The Model:

There are $i = 1, \dots, I$ agents and $l = 1, \dots, L$ commodities. Each agent has a utility function $u^i: \mathbb{R}_+^L \rightarrow \mathbb{R}$ and an endowment $e^i \in \mathbb{R}_+^L$. Each agent chooses a consumption bundle $x^i \in \mathbb{R}_+^L$ to maximize u^i subject to $p \cdot x^i \leq p \cdot e^i$, with $p \in \mathbb{R}_+^L$, or analogously we write the restriction as $x^i \in B^i(p)$. Agents are price takers.

Assumptions:

Preferences are continuous, strictly monotone and weakly convex. Hence optimal choice needs not be unique, and Marshallian demand does not need to be a function: it may be a correspondence. That is, a pair (p, w) may be associated to different values $x(p, w)$. The image of (p, w) is a subset of X. In general, we will use the notation $F: X \rightrightarrows Y$ for a correspondence.

For all agents, $e^i \gg 0$.

We define a Walrasian equilibrium as a vector of prices and consumption bundles for each agent $\left(p, \left(x^i\right)_i\right)$ such that:

1. Choices are optimal: for all i, $x^i \in arg\max\limits_{x^i \in B^i(p)} u^i(p)$
2. Markets clear: for all l, $\sum_{i=1}^{I} x_l^i = \sum_{i=1}^{I} e_l^i$

Equilibrium Existence

We need the following definition:

$G(F) = \{(x, y) \in X \times Y: y \in F(x)\}$ is the graph of F.

Let's define now a fixed point.

In a simple function $f: \mathbb{R} \longrightarrow \mathbb{R}$, a fixed point is x such that $x = f(x)$. For example, if $f(x) = x^2$, then $x = x^2 = 1$ is a fixed point.

We may generalize it for a correspondence. $F: X \rightrightarrows Y$. Now we write $x \in F(x)$ for a fixed point.

We need to find conditions that guarantee that a fixed point exists.

Kakutani's Fixed Point Theorem

Let $X \subset \mathbb{R}^n$ compact, convex, non-empty. Let $F: X \rightrightarrows X$ such that $G(F)$ is closed, and $F(x)$ is convex for all $x \in X$. Then this correspondence has a fixed point $x^* \in F(x^*)$.

There is another way to present the same result.

Consider again $X \subset \mathbb{R}^n$ compact, convex, non-empty. Let $F: X \rightrightarrows X$ be non-empty, convex, and upper-hemicontinuous (uhc). Then this correspondence has a fixed point $x^* \in F(x^*)$.

Definition of uhc. Consider $X, Y \subset \mathbb{R}^n$ compact and convex. Let $F: X \rightrightarrows Y$. Consider $\{x_n\} \subset X$, $\{y_n\} \subset Y$ and $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $y_n \in F(x_n)$ for all n. Then $y \in F(x)$.

Uhc is equivalent to closed graph only if Y is compact, which is our case.

We will also need the Maximum Theorem:

Assume $C: Q \Rightarrow \mathbb{R}^N$ is a continuous correspondence, and $c(q)$ is compact and non-empty for all $q \in Q$. Assume $f: \mathbb{R}^N \to \mathbb{R}$ is a continuous functions. Consider the problem $\max_{x \in c(q)} f(x)$. Then the maximizer $x^*(q)$ is upper-hemi continuous and the value function $f\big(x^*(q)\big)$ is continuous.

Let's go back to the consumer problem (CP). Marshallian demand: $x^l(p) = x(p, pe)$.

Notice that $x^l(\lambda p) = x(\lambda p, \lambda pe) = x(p, pe) = x^l(\lambda p)$. The first and the last equalities use the definition of $x^l(p)$ in the previous paragraph. The second equality uses the fact that Marshallian demand is unchanged if we multiply both prices and income by a positive number. This means that $x^l(p)$ is homogenous of degree zero in prices.

Moreover, strictly monotone preferences implies that the consumer spends all his income: $px^l(p, pe) = pe$, and hence $p[x^l(p, pe) - e] = 0$. This holds for any vector of prices p.

Define excess demand: $z^l(p) = \sum_{i=1}^{I} \left[x_i^l(p) - e_i^l\right]$.

