Chapter 3

Markov Chains

3.1 Recurrence in Finite Markov Chains

In the lecture we distinguished between transient and recurrent states. A state is *transient* if starting in that state there is non-zero probability never to return. A state is *recurrent* if starting in that state the probability of returning to it is 1. We said that a state is *positive recurrent* if it is recurrent and the average time to return to it is finite.

Theorem 3.1 *If S is a finite (close) irreducible set of states then every state in S is positive recurrent.*

Proof We have to show that $M_s = \sum_{k=1}^{\infty} k\Theta_{i,k}$ is finite. We know that $\sum_{k=1}^{\infty} \Theta_{i,k} = 1$ as the state is recurrent. Consider the set S. By assumption, it is finite.

Corollary 3.2 *In a finite irreducible Markov chain every state is positive recurrent.*

3.2 Exercises

^{s}1. A geometric series is a series of elements $a_0, a_1, \ldots$ such that $a_i = c \cdot q^i$ for some constants c and q. An equivalent recursive definition is given by stating $a_0 = c$ and $a_{i+1} = q \cdot a_i$ for every $i \geq 0$.

We are interested in the sum of a geometric series:

$$\begin{aligned} A_n &= \textstyle\sum_{j=0}^{n} a_j \\ A_\infty &= \textstyle\sum_{j=0}^{\infty} a_j \end{aligned}$$

(a) Write down the first few elements (e.g., a_0, ..., a_5) in the series, where $c = 1$ and for the following values of q: 1, 2, $\frac{1}{2}$, $\frac{2}{5}$.

(b) Write down the first few elements (e.g, A_0, ..., A_5) in the sum series for the same values.

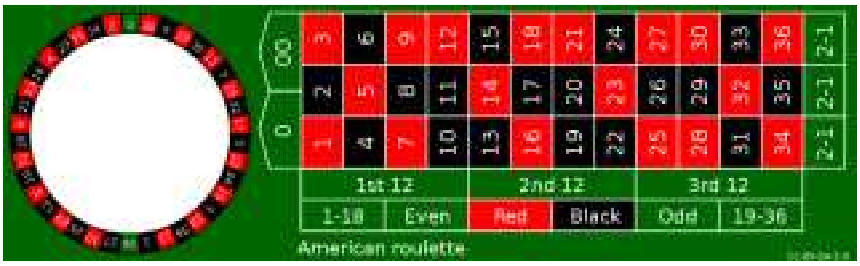
(c) Compute $(1 - q) \cdot (\sum_{j=0}^{n} q^j)$. Deduce an equation for $\sum_{j=0}^{n} q^j$.

(d) Use your formula for checking your computation of the previous sums.

For $q \geq 1$ the sum A_i tends towards infinity as i tends towards infinity. For $0 < q < 1$ the sum A_i tends towards $\frac{c}{1-q}$. In this case, $A_\infty = \frac{c}{1-q}$.

5. Compute the sums $\sum_{i=0}^{\infty} c \cdot q_i$ for $c = 1$ and $q = \frac{1}{2}$, $c = 2$ and $q = \frac{1}{3}$, and $c = 1$ and $q = 1$.

[s]2. This is the set of possible options in a normal roulette. In one partic-



ular casino the roulette was wired so that 0 and 00 are impossible. All other elementary events have equal probability.

(a) What is the probability space and what is the probability of every elementary event?

(b) What is the probability of the events:

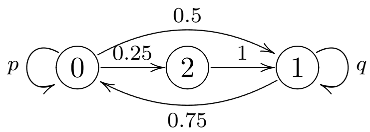
- R - getting a red number
- B - getting a black number
- Z - getting a 0 or 00
- E - getting an even number (other than 0 or 00)
- O - getting an odd number

(c) Which of these events are independent?

[s]3. We roll two dice.

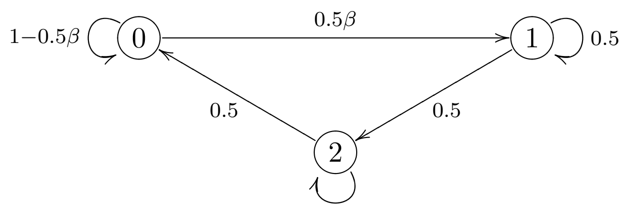
- What is the probability space and what is the probability of every elementray event?
- Define a random variable X that gets the sum of the two rolls. What is the expected value of X?
- Define two random variables X_i, where $i \in \{1, 2\}$, such that X_i is the value of the ith roll. What are the expected values of X_1 and X_2?

[s] 4. Consider the Markov chain defined by the following incomplete transition diagram:



   (a) For which values of p and q is the above diagram a probabilistic transition diagram?
   (b) What is the transition matrix of the above diagram?
   (c) Given that the system is in state 0 initially, what is the probability that it will be in state 0 again after three time steps?
   (d) Given that the system is in state 0 initially, what is the probability that it will be in state 0 again in *at most* three time steps?
   (e) Given that the system is in state 0 initially, what is the probability that it will reach state 1 without passing through state 2 first?

5. Consider the following probabilistic transition system:

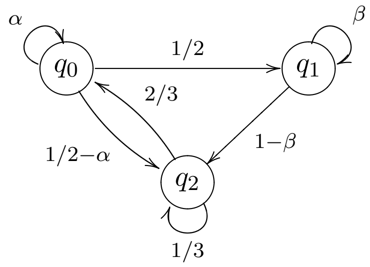


   (a) What constraints are there on the value of β (i.e. what is the possible range of β)? Justify your answer.
   (b) For which values of β is the Markov chain irreducible and all states are recurrent and aperiodic?

(c) Find the stationary probability distribution of the system for these values of β. (in terms of β.)

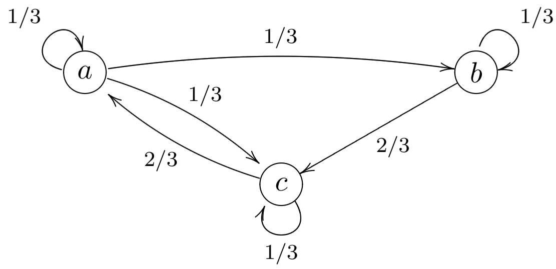
(d) Suppose that a designer of the system can control the value of β. The goal is to guarantee that, in the long run, the proportion of time spent in state 0 is less than or equal to 40%. What range of values of β is consistent with this goal?

