

# EE301 Homework-2

İsmail Enes Bülbül, Eren Meydanlı, Ahmet Caner Akar

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## Question 1

### System 1

- 1 - The system is memoryless since the value of the output  $y(t)$  depends on only the value of the input  $x(t)$  at present time instant.
- 2 - To determine the linearity of the system, apply the input  $\alpha x(t)$  and let the output  $\hat{y}(t) = \alpha x(t) \cos(2\pi f_0 t) = \alpha y(t) \rightarrow$  the system is linear.
- 3 - The system is causal since the output  $y(t)$  does not depend on the future value of the input  $x(t)$ .
- 4 - The system is time-variant. To see this, apply the input  $x(t - t_0)$  and let the output  $\hat{y}(t) = x(t - t_0) \cos(2\pi f_0 t) \neq y(t - t_0) = x(t - t_0) \cos(2\pi f_0(t - t_0))$ .
- 5 - A system is said to be stable if any bounded input creates a bounded output. Let  $x(t)$  be a bounded. That is, there exist  $M > 0$  such that  $|x(t)| \leq M$ . Also,  $-1 \leq \cos(2\pi f_0 t) \leq 1$ . Thus,  $|y(t)| = |x(t) \cos(2\pi f_0 t)| \leq |x(t)| |\cos(2\pi f_0 t)| \leq M \rightarrow$  so the system is stable.

### System 2

- 1 - The system is memoryless.
- 2 - The system is non-linear since if we apply the input  $\alpha x(t)$  then the output becomes  $\hat{y}(t) = c_1 \alpha x(t) + c_2 (\alpha x(t))^2 = \alpha c_1 x(t) + \alpha^2 c_2 x^2(t) \neq \alpha y(t) = \alpha c_1 x(t) + \alpha c_2 x^2(t)$
- 3 - The system is causal.
- 4 - The system is time-invariant consider that if we apply the input  $x(t - t_0)$  and let the output  $\hat{y}(t) = c_1 x(t - t_0) + c_2 x^2(t - t_0) = y(t - t_0)$
- 5 - Apply a bounded input  $x(t)$  such that  $|x(t)| \leq M$  then  $|y(t)| = |c_1 x(t) + c_2 x^2(t)| \leq$

$|c_1| |x(t)| + |c_2| |x^2(t)| \leq |c_1| M + |c_2| M^2 < \infty \longrightarrow$  the system is stable.

### System 3

**1** - The system is memoryless.

**2** - The system is non-linear. Apply the input  $\alpha x(t)$  and let the output  $\hat{y}(t) = \alpha x(t) + 4 \neq \alpha y(t) = \alpha x(t) + \alpha 4$

**3** - The system is causal.

**4** - Apply the input  $x(t - t_0)$ :

$\hat{y}(t) = x(t - t_0) + 4 = y(t - t_0) \longrightarrow$  the system is time-invariant.

**5** - Let  $x(t)$  be bounded input such that  $|x(t)| \leq M$ . Then,  $|y(t)| = |x(t) + 4| \leq |x(t)| + 4 \leq M + 4$  (which is a finite number). So, the system is stable.

### System 4

**1** - The system is not memoryless as it is not causal.

**2** - The system is linear.

**3** - For  $t < 0$ ,  $y(t)$  is calculated by future values of  $x(t)$ , therefore, the system is not causal.

**4** - Apply the input  $x(t - t_0)$ :

$\hat{y}(t) = x((t - t_0)/3) = y(t - t_0) \longrightarrow$  the system is time-invariant.

**5** - Let  $x(t)$  be bounded input such that  $|x(t)| \leq M$ . Then,  $|y(t)| = |x(t/3)|$  (the expanded version of  $x(t)$ )  $\leq M \longrightarrow$  the system is stable.

### System 5

**1** - The system has a memory since the value of the output  $y(t)$  depends on the future values of the input  $x(t)$ .

**2** - The system is linear.

**3** - The system is not causal because the output  $y(t)$  does not depend only on present and past values of the input.

**4** - Apply  $x(t - t_0)$ :

$\hat{y}(t) = tx(t - t_0 + 5) \neq y(t - t_0) = (t - t_0)x(t - t_0 + 5) \longrightarrow$  the system is time-variant.

