EE301 Homework-4

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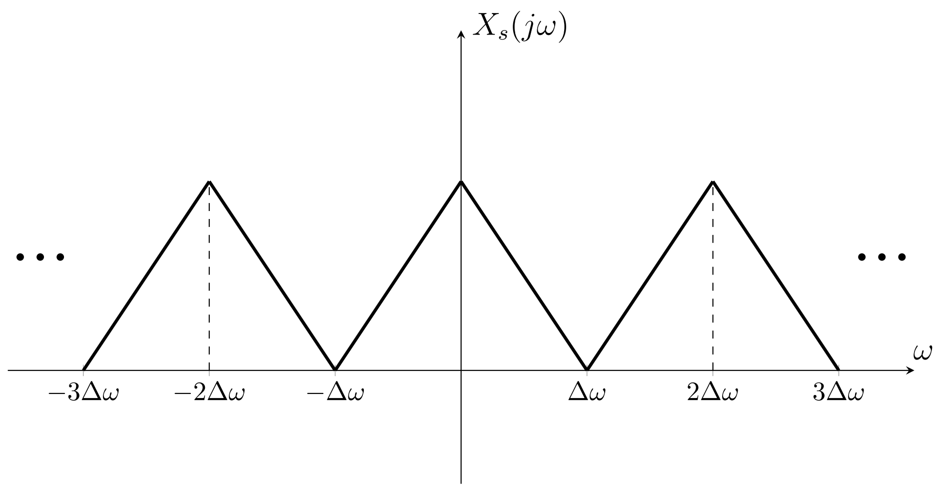
Question 1

a)

$x_s(t) = x(t)s(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$
where T_s is called the sampling period.
By the modulation property of CTFT: $X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * S(j\omega)$

s(t) is a periodic signal, therefore we should first find its CTFS representation:
$s(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}, \omega_s = \frac{2\pi}{T_s}$
$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \underbrace{s(t)}_{\delta(t)} e^{-jk\omega_s t} dt \Rightarrow a_k = \frac{1}{T_s}$
$s(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} \longrightarrow S(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$
$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} X(j(\omega - k\omega_s))$

From the Nyquist sampling theorem, the inequality $\omega_s \geq 2\Delta\omega$ should be satisfied so that the shifted replicas of $X(j\omega)$ do not overlap. Thus, it yields perfect reconstruction of the signal $x(t)$ from $x_s(t)$ with no aliasing. So, the minumum sampling rate, $\omega_s = 2\Delta\omega$. The Fourier transform of the sampling system output, $X_s(j\omega)$ can be seen below.

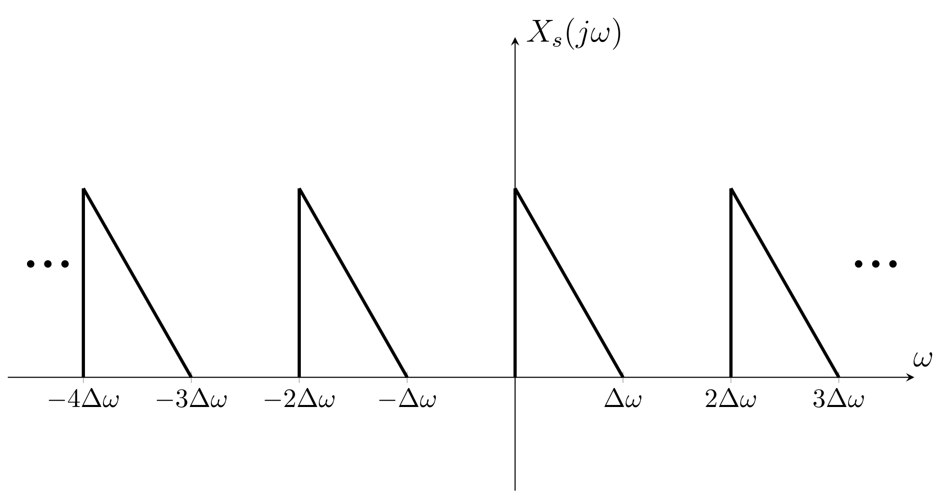


b)

$X(j\omega)$ is not symmetric with respect to y-axis, thus from the symmetry property of CTFT it can be concluded that $x(t)$ is a complex-valued signal.

