Optimal concession contract between a port authority and container-terminal operators by revenue-sharing schemes with quantity discount

---

The following is the supplemental material for the "Optimal concession contract between a port authority and container-terminal operators by revenue-sharing schemes with quantity discount". The supplemental material includes (1) differential profit function of terminal operators; (2) proof of property 1; (3) the Cournot competition model, which was used for the comparison with the model in the paper.

Supplementary document A: Expressions of $\frac{\partial \pi_1}{\partial p_1}$ and $\frac{\partial \pi_2}{\partial p_2}$.

$$\frac{\partial \pi_1}{\partial p_1} = \begin{cases} \frac{1}{1+b} + \frac{bp_2}{1-b^2} + \frac{-2p_1}{1-b^2} + \frac{c_1+R_1}{1-b^2} & q_1 < Q \\ \frac{1}{1+b} + \frac{bp_2}{1-b^2} + \frac{-2p_1}{1-b^2} + \frac{c_1}{1-b^2} + \frac{R_2}{1-b^2} & q_1 \geq Q \end{cases}$$

$$\frac{\partial \pi_2}{\partial p_2} = \begin{cases} \frac{1}{1+b} + \frac{bp_1}{1-b^2} + \frac{-2p_2}{1-b^2} + \frac{c_2+R_1}{1-b^2} & q_1 < Q \\ \frac{1}{1+b} + \frac{bp_1}{1-b^2} + \frac{-2p_2}{1-b^2} + \frac{c_2}{1-b^2} + \frac{R_2}{1-b^2} & q_1 \geq Q \end{cases}.$$

Supplementary document B: Proof of property 1

$$q_{11}^* - q_{21}^* = \frac{bc_2 - c_1(2-b^2) - bc_1 + c_2(2-b^2)}{(1-b^2)(4-b^2)}$$

$$= \frac{b(c_2-c_1) + (c_2-c_1)(2-b^2)}{(1-b^2)(4-b^2)}$$

$$= \frac{(c_2-c_1)(2+b-b^2)}{(1-b^2)(4-b^2)}$$

Due to $c_2 > c_1$ and $0 < b < 1$, hence $\frac{(c_2-c_1)(2+b-b^2)}{(1-b^2)(4-b^2)} > 0$. Thus $q_{11}^* > q_{21}^*$ holds.

$$q_{21}^* - q_{22}^* = \frac{-R_1}{(1-b^2)(4-b^2)} + \frac{bc_1 - c_2(2-b^2)}{(1-b^2)(4-b^2)} - \frac{(b^2-2)(c_2+R_1) + b(c_1+R_2)}{(4-b^2)(1-b^2)}$$

$$= \frac{-R_1(1-b)(2+b)}{(1-b^2)(4-b^2)} + \frac{bc_1 - c_2(2-b^2) + c_2(2-b^2) - R_1(b^2-2) - bc_1 - bR_2}{(1-b^2)(4-b^2)}$$

$$= \frac{-R_1(1-b)(2+b)}{(1-b^2)(4-b^2)} + \frac{-R_1(b^2-2) - bR_2}{(1-b^2)(4-b^2)}$$

$$= \frac{-R_1(2-b-b^2)}{(1-b^2)(4-b^2)} + \frac{-R_1(b^2-2) - bR_2}{(1-b^2)(4-b^2)}$$

$$= \frac{b(R_1-R_2)}{(1-b^2)(4-b^2)}$$

Due to $R_1 > R_2$, hence $\frac{b(R_1-R_2)}{(1-b^2)(4-b^2)} > 0$. Then we can conclude that $q_{21}^* > q_{22}^*$. Finally, we can conclude that $q_{11}^* > q_{21}^* > q_{22}^*$.