**5** - The system is not stable. To see this, consider the following example:

$x(t) = u(t) \Rightarrow x(t + 5) = u(t + 5)$  also  $|x(t + 5)| = |u(t + 5)| \leq 1$  which means that the

input is bounded. However, the output  $y(t) = tu(t + 5)$  is not bounded. Therefore, the system is not stable.

## System 6

$$y(t) = u(x(t)) = \begin{cases} 1, & x(t) \geq 0 \\ 0, & x(t) < 0 \end{cases}$$

**1 -** The system is memoryless.

**2 -** The system is non-linear, apply  $\alpha x(t)$ :

$\hat{y}(t) = u(\alpha x(t))$  may not be equal to  $\alpha y(t) = \alpha u(x(t))$ . For example, let the input  $x(t) = u(t)$  then for  $t \geq 0$ , the  $x(t) = 1$ . Also, choose  $\alpha = -1$ . As a result, for  $t \geq 0$ :  $u(\alpha x(t)) = 0 \neq \alpha y(t) = \alpha u(x(t)) = -1$ .

**3 -** The system is causal.

**4 -** Apply  $x(t - t_0)$ :

$\hat{y}(t) = u(x(t - t_0)) = y(t - t_0) \rightarrow$  the system is time-invariant.

**5 -** Remember that  $|u(t)| \leq 1$  so it is bounded. In this system, whatever the input  $x(t)$  is,

the output  $y(t) = u(x(t)) = \begin{cases} 1, & x(t) \geq 0 \\ 0, & x(t) < 0 \end{cases}$ . Hence, the system is stable.

## Question 2

a)

$x(t)$  and  $h(t)$  can be written as:

$$x(t) = u(t + 1) - u(t - 1)$$

$$h(t) = (1 - t)[u(t) - u(t - 1)] = \begin{cases} 1 - t, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,  $y(t) = x(t) * h(t)$

Before evaluating the convolution, consider the following:

$$\begin{aligned} \hat{y}(t) &= u(t) * h(t) = \int_{-\infty}^{\infty} u(t - \tau)h(\tau) d\tau \\ &= \int_{-\infty}^t h(\tau) d\tau = \begin{cases} 0, & t < 0 \\ \int_0^t (1 - \tau) d\tau, & 0 < t < 1 \\ \int_0^1 (1 - \tau) d\tau, & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ t - \frac{t^2}{2}, & 0 < t < 1 \\ \frac{1}{2}, & t > 1 \end{cases} \end{aligned}$$

By the properties of the LTI system:

$$\hat{y}(t+1) = u(t+1) * h(t) = \begin{cases} 0, & t+1 < 0 \\ (t+1) - \frac{(t+1)^2}{2}, & 0 < t+1 < 1 \\ \frac{1}{2}, & t+1 > 1 \end{cases} = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & t > 0 \end{cases}$$

$$\hat{y}(t-1) = u(t-1) * h(t) = \begin{cases} 0, & t-1 < 0 \\ (t-1) - \frac{(t-1)^2}{2}, & 0 < t-1 < 1 \\ \frac{1}{2}, & t-1 > 1 \end{cases} = \begin{cases} 0, & t < 1 \\ (t-1) - \frac{(t-1)^2}{2}, & 1 < t < 2 \\ \frac{1}{2}, & t > 2 \end{cases}$$

$$\text{Therefore, } y(t) = \hat{y}(t+1) - \hat{y}(t-1) = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & 0 < t < 1 \\ \frac{1}{2} - [(t-1) - \frac{(t-1)^2}{2}], & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

**b)**

$$w(t) = h(t) * g(t)$$

$$g(t) \text{ can be written as: } g(t) = x(t) + x(t-1) - x(t+1)$$

$$\text{Then, } w(t) = h(t) * [x(t) + x(t-1) - x(t+1)]$$

By the distributive property of the convolution over addition:

$$w(t) = h(t) * x(t) + h(t) * x(t-1) - h(t) * x(t+1)$$

$$= y(t) + y(t-1) - y(t+1) \text{ [by considering part a]}$$

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## Question 3

**a)**

Recall that in continuous-time systems one can apply the derivative operation to any arbitrary input signal. For example,

$$\delta(t) = \frac{d}{dt}u(t) \longrightarrow \int_{-\infty}^{\infty} \delta(t) dt$$

