EE301 Homework-2

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Question 1

a)

Let $x[n] = \delta[n]$. Then $y[n] = h[n]$
$h[n] - ah[n-1] = \delta[n] - b\delta[n-1]$
We know that $h[n] = 0$ for $n < 0$ as the system is causal.
For $n = 0$ we have
$h[0] - ah[-1] = \delta[0] - b\delta[-1]$
$h[0] = 1$
For $n = 1$, we have
$h[1] - ah[0] = \delta[1] - b\delta[0]$
$h[1] - a = -b$
$h[1] = a - b$
For $n = 2$, we have
$h[2] - ah[1] = \delta[2] - b\delta[1]$
$h[2] - a(a-b) = 0$
$h[2] = a(a-b)$
For $n = 3$, we have
$h[3] - ah[2] = \delta[3] - b\delta[2]$
$h[3] - a^2(a-b) = 0$
$h[2] = a^2(a-b)$
In general, for $n > 0$, we have $h[n] = a^{n-1}(a-b)$.

$$h[n] = \begin{cases} a^{n-1}(a-b), & n > 0 \\ 1, & n = 0 \\ 0, & n < 0 \end{cases}$$

If the system is stable, then
$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
$1 + \sum_{k=1}^{\infty} |a|^{k-1}|a-b| < \infty$
$1 + |a-b| \sum_{k=0}^{\infty} |a|^k < \infty$
Therefore, $-1 < a < 1$ if the system is stable.

b)

Question 2

a)

$x(t)$ and $h(t)$ can be written as:
$x(t) = u(t+1) - u(t-1)$
$h(t) = (1-t)[u(t) - u(t-1)] = \begin{cases} 1-t, & 0 < t < 1 \\ 0, & otherwise \end{cases}$
Then, $y(t) = x(t) * h(t)$

Before evaluating the convolution, consider the following:
$\hat{y}(t) = u(t) * h(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)\, d\tau$
$= \int_{-\infty}^{t} h(\tau)\, d\tau = \begin{cases} 0, & t < 0 \\ \int_0^t (1-\tau)\, d\tau, & 0 < t < 1 \\ \int_0^1 (1-\tau)\, d\tau, & t > 1 \end{cases} = \begin{cases} 0, & t < 0 \\ t - \frac{t^2}{2}, & 0 < t < 1 \\ \frac{1}{2}, & t > 1 \end{cases}$
By the properties of the LTI system:
$$\hat{y}(t+1) = u(t+1) * h(t) = \begin{cases} 0, & t+1 < 0 \\ (t+1) - \frac{(t+1)^2}{2}, & 0 < t+1 < 1 \\ \frac{1}{2}, & t+1 > 1 \end{cases} = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & t > 0 \end{cases}$$

$$\hat{y}(t-1) = u(t-1) * h(t) = \begin{cases} 0, & t-1 < 0 \\ (t-1) - \frac{(t-1)^2}{2}, & 0 < t-1 < 1 \\ \frac{1}{2}, & t-1 > 1 \end{cases} = \begin{cases} 0, & t < 1 \\ (t-1) - \frac{(t-1)^2}{2}, & 1 < t < 2 \\ \frac{1}{2}, & t > 2 \end{cases}$$

Therefore, $y(t) = \hat{y}(t+1) - \hat{y}(t-1) = \begin{cases} 0, & t < -1 \\ (t+1) - \frac{(t+1)^2}{2}, & -1 < t < 0 \\ \frac{1}{2}, & 0 < t < 1 \\ \frac{1}{2} - [(t-1) - \frac{(t-1)^2}{2}], & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

b)

$w(t) = h(t) * g(t)$
$g(t)$ can be written as: $g(t) = x(t) + x(t-1) - x(t+1)$
Then, $w(t) = h(t) * [x(t) + x(t-1) - x(t+1)]$

By the distributive property of the convolution over addition:
$w(t) = h(t) * x(t) + h(t) * x(t-1) - h(t) * x(t+1)$
$= y(t) + y(t-1) - y(t+1)$ [by considering part a]
Acik formulu yazilacak

Question 3

a)

