

# EE301 Homework-2

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## Question 1

a)

Let  $x[n] = \delta[n]$ . Then  $y[n] = h[n]$

$$h[n] - ah[n-1] = \delta[n] - b\delta[n-1]$$

We know that  $h[n] = 0$  for  $n < 0$  as the system is causal.

For  $n = 0$  we have

$$h[0] - ah[-1] = \delta[0] - b\delta[-1]$$

$$h[0] = 1$$

For  $n = 1$ , we have

$$h[1] - ah[0] = \delta[1] - b\delta[0]$$

$$h[1] - a = -b$$

$$h[1] = a - b$$

For  $n = 2$ , we have

$$h[2] - ah[1] = \delta[2] - b\delta[1]$$

$$h[2] - a(a - b) = 0$$

$$h[2] = a(a - b)$$

For  $n = 3$ , we have

$$h[3] - ah[2] = \delta[3] - b\delta[2]$$

$$h[3] - a^2(a - b) = 0$$

$$h[3] = a^2(a - b)$$

In general, for  $n > 0$ , we have  $h[n] = a^{n-1}(a - b)$ .

$$h[n] = \begin{cases} a^{n-1}(a - b), & n > 0 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases}$$

If the system is stable, then

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$1 + \sum_{k=1}^{\infty} |a|^{k-1} |a - b| < \infty$$

$$1 + |a - b| \sum_{k=0}^{\infty} |a|^k < \infty$$

Therefore,  $-1 < a < 1$  if the system is stable.

b)

## Question 2

a)

$x(t)$  and  $h(t)$  can be written as:

$$x(t) = u(t+1) - u(t-1)$$

$$h(t) = (1-t)[u(t) - u(t-1)] = \begin{cases} 1-t, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,  $y(t) = x(t) * h(t)$

Before evaluating the convolution, consider the following:

$$\begin{aligned} \hat{y}(t) &= u(t) * h(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau) d\tau \\ &= \int_{-\infty}^t h(\tau) d\tau = \begin{cases} 0, & t < 0 \\ \int_0^t (1-\tau) d\tau, & 0 < t < 1 \\ \int_0^1 (1-\tau) d\tau, & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ t - \frac{t^2}{2}, & 0 < t < 1 \\ \frac{1}{2}, & t > 1 \end{cases} \end{aligned}$$

By the properties of the LTI system:

$$\hat{y}(t+1) = u(t+1) * h(t) = \begin{cases} 0, & t+1 < 0 \\ (t+1) - \frac{(t+1)^2}{2}, & 0 < t+1 < 1 \\ \frac{1}{2}, & t+1 > 1 \end{cases} = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & t > 0 \end{cases}$$

$$\hat{y}(t-1) = u(t-1) * h(t) = \begin{cases} 0, & t-1 < 0 \\ (t-1) - \frac{(t-1)^2}{2}, & 0 < t-1 < 1 \\ \frac{1}{2}, & t-1 > 1 \end{cases} = \begin{cases} 0, & t < 1 \\ (t-1) - \frac{(t-1)^2}{2}, & 1 < t < 2 \\ \frac{1}{2}, & t > 2 \end{cases}$$

$$\text{Therefore, } y(t) = \hat{y}(t+1) - \hat{y}(t-1) = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & 0 < t < 1 \\ \frac{1}{2} - [(t-1) - \frac{(t-1)^2}{2}], & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

**b)**

$$w(t) = h(t) * g(t)$$

$$g(t) \text{ can be written as: } g(t) = x(t) + x(t-1) - x(t+1)$$

$$\text{Then, } w(t) = h(t) * [x(t) + x(t-1) - x(t+1)]$$

By the distributive property of the convolution over addition:

$$w(t) = h(t) * x(t) + h(t) * x(t-1) - h(t) * x(t+1)$$

$$= y(t) + y(t-1) - y(t+1) \text{ [by considering part a]}$$

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## Question 3

**a)**

$$x(t) = \dots + \delta(t+2) - 2\delta(t+1) + \delta(t) - 2\delta(t-1) + \delta(t-2) - 2\delta(t-3) + \dots$$

It is a periodic signal with fundamental period  $T_0 = 2$ .

