EE301 Homework-2

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Question 1

a)

Let $x[n] = \delta[n]$. Then $y[n] = h[n]$ and equation becomes:
$h[n] - ah[n-1] = \delta[n] - b\delta[n-1]$
Also, we know that $h[n] = 0$ for $n < 0$ since the system is said to be causal.

For $n = 0$, we have:
$h[0] - ah[-1] = \delta[0] - b\delta[-1]$
$h[0] = 1$

For $n = 1$, we have:
$h[1] - ah[0] = \delta[1] - b\delta[0]$
$h[1] - a = -b$
$h[1] = a - b$

For $n = 2$, we have:
$h[2] - ah[1] = \delta[2] - b\delta[1]$
$h[2] - a(a-b) = 0$
$h[2] = a(a-b)$

For $n = 3$, we have:
$h[3] - ah[2] = \delta[3] - b\delta[2]$
$h[3] - a^2(a-b) = 0$
$h[3] = a^2(a-b)$

In general, for $n > 0$, we have $h[n] = a^{n-1}(a-b)$.

$$h[n] = \begin{cases} a^{n-1}(a-b), & n > 0 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases}$$

If the system is stable, then the following expressions must be satisfied:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \Longleftrightarrow 1 + \sum_{k=1}^{\infty} |a|^{k-1}|a-b| < \infty \Longleftrightarrow 1 + |a-b| \sum_{k=0}^{\infty} |a|^k < \infty$$

Therefore, the system is said to be stable $\Longleftrightarrow \begin{cases} a \in \mathbb{R}, & \text{if } a = b \\ -1 < a < 1, & \text{otherwise} \end{cases}$

b)

$y[n] = h[n] * x[n]$ and $x[n] = e^{j\Omega_1 n} + e^{j\Omega_2 n}$

By distribution property of the convolution over addition:

$$y[n] = h[n] * e^{j\Omega_1 n} + h[n] * e^{j\Omega_2 n} = \sum_{k=-\infty}^{\infty} h[k] e^{j\Omega_1 (n-k)} + \sum_{k=-\infty}^{\infty} h[k] e^{j\Omega_2 (n-k)}$$

$$y[n] = e^{j\Omega_1 n} + \sum_{k=1}^{\infty} a^{k-1}(a - \frac{1}{a}) e^{j\Omega_1 (n-k)} + e^{j\Omega_2 n} + \sum_{k=1}^{\infty} a^{k-1}(a - \frac{1}{a}) e^{j\Omega_2 (n-k)}$$

$$y[n] = e^{j\Omega_1 n} + \sum_{k=1}^{\infty} a^k \frac{(a^2-1)}{a^2} e^{j\Omega_1 (n-k)} + e^{j\Omega_2 n} + \sum_{k=1}^{\infty} a^k \frac{(a^2-1)}{a^2} e^{j\Omega_2 (n-k)}$$

$$y[n] = e^{j\Omega_1 n} + \frac{(a^2-1)}{a^2} e^{j\Omega_1 n} \sum_{k=1}^{\infty} a^k e^{-j\Omega_1 k} + e^{j\Omega_2 n} + \frac{(a^2-1)}{a^2} e^{j\Omega_2 n} \sum_{k=1}^{\infty} a^k e^{-j\Omega_2 k}$$

$$y[n] = e^{j\Omega_1 n} + \frac{(a^2-1)}{a^2} e^{j\Omega_1 n} \sum_{k=1}^{\infty} (ae^{-j\Omega_1})^k + e^{j\Omega_2 n} + \frac{(a^2-1)}{a^2} e^{j\Omega_2 n} \sum_{k=1}^{\infty} (ae^{-j\Omega_2})^k$$

We know that $-1 \leq a \leq 1$ if $a = b$ for a stable system. Then we have $|ae^{-j\Omega}| \leq 1$ for any Ω.