$$q_{12}^* - q_{13}^* = \frac{R_2}{(1+b)(2-\mathrm{b})} + \frac{(b^2-2)(c_1+R_2) + b(c_2+R_1) - bc_2 + c_1(2-b^2)}{(4-b^2)(1-b^2)}$$

$$= \frac{R_2}{(1+b)(2-\mathrm{b})} + \frac{-(2-b^2)R_2 + bR_1}{(4-b^2)(1-b^2)}$$

$$= \frac{R_2(1-b)(2+b)}{(4-b^2)(1-b^2)} + \frac{-(2-b^2)R_2 + bR_1}{(4-b^2)(1-b^2)}$$

$$= \frac{R_2(2-b-b^2)}{(4-b^2)(1-b^2)} + \frac{-(2-b^2)R_2 + bR_1}{(4-b^2)(1-b^2)}$$

$$= \frac{b(R_1-R_2)}{(4-b^2)(1-b^2)}$$

Due to $R_1 > R_2$, $\frac{b(R_1-R_2)}{(4-b^2)(1-b^2)} > 0$ and thus we conclude $q_{12}^* > q_{13}^*$.

$$q_{13}^* - q_{23}^* = \frac{bc_2 - c_1(2-b^2) - bc_1 + c_2(2-b^2)}{(1-b^2)(4-b^2)}$$

$$= \frac{b(c_2-c_1) + (c_2-c_1)(2-b^2)}{(1-b^2)(4-b^2)}$$

$$= \frac{(c_2-c_1)(2+b-b^2)}{(1-b^2)(4-b^2)}$$

$$= \frac{(c_2-c_1)(2-b)(1+\mathrm{b})}{(1-b^2)(4-b^2)}$$

$$= \frac{(c_2-c_1)}{(1-\mathrm{b})(2+\mathrm{b})}$$

Due to , $\frac{(c_2-c_1)}{(1-\mathrm{b})(2+\mathrm{b})} > 0$ and $q_{13}^* > q_{23}^*$. Thus, $q_{23}^* < q_{13}^* < q_{12}^*$ holds.

From Table 1, we can conclude that: $q_{22}^* < q_{21}^* < q_{11}^* < Q < q_{23}^* < q_{13}^* < q_{12}^*$. Thus the property 1 holds. ■

Supplementary document C: Cournot competition model

The following context are the Cournot competition model, in which the annual container throughput of terminal operator is considered as decision variable to compete with each other. The Cournot competition model has the same properties compare with the Bertrand model. Due to the properties and proofs of the properties for the Cournot competition model are similar with the Bertrand model, we omitted the properties and proofs of the properties for the Cournot competition model.

1 Revenue-sharing scheme with a single rate

Chen and Liu (2014) proposed a game model for two competitive container-terminal operators with the relationship between the amount of cargo and the terminal handling charge as follows:

$$p_1 = 1 - q_1 - bq_2 \tag{S-1}$$

$$p_2 = 1 - q_2 - bq_1. \tag{S-2}$$

The profit of each container-terminal operator then becomes $\pi_i(R) = p_i q_i - (c_i + R)q_i$. Chen and Liu (2014) obtained the Nash equilibrium (NE) analytically for the competitive game as follows:

$$q_1^* = \frac{1-R}{2+b} + \frac{bc_2 - 2c_1}{4-b^2} \text{ and } q_2^* = \frac{1-R}{2+b} + \frac{bc_1 - 2c_2}{4-b^2}\,. \tag{S-3}$$

The profit becomes

$$\pi_i^*(R) = (q_i^*)^2 \text{ for } i = 1, 2. \tag{S-4}$$

Note that $q_1^* > q_2^* \geq 0$ and $\pi_1^* > \pi_2^* \geq 0$. (S-5)

From $q_2^* \geq 0$,

$$R \leq \overline{r} \equiv \frac{2(1-c_2)-b(1-c_1)}{2-b}, \tag{S-6}$$

and from $\overline{r} \geq 0$,

$$c_2 < \bar{c}_2 \equiv \frac{2-b+bc_1}{2}. \tag{S-7}$$

The revenue of the port authority, $Z(R)$, becomes $R(q_1 + q_2)$. The optimal rental fee per TEU may be derived (Chen and Liu, 2014) as follows:

$$R^* = \frac{1}{2} - \frac{c_1+c_2}{4}. \tag{S-8}$$

From $R^* < \overline{r}$, it can be shown that $c_2 \leq \hat{c}_2 \equiv \frac{3bc_1+4-2b}{6+b} (< \bar{c}_2)$, which is a tighter upper bound than $\hat{c}_2 \equiv \frac{4+3bc_1+2c_1-2b}{6+b}$ suggested by Chen and Liu (2014).

