Chapter 5. Weak Bisimulation and Observation Congruence

Goals

- Introduce the notion of *weak bisimulation* and *observation congruence*
- Properties of weak bisimulation and observation congruence
- Techniques for establishing weak bisimulation and observation congruence
- Differences and relationships between the three bisimulation relations

Overview

- In SB, every α action of one agent must be matched by an α action of the other agent, and vice versa—even for τ actions. As a result,

$$a.\tau.b.\mathbf{0} \not\sim a.b.\mathbf{0}$$

- The notion of *weak bisimulation* (WB) treats τ actions as unobservable, i.e.
  - it merely requires that each τ action of one agent be matched by **zero** or **more** τ actions of the other;
  - and that each external action l of one agent be matched by an l action accompanied, before or after, by **zero** or **more** τ actions of the other;
  - so, we should have that $a.\tau.b.\mathbf{0}$ and $a.b.\mathbf{0}$ are *weakly bisimilar*.

1. Weak Bisimulation Games

A *WB game* of interaction from a pair of agents (P_0, Q_0) is a finite or infinite sequence of the form

$$(P_0, Q_0), \ldots, (P_i, Q_i), \ldots$$

- played by two participants or observers, player I and player II such that
- player I attempts to show that an *observable* difference in behaviour is detectable, whereas player II tries to prevent this.

Rules of WB game

For each j the pair (P_{j+1}, Q_{j+1}) is determined as the result of a next step from the previous pair (P_j, Q_j) as follows:

- First player I chooses P_j (or Q_j) and a transition $P_j \xrightarrow{\alpha} P_{j+1}$ (or $Q_j \xrightarrow{\alpha} Q_{j+1}$).
- Then player II has to choose Q_j (or P_j) and respond as follows:
  - if $\alpha = \tau$, choose Q_j (or P_j) as Q_{j+1} (or P_{j+1}), or she can make one or more τ transitions
  $$Q_j \xrightarrow{\tau} \ldots \xrightarrow{\tau} Q_{j+1}$$
  $$(or \quad P_j \xrightarrow{\tau} \ldots \xrightarrow{\tau} P_{j+1})$$
  - if $\alpha \neq \tau$, choose a corresponding transition from the other agent accompanied, before or after, by zero or more τ transitions
  $$Q_j (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* Q_{j+1}$$
  $$(or \quad P_j (\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* P_{j+1})$$

Winner of WB game

If at any point a player is unable to make a move, then the other player *wins* the game:

- Player I is stuck if both agents are deadlocked.
- Player II is at a loss if no corresponding transition is available.
- If the game continues forever (is infinite) or if there is a repeated configuration, the pair (P_{j+1}, Q_{j+1}) has occurred previously, then player II also wins.

WB game equivalence

A player has a *wining strategy* from a pair (P_0, Q_0) if she is able to win any game from the pair.

Two agents P_0 and Q_0 are *WB game equivalent* if player II has a winning strategy from (P_0, Q_0).

In other words, whatever moves player I makes, player II can always match them.

Remark Clearly, P and P are WB game equivalent.

Example 1:

Consider $(P, \tau.P)$. Whenever $P \xrightarrow{\alpha} P'$ by player I, player II can response with

$$\tau.P \xrightarrow{\tau} P \xrightarrow{\alpha} P'$$

And if player I choose $\tau.P \xrightarrow{\tau} P$, player II can response by simply not making any transition on P

Thus, player II always wins, and P and $\tau.P$ are WB game equivalent.

Example 2:

Consider the following agents

$$V \stackrel{def}{=} 1p.(little.collect.V + 1p.big.collect.V)$$
$$V' \stackrel{def}{=} 1p.little.collect.V' + 1p.1p.big.collect.V'$$

Player I has a winning strategy from (V, V') as follows

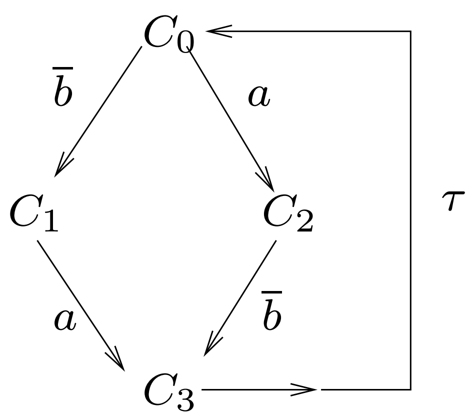
1. Player I chooses: $V' \xrightarrow{1p} 1p.big.collect.V'$
2. Play II has to make:
   $V \xrightarrow{1p} little.collect.V + 1p.big.collect.V$
3. Player I opts for $little.collect.V + 1p.big.collect.V$ and
   $$little.collect.V + 1p.big.collect.V \xrightarrow{little} collect.V$$
4. Player II cannot make a *little* transition from $1p.big.collect.V'$. Thus, Player II looses.
5. Thus, V and V' are not WB game equivalent.

