

EE301 Homework-2

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Question 1

System 1

- 1 - The system is memoryless since the value of the output $y(t)$ depends on only the value of the input $x(t)$ at present time instant.
- 2 - To determine the linearity of the system, apply the input $\alpha x(t)$ and let the output $\hat{y}(t) = \alpha x(t) \cos(2\pi f_0 t) = \alpha y(t) \rightarrow$ the system is linear.
- 3 - The system is causal since the output $y(t)$ does not depend on the future value of the input $x(t)$.
- 4 - The system is time-variant. To see this, apply the input $x(t - t_0)$ and let the output $\hat{y}(t) = x(t - t_0) \cos(2\pi f_0 t) \neq y(t - t_0) = x(t - t_0) \cos(2\pi f_0(t - t_0))$.
- 5 - A system is said to be stable if any bounded input creates a bounded output. Let $x(t)$ be a bounded. That is, there exist $M > 0$ such that $|x(t)| \leq M$. Also, $-1 \leq \cos(2\pi f_0 t) \leq 1$. Thus, $|y(t)| = |x(t) \cos(2\pi f_0 t)| \leq |x(t)| |\cos(2\pi f_0 t)| \leq M \rightarrow$ so the system is stable.

System 2

- 1 - The system is memoryless.
- 2 - The system is non-linear since if we apply the input $\alpha x(t)$ then the output becomes $\hat{y}(t) = c_1 \alpha x(t) + c_2 (\alpha x(t))^2 = \alpha c_1 x(t) + \alpha^2 c_2 x^2(t) \neq \alpha y(t) = \alpha c_1 x(t) + \alpha c_2 x^2(t)$
- 3 - The system is causal.
- 4 - The system is time-invariant consider that if we apply the input $x(t - t_0)$ and let the output $\hat{y}(t) = c_1 x(t - t_0) + c_2 x^2(t - t_0) = y(t - t_0)$
- 5 - Apply a bounded input $x(t)$ such that $|x(t)| \leq M$ then $|y(t)| = |c_1 x(t) + c_2 x^2(t)| \leq$

$|c_1| |x(t)| + |c_2| |x^2(t)| \leq |c_1| M + |c_2| M^2 < \infty \longrightarrow$ the system is stable.

System 3

1 - The system is memoryless.

2 - The system is non-linear. Apply the input $\alpha x(t)$ and let the output $\hat{y}(t) = \alpha x(t) + 4 \neq \alpha y(t) = \alpha x(t) + \alpha 4$

3 - The system is causal.

4 - Apply the input $x(t - t_0)$:

$\hat{y}(t) = x(t - t_0) + 4 = y(t - t_0) \longrightarrow$ the system is time-invariant.

5 - Let $x(t)$ be bounded input such that $|x(t)| \leq M$. Then, $|y(t)| = |x(t) + 4| \leq |x(t)| + 4 \leq M + 4$ (which is a finite number). So, the system is stable.

System 4

1 - The system is not memoryless as it is not causal.

2 - The system is linear.

3 - For $t < 0$, $y(t)$ is calculated by future values of $x(t)$, therefore, the system is not causal.

4 - Apply the input $x(t - t_0)$:

$\hat{y}(t) = x((t - t_0)/3) = y(t - t_0) \longrightarrow$ the system is time-invariant.

5 - Let $x(t)$ be bounded input such that $|x(t)| \leq M$. Then, $|y(t)| = |x(t/3)|$ (the expanded version of $x(t)$) $\leq M \longrightarrow$ the system is stable.

System 5

1 - The system has a memory since the value of the output $y(t)$ depends on the future values of the input $x(t)$.

2 - The system is linear.

3 - The system is not causal because the output $y(t)$ does not depend only on present and past values of the input.

4 - Apply $x(t - t_0)$:

$\hat{y}(t) = tx(t - t_0 + 5) \neq y(t - t_0) = (t - t_0)x(t - t_0 + 5) \longrightarrow$ the system is time-variant.

