EE301 Homework-2

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Question 1

System 1

1 - The system is memoryless since the value of the output $y(t)$ depends on only the value of the input $x(t)$ at present time instant.
2 - To determine the linearity of the system, apply the input $\alpha x(t)$ and let the output $\hat{y}(t) = \alpha x(t)cos(2\pi f_0 t) = \alpha y(t) \longrightarrow$ the system is linear.
3 - The system is causal since the output $y(t)$ does not depend on the future value of the input $x(t)$.
4 - The system is time-variant. To see this, apply the input $x(t-t_0)$ and let the output $\hat{y}(t) = x(t-t_0)cos(2\pi f_0 t) \neq y(t-t_0) = x(t-t_0)cos(2\pi f_0(t-t_0))$.
5 - A system is said to be stable if any bounded input creates a bounded output. Let $x(t)$ be a bounded. That is, there exist $M > 0$ such that $|x(t)| \leq M$. Also, $-1 \leq cos(2\pi f_0 t) \leq 1$. Thus, $|y(t)| = |x(t)cos(2\pi f_0 t)| \leq |x(t)|\,|cos(2\pi f_0 t)| \leq M \longrightarrow$ so the system is stable.

System 2

1 - The system is memoryless.
2 - The system is non-linear since if we apply the input $\alpha x(t)$ then the output becomes $\hat{y}(t) = c_1 \alpha x(t) + c_2(\alpha x(t))^2 = \alpha c_1 x(t) + \alpha^2 c_2 x^2(t) \neq \alpha y(t) = \alpha c_1 x(t) + \alpha c_2 x^2(t)$
3 - The system is causal.
4 - The system is time-invariant consider that if we apply the input $x(t-t_0)$ and let the output $\hat{y}(t) = c_1 x(t-t_0) + c_2 x^2(t-t_0) = y(t-t_0)$
5 - Apply a bounded input $x(t)$ such that $|x(t)| \leq M$ then $|y(t)| = |c_1 x(t) + c_2 x^2(t)| \leq$

$|c_1|\,|x(t)| + |c_2|\,|x^2(t)| \le |c_1|\,M + |c_2|\,M^2 < \infty \longrightarrow$ the system is stable.

System 3

1 - The system is memoryless.

2 - The system is non-linear. Apply the input $\alpha x(t)$ and let the output $\hat{y}(t) = \alpha x(t) + 4 \ne \alpha y(t) = \alpha x(t) + \alpha 4$

3 - The system is causal.

4 - Apply the input $x(t - t_0)$:
$\hat{y}(t) = x(t - t_0) + 4 = y(t - t_0) \longrightarrow$ the system is time-invariant.

5 - Let $x(t)$ be bounded input such that $|x(t)| \le M$. Then, $|y(t)| = |x(t) + 4| \le |x(t)| + 4 \le M + 4$ (which is a finite number). So, the system is stable.

System 4

1 - The system is not memoryless as it is not causal.

2 - The system is linear.

3 - For $t < 0$, $y(t)$ is calculated by future values of $x(t)$, therefore, the system is not causal.

4 - Apply the input $x(t - t_0)$:
$\hat{y}(t) = x((t - t_0)/3) = y(t - t_0) \longrightarrow$ the system is time-invariant.

5 - Let $x(t)$ be bounded input such that $|x(t)| \le M$. Then, $|y(t)| = |x(t/3)|$ (the expanded version of $x(t)$) $\le M \longrightarrow$ the system is stable.

System 5

1 - The system has a memory since the value of the output $y(t)$ depends on the future values of the input $x(t)$.

2 - The system is linear.

3 - The system is not causal because the output $y(t)$ does not depend only on present and past values of the input.

4 - Apply $x(t - t_0)$:
$\hat{y}(t) = tx(t - t_0 + 5) \ne y(t - t_0) = (t - t_0)x(t - t_0 + 5) \longrightarrow$ the system is time-variant.

5 - The system is not stable. To see this, consider the following example:
$x(t) = u(t) \Rightarrow x(t + 5) = u(t + 5)$ also $|x(t + 5)| = |u(t + 5)| \le 1$ which means that the

input is bounded. However, the output $y(t) = tu(t+5)$ is not bounded. Therefore, the system is not stable.