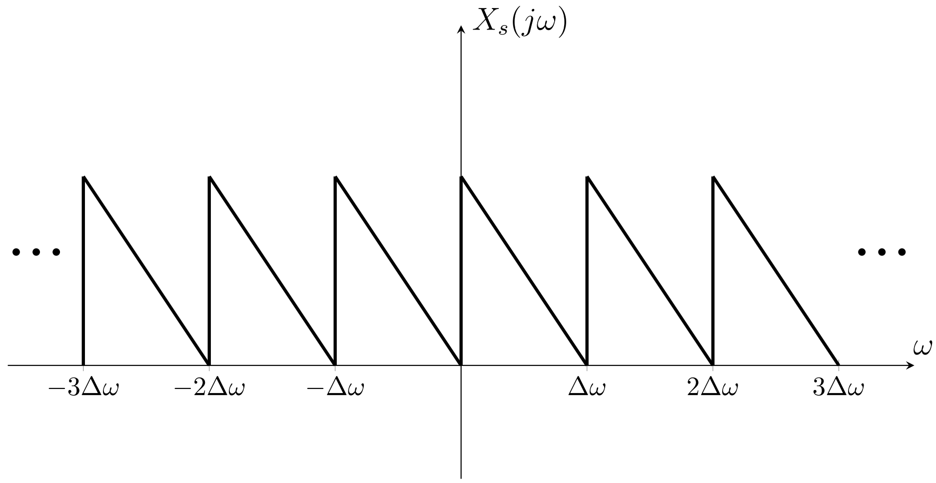
i)

Since the sampling rate of the signal is equal to the Nyquist rate, the aliasing does not occur. The Fourier transform of $X_s(j\omega)$ is plotted below for $T_s = \frac{\pi}{\Delta\omega}$.



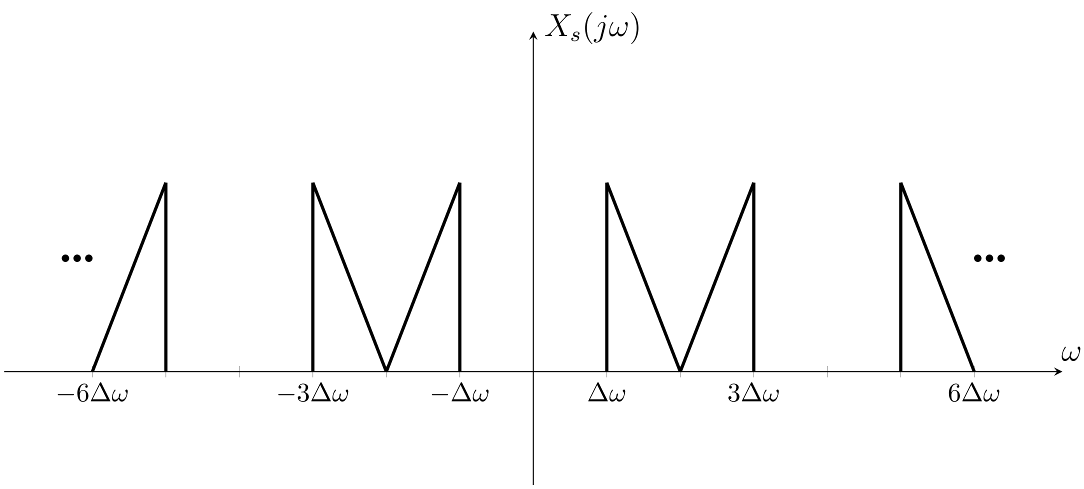
ii)

As it can be seen from the graph of the Fourier transform of $x(t)$, the graph is not symmetric with respect to y-axis, i.e., $X(j\omega)$ has no component at negative frequencies. Therefore, as it can be seen from the graph of $X_s(j\omega)$ below, the aliasing can be avoided even when $T_s = \frac{2\pi}{\Delta\omega}$ which below the Nyquist rate.



c)

The minimum sampling period, $T_s = \frac{\pi}{2\Delta\omega} \Rightarrow$ minimum sampling rate, $\frac{1}{T_s} = \frac{2\Delta\omega}{\pi}$. The Fourier transform of $X_s(j\omega)$ can be seen below. Also, the corresponding time-domain signal is band-pass as it can be seen from the graph of its Fourier transform.