5 - The system is not stable. To see this, consider the following example:

$x(t) = u(t) \Rightarrow x(t + 5) = u(t + 5)$ also $|x(t + 5)| = |u(t + 5)| \leq 1$ which means that the

input is bounded. However, the output $y(t) = tu(t + 5)$ is not bounded. Therefore, the system is not stable.

System 6

$$y(t) = u(x(t)) = \begin{cases} 1, & x(t) \geq 0 \\ 0, & x(t) < 0 \end{cases}$$

1 - The system is memoryless.

2 - The system is non-linear, apply $\alpha x(t)$:

$\hat{y}(t) = u(\alpha x(t))$ may not be equal to $\alpha y(t) = \alpha u(x(t))$. For example, let the input $x(t) = u(t)$ then for $t \geq 0$, the $x(t) = 1$. Also, choose $\alpha = -1$. As a result, for $t \geq 0$: $u(\alpha x(t)) = 0 \neq \alpha y(t) = \alpha u(x(t)) = -1$.

3 - The system is causal.

4 - Apply $x(t - t_0)$:

$\hat{y}(t) = u(x(t - t_0)) = y(t - t_0) \rightarrow$ the system is time-invariant.

5 - Remember that $|u(t)| \leq 1$ so it is bounded. In this system, whatever the input $x(t)$ is,

the output $y(t) = u(x(t)) = \begin{cases} 1, & x(t) \geq 0 \\ 0, & x(t) < 0 \end{cases}$. Hence, the system is stable.

Question 2

a)

$x(t)$ and $h(t)$ can be written as:

$$x(t) = u(t + 1) - u(t - 1)$$

$$h(t) = (1 - t)[u(t) - u(t - 1)] = \begin{cases} 1 - t, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then, $y(t) = x(t) * h(t)$

Before evaluating the convolution, consider the following:

$$\begin{aligned} \hat{y}(t) &= u(t) * h(t) = \int_{-\infty}^{\infty} u(t - \tau)h(\tau) d\tau \\ &= \int_{-\infty}^t h(\tau) d\tau = \begin{cases} 0, & t < 0 \\ \int_0^t (1 - \tau) d\tau, & 0 < t < 1 \\ \int_0^1 (1 - \tau) d\tau, & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ t - \frac{t^2}{2}, & 0 < t < 1 \\ \frac{1}{2}, & t > 1 \end{cases} \end{aligned}$$

By the properties of the LTI system:

$$\hat{y}(t+1) = u(t+1) * h(t) = \begin{cases} 0, & t+1 < 0 \\ (t+1) - \frac{(t+1)^2}{2}, & 0 < t+1 < 1 \\ \frac{1}{2}, & t+1 > 1 \end{cases} = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & t > 0 \end{cases}$$

$$\hat{y}(t-1) = u(t-1) * h(t) = \begin{cases} 0, & t-1 < 0 \\ (t-1) - \frac{(t-1)^2}{2}, & 0 < t-1 < 1 \\ \frac{1}{2}, & t-1 > 1 \end{cases} = \begin{cases} 0, & t < 1 \\ (t-1) - \frac{(t-1)^2}{2}, & 1 < t < 2 \\ \frac{1}{2}, & t > 2 \end{cases}$$

$$\text{Therefore, } y(t) = \hat{y}(t+1) - \hat{y}(t-1) = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & 0 < t < 1 \\ \frac{1}{2} - [(t-1) - \frac{(t-1)^2}{2}], & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

b)

$$w(t) = h(t) * g(t)$$

$$g(t) \text{ can be written as: } g(t) = x(t) + x(t-1) - x(t+1)$$

$$\text{Then, } w(t) = h(t) * [x(t) + x(t-1) - x(t+1)]$$

By the distributive property of the convolution over addition:

$$w(t) = h(t) * x(t) + h(t) * x(t-1) - h(t) * x(t+1)$$

$$= y(t) + y(t-1) - y(t+1) \text{ [by considering part a]}$$

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Question 3

a)

