

# EE301 Homework-4

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January 7, 2023

## Question 1

a)

$$x_s(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

where  $T_s$  is called the sampling period.

By the modulation property of CTFT:  $X_s(j\omega) = \frac{1}{2\pi}X(j\omega) * S(j\omega)$

$s(t)$  is a periodic signal, therefore we should first find its CTFS representation:

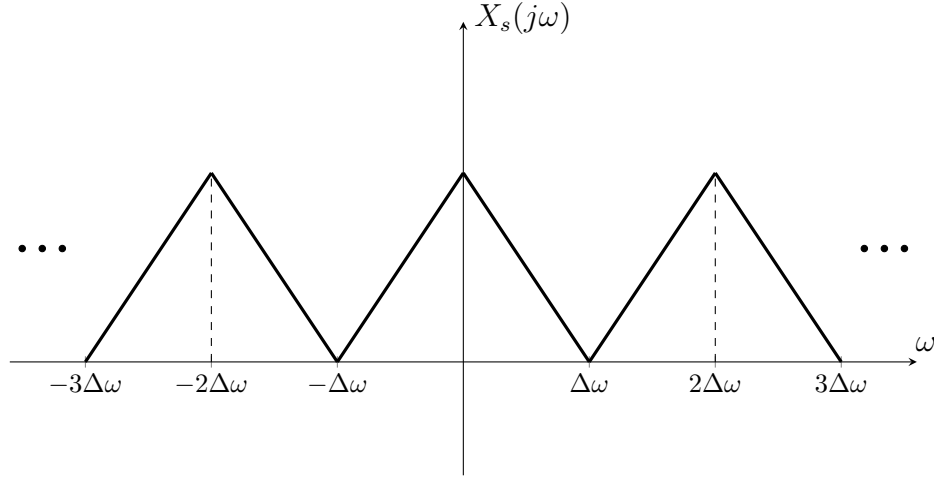
$$s(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}, \omega_s = \frac{2\pi}{T_s}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \underbrace{s(t)}_{\delta(t)} e^{-jk\omega_s t} dt \Rightarrow a_k = \frac{1}{T_s}$$

$$s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} \longrightarrow S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

$$X_s(j\omega) = \frac{1}{2\pi}X(j\omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} X(j(\omega - k\omega_s))$$

From the Nyquist sampling theorem, the inequality  $\omega_s \geq 2\Delta\omega$  should be satisfied so that the shifted replicas of  $X(j\omega)$  do not overlap. Thus, it yields perfect reconstruction of the signal  $x(t)$  from  $x_s(t)$  with no aliasing. So, the minimum sampling rate,  $\omega_s = 2\Delta\omega$ . The Fourier transform of the sampling system output,  $X_s(j\omega)$  can be seen below.

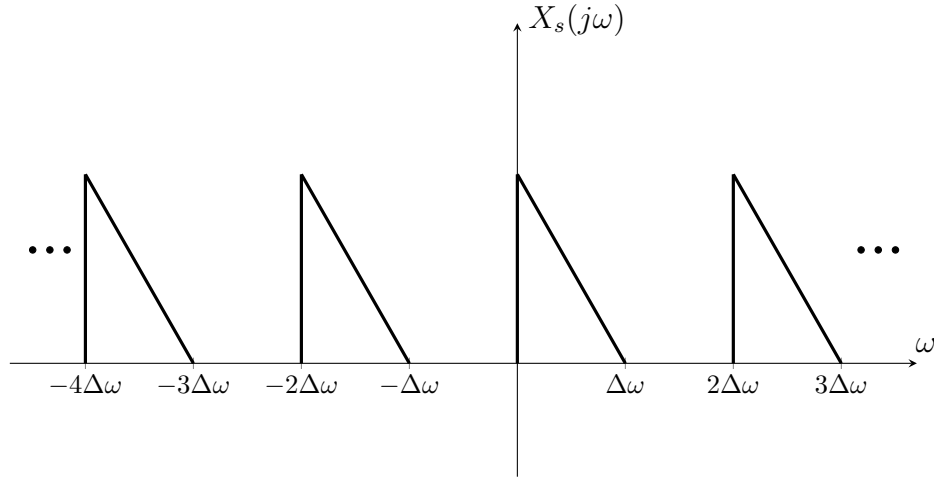


b)

$X(j\omega)$  is not symmetric with respect to y-axis, thus from the symmetry property of CTFT it can be concluded that  $x(t)$  is a complex-valued signal.

