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Algorithmic Preference Matching in the Department of Defense

Ian Shaw, Zac Dannelly, Drew Calcagno, Richard Kuzma
United States Navy, ianedwardshaw@gmail.com, dannellyz@gmail.com, drewcalcgno@gmail.com

The Department of Defense maintains a consistent matching problem in the placement of military members into jobs. The regularity of the process is due to the requirement that uniformed service members rotate positions every three years. Gale and Shapley provided a stable matching process 1962 that, if applied to this problem, would greatly improve the current, manual matching process. In this paper we go a step further noting that binary-integer programming can be applied in the matching process to sacrifice stability in favor of military leadership goals such as retention or quality spread. We formulated our prototype for the Chief of Naval Personnel. Nevertheless, our work is agnostic and can be applied not only to any service branch, but to any matching market not requiring stability. These markets are typically those where market members are legally bound to participate.

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1. Problem Description

Human talent allocation is a resource-intensive process for an institution. This paper outlines the mathematics necessary to lay the groundwork for such a task within one of the most complex institutions of American society – the DoD. We dove into this problem seeking to show the validity of the mathematics, as well as to discover the challenges that exist when creating a talent marketplace from the ground up. Furthermore, despite the small prototype sample, the find the algorithm to be widely scalable, particularly if combined with a proper front-end as our project is the back-end of a future full-stack system.

Market forces are largely the drivers behind a job marketplace – this concept is not new for the American economy. However, given a captive audience such as service members ordered to specific jobs, the mathematical benefit of a benevolent autocracy is immense. Rather than the suboptimal fate to which current servicemembers are subjected, the DoD is already shopping for a better way forward algorithmically. We improve upon what currently exists in all metrics except current implementation; nevertheless, we have a plan with stakeholders to quickly field the technology.

In the Navy, terminology of a detailing marketplace refers to the attempts to detail, or place, warfighters into their next job. This detailing marketplace is a system by which military members can transparently rank their job preferences and the people who own those jobs can supply their preference of incoming personnel. This problem is rather unique to the military due to the aforementioned captive market of many members obligated to remain in service due to contract, desiring to stay in for a pension, or perhaps wishing to stay due to a sense of public service. This market type is complemented by an inability for lateral entry – almost all members need to start from the entry level. Additionally, servicemembers typically change jobs every one to three years.

There are ontological parallels to other communities, too. This process is shared in some regards by the medical school graduates applying to the residency stage of their training. Those graduates apply to a pool of US residency programs. This pool is also rather narrow, mostly coming from US based medical schools, and the specificity of skill set required for success is better understood and measured than success as a military officer.

2. Solution Description

Our solution to the talent allocation problem is to optimize the matching of preferences from both sides: job owners (commanding officers) and job seekers (sailors). This general similarity between the military and medical markets is the reason our investigation builds off the Gale-Shapley deferred acceptance algorithm used in the National Residency Match Program (Gale and Shapley 1962).

This paper proposes an optimization-based solution to the matching process. The optimization is not found in the National Residency Match Program due to the need for stability. Stability here is defined as a system where no two individuals from opposite sides of a market (male and female in the stable marriage problem) can leave their assigned partners and be better off for it. Stability is necessary for medical students who have the ability to reject their assigned position, a choice not available to most military members. Thus preference matching in the military context does not require stability. Unburdened by the constraint of stability, we can leverage mathematical optimization (Roth 1984) (Roth 1985a) (Roth and Sotomayor 1989); this insight is the key to the rest of our investigation.

We found that a traditional algorithm, like Gale-Shapley deferred acceptance, was unconducive to the creation of modular constraints that common in US Navy personnel policy. One such constraint is the Chief of Naval Personnel's goal to have at least 95% of dual-military couples stationed within 50 miles of their partner. To gain modularity we turn to binary-integer programming. We later show that this approach can outperform the deferred acceptance iteration in our desired metrics, in addition to its ease of introducing constraints without massive re-coding.

2.1. Strategy Proofness

An important note is the attempt at strategy proofness of the algorithm (Budish 2011) (A. Abdulkadiroglu and T 2006), thus creating an incentive for honesty (Roth 1982). Proper market construction ensures that there is no benefit for a participant to provide incomplete preferences. Likewise, we must acknowledge that there can be difficulty associated with a market participant ranking an incredibly large number of opportunities. To meet this need, we complete any incompletely provided preferences using the 'implied preference method(Shaw 2019).

