EE301 Homework-3

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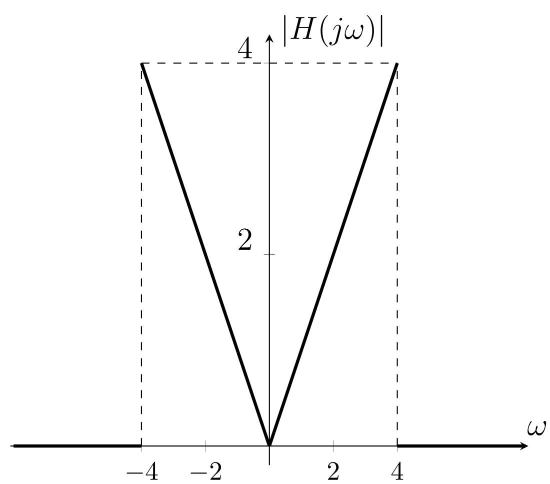
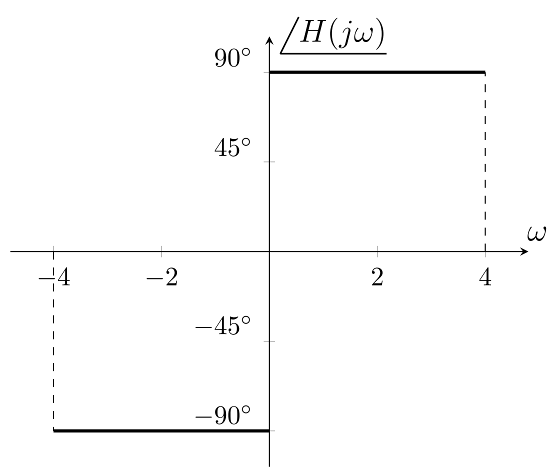
Question 1

a)

The magnitude and phase responses of $H(j\omega)$ can be seen below:

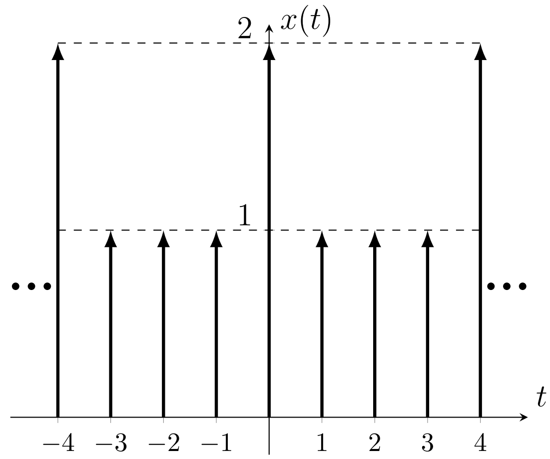




By differentiation property, we know that CTFT of a signal is multiplied by $j\omega$ if the signal is differentiated in time domain. Therefore, this system acts like an differentiator and since it has nonzero values only between $\omega = -4$ and $\omega = 4$, it is named as ideal band-limited differentiator.

b)

x(t) can be plotted as:



x(t) is a periodic signal with fundamental period $T_0 = 4$. Also, $x(t)$ can be written in CTFS representation: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, where $\omega_0 = \dfrac{2\pi}{T_0} = \dfrac{\pi}{2}$.

Let's first calculate the FS cofficients: $a_k = \dfrac{1}{T_0}\displaystyle\int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}dt = \dfrac{1}{4}\displaystyle\int_{0^-}^{4^-} x(t)e^{-jk\frac{\pi}{2}t}$

$\Rightarrow a_0 = \dfrac{5}{4},\ a_1 = \dfrac{1}{4},\ a_{-1} = \dfrac{1}{4},\ a_2 = \dfrac{1}{4},\ a_{-2} = \dfrac{1}{4}$

Also, recall that:

$$e^{jk\omega_0 t} \longrightarrow \boxed{\text{LTI system}} \longrightarrow H(jk\omega_0)e^{jk\omega_0 t}$$

By linearity:

$$\underbrace{\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}}_{x(t)} \longrightarrow \boxed{\text{LTI system}} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}}_{y(t)}$$

Therefore, $y(t) = \displaystyle\sum_{k=-2}^{2} a_k H(jk\omega_0)e^{jk\omega_0 t} = -\frac{1}{4}\pi j e^{-j\pi t} - \frac{1}{8}\pi j e^{-j\frac{\pi}{2}t} + 0 + \frac{1}{8}\pi j e^{j\frac{\pi}{2}t} + \frac{1}{4}\pi j e^{j\pi t}$

$y(t) = -\dfrac{\pi}{2}\left(\dfrac{1}{2j}e^{j\pi t} - \dfrac{1}{2j}e^{-j\pi t}\right) - \dfrac{\pi}{4}\left(\dfrac{1}{2j}e^{j\frac{\pi}{2}t} - \dfrac{1}{2j}e^{-j\frac{\pi}{2}t}\right) = -\dfrac{\pi}{2}sin(\pi t) - \dfrac{\pi}{4}sin(\dfrac{\pi}{2}t)$

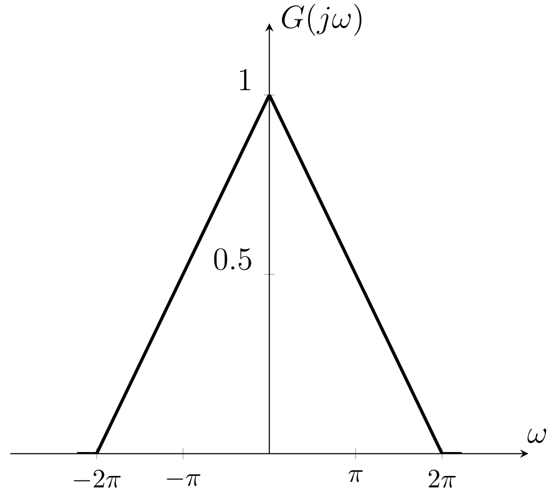
Question 2

a)