$y[n] = e^{j\Omega_1 n} + \frac{(a^2-1)}{a^2} e^{j\Omega_1 n} \frac{ae^{-j\Omega_1}}{1-ae^{-j\Omega_1}} + e^{j\Omega_2 n} + \frac{(a^2-1)}{a^2} e^{j\Omega_2 n} \frac{ae^{-j\Omega_2}}{1-ae^{-j\Omega_2}}$

Question 2

a)

$x(t)$ and $h(t)$ can be written as:

$x(t) = u(t+1) - u(t-1)$

$$h(t) = (1-t)[u(t) - u(t-1)] = \begin{cases} 1-t, & 0 < t < 1 \\ 0, & otherwise \end{cases}$$

Then, $y(t) = x(t) * h(t)$

Before evaluating the convolution, consider the following:

$\hat{y}(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau$

$$= \int_{-\infty}^{t} h(\tau)\,d\tau = \begin{cases} 0, & t < 0 \\ \int_0^t (1-\tau)\,d\tau, & 0 < t < 1 \\ \int_0^1 (1-\tau)\,d\tau, & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ t - \frac{t^2}{2}, & 0 < t < 1 \\ \frac{1}{2}, & t > 1 \end{cases}$$

By the properties of the LTI system:

$$\hat{y}(t+1) = u(t+1) * h(t) = \begin{cases} 0, & t+1 < 0 \\ (t+1) - \frac{(t+1)^2}{2}, & 0 < t+1 < 1 \\ \frac{1}{2}, & t+1 > 1 \end{cases} = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & t > 0 \end{cases}$$

$$\hat{y}(t-1) = u(t-1) * h(t) = \begin{cases} 0, & t-1 < 0 \\ (t-1) - \frac{(t-1)^2}{2}, & 0 < t-1 < 1 \\ \frac{1}{2}, & t-1 > 1 \end{cases} = \begin{cases} 0, & t < 1 \\ (t-1) - \frac{(t-1)^2}{2}, & 1 < t < 2 \\ \frac{1}{2}, & t > 2 \end{cases}$$

$$\text{Therefore, } y(t) = \hat{y}(t+1) - \hat{y}(t-1) = \begin{cases} 0, & t < -1 \\ \frac{1-t^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & 0 < t < 1 \\ \frac{1}{2} - (\frac{t^2-4t+3}{2}), & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

Also, $y(t)$ can be written as:

$$y(t) = (\tfrac{1-t^2}{2}) \cdot u(t+1) + (\tfrac{t^2}{2}) \cdot u(t) - (\tfrac{t^2-4t+3}{2}) \cdot u(t-1) + (\tfrac{t^2-4t+2}{2}) \cdot u(t-2)$$

b)

$w(t) = h(t) * g(t)$

$g(t)$ can be written as: $g(t) = x(t) + x(t-1) - x(t+1)$

Then, $w(t) = h(t) * [x(t) + x(t-1) - x(t+1)]$

By the distributive property of the convolution over addition:

$w(t) = h(t) * x(t) + h(t) * x(t-1) - h(t) * x(t+1)$

$w(t) = y(t) + y(t-1) - y(t+1)$ [by considering part a]

By the time-invarince property of the LTI system:
$y(t-1) = (\frac{2t-t^2}{2}) \cdot u(t) + (\frac{t^2-2t+1}{2}) \cdot u(t-1) - (\frac{t^2-6t+8}{2}) \cdot u(t-2) + (\frac{t^2-6t+7}{2}) \cdot u(t-3)$
$y(t+1) = (\frac{-t^2-2t}{2}) \cdot u(t+2) + (\frac{t^2+2t+1}{2}) \cdot u(t+1) - (\frac{t^2-2t}{2}) \cdot u(t) + (\frac{t^2-2t-1}{2}) \cdot u(t-1)$

As a result:
$w(t) = (\frac{-t^2-2t}{2}) \cdot u(t+2) + (-t^2-t) \cdot u(t+1) + (\frac{t^2}{2}) \cdot u(t) + (\frac{-t^2+4t-1}{2}) \cdot u(t-1) + (t-3) \cdot u(t-2) + (\frac{t^2-6t+7}{2}) \cdot u(t-3)$

Question 3

a)

$x(t) = \ \cdots + \delta(t+2) - 2\delta(t+1) + \delta(t) - 2\delta(t-1) + \delta(t-2) - 2\delta(t-3) + \ \cdots$
It is a perodic signal with fundamental period $T_0 = 2$.