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{3}{2}} x(t) e^{-jk\pi t} dt$$

$$a_k = \frac{1}{2} \underbrace{\int_{0^-}^{0^+} \delta(t) e^{-jk\pi t} dt}_{e^{-jk\pi(0)}} + \frac{1}{2} \underbrace{\int_{1^-}^{1^+} -2\delta(t-1) e^{-jk\pi t} dt}_{e^{-jk\pi(1)}}$$

Also, recall that  $\int_{-\infty}^{\infty} \delta(t-t_0)x(t) dt = x(t_0)$

$$\Rightarrow a_k = \frac{1}{2} - e^{-jk\pi}$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \left( \frac{1}{2} - e^{-jk\pi} \right) e^{jk\pi t}$$

b)

c)

$$x[n] = (-1)^n + j^n + \cos\left(\frac{2\pi n}{3}\right) = e^{j\pi n} + e^{j\frac{\pi}{2}n} + \cos\left(\frac{2\pi n}{3}\right)$$

Therefore,  $x[n]$  can be written as a summation of 3 distinct periodic signals:

$$x[n] = \underbrace{e^{j\pi n}}_{x_1[n]} + \underbrace{e^{j\frac{\pi}{2}n}}_{x_2[n]} + \underbrace{\cos\left(\frac{2\pi n}{3}\right)}_{x_3[n]} = x_1[n] + x_2[n] + x_3[n]$$

$x_1[n]$  is a periodic signal with fundamental period  $N_0 = 2$ .

$x_2[n]$  is a periodic signal with fundamental period  $N_0 = 4$ .

$x_3[n]$  is a periodic signal with fundamental period  $N_0 = 3$ .

d)

## Question 4

a)

Note that, any arbitrary periodic signal  $x(t)$  (or,  $x[n]$ ) with Fourier series coefficients  $a_k$  is real-valued if  $a_k^* = a_{-k}$ .

$$\text{For } x_1(t): a_k^* = \left[\left(\frac{1}{2}\right)^{-k}\right]^* = \left(\frac{1}{2}\right)^{-k} \neq \left(\frac{1}{2}\right)^{-(-k)} = \left(\frac{1}{2}\right)^k = a_{-k}$$

$$\text{For } x_2(t): a_k^* = (\cos(k\pi))^* = \cos(k\pi) = \cos(-k\pi) = a_{-k}$$

$$\text{For } x_3[n]: a_k^* = (j\sin(\frac{k\pi}{2}))^* = -j\sin(\frac{k\pi}{2}) = j\sin(\frac{-k\pi}{2}) = a_{-k}$$

Thus,  $x_2(t)$  and  $x_3[n]$  are real-valued signals.

b)

Any arbitrary periodic signal  $x(t)$  (or,  $x[n]$ ) with Fourier series coefficients  $a_k$  is even if  $a_k$  is a real-valued and even function.

$$\text{For } x_1(t): a_k = \left(\frac{1}{2}\right)^{-k} \Rightarrow \text{is a real-valued but not even function, i.e., } \left(\frac{1}{2}\right)^{-k} \Big|_{k=1} \neq \left(\frac{1}{2}\right)^{-k} \Big|_{k=-1}$$

$$\text{For } x_2(t): a_k = \cos(k\pi) \Rightarrow \text{is a real-valued and also an even function since consider that } -1 \leq \cos(k\pi) \leq 1 \text{ and } \cos(k\pi) \Big|_{k=1} = \cos(k\pi) \Big|_{k=-1}$$

$$\text{For } x_3[n]: a_k = j\sin(\frac{k\pi}{2}) \Rightarrow \text{neither real-valued nor even function.}$$