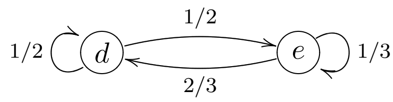
6. What are the possible pairs of values for (α, β) in the following Markov chain:



Which pairs give rise to a Markov chain that has a unique stationary behavior that does not depend on the initial distribution?

7. Consider the following two Markov chains:





(a) Assume that the two follow independent events. Construct a Markov chain that describes the status of these two events simultaneously.

(b) What is the largest irreducible set in the resulting Markov chain?

(c) Find the stationary probability distribution of this irreducible set (assuming that the probability to start in this irreducible set is 1).

[s]8. The stock level of a production plant can be either high, low or out of resources. The following transition probabilities are known:

- the probability of running low on resources if the resource level is high is 0.1; the resource level never changes from high to zero;
- the probability of an arriving delivery if the stock level is low or the plant has run out of resources is 0.5; in both cases, the stock levels will be high again;
- it never happens that the resource level changes from out of resources to low on resources;
- the probability of running out of resources, given that the plant is low on resources, is 0.2.

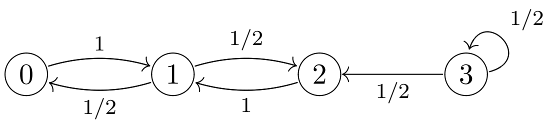
(a) Draw a probabilistic transition diagram representing the system that has been described above. Determine the probabilities of the transitions that have not been specified in the above description.

(b) Determine the transition matrix of the system

(c) Use the transition matrix to compute the probabilities that the plant will be high on resources, low on resources or out of resources after two time steps assuming that it is initially high on resources.

[s]9. In the probabilistic transition system shown below, which states are:

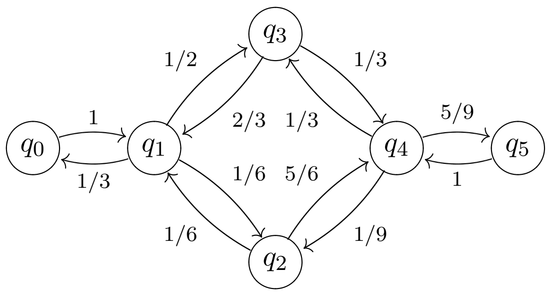
(i) periodic; (ii) aperiodic; (iii) transient; (iv) recurrent.



Give justifications for your answers.

Given the initial probability distribution $(1/2, 0, 0, 1/2)$, calculate the probability distribution after three steps.

10. Consider the probabilistic transition system shown below. Which states are periodic and which states are aperiodic? Which states are transient and which states are recurrent?

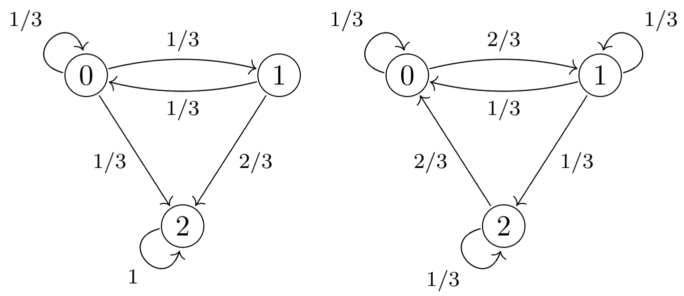


Give justifications for your answers.

Given the initial probability distribution $(1, 0, 0, 0, 0, 0)$, calculate the following.

- What is the probability distribution after four steps.
- What is the probability to get from state q_0 to itself in at most six steps.

[s] 11. Consider the two probabilistic transition systems shown below:

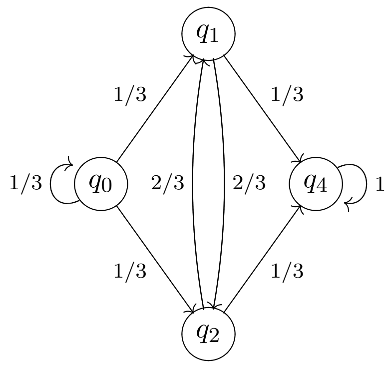


(a) Explain why all the states in both systems are aperiodic.

(b) Explain why one system is irreducible and the other is not.

(c) Find the unique stationary probability distribution of the irreducible system.

12. Consider the probabilistic transition system shown below. Which states are periodic and which states are aperiodic? Which states are transient and which states are recurrent?



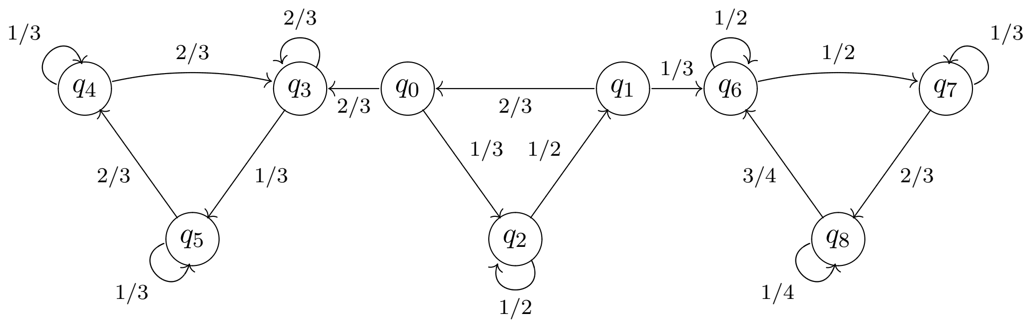
Give justifications for your answers.

Given the initial probability distribution $(1, 0, 0, 0)$, calculate the probability distribution after four steps.

13. For the probabilistic transition systems shown below:

    (a) explain briefly which of the states of the system are transient and which are recurrent. Which subsets of the states of the system are aperiodic and irreducible?

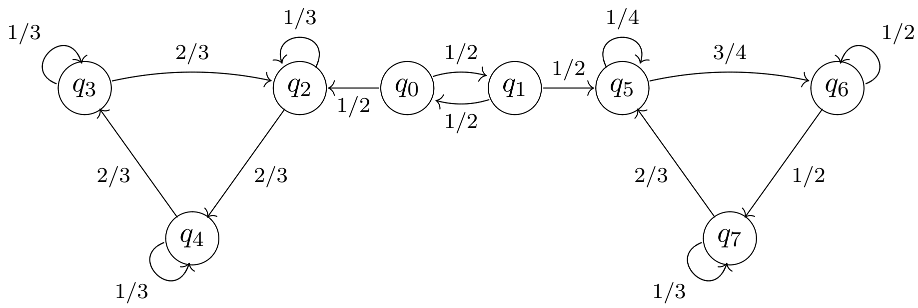
    (b) find the unique stationary probability distribution of the system starting from $(1/2, 1/2, 0, 0, 0, 0, 0, 0, 0)$.