The complete strategy proofness investigation of this formulation is left as an aspect of future work, but we acknowledge that there could be benefits to coalitions or awareness of the preference landscape of competitors. There is also a possibility of rejecting the implied preferences, as one market maker concluded unexpressed preferences indicate indifference (Irving 1994) – an expression of preference all its own. Nevertheless, the strategy proofness of Irvings approach in an MIP matching process has yet to be explored. In this same vein, the aformentioned 'implied preference' method has not been expanded to to encompass the 'separation' preference described in Section 2.2.5.

2.2. Binary-Integer Program

The constraints we apply are the same as the college admissions problem (Roth 1985b). The line numbers refer to their mathematical formulation in the binary-integer program of Section **??**.

1. Each sailor can only be assigned to one position (2)
2. Either all the jobs are filled or every job seeker is placed (3)
3. Each job can fill up to but not exceed the allocated number of positions (4)

For the sake of the prototype improving upon deferred acceptance in our desired metrics (explained in Section 3.1) we also include the constraints in line (5, 6). This constellation of constraints has the added benefit of guaranteed feasibility since the deferred acceptance output is a feasible solution.

Though a comparison to deferred acceptance is of academic interest, we also provide alternate formulations that may be of more pragmatic interest to military leaders. These include co-location (Section 2.2.2), quality spread (Section 2.2.3), weighting the importance of certain command's preferences over others (Section 2.2.4), and retention (Section 2.2.5).

2.2.1. Prototype Formulation The binary-integer program used in our takes the form:

$$\min \quad f(X) \tag{1}$$

$$\text{such that} \quad \sum_{j=1}^{m} X_{i,j} \leq 1 \quad \forall i \in \{1, \dots n\} \tag{2}$$

$$\sum_{i=1}^{i=n}\sum_{j=1}^{m} X_{ij} = \min\left(n, \sum_{j=1}^{m} a_j\right) \quad \forall j \in \{1,\dots m\} \tag{3}$$

$$\sum_{i=1}^{n} X_{ij} \le a_j \quad \forall j \in \{1,\dots m\} \tag{4}$$

$$\sum\sum X_{ij}\mathbf{1}(X_{ij}P_i^S \le w) \ge \sum\sum X_{ij}\mathbf{1}(X_{ij}^{DA}P_i^S \le w) \quad \forall w \in \{1,5,10\} \tag{5}$$

$$\sum\sum X_{ij}\mathbf{1}(X_{ij}P_{ji}^O \le w) \ge \sum\sum X_{ij}\mathbf{1}(X_{ij}^{DA}P_{ji}^O \le w) \quad \forall w \in \{1,5,10\} \tag{6}$$

The objective function f below is designed to improve over deferred acceptance in average preference sum, a metric described in Section 3.2.

$$f(X) = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}(P_{ij}^S + P_{ji}^O)$$

2.2.2. Co-Location The co-location constraint takes the form

$$\frac{1}{n_c}\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mathcal{D}(C_{ij}) \ge 0.95 \quad \texttt{at least 95\% of couples are co-located}$$

Distance Function $\mathcal{D}$ returns 1 if the couple is considered co-located, 0 if not or if single. Here we choose 50 miles between job locations to be consider co-located because that is the threshold for receiving dislocation allowance (DLA) for a permanent change of station (PCS) according to the Joint Travel Regulations (JTR). The function $L(S_i)$ returns the location of the proposed station for Seeker i.

$$\mathcal{D}(C_{ij}) = \begin{cases} 0 & \text{if } C_{ij} == 0 \\ \mathbf{1}\big(||L(S_i) - L(S_j)|| \le 50\big) & \text{otherwise} \end{cases}$$

2.2.3. Quality Spread There is precedent in the DoD to do assignments in a manner by which the distribution of quality officers is even accross some sort of grouping. This is the assignment policy for US Navy submariners to boats after initial training and US Marines to MArine Occupational Spectialties (MOS). The purpose of this is to ensure that no one unit or community within the force falls behind in competency due to a preponderance of low-performing personnel.

To incorporate this into the matching algorithm would be easy. We would simply make the below formulation the objective function at (1).

$$f(X) = \gamma\sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}\left(P_{ij}^S + P^O - ji\right) + \lambda(\sigma^2(X))$$

The explanation of the objective function can be found below.