Question 2

The input-output relation of the DT System is given. Thus, if we take the Fourier transform of the both sides:

$Y(e^{j\Omega}) - \frac{1}{3}Y(e^{j\Omega})e^{-j\Omega} = \frac{2}{3}X(e^{j\Omega}) - 2X(e^{j\Omega})e^{-j\Omega}$

$Y(e^{j\Omega})[1 - \frac{e^{-j\Omega}}{3}] = X(e^{j\Omega})[\frac{2}{3} - 2e^{-j\Omega}] \Rightarrow H(e^{j\Omega}) = \frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{2-6e^{-j\Omega}}{3-e^{-j\Omega}}$

$x_c(t) = sin(1000\pi t)$ and $x[n] = x_c(nT)$ $\boxed{\omega_c = 1000\pi}$

$X_c(j\omega) = \frac{\pi}{j}(\delta(\omega - 1000\pi) - \delta(\omega + 1000\pi))$

$$X(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\frac{\Omega-2\pi k}{T})\right) = \frac{\pi}{jT}$$

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \delta(\frac{\Omega-2\pi k}{T} - 1000\pi) - \delta(\frac{\Omega-2\pi k}{T} + 1000\pi)$$

$$\boxed{\underbrace{\frac{2\pi}{T} > 2\omega_c \Rightarrow \frac{1}{T} > 1000}_{\text{to avoid aliasing}}}$$

If there is no aliasing, the below equation holds:

$$Y_r(j\omega) = \begin{cases} H(e^{j\omega T})X_c(j\omega), & |\omega| \leq \frac{\pi}{T} \\ 0, & otherwise \end{cases}$$

$$Y_r(j\omega) = \begin{cases} \frac{\pi}{j}\frac{2-6e^{-j\omega T}}{3-e^{-j\omega T}}[\delta(\omega-1000\pi) - \delta(\omega+1000\pi)], & |\omega| \leq \frac{\pi}{T} \\ 0, & otherwise \end{cases}$$

$$y_r(t) = \frac{1}{2\pi}\int_{\infty}^{\infty} Y_r(j\omega)e^{j\omega t}d\omega$$

i)

$\frac{1}{T} = 2\text{kHz} \Rightarrow \frac{2\pi}{T} > 2\omega_c$ (No aliasing) $\Rightarrow \omega_c T = \frac{\pi}{2}$

$y_r(t) = \frac{1}{2j}\left[\frac{2-6e^{-j\pi/2}}{3-e^{-j\pi/2}}e^{j1000\pi t} - \frac{2-6e^{j\pi/2}}{3-e^{j\pi/2}}e^{-j1000\pi t}\right] = \frac{1}{2j}\left[(1.2+1.6j)e^{j1000\pi t} - (1.2-1.6j)e^{-j1000\pi t}\right]$

$\boxed{y_r(t) = 1.2sin(1000\pi t) + 1.6cos(1000\pi t)}$

ii)

$\frac{1}{T} = 1\text{kHz} \Rightarrow \frac{2\pi}{T} = 2\omega_c$ (No aliasing) $\Rightarrow \omega_c T = \pi$

$y_r(t) = \frac{1}{2j}\left[\frac{2-6e^{-j\pi}}{3-e^{-j\pi}}e^{j1000\pi t} - \frac{2-6e^{j\pi}}{3-e^{j\pi}}e^{-j1000\pi t}\right] = \frac{1}{2j}(2e^{j1000\pi t} - 2e^{-j1000\pi t})$

$\boxed{y_r(t) = 2sin(1000\pi t)}$

Question 3

a)