$$a_k = \tfrac{1}{T_0}\int_{t_o}^{t_0+T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{3}{2}} x(t)e^{-jk\pi t}dt$$

$$a_k = \frac{1}{2}\underbrace{\int_{0^-}^{0^+} \delta(t)e^{-jk\pi t}dt}_{e^{-jk\pi(0)}} + \frac{1}{2}\underbrace{\int_{1^-}^{1^+} -2\delta(t-1)e^{-jk\pi t}dt}_{e^{-jk\pi(1)}}$$

$$\boxed{\text{Also, recall that } \int_{-\infty}^{\infty} \delta(t-t_0)x(t) = x(t_0)}$$

$$\Rightarrow a_k = \frac{1}{2} - e^{-jk\pi}$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2} - e^{-jk\pi}\right) e^{jk\pi t}$$

b)

$x(t)$ is a perodic signal with fundamental period, $T_0 = 4$.

$$a_k = \tfrac{1}{4}\int_{-2}^{2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{4}\int_0^1 cos(\frac{\pi t}{2})e^{-jk\frac{\pi}{2}t}dt$$

$$a_k = \frac{1}{4}\int_0^1 \frac{1}{2}\left(e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}\right)e^{-jk\frac{\pi}{2}t}dt = \frac{1}{8}\int_0^1 \left(e^{-j\frac{\pi}{2}(k-1)t} + e^{-j\frac{\pi}{2}(k+1)t}\right)dt$$

$$a_k = -\frac{1}{8}\left(\frac{e^{-j\frac{\pi}{2}(k-1)t}}{j(k-1)\frac{\pi}{2}}\Big|_0^1 + \frac{e^{-j\frac{\pi}{2}(k+1)t}}{j(k+1)\frac{\pi}{2}}\Big|_0^1\right) = \frac{1}{8}\left[\frac{1-e^{-j\frac{\pi}{2}(k-1)}}{j(k-1)\frac{\pi}{2}}\right] + \frac{1}{8}\left[\frac{1-e^{-j\frac{\pi}{2}(k+1)}}{j(k+1)\frac{\pi}{2}}\right]$$

$$a_k = \frac{1}{8}\left[\frac{1-e^{-j\frac{\pi}{2}k}e^{j\frac{\pi}{2}}}{j(k-1)\frac{\pi}{2}}\right] + \frac{1}{8}\left[\frac{1-e^{-j\frac{\pi}{2}k}e^{-j\frac{\pi}{2}}}{j(k+1)\frac{\pi}{2}}\right]$$

$$a_k = \frac{1}{8}\left[\frac{1-je^{-j\frac{\pi}{2}k}}{j(k-1)\frac{\pi}{2}}\right] + \frac{1}{8}\left[\frac{1+je^{-j\frac{\pi}{2}k}}{j(k+1)\frac{\pi}{2}}\right] = \frac{1}{4}\left[\frac{k-je^{-j\frac{\pi}{2}k}}{j(k^2-1)\frac{\pi}{2}}\right] = -\frac{1}{4}\left[\frac{e^{-j\frac{\pi}{2}k}+jk}{(k^2-1)\frac{\pi}{2}}\right]$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} -\frac{1}{4}\left(\frac{e^{-j\frac{\pi}{2}k}+jk}{(k^2-1)\frac{\pi}{2}}\right) e^{jk\frac{\pi}{2}t}$$

c)

$x[n] = (-1)^n + j^n + cos(\frac{2\pi n}{3}) = e^{j\pi n} + e^{j\frac{\pi}{2}n} + cos(\frac{2\pi n}{3})$

Therefore, $x[n]$ can be written as a summation of 3 distinct perodic signals:

$$x[n] = \underbrace{e^{j\pi n}}_{x_1[n]} + \underbrace{e^{j\frac{\pi}{2}n}}_{x_2[n]} + \underbrace{cos\left(\frac{2\pi n}{3}\right)}_{x_3[n]} = x_1[n] + x_2[n] + x_3[n]$$

$x_1[n]$ is a periodic signal with fundamental period $N_0 = 2$.

$x_2[n]$ is a periodic signal with fundamental period $N_0 = 4$.

$x_3[n]$ is a periodic signal with fundamental period $N_0 = 3$.