System 6

$$y(t) = u(x(t)) = \begin{cases} 1, & x(t) \geq 0 \\ 0, & x(t) < 0 \end{cases}$$

1 - The system is memoryless.
2 - The system is non-linear, apply $\alpha x(t)$:
$\hat{y}(t) = u(\alpha x(t))$ may not be equal to $\alpha y(t) = \alpha u(x(t))$. For example, let the input $x(t) = u(t)$ then for $t \geq 0$, the $x(t) = 1$. Also, choose $\alpha = -1$. As a result, for $t \geq 0$: $u(\alpha x(t)) = 0 \neq \alpha y(t) = \alpha u(x(t)) = -1$.
3 - The system is causal.
4 - Apply $x(t - t_0)$:
$\hat{y}(t) = u(x(t - t_0)) = y(t - t_0) \longrightarrow$ the system is time-invariant.
5 - Remember that $|u(t)| \leq 1$ so it is bounded. In this system, whatever the input $x(t)$ is, the output $y(t) = u(x(t)) = \begin{cases} 1, & x(t) \geq 0 \\ 0, & x(t) < 0 \end{cases}$. Hence, the system is stable.

Question 2

a)

$x(t)$ and $h(t)$ can be written as:
$x(t) = u(t+1) - u(t-1)$

$$h(t) = (1-t)[u(t) - u(t-1)] = \begin{cases} 1-t, & 0 < t < 1 \\ 0, & otherwise \end{cases}$$

Then, $y(t) = x(t) * h(t)$

Before evaluating the convolution, consider the following:
$\hat{y}(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(t-\tau) h(\tau)\, d\tau$

$$= \int_{-\infty}^{t} h(\tau)\, d\tau = \begin{cases} 0, & t < 0 \\ \int_0^t (1-\tau)\, d\tau, & 0 < t < 1 \\ \int_0^1 (1-\tau)\, d\tau, & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ t - \frac{t^2}{2}, & 0 < t < 1 \\ \frac{1}{2}, & t > 1 \end{cases}$$

By the properties of the LTI system:

$$\hat{y}(t+1) = u(t+1) * h(t) = \begin{cases} 0, & t+1<0 \\ (t+1) - \frac{(t+1)^2}{2}, & 0<t+1<1 \\ \frac{1}{2}, & t+1>1 \end{cases} = \begin{cases} 0, & t<-1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1<t<0 \\ \frac{1}{2}, & t>0 \end{cases}$$

$$\hat{y}(t-1) = u(t-1) * h(t) = \begin{cases} 0, & t-1<0 \\ (t-1) - \frac{(t-1)^2}{2}, & 0<t-1<1 \\ \frac{1}{2}, & t-1>1 \end{cases} = \begin{cases} 0, & t<1 \\ (t-1) - \frac{(t-1)^2}{2}, & 1<t<2 \\ \frac{1}{2}, & t>2 \end{cases}$$

Therefore, $y(t) = \hat{y}(t+1) - \hat{y}(t-1) = \begin{cases} 0, & t<-1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1<t<0 \\ \frac{1}{2}, & 0<t<1 \\ \frac{1}{2} - [(t-1) - \frac{(t-1)^2}{2}], & 1<t<2 \\ 0, & t>2 \end{cases}$

b)

$w(t) = h(t) * g(t)$

$g(t)$ can be written as: $g(t) = x(t) + x(t-1) - x(t+1)$

Then, $w(t) = h(t) * [x(t) + x(t-1) - x(t+1)]$

By the distributive property of the convolution over addition:

$w(t) = h(t) * x(t) + h(t) * x(t-1) - h(t) * x(t+1)$

$= y(t) + y(t-1) - y(t+1)$ [by considering part a]

Acik formulu yazilacak

Question 3

a)

Recall that in continuous-time systems one can apply the derivative operation to any arbitrary input signal. For example,
$\delta(t) = \frac{d}{dt}u(t) \longrightarrow \int_{-\infty}^{\infty} \delta(t)\, dt$
Also, it can be applied by using the formal definition of the derivative:
$\lim_{h \to 0} \frac{x(t+h)-x(t)}{h}$
However, in discrete-time systems h cannot go to zero and the minimum value for h can be one. So, in discrete time the derivative expression becomes
$\lim_{h \to 1} \frac{x(n+h)-x(n)}{h} = x[n+1] - x[n]$
Thus, the derivative operation in continuous-time systems is analogous of the difference operation in discrete-time. Hence, we can obtain the impulse response of the difference operation as: $h[n] = \delta[n] - \delta[n-1]$

b)

By convolution, $y[n] = x[n] * h[n] = x[n] * (\delta[n] - \delta[n-1])$
By distributive property of the convolution operation,
$x[n] * (\delta[n] – \delta[n-1]) = (x[n] * \delta[n]) – (x[n] * \delta[n-1])$
$y[n] = x[n] – x[n-1]$

c)

