EE301 Homework-3

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Question 1

a)

b)

Question 2

a)

b)

c)

Question 3

a)

i)

$x(t) = \frac{sin(4\pi t)}{\pi t} cos(2\pi t) = \frac{4sin(4\pi t)}{4\pi t} cos(2\pi t) = x_1(t)x_2(t) \Big($ where $x_1(t) = \frac{4sin(4\pi t)}{4\pi t}$, $x_2(t) = cos(2\pi t)\Big)$

Recall that: $\mathscr{F}\{rect(\theta)\} = \frac{sin(\omega/2)}{\omega/2}$
By duality property of CTFT: $\frac{sin(t/2)}{t/2} \longleftrightarrow 2\pi rect(-\omega) = 2\pi rect(\omega)$
By scaling property of CTFT: $\frac{sin(4\pi t)}{4\pi t} \longleftrightarrow \frac{rect(\frac{\omega}{8\pi})}{4}$
By linearity: $\frac{4sin(4\pi t)}{4\pi t} \longleftrightarrow rect(\frac{\omega}{8\pi}) = X_1(j\omega)$

$x_2(t) = cos(2\pi t) = \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$

$\mathscr{F}\{\frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}\} \longleftrightarrow \pi[\delta(\omega + 2\pi) + \delta(\omega - 2\pi)] = X_2(j\omega)$

By modulation property of CTFT:
$x(t) = x_1(t)x_2(t) \longleftrightarrow X(j\omega) = \frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
$X(j\omega) = \frac{1}{2\pi}rect(\frac{\omega}{8\pi}) * \pi[\delta(\omega+2\pi)+\delta(\omega-2\pi)] = \frac{1}{2}[rect(\frac{\omega}{8\pi}) * \delta(\omega+2\pi) + rect(\frac{\omega}{8\pi}) * \delta(\omega-2\pi)]$
$X(j\omega) = \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi})$

ii)

$y(t) = h(t) * x(t)$
By convolution property of CTFT: $y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$
$Y(j\omega) = \left(1 - rect(\frac{\omega}{4\pi})\right)\left(\frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi})\right)$
$Y(j\omega) = \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) - \frac{1}{2}\left(rect(\frac{\omega}{4\pi})rect(\frac{\omega-2\pi}{8\pi}) + rect(\frac{\omega}{4\pi})rect(\frac{\omega+2\pi}{8\pi})\right)$
$Y(j\omega) = rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})$

b)

i)

By modulation property of CTFT:
$z(t) \longleftrightarrow Z(j\omega) = \frac{1}{2\pi}Y(j\omega) * \mathscr{F}\{\frac{sin(2\pi t)}{\pi t}\} = \frac{1}{2\pi}Y(j\omega) * rect(\frac{\omega}{4\pi})$

$Y(j\omega) = \frac{1}{\pi}rect(\frac{\omega}{4\pi}) * \pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$
$\Rightarrow Z(j\omega) = \frac{1}{2\pi}\left(rect(\frac{\omega}{4\pi}) * rect(\frac{\omega}{4\pi})\right) * [\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$

ii)

$y(t) \longleftrightarrow Y(j\omega) = rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})$
$Y(j\omega) = \frac{1}{\pi}rect(\frac{\omega}{4\pi}) * \pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$
By modulation property of CTFT:
$\mathscr{F}^{-1}\{Y(j\omega)\} = y(t) = \frac{2sin(2\pi t)}{\pi t}cos(4\pi t)$

Question 4

a)

i)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n]e^{-j\Omega n}}_{\delta[n]e^{-j\Omega 0}} = \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

ii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} (2\delta[n-3]-\delta[n-10])e^{-j\Omega n} \overset{\text{(by linearity)}}{=} 2\sum_{n=-\infty}^{\infty} \delta[n-3]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} \delta[n-10]e^{-j\Omega n}$$

By time-shift property of DTFT:
$X(e^{j\Omega}) = 2e^{-j3\Omega} - e^{-j10\Omega}$

iii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=1}^{4} \frac{1}{n^2}e^{-j\Omega n} = e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega} + \frac{1}{9}e^{-j3\Omega} + \frac{1}{16}e^{-j4\Omega}$$

iv)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]\right) e^{-j\Omega n}$$

$$\overset{\text{(by linearity)}}{=} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} 3^n u[-n-1]e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n - \sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} \quad \left(\text{since } \left|\frac{1}{2}e^{-j\Omega}\right| = \frac{1}{2} < 1, \text{so the expression is convergent}\right)$$