Recall that in continuous-time systems one can apply the derivative operation to any arbitrary input signal. For example,
$\delta(t) = \frac{d}{dt}u(t) \longrightarrow \int_{-\infty}^{\infty} \delta(t)\, dt$
Also, it can be applied by using the formal definition of the derivative:
$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$
However, in discrete-time systems h cannot go to zero and the minimum value for h can be one. So, in discrete time the derivative expression becomes
$\lim_{h \to 1} \frac{x(n+h) - x(n)}{h} = x[n+1] - x[n]$
Thus, the derivative operation in continuous-time systems is analogous of the difference operation in discrete-time. Hence, we can obtain the impulse response of the difference operation as: $h[n] = \delta[n] - \delta[n-1]$

b)

By convolution, $y[n] = x[n] * h[n] = x[n] * (\delta[n] - \delta[n-1])$
By distributive property of the convolution operation,
$x[n] * (\delta[n] – \delta[n-1]) = (x[n] * \delta[n]) – (x[n] * \delta[n-1])$
$y[n] = x[n] – x[n-1]$

c)

$e^{j\Omega_0 n}(1 - e^{-j\Omega_0}) = (cos(\Omega_0 n) + jsin(\Omega_0 n))(1 - e^{-j\Omega_0})$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - (cos(\Omega_0 n) + jsin(\Omega_0 n))(cos(\Omega_0 n) - jsin(\Omega_0 n))$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - cos(\Omega_0 n)cos(\Omega_0) + jcos(\Omega_0 n)sin(\Omega_0) - jsin(\Omega_0 n)cos(\Omega_0) - sin(\Omega_0 n)sin(\Omega_0)$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - (cos(\Omega_0 n)cos(\Omega_0) + sin(\Omega_0 n)sin(\Omega_0)) - j(sin(\Omega_0 n)cos(\Omega_0) - cos(\Omega_0 n)sin(\Omega_0))$
$= cos(\Omega_0 n) + jsin(\Omega_0 n) - cos(\Omega_0 n - \Omega_0) - jsin(\Omega_0 n - \Omega_0)$
$= cos(\Omega_0 n) - cos(\Omega_0(n-1)) + j(sin(\Omega_0 n) - sin(\Omega_0(n-1)))$

$|y[n]| = [(cos(\Omega_0 n) - cos(\Omega_0(n-1)))^2 + (sin(\Omega_0 n) - sin(\Omega_0(n-1)))^2]^{1/2}$
$= [cos^2(\Omega_0 n) - 2cos(\Omega_0 n)cos(\Omega_o(n-1)) + cos^2(\Omega_0(n-1)) + sin^2(\Omega_0 n) - 2sin(\Omega_0 n)sin(\Omega_o(n-1)) + sin^2(\Omega_0(n-1))]^{1/2}$
$= \sqrt{2 - 2[cos(\Omega_0 n)cos(\Omega_0(n-1)) + sin(\Omega_0 n)sin(\Omega_0(n-1))]}$
$= \sqrt{2 - 2cos(\Omega_0 n - \Omega_0(n-1))}$
$= \sqrt{2 - 2cos(\Omega_0)}$

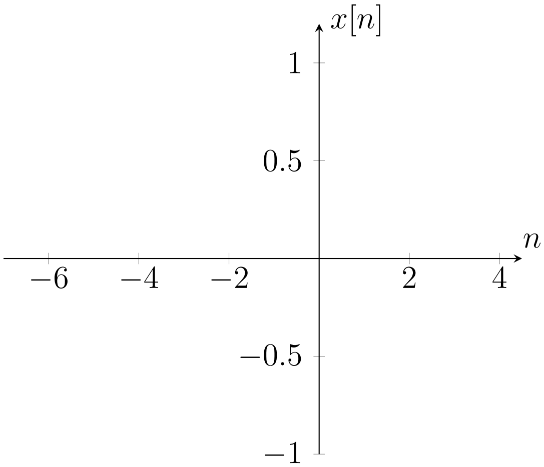
d)

$\Omega_0 = 0 \longrightarrow |y[n]| = \sqrt{2 - 2cos(0)} = 0$
$\Omega_0 = \frac{2\pi}{8} \longrightarrow |y[n]| = \sqrt{2 - 2cos(\frac{2\pi}{8})} = \sqrt{2 - \sqrt{2}}$
$\Omega_0 = \frac{2\pi}{4} \longrightarrow |y[n]| = \sqrt{2 - 2cos(\frac{2\pi}{4})} = \sqrt{2}$
$\Omega_0 = \frac{2\pi}{2} \longrightarrow |y[n]| = \sqrt{2 - 2cos(\frac{2\pi}{2})} = 2$