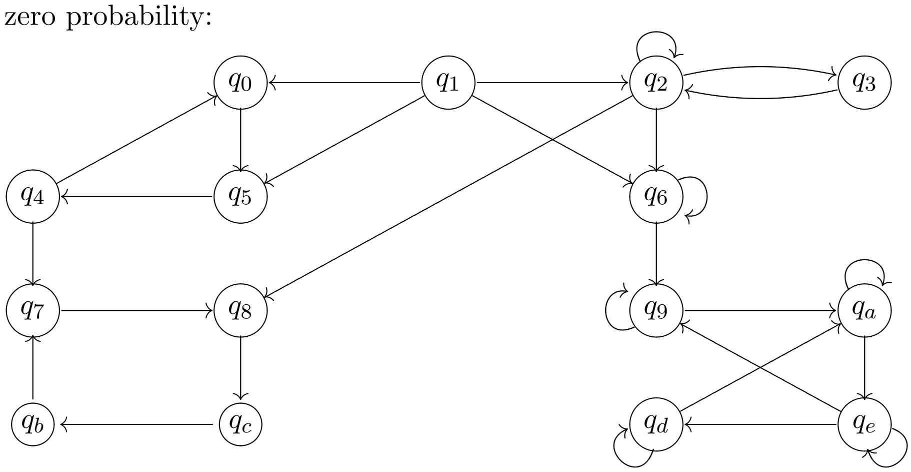
14. For the probabilistic transition systems shown below:

    (a) explain briefly which of the states of the system are transient and which are recurrent. Which subsetes of the states of the system are aperiodic and irreducible?

(b) find the unique stationary probability distribution of the system starting from $(1/2, 1/2, 0, 0, 0, 0, 0, 0)$.



15. Consider the following Markov chain, where all drawn edges have non-zero probability:



(a) Which states are periodic and which states are aperiodic?

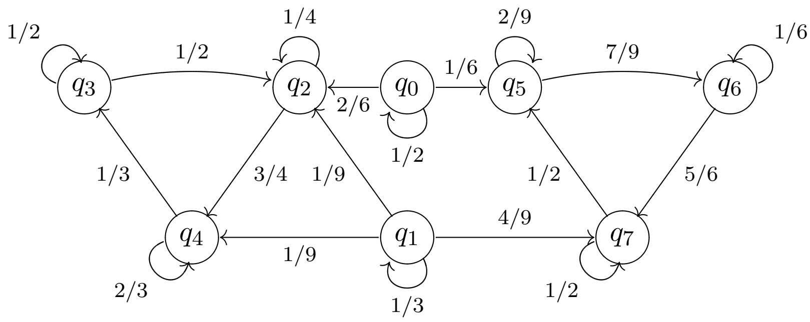
(b) Which states are transient and which states are recurrent?

(c) Find all the minimial closed sets in the Markov chain. Identify which are irreducible.

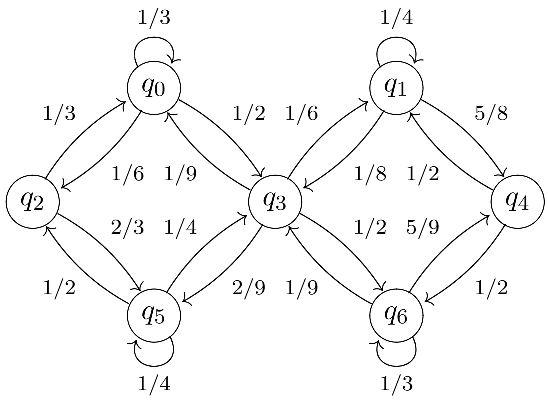
16. For the probabilistic transition systems shown below:

(a) explain briefly which of the states of the system are transient and which are recurrent. Which subsets of the states of the system are aperiodic and irreducible?

(b) find the unique stationary probability distribution of the system starting from $(1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$.



17. Consider the probabilistic transition system shown below.



Give justifications for your answers.

Given the initial probability distribution $(1/6, 1/9, 2/9, 1/6, 1/9, 1/9, 1/9)$, calculate the following.

- What is the probability distribution after five steps.
- What is the probability to get from state q_0 to itself in at most six steps.

[s] 18. Consider a variant of the packet transmission example shown in class. Here, the success of the next transmission does not depend on the success/failure of the previous transmission. Depicted as a Markov chain, the process looks like this:

(a) Let X_i be a random variable whose value is 1 in case that the ith transmission succeeds and 0 otherwise. What is the expected value of X_i, for every $i \geq 0$.

(b) Let Y be a random variable whose value is the number of successful transmissions in the first 40 tries. What is the expected value of Y.

^{s}19. Construct a Markov chain to model the following queue.

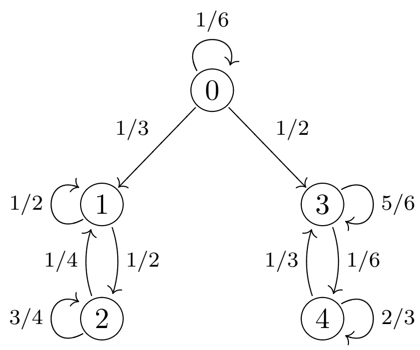
- The queue has two locations.
- Items arrive and depart from the queue one by one.
- If the queue is empty it will remain empty with probabilty half.
- If the queue is one location full, then a new item will arrive with probability 1/3 and an item will be removed from the queue with probability 1/6.
- If the queue is full, then an item will be removed with probability 1/4.

(a) Draw a diagram of the Markov chain model.

(b) Write down the transition probability matrix.

(c) If the queue starts empty, what is the probability that in at most 4 steps, it will reach a full state?

^{s}20. What is the stationary distribution for the following Markov chain starting from distribution $(1, 0, 0, 0, 0)$.



^{s}21. In an $M/M/1$ queue customers arrive (on average) every two minutes and the average service time is 40 seconds.

(a) What is the expected number of items in the system?

(b) What is the average turnaround time?

(c) What is the probability that there are at least five items in the system?

