

# EE301 Homework-2

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## Question 1

a)

Let  $x[n] = \delta[n]$ . Then  $y[n] = h[n]$  and equation becomes:

$$h[n] - ah[n-1] = \delta[n] - b\delta[n-1]$$

Also, we know that  $h[n] = 0$  for  $n < 0$  since the system is said to be causal.

For  $n = 0$ , we have:

$$h[0] - ah[-1] = \delta[0] - b\delta[-1]$$

$$h[0] = 1$$

For  $n = 1$ , we have:

$$h[1] - ah[0] = \delta[1] - b\delta[0]$$

$$h[1] - a = -b$$

$$h[1] = a - b$$

For  $n = 2$ , we have:

$$h[2] - ah[1] = \delta[2] - b\delta[1]$$

$$h[2] - a(a - b) = 0$$

$$h[2] = a(a - b)$$

For  $n = 3$ , we have:

$$h[3] - ah[2] = \delta[3] - b\delta[2]$$

$$h[3] - a^2(a - b) = 0$$

$$h[3] = a^2(a - b)$$

In general, for  $n > 0$ , we have  $h[n] = a^{n-1}(a - b)$ .

$$h[n] = \begin{cases} a^{n-1}(a - b), & n > 0 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases}$$

If the system is stable, then the following expressions must be satisfied:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \iff 1 + \sum_{k=1}^{\infty} |a|^{k-1}|a - b| < \infty \iff 1 + |a - b| \sum_{k=0}^{\infty} |a|^k < \infty$$

$$\text{Therefore, the system is said to be stable} \iff \begin{cases} a \in \mathbb{R}, & \text{if } a = b \\ -1 < a < 1, & \text{otherwise} \end{cases}$$

**b)**

$$y[n] = h[n] * x[n] \quad \text{and} \quad x[n] = e^{j\Omega_1 n} + e^{j\Omega_2 n}$$

By distribution property of the convolution over addition:

$$y[n] = h[n] * e^{j\Omega_1 n} + h[n] * e^{j\Omega_2 n} = \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega_1(n-k)} + \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega_2(n-k)}$$

$$y[n] = e^{j\Omega_1 n} + \sum_{k=1}^{\infty} a^{k-1}\left(a - \frac{1}{a}\right)e^{j\Omega_1(n-k)} + e^{j\Omega_2 n} + \sum_{k=1}^{\infty} a^{k-1}\left(a - \frac{1}{a}\right)e^{j\Omega_2(n-k)}$$

$$y[n] = e^{j\Omega_1 n} + \sum_{k=1}^{\infty} a^k \frac{(a^2 - 1)}{a^2} e^{j\Omega_1(n-k)} + e^{j\Omega_2 n} + \sum_{k=1}^{\infty} a^k \frac{(a^2 - 1)}{a^2} e^{j\Omega_2(n-k)}$$

$$y[n] = e^{j\Omega_1 n} + \frac{(a^2 - 1)}{a^2} e^{j\Omega_1 n} \sum_{k=1}^{\infty} a^k e^{-j\Omega_1 k} + e^{j\Omega_2 n} + \frac{(a^2 - 1)}{a^2} e^{j\Omega_2 n} \sum_{k=1}^{\infty} a^k e^{-j\Omega_2 k}$$

$$y[n] = e^{j\Omega_1 n} + \frac{(a^2 - 1)}{a^2} e^{j\Omega_1 n} \sum_{k=1}^{\infty} (ae^{-j\Omega_1})^k + e^{j\Omega_2 n} + \frac{(a^2 - 1)}{a^2} e^{j\Omega_2 n} \sum_{k=1}^{\infty} (ae^{-j\Omega_2})^k$$

We know that  $-1 \leq a \leq 1$  if  $a = b$  for a stable system. Then we have  $|ae^{-j\Omega}| \leq 1$  for any  $\Omega$ .

$$y[n] = e^{j\Omega_1 n} + \frac{(a^2 - 1)}{a^2} e^{j\Omega_1 n} \frac{ae^{-j\Omega_1}}{1 - ae^{-j\Omega_1}} + e^{j\Omega_2 n} + \frac{(a^2 - 1)}{a^2} e^{j\Omega_2 n} \frac{ae^{-j\Omega_2}}{1 - ae^{-j\Omega_2}}$$

