EE301 Homework-3

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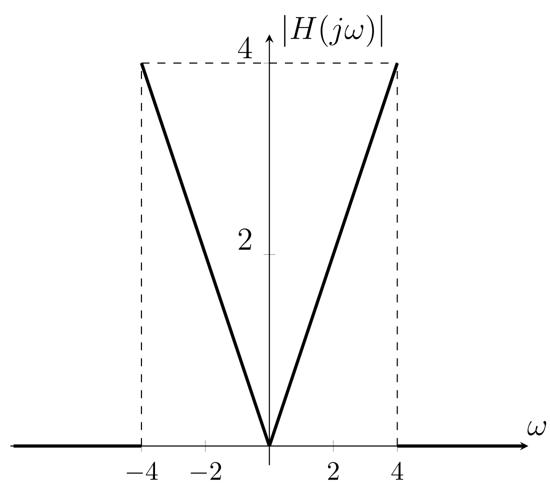
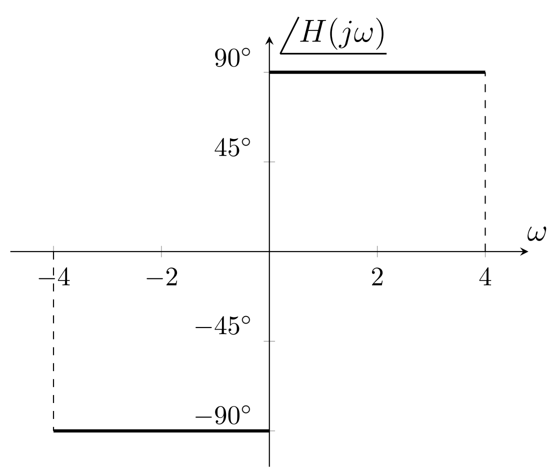
Question 1

a)

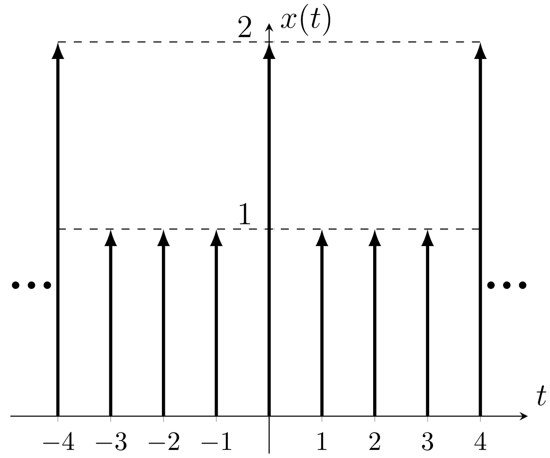
The magnitude and phase responses of $H(j\omega)$ can be seen below:





b)

x(t) can be plotted as:



x(t) is a periodic signal with fundamental period $T_0 = 4$. Also, $x(t)$ can be written in CTFS representation: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, where $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$.

Let's first calculate the FS cofficients: $a_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{4}\int_{0^-}^{4^-} x(t)e^{-jk\frac{\pi}{2}t}$

$\Rightarrow a_0 = \frac{5}{4},\ a_1 = \frac{1}{4},\ a_{-1} = \frac{1}{4},\ a_2 = \frac{3}{4},\ a_{-2} = \frac{3}{4}$

Also, recall that:

$$e^{jk\omega_0 t} \longrightarrow \boxed{\text{LTI system}} \longrightarrow H(jk\omega_0)e^{jk\omega_0 t}$$

By linearity:

$$\underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}_{x(t)} \longrightarrow \boxed{\text{LTI system}} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}}_{y(t)}$$

Therefore, $y(t) = \sum_{k=-2}^{2} a_k H(jk\omega_0)e^{jk\omega_0 t} = -\frac{3}{4}\pi j e^{-j\pi t} - \frac{3}{8}\pi j e^{-j\frac{\pi}{2}t} + 0 + \frac{3}{8}\pi j e^{j\frac{\pi}{2}t} + \frac{3}{4}\pi j e^{j\pi t}$

