EE301 Homework-3

İsmail Enes Bülbül, Eren Meydanlı, Ahmet Caner Akar

December 11, 2022

Question 1

a)

b)

Question 2

a)

b)

c)

Question 3

a)

b)

Question 4

a)

i)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n]e^{-j\Omega n}}_{\delta[n]e^{-j\Omega 0}} = \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

ii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} (2\delta[n-3]-\delta[n-10])e^{-j\Omega n} \overset{\text{(by linearity)}}{=} 2\sum_{n=-\infty}^{\infty} \delta[n-3]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} \delta[n-10]e^{-j\Omega n}$$

By time-shifting property of DTFT:
$X(e^{j\Omega}) = 2e^{-j3\Omega} - e^{-j10\Omega}$

iii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=1}^{4} \frac{1}{n^2}e^{-j\Omega n} = e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega} + \frac{1}{9}e^{-j3\Omega} + \frac{1}{16}e^{-j4\Omega}$$

iv)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]\right) e^{-j\Omega n}$$

$$\overset{\text{(by linearity)}}{=} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} 3^n u[-n-1]e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n - \sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} \quad \left(\text{since } \left|\frac{1}{2}e^{-j\Omega}\right| = \frac{1}{2} < 1, \text{so the expression is convergent}\right)$$

Let $m = -n$: $\displaystyle\sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n} = \sum_{m=1}^{\infty} 3^{-m} e^{j\Omega m} = \sum_{m=1}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m = \underbrace{\left[\sum_{m=0}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m\right]}_{\frac{1}{1-\frac{1}{3}e^{j\Omega}}} - 1$

$$\Rightarrow X(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$$

v)

Say that, $\hat{x}[n] = \left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]$ and $\mathscr{F}\{\hat{x}[n]\} = \hat{X}(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$
By time-shifting property of DTFT:
$x[n] = \hat{x}[n-7] \longleftrightarrow X(e^{j\Omega}) = \hat{X}(e^{j\Omega})e^{-j7\Omega}$

$$\Rightarrow X(e^{j\Omega}) = \frac{e^{-j7\Omega}}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{e^{-j7\Omega}}{1-\frac{1}{3}e^{j\Omega}} - e^{-j7\Omega}\right)$$

vi)

Let x[n] be a periodic signal with fundamental period N. Then,

$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} \sum_{k=k_0}^{k_0+N-1} a_k 2\pi\delta(\Omega - k\frac{2\pi}{N} - 2\pi m)$$

Note that for this signal if we consider the interval $0 \leq \Omega < 2\pi$, DTFT of the signal will be written as:

$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3})$$

First, find the DTFS coefficients of x[n]: $a_k = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-jk\frac{2\pi}{3}n}$

$$\Rightarrow a_0 = \frac{1}{3}\sum_{n=0}^{2} x[n] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow a_1 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{2\pi}{3}n} = \frac{1}{3}e^{-j\frac{2\pi}{3}} + \frac{1}{3}e^{-j\frac{4\pi}{3}} = \frac{-1}{3}$$

$$\Rightarrow a_2 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{4\pi}{3}n} = \frac{1}{3}e^{-j\frac{4\pi}{3}} + \frac{1}{3}e^{-j\frac{8\pi}{3}} = \frac{-1}{3}$$

$$X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3}) = a_0 2\pi\delta(\Omega) + a_1 2\pi\delta(\Omega - \frac{2\pi}{3}) + a_2 2\pi\delta(\Omega - \frac{4\pi}{3})$$

$$X(e^{j\Omega}) = \frac{2\pi}{3}\left(2\delta(\Omega) - \delta(\Omega - \frac{2\pi}{3}) - \delta(\Omega - \frac{4\pi}{3})\right)$$

b)

i)

$$x[n] = \tfrac{1}{2\pi}\int_{\Omega_0}^{\Omega_0+2\pi} \underbrace{X(e^{j\Omega})}_{=1} e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega n}d\Omega = \frac{1}{2\pi jn}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

$$x[n] = \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\pi n} - e^{-j\pi n}\right) = \frac{sin(\pi n)}{\pi n}$$

$$\boxed{\text{Recall that, } sinc(t) = \begin{cases} \frac{sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}}$$

Here, n is an integer. So, $\frac{sin(\pi n)}{\pi n} = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases} \Rightarrow x[n] = \delta[n]$

ii)

We know that, $\mathscr{F}\{1\} = 2\pi\sum_{m=-\infty}^{\infty}\delta(\Omega - 2\pi m)$

By frequency-shifting property of DTFT:

$$\mathscr{F}\{e^{j\Omega_0 n}\} = 2\pi\sum_{m=-\infty}^{\infty}\delta(\Omega - \Omega_0 - 2\pi m)$$

$$\Rightarrow \mathscr{F}\{e^{j\frac{\pi}{3}n}\} = 2\pi\sum_{m=-\infty}^{\infty}\delta(\Omega - \frac{\pi}{3} - 2\pi m) = 2\pi X(e^{j\Omega})$$

$$\Rightarrow \mathscr{F}^{-1}\{\sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m)\} = \frac{e^{j\frac{\pi}{3}n}}{2\pi} = x[n]$$

iii)

iv)

Question 5

a)

b)

c)

d)

Question 6