We know that $X(j\omega) = \int_{\infty}^{-\infty} x(t)e^{-j\omega t}\, dt$. Then
$\frac{d}{d\omega}X(j\omega) = \frac{d}{d\omega}\int_{\infty}^{-\infty} x(t)e^{-j\omega t}\, dt$
$= \int_{\infty}^{-\infty} \frac{d}{d\omega}(x(t)e^{-j\omega t})\, dt$
$= \int_{\infty}^{-\infty} x(t)(-jt)e^{-j\omega t}\, dt$
$= \mathscr{F}\{x(t)(-jt)\}$
Therefore, $\mathscr{F}^{-1}\{\frac{d}{d\omega}X(j\omega)\} = x(t)(-jt)$

b)

Let $g_1(t) = \frac{sin(\pi t)}{\pi t}$. Then we know that $\mathscr{F}\{g_1(t)\} = G_1(j\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & else \end{cases}$
Since $g(t) = g_1(t)g_1(t)$, by multiplication property, we know that $G(j\omega) = \frac{1}{2\pi}G_1(j\omega)*G_1(j\omega)$
$G(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\theta)G(j(\omega-\theta))\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(j(\omega-\theta))\, d\theta$
Let $\theta' = \omega - \theta$. Then $d\theta' = -d\theta$ and
$G(j\omega) = \frac{1}{2\pi}\int_{\omega+\pi}^{\omega-\pi} G_1(j\theta')\, d(-\theta') = \frac{1}{2\pi}\int_{\omega-\pi}^{\omega+\pi} G_1(j\theta')\, d\theta'$
For $\omega < -2\pi$ or $\omega > 2\pi$, $G(j\omega) = 0$.
For $-2\pi < \omega < 0$
$G(j\omega) = \frac{1}{2\pi}\int_{-\pi}^{\omega+\pi} d\theta' = \frac{\omega+2\pi}{2\pi}$
For $0 < \omega < 2\pi$
$G(j\omega) = \frac{1}{2\pi}\int_{\omega-\pi}^{\pi} d\theta' = \frac{2\pi-\omega}{2\pi}$
Therefore, $G(j\omega) = \begin{cases} \frac{\omega+2\pi}{2\pi}, & -2\pi < \omega < 0 \\ \frac{2\pi-\omega}{2\pi}, & 0 < \omega < 2\pi \\ 0, & else \end{cases}$, and $G(j\omega)$ is plotted below



c)

i)

$H(j\omega) = \int_{-\infty}^{\infty} f^*(-t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}(f(-t)e^{j\omega t})^*\,dt$
Let $t' = -t$. Then $dt' = -dt$ and
$H(j\omega) = \int_{\infty}^{-\infty}(f(t')e^{-j\omega t'})^*\,d(-t') = (\int_{-\infty}^{\infty} f(t')e^{-j\omega t'}\,d(t'))^* = F^*(j\omega)$

ii)

Let $\mathscr{F}\{y(t)\} = Y(j\omega)$. Then $Y(j\omega) = F^*(j\omega)F(j\omega) = |F(j\omega)|^2$
$Y^*(j\omega) = |F(j\omega)|$, therefore we can say that $Y(j\omega)$ is even and $y(t)$ is real valued.
$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}\,d\omega$
$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 e^{j\omega t}\,d\omega$
$= \frac{1}{2\pi}(\int_{-\infty}^{\infty} |F(j\omega)|^2 cos(\omega t)\,d\omega + j\int_{-\infty}^{\infty} |F(j\omega)|^2 sin(\omega t)\,d\omega)$
Since $y(t)$ is real valued, $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 cos(\omega t)\,d\omega$.
$y(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\,d\omega$
$|y(t)| < \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2 |cos(\omega t)|\,d\omega < y(0)$
$|y(t)| < |y(0)|$ for all t.

Question 3

a)

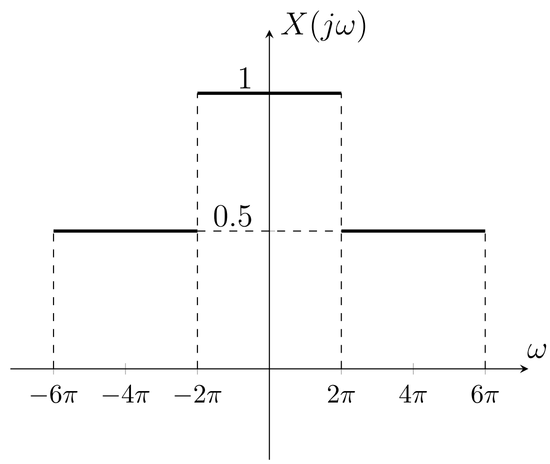
i)

$x(t) = \frac{sin(4\pi t)}{\pi t} cos(2\pi t) = \frac{4sin(4\pi t)}{4\pi t} cos(2\pi t) = x_1(t)x_2(t) \Big($ where $x_1(t) = \frac{4sin(4\pi t)}{4\pi t}$, $x_2(t) = cos(2\pi t)\Big)$

Recall that: $\mathscr{F}\{rect(\theta)\} = \frac{sin(\omega/2)}{\omega/2}$
By duality property of CTFT: $\frac{sin(t/2)}{t/2} \longleftrightarrow 2\pi rect(-\omega) = 2\pi rect(\omega)$
By scaling property of CTFT: $\frac{sin(4\pi t)}{4\pi t} \longleftrightarrow \frac{rect(\frac{\omega}{8\pi})}{4}$
By linearity: $\frac{4sin(4\pi t)}{4\pi t} \longleftrightarrow rect(\frac{\omega}{8\pi}) = X_1(j\omega)$

$x_2(t) = cos(2\pi t) = \frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}$
$\mathscr{F}\{\frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}\} \longleftrightarrow \pi[\delta(\omega + 2\pi) + \delta(\omega - 2\pi)] = X_2(j\omega)$