[s]22. An enquiry desk is handling questions from customers that approach it. There is one person on duty and it is observed that, on average, it takes her three minutes to deal with an enquiry. Assuming that the situation is modeled by an $M/M/1$ queue, what is the maximum arrival rate that can occur so that the expected waiting time before someone starts being served at the desk does not exceed five minutes?

[s]23. In an $M/M/1/4$ queue items arrive (on average) every two minutes and the average service time is 40 seconds.

(a) What is the expected number of items in the system?

(b) What is the average turnaround time?

Compare your results with your solution to Exercise 20. Does this seem reasonable? Briefly explain your answer.

[s]24. In an $M/M/1/N$ queue items arrive (on average) every four minutes and the average service time is three minutes. If an item arrives and the system is full (i.e. if there are N items already in the system) then the item is rejected. What is the minimum value of N that will keep the percentage of rejected items below 20%?

[s]25. In Lecture 21 we showed that the expected number of items in an $M/M/1/N$ queue is

$$L = \frac{\rho}{1-\rho^{N+1}}\left[\frac{1-\rho^N}{1-\rho} - N\rho^N\right]$$

provided that $\rho \neq 1$. Derive a formula for L (in terms of N) for the case $\rho = 1$.

26. In an $M/M/1$ queue items arrive (on average) every 3 minutes and the average service time is 60 seconds.

(a) What is the expected number of items in the system?

(b) What is the average turnaround time?

(c) What is the probability that there are at least five items in the system?

27. In an $M/M/1$ queue items arrive (on average) every 4 minutes and the average service time is 80 seconds.

    (a) What is the expected number of items in the system?

    (b) What is the average turnaround time?

    (c) What is the probability that there are at least five items in the system?

28. In an $M/M/1$ queue items arrive (on average) every six minutes and the average service time is 90 seconds.

    (a) What is the expected number of items in the system?

    (b) What is the average turnaround time?

    (c) What is the probability that there are at least four items in the system?

29. In an $M/M/1/3$ queue items are serviced (on average) every four minutes. If an item arrives and the system is full (i.e. if there are 3 items already in the system) then the incoming item is rejected. What is maximum arrival rate that would result in at most one item in seven being rejected by the system?

    Important: Distinguish between the cases $\lambda = \mu$ and $\lambda \neq \mu$.

30. In an $M/M/1/3$ queue items are serviced (on average) every three minutes. If an item arrives and the system is full (i.e. if there are 3 items already in the system) then the incoming item is rejected. What is maximum arrival rate that would result in at most one item in six being rejected by the system?

    Important: Distinguish between the cases $\lambda = \mu$ and $\lambda \neq \mu$.

31. Show that, if we have an $M/M/4$ queue where the average arrival rate is λ and the average service rate is μ, then the average number of items in the system is given by

$$L = 4\sigma + \frac{32\sigma^4}{(1-\sigma)(3 + 9\sigma + 12\sigma^2 + 8\sigma^3)}$$

where

$$\sigma = \frac{\lambda}{4\mu}.$$

32. In an $M/M/1/2$ queue items are serviced (on average) every four minutes. If an item arrives and the system is full (i.e. if there are 2 items already in the system) then the incoming item is rejected. What is maximum arrival rate that would result in at most one item in seven being rejected by the system?

    Important: Distinguish between the cases $\lambda = \mu$ and $\lambda \neq \mu$.

[s]33. Use the formula

$$T = \frac{1}{\mu} + \frac{1}{\mu}\frac{(m\sigma)^m}{m!}\frac{1}{m(1-\sigma)^2}P_0$$

from Lecture 22 for the average turnaround for an item in an $M/M/m$ queue to show that, in an $M/M/2$ queue, we have

$$T = \frac{1}{\mu(1-\sigma^2)}$$

where

$$\sigma = \frac{\lambda}{2\mu}.$$

[s]34. Show that, if we have an $M/M/3$ queue where the average arrival rate is λ and the average service rate is μ, then the average number of items in the system is given by

$$L = 3\sigma + \frac{9\sigma^4}{(1-\sigma)(2+4\sigma+3\sigma^2)}$$

where

$$\sigma = \frac{\lambda}{3\mu}.$$

35. We define a variant of Markov chains as acceptor of languages. A Markov chain is $M = (Q, p)$, where Q is a set of states and $p : Q \times Q \to [0, 1]$ is a transition probability function such that for every $q \in Q$ we have $\sum_{q' \in Q} p(q, q') = 1$.

    A Markov acceptor is $M = (Q, \Sigma, p, q_0, F)$, where Q is a *finite* set of states, Σ a finite alphabet, $q_0 \in Q$ is an initial state, $F \subseteq Q$ is a set of accepting states, $t \in [0, 1]$ is acceptance threshold, and $p : Q \times \Sigma \times Q \to$

$[0,1]$ is a transition matrix such that for every $q \in Q$ and every $\sigma \in \Sigma$ we have $\sum_{q' \in Q} p(q, \sigma, q') = 1$.

We can think of p as a set of $|\Sigma|$ transition probability matrices, each matching a single letter in Σ. We denote by $p(\sigma)$ the matrix corresponding to letter σ. Let $\mu : Q \to [0,1]$ be a probability distribution such that $\sum_{q \in Q} \mu(q) = 1$. Then, reading σ from μ corresponds to taking one step according to the Matrix $p(\sigma)$. Equally, reading σ from μ is the vector that results from multiplying the vector μ by the matrix $p(\sigma)$. Generalizing this definition to words, we get that the result of reading $w = \sigma_0, \ldots, \sigma_n$ from vector μ is

$$\mu \cdot p(\sigma_0) \cdots p(\sigma_n).$$

That is, we take the product of the matrices $p(\sigma_0)$, …, $p(\sigma_n)$ and multiply the vector μ by the result.

Let μ_0 denote the probability distribution that associates $\mu_0(q_0) = 1$ and $\mu_0(q) = 0$ for $q \neq q_0$. Then, the result of reading word w starting from μ_0 defines a distribution on the states in Q. Denote this distribution by $M(w)$. We say that M *accepts* w if $\sum_{f \in F} M(w)(f) > 0$. That is, we sum the probability that the word w leads from q_0 to an accepting state. If this probability is non-zero then the word is accepted.

Show that Markov acceptors accept exactly the regular languages.