Define the following notation:

$$Z(p) = \begin{bmatrix} z^1(p) \\ z^2(p) \\ \vdots \\ z^l(p) \end{bmatrix}$$

This is a vector of excess demand, for each good, at prices p.

Another way to define a Walrasian equilibrium: it is a vector $\left(p, \{x_l^i\}_{l,i}\right)$ such that $Z(p) = 0$. This summarizes two conditions: consumers optimize (implicit in the use of Marshallian demands) and markets clear.

(One may also use $Z(p) \leq 0$, with $Z(p) < 0$ for prices equal to zero. Not relevant for us.)

Notice that if $Z(p) = 0$, then $Z(\lambda p) = 0$ for all $\lambda > 0$. Excess demand is also homogenous of degree zero in prices: this property is inherited from $x^l(\lambda p) = x^l(\lambda p)$, seen above, since endowment e_i^l is constant (i.e., does not depend on prices). This also implies $p \cdot Z(p) = 0$ for any vector of prices: this is Walras' Law.

This has an important interpretation: we only find relative prices in equilibrium.

Turn now to our equilibrium result.

The function $Z(p)$ has the following properties:

1- Uhc and non-empty (follows from Theorem of the Maximum. There is only one detail: we cannot apply the theorem directly to CP because $B^i(p)$ is not compact for prices going to zero. To solve this, simply define that the budget set is $B^i(p) \cap T$, in which $T = \left\{x \in \mathbb{R}_+^L : x \leq 2 \cdot \sum_{i=1}^{I} e^i\right\}$. CP is unchanged by now we have a compact set, and can apply the Maximum Theorem to conclude that the solution $x^i(p)$is Uhc and non-empty.)

2- Convex for all p (direct consequence of result from consumer theory: solution is convex if preferences are convex)

3- Bounded below: there is $Z > 0$ such that for all l and all p, $z_l(p) > -Z$. (Lowest possible demand is $x_l = 0$, and hence $z_l \geq -\sum_{i=1}^{I} e_i^l$ for all goods.)

4- If $p_n \to p \neq 0$ with some coordinate $p_l = 0$, then $max\{z_1(p_n), \ldots, z_L(p_n)\} \to \infty$. (If some but not all prices go to zero, some consumer must have wealth going to infinity, and then strongly monotone preferences imply that his demand for the good $p_l = 0$ goes to infinity.)

5- Homogenous of degree zero (shown above)

6- $pZ(p)$=0 for all p (Walras' Law – shown above)

And these properties imply that there is a Walrasian equilibrium as defined above.

Let us show this.

Begin normalizing prices. We can always do this in general equilibrium because, as we saw, we only look at relative prices. Choose the following normalization:

$$p_1 + p_2 + \cdots + p_L = 1$$

That is, divide all prices by the sum of prices, which must be non-negative: $p_l \geq 0$ for all l.

This is a simplex. The set of all prices that respect these restrictions is Δ^L. That is, $p \in \Delta^L$. This set is closed, bounded (hence compact), and convex.

Consider the case of two goods for visualization.

This simplex will be our domain and codomain X.
Now let us define the following function:

$$Z: \Delta^L \rightrightarrows \mathbb{R}^L$$

For each price vector p, $Z(p)$ informs the excess demand for each good l.

This is a correspondence that takes from Δ^L to $\mathbb{R}^L$. Not enough to use Kakutani's theorem because domain and codomain are not the same.

Define the following function:

$$\boldsymbol{m(Z(p)) = argmax_{\hat{p} \in \Delta^L} \hat{p} Z(p)}$$

Notice that $m(Z)$ is a vector of prices. It is the vector of prices that maximizes $\hat{p}Z(p)$. Observe that we first have $\hat{p}$ and then have p: we are choosing $\hat{p}$ to maximize the value of excess demand at the original vector price p. This is a continuous function on a compact domain, and hence the solution set is non-empty.

So we use the correspondence Z to go from Δ^L to $\mathbb{R}^L$, and then apply $m(Z)$ to go from $\mathbb{R}^L$ to Δ^L. This is a correspondence composition: $\Delta^L \rightrightarrows \mathbb{R}^L \rightrightarrows \Delta^L$.

Let us present now a series of results.