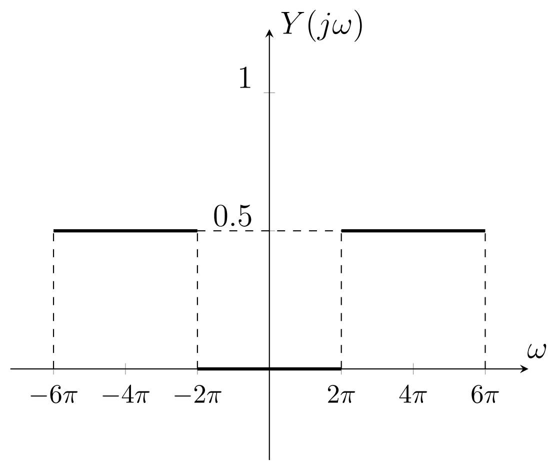
By modulation property of CTFT:
$x(t) = x_1(t)x_2(t) \longleftrightarrow X(j\omega) = \frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
$X(j\omega) = \frac{1}{2\pi}rect(\frac{\omega}{8\pi}) * \pi[\delta(\omega+2\pi) + \delta(\omega-2\pi)] = \frac{1}{2}[rect(\frac{\omega}{8\pi}) * \delta(\omega+2\pi) + rect(\frac{\omega}{8\pi}) * \delta(\omega-2\pi)]$
$X(j\omega) = \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi})$



ii)

$y(t) = h(t) * x(t)$
By convolution property of CTFT: $y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$
$Y(j\omega) = \left(1 - rect(\frac{\omega}{4\pi})\right)\left(\frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi})\right)$
$Y(j\omega) = \frac{1}{2}rect(\frac{\omega-2\pi}{8\pi}) + \frac{1}{2}rect(\frac{\omega+2\pi}{8\pi}) - \frac{1}{2}\left(rect(\frac{\omega}{4\pi})rect(\frac{\omega-2\pi}{8\pi}) + rect(\frac{\omega}{4\pi})rect(\frac{\omega+2\pi}{8\pi})\right)$

$Y(j\omega) = \frac{1}{2}\left(rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})\right)$



b)

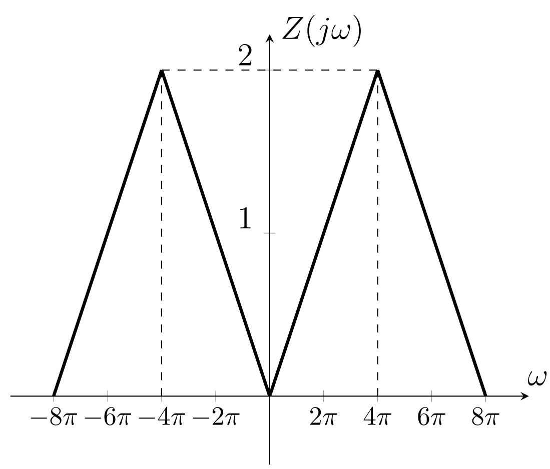
i)

By modulation property of CTFT:

$z(t) \longleftrightarrow Z(j\omega) = \frac{1}{2\pi}Y(j\omega) * \mathscr{F}\{\frac{sin(2\pi t)}{\pi t}\} = \frac{1}{2\pi}Y(j\omega) * rect(\frac{\omega}{4\pi})$

$\Rightarrow Z(j\omega) = \frac{1}{2\pi}\left(rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})\right) * rect(\frac{\omega}{4\pi})$

$$\Rightarrow Z(j\omega) = \begin{cases} \frac{8\pi-\omega}{2\pi}, & 4\pi < \omega < 8\pi \\ \frac{\omega}{2\pi}, & 0 < \omega \leq 4\pi \\ \frac{-\omega}{2\pi}, & -4\pi < \omega < 0 \\ \frac{8\pi+\omega}{2\pi}, & -8\pi < \omega \leq -4\pi \end{cases}$$



ii)

$y(t) \longleftrightarrow Y(j\omega) = rect(\frac{\omega+4\pi}{4\pi}) + rect(\frac{\omega-4\pi}{4\pi})$
$Y(j\omega) = \frac{1}{\pi}rect(\frac{\omega}{4\pi}) * \pi[\delta(\omega-4\pi)+\delta(\omega+4\pi)]$
By modulation property of CTFT:
$\mathscr{F}^{-1}\{Y(j\omega)\} = y(t) = \frac{2sin(2\pi t)}{\pi t}cos(4\pi t)$

Question 4

a)

i)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \underbrace{\delta[n]e^{-j\Omega n}}_{\delta[n]e^{-j\Omega 0}} = \sum_{n=-\infty}^{\infty} \delta[n] = 1$$

ii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} (2\delta[n-3]-\delta[n-10])e^{-j\Omega n} \overset{\text{(by linearity)}}{=} 2\sum_{n=-\infty}^{\infty} \delta[n-3]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} \delta[n-10]e^{-j\Omega n}$$

By time-shift property of DTFT:
$X(e^{j\Omega}) = 2e^{-j3\Omega} - e^{-j10\Omega}$

iii)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = \sum_{n=1}^{4} \frac{1}{n^2}e^{-j\Omega n} = e^{-j\Omega} + \frac{1}{4}e^{-j2\Omega} + \frac{1}{9}e^{-j3\Omega} + \frac{1}{16}e^{-j4\Omega}$$

iv)

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \left(\left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]\right) e^{-j\Omega n}$$

$$\overset{\text{(by linearity)}}{=} \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n]e^{-j\Omega n} - \sum_{n=-\infty}^{\infty} 3^n u[-n-1]e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n - \sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}e^{-j\Omega}\right)^n = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} \quad \left(\text{since } \left|\frac{1}{2}e^{-j\Omega}\right| = \frac{1}{2} < 1, \text{so the expression is convergent}\right)$$

Let $m = -n$: $\displaystyle\sum_{n=-\infty}^{-1} 3^n e^{-j\Omega n} = \sum_{m=1}^{\infty} 3^{-m}e^{j\Omega m} = \sum_{m=1}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m = \underbrace{\left[\sum_{m=0}^{\infty} \left(\frac{1}{3}e^{j\Omega}\right)^m\right]}_{\frac{1}{1-\frac{1}{3}e^{j\Omega}}} - 1$

$$\Rightarrow X(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$$

v)

Say that, $\hat{x}[n] = \left(\frac{1}{2}\right)^n u[n] - 3^n u[-n-1]$ and $\mathscr{F}\{\hat{x}[n]\} = \hat{X}(e^{j\Omega}) = \frac{1}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{1}{1-\frac{1}{3}e^{j\Omega}} - 1\right)$
By time-shift property of DTFT:
$x[n] = \hat{x}[n-7] \longleftrightarrow X(e^{j\Omega}) = \hat{X}(e^{j\Omega})e^{-j7\Omega}$

$$\Rightarrow X(e^{j\Omega}) = \frac{e^{-j7\Omega}}{1-\frac{1}{2}e^{-j\Omega}} - \left(\frac{e^{-j7\Omega}}{1-\frac{1}{3}e^{j\Omega}} - e^{-j7\Omega}\right)$$

vi)