3.3 Solutions to Selected Exercises

1. 1,2. The values of a_0, …, a_5 and A_0, …, A_5 for $c = 1$ and $q = 1, 2, \frac{1}{2}, \frac{2}{5}$ are as follows.

q	a_0	a_1	a_2	a_3	a_4	a_5
1	1	1	1	1	1	1
2	1	2	4	8	16	32
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$\frac{2}{5}$	1	$\frac{2}{5}$	$\frac{4}{25}$	$\frac{8}{125}$	$\frac{16}{625}$	$\frac{32}{3125}$

q	A_0	A_1	A_2	A_3	A_4	A_5
1	1	2	3	4	5	6
2	1	3	7	15	31	63
$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{7}{4}$	$\frac{15}{8}$	$\frac{31}{16}$	$\frac{63}{32}$
$\frac{2}{5}$	1	$1\frac{2}{5}$	$1\frac{14}{25}$	$1\frac{78}{125}$	$1\frac{406}{625}$	$1\frac{2062}{3125}$

3. Notice that in order to compute $(1-q)\cdot(\sum \cdots)$ we have to assume that $q \neq 1$. If $q = 1$ then $(1-q) \cdot (\sum \cdots) = 0$. If $q \neq 1$, then:

$$(1-q)\cdot(\sum_{j=0}^{i} q^j) = (\sum_{j=0}^{i} q^j) - (\sum_{j=0}^{i} q^{j+1}) = (\sum_{j=0}^{i} q^j) - (\sum_{j=1}^{i+1} q^j) = 1 - q^{i+1}$$

It follows that, for $q \neq 1$, $\sum_{j=0}^{i} q^j = \frac{1-q^{i+1}}{1-q}$.

4,5. The values of A_5 and A_∞ are:

q	A_5
2	$\frac{1-2^6}{1-2} = \frac{-63}{-1} = 63$
$\frac{1}{2}$	$\frac{1-(\frac{1}{2})^6}{1-\frac{1}{2}} = \frac{\frac{63}{64}}{\frac{1}{2}} = \frac{63}{32}$
$\frac{2}{5}$	$\frac{1-(\frac{2}{5})^6}{1-\frac{1}{2}} = \frac{1-\frac{64}{15625}}{1-\frac{2}{5}} = \frac{\frac{15561}{15625}}{\frac{3}{5}} = \frac{5187}{3125} = 1\frac{2062}{3125}$

c	q	A_∞
1	$\frac{1}{2}$	$\frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$
2	$\frac{1}{3}$	$\frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$
1	1	∞

2. (a) The probability space is $\Omega = \{0, 00, 1, 2, \ldots, 36\}$. The probability of the elementary events 0 and 00 is 0 and the probability of all other elementary events is $\frac{1}{36}$.

(b) The probabilities of these events are: $P(R) = P(B) = P(E) = P(O) = \frac{1}{2}$ and $P(Z) = 0$.

(c) The following table summarizes the independence issues:

	R			B			Z		
	$P(\cap)$	$P(\cdot)\cdot P(\cdot)$	Ind	$P(\cap)$	$P(\cdot)\cdot P(\cdot)$	Ind	$P(\cap)$	$P(\cdot)\cdot P(\cdot)$	Ind
R	$\frac{1}{2}$	$\frac{1}{4}$	$\times$	0	$\frac{1}{4}$	$\times$	0	0	$\surd$
B	0	$\frac{1}{4}$	$\times$	$\frac{1}{2}$	$\frac{1}{4}$	$\times$	0	0	$\surd$
Z	0	0	$\surd$	0	0	$\surd$	0	0	$\surd$
E	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$	0	0	$\surd$
O	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$	0	0	$\surd$

	E			O		
	$P(\cap)$	$P(\cdot)\cdot P(\cdot)$	Ind	$P(\cap)$	$P(\cdot)\cdot P(\cdot)$	Ind
R	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$
B	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$	$\frac{1}{4}$	$\frac{1}{4}$	$\surd$
Z	0	0	$\surd$	0	0	$\surd$
E	$\frac{1}{2}$	$\frac{1}{4}$	$\times$	0	$\frac{1}{4}$	$\times$
O	0	$\frac{1}{4}$	$\times$	$\frac{1}{2}$	$\frac{1}{4}$	$\times$

3. • The probability space is $\{(i,j) \mid 1 \leq i,j \leq 6\}$. That is, all pairs (i,j) such that both i and j are between 1 and 6 (inclusive).

   • What are the possible values of X? These are 2, …, 12. What is the probability of each of these values?

	2	3	4	5	6	7	8	9	10	11	12
$P(X=i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
$i \cdot P(X=i)$	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{12}{36}$	$\frac{20}{36}$	$\frac{30}{36}$	$\frac{42}{36}$	$\frac{40}{36}$	$\frac{36}{36}$	$\frac{30}{36}$	$\frac{22}{36}$	$\frac{12}{36}$

   The expected value of X is $E(X) = \sum_{i=2}^{12} i \cdot P(X=i)$. Thus, $E(X) = \frac{252}{36} = 7$.

   • The expected value of X_i is

$$E(X_i) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6} = 3\frac{1}{2}$$

   So $E(X) = E(X_1) + E(X_2)$.
   In general, in case of independent events if $X = f(X_1) + g(X_2)$ and X_1 and X_2 are independent then $E(X) = E(f(X_1)) + E(g(X_2))$, where f and g are some functions.

4. (a) Adding up the probabilities at state 0 yields that

$$p + 0.5 + 0.25 = 1,$$

   so that $p = 0.25$. Adding up the probabilities at state 1 yields that

$$q + 0.75 = 1,$$

   so that $q = 0.25$.

   (b) The transition matrix is

$$\begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

   (c) The probabilities after one time step are given by

$$(1,0,0) \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (0.25, 0.5, 0.25).$$

The probabilities after two time steps are then given by

$$(0.25, 0.5, 0.25)\begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (0.4375, 0.5, 0.0625).$$

The probabilities after three time steps are then given by

$$(0.4375, 0.5, 0.0625)\begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.75 & 0.25 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (0.484375, 0.40625, 0.109375).$$

So the probability of being back in state 0 after three time steps is 0.484375.

(d) The probability of reaching state 0 in one step is 0.25. The probability of reaching state 0 in two steps is $0.75 \times 0.5 + 0.25 \times 0.25$. However, we are only interested in paths that reach state 0 in two steps *without passing through* 0 *in between*. So we are only interested in the path 010, whose probability is 0.375. The paths that reach state 0 in three steps without passing through state 0 before are 0110 and 0210. So the probability of both of them is $0.5 \times 0.25 \times 0.75 + 0.25 \times 1 \times 0.75 = 0.0935 + 0.1875 = 0.281$. So, overall, the probability to reach from state 0 to itself in at most 3 steps is 0.906.