2 Revenue-sharing scheme with a quantity discount

2.1 Revenue-sharing scheme with an incremental discount

2.1.1 Optimal behaviours of terminal operators

The profit function for container-terminal operator i may be expressed as follows:

$$\pi_i = \begin{cases} p_i q_i - (c_i + R_1) q_i & q_i < Q \\ p_i q_i - c_i q_i - R_1 Q - R_2 (q_i - Q) & q_i \geq Q. \end{cases} \tag{S-11}$$

Given (R_1, R_2, Q), which is provided by the port authority, the container-terminal operators compete with each other to decide their own optimal cargo amounts (q_1^*, q_2^*) by solving the following problems:

$$\max_{q_1 \geq 0} \pi_1 = \begin{cases} p_1 q_1 - (c_1 + R_1) q_1 & q_1 < Q \\ p_1 q_1 - c_1 q_1 - R_1 Q - R_2 (q_1 - Q) & q_1 \geq Q \end{cases} \tag{S-12}$$

$$\max_{q_2 \geq 0} \pi_2 = \begin{cases} p_2 q_2 - (c_2 + R_1) q_2 & q_2 < Q \\ p_2 q_2 - c_2 q_2 - R_1 Q - R_2 (q_2 - Q) & q_2 \geq Q. \end{cases} \tag{S-13}$$

From the first order necessary conditions for maximizing π_i ($\frac{\partial \pi_1}{\partial q_1} = \frac{\partial \pi_2}{\partial q_2} = 0$),

$$\frac{\partial \pi_1}{\partial q_1} = \begin{cases} 1 - 2q_1 - bq_2 - c_1 - R_1 & q_1 < Q \\ 1 - 2q_1 - bq_2 - c_1 - R_2 & q_1 \geq Q \end{cases} \tag{S-14}$$

$$\frac{\partial \pi_2}{\partial q_2} = \begin{cases} 1 - 2q_2 - bq_1 - c_2 - R_1 & q_2 < Q \\ 1 - 2q_2 - bq_1 - c_2 - R_2 & q_2 \geq Q. \end{cases} \tag{S-15}$$

Hence, $\frac{\partial^2 \pi_i}{\partial (q_i)^2} = -2$ and $\frac{\partial^2 \pi_i}{\partial q_1 q_2} = -b$. Thus, the Hessian matrix, H>0, implies that π_i is convex with respect to q_1 and q_2. If the equations $\frac{\partial \pi_1}{\partial q_1} = 0$ and $\frac{\partial \pi_2}{\partial q_2} = 0$ are solved simultaneously, the NE solution of the cargo amounts (q_1^*, q_2^*) can be obtained.

Considering whether two terminal operators are entitled to a concession discount or not, four different cases are shown as follows:

Case 1: Both container-terminal operators 1 and 2 are not entitled to the discount.

Case 2 (primary): Container-terminal operator 1 is not entitled to the discount but container-terminal operator 2 is.

Case 3: Container-terminal operator 2 is not entitled to the discount but container-terminal operator 1 is.

Case 4: Both container-terminal operators 1 and 2 are entitled to the discount.

By solving the first order necessary conditions for cases 1, 2, 3, and 4, the optimal cargo amount (q_1^*, q_2^*) and optimal profit (π_1^*, π_2^*) can be obtained, as shown in Supplementary Table 1.

Supplementary Table 1. Optimal cargo amounts (q_1^*, q_2^*) and optimal profit (π_1^*, π_2^*) for container-terminal operators 1 and 2 for revenue-sharing scheme with an incremental discount.

