

# EE301 Homework-4

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## Question 1

a)

$$x_s(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

where  $T_s$  is called the sampling period.

By the modulation property of CTFT:  $X_s(j\omega) = \frac{1}{2\pi}X(j\omega) * S(j\omega)$

$s(t)$  is a periodic signal, therefore we should first find its CTFS representation:

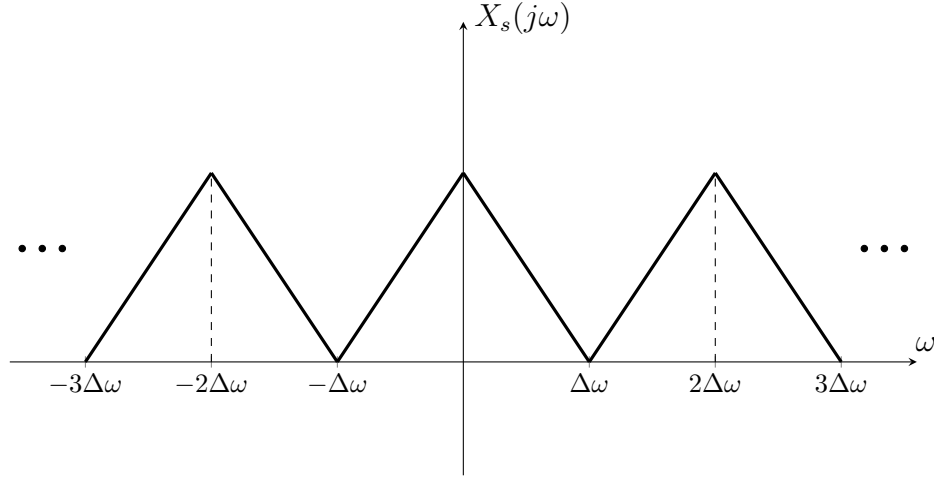
$$s(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}, \omega_s = \frac{2\pi}{T_s}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \underbrace{s(t)}_{\delta(t)} e^{-jk\omega_s t} dt \Rightarrow a_k = \frac{1}{T_s}$$

$$s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} \longrightarrow S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

$$X_s(j\omega) = \frac{1}{2\pi}X(j\omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} X(j(\omega - k\omega_s))$$

From the Nyquist sampling theorem, the inequality  $\omega_s \geq 2\Delta\omega$  should be satisfied so that the shifted replicas of  $X(j\omega)$  do not overlap. Thus, it yields perfect reconstruction of the signal  $x(t)$  from  $x_s(t)$  with no aliasing. So, the minimum sampling rate,  $\omega_s = 2\Delta\omega$ . The Fourier transform of the sampling system output,  $X_s(j\omega)$  can be seen below.

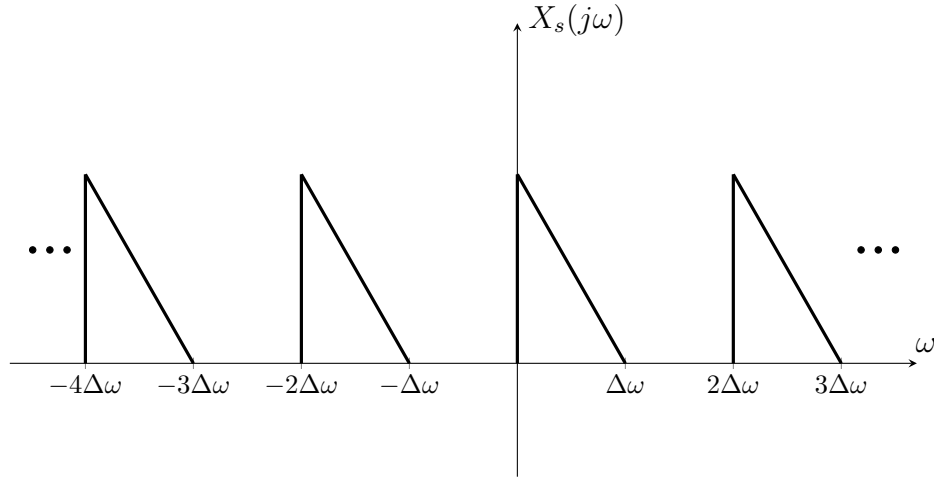


b)

$X(j\omega)$  is not symmetric with respect to y-axis, thus from the symmetry property of CTFT it can be concluded that  $x(t)$  is a complex-valued signal.