Therefore, only the signal  $x_2(t)$  is even.

c)

By the time-shifting property of Continuous Time Fourier Series:

$$x_2(t-5) = \sum_{k=-100}^{100} \cos(k\pi) e^{-jk\frac{\pi}{5}} e^{jk\frac{2\pi}{50}t} \quad \text{and, also note that} \quad \cos(k\pi) = \frac{1}{2}(e^{jk\pi} + e^{-jk\pi})$$

$$\Rightarrow x_2(t-5) = \sum_{k=-100}^{100} \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}) e^{jk\frac{2\pi}{50}t}$$

Thus,  $x_2(t-5)$  is another periodic signal with Fourier series coefficients  $b_k = \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}})$ .

Also, remember the theorem mentioned above, the signal  $x_2(t-5)$  is real if  $b_k^* = b_{-k}$ .

$$b_k^* = (\frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}))^* = \frac{1}{2}(e^{-jk\frac{4\pi}{5}} + e^{jk\frac{6\pi}{5}}) = b_{-k} \Rightarrow \text{the signal } x_2(t-5) \text{ is real.}$$

To check whether the signal  $x_2(t-5)$  is even,  $b_k$  can be written as:

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + j\sin(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5}) - j\sin(k\frac{6\pi}{5})]$$

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - \frac{1}{2}j[\sin(k\frac{6\pi}{5}) - \sin(k\frac{4\pi}{5})]$$

Also, note that  $\boxed{\sin a - \sin b = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})}$

$$\text{Thus, } b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - j[\cos(2\pi k)\sin(k\frac{2\pi}{5})]$$

$$\Rightarrow b_k = \frac{1}{2}(\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})) - j\sin(k\frac{2\pi}{5})$$

As a result, it can be seen that  $b_k$  is not real-valued, so the signal  $x_2(t-5)$  is not even.

d)

By differentiation property of Continuous Time Fourier Series:

$$\frac{d}{dt}x_2(t) = \sum_{k=-\infty}^{\infty} \cos(k\pi) jk \frac{2\pi}{50} e^{jk\frac{2\pi}{50}t} \Rightarrow \text{is a periodic signal with Fourier series coefficients } c_k$$

$$\text{such that } c_k = \cos(k\pi) jk \frac{2\pi}{50}.$$

$$c_k^* = -jk \frac{2\pi}{50} \cos(k\pi) \quad \text{and} \quad c_{-k} = j(-k) \frac{2\pi}{50} \cos(-k\pi) = -jk \frac{2\pi}{50} \cos(k\pi)$$

$$c_k^* = c_{-k} \Rightarrow \frac{d}{dt}x_2(t) \text{ is real-valued.}$$

Note that,  $\cos(k\pi)$  take only the values  $+1$  or  $-1$ , so it is always real-valued. However,  $-jk\frac{2\pi}{50}$  is purely imaginary. Therefore,  $\frac{d}{dt}x_2(t)$  is not even.

e)

**Parseval's Identity:** For any continuous-time periodic signal  $x(t)$  with the Fourier series coefficients  $a_k$ ,

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)x^*(t)dt = \sum_{k=-\infty}^{\infty} a_k a_k^* \Rightarrow \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

In this question,  $x_1(t)$  is a periodic signal whose fundamental period,  $T_0 = 50$  and its Fourier series coefficients  $a_k = (\frac{1}{2})^{-k}$ . Please also note that, for this signal, the coefficients are defined only for  $0 \leq k \leq 100$  since outside this domain all of the coefficients are equal to 0. So, by using Parseval's identity, the average power of  $x_1(t)$  in one period:

$$\frac{1}{50} \int_0^{50} |x_1(t)|^2 dt = \sum_{k=0}^{100} |a_k|^2 = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^{-2k} = \sum_{k=0}^{100} 4^k = 1 + 4^1 + 4^2 + 4^3 + \dots + 4^{100}$$

## Question 5

a)

b)