Middle East Technical University
Department of Electrical and Electronics Engineering
EE301: Signals and Systems I

Homework 3
Due: Sunday December 11, 2022

1. The frequency response of an ideal *band-limited differentiator* is given below

$$H(j\omega) = \begin{cases} j\omega, & \text{if } |\omega| < 4 \\ 0, & \text{elsewhere} \end{cases} . \tag{1}$$

   (a) Plot the magnitude and phase responses, and explain why this system is named as an *ideal band-limited differentiator*.

   (b) The following signal is the input signal of this system:

$$x(t) = \sum_{n=-\infty}^{\infty} \left[\delta(t-n-1) + \delta(t-4n)\right]. \tag{2}$$

   Find the time-domain output signal, namely $y(t)$.

2. Answer the followings

   (a) Let $X(j\omega)$ be the Fourier transform of continuous time signal $x(t)$. Derive the expression for the *Inverse Fourier Transform* of $\frac{\mathrm{d}(X(j\omega))}{\mathrm{d}\,\omega}$ in terms of $x(t)$.

   (b) Determine and plot the Fourier Transform, namely $G(j\omega)$, of $g(t) = \left[\frac{\sin(\pi t)}{\pi t}\right]^2$.

   (c) A filter with impulse response $h(t) = f^*(-t)$ is said to be *matched* to $f(t)$.

      i. Find the frequency response of the filter, namely $H(j\omega)$, in terms of $F(j\omega)$.

      ii. Define a signal $y(t) = h(t) * f(t)$. Show that $|y(t)| \leq |y(0)|\ \forall t$.

3. Consider the following *high-pass filter* whose frequency response is given below

$$H(j\omega) = 1 - \mathrm{rect}\left(\frac{\omega}{4\pi}\right). \tag{3}$$

   (a) The following signal is the input signal of this system: $x(t) = \frac{\sin(4\pi t)}{\pi t}\cos(2\pi t)$.

      i. Let $X(j\omega)$ be the Fourier transform of input signal $x(t)$. Sketch and clearly label $X(j\omega)$.

      ii. Let $Y(j\omega)$ be the Fourier transform of output signal $y(t)$. Sketch and clearly label $Y(j\omega)$.

   (b) The output signal $y(t)$ is processed to obtain a new signal: $z(t) = \frac{\sin(2\pi t)}{\pi t} y(t)$.

      i. Let $Z(j\omega)$ be the Fourier transform of $z(t)$. Sketch and clearly label $Z(j\omega)$.

      ii. Find the time-domain expression for $z(t)$.

4. (a) Find the Discrete-Time Fourier Transform (DTFT) $X(e^{j\Omega})$ of the following signals:

      i. $x[n] = \delta[n]$

      ii. $x[n] = 2\delta[n-3] - \delta[n-10]$

      iii. $x[n] = \begin{cases} \frac{1}{n^2}, & \text{if } 1 \leq n \leq 4 \\ 0, & \text{otherwise} \end{cases}$

      iv. $x[n] = \left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]$

      v. $x[n] = \left(\frac{1}{2}\right)^{(n-7)} u[n-7] - 3^{(n-7)} u[-n+6]$ (**Hint:** Use your answer to part (iv) and DTFT properties)

vi. $x[n]$ is a periodic signal with period $N = 3$, defined over a period as $x[n] = \begin{cases} 0, & \text{if } n = 0 \\ 1, & \text{if } n = 1, 2. \end{cases}$

(b) Find the signals $x[n]$ whose DTFTs are given as follows:

i. $X(e^{j\Omega}) = 1$ for all $\Omega \in \mathbb{R}$.

ii. $X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi k)$

iii. $X(e^{j\Omega}) = \begin{cases} 1, & \text{if } \frac{\pi}{2} \le |\Omega| \le \pi \\ 0, & \text{if } |\Omega| < \frac{\pi}{2} \end{cases}$

(**Hint:** Draw $X(e^{j\Omega})$ over the whole real line, and find the result by evaluating a single integral with conveniently chosen integral limits.)

iv. Find $y[n]$ if $Y(e^{j\Omega}) = X(e^{j(\Omega - \frac{\pi}{4})})$ where $X(e^{j\Omega})$ is as in part (iii).
(**Hint:** Use DTFT properties.)