Then, the fundamental period of $x[n]$ is the least common multiple of the fundamental periods of these three signals, so $N_0 = 12$.

$$a_k = \frac{1}{T_0}\sum_{n=n_0}^{n_0+N_0-1} x[n]e^{-jk\frac{2\pi}{N_0}n} = \frac{1}{12}\sum_{n=-6}^{5} x[n]e^{-jk\frac{\pi}{6}n}$$

d)

$x(t)$ is periodic signal with fundamental period $T_0 = 1$.

$$a_k = \frac{1}{T_0}\int_{t_o}^{t_0+T_0} x(t)e^{-jk\omega_0 t}dt = \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-t}e^{-jk2\pi t}dt = \int_{\frac{-1}{2}}^{\frac{1}{2}} e^{-t(1+jk2\pi)}dt$$

$$= \frac{-1}{1+jk2\pi}\left[e^{\frac{-1}{2}}\underbrace{e^{-jk\pi}}_{cos(k\pi)} - e^{\frac{1}{2}}\underbrace{e^{jk\pi}}_{cos(k\pi)}\right] = \frac{cos(k\pi)}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)$$

$$\Rightarrow x(t) = \sum_{k=-\infty}^{\infty} \frac{cos(k\pi)}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi t}$$

e)

Recall that: $\boxed{y(t) = h(t) * x(t)}$ $\boxed{x(t) * \delta(t) = x(t)}$ $\boxed{x(t) * \delta(t-t_0) = x(t-t_0)}$

$$\Rightarrow y(t) = x(t) * (\delta(t) - \delta(t-\tfrac{1}{2})) \overset{\text{(by distribution property)}}{=} x(t) * \delta(t) - x(t) * \delta(t-\tfrac{1}{2})$$

$$\Rightarrow y(t) = x(t) - x(t-\tfrac{1}{2})$$

$$= \sum_{k=-\infty}^{\infty} \frac{cos(k\pi)}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi t} - \sum_{k=-\infty}^{\infty} \frac{cos(k\pi)}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi(t-\frac{1}{2})}$$
$$= \sum_{k=-\infty}^{\infty} \frac{cos(k\pi)}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi t} - \sum_{k=-\infty}^{\infty} \frac{cos(k\pi)}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi t}\underbrace{e^{-jk\pi}}_{cos(k\pi)}$$
$$= \sum_{k=-\infty}^{\infty} \frac{cos(k\pi)}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi t}\left(1 - cos(k\pi)\right)$$
$$= \sum_{k=-\infty}^{\infty} \frac{cos(k\pi) - \overbrace{cos(k\pi)^2}^{1}}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi t}$$
$$y(t) = \sum_{k=-\infty}^{\infty} \frac{cos(k\pi) - 1}{1+jk2\pi}\left(e^{\frac{1}{2}} - e^{\frac{-1}{2}}\right)e^{jk2\pi t}$$

Question 4

a)

Note that, any arbitrary periodic signal x(t) (or, x[n]) with Fourier series coefficients a_k is real-valued if $a_k^* = a_{-k}$.

For $x_1(t)$: $a_k^* = [(\frac{1}{2})^{-k}]^* = (\frac{1}{2})^{-k} \neq (\frac{1}{2})^{-(-k)} = (\frac{1}{2})^{k} = a_{-k}$
For $x_2(t)$: $a_k^* = (cos(k\pi))^* = cos(k\pi) = cos(-k\pi) = a_{-k}$
For $x_3[n]$: $a_k^* = (jsin(\frac{k\pi}{2}))^* = -jsin(\frac{k\pi}{2}) = jsin(\frac{-k\pi}{2}) = a_{-k}$

Thus, $x_2(t)$ and $x_3[n]$ are real-valued signals.

b)

Any arbitrary periodic signal x(t) (or, x[n]) with Fourier series coefficients a_k is even if a_k is a real-valued and even function.
For $x_1(t)$: $a_k = (\frac{1}{2})^{-k} \Rightarrow$ is a real-valued but not even function, i.e., $(\frac{1}{2})^{-k}\Big|_{k=1} \neq (\frac{1}{2})^{-k}\Big|_{k=-1}$
For $x_2(t)$: $a_k = cos(k\pi) \Rightarrow$ is a real-valued and also an even function since consider that $-1 \leq cos(k\pi) \leq 1$ and $cos(k\pi)\Big|_{k=1} = cos(k\pi)\Big|_{k=-1}$
For $x_3[n]$: $a_k = jsin(\frac{k\pi}{2}) \Rightarrow$ neither real-valued nor even function.