Recall that in continuous-time systems one can apply the derivative operation to any arbitrary input signal. For example,

$$\delta(t) = \frac{d}{dt}u(t) \longrightarrow \int_{-\infty}^{\infty} \delta(t) dt$$

Also, it can be applied by using the formal definition of the derivative:

$$\lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h}$$

However, in discrete-time systems h cannot go to zero and the minimum value for h can be one. So, in discrete time the derivative expression becomes

$$\lim_{h \rightarrow 1} \frac{x(n+h) - x(n)}{h} = x[n+1] - x[n]$$

Thus, the derivative operation in continuous-time systems is analogous of the difference operation in discrete-time. Hence, we can obtain the impulse response of the difference operation as: $h[n] = \delta[n] - \delta[n-1]$

b)

By convolution, $y[n] = x[n] * h[n] = x[n] * (\delta[n] - \delta[n-1])$

By distributive property of the convolution operation,

$$x[n] * (\delta[n] - \delta[n-1]) = (x[n] * \delta[n]) - (x[n] * \delta[n-1])$$

$$y[n] = x[n] - x[n-1]$$

c)

$$\begin{aligned} e^{j\Omega_0 n} (1 - e^{-j\Omega_0}) &= (\cos(\Omega_0 n) + j\sin(\Omega_0 n))(1 - e^{-j\Omega_0}) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - (\cos(\Omega_0 n) + j\sin(\Omega_0 n))(\cos(\Omega_0) - j\sin(\Omega_0)) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - \cos(\Omega_0 n)\cos(\Omega_0) + j\cos(\Omega_0 n)\sin(\Omega_0) - j\sin(\Omega_0 n)\cos(\Omega_0) - \sin(\Omega_0 n)\sin(\Omega_0) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - (\cos(\Omega_0 n)\cos(\Omega_0) + \sin(\Omega_0 n)\sin(\Omega_0)) - j(\sin(\Omega_0 n)\cos(\Omega_0) - \cos(\Omega_0 n)\sin(\Omega_0)) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - \cos(\Omega_0 n - \Omega_0) - j\sin(\Omega_0 n - \Omega_0) \\ &= \cos(\Omega_0 n) - \cos(\Omega_0(n-1)) + j(\sin(\Omega_0 n) - \sin(\Omega_0(n-1))) \end{aligned}$$

$$\begin{aligned} |y[n]| &= [(\cos(\Omega_0 n) - \cos(\Omega_0(n-1)))^2 + (\sin(\Omega_0 n) - \sin(\Omega_0(n-1)))^2]^{1/2} \\ &= [\cos^2(\Omega_0 n) - 2\cos(\Omega_0 n)\cos(\Omega_0(n-1)) + \cos^2(\Omega_0(n-1)) + \sin^2(\Omega_0 n) - 2\sin(\Omega_0 n)\sin(\Omega_0(n-1)) + \sin^2(\Omega_0(n-1))]^{1/2} \\ &= \sqrt{2 - 2[\cos(\Omega_0 n)\cos(\Omega_0(n-1)) + \sin(\Omega_0 n)\sin(\Omega_0(n-1))]} \\ &= \sqrt{2 - 2\cos(\Omega_0 n - \Omega_0(n-1))} \\ &= \sqrt{2 - 2\cos(\Omega_0)} \end{aligned}$$

d)

$$\Omega_0 = 0 \longrightarrow |y[n]| = \sqrt{2 - 2\cos(0)} = 0$$

$$\Omega_0 = \frac{2\pi}{8} \longrightarrow |y[n]| = \sqrt{2 - 2\cos(\frac{2\pi}{8})} = \sqrt{2 - \sqrt{2}}$$