5. A discrete-time LTI system with impulse response $h[n]$ is excited with an input signal $x[n]$, and the signal $y[n]$ is observed at the output. The DTFTs of the input and the output signals are sketched below for the interval $\Omega \in [-\pi, \pi]$.

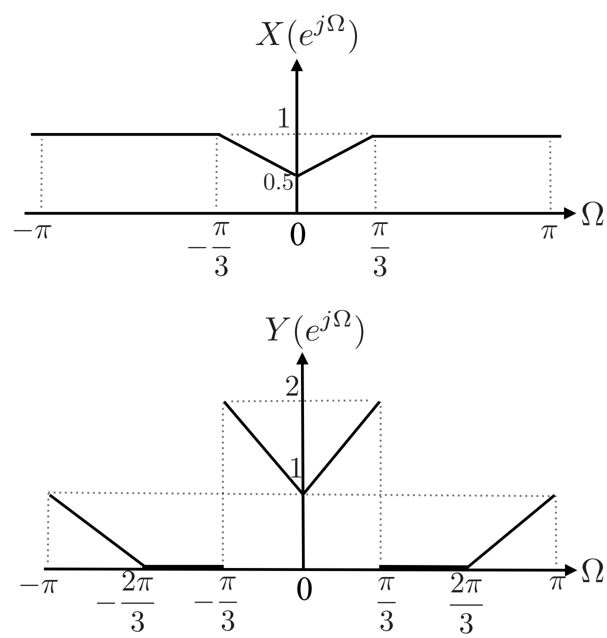

Figure 1: Spectra of the input and output signals for Question 5

(a) Find and sketch the frequency response $H(e^{j\Omega})$ of the system in the interval $\Omega \in [-2\pi, 2\pi]$.

Solve parts (b) and (c) by using DTFT properties (do not evaluate the inverse DTFT integral):

(b) Justifying your answer briefly, determine whether $h[n]$

i. is real

ii. has even symmetry

iii. has odd symmetry

iv. has finite energy

v. is periodic

(c) Find the following:

i. $h[0]$

ii. $\sum_{n=-\infty}^{\infty} (h[n])^2$

iii. $\sum_{n=1}^{\infty} (h[n])^2$

(d) Find the output $v[n]$ if the system is excited by the following input signals $w[n]$. (**Hint:** Think in the frequency domain!)

i. $w[n] = 1$ for all n.

ii. $w[n] = e^{j\frac{\pi}{6}n}$

iii. $w[n] = 3\cos(\frac{\pi}{6}n) + 5\sin(\frac{\pi}{2}n)$

iv. $w[n] = (-1)^n$ (**Hint:** Can this signal be written in form of a complex exponential?)

6. In this question, we will study a digital signal processing application. We will work on a noisy audio signal and enhance its quality by removing the noise with a low-pass filter.

(a) The signal $x[n]$ is a noisy audio signal that is severely contaminated with additive high-frequency noise. The signal $x[n]$ is available in ODTUClass under the file name `x.mat`. Load the signal in MATLAB and obtain its length using the following commands:

```
load x.mat
N=length(x);
```

Plot the noisy signal $x[n]$. Listen to the signal with the following command:

```
sound(x,44100);
```

(The number 44100 we specify here is a parameter related to the recording settings of the signal). Comment on what you hear. Is it possible to recognize the song in the audio signal?

(b) We will now compute the DTFT $X(e^{j\Omega})$ of the signal $x[n]$. In fact, since the frequency variable Ω is a continuous variable, it is not possible to represent the DTFT $X(e^{j\Omega})$ in an exact way in MATLAB. Instead, we will use the `fft` function to take the DFT (Discrete Fourier Transform) of $x[n]$, which will give us a regularly sampled version of the DTFT. (You can find more detailed information on the concept of DFT in the supplementary lecture video here). For the practical purpose of this question, we can simply consider that we are computing and plotting an approximate version of the DTFT.