$$
\begin{aligned}
f(X) &= \gamma \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}\left(P_{ij}^{S} + P^{O} - ji\right) + \lambda(\sigma^2(X)) \\
\gamma, \lambda &= \texttt{weighting coefficient}, \in [0,1] \cap \mathbb{R} \\
\sigma^2(X) &= \texttt{Variance of the quality assignments} \\
\sigma^2(X) &= \frac{\sum_{j=1}^{m}(\mu_j^q - \bar{\mu^q})^2}{m-1} \\
\mu_j^q &= \texttt{Average Quality score of person assigned to position } j \\
\mu_j^q &= \sum_{i=1}^{n} \frac{X_{ij} Q_i}{A_j} \\
\bar{\mu^q} &= \sum_{j=1}^{m} \frac{\mu_j^q}{m} \\
Q &= \texttt{Quality scores for each seeker }, \mathbb{R}^{n\times 1}
\end{aligned}
$$

2.2.4. Weighted Importance Suppose service leadership were to favor some sort of command, such giving high-tempo operational commands twice the value preference over administrative commands This can easily be incorporated by a tweak to the objective function (1).

$$
f(X) = \sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}(P_{ij}^{S} + W_j P_{ji}^{O})
$$

Here the vector $W \in \mathbb{R}^{m\times 1}$ is a weighting vector. For example, if the service leadership were to value a command #1 twice that of any other command then:

$$
W_i = \begin{cases} 2 & i = 1 \\ 1 & \text{otherwise} \end{cases}
$$

2.2.5. Retention Retention is often and important goal of military leaders wanting to maintain manpower numbers in order to keep the force ready for possible operations. The difficulty is, with an all volunteer force, if a service member is beyond contractual obligation, they need to have incentive to stay in. For example, a sailor may be willing to stay in for another 3 years if they are guaranteed to stay in San Diego so that their child can finish out high school there. If asked to move across the countryi, then they will opt to leave the service and seek local, civilian employment.

For the purpose of maximizing retention (minimization is just a sign change), we can have the objective function (1) take the form below.

$$
f(X) = \gamma\left(\sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}(P_{ij}^{S} + P_{ji}^{O})\right) + \lambda(R(X))
$$

$$P_j^S = \begin{cases} m+1 & \text{Would rather separate than accept assignment to } j \\ [1,m]\cap\mathbb{Z} & \text{otherwise expressed preference} \end{cases}$$

$$\gamma, \lambda = \texttt{weighting coefficients}, \in [0,1]\cap\mathbb{R}$$

$$R(X) = \texttt{Retention rate of assignment set } X$$

$$R(X) = \frac{\sum_{i=1}^{n} \mathbf{1}\left(X \bullet P^S \langle m+1\right)}{n}$$

We can also incorpoate retention as a constraint. Say a service leader were to desire retention above 90%, then the following constraint would be necessary.

$$R(X) \geq \texttt{Desired retention rate between 0 and 1}$$

Keep in mind, there is maximum rate on retention that is not necessarily 100%, therefore when employing this constraint special care must be taken to guarantee feasibility.

The maximum retention rate given the constraints of the college admission formulation (defined in lines (2, 3, 4)), is the value of the objective function for the program described below. Keep in mind these constraints must be feasible for this analysis to be worthwhile; this requires that there is at least one solution where there are is at least one job seeker willing to fill each position in the market.

$$\begin{aligned} \max \quad & R(X) \\ \text{such that} \quad & \sum_{j=1}^{m} X_{i,j} \leq 1 \quad \forall i \in \{1, \dots n\} \\ & \sum_{i=1}^{i=n} \sum_{j=1}^{m} X_{ij} = \min\left(n, \sum_{j=1}^{m} a_j\right) \quad \forall j \in \{1, \dots m\} \\ & \sum_{i=1}^{n} X_{ij} \leq a_j \quad \forall j \in \{1, \dots m\} \end{aligned}$$

We suspect the implementation of these retention in the objective function and/or constraints to be strategy proof (indicating it is in the best interest of market participants to honestly express their ordinal preferences) so long as the placements described in the output are enforced (as opposed to used for negotiation). Yet, we have not explored the strategy proofness rigorously and leave that investigation to future work.

3. Post-Match Metrics

The below metrics are how military leaders usually describe placement performance. Not uncommon is the announcement after service selection at the US Naval Academy, “This year XX midshipman got their first choice, YY in their top five, and an average preference assignment of ZZ.”

3.1. Preference Allocation Windows

Preference allocation indicates how many individuals received their top preference, how many individuals got a top one, top five, and top ten preference, and how many failed to be matched with any preference.

3.2. Average Preference Sum

Preference sum is the addition of the ordinal preference of a sailor for their assigned command with the ordinal preference of the command for their assigned sailor. For example if a sailor is assigned their most preferred command but the command ranked the sailor third, the preference sum would have a value of 4. The average of this sum accross the market provides a high level understanding of how the matching system performs in aggregate.