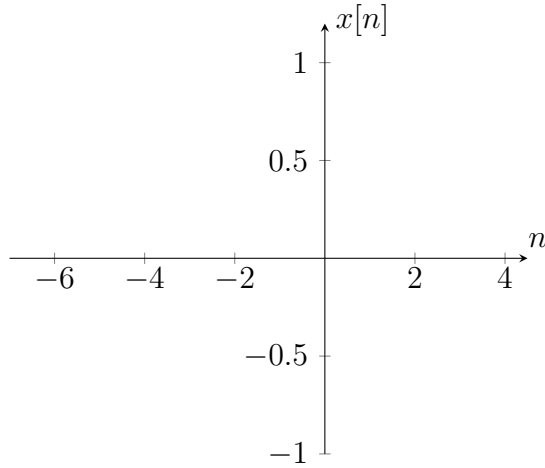
$$\Omega_0 = \frac{2\pi}{4} \longrightarrow |y[n]| = \sqrt{2 - 2\cos(\frac{2\pi}{4})} = \sqrt{2}$$

$$\Omega_0 = \frac{2\pi}{2} \longrightarrow |y[n]| = \sqrt{2 - 2\cos(\frac{2\pi}{2})} = 2$$

As we have shown in part a), the given discrete-time LTI system is analogous to the derivative operation in continuous-time systems. Also, please remember that the derivative is defined as the rate of change of a function with respect to a variable. Here, as the frequency of the input $x[n]$ is increased, the period becomes smaller and the values of the input function are changed more rapidly. Therefore, the rate of change of a input function is increased as the frequency is increased, and the modulus of the output $|y[n]|$ is getting larger values.

e)

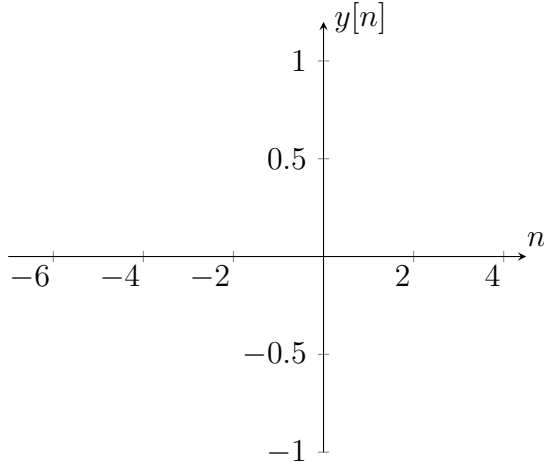
$x[n] = u[n + 5] - u[n - 2]$ can be plotted as:



We can calculate $y[n]$ for $x[n] = u[n + 5] - u[n - 2]$ by using the answer in (b) as:

$$y[n] = u[n + 5] - u[n + 2] - (u[n + 4] + u[n - 3])$$

$y[n]$ can be plotted as:



From the graph of $y[n]$, we can say that $x[n]$ has edges at $n = -5$ and $n = 2$.

f)

From (d), we can say that magnitude of the output signal of the system increases as the frequency increases. In (f), $x[n]$ has rapid changes at $n = -5$ and $n = 2$, and as a result, $y[n]$ has magnitude 1 at $n = -5$ and $n = 2$, and 0 elsewhere.

Question 4

a)

Note that, any arbitrary periodic signal $x(t)$ (or, $x[n]$) with Fourier series coefficients a_k is real-valued if $a_k^* = a_{-k}$.

$$\text{For } x_1(t): a_k^* = \left[\left(\frac{1}{2}\right)^{-k}\right]^* = \left(\frac{1}{2}\right)^{-k} \neq \left(\frac{1}{2}\right)^{-(-k)} = \left(\frac{1}{2}\right)^k = a_{-k}$$

$$\text{For } x_2(t): a_k^* = (\cos(k\pi))^* = \cos(k\pi) = \cos(-k\pi) = a_{-k}$$

$$\text{For } x_3[n]: a_k^* = (j\sin(\frac{k\pi}{2}))^* = -j\sin(\frac{k\pi}{2}) = j\sin(\frac{-k\pi}{2}) = a_{-k}$$