Also, it can be applied by using the formal definition of the derivative:

$$\lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h}$$

However, in discrete-time systems  $h$  cannot go to zero and the minimum value for  $h$  can be one. So, in discrete time the derivative expression becomes

$$\lim_{h \rightarrow 1} \frac{x(n+h) - x(n)}{h} = x[n+1] - x[n]$$

Thus, the derivative operation in continuous-time systems is analogous of the difference operation in discrete-time. Hence, we can obtain the impulse response of the difference operation as:  $h[n] = \delta[n] - \delta[n-1]$

**b)**

By convolution,  $y[n] = x[n] * h[n] = x[n] * (\delta[n] - \delta[n-1])$

By distributive property of the convolution operation,

$$x[n] * (\delta[n] - \delta[n-1]) = (x[n] * \delta[n]) - (x[n] * \delta[n-1])$$

$$y[n] = x[n] - x[n-1]$$

**c)**

$$\begin{aligned} e^{j\Omega_0 n} (1 - e^{-j\Omega_0}) &= (\cos(\Omega_0 n) + j\sin(\Omega_0 n))(1 - e^{-j\Omega_0}) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - (\cos(\Omega_0 n) + j\sin(\Omega_0 n))(\cos(\Omega_0) - j\sin(\Omega_0)) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - \cos(\Omega_0 n)\cos(\Omega_0) + j\cos(\Omega_0 n)\sin(\Omega_0) - j\sin(\Omega_0 n)\cos(\Omega_0) - \sin(\Omega_0 n)\sin(\Omega_0) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - (\cos(\Omega_0 n)\cos(\Omega_0) + \sin(\Omega_0 n)\sin(\Omega_0)) - j(\sin(\Omega_0 n)\cos(\Omega_0) - \cos(\Omega_0 n)\sin(\Omega_0)) \\ &= \cos(\Omega_0 n) + j\sin(\Omega_0 n) - \cos(\Omega_0 n - \Omega_0) - j\sin(\Omega_0 n - \Omega_0) \\ &= \cos(\Omega_0 n) - \cos(\Omega_0(n-1)) + j(\sin(\Omega_0 n) - \sin(\Omega_0(n-1))) \end{aligned}$$

$$\begin{aligned} |y[n]| &= [(\cos(\Omega_0 n) - \cos(\Omega_0(n-1)))^2 + (\sin(\Omega_0 n) - \sin(\Omega_0(n-1)))^2]^{1/2} \\ &= [\cos^2(\Omega_0 n) - 2\cos(\Omega_0 n)\cos(\Omega_0(n-1)) + \cos^2(\Omega_0(n-1)) + \sin^2(\Omega_0 n) - 2\sin(\Omega_0 n)\sin(\Omega_0(n-1)) + \sin^2(\Omega_0(n-1))]^{1/2} \\ &= \sqrt{2 - 2[\cos(\Omega_0 n)\cos(\Omega_0(n-1)) + \sin(\Omega_0 n)\sin(\Omega_0(n-1))]} \\ &= \sqrt{2 - 2\cos(\Omega_0 n - \Omega_0(n-1))} \\ &= \sqrt{2 - 2\cos(\Omega_0)} \end{aligned}$$

d)

$$\Omega_0 = 0 \longrightarrow |y[n]| = \sqrt{2 - 2\cos(0)} = 0$$

$$\Omega_0 = \frac{2\pi}{8} \longrightarrow |y[n]| = \sqrt{2 - 2\cos(\frac{2\pi}{8})} = \sqrt{2 - \sqrt{2}}$$

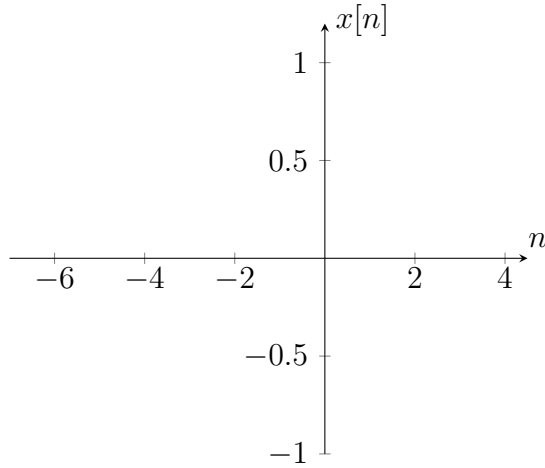
$$\Omega_0 = \frac{2\pi}{4} \longrightarrow |y[n]| = \sqrt{2 - 2\cos(\frac{2\pi}{4})} = \sqrt{2}$$

$$\Omega_0 = \frac{2\pi}{2} \longrightarrow |y[n]| = \sqrt{2 - 2\cos(\frac{2\pi}{2})} = 2$$