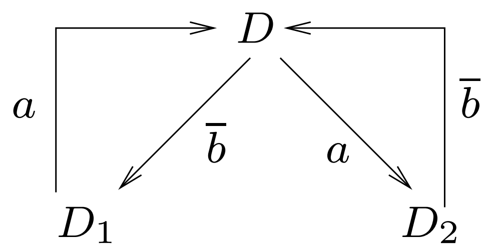
Example 3:

Let

$$C_0 \stackrel{def}{=} \overline{b}.C_1 + a.C_2 \qquad C_1 \stackrel{def}{=} a.C_3$$
$$C_2 \stackrel{def}{=} \overline{b}.C_3 \qquad C_3 \stackrel{def}{=} \tau.C_0$$

$$D \stackrel{def}{=} a.D_2 + \overline{b}.D_1$$
$$D_1 \stackrel{def}{=} a.D \qquad D_2 \stackrel{def}{=} \overline{b}.D$$





Then C_0 and D are WB game equivalent as any game will go through the following pairs of states (not particularly in this order):

$(C_0, D), (C_1, D_1), (C_2, D_2), (C_3, D)$

The associated weak bisimulation relation for (C_0, D) is

$\{(C_0, D), (C_1, D_1), (C_2, D_2), (C_3, D)\}$

Weak Bisimulation and Observation Congruence

Goals

- Brief motivation of *weak bisimulation* and *observation congruence*
- Definitions of *weak bisimulation* and *observation congruence*
- Properties of weak bisimulation and observation congruence
- Techniques for establishing weak bisimulation and observation congruence
- Differences and relationships between the three bisimulation relations

2. Weak Bisimulation

Preliminary definitions

- **Definition 1** Act^* is the set of all finite sequences of actions in Act; $\varepsilon \in Act^*$ is the empty sequence; α^n is the sequence of n actions α.
- **Definition 2** For $t \in Act^*$, $\widehat{t}$ is the sequence gained by deleting all occurrences of τ from t.

  Note: $\widehat{\tau^n} = \varepsilon$
- **Definition 3** For $t = \alpha_1 \ldots \alpha_n \in Act^*$, we write $E \stackrel{t}{\longrightarrow} E'$ instead of

  $$E \stackrel{\alpha_1}{\longrightarrow} \cdots \stackrel{\alpha_n}{\longrightarrow} E'$$
- **Definition 4** For $t = \alpha_1 \ldots \alpha_n \in Act^*$, we write $E \stackrel{t}{\Longrightarrow} E'$ instead of

  $$E(\stackrel{\tau}{\longrightarrow})^* \stackrel{\alpha_1}{\longrightarrow} (\stackrel{\tau}{\longrightarrow})^* \cdots (\stackrel{\tau}{\longrightarrow})^* \stackrel{\alpha_n}{\longrightarrow} (\stackrel{\tau}{\longrightarrow})^* E'$$

  For example $E \stackrel{ab}{\Longrightarrow} E'$ means that there exist $p, q, r \geq 0$ such that

  $$E \stackrel{\tau^p}{\longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^q}{\longrightarrow} \stackrel{b}{\longrightarrow} \stackrel{\tau^r}{\longrightarrow} E'$$

Properties of $\xrightarrow{t}$, $\overset{t}{\Rightarrow}$, $\overset{\widehat{t}}{\Rightarrow}$

- Each specifies an action sequence with **exactly the same observable actions**, namely those in t, but they are different w.r.t. τ actions:
  - $\xrightarrow{t}$ specifies **exactly** the τ actions occurring in t.
  - $\overset{t}{\Rightarrow}$ specifies **at least** the τ actions occurring in t.
  - $\overset{\widehat{t}}{\Rightarrow}$ specifies **nothing** about τ actions.
- $P \xrightarrow{t} P'$ implies $P \overset{t}{\Rightarrow} P'$, and $P \overset{t}{\Rightarrow} P'$ implies $P \overset{\widehat{t}}{\Rightarrow} P'$.

Weak Bisimulation

Definition 5 A relation $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ is a *weak bisimulation* (WB) if, whenever $P\mathcal{S}Q$ and $\alpha \in Act$, then

1. if $P \xrightarrow{\alpha} P'$, then, for some Q', $Q \stackrel{\widehat{\alpha}}{\Rightarrow} Q'$ and $P'\mathcal{S}Q'$, and
2. if $Q \xrightarrow{\alpha} Q'$, then, for some P', $P \stackrel{\widehat{\alpha}}{\Rightarrow} P'$ and $P'\mathcal{S}Q'$.