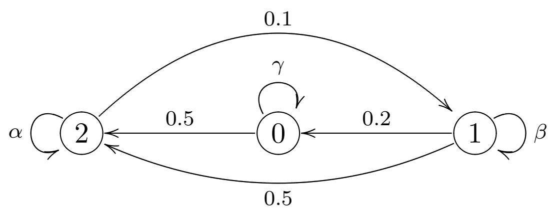
(e) The probability to reach from state 0 to state 1 without passing through state 2 on the way is as follows. All the possible ways to reache state 1 are either to go directly to state 1 or to stay in state 0 some number of times and then go to state 1.

The probability of reaching state 1 in one step is 0.5. The probability of reaching state 1 in two steps is 0.25×0.5. Generally, the probability of staying in 0 i times and then going to state 1 is $(0.25)^i \times 0.5$. It follows that the probability that interests us is:

$$P = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots = \sum_{i=0}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{4}\right)^i = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

8. (a) Let state 0 denote "out of resources", state 1 denote "low on resources" and state 2 denote "high on resources". Then we have

the diagram



We could have added lines labelled 0 from state 2 to state 0 and from state 0 to state 1; we have omitted these.

(b) Adding up the probabilities at state 2 yields that

$$\alpha + 0.1 = 1,$$

so that $\alpha = 0.9$. Adding up the probabilities at state 1 yields that

$$\beta + 0.5 + 0.2 = 1,$$

so that $\beta = 0.3$. Adding up the probabilities at state 0 yields that

$$\gamma + 0.5 = 1,$$

so that $\gamma = 0.5$. The transition matrix of the system is therefore

$$\begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.2 & 0.3 & 0.5 \\ 0 & 0.1 & 0.9 \end{pmatrix}.$$

(c) The probabilities after one time step are given by

$$(0,0,1)\begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.2 & 0.3 & 0.5 \\ 0 & 0.1 & 0.9 \end{pmatrix} = (0, 0.1, 0.9).$$

The probabilities after two time steps are then given by

$$(0,0.1,0.9)\begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.2 & 0.3 & 0.5 \\ 0 & 0.1 & 0.9 \end{pmatrix} = (0.02, 0.12, 0.86).$$

So the probability that the plant will be high on resources is 0.86, low on resources is 0.12 and out of resources is 0.02.

9. State 3 is aperiodic; there is a transition from state 3 to itself and so we can revisit state 3 after any number of steps.

   States 0, 1 and 2 form a closed set (there is no transition from any of these states to state 3). Every transition concerning states 0, 1 and 2 takes us from the set $\{0, 2\}$ to 1 or vice-versa; so we can only revisit any of these three states after an even number of steps, and hence these states are all periodic.

   For state 3 we have $\theta_{3,1} = 1/2$, $\theta_{3,k} = 0$ for $k \geqslant 2$. So

$$\theta_3 = \sum_{k=1}^{\infty} \theta_{3,k} = 1/2 < 1,$$

   and so state 3 is transient.

   State 1 is clearly recurrent as $\theta_{1,1} = 0$ and $\theta_{1,2} = 1$; So $\theta_1 = \sum_{k=1}^{\infty} \theta_{1,k} = 1$. States 0 and 2 are also recurrent. For state 0 we have that

$$\theta_{0,1} = 0,\ \theta_{0,2} = 1/2,\ \theta_{0,3} = 0,\ \theta_{0,4} = 1/4,\ \ \ldots,$$

   and so

$$\theta_0 = \sum_{k=1}^{\infty} \theta_{1,k} = 1/2 + 1/4 + 1/8 + \ldots = 1.$$

   We have the same argument for state 2.

   For the probability distribution after one step we have:

$$(1/2, 0, 0, 1/2) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} = (0, 1/2, 1/4, 1/4).$$

   After two steps we have:

$$(0, 1/2, 1/4, 1/4) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} = (1/4, 1/4, 3/8, 1/8).$$

   After three steps we have:

$$(1/4, 1/4, 3/8, 1/8) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} = (1/8, 5/8, 3/16, 1/16).$$

11. (a) In the left-hand system we can revisit states 0 and 2 after any number of steps (as there is a loop at each); so both are aperiodic. We can visit state 1 after 2 steps ($1 \to 0 \to 1$) and after three steps ($1 \to 0 \to 0 \to 1$); since 2 and 3 are coprime, state 1 is also aperiodic.

    In the right-hand system we can revisit any of the states after any number of steps (as there is a loop at each state); so all the states are aperiodic.

    (b) The left-hand system is not irreducible as we cannot reach states 0 and 1 from state 2.

    The right-hand system is irreducible; as we have a circuit $0 \to 1 \to 2 \to 0$, we can reach any state from any other state.

    (c) To find the unique stationary probability distribution of the irreducible system we will solve the equation

$$(x, y, z) \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 2/3 & 0 & 1/3 \end{pmatrix} = (x, y, z),$$

    where $x + y + z = 1$. This gives:

$$\begin{aligned} x/3 + y/3 + 2z/3 &= x \\ 2x/3 + y/3 &= y \\ y/3 + z/3 &= z \\ x + y + z &= 1 \end{aligned}$$

    The second equation gives that $2x/3 = 2y/3$ and hence that $x = y$. The third equation gives that $y/3 = 2z/3$ and hence that $y = 2z$ (and so $x = y = 2z$). Feeding these into the third equation gives that

$$2z + 2z + z = 1,$$

    so that $z = 1/5$ and then $x = y = 2/5$. So the unique stationary probability distribution is $(2/5, 2/5, 1/5)$.