## Question 2

**a)**

$x(t)$  and  $h(t)$  can be written as:

$$x(t) = u(t + 1) - u(t - 1)$$

$$h(t) = (1-t)[u(t) - u(t-1)] = \begin{cases} 1-t, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

Then,  $y(t) = x(t) * h(t)$

Before evaluating the convolution, consider the following:

$$\begin{aligned} \hat{y}(t) &= u(t) * h(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau) d\tau \\ &= \int_{-\infty}^t h(\tau) d\tau = \begin{cases} 0, & t < 0 \\ \int_0^t (1-\tau) d\tau, & 0 < t < 1 \\ \int_0^1 (1-\tau) d\tau, & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ t - \frac{t^2}{2}, & 0 < t < 1 \\ \frac{1}{2}, & t > 1 \end{cases} \end{aligned}$$

By the properties of the LTI system:

$$\hat{y}(t+1) = u(t+1) * h(t) = \begin{cases} 0, & t+1 < 0 \\ (t+1) - \frac{(t+1)^2}{2}, & 0 < t+1 < 1 \\ \frac{1}{2}, & t+1 > 1 \end{cases} = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & t > 0 \end{cases}$$

$$\hat{y}(t-1) = u(t-1) * h(t) = \begin{cases} 0, & t-1 < 0 \\ (t-1) - \frac{(t-1)^2}{2}, & 0 < t-1 < 1 \\ \frac{1}{2}, & t-1 > 1 \end{cases} = \begin{cases} 0, & t < 1 \\ (t-1) - \frac{(t-1)^2}{2}, & 1 < t < 2 \\ \frac{1}{2}, & t > 2 \end{cases}$$

$$\text{Therefore, } y(t) = \hat{y}(t+1) - \hat{y}(t-1) = \begin{cases} 0, & t < -1 \\ \frac{1-t^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & 0 < t < 1 \\ \frac{1}{2} - \left(\frac{t^2-4t+3}{2}\right), & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

Also,  $y(t)$  can be written as:

$$y(t) = \left(\frac{1-t^2}{2}\right) \cdot u(t+1) + \left(\frac{t^2}{2}\right) \cdot u(t) - \left(\frac{t^2-4t+3}{2}\right) \cdot u(t-1) + \left(\frac{t^2-4t+2}{2}\right) \cdot u(t-2)$$

**b)**

$$w(t) = h(t) * g(t)$$

$g(t)$  can be written as:  $g(t) = x(t) + x(t-1) - x(t+1)$

Then,  $w(t) = h(t) * [x(t) + x(t-1) - x(t+1)]$

By the distributive property of the convolution over addition:

$$w(t) = h(t) * x(t) + h(t) * x(t-1) - h(t) * x(t+1)$$

$$w(t) = y(t) + y(t-1) - y(t+1) \text{ [by considering part a]}$$

By the time-invariance property of the LTI system:

$$y(t-1) = \left(\frac{2t-t^2}{2}\right) \cdot u(t) + \left(\frac{t^2-2t+1}{2}\right) \cdot u(t-1) - \left(\frac{t^2-6t+8}{2}\right) \cdot u(t-2) + \left(\frac{t^2-6t+7}{2}\right) \cdot u(t-3)$$

$$y(t+1) = \left(\frac{-t^2-2t}{2}\right) \cdot u(t+2) + \left(\frac{t^2+2t+1}{2}\right) \cdot u(t+1) - \left(\frac{t^2-2t}{2}\right) \cdot u(t) + \left(\frac{t^2-2t-1}{2}\right) \cdot u(t-1)$$

As a result:

$$w(t) = \left(\frac{-t^2-2t}{2}\right) \cdot u(t+2) + (-t^2-t) \cdot u(t+1) + \left(\frac{t^2}{2}\right) \cdot u(t) + \left(\frac{-t^2+4t-1}{2}\right) \cdot u(t-1) + (t-3) \cdot u(t-2) + \left(\frac{t^2-6t+7}{2}\right) \cdot u(t-3)$$

## Question 3

a)

$$x(t) = \dots + \delta(t+2) - 2\delta(t+1) + \delta(t) - 2\delta(t-1) + \delta(t-2) - 2\delta(t-3) + \dots$$

It is a periodic signal with fundamental period  $T_0 = 2$ .