Cases	q_1^*, q_2^*	π_1^*, π_2^*
1	$q_1^*(R_1, R_1, q_1, q_{21}^*) = q_{11}^*$ $= \dfrac{bc_2 - 2c_1 + 2 - b - 2R_1 + bR_1}{4 - b^2}$ $q_2^*(R_1, R_1, q_{11}^*, q_2) = q_{21}^*$ $= \dfrac{bc_1 - 2c_2 + 2 - b - 2r_1 + br_1}{4 - b^2}$	$\pi_1^*(R_1, R_1, q_{11}^*, q_{21}^*) = \pi_{11}^* = (q_{11}^*)^2$ $\pi_2^*(R_1, R_1, q_{11}^*, q_{21}^*) = \pi_{21}^* = (q_{21}^*)^2$
2	$q_1^*(R_2, R_1, q_1, q_{22}^*) = q_{12}^*$ $= \dfrac{bc_2 - 2c_1 + 2 - b - 2R_2 + bR_1}{4 - b^2}$ $q_2^*(R_2, R_1, q_{12}^*, q_2) = q_{22}^*$ $= \dfrac{bc_1 - 2c_2 + 2 - b - 2R_1 + bR_2}{4 - b^2}$	$\pi_1^*(R_2, R_1, q_{12}^*, q_{22}^*) = \pi_{12}^*$ $= (q_{12}^*)^2 + R_2 Q - R_1 Q$ $\pi_2^*(R_2, R_1, q_{12}^*, q_{22}^*) = \pi_{22}^* = (q_{22}^*)^2$
3	$q_1^*(R_2, R_2, q_1, q_{23}^*) = q_{13}^*$ $= \dfrac{bc_2 - 2c_1 + 2 - b - 2R_2 + bR_2}{4 - b^2}$ $q_2^*(R_2, R_2, q_{13}^*, q_2) = q_{23}^*$ $= \dfrac{bc_1 - 2c_2 + 2 - b - 2R_2 + bR_2}{4 - b^2}$	$\pi_1^*(R_2, R_2, q_{13}^*, q_{23}^*) = \pi_{13}^*$ $= (q_{13}^*)^2 + R_2 Q - R_1 Q$ $\pi_2^*(R_2, R_2, q_{13}^*, q_{23}^*) = \pi_{23}^*$ $= (q_{23}^*)^2 + R_2 Q - R_1 Q$

2.1.2 Optimal concession contracts

Suppose that (R_1, R_2) are given. When Q is very large, both terminal operators do not want to accept the discount rate, which is Case 1. As Q becomes smaller, there are two possible routes for the transition of the cases: Case 1 → Case 2 → Case 4 or Case 1 → Case 3 → Case 4. Table 2 lists the boundary values of Q, which changes the situation from one case to another. For example, suppose that the current situation corresponds to case 1, which means Q is too large for both terminals to utilize the discounted unit rate. When Q decreases, π_{13}^* increases and then exceeds π_{11}^* at a specific boundary value of Q. The boundary value of Q, which is denoted as

Q_{11}, may be obtained by solving $\pi_{11}^{*}=\pi_{13}^{*}$. In the same way, the other boundary values may be obtained, as shown in Supplementary Table 2.

Supplementary Table 2 . Boundary values of Q between cases.

From	To	Boundary value of Q
Case 1	Case 2	From $\pi_{11}^{*}=\pi_{13}^{*}$, $Q=Q_1\equiv\frac{4(bc_2-2c_1+2-b-R_1-R_2+bR_1)}{(4-b^2)^2}$
Case 2	Case 3	From $\pi_{23}^{*}=\pi_{24}^{*}$, $Q=Q_2\equiv\frac{4(bc_1-2c_2+2-b-R_1-R_2+bR_2)}{(4-b^2)^2}$

The properties 1, 2, and 3 are hold for the revenue-sharing scheme with an incremental discount by using Cournot competition model. Due to the method to proof for these properties are similar, we omit the proofs for these properties.

2.2 *Revenue-sharing scheme with an all-unit discount*

2.2.1 Optimal behaviours of terminal operators

The NE quantity may be derived easily using the first order necessary conditions, as listed in Table 3.

Supplementary Table 3. Optimal cargo amounts (q_1^*, q_2^*) and optimal profit (π_1^*, π_2^*) for container-terminal operator 1 and 2 for a revenue-sharing scheme with all-unit discount.