Lemma 1. $m(Z)$ is convex.

Proof. Consider $p_1, p_2 \in \Delta^L$ such that $p_1, p_2 \in argmax_{\hat{p} \in \Delta^L} \hat{p} Z(p)$.

Hence $p_1 Z(p) = p_2 Z(p)$. Then for all $\lambda \in [0,1]$:

$$\lambda p_1 Z(p) + (1-\lambda) p_2 Z(p) = p_1 Z(p) = p_2 Z(p)$$

That is, $\lambda p_1 + (1-\lambda) p_2$ also maximizes $\boldsymbol{\hat{p} Z(p)}$. Hence $\lambda p_1 + (1-\lambda) p_2 \in m(Z)$. We conclude that $m(Z)$ is convex, since it contains any convex combination of two of its elements. QED.

Lemma 2. $m(Z)$ is uhc.

Proof. Take some sequence $p^n \to p^*$. Consider $Z^n \to Z^* = Z(p^*)$ and $p^n \in m(Z^n)$. We need to show that $p^* \in m(Z^*)$.

Assume $p^* \notin m(Z^*)$.

Then there is some $\bar{p} \neq p^*$ such that $\bar{p}Z^* > p^*Z^*$. This is simply the definition of $m(Z^*)$.

But $Z^n \to Z^*$ and $p^n \to p^*$. Hence $\bar{p}Z^n \to \bar{p}Z^*$. Also $p^nZ^n \to p^*Z^*$.

So we may choose n big enough such that:

$$|\bar{p}Z^n - \bar{p}Z^*| < \frac{\varepsilon}{2}$$
$$|p^nZ^n - p^*Z^*| < \frac{\varepsilon}{2}$$

Remember now that $\bar{p}Z^* > p^*Z^*$. This allows us to conclude:

$$\bar{p}Z^n > p^*Z^* + \frac{\varepsilon}{2}$$
$$p^*Z^* + \frac{\varepsilon}{2} > p^nZ^n$$

The first line is implied by two facts: $\bar{p}Z^n$ is close to $\bar{p}Z^*$ (inequality in yellow), and $\bar{p}Z^* > p^*Z^*$. The second line is a direct implication of the inequality in green.

These two last lines imply:

$$\bar{p}Z^n > p^nZ^n$$

That is, p^n does not maximize p^nZ^n, which is absurd: we assumed that p^n maximizes p^nZ^n. QED.

This is argument is similar to: $a_n \geq 0$ and $a_n \to a$ imply $a \geq 0$. This argument is simple: the difficulty is basically notation and an unusual environment, not mathematics.

Now let us build the following correspondence $g: \Delta^L \rightrightarrows \Delta^L$ as follows:

$$g(p) = m(Z(p))$$

That is, we have $\Delta^L \rightrightarrows \mathbb{R}^L \rightrightarrows \Delta^L$. We may apply Kakutani on the correspondence g.

We have the following result: the composition of non-empty, uhc and convex correspondences is also non-empty, uhc and convex. (Exercise.)

Hence we may apply Kakutani's fixed point theorem to conclude that there is $p^* \in g(p^*)$.

Lastly, we need to show that p^* is a vector of equilibrium prices.

Write the definition of $p^* \in g(p^*)$:

$$p^* \in argmax_{\hat{p} \in \Delta^L} \hat{p}Z(p^*)$$

This means:

$$p^*Z(p^*) \geq pZ(p^*)$$

For all $p \in \Delta^L$. Keep this in mind: we will use it in the last line of the proof below.

Let us prove our last lemma.

Lemma 3. $Z(p^*) \leq 0$.

Proof. Suppose that there is a commodity l such that $Z_l(p^*) > 0$.

Choose $\tilde{p} = (0, \dots 0,1,0, \dots ,0)$: that is, $\tilde{p}_l = 1$, and $\tilde{p}_k = 0$ for $k \neq l$.

Then:

$$\tilde{p}Z(p^*) = Z_l(p^*) > 0 = p^*Z(p^*)$$

The last equality is Walras' Law, which applies for any vector of prices – in particular, it applies for p^*.

This is absurd because $p^*Z(p^*) \geq pZ(p^*)$ for all $p \in \Delta^L$. QED.