Let x[n] be a periodic signal with fundamental period N. Then,
$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} \sum_{k=k_0}^{k_0+N-1} a_k 2\pi\delta(\Omega - k\frac{2\pi}{N} - 2\pi m)$$
Note that for this signal if we consider the interval $0 \le \Omega < 2\pi$, DTFT of the signal will be written as:
$$\mathscr{F}\{x[n]\} = X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3})$$
First, find the DTFS coefficients of x[n]: $a_k = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-jk\frac{2\pi}{3}n}$
$$\Rightarrow a_0 = \frac{1}{3}\sum_{n=0}^{2} x[n] = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$
$$\Rightarrow a_1 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{2\pi}{3}n} = \frac{1}{3}e^{-j\frac{2\pi}{3}} + \frac{1}{3}e^{-j\frac{4\pi}{3}} = \frac{-1}{3}$$
$$\Rightarrow a_2 = \frac{1}{3}\sum_{n=0}^{2} x[n]e^{-j\frac{4\pi}{3}n} = \frac{1}{3}e^{-j\frac{4\pi}{3}} + \frac{1}{3}e^{-j\frac{8\pi}{3}} = \frac{-1}{3}$$
$$X(e^{j\Omega}) = \sum_{k=0}^{2} a_k 2\pi\delta(\Omega - k\frac{2\pi}{3}) = a_0 2\pi\delta(\Omega) + a_1 2\pi\delta(\Omega - \frac{2\pi}{3}) + a_2 2\pi\delta(\Omega - \frac{4\pi}{3})$$
$$X(e^{j\Omega}) = \frac{2\pi}{3}\left(2\delta(\Omega) - \delta(\Omega - \frac{2\pi}{3}) - \delta(\Omega - \frac{4\pi}{3})\right)$$

b)

i)

$$x[n] = \tfrac{1}{2\pi}\int_{\Omega_0}^{\Omega_0+2\pi} \underbrace{X(e^{j\Omega})}_{=1} e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega n} d\Omega = \frac{1}{2\pi j n}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

$$x[n] = \frac{1}{\pi n}\frac{1}{2j}\left(e^{j\pi n} - e^{-j\pi n}\right) = \frac{sin(\pi n)}{\pi n}$$

$$\boxed{\text{Recall that, } sinc(t) = \begin{cases} \frac{sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}}$$

Here, n is an integer. So, $\dfrac{sin(\pi n)}{\pi n} = \begin{cases} 1, & n = 0 \\ 0, & otherwise \end{cases} \Rightarrow x[n] = \delta[n]$

ii)

We know that, $\mathscr{F}\{1\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$

By frequency-shift property of DTFT:

$$\mathscr{F}\{e^{j\Omega_0 n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m)$$

$$\Rightarrow \mathscr{F}\{e^{j\frac{\pi}{3}n}\} = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m) = 2\pi X(e^{j\Omega})$$

$$\Rightarrow \mathscr{F}^{-1}\{\sum_{m=-\infty}^{\infty} \delta(\Omega - \frac{\pi}{3} - 2\pi m)\} = \frac{e^{j\frac{\pi}{3}n}}{2\pi} = x[n]$$

iii)

$$x[n] = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})e^{j\Omega n} d\Omega = \frac{1}{2\pi} \underbrace{\int_{-\pi}^{\frac{-\pi}{2}} e^{j\Omega n} d\Omega}_{= \int_{\frac{\pi}{2}}^{\pi} e^{-j\Omega n} d\Omega} + \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n} d\Omega = \frac{1}{2\pi}\int_{\frac{\pi}{2}}^{\pi} e^{j\Omega n} + e^{-j\Omega n} d\Omega$$

$$x[n] = \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} cos(\Omega n) = \frac{1}{\pi n} sin(\Omega n)\Big|_{\frac{\pi}{2}}^{\pi} = \frac{sin(\pi n)}{\pi n} - \frac{sin(\frac{\pi}{2}n)}{\pi n} = \frac{sin(\pi n)}{\pi n} - \frac{1}{2}\frac{sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n}$$

$$x[n] = sinc(n) - \frac{1}{2}sinc(\frac{n}{2})$$

iv)

By frequency-shift property of DTFT:

$x[n] \longleftrightarrow X(e^{j\Omega})$

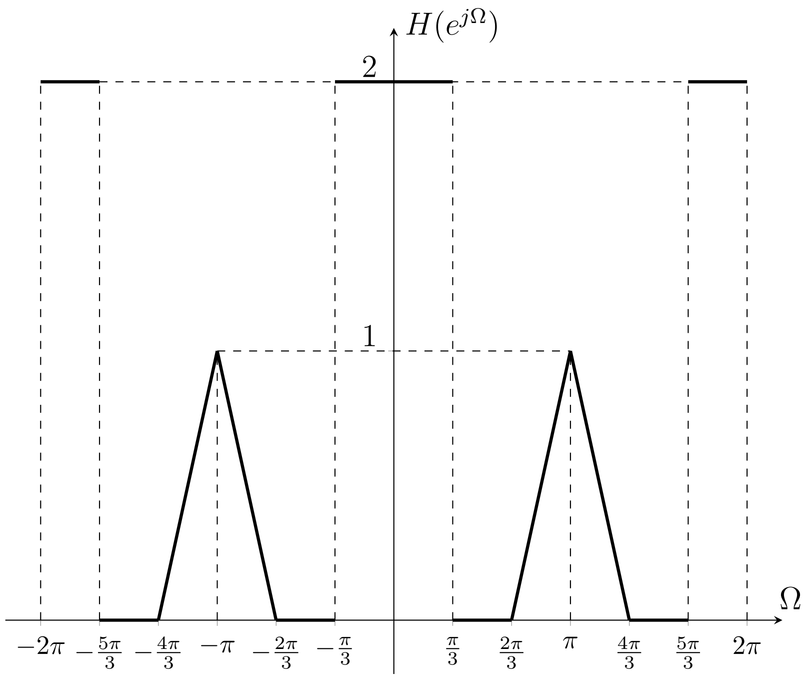
$x[n]e^{j\frac{\pi}{4}n} \longleftrightarrow X(e^{j(\Omega - \frac{\pi}{4})}) = Y(e^{j\Omega})$