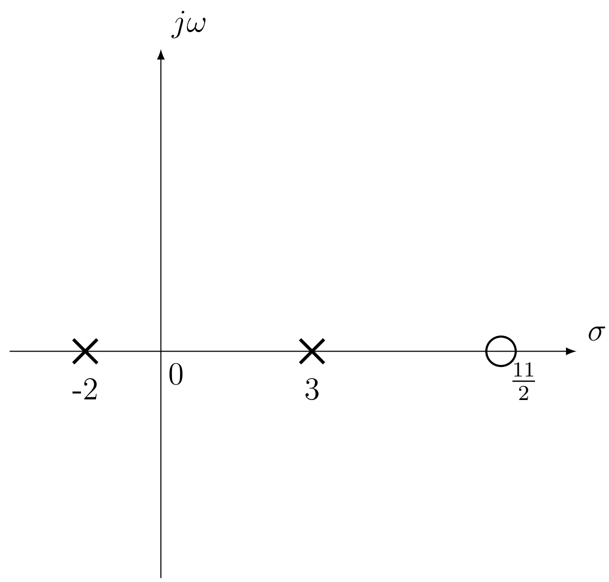
If we take the Laplace transform of the both sides of the equation:

$s^2Y(s) - sY(s) - 6Y(s) = 2sX(s) - 11X(s)$

$Y(s)(s^2 - s - 6) = X(s)(2s - 11)$

$H_1(s) = \frac{Y(s)}{X(s)} = \frac{2s-11}{(s-3)(s+2)} = \frac{-1}{s-3} + \frac{3}{s+2}$

The pole-zero diagram of $H_1(s)$ is drawn below.



We cannot determine $h_1(t)$ unless we know the ROC.

b)

i)

If the system is stable, then ROC includes $j\omega$-axis. Therefore, we can determine ROC as $-2 < \sigma < 3$.
$-2 < \sigma$ implies $\frac{3}{s+2}$ corresponds to right-sided time function which is $3e^{-2t}u(t)$.
$\sigma < 3$ implies $\frac{-1}{s-3}$ corresponds to left-sided time function which is $e^{3t}u(-t)$.
$h_1(t) = e^{3t}u(-t) + 3e^{-2t}u(t)$

ii)

If the system is causal, then ROC is the right side of the right-most pole. Therefore, we can determine ROC as $\sigma > 3$.
$\sigma > -2$ implies $\frac{3}{s+2}$ corresponds to right-sided time function which is $3e^{-2t}u(t)$.
$\sigma > 3$ implies $\frac{-1}{s-3}$ corresponds to right-sided time function which is $-e^{3t}u(t)$.
$h_1(t) = (-e^{3t} + 3e^{-2t})u(t)$

iii)

If the system is anti-causal, then ROC is the left side of the left-most pole. Therefore, we can determine ROC as $\sigma < -2$.
$\sigma < -2$ implies $\frac{3}{s+2}$ corresponds to right-sided time function which is $-3e^{-2t}u(-t)$.

$\sigma < 3$ implies $\frac{-1}{s-3}$ corresponds to right-sided time function which is $e^{3t}u(-t)$.
$h_1(t) = (e^{3t} + -3e^{-2t})u(-t)$

c)

i)

The ROC is the whole s-plane.

ii)

$H(s) = \frac{Y(s)}{X(s)} = s - 3$
$Y(s) = X(s)(s-3) = sX(s) - 3X(s)$
$y(t) = \frac{d}{dt}x(t) - 3x(t)$

d)

i)

Assume $h_1(t)$ is causal. Then it is not stable since the ROC does not include $j\omega$-axis.

ii)

$H(s) = H_1(s)H_2(s) = \frac{2s-11}{(s-3)(s+2)}(s-3)H(s) = \frac{2s-11}{s+2} = 2 - \frac{15}{s+2} \quad ROC : \sigma > -2$
$h(t) = 2\delta(t) - 15e^{-2t}u(t)$

iii)

The cascaded system has a ROC which includes $j\omega$-axis. Therefore, we can say that the cascaded system is stable although the first system is not stable. The second system is used to make the first system stable.