18. (a) The expected value of X_i is computed as follows:

$$E(X_i) = 0 \cdot P(X_i = 0) + 1 \cdot P(X_i) = 1 = 1 \cdot 0.8 = 0.8$$

(b) By definition the expected value of Y is:

$$E(Y) = \sum_{j=0}^{40} j \cdot P(Y = j)$$

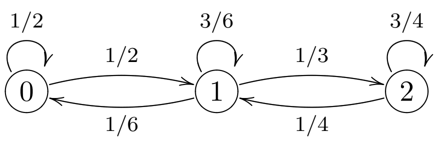
But there is a different way to write the same summation. Let $I = \{1, \ldots, 40\}$ denote the set of experiments. For $I' \subseteq I$ let $E_{I'}$ denote the event in which all experiments for $i \in I'$ are successful and all experiments for $i \notin I'$ are unsuccessful. It is simple to see that $P(E_{I'}) = 0.8^{|I'|} \cdot 0.2^{40-|I'|}$. Also, for $I' \neq I''$ it is clear that $P(E_{I'} \cap E_{I''}) = 0$. So, it follows that the sum above can be written also as:

$$E(Y) = \sum_{I' \subseteq I} |I'| \cdot P(E_{I'})$$

But this is quite simple to calculate:

$$\begin{aligned} E(Y) = \textstyle\sum_{j=0}^{40} \sum_{I' \subseteq I : |I'| = j} j \cdot 0.8^j \cdot 0.2^{40-j} = \sum_{j=0}^{40} j \cdot \binom{40}{j} \cdot 0.8^j \cdot 0.2^{40-j} = \\ = \textstyle\sum_{j=0}^{40} j \cdot \left(\frac{40!}{j! \cdot (40-j)!}\right) \cdot 0.8^j \cdot 0.2^{40-j} = \sum_{j=1}^{40} \left(\frac{40!}{(j-1)!(40-j)!}\right) \cdot 0.8^j \cdot 0.2^{40-j} = \\ = 40 \cdot 0.8 \cdot \textstyle\sum_{j=1}^{40} \left(\frac{39!}{(j-1)!(39-(j-1))!}\right) \cdot 0.8^{j-1} \cdot 0.2^{39-(j-1)} = \\ = 40 \cdot 0.8 \cdot \textstyle\sum_{k=0}^{39} \left(\frac{39!}{k!(39-k)!}\right) \cdot 0.8^k \cdot 0.2^{39-k} = 40 \cdot 0.8 \cdot (0.8 + 0.2)^{39} \end{aligned}$$

19. (a) The Markov chain is:



(b) The transition probability matrix is:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{3}{6} & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

(c) The only path of length 2 reching the full state is $0, 1, 2$, whose probability is $\frac{1}{6}$. The paths of length 3 reaching the full state are $0, 0, 1, 2$ and $0, 1, 1, 2$, whose probability is $\frac{1}{12} + \frac{1}{12}$. The paths of length 4 reaching the full state are $0, 0, 0, 1, 2$, $0, 0, 1, 1, 2$, $0, 1, 0, 1, 2$, and $0, 1, 1, 1, 2$. Their probability is $\frac{1}{24} + \frac{1}{24} + \frac{1}{72} + \frac{1}{24} = \frac{10}{72}$. The total probability is the sum of the three.

20. First we compute what is the probability to end up on the left (states 1 and 2) and what is the probability to end up on the right (states 3 and 4). The probablity to reach state 3 is:

$$\sum_{i=0}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{6}\right)^i = \frac{1}{2}\frac{1}{1-\frac{1}{6}} = \frac{3}{5}$$

This implies that the probability to reach state 1 is $\frac{2}{5}$. The same result achieved by $\frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{3}} = \frac{2}{5}$.

It follows that the following equations need to be solved.

$$\begin{aligned} p_1 + p_2 &= \tfrac{2}{5} \\ p_3 + p_4 &= \tfrac{3}{5} \\ p_1 &= \tfrac{1}{2}p_1 + \tfrac{1}{4}p_2 \\ p_2 &= \tfrac{1}{2}p_1 + \tfrac{3}{4}p_2 \\ p_3 &= \tfrac{5}{6}p_3 + \tfrac{1}{3}p_4 \\ p_4 &= \tfrac{1}{6}p_3 + \tfrac{2}{3}p_4 \end{aligned}$$

We get that $\frac{3}{2}p_2 = \frac{2}{5}$ implying $p_2 = \frac{4}{15}$ and $p_1 = \frac{2}{15}$. Also, $3p_4 = \frac{3}{5}$ implying $p_4 = \frac{1}{5}$ and $p_3 = \frac{2}{5}$.

21. (a) Here $\lambda = 1/2$ and $\mu = 3/2$; so $\rho = \lambda/\mu = 1/3$. The expected number of items in the system is then

$$L = \frac{\rho}{1-\rho} = \frac{1/3}{2/3} = \frac{1}{2}.$$

(b) The average turnaround time is

$$T = \frac{1}{\mu - \lambda} = \frac{1}{3/2 - 1/2} = 1 \text{ minute}.$$

(c) The probability that there are at least five items in the system is

$$\sum_{n=5}^{\infty} P_n = \sum_{n=5}^{\infty} \rho^n(1-\rho) = \rho^5(1-\rho)\sum_{n=0}^{\infty} \rho^n = \rho^5(1-\rho)\frac{1}{1-\rho} = \rho^5 = \frac{1}{3^5} = 0.004115.$$

22. Here $\mu = 1/3$. Given that the average time spent in the queue waiting for service is $W = \lambda/\mu(\mu - \lambda)$, we want

$$\frac{\lambda}{\frac{1}{3}(\frac{1}{3} - \lambda)} \leqslant 5,$$

i.e. $3\lambda/(\frac{1}{3}-\lambda) \leqslant 5$, i.e. $9\lambda/(1-3\lambda) \leqslant 5$. This gives that

$$9\lambda \leqslant 5 - 15\lambda,$$

and so we want $24\lambda \leqslant 5$, i.e. $\lambda \leqslant \frac{5}{24}$. So the maximum arrival rate that can occur is $\frac{5}{24}$ arrivals per minute, i.e. an arrival (on average) every $24/5 \times 60 = 288$ seconds.