$e^{j\Omega_0 n}(1 - e^{-j\Omega_0}) = (cos(\Omega_0 n) + jsin(\Omega_0 n))(1 - e^{-j\Omega_0})$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - (cos(\Omega_0 n) + jsin(\Omega_0 n))(cos(\Omega_0 n) - jsin(\Omega_0 n))$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - cos(\Omega_0 n)cos(\Omega_0) + jcos(\Omega_0 n)sin(\Omega_0) - jsin(\Omega_0 n)cos(\Omega_0) - sin(\Omega_0 n)sin(\Omega_0)$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - (cos(\Omega_0 n)cos(\Omega_0) + sin(\Omega_0 n)sin(\Omega_0)) - j(sin(\Omega_0 n)cos(\Omega_0) - cos(\Omega_0 n)sin(\Omega_0))$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - cos(\Omega_0 n - \Omega_0) - jsin(\Omega_0 n - \Omega_0)$
$= cos(\Omega_0 n) - cos(\Omega_0(n-1)) + j(sin(\Omega_0 n) - sin(\Omega_0(n-1)))$

$|y[n]| = [(cos(\Omega_0 n) - cos(\Omega_0(n-1)))^2 + (sin(\Omega_0 n) - sin(\Omega_0(n-1)))^2]^{1/2}$
$= [cos^2(\Omega_0 n) - 2cos(\Omega_0 n)cos(\Omega_o(n-1)) + cos^2(\Omega_0(n-1)) + sin^2(\Omega_0 n) - 2sin(\Omega_0 n)sin(\Omega_o(n-1)) + sin^2(\Omega_0(n-1))]^{1/2}$
$= \sqrt{2 - 2[cos(\Omega_0 n)cos(\Omega_0(n-1)) + sin(\Omega_0 n)sin(\Omega_0(n-1))]}$

$= \sqrt{2 - 2cos(\Omega_0 n - \Omega_0(n-1))}$
$= \sqrt{2 - 2cos(\Omega_0)}$

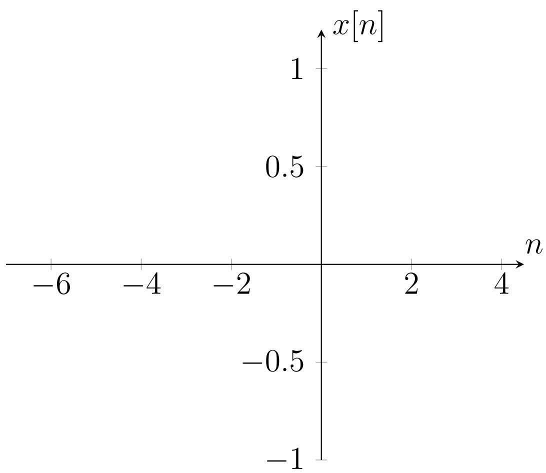
d)

$\Omega_0 = 0 \longrightarrow |y[n]| = \sqrt{2 - 2cos(0)} = 0$
$\Omega_0 = \frac{2\pi}{8} \longrightarrow |y[n]| = \sqrt{2 - 2cos(\frac{2\pi}{8})} = \sqrt{2 - \sqrt{2}}$
$\Omega_0 = \frac{2\pi}{4} \longrightarrow |y[n]| = \sqrt{2 - 2cos(\frac{2\pi}{4})} = \sqrt{2}$
$\Omega_0 = \frac{2\pi}{2} \longrightarrow |y[n]| = \sqrt{2 - 2cos(\frac{2\pi}{2})} = 2$

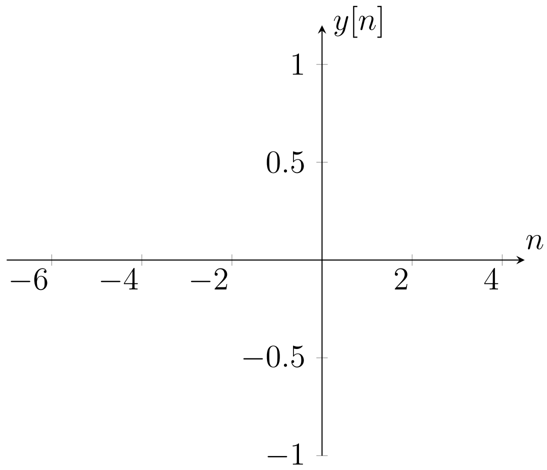
As we have shown in part a), the given discrete-time LTI system is analagous to the derivative operation in continuous-time systems. Also, please remember that the derivative is defined as the rate of change of a function with respect to a variable. Here, as the frequency of the input x[n] is increased, the period becomes smaller and the values of the input function are changed more rapidly. Therefore, the rate of change of a input function is increased as the frequency is increased, and the modulus of the output $|y[n]|$ is getting larger values.