Therefore, $y[n] = x[n]e^{j\frac{\pi}{4}n} = sinc(n)e^{j\frac{\pi}{4}n} - \frac{1}{2}sinc(\frac{n}{2})e^{j\frac{\pi}{4}n}$

Question 5

a)

Recall that, $Y(e^{j\Omega}) = X(e^{j\Omega})H(e^{j\Omega}) \Rightarrow H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})}$



b)

i)

$h[n]$ is real by symmetry property of DTFT since $H(e^{-j\Omega}) = H^*(e^{j\Omega})$.

ii)

$h[n]$ is even by symmetry property of DTFT since $H(e^{j\Omega})$ is real-valued and even as it can be seen from its graph, above.

iii)

$h[n]$ has no odd symmetry since its real-valued and even signal.

iv)

The impulse response $h[n]$ has finite energy because for $H(e^{j\Omega})$ to exist, $h[n]$ must be absolutely summable or have finite energy. Thus, since $H(e^{j\Omega})$ exists, $h[n]$ has finite energy.

v)

$h[n]$ is perodic since its DTFT $H(e^{j\Omega})$ is perodic with period 2π.

c)

i)

$h[n]$ can be obtained from $H(e^{j\Omega})$ by the inverse DTFT integral as follows:
$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\Omega})e^{j\Omega n}d\Omega$
$h[0] = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} H(e^{j\Omega})d\Omega = \dfrac{5}{6}$

ii)

By Parseval's identity: $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\Omega})|^2 d\Omega$

$\displaystyle\sum_{n=-\infty}^{\infty} |h[n]|^2 = \sum_{n=-\infty}^{\infty} (h[n])^2,$ since $h[n]$ is real-valued.

Also, $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]|^2 = (h[0])^2 + 2\sum_{n=1}^{\infty} |h[n]|^2$ since $h[n]$ is even function.

$$|H(e^{j\Omega})|^2 = \begin{cases} (\frac{-3\Omega}{\pi} - 2)^2, & -\pi < \Omega < -\frac{2\pi}{3} \\ 0, & -\frac{2\pi}{3} < \Omega < -\frac{\pi}{3} \\ 2^2, & -\frac{\pi}{3} < \Omega < 0 \\ 2^2, & 0 < \Omega < \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < \Omega < \frac{2\pi}{3} \\ (\frac{3\Omega}{\pi} - 2)^2, & \frac{2\pi}{3} < \Omega < \pi \end{cases}$$

$$\int_{-\pi}^{\pi} |H(e^{j\Omega})|^2 d\Omega = \left(\frac{3\Omega^3}{\pi^2} + \frac{6\Omega^2}{\pi} + 4\Omega\right)\Big|_{-\pi}^{-\frac{2\pi}{3}} + (4\Omega)\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} + \left(\frac{3\Omega^3}{\pi^2} - \frac{6\Omega^2}{\pi} + 4\Omega\right)\Big|_{\frac{2\pi}{3}}^{\pi} = \frac{26\pi}{9}$$

$$\sum_{n=-\infty}^{\infty} (h[n])^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\Omega})|^2 d\Omega = \frac{13}{9}$$

iii)

$$(h[0])^2 + 2\sum_{n=1}^{\infty}(h[n])^2 = \sum_{n=-\infty}^{\infty}(h[n])^2$$
$$\Rightarrow \sum_{n=1}^{\infty}(h[n])^2 = \frac{1}{2}(\frac{13}{9} - \frac{25}{36}) = \frac{3}{8}$$

d)

i)

The given LTI system is stable since $h[n]$ is absolutely summable as it can be seen in part c). Also, recall that for a stable LTI system:

$$e^{j\Omega n} \longrightarrow \boxed{\text{LTI system}} \longrightarrow H(e^{j\Omega})e^{j\Omega n}$$

$w[n] = 1 = e^{j0n} \Rightarrow v[n] = H(e^{j0})e^{j0n} = 2$

ii)

$w[n] = e^{j\frac{\pi}{6}n} \Rightarrow v[n] = H(e^{j\frac{\pi}{6}})e^{j\frac{\pi}{6}n} = 2e^{j\frac{\pi}{6}n}$

iii)

$w[n] = 3cos(\frac{\pi}{6}n) + 5sin(\frac{\pi}{2}n) = \frac{3}{2}(e^{j\frac{\pi}{6}n} + e^{-j\frac{\pi}{6}n}) + \frac{5}{2j}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n})$
$v[n] = \frac{3}{2}H(e^{j\frac{\pi}{6}})e^{j\frac{\pi}{6}n} + \frac{3}{2}H(e^{-j\frac{\pi}{6}})e^{-j\frac{\pi}{6}n} + \frac{5}{2j}H(e^{j\frac{\pi}{2}})e^{j\frac{\pi}{2}n} - \frac{5}{2j}H(e^{-j\frac{\pi}{2}})e^{-j\frac{\pi}{2}n}$
$v[n] = 3(e^{j\frac{\pi}{6}n} + e^{-j\frac{\pi}{6}n}) + 0 = 6cos(\frac{\pi}{6}n)$