Case	Simplified notation	Formal notation	Expression
1	q_{11}	$q_1^*(R_1, R_1, q_1, q_{21})$	$\frac{bc_2-2c_1+2-b-2R_1+bR_1}{4-b^2}$
	q_{21}	$q_2^*(R_1, R_1, q_{11}, q_2)$	$\frac{bc_1-2c_2+2-b-2R_1+bR_1}{4-b^2}$
2	q_{12}	$q_1^*(R_2, R_2, q_1, q_{22})$	$\frac{bc_2-2c_1+2-b-2R_2+bR_2}{4-b^2}$
	q_{22}	$q_2^*(R_2, R_2, q_{12}, q_2)$	$\frac{bc_1-2c_2+2-b-2R_2+bR_2}{4-b^2}$
3	q_{13}	$q_1^*(R_2, R_1, q_1, q_{23})$	$\frac{bc_2-2c_1+2-b-2R_2+bR_1}{4-b^2}$
	q_{23}	$q_2^*(R_2, R_1, q_{13}, q_2)$	$\frac{bc_1-2c_2+2-b-2R_1+bR_2}{4-b^2}$
4	q'_{13}	$q_1^*(R_2, R_1, Q, q'_{23})$	Q

	q'_{23}	$q_2^*(R_2, R_1, Q, q_2)$	$\frac{1-bQ-c_2-R_1}{2}$
5	q'_{12}	$q_1^*(R_2, R_2, q_1, Q)$	$\frac{1-bQ-c_1-R_2}{2}$
	q'_{22}	$q_2^*(R_2, R_2, q'_{12}, Q)$	Q
6	q''_{12}	$q_1^*(R_2, R_2, Q, Q)$	Q
	q''_{22}	$q_2^*(R_2, R_2, Q, Q)$	Q

From $c_2 > c_1$, $R_2 < R_1$, and $0 < b < 1$, the following inequalities follow:

$q_{11} < q_{12} < q_{13}$, $q_{23} < q_{21} < q_{22}$, $q_{11} > q_{21}$, $q_{12} > q_{22}$.

4.2.2 Optimal concession contracts

Supplementary Table 4. Change of the revenue of the port authority of each case for change of Q

At	Revenue increases as Q	Until it changes to	At boundary value of Q	If
Case 4	Increase	Case 1	Q_1	$R_2 \geq bR_1/2$
Case 4	Decrease	Case 3	Q_2	$R_2 < bR_1/2$
Case 6	Increase	Case 4	Q_3	-
Case 5	Increase	Case 3	Q_4	-

- Denotes that for given any R_1 and R_2

The details for deriving Q_i is presented as follows. Q_1 is the value of Q satisfying that $\pi_{13}(R_2, R_1, Q, q'_{23}) = \pi_{11}(R_1, R_1, q_{11}, q_{21})$ and Q_2 is the value of q_{13}. Q_3 is the value of Q satisfying that $\pi_{22}(R_2, R_2, Q, Q) = \pi_{23}(R_2, R_1, Q, q'_{23})$. Q_4 is the value of Q satisfying that $\pi_{22}(R_2, R_2, q_{12}, Q) = \pi_{23}(R_2, R_1, q_{13}, q_{23})$. By solving the above equations of Q, we can derive the Q_1, Q_2, Q_3, and Q_4 which are presented as follows:

$$Q_1 = \frac{bc_2-2c_1+2-b-2R_2+bR_1}{4-2b^2} - \frac{\sqrt[2]{\left(\frac{bc_2-2c_1+2-b-2R_2+bR_1}{2}\right)^2+2(b^2-2)\left(\frac{bc_2-2c_1+2-b-2R_1+bR_1}{4-b^2}\right)^2}}{b^2-2}$$

$$Q_2 = \frac{bc_2-2c_1+2-b-2R_2+bR_1}{4-b^2}$$

$$Q_3 = \frac{(b-bc_2-bR_1-2c_2-2R_2+2)}{(b^2+4b+4)} + \frac{\sqrt{(b-bc_2-bR_1-2c_2-2R_2+2)^2-\left((b+2)(1-c_2-R_1)\right)^2}}{(b^2+4b+4)},$$

$$Q_4 = \frac{bc_1-2c_2+2-b-2R_2+bR_2}{4-2b^2} - \frac{\sqrt[2]{\left(\frac{bc_1-2c_2+2-b-2R_2+bR_2}{2}\right)^2+2(b^2-2)(\frac{bc_1-2c_2+2-b-2R_1+bR_2}{4-b^2})^2}}{b^2-2}.$$

The properties 7, 8, and 9 hold for the revenue-sharing scheme with all unit discount by using Cournot competition model. Due to the method to prove these properties are similar, we omit the proofs for these properties.