Agents P and Q are *weakly bisimilar*, written $P \approx Q$, if there is a WB $\mathcal{S}$ such that $P\mathcal{S}Q$.

Proposition 1

A relation $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ is a WB iff whenever $P\mathcal{S}Q$ then

1. if $P \xrightarrow{l} P'$ then for some Q', $Q \overset{l}{\Rightarrow} Q'$ and $P'\mathcal{S}Q'$,
2. if $P \xrightarrow{\tau} P'$ then for some Q', $Q(\xrightarrow{\tau})^*Q'$ and $P'\mathcal{S}Q'$,
3. if $Q \xrightarrow{l} Q'$ then for some P', $P \overset{l}{\Rightarrow} P'$ and $P'\mathcal{S}Q'$,
4. if $Q \xrightarrow{\tau} Q'$ then for some P', $P(\xrightarrow{\tau})^*P'$ and $P'\mathcal{S}Q'$,

Corollary 2 For all P and Q, $P \sim Q$ implies $P \approx Q$.

The converse of Corollary 2 is clearly not valid.

By Corollary 2 all the **equational** laws for $\sim$ hold for $\approx$.
Moreover, all three τ laws hold for $\approx$.
Thus, all equational laws from Chapter 3 hold for $\approx$:

$$\forall P, Q.\ P = Q \text{ implies } P \approx Q$$

But the converse is not valid! There are agents P and Q, as in Example 4, such that

$$P \approx Q \text{ and } P \neq Q$$

Example 4

1. $P \approx \tau.P$ since the following is a WB
$$\{(P, \tau.P), (P, P), (P', P') \mid P \xrightarrow{t} P'\}$$
Recall that $P \neq \tau.P$.

2. $\mu.\tau.P \approx \mu.P$ since the following is a WB
$$\{(\mu.\tau.P, \mu.P), (\tau.P, P), (P, P), (P', P') \mid P \xrightarrow{t} P'\}$$

3. $P + \tau.P \approx \tau.P$ since the following is a WB
$$\{(P + \tau.P, \tau.P), (P, P), (P', P') \mid P \xrightarrow{t} P'\}$$

4. $\alpha.(P + \tau.Q) + \alpha.Q \approx \alpha.(P + \tau.Q)$ since the following is a WB

$$\{(\alpha.(P + \tau.Q) + \alpha.Q,\ \alpha.(P + \tau.Q)),$$

$$(P, P), (Q, Q), (P', P'), (Q', Q') \mid P \xrightarrow{t} P', Q \xrightarrow{s} Q'\}$$

Example 5 For C_0 and D as in Example 3, $C_0 \approx D$ since the following is a WB

$$\{(C_0, D), (C_1, D_1), (C_2, D_2), (C_3, D)\}$$

Example 6 $a.\tau.b.\mathbf{0} \approx a.b.\mathbf{0}$. The following is a WB

$$\{(a.\tau.b.\mathbf{0}, a.b.\mathbf{0}), (\tau.b.\mathbf{0}, b.\mathbf{0}), (b.\mathbf{0}, b.\mathbf{0}), (\mathbf{0}, \mathbf{0})\}$$

Example 7 (**$\approx$ is not a congruence relation**)

Although $b.\mathbf{0} \approx \tau.b.\mathbf{0}$, but if $a \neq b$, then

$$a.\mathbf{0} + b.\mathbf{0} \not\approx a.\mathbf{0} + \tau.b.\mathbf{0}$$

Proof: If there is a WB $\mathcal{S}$ such that $(LHS, RHS) \in \mathcal{S}$, then

since	$RHS \xrightarrow{\tau} b.\mathbf{0}$	
we need	$LHS \overset{\hat{\tau}}{\Rightarrow} P'$	for some P',
and	$(P', b.\mathbf{0}) \in \mathcal{S}$;	
in fact	$P' \equiv LHS$	and clearly
	$(a.\mathbf{0} + b.\mathbf{0},\ b.\mathbf{0}) \notin \mathcal{S}$	

Thus, $\approx$ is not a congruence relation:

although $b.\mathbf{0} \approx \tau.b.\mathbf{0}$ but not $a.\mathbf{0} + b.\mathbf{0} \approx a.\mathbf{0} + \tau.b.\mathbf{0}$!

3. Properties of Weak Bisimulation

Weak bisimulation shares many properties with strong bisimulation.