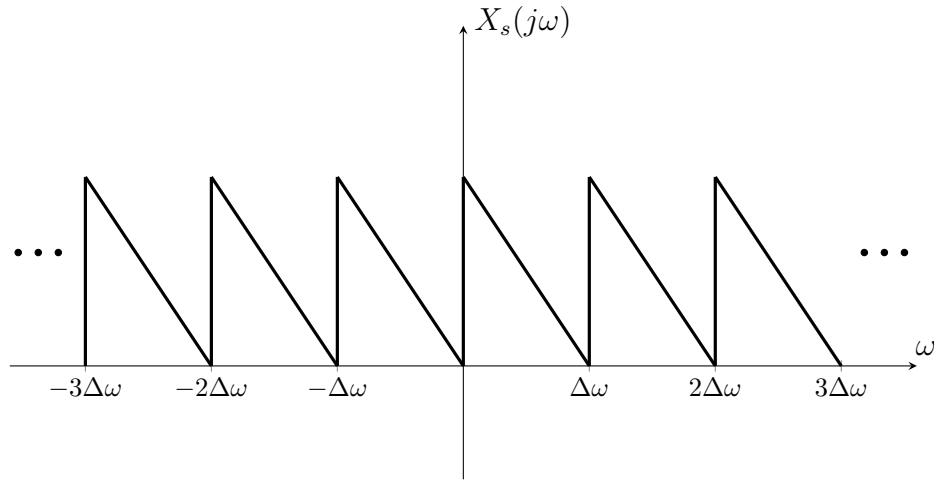
i)

Since the sampling rate of the signal is equal to the Nyquist rate, the aliasing does not occur. The Fourier transform of  $X_s(j\omega)$  is plotted below for  $T_s = \frac{\pi}{\Delta\omega}$ .



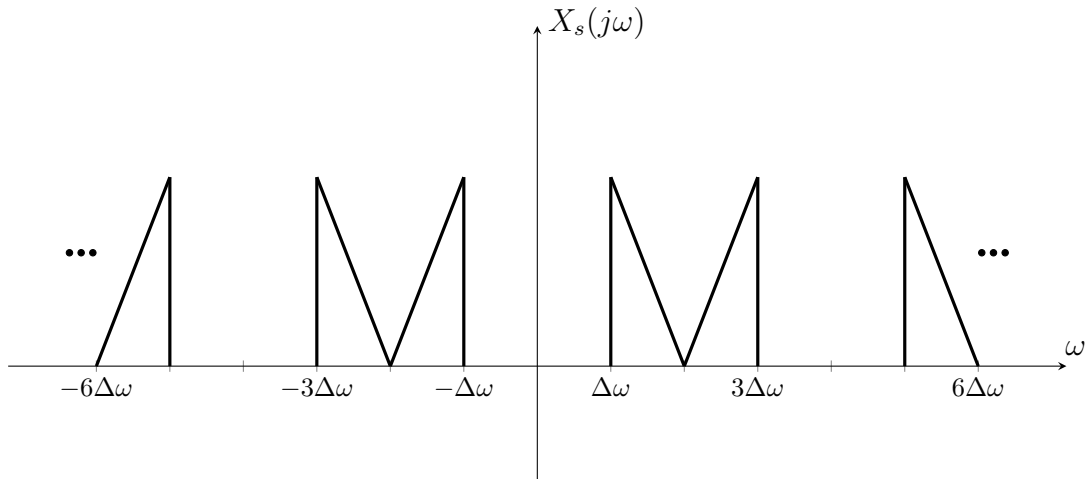
ii)

As it can be seen from the graph of the Fourier transform of  $x(t)$ , the graph is not symmetric with respect to y-axis, i.e.,  $X(j\omega)$  has no component at negative frequencies. Therefore, as it can be seen from the graph of  $X_s(j\omega)$  below, the aliasing can be avoided even when  $T_s = \frac{2\pi}{\Delta\omega}$  which below the Nyquist rate.



**c)**

The minimum sampling period,  $T_s = \frac{\pi}{2\Delta\omega} \Rightarrow$  minimum sampling rate,  $\frac{1}{T_s} = \frac{2\Delta\omega}{\pi}$ . The Fourier transform of  $X_s(j\omega)$  can be seen below. Also, the corresponding time-domain signal is band-pass as it can be seen from the graph of its Fourier transform.