e)

$x[n] = u[n+5] - u[n-2]$ can be plotted as:



We can calculate $y[n]$ for $x[n] = u[n+5] - u[n-2]$ by using the answer in (b) as:
$y[n] = u[n+5] - u[n+2] - (u[n+4] + u[n-3])$
$y[n]$ can be plotted as:



From the graph of $y[n]$, we can say that $x[n]$ has edges at $n = -5$ and $n = 2$.

f)

From (d), we can say that magnitude of the output signal of the system increases as the frequency increases. In (f), $x[n]$ has rapid changes at $n = -5$ and $n = 2$, and as a result, $y[n]$ has magnitude 1 at $n = -5$ and $n = 2$, and 0 elsewhere.

Question 4

a)

Note that, any arbitrary periodic signal x(t) (or, x[n]) with Fourier series coefficients a_k is real-valued if $a_k^* = a_{-k}$.

For $x_1(t)$: $a_k^* = [(\frac{1}{2})^{-k}]^* = (\frac{1}{2})^{-k} \neq (\frac{1}{2})^{-(-k)} = (\frac{1}{2})^{k} = a_{-k}$
For $x_2(t)$: $a_k^* = (cos(k\pi))^* = cos(k\pi) = cos(-k\pi) = a_{-k}$
For $x_3[n]$: $a_k^* = (jsin(\frac{k\pi}{2}))^* = -jsin(\frac{k\pi}{2}) = jsin(\frac{-k\pi}{2}) = a_{-k}$

Thus, $x_2(t)$ and $x_3[n]$ are real-valued signals.

b)

Any arbitrary periodic signal x(t) (or, x[n]) with Fourier series coefficients a_k is even if a_k is a real-valued and even function.
For $x_1(t)$: $a_k = (\frac{1}{2})^{-k} \Rightarrow$ is a real-valued but not even function, i.e., $(\frac{1}{2})^{-k}\Big|_{k=1} \neq (\frac{1}{2})^{-k}\Big|_{k=-1}$
For $x_2(t)$: $a_k = cos(k\pi) \Rightarrow$ is a real-valued and also an even function since consider that

$-1 \leq cos(k\pi) \leq 1$ and $cos(k\pi)\Big|_{k=1} = cos(k\pi)\Big|_{k=-1}$
For $x_3[n]$: $a_k^* = (jsin(\frac{k\pi}{2}))^* = -jsin(\frac{k\pi}{2}) = jsin(\frac{-k\pi}{2}) = a_{-k}$

Thus, $x_2(t)$ and $x_3[n]$ are real-valued signals.

c)

i.

The systems are connected in parallel, so we can calculate the impulse response of the overall system $h[n]$ as:
$h[n] = h_1[n] - h_2[n]$
$h[n] = u[n+2] - u[n-2]$

ii.

We can calculate the output signal of the overall system $y[n]$ for an arbitrary input signal $x[n]$ as:
$y[n] = x[n-1] + x[n] + x[n+1] + x[n+2]$
The system is neither memoryless nor causal since $y[n]$ depends future values of $x[n]$.
For a bounded input signal, we can say that there exist $M > 0$ such that $|x[n]| \leq M$. Then $|y[n]| \leq 4M$ and $y[n]$ is bounded, therefore the system is stable.

Question 5

a)

In calculating part of the question, we added zero vectors to end and beginning of the main vector to obtain correct results in range of $0 \leq n \leq N-1$, which can be seen in Appendix. Since we manually did that zero adding part, we deviated the result for $n > N$ range. Because of this situation, our calculated result has only N value in range. On the other hand, MATLAB's built-in function, *conv()*, evaluate the convolution so that the size of the output is equal to $N + L - 1$. That's why the output graphs of convolution operation are different by using our function and MATLAB's built-in function as shown in Appendix. In fact, to get the same result that we got by using our function, one can specify an input argument, *same*, to MATLAB's function so that the result gives the central part of the convolution operation. Otherwise, the built-in function gives the full convolution whose length is equal to $N + L - 1$.

b)

The output signal looks more smoother in larger values of L. As we can observe that h[n] impulse response created accordingly to L which directly affects the convolution range. By saying range, we mean that how many $x*h$ multiplication we will use to calculate one element of the output. So, this is why that the output signal gain more elements and more precise values when L gets larger. And also, we can observe that the L value expands the output range by expanding h[n] size. The MATLAB code and the resulting graphs for different values of L can be seen in Appendix.