4. Running the Prototype

4.1. Practical Considerations

The essential effort of this paper is to explore mathematically rigorous personnel placement for the DoD. We aggregated and prototyped on real preference data. Though not necessary to prove the viability of these formulations (only feasibility and argument are sufficient), displaying results on actual sailor preferences helped bring theoretical results to a practical display. These displays were vital in convincing Navy leadership in improvement over the status quo.

At the most foundational level, the algorithm solely requires preferences from job seekers (sailors) and job owners (commanding officers). In order for these preferences to be well informed, the system ought to have standardized job descriptions and sailor resumes. Obtaining this central information proved to be the most difficult aspect of piloting the algorithm rather than the mathematics, no matter how novel the MIP approach or exquisite the introduction of implied preferences.

Even Alvin Roth, who won a Nobel Prize for his foundational matching algorithm, experienced the same human and bureaucratic difficulties. Roth suggests, “overall, one lesson from the [matching project] is that mechanism design in a political environment requires that not only policy makers themselves be persuaded of the virtues of a new design, but that they be able to explain and defend the mechanism to the various constituencies they serve.” (A. Abdulkadiroglu and T 2006) The explainability of our algorithm is relatively simple in that it matches preferences from two sides – seekers and owners – yet any sort of deviation from existing processes creates concern and risk for senior stakeholders, real or imagined. Like building a boat in a bottle, the art of the prototype was not so much the algorithm itself, but the delicacy needed to create it within a debilitatingly constrained environment.

To best alleviate these issues, our team leveraged a web-based polling tool that allows both sailors and commanding officers to input their preferences. Sailors were given job descriptions and commanding officers were given sailor bios in order to inform their respective preferences.

4.2. Data Sources

Our algorithm was run against three total data-sets.

4.2.1. Naval Cyber Officers The Navy has various types of service. Some officers work with computers rather than operating aircraft or driving ships. We worked with one subset of cyber-oriented cryptologic warfare officers to attempt a bottom-to-top pilot of the matching algorithm. That required an novel data collection effort because the previous method of matching sailors to jobs was completed non-optimally, largely based on who was available at the time. This dataset proved to be one of the most difficult to collect because there was no collection infrastructure or existing practice of doing so.

4.2.2. Naval Explosive Ordinance Disposal Officers (EOD) EOD officers are Naval expeditionary special forces who deal primarily with the handling, disarming, and disposing of explosive materials across the world. Given the specialization of their work, the small communitys leadership have expressed a desire for a matching mechanism to identify and manage its peoples talents. Working closely with that leadership, our team was able to gather a large amount of preferences on both sides of the marketplace (seekers and owners).

Tangentially, as evidenced by the near impossibility of obtaining data in the cyber community due to a lack of precedent, the EODs precedent extended to their funding a front end collection mechanism. Later in this piece we will advise the Navy to combine our algorithmic back end to the EOD communitys budding front end to create a full stack system with potential for scalability.

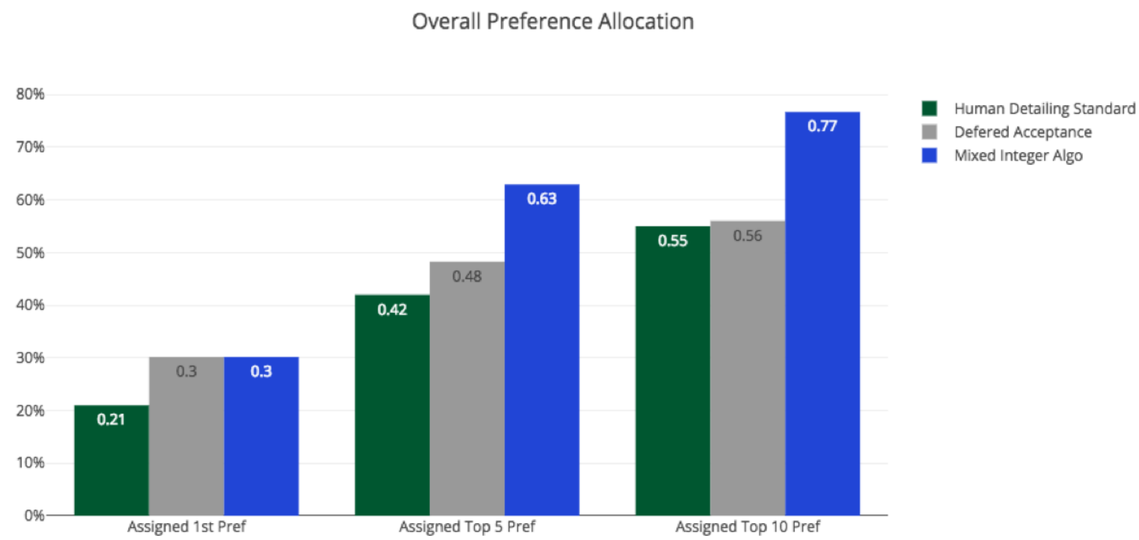
4.2.3. Naval Doctors (Medical Corps) The Navy has its own doctors. These doctors may go to civilian medical schools or the governments own medical school (Uniformed Services University Hebert School of Medicine, or USUHS). Additionally, they conduct their residencies at public or private hospitals. As a result of that process, they are involved in the National Resident Matching Program, a foundational aspect of this paper.