## Question 2

i)

ii)

## Question 3

a)

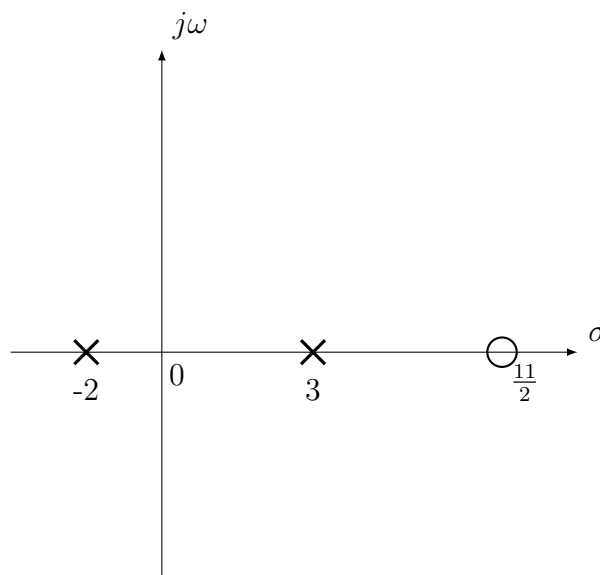
If we take the Laplace transform of the both sides of the equation:

$$s^2Y(s) - sY(s) - 6Y(s) = 2sX(s) - 11X(s)$$

$$Y(s)(s^2 - s - 6) = X(s)(2s - 11)$$

$$H_1(s) = \frac{Y(s)}{X(s)} = \frac{2s-11}{(s-3)(s+2)} = \frac{-1}{s-3} + \frac{3}{s+2}$$

The pole-zero diagram of  $H_1(s)$  is drawn below.



We cannot determine  $h_1(t)$  unless we know the ROC.

b)

i)

If the system is stable, then ROC includes  $j\omega$ -axis. Therefore, we can determine ROC as  $-2 < \sigma < 3$ .

$-2 < \sigma$  implies  $\frac{3}{s+2}$  corresponds to right-sided time function which is  $3e^{-2t}u(t)$ .

$\sigma < 3$  implies  $\frac{-1}{s-3}$  corresponds to left-sided time function which is  $e^{3t}u(-t)$ .

$$h_1(t) = e^{3t}u(-t) + 3e^{-2t}u(t)$$

ii)

If the system is causal, then ROC is the right side of the right-most pole. Therefore, we can determine ROC as  $\sigma > 3$ .

$\sigma > -2$  implies  $\frac{3}{s+2}$  corresponds to right-sided time function which is  $3e^{-2t}u(t)$ .

$\sigma > 3$  implies  $\frac{-1}{s-3}$  corresponds to right-sided time function which is  $-e^{3t}u(t)$ .

$$h_1(t) = (-e^{3t} + 3e^{-2t})u(t)$$

iii)

If the system is anti-causal, then ROC is the left side of the left-most pole. Therefore, we can determine ROC as  $\sigma < -2$ .

$\sigma < -2$  implies  $\frac{3}{s+2}$  corresponds to right-sided time function which is  $-3e^{-2t}u(-t)$ .

$\sigma < 3$  implies  $\frac{-1}{s-3}$  corresponds to right-sided time function which is  $e^{3t}u(-t)$ .

$$h_1(t) = (e^{3t} + -3e^{-2t})u(-t)$$

c)

i)

The ROC is the whole s-plane.

ii)

$$H(s) = \frac{Y(s)}{X(s)} = s - 3$$

$$Y(s) = X(s)(s - 3) = sX(s) - 3X(s)$$

$$y(t) = \frac{d}{dt}x(t) - 3x(t)$$

d)

i)

Assume  $h_1(t)$  is causal. Then it is not stable since the ROC does not include  $j\omega$ -axis.

ii)

$$H(s) = H_1(s)H_2(s) = \frac{2s-11}{(s-3)(s+2)}(s-3)H(s) = \frac{2s-11}{s+2} = 2 - \frac{15}{s+2} \quad ROC : \sigma > -2$$

$$h(t) = 2\delta(t) - 15e^{-2t}u(t)$$

iii)

The cascaded system has a ROC which includes  $j\omega$ -axis. Therefore, we can say that the cascaded system is stable although the first system is not stable. The second system is used to make the first system stable.

## Question 4

a)

b)

c)

d)