$y(t) = -\frac{3\pi}{2}\left(\frac{1}{2j}e^{j\pi t} - \frac{1}{2j}e^{-j\pi t}\right) - \frac{3\pi}{4}\left(\frac{1}{2j}e^{j\frac{\pi}{2}t} - \frac{1}{2j}e^{-j\frac{\pi}{2}t}\right) = -\frac{3\pi}{2}sin(\pi t) - \frac{3\pi}{4}sin(\frac{\pi}{2}t)$

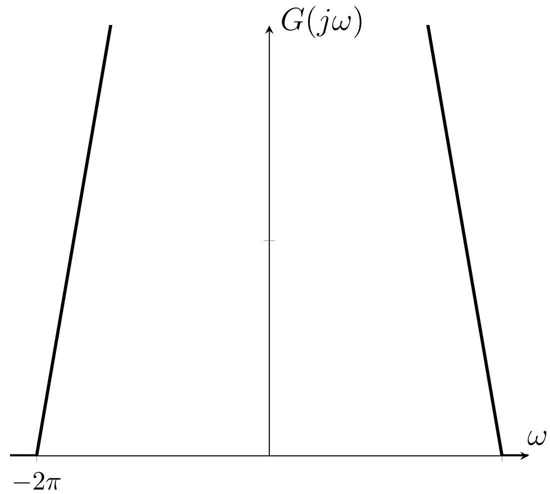
Question 2

a)

We know that $X(j\omega) = \int_{\infty}^{-\infty} x(t)e^{-j\omega t}\,dt$. Then
$\frac{d}{d\omega}X(j\omega) = \frac{d}{d\omega}\int_{\infty}^{-\infty} x(t)e^{-j\omega t}\,dt = \int_{\infty}^{-\infty} \frac{d}{d\omega}(x(t)e^{-j\omega t})\,dt = \int_{\infty}^{-\infty} x(t)(-jt)e^{-j\omega t}\,dt = \mathscr{F}\{x(t)(-jt)\}$
Therefore, $\mathscr{F}^{-1}\{\frac{d}{d\omega}X(j\omega)\} = x(t)(-jt)$

b)

Let $g_1(t) = \frac{sin(\pi t)}{\pi t}$. Then we know that $\mathscr{F}\{g_1(t)\} = \begin{cases} 1, & |\omega| < \pi \\ 0, & else \end{cases}$
Since $g(t) = g_1(t)g_1(t)$, by multiplication property, we know that $G(j\omega) = \frac{1}{2\pi}G_1(j\omega) * G_1(j\omega)$
$G(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\theta)G(j(\omega-\theta))\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(j(\omega-\theta))\,d\theta$
Let $\theta' = \omega - \theta$. Then $d\theta' = -d\theta$ and
$G(j\omega) = \frac{1}{2\pi}\int_{\omega+\pi}^{\omega-\pi} G_1(j\theta')\,d(-\theta') = \frac{1}{2\pi}\int_{\omega-\pi}^{\omega+\pi} G_1(j\theta')\,d\theta'$
For $\omega < -2\pi$ or $\omega > 2\pi$, $G(j\omega) = 0$. For $\omega < 0$
$G(j\omega) = \frac{1}{2\pi}\int_{-\pi}^{\omega+\pi}\,d\theta' = \frac{\omega+\pi}{2\pi}$
For $\omega > 0$
$G(j\omega) = \frac{1}{2\pi}\int_{\omega-\pi}^{\pi}\,d\theta' = \frac{2\pi-\omega}{2\pi}$
Therefore, $G(j\omega) = \begin{cases} \frac{\omega+\pi}{2\pi}, & -2\pi < \omega < 0 \\ \frac{2\pi-\omega}{2\pi}, & 0 < \omega < 2\pi \\ 0, & else \end{cases}$, and $G(j\omega)$ is plotted below



c)

i)