Their familiarity with the Gale-Shapley matching algorithm, used by some Navy medical disciplines for placement and the National Residency, the culture of the medical corps is such that ordinal preference data is collected frequently. As a result, our request for data from this community was met willingly and completely, offering a great opportunity to run clean data as well as provide our algorithm back to that excitepd customer most seamlessly.

4.3. Prototype Results

Given that our algorithm does not need to maintain stability, we are able to further optimize as compared to a deferred acceptance approach (the one used for hospital residents). Our algorithm is shown to measurably improve placement at each incremental preference level. The improvements

Figure 1 Medical Corps



Note. Results for Medical Corps Data.

are even more notable when compared to the current non-algorithmic matching effort conducted by humans with imperfect information and an overwhelming amount of data. This human standard was measured by an officer who conducted a similar investigation for the Air Force (Lepird 2018).

5. Future Work

5.1. Inspired from Academia

Ongoing work should continue to explore the best ways in which markets can be created. This includes consideration of partial matches or lotteries in MIP.(A. Roth and Vate 1993) One may also consider the ability to explore the viability of leveraging Budish's wagering formulation of approximate competitive equilibrium from equal incomes. (Budish 2011) Additional research would include how to incorporate synergy of selection preferences by Job owners if allowed more than one person. (Roth 1985a) Furthermore, service members could be placed in multiple jobs, such as selecting their main role and their collateral duties in the same matching process. (Roth 1982) Lastly, an open question the authors are curious about is the opportunity for unsupervised learning (such as K-means clustering) on the preference data to see if there are clusters of service members with regards to their preferences, and what theres clusters indicate.

5.2. Requests from DoD Personnel

From interviewing DoD leadership there are several institution specific asks. Many of our considered markets have an on-line placement procedure (S. Khuller and Vazirani 1994), such as that which happens at many information-sensitive commands due to the trickle of clearance issuance as opposed to bulk assignment.

In the same spirit of temporal considerations, military members have set time-lines at each command and are given expected rotation dates. Consideration of overlapping rotation and end strength could give another direction to objective function formulation.

Tailored Compensation decisions could be made for uncompetitive positions the DoD needs filled.

Appendix A: Notation

Throughout the paper, we will reference the notation listed in this section.

$$
\begin{aligned}
m &= \text{number of different jobs available} \\
n &= \text{number of persons} \\
\vec{P_i^S} &= \text{Preference vector of job seeker } i, \ \in \mathbb{Z}^{+,m\times 1} \\
P^S &= [\vec{P_i^S}|\dots|\vec{P_n^S}] \in \mathbb{Z}^{+,m\times n}, \\
&\quad \text{Preference Matrix of Seekers} \\
\vec{P_i^O} &= \text{Preference vector of job owner } j, \ \in \mathbb{Z}^{+,n\times 1} \\
P^O &= [\vec{P_i^O}|\dots|\vec{P_n^O}] \in \mathbb{Z}^{+,n\times m}, \\
&\quad \text{Preference Matrix of Job Owners} \\
\vec{A} &= \text{Position Available vector } \in \mathbb{Z}^{+,m\times 1} \\
a_j &= \text{Amount of positions for job } j, \ \in \mathbb{Z}^{+} \\
M &= \sum_i a_j \\
&= \text{ number of openings across Navy} \\
M &\geq m \\
X &= \text{Placement Matrix} \in \{0,1\}^{n\times m} \\
x_{i,j} &= \begin{cases} 1 & \text{if } S_i \text{ is slated for job } j \\ 0 & \text{otherwise} \end{cases} \\
C &= \text{Co-Location Matrix, symmetric} \\
C_{ij} &= \begin{cases} 1 & \text{Seekers } i\text{i and } j \text{ requests co-location} \\ 0 & \text{otherwise} \end{cases} \\
n_c &= \text{number of couples requesting co-location} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}
\end{aligned}
$$

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