Therefore, only the signal $x_2(t)$ is even.

c)

By the time-shifting property of Continuous Time Fourier Series:

$x_2(t-5) = \sum_{k=-100}^{100} cos(k\pi)e^{-jk\frac{\pi}{5}}e^{jk\frac{2\pi}{50}t}$ and, also note that $cos(k\pi) = \frac{1}{2}(e^{jk\pi} + e^{-jk\pi})$

$\Rightarrow x_2(t-5) = \sum_{k=-100}^{100} \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}})e^{jk\frac{2\pi}{50}t}$

Thus, $x_2(t-5)$ is another periodic signal with Fourier series coefficients $b_k = \frac{1}{2}(e^{jk\frac{4\pi}{5}}+e^{-jk\frac{6\pi}{5}})$.
Also, remember the theorem mentioned above, the signal $x_2(t-5)$ is real if $b_k^* = b_{-k}$.
$b_k^* = (\frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}))^* = \frac{1}{2}(e^{-jk\frac{4\pi}{5}} + e^{jk\frac{6\pi}{5}}) = b_{-k} \Rightarrow$ the signal $x_2(t-5)$ is real.

To check whether the signal $x_2(t-5)$ is even, b_k can be written as:
$b_k = \frac{1}{2}[cos(k\frac{4\pi}{5}) + jsin(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5}) - jsin(k\frac{6\pi}{5})]$
$b_k = \frac{1}{2}[cos(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5})] - \frac{1}{2}j[sin(k\frac{6\pi}{5}) - sin(k\frac{4\pi}{5})]$

Also, note that $\boxed{sina - sinb = 2cos(\frac{a+b}{2})sin(\frac{a-b}{2})}$

Thus, $b_k = \frac{1}{2}[cos(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5})] - j[cos(2\pi k)sin(k\frac{2\pi}{5})]$
$\Rightarrow b_k = \frac{1}{2}(cos(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5})) - jsin(k\frac{2\pi}{5})$
As a result, it can be seen that b_k is not real-valued, so the signal $x_2(t-5)$ is not even.

d)

By differentiation property of Continuous Time Fourier Series:

$\frac{d}{dt}x_2(t) = \sum_{k=-\infty}^{\infty} cos(k\pi)jk\frac{2\pi}{50}e^{jk\frac{2\pi}{50}} \Rightarrow$ is a periodic signal with Fourier series coefficients c_k such that $c_k = cos(k\pi)jk\frac{2\pi}{50}$.

$c_k^* = -jk\frac{2\pi}{50}cos(k\pi)$ and $c_{-k} = j(-k)\frac{2\pi}{50}cos(-k\pi) = -jk\frac{2\pi}{50}cos(k\pi)$

$c_k^* = c_{-k} \Rightarrow \frac{d}{dt}x_2(t)$is real-valued.

Note that, $cos(k\pi)$ take only the values $+1$ or -1, so it is always real-valued. However, $-jk\frac{2\pi}{50}$ is purely imaginary. Therefore, $\frac{d}{dt}x_2(t)$ is not even.

e)

Parseval's Identity: For any continuous-time periodic signal $x(t)$ with the Fourier series coefficients a_k,

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)x^*(t)dt = \sum_{k=-\infty}^{\infty} a_k a_k^* \Rightarrow \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

In this question, $x_1(t)$ is a periodic signal whose fundamental period, $T_0 = 50$ and its Fourier series coefficients $a_k = (\frac{1}{2})^{-k}$. Please also note that, for this signal, the coefficients are defined only for $0 \leq k \leq 100$ since outside this domain all of the coefficients are equal to 0. So, by using Parseval's identity, the average power of $x_1(t)$ in one period:

$$\frac{1}{50}\int_{0}^{50} |x_1(t)|^2 dt = \sum_{k=0}^{100} |a_k|^2 = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^{-2k} = \sum_{k=0}^{100} 4^k = 1 + 4^1 + 4^2 + 4^3 + ... + 4^{100}$$

Question 5

a)

b)