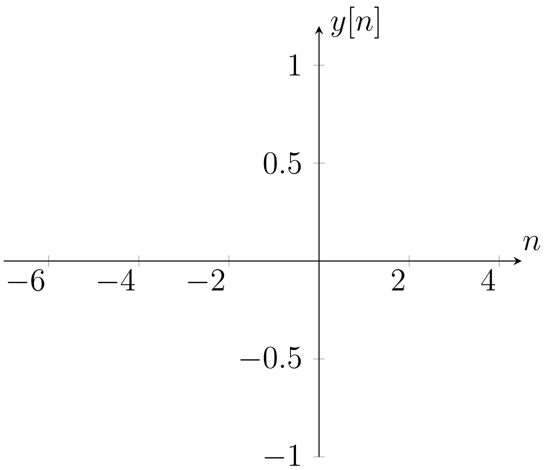
As we have shown in part a), the given discrete-time LTI system is analagous to the derivative operation in continuous-time systems. Also, please remember that the derivative is defined as the rate of change of a function with respect to a variable. Here, as the frequency of the input x[n] is increased, the period becomes smaller and the values of the input function are changed more rapidly. Therefore, the rate of change of a input function is increased as the frequency is increased, and the modulus of the output $|y[n]|$ is getting larger values.

e)

$x[n] = u[n+5] - u[n-2]$ can be plotted as:



We can calculate $y[n]$ for $x[n] = u[n+5] - u[n-2]$ by using the answer in (b) as:
$y[n] = u[n+5] - u[n+2] - (u[n+4] + u[n-3])$
$y[n]$ can be plotted as:



From the graph of $y[n]$, we can say that $x[n]$ has edges at $n = -5$ and $n = 2$.

f)

From (d), we can say that magnitude of the output signal of the system increases as the frequency increases. In (f), $x[n]$ has rapid changes at $n = -5$ and $n = 2$, and as a result,

$y[n]$ has magnitude 1 at $n = -5$ and $n = 2$, and 0 elsewhere.

Question 4

a)

Note that, any arbitrary periodic signal x(t) (or, x[n]) with Fourier series coefficients a_k is real-valued if $a_k^* = a_{-k}$.

For $x_1(t)$: $a_k^* = [(\frac{1}{2})^{-k}]^* = (\frac{1}{2})^{-k} \neq (\frac{1}{2})^{-(-k)} = (\frac{1}{2})^{k} = a_{-k}$
For $x_2(t)$: $a_k^* = (cos(k\pi))^* = cos(k\pi) = cos(-k\pi) = a_{-k}$
For $x_3[n]$: $a_k^* = (jsin(\frac{k\pi}{2}))^* = -jsin(\frac{k\pi}{2}) = jsin(\frac{-k\pi}{2}) = a_{-k}$

Thus, $x_2(t)$ and $x_3[n]$ are real-valued signals.

b)

Any arbitrary periodic signal x(t) (or, x[n]) with Fourier series coefficients a_k is even if a_k is a real-valued and even function.
For $x_1(t)$: $a_k = (\frac{1}{2})^{-k} \Rightarrow$ is a real-valued but not even function, i.e., $(\frac{1}{2})^{-k}\Big|_{k=1} \neq (\frac{1}{2})^{-k}\Big|_{k=-1}$
For $x_2(t)$: $a_k = cos(k\pi) \Rightarrow$ is a real-valued and also an even function since consider that $-1 \leq cos(k\pi) \leq 1$ and $cos(k\pi)\Big|_{k=1} = cos(k\pi)\Big|_{k=-1}$
For $x_3[n]$: $a_k = jsin(\frac{k\pi}{2}) \Rightarrow$ neither real-valued nor even function.

Therefore, only the signal $x_2(t)$ is even.

c)

By the time-shifting property of Continuous Time Fourier Series:

$x_2(t-5) = \sum_{k=-100}^{100} cos(k\pi)e^{-jk\frac{\pi}{5}}e^{jk\frac{2\pi}{50}t}$ and, also note that $cos(k\pi) = \frac{1}{2}(e^{jk\pi} + e^{-jk\pi})$

$\Rightarrow x_2(t-5) = \sum_{k=-100}^{100} \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}})e^{jk\frac{2\pi}{50}t}$

Thus, $x_2(t-5)$ is another periodic signal with Fourier series coefficients $b_k = \frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}})$.
Also, remember the theorem mentioned above, the signal $x_2(t-5)$ is real if $b_k^* = b_{-k}$.
$b_k^* = (\frac{1}{2}(e^{jk\frac{4\pi}{5}} + e^{-jk\frac{6\pi}{5}}))^* = \frac{1}{2}(e^{-jk\frac{4\pi}{5}} + e^{jk\frac{6\pi}{5}}) = b_{-k} \Rightarrow$ the signal $x_2(t-5)$ is real.