$H(j\omega) = \int_{-\infty}^{\infty} f^*(-t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} (f(-t)e^{j\omega t})^*\,dt$
Let $t' = -t$. Then $dt' = -dt$ and
$H(j\omega) = \int_{\infty}^{-\infty} (f(t')e^{-j\omega t'})^*\,d(-t') = \int_{-\infty}^{\infty} (f(t')e^{-j\omega t'})^*\,d(t') = (\int_{-\infty}^{\infty} f(t')e^{-j\omega t'}\,d(t'))^* = F^*(j\omega)$

ii)

Let $\mathscr{F}\{y(t)\} = Y(j\omega)$. Then $Y(j\omega) = F^*(j\omega)F(j\omega) = |F(j\omega)|^2$
$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}\,d\omega =$

Question 3

a)

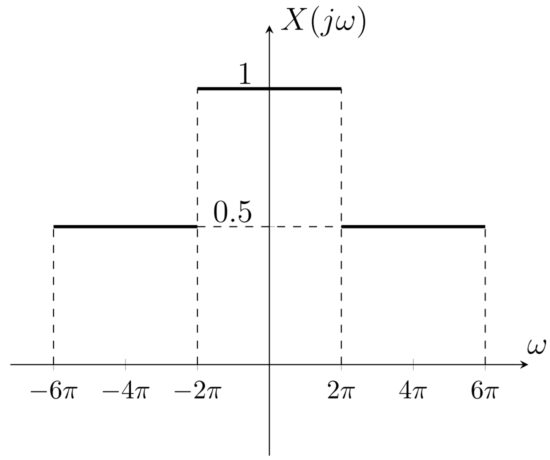
i)

$x(t) = \frac{sin(4\pi t)}{\pi t}cos(2\pi t) = \frac{4sin(4\pi t)}{4\pi t}cos(2\pi t) = x_1(t)x_2(t)\left(\text{ where } x_1(t) = \frac{4sin(4\pi t)}{4\pi t},\, x_2(t) = cos(2\pi t)\right)$

Recall that: $\mathscr{F}\{rect(\theta)\} = \frac{sin(\omega/2)}{\omega/2}$
By duality property of CTFT: $\frac{sin(t/2)}{t/2} \longleftrightarrow 2\pi rect(-\omega) = 2\pi rect(\omega)$
By scaling property of CTFT: $\frac{sin(4\pi t)}{4\pi t} \longleftrightarrow \frac{rect(\frac{\omega}{8\pi})}{4}$
By linearity: $\frac{4sin(4\pi t)}{4\pi t} \longleftrightarrow rect(\frac{\omega}{8\pi}) = X_1(j\omega)$

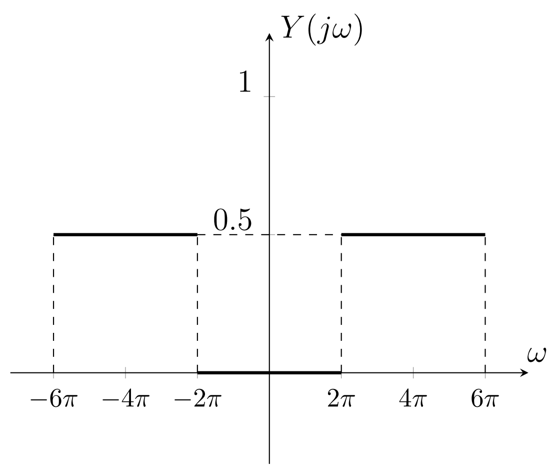
$x_2(t) = cos(2\pi t) = \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$
$\mathscr{F}\{\frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}\} \longleftrightarrow \pi[\delta(\omega + 2\pi) + \delta(\omega - 2\pi)] = X_2(j\omega)$