Compute the DTFT $X(e^{j\Omega})$ of the signal $x[n]$ with the following command:

```
X=fftshift(fft(x));
```

Then generate the frequency variable Ω with the following commands:

```
Omega=linspace(-pi,pi,N+1);
Omega=Omega(2:end);
```

i. Plot the magnitude $|X(e^{j\Omega})|$ of the DTFT of $x[n]$ as a function of Ω by typing

```
figure; plot(Omega,abs(X));
xlabel('\Omega');
ylabel('|X(e^{j\Omega})|');
```

Here, it is important to remember that we should use the `abs` function to take the magnitude of the DTFT. The signal $x[n]$ is known to contain a low-frequency component corresponding to the original clean audio signal, whose magnitude decays as the frequency increases. On the other hand, the noise component of $x[n]$ has a strong magnitude and a flat spectrum that covers high-frequencies. By examining the spectrum $|X(e^{j\Omega})|$ of the noisy signal, can you guess within which frequency range $\Omega \in [-B, B]$ the original audio signal lies?

ii. Plot also the real part $Re\{X(e^{j\Omega})\}$ and the imaginary part $Im\{X(e^{j\Omega})\}$ of the spectrum of $x[n]$. Noticing that $x[n]$ is a real signal, what kind of symmetry do we expect $Re\{X(e^{j\Omega})\}$ and $Im\{X(e^{j\Omega})\}$ to have? Verify this by examining your plots and comment.

(c) Our purpose now is to design a low-pass filter that will remove the high-frequency noise component of $x[n]$, so that we obtain back the clean audio signal. The low-pass filtering is done by passing the signal $x[n]$ through an LTI system with impulse response $h[n]$, so that we obtain the clean signal $y[n]$ as

$$y[n] = x[n] * h[n].$$

The input signal $x[n]$ and the output signal $y[n]$ are related in the frequency domain as

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

where $H(e^{j\Omega})$ and $Y(e^{j\Omega})$ are the DTFTs of $h[n]$ and $y[n]$.
We will use an ideal low-pass filter $H(e^{j\Omega})$ given by

$$H(e^{j\Omega}) = \begin{cases} 1, & \text{if } |\Omega| < B \\ 0, & \text{otherwise} . \end{cases}$$

Choose the cut-off frequency B of your low-pass filter based on the guess you made in part (b). Then, generate a vector `H` in MATLAB, which represents your ideal low-pass filter $H(e^{j\Omega})$ in the frequency domain. (Remember that we defined the vector `Omega` representing the frequency variable Ω in the interval $(-\pi, \pi]$.) Plot the spectrum $H(e^{j\Omega})$ of the low-pass filter with respect to Ω.

(d) We will now remove the noise in the noisy signal $x[n]$ by passing it through the ideal low-pass filter we generated in part (c). Perform the filtering operation

$$Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega})$$

in the frequency domain as

```
Y=H.*X;
```

Plot the magnitude $|Y(e^{j\Omega})|$ of the spectrum of the output signal $y[n]$ against the frequency variable Ω. Try different values for the cut-off frequency B and observe its effect on the spectrum of the output signal.

(e) We can finally obtain the time-domain representation of the output signal $y[n]$ by taking the inverse DTFT of $Y(e^{j\Omega})$ as follows:

```
y=ifft(ifftshift(Y));
```

Plot the signal $y[n]$. (Note that you might observe a small imaginary part due to numerical errors, so make sure to plot only the real part of the signal.) Compare the output signal $y[n]$ to the noisy input signal $x[n]$ and comment on the results.

(f) Now it is time to listen to our output signal $y[n]$. Type the following command:

```
sound(real(y),44100);
```

If you have chosen B properly, you should now be able to hear a clean audio signal. Can you say which song it is? ☺
Try different values for the cut-off frequency B (e.g., too large, too small) and comment on their effect on the quality of the output signal.