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{3}{2}} x(t) e^{-jk\pi t} dt$$

$$a_k = \frac{1}{2} \underbrace{\int_{0^-}^{0^+} \delta(t) e^{-jk\pi t} dt}_{e^{-jk\pi(0)}} + \frac{1}{2} \underbrace{\int_{1^-}^{1^+} -2\delta(t-1) e^{-jk\pi t} dt}_{e^{-jk\pi(1)}}$$

Also, recall that  $\int_{-\infty}^{\infty} \delta(t-t_0)x(t) dt = x(t_0)$

$$\Rightarrow a_k = \frac{1}{2} - e^{-jk\pi}$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2} - e^{-jk\pi}\right) e^{jk\pi t}$$

b)

$x(t)$  is a periodic signal with fundamental period,  $T_0 = 4$ .

$$a_k = \frac{1}{4} \int_{-2}^2 x(t) e^{-jk\omega_0 t} dt = \frac{1}{4} \int_0^1 \cos\left(\frac{\pi t}{2}\right) e^{-jk\frac{\pi}{2}t} dt$$

$$a_k = \frac{1}{4} \int_0^1 \frac{1}{2} (e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}) e^{-jk\frac{\pi}{2}t} dt = \frac{1}{8} \int_0^1 (e^{-j\frac{\pi}{2}(k-1)t} + e^{-j\frac{\pi}{2}(k+1)t}) dt$$

$$a_k = -\frac{1}{8} \left( \left. \frac{e^{-j\frac{\pi}{2}(k-1)t}}{j(k-1)\frac{\pi}{2}} \right|_0^1 + \left. \frac{e^{-j\frac{\pi}{2}(k+1)t}}{j(k+1)\frac{\pi}{2}} \right|_0^1 \right) = \frac{1}{8} \left[ \frac{1 - e^{-j\frac{\pi}{2}(k-1)}}{j(k-1)\frac{\pi}{2}} \right] + \frac{1}{8} \left[ \frac{1 - e^{-j\frac{\pi}{2}(k+1)}}{j(k+1)\frac{\pi}{2}} \right]$$

$$\begin{aligned}
a_k &= \frac{1}{8} \left[ \frac{1 - e^{-j\frac{\pi}{2}k} e^{j\frac{\pi}{2}}}{j(k-1)\frac{\pi}{2}} \right] + \frac{1}{8} \left[ \frac{1 - e^{-j\frac{\pi}{2}k} e^{-j\frac{\pi}{2}}}{j(k+1)\frac{\pi}{2}} \right] \\
a_k &= \frac{1}{8} \left[ \frac{1 - j e^{-j\frac{\pi}{2}k}}{j(k-1)\frac{\pi}{2}} \right] + \frac{1}{8} \left[ \frac{1 + j e^{-j\frac{\pi}{2}k}}{j(k+1)\frac{\pi}{2}} \right] = \frac{1}{4} \left[ \frac{k - j e^{-j\frac{\pi}{2}k}}{j(k^2-1)\frac{\pi}{2}} \right] = -\frac{1}{4} \left[ \frac{e^{-j\frac{\pi}{2}k} + jk}{(k^2-1)\frac{\pi}{2}} \right] \\
\Rightarrow x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} -\frac{1}{4} \left( \frac{e^{-j\frac{\pi}{2}k} + jk}{(k^2-1)\frac{\pi}{2}} \right) e^{jk\frac{\pi}{2}t}
\end{aligned}$$

c)

$$x[n] = (-1)^n + j^n + \cos\left(\frac{2\pi n}{3}\right) = e^{j\pi n} + e^{j\frac{\pi}{2}n} + \cos\left(\frac{2\pi n}{3}\right)$$

Therefore,  $x[n]$  can be written as a summation of 3 distinct periodic signals:

$$x[n] = \underbrace{e^{j\pi n}}_{x_1[n]} + \underbrace{e^{j\frac{\pi}{2}n}}_{x_2[n]} + \underbrace{\cos\left(\frac{2\pi n}{3}\right)}_{x_3[n]} = x_1[n] + x_2[n] + x_3[n]$$

$x_1[n]$  is a periodic signal with fundamental period  $N_0 = 2$ .

$x_2[n]$  is a periodic signal with fundamental period  $N_0 = 4$ .

$x_3[n]$  is a periodic signal with fundamental period  $N_0 = 3$ .