Thus, $x_2(t)$ and $x_3[n]$ are real-valued signals.

b)

Any arbitrary periodic signal $x(t)$ (or, $x[n]$) with Fourier series coefficients a_k is even if a_k is a real-valued and even function.

$$\text{For } x_1(t): a_k = \left(\frac{1}{2}\right)^{-k} \Rightarrow \text{is a real-valued but not even function, i.e., } \left(\frac{1}{2}\right)^{-k} \Big|_{k=1} \neq \left(\frac{1}{2}\right)^{-k} \Big|_{k=-1}$$

$$\text{For } x_2(t): a_k = \cos(k\pi) \Rightarrow \text{is a real-valued and also an even function since consider that}$$

$-1 \leq \cos(k\pi) \leq 1$ and $\cos(k\pi)\Big|_{k=1} = \cos(k\pi)\Big|_{k=-1}$
For $x_3[n]$: $a_k = j\sin(\frac{k\pi}{2}) \Rightarrow$ neither real-valued nor even function.

Therefore, only the signal $x_2(t)$ is even.

c)

By the time-shifting property of Continuous Time Fourier Series:

$$x_2(t-5) = \sum_{k=-100}^{100} \cos(k\pi) e^{-jk\frac{\pi}{5}} e^{jk\frac{2\pi}{50}t} \quad \text{and, also note that} \quad \cos(k\pi) = \frac{1}{2}(e^{jk\pi} + e^{-jk\pi})$$

$$\Rightarrow x_2(t-5) = \sum_{k=-100}^{100} \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}) e^{jk\frac{2\pi}{50}t}$$

Thus, $x_2(t-5)$ is another periodic signal with Fourier series coefficients $b_k = \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}})$.

Also, remember the theorem mentioned above, the signal $x_2(t-5)$ is real if $b_k^* = b_{-k}$.

$b_k^* = (\frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}))^* = \frac{1}{2}(e^{-jk\frac{4\pi}{5}} + e^{jk\frac{6\pi}{5}}) = b_{-k} \Rightarrow$ the signal $x_2(t-5)$ is real.

To check whether the signal $x_2(t-5)$ is even, b_k can be written as:

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + j\sin(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5}) - j\sin(k\frac{6\pi}{5})]$$

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - \frac{1}{2}j[\sin(k\frac{6\pi}{5}) - \sin(k\frac{4\pi}{5})]$$

Also, note that $\boxed{\sin a - \sin b = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})}$

$$\text{Thus, } b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - j[\cos(2\pi k)\sin(k\frac{2\pi}{5})]$$

$$\Rightarrow b_k = \frac{1}{2}(\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})) - j\sin(k\frac{2\pi}{5})$$

As a result, it can be seen that b_k is not real-valued, so the signal $x_2(t-5)$ is not even.

Question 5

a)

In calculating part of the question, we added zero vectors to end and beginning of the main vector to obtain correct results in range of $0 \leq n \leq N-1$, which can be seen in Appendix. Since we manually did that zero adding part, we deviated the result for $n > N$ range. Because of this situation, our calculated result has only N value in range. On the other hand, MATLAB's built-in function, *conv()*, evaluate the convolution so that the size of the

output is equal to $N + L - 1$. That's why the output graphs of convolution operation are different by using our function and MATLAB's built-in function as shown in Appendix. In fact, to get the same result that we got by using our function, one can specify an input argument, *same*, to MATLAB's function so that the result gives the central part of the convolution operation. Otherwise, the built-in function gives the full convolution whose length is equal to $N + L - 1$.

b)

The output signal looks more smoother in larger values of L . As we can observe that $h[n]$ impulse response created accordingly to L which directly affects the convolution range. By saying range, we mean that how many $x*h$ multiplication we will use to calculate one element of the output. So, this is why that the output signal gain more elements and more precise values when L gets larger. And also, we can observe that the L value expands the output range by expanding $h[n]$ size. The MATLAB code and the resulting graphs for different values of L can be seen in Appendix.