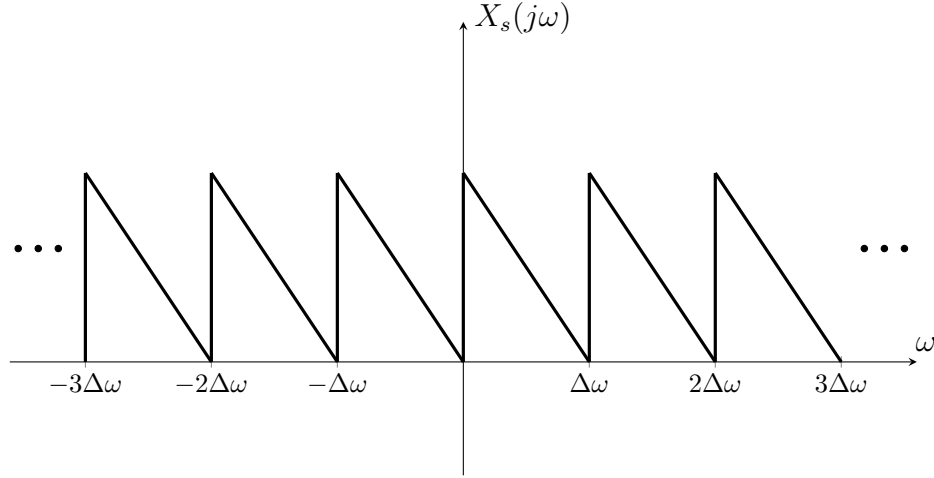
i)

Since the sampling rate of the signal is equal to the Nyquist rate, the aliasing does not occur. The Fourier transform of  $X_s(j\omega)$  is plotted below for  $T_s = \frac{\pi}{\Delta\omega}$ .



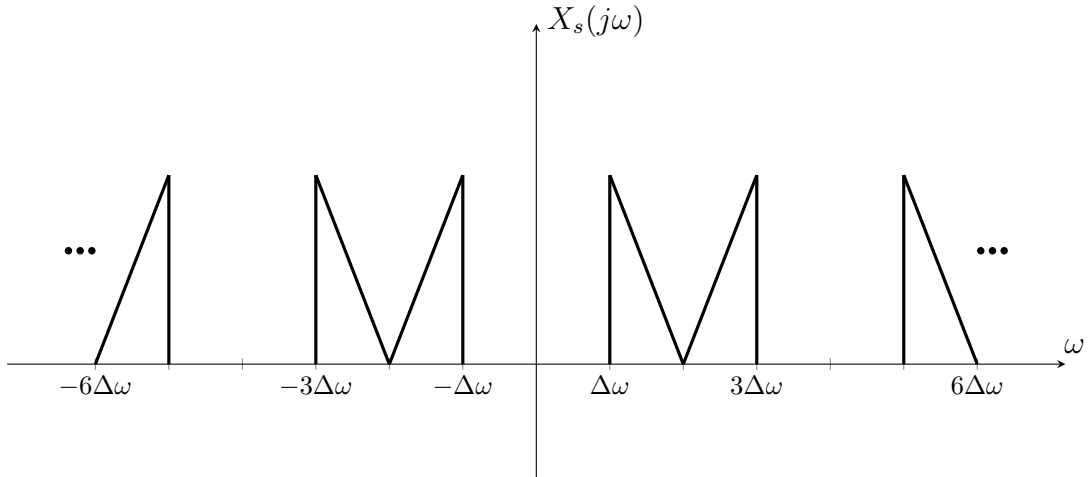
ii)