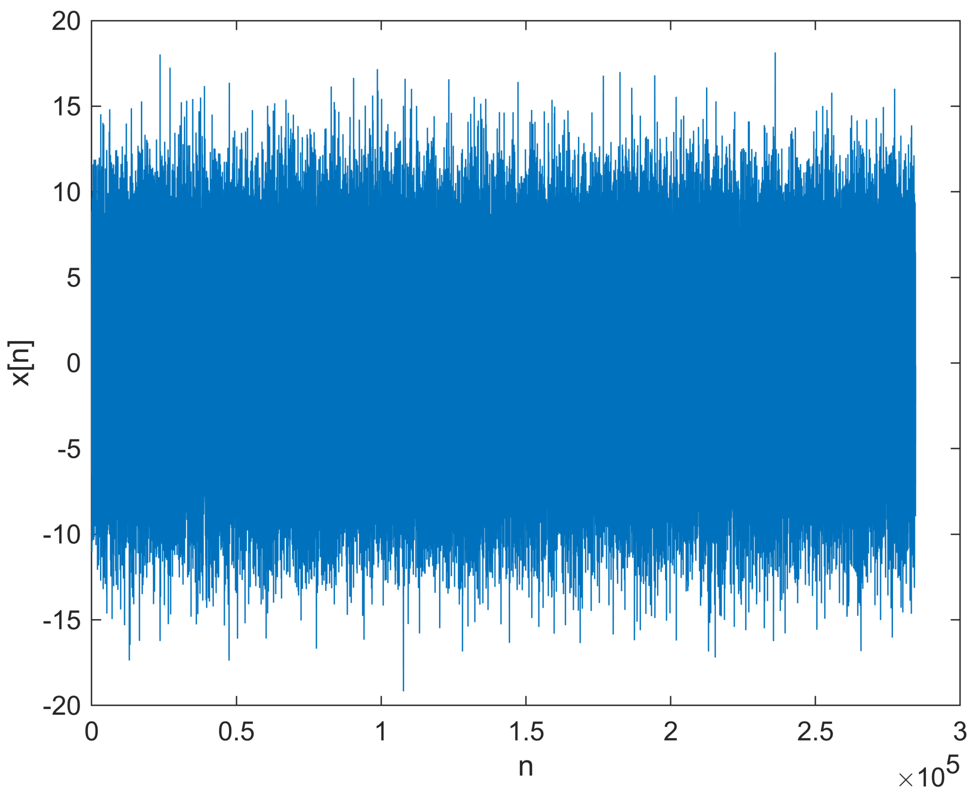
Question-6

Step-a: Load the x.mat and sound the input signal.

```
close all;
clear;
load('x.mat');


N = length(x);
sound(x,44100);

figure; plot(x);
xlabel('n');
ylabel('x[n]');
```



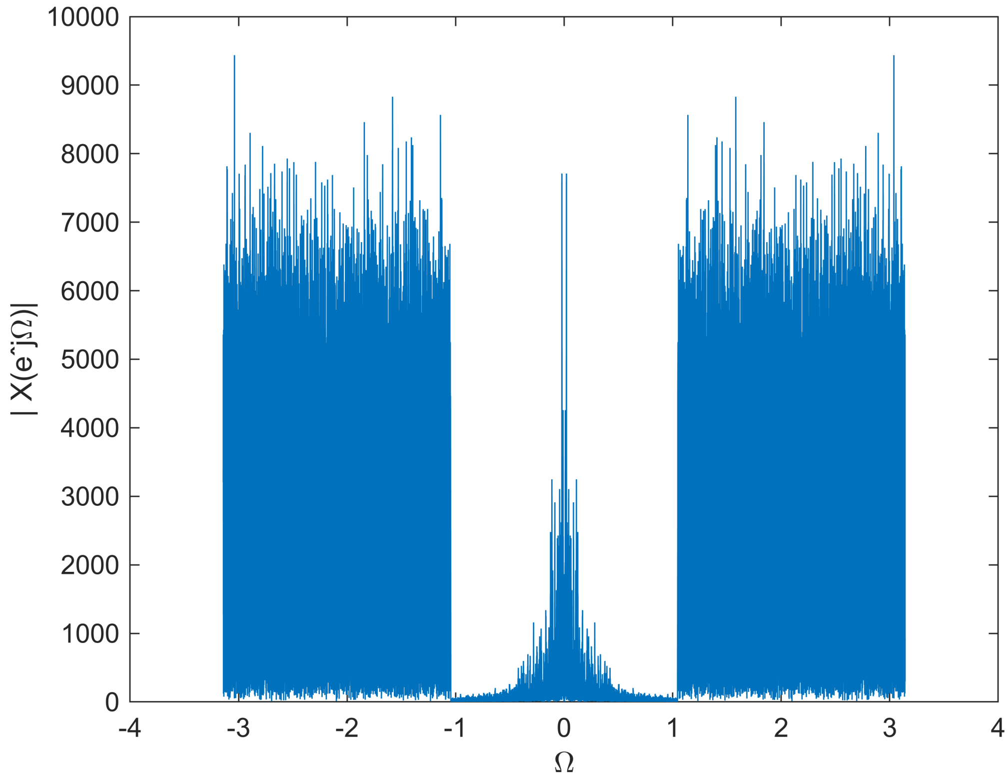
By listening the x signal, it can be said that the original clean sound cannot be determined by just hearing this noisy sound.

Step-b: Apply the Fourier Tranform commands and plot the input signal's FT.

```
X=fftshift(fft(x));
Omega=linspace(-pi,pi,N+1);
Omega=Omega(2:end);

figure; plot(Omega,abs(X));
```

```
xlabel('\Omega');
ylabel('| X(e^{j\Omega})|');
```