Then, the fundamental period of  $x[n]$  is the least common multiple of the fundamental periods of these three signals, so  $N_0 = 12$ .

$$a_k = \frac{1}{T_0} \sum_{n=n_0}^{n_0+N_0-1} x[n] e^{-jk\frac{2\pi}{N_0}n} = \frac{1}{12} \sum_{n=-6}^5 x[n] e^{-jk\frac{\pi}{6}n}$$

d)

$x(t)$  is periodic signal with fundamental period  $T_0 = 1$ .

$$\begin{aligned}
a_k &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\omega_0 t} dt = \int_0^1 e^{-t} e^{-jk2\pi t} dt = \int_0^1 e^{-t(1+jk2\pi)} dt \\
&= \frac{-1}{1+jk2\pi} \left[ e^{-1} \underbrace{e^{-jk2\pi}}_{\cos(2k\pi)} - 1 \right] = \frac{1}{1+jk2\pi} (1 - e^{-1}) \\
\Rightarrow x(t) &= \sum_{k=-\infty}^{\infty} \frac{1}{1+jk2\pi} (1 - e^{-1}) e^{jk2\pi t}
\end{aligned}$$

e)

Recall that:  $\boxed{y(t) = h(t) * x(t)}$   $\boxed{x(t) * \delta(t) = x(t)}$   $\boxed{x(t) * \delta(t - t_0) = x(t - t_0)}$

$$\Rightarrow y(t) = x(t) * (\delta(t) - \delta(t - \frac{1}{2})) \stackrel{\text{(by distribution property)}}{=} x(t) * \delta(t) - x(t) * \delta(t - \frac{1}{2})$$

$$\begin{aligned}
&\Rightarrow y(t) = x(t) - x(t - \tfrac{1}{2}) \\
&= \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi)}{1 + jk2\pi} \left( e^{\frac{1}{2}} - e^{\frac{-1}{2}} \right) e^{jk2\pi t} - \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi)}{1 + jk2\pi} \left( e^{\frac{1}{2}} - e^{\frac{-1}{2}} \right) e^{jk2\pi(t-\frac{1}{2})} \\
&= \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi)}{1 + jk2\pi} \left( e^{\frac{1}{2}} - e^{\frac{-1}{2}} \right) e^{jk2\pi t} - \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi)}{1 + jk2\pi} \left( e^{\frac{1}{2}} - e^{\frac{-1}{2}} \right) e^{jk2\pi t} \underbrace{e^{-jk\pi}}_{\cos(k\pi)} \\
&= \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi)}{1 + jk2\pi} \left( e^{\frac{1}{2}} - e^{\frac{-1}{2}} \right) e^{jk2\pi t} (1 - \cos(k\pi)) \\
&= \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi) - \overbrace{\cos(k\pi)^2}^1}{1 + jk2\pi} \left( e^{\frac{1}{2}} - e^{\frac{-1}{2}} \right) e^{jk2\pi t} \\
y(t) &= \sum_{k=-\infty}^{\infty} \frac{\cos(k\pi) - 1}{1 + jk2\pi} \left( e^{\frac{1}{2}} - e^{\frac{-1}{2}} \right) e^{jk2\pi t}
\end{aligned}$$

## Question 4

a)

Note that, any arbitrary periodic signal  $x(t)$  (or,  $x[n]$ ) with Fourier series coefficients  $a_k$  is real-valued if  $a_k^* = a_{-k}$ .

For  $x_1(t)$ :  $a_k^* = [(\frac{1}{2})^{-k}]^* = (\frac{1}{2})^{-k} \neq (\frac{1}{2})^{-(-k)} = (\frac{1}{2})^k = a_{-k}$

For  $x_2(t)$ :  $a_k^* = (\cos(k\pi))^* = \cos(k\pi) = \cos(-k\pi) = a_{-k}$

For  $x_3[n]$ :  $a_k^* = (j\sin(\frac{k\pi}{2}))^* = -j\sin(\frac{k\pi}{2}) = j\sin(\frac{-k\pi}{2}) = a_{-k}$

Thus,  $x_2(t)$  and  $x_3[n]$  are real-valued signals.

b)

Any arbitrary periodic signal  $x(t)$  (or,  $x[n]$ ) with Fourier series coefficients  $a_k$  is even if  $a_k$  is a real-valued and even function.