By modulation property of CTFT:
$x(t) = x_1(t)x_2(t) \longleftrightarrow X(j\omega) = \frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
$X(j\omega) = \frac{1}{2\pi}rect(\frac{\omega}{8\pi}) * \pi[\delta(\omega+2\pi)+\delta(\omega-2\pi)] = \frac{1}{2}[rect(\frac{\omega}{8\pi}) * \delta(\omega+2\pi) + rect(\frac{\omega}{8\pi}) * \delta(\omega-2\pi)]$
$X(j\omega) = \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi})$



ii)

$y(t) = h(t) * x(t)$
By convolution property of CTFT: $y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$
$Y(j\omega) = \left(1 - rect(\frac{\omega}{4\pi})\right)\left(\frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi})\right)$
$Y(j\omega) = \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) - \frac{1}{2}\left(rect(\frac{\omega}{4\pi})rect(\frac{\omega-2\pi}{8\pi}) + rect(\frac{\omega}{4\pi})rect(\frac{\omega+2\pi}{8\pi})\right)$
$Y(j\omega) = \frac{1}{2}\left(rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})\right)$

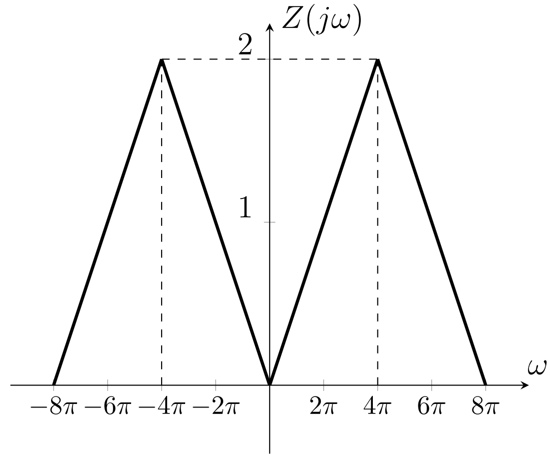


b)

i)

By modulation property of CTFT:
$z(t) \longleftrightarrow Z(j\omega) = \frac{1}{2\pi}Y(j\omega) * \mathscr{F}\{\frac{sin(2\pi t)}{\pi t}\} = \frac{1}{2\pi}Y(j\omega) * rect(\frac{\omega}{4\pi})$
$\Rightarrow Z(j\omega) = \frac{1}{2\pi}\left(rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})\right) * rect(\frac{\omega}{4\pi})$

$$\Rightarrow Z(j\omega) = \begin{cases} \frac{8\pi-\omega}{2\pi}, & 4\pi < \omega < 8\pi \\ \frac{\omega}{2\pi}, & 0 < \omega \leq 4\pi \\ \frac{-\omega}{2\pi}, & -4\pi < \omega < 0 \\ \frac{8\pi+\omega}{2\pi}, & -8\pi < \omega \leq -4\pi \end{cases}$$



ii)

$y(t) \longleftrightarrow Y(j\omega) = rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})$
$Y(j\omega) = \frac{1}{\pi} rect(\frac{\omega}{4\pi}) * \pi[\delta(\omega - 4\pi) + \delta(\omega + 4\pi)]$
By modulation property of CTFT:
$\mathscr{F}^{-1}\{Y(j\omega)\} = y(t) = \frac{2sin(2\pi t)}{\pi t} cos(4\pi t)$

Question 4

a)

i)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n]e^{-j\Omega n}}_{\delta[n]e^{-j\Omega 0}} = \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

ii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} (2\delta[n-3] - \delta[n-10])e^{-j\Omega n} \overset{\text{(by linearity)}}{=} 2\sum_{n=-\infty}^{\infty} \delta[n-3]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} \delta[n-10]e^{-j\Omega n}$$