Proposition 3 Assume that each $\mathcal{S}_i$ $(i = 1, 2, \ldots)$ is a WB. Then the following relations are WBs

(1)	$Id_{\mathcal{P}}$	(3)	$\mathcal{S}_1\mathcal{S}_2$
(2)	$\mathcal{S}_i^{-1}$	(4)	$\bigcup_{i \in I} \mathcal{S}_i$

Proposition 4

1. $\approx$ is the largest WB.
2. $\approx$ is an equivalence relation.
3. $P \approx Q$ iff, for all $\alpha \in Act$
   (a) Whenever $P \xrightarrow{\alpha} P'$ then, for some Q',
   $$Q \overset{\widehat{\alpha}}{\Rightarrow} Q' \text{ and } P' \approx Q'$$
   (b) Whenever $Q \xrightarrow{\alpha} Q'$ then, for some P'
   $$P \overset{\widehat{\alpha}}{\Rightarrow} P' \text{ and } P' \approx Q'$$

Note:

$\approx$ **is not** a congruence relation since the choice operator does not preserve it:

- it is not in general the case that if $P \approx Q$ then $P + R \approx Q + R$, as shown in Example 7.

Proposition 5

If $P \approx Q$, $P_1 \approx P_2$ and $P_i \approx Q_i$ for $i \in I$, then

1. $\alpha.P \approx \alpha.Q$
2. $\sum_{i \in I} \alpha_i.P_i \approx \sum_{i \in I} \alpha_i.Q_i$
3. $P_1|Q \approx P_2|Q$
4. $P_1 \backslash L \approx P_2 \backslash L$
5. $P_1[f] \approx P_2[f]$

This proposition tells us that prefixing, parallel, restriction and relabelling operators preserve $\approx$.

Also, a combined operation of a choice of prefixed agents preserves $\approx$.

But the choice operator **does not** preserve $\approx$.

5. Observation Congruence

Observation congruence is very closely related to $\approx$.

Definition 7 Agents P, Q are *observation congruent*, denoted by $P \approx_o Q$, if for all $\alpha \in Act$,

1. if $P \stackrel{\alpha}{\rightarrow} P'$, then, for some Q', $Q \stackrel{\alpha}{\Rightarrow} Q'$ and $P' \approx Q'$, and
2. if $Q \stackrel{\alpha}{\rightarrow} Q'$, then, for some P', $P \stackrel{\alpha}{\Rightarrow} P'$ and $P' \approx Q'$

Remarks

- $\approx_o$ differs from $\approx$ only in one respect: $\stackrel{\alpha}{\Rightarrow}$ takes the place of $\stackrel{\widehat{\alpha}}{\Rightarrow}$ for the first actions of P and Q only.
- Thus, each action of P or Q must be matched by *at least* one action of the other—this only applies to the first actions of P and Q.

This becomes more clear if we compare the above definition with Proposition 4.3:

(Recall **Proposition 4.3**: $P \approx Q$ iff, for all $\alpha \in Act$

1. if $P \stackrel{\alpha}{\rightarrow} P'$, then, for some Q', $Q \stackrel{\widehat{\alpha}}{\Rightarrow} Q'$ and $P' \approx Q'$
2. if $Q \stackrel{\alpha}{\rightarrow} Q'$, then, for some P' $P \stackrel{\widehat{\alpha}}{\Rightarrow} P'$ and $P' \approx Q'$)

It easy to show using the definitions that $\approx_o \subseteq \approx$.

The following result tells us clearly how to check that two agents are observation congruent.

Proposition 7

$P \approx_o Q$ iff

1. if $P \xrightarrow{l} P'$ then for some Q', $Q \stackrel{l}{\Rightarrow} Q'$ and $P' \approx Q'$,
2. if $P \xrightarrow{\tau} P'$ then for some Q', $Q \stackrel{\tau}{\Rightarrow} Q'$ and $P' \approx Q'$,
3. if $Q \xrightarrow{l} Q'$ then for some P', $P \stackrel{l}{\Rightarrow} P'$ and $P' \approx Q'$,
4. if $Q \xrightarrow{\tau} Q'$ then for some P', $P \stackrel{\tau}{\Rightarrow} P'$ and $P' \approx Q'$.

6. Fundamental Properties of $\approx_o$

Proposition 8

1. $\approx_o$ is an equivalence relation and a congruence relation.
2. All the equational laws for $=$ in Chapter 3 are valid for $\approx_o$.

Part 2 means that if $P = Q$ can be proved using the laws from Chapter 3, then $P \approx_o Q$.