Question 4

a)

If $h[n]$ is real, then the following condition must be satisfied: $H(z) = H^*(z^*)$
$H(z) = \frac{z(z-1)}{\left(z-a(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}})\right)\left(z-a(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}})\right)} = \left(\frac{z^*(z^*-1)}{\left(z^*-a(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}})\right)\left(z^*-a(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}})\right)}\right)^* = H^*(z^*)$

Thus, $h[n]$ is a real-valued signal.

b)

Since the DT system is causal, i.e., $h[n] = 0$ for $n < 0$, the inverse z-transform of $H(z)$ is right-sided.
Also, if the DTFT of h[n] exists, the ROC of H(z) contains the jw-axis ($|z| = 1$).
$H(z) = \frac{1-z^{-1}}{\left(1-a(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}})z^{-1}\right)\left(1-a(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}})z^{-1}\right)} = \frac{c_1}{\left(1-a(\frac{1}{\sqrt{2}}-\frac{j}{\sqrt{2}})z^{-1}\right)} + \frac{c_1}{\left(1-a(\frac{1}{\sqrt{2}}+\frac{j}{\sqrt{2}})z^{-1}\right)}$

$h[n] = c_1[a(\frac{1-j}{\sqrt{2}})]^n u[n] + c_2[a(\frac{1+j}{\sqrt{2}})]^n u[n]$, ROC: $|z| > |a(\frac{1-j}{\sqrt{2}})| = |a(\frac{1+j}{\sqrt{2}})| = a^2$
Thus, the following inequality should be satisfied: $|z| < 1 \Rightarrow a^2 < 1 \Rightarrow \boxed{0 < a < 1}$

c)

If $a = \sqrt{2} \Rightarrow H(z) = \frac{z(z-1)}{z^2-2z+2}$

$$x[n] = u[n-1] \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} u[n-1]z^{-n} = \sum_{n=1}^{\infty} z^{-n} = \sum_{n=0}^{\infty} z^{-n} - 1 = \frac{1}{z-1}, |z| < 1$$

$$Y(z) = X(z)H(z) = \frac{z}{z^2-2z+2} = \frac{z}{(z-1+j)(z-1-j)}, \text{ ROC: } |z| > \sqrt{2}$$

$$Y(z) = \frac{z^{-1}}{(1-(1-j)z^{-1})(1-(1+j)z^{-1})} = \frac{c_1}{1-(1-j)z^{-1}} + \frac{c_2}{1-(1+j)z^{-1}}$$

$$Y(z) = \frac{j/2}{1-(1-j)z^{-1}} + \frac{-j/2}{1-(1+j)z^{-1}} \Longrightarrow y[n] = \frac{j}{2}(1-j)^n u[n] - \frac{j}{2}(1+j)^n u[n]$$

d)

By the scaling property in z-domain: $h'[n] = (\frac{1}{2})^n h[n] \longleftrightarrow H'(z) = H(2z)$, ROC: $|z| > \frac{\sqrt{2}}{2}$
Since the ROC of $H'(z)$ contains $j\omega$-axis ($|z| = 1$), the DTFT of $H'(z)$ exists:
$H'(e^{j\Omega}) = H(2e^{j\Omega}) = \frac{z(2z-1)}{2z^2-2z+1}\Big|_{z=e^{j\Omega}}$
$x[n] = cos(\frac{2000\pi n}{2000}) = cos(\pi n)$

$$X(e^{j\Omega}) = \sum_{k=-\infty}^{\infty} \pi\delta(\Omega - \pi - 2\pi k) + \pi\delta(\Omega + \pi - 2\pi k)$$

$$Y(e^{j\Omega}) = X(e^{j\Omega})H'(e^{j\Omega}) = \left(\sum_{k=-\infty}^{\infty} \pi\delta(\Omega - \pi - 2\pi k) + \pi\delta(\Omega + \pi - 2\pi k)\right) H'(e^{j\Omega})$$

$$y[n] = \mathscr{F}^{-1}\{Y(e^{j\Omega})\} = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=-\infty}^{\infty} \pi\delta(\Omega - \pi - 2\pi k) + \pi\delta(\Omega + \pi - 2\pi k)\right) H'(e^{j\Omega})e^{j\Omega n}d\Omega$$

$$y[n] = \frac{1}{2}H'(e^{j\pi})e^{j\pi n} = \frac{3}{10}e^{j\pi n}$$