For  $x_1(t)$ :  $a_k = (\frac{1}{2})^{-k} \Rightarrow$  is a real-valued but not even function, i.e.,  $(\frac{1}{2})^{-k} \Big|_{k=1} \neq (\frac{1}{2})^{-k} \Big|_{k=-1}$

For  $x_2(t)$ :  $a_k = \cos(k\pi) \Rightarrow$  is a real-valued and also an even function since consider that  $-1 \leq \cos(k\pi) \leq 1$  and  $\cos(k\pi) \Big|_{k=1} = \cos(k\pi) \Big|_{k=-1}$

For  $x_3[n]$ :  $a_k = j\sin(\frac{k\pi}{2}) \Rightarrow$  neither real-valued nor even function.

Therefore, only the signal  $x_2(t)$  is even.

c)

By the time-shifting property of Continuous Time Fourier Series:

$$x_2(t-5) = \sum_{k=-100}^{100} \cos(k\pi) e^{-jk\frac{\pi}{5}} e^{jk\frac{2\pi}{50}t} \quad \text{and, also note that} \quad \cos(k\pi) = \frac{1}{2}(e^{jk\pi} + e^{-jk\pi})$$

$$\Rightarrow x_2(t-5) = \sum_{k=-100}^{100} \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}) e^{jk\frac{2\pi}{50}t}$$

Thus,  $x_2(t-5)$  is another periodic signal with Fourier series coefficients  $b_k = \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}})$ .

Also, remember the theorem mentioned above, the signal  $x_2(t-5)$  is real if  $b_k^* = b_{-k}$ .

$$b_k^* = (\frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}))^* = \frac{1}{2}(e^{-jk\frac{4\pi}{5}} + e^{jk\frac{6\pi}{5}}) = b_{-k} \Rightarrow \text{the signal } x_2(t-5) \text{ is real.}$$

To check whether the signal  $x_2(t-5)$  is even,  $b_k$  can be written as:

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + j\sin(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5}) - j\sin(k\frac{6\pi}{5})]$$

$$b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - \frac{1}{2}j[\sin(k\frac{6\pi}{5}) - \sin(k\frac{4\pi}{5})]$$

Also, note that  $\boxed{\sin a - \sin b = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})}$

$$\text{Thus, } b_k = \frac{1}{2}[\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})] - j[\cos(2\pi k)\sin(k\frac{2\pi}{5})]$$

$$\Rightarrow b_k = \frac{1}{2}(\cos(k\frac{4\pi}{5}) + \cos(k\frac{6\pi}{5})) - j\sin(k\frac{2\pi}{5})$$

As a result, it can be seen that  $b_k$  is not real-valued, so the signal  $x_2(t-5)$  is not even.

d)

By differentiation property of Continuous Time Fourier Series:

$$\frac{d}{dt}x_2(t) = \sum_{k=-\infty}^{\infty} \cos(k\pi) jk \frac{2\pi}{50} e^{jk\frac{2\pi}{50}t} \Rightarrow \text{is a periodic signal with Fourier series coefficients } c_k$$

$$\text{such that } c_k = \cos(k\pi) jk \frac{2\pi}{50}.$$

$$c_k^* = -jk \frac{2\pi}{50} \cos(k\pi) \quad \text{and} \quad c_{-k} = j(-k) \frac{2\pi}{50} \cos(-k\pi) = -jk \frac{2\pi}{50} \cos(k\pi)$$

$$c_k^* = c_{-k} \Rightarrow \frac{d}{dt}x_2(t) \text{ is real-valued.}$$

Note that,  $\cos(k\pi)$  take only the values +1 or -1, so it is always real-valued. However,  $-jk\frac{2\pi}{50}$  is purely imaginary. Therefore,  $\frac{d}{dt}x_2(t)$  is not even.

e)

**Parseval's Identity:** For any continuous-time periodic signal  $x(t)$  with the Fourier series coefficients  $a_k$ ,

$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t)x^*(t)dt = \sum_{k=-\infty}^{\infty} a_k a_k^* \Rightarrow \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

In this question,  $x_1(t)$  is a periodic signal whose fundamental period,  $T_0 = 50$  and its Fourier series coefficients  $a_k = (\frac{1}{2})^{-k}$ . Please also note that, for this signal, the coefficients are defined only for  $0 \leq k \leq 100$  since outside this domain all of the coefficients are equal to 0. So, by using Parseval's identity, the average power of  $x_1(t)$  in one period:

$$\frac{1}{50} \int_0^{50} |x_1(t)|^2 dt = \sum_{k=0}^{100} |a_k|^2 = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^{-2k} = \sum_{k=0}^{100} 4^k = 1 + 4^1 + 4^2 + 4^3 + \dots + 4^{100}$$