We will assume throughout that the utility function for every agent is continuous, increasing ($x' \gg x \Longrightarrow u(x') > u(x)$), and concave, and also that everyone has a strictly positive endowment of every commodity ($e^i \gg 0$).

Definition: Walrasian Equilibrium, pure exchange economy: every consumer optimizes and markets clear.

<u>First welfare theorem</u>: If $\left((p), (x^i)_i\right)$ is a Walrasian equilibrium, then $(x^i)_i$ is Pareto optimal.

Proof:

Assume there is some feasible $(\hat{x}^i)_i$ such that $u(\hat{x}^i) \geq u(x^i)$ for all i and $u(\hat{x}^i) > u(x^i)$ for some i. Then $p \cdot \hat{x}^i \geq p \cdot x^i$ for all i and $p \cdot \hat{x}^i > p \cdot x^i$ for some i due to monotonicity. Since prices are non-negative, there must be a good l such that, summing over i, one has $\sum_{i=1}^{I} \hat{x}_l^i > \sum_{i=1}^{I} x_l^i = \sum_{i=1}^{I} e^i$. Hence $(\hat{x}^i)_i$ is not feasible, which is absurd. QED.

<u>Second welfare theorem</u>:

If $(e^i)_i$ is Pareto optimal, then there is a price vector $p \in \mathbb{R}_+^L$ such that $\left((p), (e^i)_i\right)$ is a Walrasian equilibrium.

Proof:

1. We will use (a version of) the separating hyperplane theorem: if $A \subseteq \mathbb{R}^n$ is convex and $x \notin A$, then there exists $p \neq 0$ such that $p \cdot a \geq p \cdot x$ for all $a \in A$.

2. Define the following set: $A^i = \left\{a \in \mathbb{R}^L : e^i + a \geq 0 \ \ and \ \ u^i(e^i + a) > u^i(e^i)\right\}$. This is a set of redistributions a that make at least one agent strictly better off.

3. Preferences are convex (utility is concave), and hence A^i is convex.

4. Define now $A = \sum_{i=1}^{I} A^i = \left\{a \in \mathbb{R}^L : a = \sum_{i=1}^{I} a^i \ \ with \ a^i \in A_i\right\}$. This is a sum of redistributions.

5. A^i convex implies A convex (exercise).

6. $0 \notin A$. Otherwise there would be $(a^i)_i$ with $\sum_{i=1}^{I} a^i = 0$ and $u^i(e^i + a) > u^i(e^i)$ for all i, meaning that $(e^i)_i$ is not Pareto optimal.

7. The separating hyperplane theorem now implies that there exists $p \neq 0$ such that $p \cdot a \geq p \cdot 0 = 0$ for all $a \in A$. (Point 6 implies that we can take $x = 0$ in point 1.) In short, $p \cdot a \geq 0$

8. $a \gg 0$ implies that $a \in A$ by monotonicity: we can just split the strictly positive amount of each commodity among consumers and make everyone strictly better off due to monotonicity.

9. The two previous points ($p \cdot a \geq 0$ and $a \gg 0$) imply $p \geq 0$. If there were some $p_l < 0$, then we could take a_l very large, a_k very small for all $k \neq l$, and get $p \cdot a < 0$, a contradiction.

10. Points 7 and 9 give us $p \neq 0$ and $p \geq 0$. Hence $p > 0$.

11. Now we need to show that $\left((p), \left(e^i\right)_i\right)$ is a Walrasian equilibrium. That is: consumers optimize and markets clear.

12. Market clearing holds by definition since $\left(e^i\right)_i$ is the initial allocation.

13. We need to show that $\left(e^i\right)_i$ is the optimal demand for prices p. To do that, we will show that if $u^i\left(x^i\right) > u^i\left(e^i\right)$, then necessarily $p \cdot x^i > p \cdot e^i$: that is, it is not in the budget set, and hence cannot be the solution to the consumer problem.

14. If $u^i\left(x^i\right) > u^i\left(e^i\right)$, then $p \cdot x^i \geq p \cdot e^i$. If not, then $p \cdot x^i < p \cdot e^i$, and by monotonicity (or simply local non-satiation), it would be possible to find another allocation strictly better than x^i and still affordable.

15. By continuity of the utility function, there is $\lambda < 1$, but very close to one, such that $u^i\left(\lambda x^i\right) > u^i\left(e^i\right)$ still holds.

16. Repeating the argument in 14 now to λx^i, one has $p \cdot \lambda x^i \geq p \cdot e^i$. But $\lambda < 1$ implies that $p \cdot x^i > p \cdot \lambda x^i$. Putting these two inequalities together, we get $p \cdot x^i > p \cdot e^i$. This is what we needed to show in 13, concluding the proof.