To check whether the signal $x_2(t-5)$ is even, b_k can be written as:
$b_k = \frac{1}{2}[cos(k\frac{4\pi}{5}) + jsin(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5}) - jsin(k\frac{6\pi}{5})]$
$b_k = \frac{1}{2}[cos(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5})] - \frac{1}{2}j[sin(k\frac{6\pi}{5}) - sin(k\frac{4\pi}{5})]$

Also, note that $\boxed{sina - sinb = 2cos(\frac{a+b}{2})sin(\frac{a-b}{2})}$

Thus, $b_k = \frac{1}{2}[cos(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5})] - j[cos(2\pi k)sin(k\frac{2\pi}{5})]$
$\Rightarrow b_k = \frac{1}{2}(cos(k\frac{4\pi}{5}) + cos(k\frac{6\pi}{5})) - jsin(k\frac{2\pi}{5})$
As a result, it can be seen that b_k is not real-valued, so the signal $x_2(t-5)$ is not even.

d)

By differentiation property of Continuous Time Fourier Series:
$\frac{d}{dt}x_2(t) = \sum_{k=-\infty}^{\infty} cos(k\pi)jk\frac{2\pi}{50}e^{jk\frac{2\pi}{50}} \Rightarrow$ is a periodic signal with Fourier series coefficients c_k
such that $c_k = cos(k\pi)jk\frac{2\pi}{50}$.
$c_k^* = -jk\frac{2\pi}{50}cos(k\pi)$ and $c_{-k} = j(-k)\frac{2\pi}{50}cos(-k\pi) = -jk\frac{2\pi}{50}cos(k\pi)$
$c_k^* = c_{-k} \Rightarrow \frac{d}{dt}x_2(t)$is real-valued.

Note that, $cos(k\pi)$ take only the values $+1$ or -1, so it is always real-valued. However, $-jk\frac{2\pi}{50}$ is purely imaginary. Therefore, $\frac{d}{dt}x_2(t)$ is not even.

e)

Parseval's Identity: For any continuous-time periodic signal $x(t)$ with the Fourier series coefficients a_k,
$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)x^*(t)dt = \sum_{k=-\infty}^{\infty} a_k a_k^* \Rightarrow \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$

In this question, $x_1(t)$ is a periodic signal whose fundamental period, $T_0 = 50$ and its Fourier series coefficients $a_k = (\frac{1}{2})^{-k}$. Please also note that, for this signal, the coefficients are defined only for $0 \le k \le 100$ since outside this domain all of the coefficients are equal to 0. So, by using Parseval's identity, the average power of $x_1(t)$ in one period:
$\frac{1}{50}\int_0^{50} |x_1(t)|^2 dt = \sum_{k=0}^{100} |a_k|^2 = \sum_{k=0}^{100} \left(\frac{1}{2}\right)^{-2k} = \sum_{k=0}^{100} 4^k = 1 + 4^1 + 4^2 + 4^3 + ... + 4^{100}$

Question 5

a)

In calculating part of the question, we added zero vectors to end and beginning of the main vector to obtain correct results in range of $0 \leq n \leq N-1$, which can be seen in Appendix. Since we manually did that zero adding part, we deviated the result for $n > N$ range. Because of this situation, our calculated result has only N value in range. On the other hand, MATLAB's built-in function, *conv()*, evaluate the convolution so that the size of the output is equal to $N + L - 1$. That's why the output graphs of convolution operation are different by using our function and MATLAB's built-in function as shown in Appendix. In fact, to get the same result that we got by using our function, one can specify an input argument, *same*, to MATLAB's function so that the result gives the central part of the convolution operation. Otherwise, the built-in function gives the full convolution whose length is equal to $N + L - 1$.

b)

The output signal looks more smoother in larger values of L. As we can observe that h[n] impulse response created accordingly to L which directly affects the convolution range. By saying range, we mean that how many $x*h$ multiplication we will use to calculate one element of the output. So, this is why that the output signal gain more elements and more precise values when L gets larger. And also, we can observe that the L value expands the output range by expanding h[n] size. The MATLAB code and the resulting graphs for different values of L can be seen in Appendix.