Definition 1. Let R be a binary relation on a set S. We say that R^+ is the reflexive transitive closure of R if the following conditions are satisfied:

1. $R \subseteq R^+$ and
2. for every transitive reflexive relation R^* such that $R \subseteq R^*$, we have $R^+ \subseteq R^*$ and
3. R^+ is reflexive and transitive.

□

Theorem 1. Let R be a binary relation on a a set S. Let $R^@$ be another binary relation on S such that for every $A, B \in S$ we have $R^@(A, B)$ iff there exists a sequence $C_1, \ldots, C_n$ of elements of S such that

1. $C_1 = A$
2. $C_n = B$
3. for every $i \in \{1..n-1\}$, $R(C_i, C_{i+1})$

We have:

$$R^@ = R^+$$

□

Proof. We first show

$$R \subseteq R^@ \tag{1}$$

Indeed, suppose $(A, B) \in R$. Then we have a two-element sequence $C_1 = A, C_2 = B$ such that $(C_1, C_2) \in R$. Thus, by definition of $R^@$, $(A, B) \in R^@$.
Next, we show

$$R^@ \text{ is reflexive} \tag{2}$$

Indeed, consider an element $A \in S$. Then we have a sequence consisting of one element, $C_1 = A$. It is easy to see that conditions 1-3 of the definition of $R^@$ are satisfied for the pair (A, A). Therefore, $R^@(A, A)$ holds for every $A \in S$ and we have (2).
Next we show

$$R^@ \text{ is transitive} \tag{3}$$

Let A, B, C be three elements of S (not necessarily distinct) such that

$$R^@(A, B) \tag{4}$$

and

$$R^{@}(B, C) \tag{5}$$

To prove (2), it is sufficent to show

$$R^{@}(A, C) \tag{6}$$

From (4), by definition of $R^{@}$, there exists a sequence $C_1^1, \dots, C_n^1$ of elements of S such that

$$C_1^1 = A \tag{7}$$

$$C_n^1 = B \tag{8}$$

$$\text{for every } i \in \{1..n-1\}, R(C_i^1, C_{i+1}^1) \tag{9}$$

From (5), by definition of $R^{@}$, there exists a sequence $C_1^2, \dots, C_m^2$ of elements of S such that

$$C_1^2 = B \tag{10}$$

$$C_m^2 = C \tag{11}$$

$$\text{for every } i \in \{1..m-1\}, R(C_i^2, C_{i+1}^2) \tag{12}$$

Consider the sequence $D = C_1^1, \dots, C_n^1, \dots, C_m^2$. It is easy to check using (7) - (12) that D satisfies conditions from the definition of $R^{@}$ for the pair (A, C). Therefore, (6) holds and $R^{@}$ is transitive. Therefore, from (1) - (3), $R^{@}$ is a reflexive transitive relation containing R. By definition of R^{+}, we must have:

$$R^{+} \subseteq R^{@} \tag{13}$$

We next show

$$R^{@} \subseteq R^{+} \tag{14}$$

Let $(A, B) \in R^{@}$. Then, by definition of $R^{@}$, there exists a sequence $C_1, \dots, C_k$ of elements of S such that

$$C_1 = A \tag{15}$$

$$C_k = B \tag{16}$$

$$\text{for every } i \in \{1..k-1\}, R(C_i, C_{i+1}) \tag{17}$$

We prove the following:

$$R^{+}(C_1, C_r) \text{ for every } r \in \{1, \dots, k\} \tag{18}$$

We prove (18) by induction on r.

Base Case For $r = 1$, $R^+(C_1, C_1)$ holds because R^+ is a reflexive relation on S by clause 3 of the definition.

Induction Hypothesis Suppose $R^+(C_1, C_l)$ for some $l \in \{1..k-1\}$

Inductive Step We need to show

$$R^+(C_1, C_{l+1}) \tag{19}$$

From (17) we have

$$R^+(C_l, C_{l+1}) \tag{20}$$

By inductive hypothesis, we have:

$$R^+(C_1, C_l) \tag{21}$$

Since R^+ is a transitive relation by clause 3 of the definition, from (20) and (21) we have:

$$R^+(C_1, C_{l+1}) \tag{22}$$

Therefore, (18) holds, and, in particular, we have

$$R^+(C_1, C_k) \tag{23}$$

Therefore, (14) holds. From (14) and (13) we have

$$R^@ = R^+$$

which concludes the proof.

□