23. As in Exercise 1 we have $\lambda = \frac{1}{2}$, $\mu = \frac{3}{2}$ and $\rho = \frac{1}{3}$. The expected number of items in the system for an $M/M/1/N$ queue is

$$L = \frac{\rho}{1-\rho^{N+1}}\left[\frac{1-\rho^N}{1-\rho} - N\rho^N\right].$$

Here $N = 4$ and so we have

$$L = \frac{\rho}{1-\rho^5}\left[\frac{1-\rho^4}{1-\rho} - 4\rho^4\right] = \frac{\rho}{1-\rho^5}\left[\frac{1-\rho^4-4\rho^4(1-\rho)}{1-\rho}\right] = \frac{\rho}{1-\rho^5}\left[\frac{1-5\rho^4+4\rho^5}{1-\rho}\right].$$

We have that $\rho = \frac{1}{3}$ and so

$$L = \frac{\frac{1}{3}}{1-\left(\frac{1}{3}\right)^5}\left(\frac{1-5\left(\frac{1}{3}\right)^4+4\left(\frac{1}{3}\right)^5}{\frac{2}{3}}\right) = \left(\frac{3^5}{3^5-1}\right)\left(\frac{3^5-15+4}{2\times 3^5}\right) = \frac{1}{2}\left(\frac{3^5-11}{3^5-1}\right) = 0.4793.$$

The average turnaround time is

$$T = \frac{1}{\lambda(1-P_N)}L = \frac{1}{\frac{1}{2}(1-P_N)}\frac{1}{2}\left(\frac{3^5-11}{3^5-1}\right) = \frac{1}{(1-P_N)}\left(\frac{3^5-11}{3^5-1}\right)$$

where

$$P_N = \rho^N\left(\frac{1-\rho}{1-\rho^{N+1}}\right).$$

Here $N = 4$ and $\rho = \frac{1}{3}$ and so we have

$$P_4 = \rho^4\left(\frac{1-\rho}{1-\rho^5}\right) = \left(\frac{1}{3}\right)^4\left(\frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^5}\right) = \frac{2}{3^5-1} \text{ and so } 1-P_4 = \frac{(3^5-1)-2}{3^5-1} = \frac{3^5-3}{3^5-1}.$$

So

$$T = \frac{1}{(1-P_4)}\left(\frac{3^5-11}{3^5-1}\right) = \left(\frac{3^5-1}{3^5-3}\right)\left(\frac{3^5-11}{3^5-1}\right) = \frac{3^5-11}{3^5-3} = 0.9667.$$

For the $M/M/1$ queue (with the same values of λ and μ) in Exercise 1 we had $L = \frac{1}{2}$ and $T = 1$, which are very similar values to those we have here. This is explained by the fact that the blocking probability P_4 is very small and so the bound on the number of items in the system does not have a significant effect on the average behaviour of the system.

24. In an $M/M/1/N$ queue, the probability that an item is rejected is P_N which is the probability that there are N items in the system; this is given by
$$P_N = \rho^N \left(\frac{1-\rho}{1-\rho^{N+1}} \right).$$
In our case we have $\lambda = \frac{1}{4}$ and $\mu = \frac{1}{3}$ so that $\rho = \frac{\lambda}{\mu} = \frac{3}{4}$. So
$$P_N = \left(\frac{3}{4}\right)^N \left(\frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^{N+1}} \right) = \frac{3^N}{4^{N+1} - 3^{N+1}}. \tag{3.1}$$
We want $P_N < 0.2$. If $N = 2$ then (3.1) gives that $P_N = P_2 = 0.2432$ and, if $N = 3$ then (3.1) gives that $P_N = P_3 = 0.1543$. So the minimum value of N that will keep the percentage of customers leaving in this way below 20% is $N = 3$.

25. We have that $P_{n+1} = \frac{\lambda}{\mu} P_n = \rho P_n$ for all n (as in Lecture 21). If $\rho = 1$ then we have that
$$P_0 = P_1 = P_2 = \ldots\ldots = P_{N-1} = P_N.$$
Since $P_0 + P_1 + P_2 + \ldots + P_{N-1} + P_N = 1$ we have that $(N+1)P_0 = 1$ and hence that
$$P_n = \frac{1}{N+1}$$
for $0 \leqslant n \leqslant N$. Then
$$L = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} \frac{n}{N+1} = \frac{1}{N+1} \sum_{n=0}^{N} n = \frac{1}{N+1} \times \frac{1}{2} N(N+1) = \frac{N}{2}.$$

33. We saw in Lecture 22 that, for the $M/M/2$ queue, we have that
$$P_0 = \frac{1-\sigma}{1+\sigma}.$$
Using this and the fact that $m = 2$ in the formula
$$T = \frac{1}{\mu} + \frac{1}{\mu} \frac{(m\sigma)^m}{m!} \frac{1}{m(1-\sigma)^2} P_0$$

gives that

$$\begin{aligned}
T &= \frac{1}{\mu} + \frac{1}{\mu}\frac{(2\sigma)^2}{2!}\frac{1}{2(1-\sigma)^2}\left(\frac{1-\sigma}{1+\sigma}\right) \\
&= \frac{1}{\mu}\left[1 + \frac{\sigma^2}{(1-\sigma)(1+\sigma)}\right] \\
&= \frac{1}{\mu}\left[1 + \frac{\sigma^2}{1-\sigma^2}\right] \\
&= \frac{1}{\mu}\left[\frac{(1-\sigma^2)+\sigma^2}{1-\sigma^2}\right] \\
&= \frac{1}{\mu(1-\sigma^2)}
\end{aligned}$$

as required.

34. The average number of items in the system for an $M/M/3$ queue, where the average arrival rate is λ and the average service rate is μ, is given by

$$L = 3\sigma + \frac{(3\sigma)^3}{3!}\frac{\sigma}{(1-\sigma)^2}P_0 = 3\sigma + \frac{9\sigma^4}{2(1-\sigma)^2}P_0$$

where

$$\begin{aligned}
P_0 &= \frac{1}{\left[1 + \sum_{n=1}^{m-1}\frac{(m\sigma)^n}{n!} + \frac{(m\sigma)^m}{m!}\frac{1}{(1-\sigma)}\right]} \\
&= \frac{1}{1 + 3\sigma + \frac{(3\sigma)^2}{2!} + \frac{(3\sigma)^3}{3!}\frac{1}{(1-\sigma)}} \\
&= \frac{6(1-\sigma)}{6(1-\sigma)(1+3\sigma) + 3(1-\sigma)(3\sigma)^2 + (3\sigma)^3} \\
&= \frac{6(1-\sigma)}{6 + 12\sigma - 18\sigma^2 + 27\sigma^2 - 27\sigma^3 + 27\sigma^3} \\
&= \frac{6(1-\sigma)}{6 + 12\sigma + 9\sigma^2} \\
&= \frac{2(1-\sigma)}{2 + 4\sigma + 3\sigma^2}.
\end{aligned}$$

So

$$L = 3\sigma + \frac{9\sigma^4}{2(1-\sigma)^2}\frac{2(1-\sigma)}{(2+4\sigma+3\sigma^2)} = 3\sigma + \frac{9\sigma^4}{(1-\sigma)(2+4\sigma+3\sigma^2)}.$$