n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m) = 2\pi X(e^{j\Omega})$$

$$\Rightarrow \mathcal{F}^{-1}\left\{\sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m)\right\} = \frac{e^{j\frac{\pi}{3}n}}{2\pi} = x[n]$$

iii)

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}) e^{j\Omega n} d\Omega = \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\frac{-\pi}{2}} e^{j\Omega n} d\Omega}_{= \int_{\frac{\pi}{2}}^{\pi} e^{-j\Omega n} d\Omega} + \frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n} d\Omega = \frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n} + e^{-j\Omega n} d\Omega$$

$$x[n] = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\pi} \cos(\Omega n) d\Omega = \frac{1}{\pi n} \sin(\Omega n) \Big|_{\frac{\pi}{2}}^{\pi} = \frac{\sin(\pi n)}{\pi n} - \frac{\sin(\frac{\pi}{2}n)}{\pi n} = \frac{\sin(\pi n)}{\pi n} - \frac{1}{2} \frac{\sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n}$$

$$x[n] = \text{sinc}(n) - \frac{1}{2} \text{sinc}\left(\frac{n}{2}\right)$$

iv)

By frequency-shift property of DTFT:

$$x[n] \longleftrightarrow X(e^{j\Omega})$$

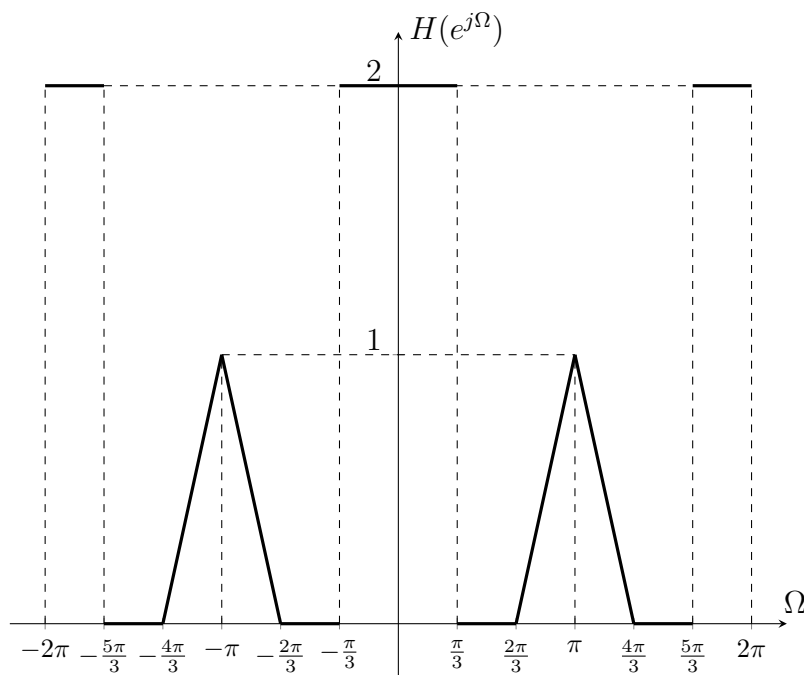
$$x[n] e^{j\frac{\pi}{4}n} \longleftrightarrow X(e^{j(\Omega - \frac{\pi}{4})}) = Y(e^{j\Omega})$$

Therefore, $y[n] = x[n]e^{j\frac{\pi}{4}n} = \text{sinc}(n)e^{j\frac{\pi}{4}n} - \frac{1}{2}\text{sinc}(\frac{n}{2})e^{j\frac{\pi}{4}n}$

Question 5

a)

Recall that, $Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) \Rightarrow H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$



b)

c)

d)

Question 6