As we have shown in part a), the given discrete-time LTI system is analogous to the derivative operation in continuous-time systems. Also, please remember that the derivative is defined as the rate of change of a function with respect to a variable. Here, as the frequency of the input  $x[n]$  is increased, the period becomes smaller and the values of the input function are changed more rapidly. Therefore, the rate of change of a input function is increased as the frequency is increased, and the modulus of the output  $|y[n]|$  is getting larger values.

e)

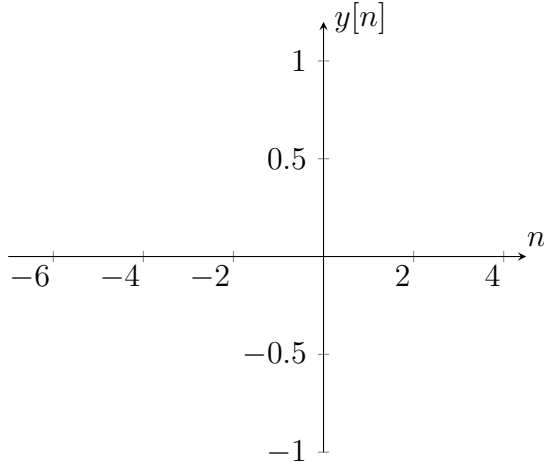
$x[n] = u[n + 5] - u[n - 2]$  can be plotted as:



We can calculate  $y[n]$  for  $x[n] = u[n + 5] - u[n - 2]$  by using the answer in (b) as:

$$y[n] = u[n + 5] - u[n + 2] - (u[n + 4] + u[n - 3])$$

$y[n]$  can be plotted as:



From the graph of  $y[n]$ , we can say that  $x[n]$  has edges at  $n = -5$  and  $n = 2$ .

**f)**

From (d), we can say that magnitude of the output signal of the system increases as the frequency increases. In (f),  $x[n]$  has rapid changes at  $n = -5$  and  $n = 2$ , and as a result,  $y[n]$  has magnitude 1 at  $n = -5$  and  $n = 2$ , and 0 elsewhere.

## Question 4

**a)**

Note that, any arbitrary periodic signal  $x(t)$  (or,  $x[n]$ ) with Fourier series coefficients  $a_k$  is real-valued if  $a_k^* = a_{-k}$ .

$$\text{For } x_1(t): a_k^* = \left[\left(\frac{1}{2}\right)^{-k}\right]^* = \left(\frac{1}{2}\right)^{-k} \neq \left(\frac{1}{2}\right)^{-(-k)} = \left(\frac{1}{2}\right)^k = a_{-k}$$

$$\text{For } x_2(t): a_k^* = (\cos(k\pi))^* = \cos(k\pi) = \cos(-k\pi) = a_{-k}$$

$$\text{For } x_3[n]: a_k^* = (j\sin(\frac{k\pi}{2}))^* = -j\sin(\frac{k\pi}{2}) = j\sin(\frac{-k\pi}{2}) = a_{-k}$$

Thus,  $x_2(t)$  and  $x_3[n]$  are real-valued signals.

**b)**

Any arbitrary periodic signal  $x(t)$  (or,  $x[n]$ ) with Fourier series coefficients  $a_k$  is even if  $a_k$  is a real-valued and even function.

$$\text{For } x_1(t): a_k = \left(\frac{1}{2}\right)^{-k} \Rightarrow \text{is a real-valued but not even function, i.e., } \left(\frac{1}{2}\right)^{-k} \Big|_{k=1} \neq \left(\frac{1}{2}\right)^{-k} \Big|_{k=-1}$$

$$\text{For } x_2(t): a_k = \cos(k\pi) \Rightarrow \text{is a real-valued and also an even function since consider that}$$

$-1 \leq \cos(k\pi) \leq 1$  and  $\cos(k\pi)\Big|_{k=1} = \cos(k\pi)\Big|_{k=-1}$   
For  $x_3[n]$ :  $a_k = j\sin(\frac{k\pi}{2}) \Rightarrow$  neither real-valued nor even function.