As it can be seen from the graph of the Fourier transform of  $x(t)$ , the graph is not symmetric with respect to y-axis, i.e.,  $X(j\omega)$  has no component at negative frequencies. Therefore, as it can be seen from the graph of  $X_s(j\omega)$  below, the aliasing can be avoided even when  $T_s = \frac{2\pi}{\Delta\omega}$  which below the Nyquist rate.



c)

The minimum sampling period,  $T_s = \frac{\pi}{2\Delta\omega} \Rightarrow$  minimum sampling rate,  $\frac{1}{T_s} = \frac{2\Delta\omega}{\pi}$ . The Fourier transform of  $X_s(j\omega)$  can be seen below. Also, the corresponding time-domain signal is band-pass as it can be seen from the graph of its Fourier transform.



## Question 2

The input-output relation of the DT System is given. Thus, if we take the Fourier transform of the both sides:

$$Y(e^{j\Omega}) - \frac{1}{3}Y(e^{j\Omega})e^{-j\Omega} = \frac{2}{3}X(e^{j\Omega}) - 2X(e^{j\Omega})e^{-j\Omega}$$

$$Y(e^{j\Omega})[1 - \frac{e^{-j\Omega}}{3}] = X(e^{j\Omega})[\frac{2}{3} - 2e^{-j\Omega}] \Rightarrow H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{2-6e^{-j\Omega}}{3-e^{-j\Omega}}$$

$$x_c(t) = \sin(1000\pi t) \text{ and } x[n] = x_c(nT) \quad \boxed{\omega_c = 1000\pi}$$

$$X_c(j\omega) = \frac{\pi}{j}(\delta(\omega - 1000\pi) - \delta(\omega + 1000\pi))$$

$$X(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c \left( j \left( \frac{\Omega - 2\pi k}{T} \right) \right) = \frac{\pi}{jT}$$

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \delta \left( \frac{\Omega - 2\pi k}{T} - 1000\pi \right) - \delta \left( \frac{\Omega - 2\pi k}{T} + 1000\pi \right)$$

$\underbrace{\frac{2\pi}{T} > 2\omega_c \Rightarrow \frac{1}{T} > 1000}_{\text{to avoid aliasing}}$
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If there is no aliasing, the below equation holds:

$$Y_r(j\omega) = \begin{cases} H(e^{j\omega T})X_c(j\omega), & |\omega| \leq \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

$$Y_r(j\omega) = \begin{cases} \frac{\pi}{j} \frac{2-6e^{-j\omega T}}{3-e^{-j\omega T}} [\delta(\omega - 1000\pi) - \delta(\omega + 1000\pi)], & |\omega| \leq \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases}$$

$$y_r(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y_r(j\omega) e^{j\omega t} d\omega$$

i)

$$\frac{1}{T} = 2\text{kHz} \Rightarrow \frac{2\pi}{T} > 2\omega_c \text{ (No aliasing)} \Rightarrow \omega_c T = \frac{\pi}{2}$$

$$y_r(t) = \frac{1}{2j} \left[ \frac{2-6e^{-j\pi/2}}{3-e^{-j\pi/2}} e^{j1000\pi t} - \frac{2-6e^{j\pi/2}}{3-e^{j\pi/2}} e^{-j1000\pi t} \right] = \frac{1}{2j} [(1.2 + 1.6j)e^{j1000\pi t} - (1.2 - 1.6j)e^{-j1000\pi t}]$$

$$y_r(t) = 1.2\sin(1000\pi t) + 1.6\cos(1000\pi t)$$

ii)

$$\frac{1}{T} = 1\text{kHz} \Rightarrow \frac{2\pi}{T} = 2\omega_c \text{ (No aliasing)} \Rightarrow \omega_c T = \pi$$

$$y_r(t) = \frac{1}{2j} \left[ \frac{2-6e^{-j\pi}}{3-e^{-j\pi}} e^{j1000\pi t} - \frac{2-6e^{j\pi}}{3-e^{j\pi}} e^{-j1000\pi t} \right] = \frac{1}{2j} (2e^{j1000\pi t} - 2e^{-j1000\pi t})$$

$$y_r(t) = 2\sin(1000\pi t)$$

## Question 3

a)