Importantly, the converse is also true, i.e. if $P \approx_o Q$, then $P = Q$ can be proved using the laws from Chapter 3. Hence,

$$\forall P, Q.\ \ P \approx_o Q \ \text{ iff } \ P = Q$$

7. Relationship between $\sim$, $\approx$ and $\approx_o$

Proposition 9

1. If $P \sim Q$ then $P \approx Q$,
2. if $P \sim Q$ then $P \approx_o Q$,
3. if $P \approx_o Q$ then $P \approx Q$.
4. So, $\sim \subseteq \approx_o$ (which is the same as $=$) $\subseteq \approx$.
5. None of the inverses above are generally valid:

   $$b.\mathbf{0} \approx \tau.b.\mathbf{0}, \text{ but } b.\mathbf{0} \not\approx_o \tau.b.\mathbf{0}$$

   $$\tau.b.\mathbf{0} \approx_o \tau.\tau.b.\mathbf{0}, \text{ but } \tau.b.\mathbf{0} \not\sim \tau.\tau.b.\mathbf{0}$$

6. Thus, $\sim \subset \approx_o$ (same as $=$) $\subset \approx$.
7. Thus, all the **equational** laws for $\sim$ hold for $\approx_o$; and all the **equational** laws for $\approx_o$ hold for $\approx$.

Proposition 10

1. If $P \approx Q$ and P and Q are *stable*, i.e. have no immediate τ-transitions, then $P = Q$.
2. If $P \approx Q$, then $\alpha.P = \alpha.Q$

Further reading: Milner's book, Chapter 4, 5 and 7.

Equational laws

- **Monoid laws**
  1. $P + Q = Q + P$ — commutativity
  2. $P + (Q + R) = (P + Q) + R$ — associativity
  3. $P + P = P$ — Idempotence
  4. $P + \mathbf{0} = P$ — $\mathbf{0}$ is the zero element of $+$
- **The τ laws**
  1. $\alpha.\tau.P = \alpha.P$ — Drop any τ except the first one
  2. $P + \tau.P = \tau.P$ — Add a first τ
  3. $\alpha.(P + \tau.Q) + \alpha.Q = \alpha.(P + \tau.Q)$
- **Laws for Agent constants and equations**
  1. If $A \stackrel{def}{=} P$, then $A = P$.
  2. Let E_i $(i \in I)$ contain at most the variables $\{X_j : j \in I\}$, and let these variables are guarded and sequential in each E_i. Then

     If $\tilde{P} = \tilde{E}\{\tilde{P}/\tilde{X}\}$ and $\tilde{Q} = \tilde{E}\{\tilde{Q}/\tilde{X}\}$
     then $\tilde{P} = \tilde{Q}$
- **The expansion law**

  Let $P \equiv (P_1[f_1]| \ldots |P_n[f_n])\backslash L$. Then

$$
\begin{aligned}
P \quad = \quad & \sum\{f_i(\alpha).(P_1[f_1]| \ldots |P_i'[f_i]| \ldots |P_n[f_n])\backslash L : \\
& P_i \xrightarrow{\alpha} P_i', f_i(\alpha) \notin L \cup \overline{L}\} \\
+ \quad & \sum\{\tau.(P_1[f_1]| \ldots |P_i'[f_i]| \ldots |P_j'[f_j]| \ldots |P_n[f_n])\backslash L \\
& P_i \xrightarrow{l_1} P_i', P_j \xrightarrow{l_2} P_j', f_i(l_1) = \overline{f_j(l_2)}, i < j\}
\end{aligned}
$$

- **Composition laws**
  1. $P|Q = Q|P$ – commutativity
  2. $P|(Q|R) = (P|Q)|R$ – associativity
  3. $P|\mathbf{0}$ =P – $\mathbf{0}$ is an unit
- **Restriction laws**
  1. $P\backslash L = P$ if $\mathcal{L}(P) \cap (L \cup \overline{L}) = \emptyset$
  2. $P\backslash K\backslash L = P\backslash(K \cup L)$
  3. $P[f]\backslash L = (P\backslash f^{-1}(L))[f]$
  4. $(P|Q)\backslash L = P\backslash L|Q\backslash L$ if
  $$\mathcal{L}(P) \cap \overline{\mathcal{L}(Q)} \cap (L \cup \overline{L}) = \emptyset$$
- **Relabelling laws**
  1. $P[Id] = P$
  2. $P[f] = P[f']$ if $f \upharpoonright \mathcal{L}(P) = f' \upharpoonright \mathcal{L}(P)$
  3. $P[f][f'] = P[f' \circ f]$
  4. $(P|Q)[f] = P[f]|Q[f]$ if $f \upharpoonright (L \cup \overline{L})$ is one-one, where $L = \mathcal{L}(P|Q)$