Therefore, only the signal  $x_2(t)$  is even.

c)

By the time-shifting property of Continuous Time Fourier Series:

$$x_2(t-5) = \sum_{k=-100}^{100} \cos(k\pi) e^{-jk\frac{\pi}{5}} e^{jk\frac{2\pi}{50}t} \quad \text{and, also note that} \quad \cos(k\pi) = \frac{1}{2}(e^{jk\pi} + e^{-jk\pi})$$

$$\Rightarrow x_2(t-5) = \sum_{k=-100}^{100} \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}) e^{jk\frac{2\pi}{50}t}$$

Thus,  $x_2(t-5)$  is another periodic signal with Fourier series coefficients  $b_k = \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}})$ .

Also, remember the theorem mentioned above, the signal  $x_2(t-5)$  is real if  $b_k^* = b_{-k}$ .

$b_k^* = (\frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}))^* = \frac{1}{2}(e^{-jk\frac{4\pi}{5}} + e^{jk\frac{6\pi}{5}}) = b_{-k} \Rightarrow$  the signal  $x_2(t-5)$  is real.

To check whether the signal  $x_2(t-5)$  is even,  $b_k$  can be written as:

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + j\sin(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5}) - j\sin(k\frac{6\pi}{5})]$$

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - \frac{1}{2}j[\sin(k\frac{6\pi}{5}) - \sin(k\frac{4\pi}{5})]$$

Also, note that  $\boxed{\sin a - \sin b = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})}$

$$\text{Thus, } b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - j[\cos(2\pi k)\sin(k\frac{2\pi}{5})]$$

$$\Rightarrow b_k = \frac{1}{2}(\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})) - j\sin(k\frac{2\pi}{5})$$

As a result, it can be seen that  $b_k$  is not real-valued, so the signal  $x_2(t-5)$  is not even.

d)

By differentiation property of Continuous Time Fourier Series:

$$\frac{d}{dt}x_2(t) = \sum_{k=-\infty}^{\infty} \cos(k\pi) jk \frac{2\pi}{50} e^{jk\frac{2\pi}{50}t} \Rightarrow \text{is a periodic signal with Fourier series coefficients } c_k$$

such that  $c_k = \cos(k\pi) jk \frac{2\pi}{50}$ .

$$c_k^* = -jk \frac{2\pi}{50} \cos(k\pi) \quad \text{and} \quad c_{-k} = j(-k) \frac{2\pi}{50} \cos(-k\pi) = -jk \frac{2\pi}{50} \cos(k\pi)$$

$$c_k^* = c_{-k} \Rightarrow \frac{d}{dt}x_2(t) \text{ is real-valued.}$$



Note that,  $\cos(k\pi)$  take only the values  $+1$  or  $-1$ , so it is always real-valued. However,  $-jk\frac{2\pi}{50}$  is purely imaginary. Therefore,  $\frac{d}{dt}x_2(t)$  is not even.

e)

## Question 5

a)

In calculating part of the question, we added zero vectors to end and beginning of the main vector to obtain correct results in range of  $0 \leq n \leq N - 1$ , which can be seen in Appendix. Since we manually did that zero adding part, we deviated the result for  $n > N$  range. Because of this situation, our calculated result has only  $N$  value in range. On the other hand, MATLAB's built-in function, *conv()*, evaluate the convolution so that the size of the output is equal to  $N + L - 1$ . That's why the output graphs of convolution operation are different by using our function and MATLAB's built-in function as shown in Appendix. In fact, to get the same result that we got by using our function, one can specify an input argument, *same*, to MATLAB's function so that the result gives the central part of the convolution operation. Otherwise, the built-in function gives the full convolution whose length is equal to  $N + L - 1$ .

b)

The output signal looks more smoother in larger values of  $L$ . As we can observe that  $h[n]$  impulse response created accordingly to  $L$  which directly affects the convolution range. By saying range, we mean that how many  $x*h$  multiplication we will use to calculate one element of the output. So, this is why that the output signal gain more elements and more precise values when  $L$  gets larger. And also, we can observe that the  $L$  value expands the output range by expanding  $h[n]$  size. The MATLAB code and the resulting graphs for different values of  $L$  can be seen in Appendix.