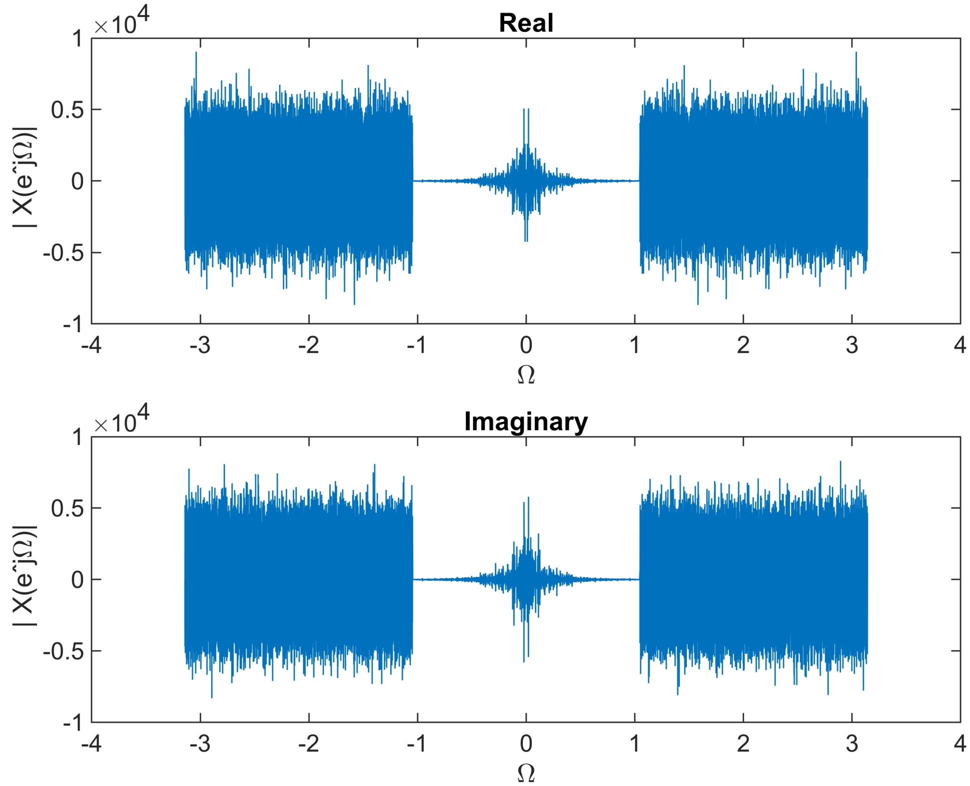


We know from the informtion given in the question, noise component of the signal has a strong magnitude and a flat spectrum that covers high-frequencies. Also, that means the clean sound has lower frequencies. It can be seen from the plot that there is a certain change both in Omega = 1 & -1. These 1 & -1 points have lower frequencies compared in one period which lead us to take B=1.

Real & Imaginary Part plotting:

```
figure;
subplot(2,1,1)
plot(Omega,real(X));
xlabel('\Omega'); ylabel(' | X(e^{j\Omega})|');
title('Real');

subplot(2,1,2);
plot(Omega,imag(X));
xlabel('\Omega'); ylabel(' | X(e^{j\Omega})|');
title('Imaginary');
```



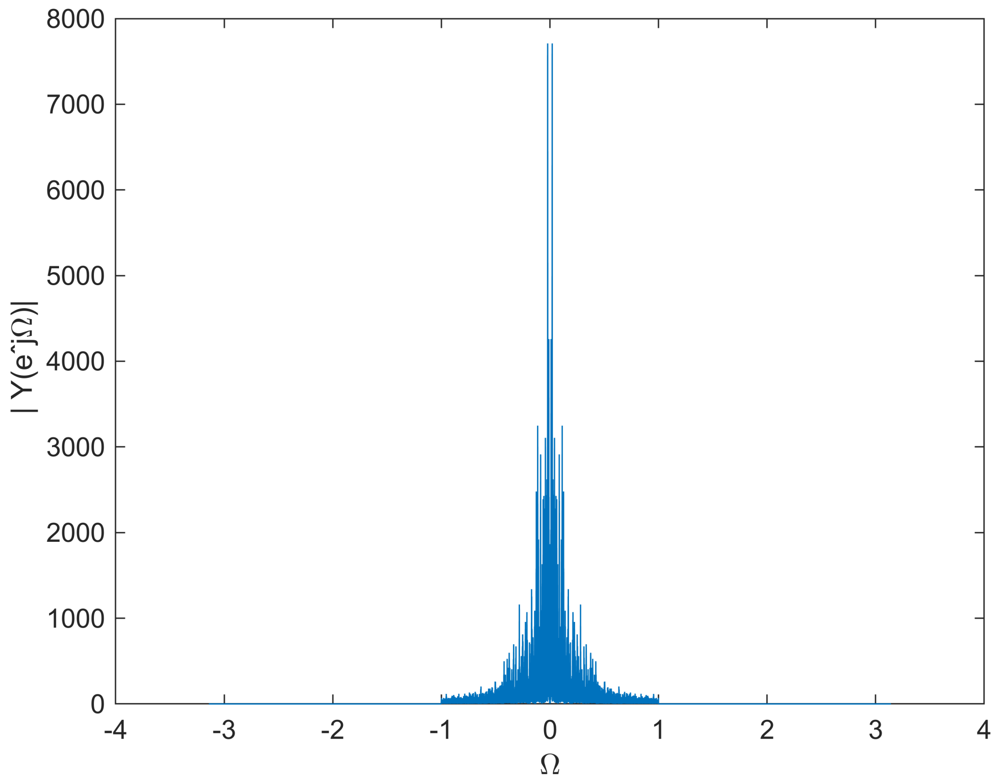
Step-c: Obtain the low-pass filter.

```
H = zeros(1,N);
B = 1;
index1 = find(Omega >-(B-1e-6),1);
index2 = find(Omega > (B-1e-6),1);
H(index1:index2) = 1;
```

In this part H(exp(j*omega)) works as a low-pass filter. So, by selecting |B|=1 we will obtain the above function.

Step-d: Determine Y(exp(j*omega)) by convolution property Y(exp(j*omega)) = H(exp(j*omega))*X(exp(j*omega));

```
Y=H.*X;
figure; plot(Omega,abs(Y));
xlabel('\Omega');
ylabel(' | Y(e^{j\Omega})|');
```



As asked in the question, we assigned different B values to obtain different output signals as well.

```
H_different = zeros(3,N);
B_different = [0.25 , 1.7 , 2.5];
for i=1:3

    index1_D(i) = find(Omega >-B_different(i),1);
    index2_D(i) = find(Omega > B_different(i),1);

    H_different(i,index1_D(i):index2_D(i)) = 1;
    Y_different(i,:) = H_different(i,:) .* X;

end
```