By time-shift property of DTFT:
$X(e^{j\Omega}) = 2e^{-j3\Omega} - e^{-j10\Omega}$

iii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=1}^{4} \frac{1}{n^2}e^{-j\Omega n} = e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega} + \frac{1}{9}e^{-j3\Omega} + \frac{1}{16}e^{-j4\Omega}$$

iv)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]\right) e^{-j\Omega n}$$

$$\overset{\text{(by linearity)}}{=} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} 3^n u[-n-1]e^{-j\Omega n} = \sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\Omega}\right)^n - \sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n}$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\Omega}\right)^n = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} \quad \left(\text{since } \left|\frac{1}{2}e^{-j\Omega}\right| = \frac{1}{2} < 1, \text{so the expression is convergent}\right)$$

Let $m = -n$: $\displaystyle\sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n} = \sum_{m=1}^{\infty} 3^{-m}e^{j\Omega m} = \sum_{m=1}^{\infty}\left(\frac{1}{3}e^{j\Omega}\right)^m = \underbrace{\left[\sum_{m=0}^{\infty}\left(\frac{1}{3}e^{j\Omega}\right)^m\right]}_{\frac{1}{1-\frac{1}{3}e^{j\Omega}}} - 1$

$$\Rightarrow X(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$$

v)

Say that, $\hat{x}[n] = \left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]$ and $\mathscr{F}\{\hat{x}[n]\} = \hat{X}(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$
By time-shift property of DTFT:
$x[n] = \hat{x}[n-7] \longleftrightarrow X(e^{j\Omega}) = \hat{X}(e^{j\Omega})e^{-j7\Omega}$

$$\Rightarrow X(e^{j\Omega}) = \frac{e^{-j7\Omega}}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{e^{-j7\Omega}}{1-\frac{1}{3}e^{j\Omega}} - e^{-j7\Omega}\right)$$

vi)

Let x[n] be a periodic signal with fundamental period N. Then,
$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{m=-\infty}^{\infty}\sum_{k=k_0}^{k_0+N-1} a_k 2\pi\delta(\Omega - k\frac{2\pi}{N} - 2\pi m)$$
Note that for this signal if we consider the interval $0 \le \Omega < 2\pi$, DTFT of the signal will be written as:
$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3})$$
First, find the DTFS coefficients of x[n]: $a_k = \dfrac{1}{3}\displaystyle\sum_{n=0}^{2} x[n]e^{-jk\frac{2\pi}{3}n}$

$$\Rightarrow a_0 = \frac{1}{3}\sum_{n=0}^{2} x[n] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$
$$\Rightarrow a_1 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{2\pi}{3}n} = \frac{1}{3}e^{-j\frac{2\pi}{3}} + \frac{1}{3}e^{-j\frac{4\pi}{3}} = \frac{-1}{3}$$
$$\Rightarrow a_2 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{4\pi}{3}n} = \frac{1}{3}e^{-j\frac{4\pi}{3}} + \frac{1}{3}e^{-j\frac{8\pi}{3}} = \frac{-1}{3}$$
$$X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3}) = a_0 2\pi\delta(\Omega) + a_1 2\pi\delta(\Omega - \frac{2\pi}{3}) + a_2 2\pi\delta(\Omega - \frac{4\pi}{3})$$
$$X(e^{j\Omega}) = \frac{2\pi}{3}\left(2\delta(\Omega) - \delta(\Omega - \frac{2\pi}{3}) - \delta(\Omega - \frac{4\pi}{3})\right)$$

b)

i)

$$x[n] = \tfrac{1}{2\pi}\int_{\Omega_0}^{\Omega_0+2\pi} \underbrace{X(e^{j\Omega})}_{=1} e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega n} d\Omega = \frac{1}{2\pi jn}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

$$x[n] = \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\pi n} - e^{-j\pi n}\right) = \frac{sin(\pi n)}{\pi n}$$

$$\boxed{\text{Recall that, } sinc(t) = \begin{cases} \frac{sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}}$$