Let $m = -n$: $\displaystyle\sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n} = \sum_{m=1}^{\infty} 3^{-m} e^{j\Omega m} = \sum_{m=1}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m = \underbrace{\left[\sum_{m=0}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m\right]}_{\frac{1}{1-\frac{1}{3}e^{j\Omega}}} - 1$

$$\Rightarrow X(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$$

v)

Say that, $\hat{x}[n] = \left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]$ and $\mathscr{F}\{\hat{x}[n]\} = \hat{X}(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$
By time-shift property of DTFT:
$x[n] = \hat{x}[n-7] \longleftrightarrow X(e^{j\Omega}) = \hat{X}(e^{j\Omega})e^{-j7\Omega}$

$\Rightarrow X(e^{j\Omega}) = \frac{e^{-j7\Omega}}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{e^{-j7\Omega}}{1-\frac{1}{3}e^{j\Omega}} - e^{-j7\Omega}\right)$

vi)

Let x[n] be a periodic signal with fundamental period N. Then,

$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} \sum_{k=k_0}^{k_0+N-1} a_k 2\pi\delta(\Omega - k\frac{2\pi}{N} - 2\pi m)$$

Note that for this signal if we consider the interval $0 \leq \Omega < 2\pi$, DTFT of the signal will be written as:

$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3})$$

First, find the DTFS coefficients of x[n]: $a_k = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-jk\frac{2\pi}{3}n}$

$$\Rightarrow a_0 = \frac{1}{3}\sum_{n=0}^{2} x[n] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow a_1 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{2\pi}{3}n} = \frac{1}{3}e^{-j\frac{2\pi}{3}} + \frac{1}{3}e^{-j\frac{4\pi}{3}} = \frac{-1}{3}$$

$$\Rightarrow a_2 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{4\pi}{3}n} = \frac{1}{3}e^{-j\frac{4\pi}{3}} + \frac{1}{3}e^{-j\frac{8\pi}{3}} = \frac{-1}{3}$$

$$X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3}) = a_0 2\pi\delta(\Omega) + a_1 2\pi\delta(\Omega - \frac{2\pi}{3}) + a_2 2\pi\delta(\Omega - \frac{4\pi}{3})$$

$$X(e^{j\Omega}) = \frac{2\pi}{3}\left(2\delta(\Omega) - \delta(\Omega - \frac{2\pi}{3}) - \delta(\Omega - \frac{4\pi}{3})\right)$$

b)

i)

$$x[n] = \tfrac{1}{2\pi}\int_{\Omega_0}^{\Omega_0+2\pi} \underbrace{X(e^{j\Omega})}_{=1} e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega n} d\Omega = \frac{1}{2\pi jn}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

$$x[n] = \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\pi n} - e^{-j\pi n}\right) = \frac{sin(\pi n)}{\pi n}$$

$$\boxed{\text{Recall that, } sinc(t) = \begin{cases} \frac{sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}}$$

Here, n is an integer. So, $\frac{sin(\pi n)}{\pi n} = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases} \Rightarrow x[n] = \delta[n]$

ii)

We know that, $\mathscr{F}\{1\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$

By frequency-shift property of DTFT:

$\mathscr{F}\{e^{j\Omega_0 n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$

$\Rightarrow \mathscr{F}\{e^{j\frac{\pi}{3}n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m) = 2\pi X(e^{j\Omega})$

$\Rightarrow \mathscr{F}^{-1}\{\sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m)\} = \frac{e^{j\frac{\pi}{3}n}}{2\pi} = x[n]$

iii)

$$x[n] = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\frac{-\pi}{2}} e^{j\Omega n}d\Omega}_{=\int_{\frac{\pi}{2}}^{\pi} e^{-j\Omega n}d\Omega} + \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n} + e^{-j\Omega n}d\Omega$$

$$x[n] = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} cos(\Omega n) = \frac{1}{\pi n} sin(\Omega n)\Big|_{\frac{\pi}{2}}^{\pi} = \frac{sin(\pi n)}{\pi n} - \frac{sin(\frac{\pi}{2}n)}{\pi n}$$

iv)

By frequency-shift property of DTFT:

$x[n] \longleftrightarrow X(e^{j\Omega})$

$x[n]e^{j\frac{\pi}{4}n} \longleftrightarrow X(e^{j(\Omega-\frac{\pi}{4})}) = Y(e^{j\Omega})$

Therefore, $y[n] = x[n]e^{j\frac{\pi}{4}n} = \frac{sin(\pi n)e^{j\frac{\pi}{4}n}}{\pi n} - \frac{sin(\frac{\pi}{2}n)e^{j\frac{\pi}{4}n}}{\pi n}$

Question 5

a)

b)

c)

d)

Question 6