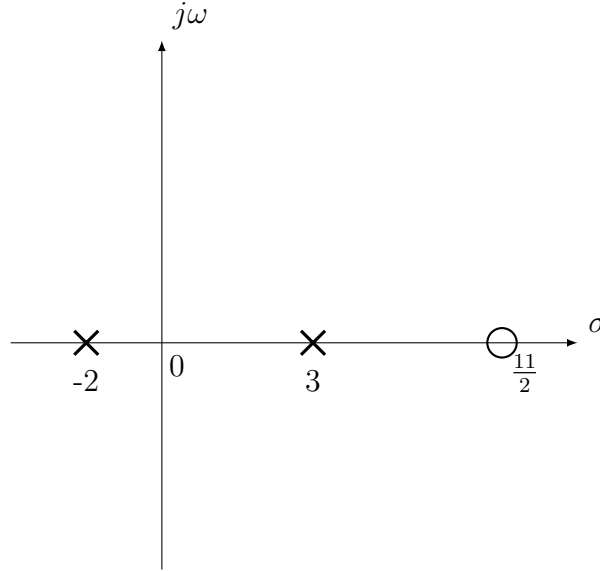
If we take the Laplace transform of the both sides of the equation:

$$s^2 Y(s) - sY(s) - 6Y(s) = 2sX(s) - 11X(s)$$

$$Y(s)(s^2 - s - 6) = X(s)(2s - 11)$$

$$H_1(s) = \frac{Y(s)}{X(s)} = \frac{2s-11}{(s-3)(s+2)} = \frac{-1}{s-3} + \frac{3}{s+2}$$

The pole-zero diagram of  $H_1(s)$  is drawn below.



We cannot determine  $h_1(t)$  unless we know the ROC.

**b)**

**i)**

If the system is stable, then ROC includes  $j\omega$ -axis. Therefore, we can determine ROC as  $-2 < \sigma < 3$ .

$-2 < \sigma$  implies  $\frac{3}{s+2}$  corresponds to right-sided time function which is  $3e^{-2t}u(t)$ .

$\sigma < 3$  implies  $\frac{-1}{s-3}$  corresponds to left-sided time function which is  $e^{3t}u(-t)$ .

$$h_1(t) = e^{3t}u(-t) + 3e^{-2t}u(t)$$

**ii)**

If the system is causal, then ROC is the right side of the right-most pole. Therefore, we can determine ROC as  $\sigma > 3$ .

$\sigma > -2$  implies  $\frac{3}{s+2}$  corresponds to right-sided time function which is  $3e^{-2t}u(t)$ .

$\sigma > 3$  implies  $\frac{-1}{s-3}$  corresponds to right-sided time function which is  $-e^{3t}u(t)$ .

$$h_1(t) = (-e^{3t} + 3e^{-2t})u(t)$$

**iii)**

If the system is anti-causal, then ROC is the left side of the left-most pole. Therefore, we can determine ROC as  $\sigma < -2$ .

$\sigma < -2$  implies  $\frac{3}{s+2}$  corresponds to right-sided time function which is  $-3e^{-2t}u(-t)$ .

$\sigma < 3$  implies  $\frac{-1}{s-3}$  corresponds to right-sided time function which is  $e^{3t}u(-t)$ .  
 $h_1(t) = (e^{3t} + -3e^{-2t})u(-t)$

c)

i)

The ROC is the whole s-plane.

ii)

$$H(s) = \frac{Y(s)}{X(s)} = s - 3$$

$$Y(s) = X(s)(s - 3) = sX(s) - 3X(s)$$

$$y(t) = \frac{d}{dt}x(t) - 3x(t)$$

d)

i)

Assume  $h_1(t)$  is causal. Then it is not stable since the ROC does not include  $j\omega$ -axis.

ii)

$$H(s) = H_1(s)H_2(s) = \frac{2s-11}{(s-3)(s+2)}(s-3)H(s) = \frac{2s-11}{s+2} = 2 - \frac{15}{s+2} \quad \text{ROC} : \sigma > -2$$

$$h(t) = 2\delta(t) - 15e^{-2t}u(t)$$

iii)

The cascaded system has a ROC which includes  $j\omega$ -axis. Therefore, we can say that the cascaded system is stable although the first system is not stable. The second system is used to make the first system stable.