Here, n is an integer. So, $\dfrac{sin(\pi n)}{\pi n} = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases} \Rightarrow x[n] = \delta[n]$

ii)

We know that, $\mathscr{F}\{1\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$

By frequency-shift property of DTFT:

$$\mathscr{F}\{e^{j\Omega_0 n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$$
$$\Rightarrow \mathscr{F}\{e^{j\frac{\pi}{3}n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m) = 2\pi X(e^{j\Omega})$$
$$\Rightarrow \mathscr{F}^{-1}\{\sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m)\} = \frac{e^{j\frac{\pi}{3}n}}{2\pi} = x[n]$$

iii)

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{1}{2\pi}\underbrace{\int_{-\pi}^{\frac{-\pi}{2}} e^{j\Omega n}d\Omega}_{=\int_{\frac{\pi}{2}}^{\pi} e^{-j\Omega n}d\Omega} + \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n} + e^{-j\Omega n}d\Omega$$

$$x[n] = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} cos(\Omega n) = \frac{1}{\pi n}sin(\Omega n)\Big|_{\frac{\pi}{2}}^{\pi} = \frac{sin(\pi n)}{\pi n} - \frac{sin(\frac{\pi}{2}n)}{\pi n} = \frac{sin(\pi n)}{\pi n} - \frac{1}{2}\frac{sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n}$$

$$x[n] = sinc(n) - \frac{1}{2}sinc(\frac{n}{2})$$

iv)

By frequency-shift property of DTFT:

$x[n] \longleftrightarrow X(e^{j\Omega})$

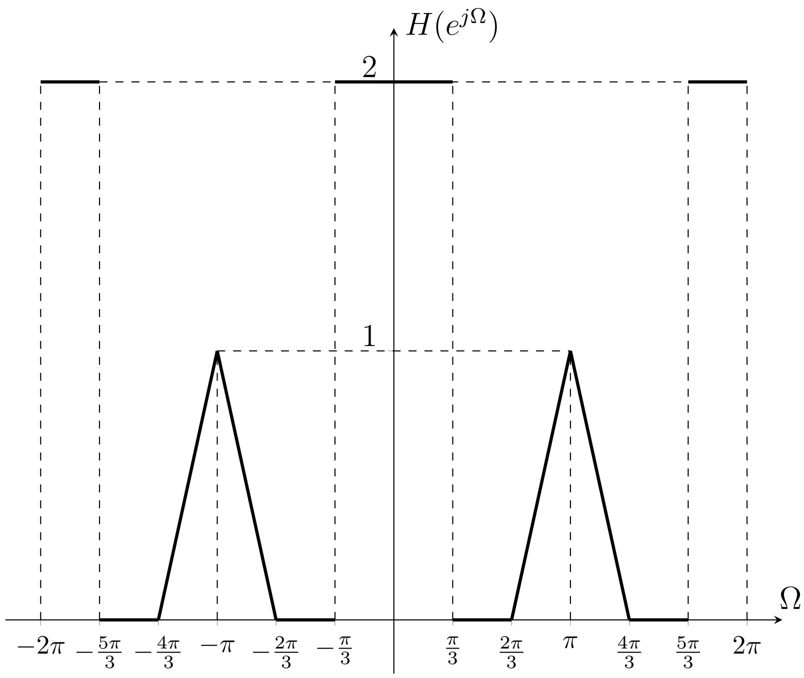
$x[n]e^{j\frac{\pi}{4}n} \longleftrightarrow X(e^{j(\Omega-\frac{\pi}{4})}) = Y(e^{j\Omega})$

Therefore, $y[n] = x[n]e^{j\frac{\pi}{4}n} = sinc(n)e^{j\frac{\pi}{4}n} - \frac{1}{2}sinc(\frac{n}{2})e^{j\frac{\pi}{4}n}$

Question 5

a)

Recall that, $Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) \Rightarrow H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$



b)

c)

d)

Question 6