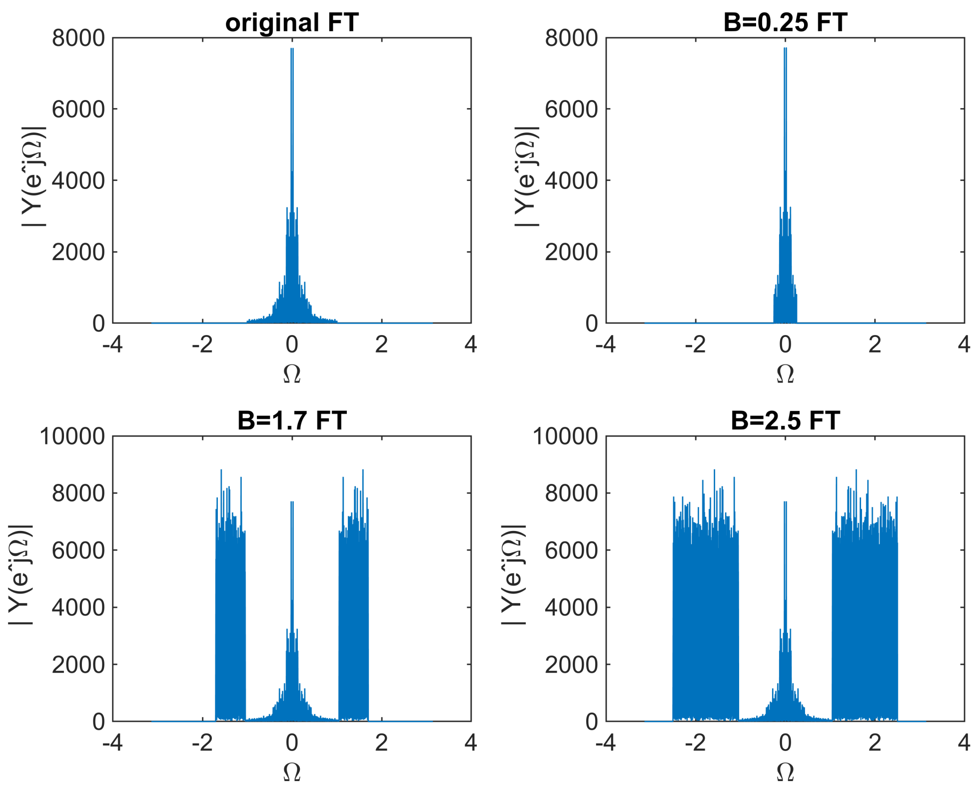
Plot of the output signals' Fourier Transforms is given below.

```
figure;
subplot(2,2,1);
plot(Omega,abs(Y));
xlabel('\Omega');
ylabel(' | Y(e^{j\Omega})|'); title('original FT');

subplot(2,2,2);
plot(Omega,abs(Y_different(1,:)));
xlabel('\Omega');
ylabel(' | Y(e^{j\Omega})|'); title('B=0.25 FT');
```

```
subplot(2,2,3);
plot(Omega,abs(Y_different(2,:)));
xlabel('\Omega');
ylabel(' | Y(e^{j\Omega})|'); title('B=1.7 FT');

subplot(2,2,4);
plot(Omega,abs(Y_different(3,:)));
xlabel('\Omega');
ylabel(' | Y(e^{j\Omega})|'); title('B=2.5 FT');
```



Part-e: Calculate the output signal by using its fourier transform.

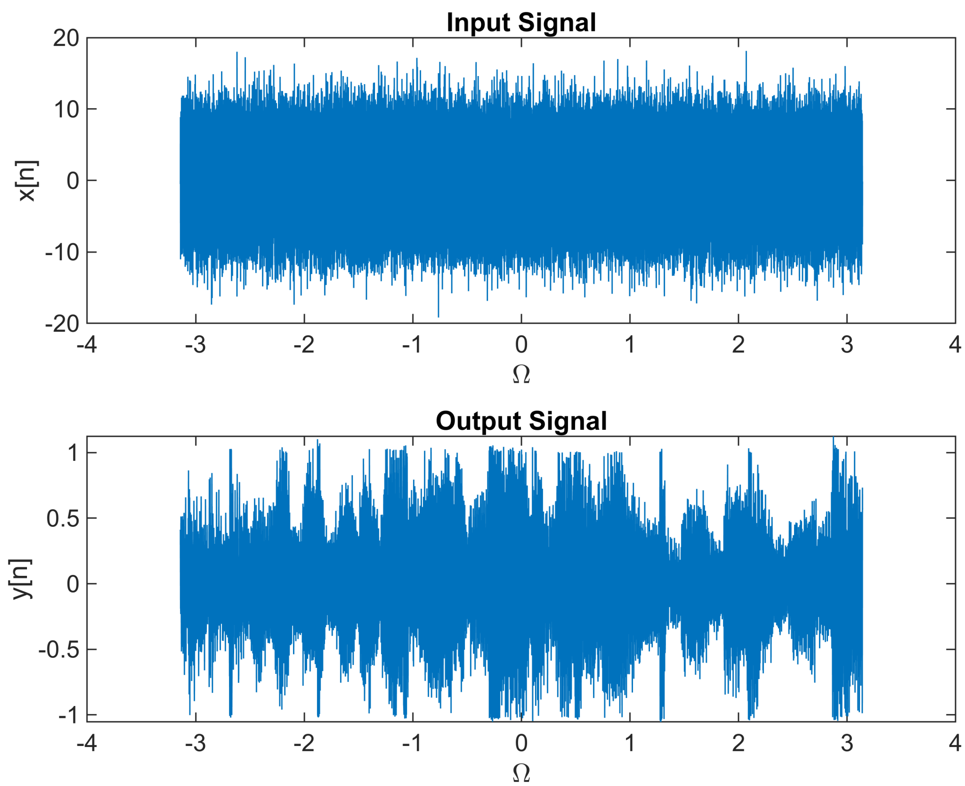
```
y=ifft(ifftshift(Y));

y_different(1,:) = ifft(ifftshift(Y_different(1,:)));
y_different(2,:) = ifft(ifftshift(Y_different(2,:)));
y_different(3,:) = ifft(ifftshift(Y_different(3,:)));
```

Output signal vs Input signal plot is given below as well.

```
figure;
subplot(2,1,1);
plot(Omega,x);
xlabel('\Omega');
ylabel('x[n]'); title('Input Signal');
```

```
subplot(2,1,2);
plot(Omega,real(y));
xlabel('\Omega');
ylabel('y[n]'); title('Output Signal');
```



Part-f: Listening part.

```
sound(real(y),44100);
```

Obviously, the song is "Hotel California" :)

Since we can clearly hear the song hidden in the noisy signal, it is possible to say that we have selected the correct B value.

In the below block, there are sounds for the different B values.

```
sound(real(y_different(1,:)),44100);
sound(real(y_different(2,:)),44100);
sound(real(y_different(3,:)),44100);
```

For the lower frequency than B has still clear sound, it can be easily understandible. However, there is a problem with the quality of the sound, the noise might be gone but there is still a corruption in the signal.

And for the higher frequency values, it becomes harder to understand by drifting apart the exact B value.