## Question 4

a)

If  $h[n]$  is real, then the following condition must be satisfied:  $H(z) = H^*(z^*)$

$$H(z) = \frac{z(z-1)}{\left(z-a\left(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)\right)\left(z-a\left(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)\right)} = \left(\frac{z^*(z^*-1)}{\left(z^*-a\left(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)\right)\left(z^*-a\left(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)\right)}\right)^* = H^*(z^*)$$

Thus,  $h[n]$  is a real-valued signal.

**b)**

Since the DT system is causal, i.e.,  $h[n] = 0$  for  $n < 0$ , the inverse z-transform of  $H(z)$  is right-sided.

Also, if the DTFT of  $h[n]$  exists, the ROC of  $H(z)$  contains the  $j\omega$ -axis ( $|z| = 1$ ).

$$H(z) = \frac{1-z^{-1}}{\left(1-a\left(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)z^{-1}\right)\left(1-a\left(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)z^{-1}\right)} = \frac{c_1}{\left(1-a\left(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}}\right)z^{-1}\right)} + \frac{c_2}{\left(1-a\left(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)z^{-1}\right)}$$

$$h[n] = c_1\left[a\left(\frac{1-j}{\sqrt{2}}\right)\right]^n u[n] + c_2\left[a\left(\frac{1+j}{\sqrt{2}}\right)\right]^n u[n], \text{ ROC: } |z| > \left|a\left(\frac{1-j}{\sqrt{2}}\right)\right| = \left|a\left(\frac{1+j}{\sqrt{2}}\right)\right| = a^2$$

Thus, the following inequality should be satisfied:  $|z| < 1 \Rightarrow a^2 < 1 \Rightarrow \boxed{0 < a < 1}$

**c)**

$$\text{If } a = \sqrt{2} \Rightarrow H(z) = \frac{z(z-1)}{z^2-2z+2}$$

$$x[n] = u[n-1] \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} u[n-1]z^{-n} = \sum_{n=1}^{\infty} z^{-n} = \sum_{n=0}^{\infty} z^{-n} - 1 = \frac{1}{z-1}, |z| < 1$$

$$Y(z) = X(z)H(z) = \frac{z}{z^2-2z+2} = \frac{z}{(z-1+j)(z-1-j)}, \text{ ROC: } |z| > \sqrt{2}$$

$$Y(z) = \frac{z^{-1}}{(1-(1-j)z^{-1})(1-(1+j)z^{-1})} = \frac{c_1}{1-(1-j)z^{-1}} + \frac{c_2}{1-(1+j)z^{-1}}$$

$$Y(z) = \frac{j/2}{1-(1-j)z^{-1}} + \frac{-j/2}{1-(1+j)z^{-1}} \Rightarrow y[n] = \frac{j}{2}(1-j)^n u[n] - \frac{j}{2}(1+j)^n u[n]$$

**d)**

By the scaling property in z-domain:  $h'[n] = \left(\frac{1}{2}\right)^n h[n] \longleftrightarrow H'(z) = H(2z)$ , ROC:  $|z| > \frac{\sqrt{2}}{2}$

Since the ROC of  $H'(z)$  contains  $j\omega$ -axis ( $|z| = 1$ ), the DTFT of  $H'(z)$  exists:

$$H'(e^{j\Omega}) = H(2e^{j\Omega}) = \left. \frac{z(2z-1)}{2z^2-2z+1} \right|_{z=e^{j\Omega}}$$