Chapter 2

Games as trees and in strategic form

This chapter introduces the main concepts of non-cooperative game theory: game trees and games in strategic form and the ways to analyse them.

We occasionally refer to concepts discussed in chapter 1, such as "game states", to clarify the connection, but chapter 1 is not a requirement to study this chapter.

2.1 Learning objectives

After studying this chapter, you should be able to:

- interpret game trees and games in strategic form;
- explain the concepts of move in a game tree, strategy (and how it differs from move), strategy profile, backward induction, symmetric games, dominance and weak dominance, dominance solvable, Nash equilibrium, reduced strategies and reduced strategic form, subgame perfect Nash equilibrium, and commitment games;
- apply these concepts to specific games.

2.2 Further reading

The presented concepts are standard in game theory. They can be found, for example, in the following book:

- Osborne, Martin J., and Ariel Rubinstein *A Course in Game Theory.* (MIT Press, 1994) [ISBN 0262650401].

Osborne and Rubinstein treat game theory as it is used in economics. Rubinstein is also a pioneer of bargaining theory. He invented the alternating-offers model treated in chapter 5. On the other hand, the book uses some non-standard descriptions. For example, Osborne and Rubinstein define games of perfect information via "histories" and not game trees; we prefer the latter because they are less abstract. Hence, keep in mind that the terminology used by Osborne and Rubinstein may not be standard and may differ from ours.

Another possible reference is:

- Gibbons, Robert *A Primer in Game Theory* [in the United States sold under the title *Game Theory for Applied Economists*]. (Prentice Hall / Harvester Wheatsheaf, 1992) [ISBN 0745011594].

In particular, this book looks at the commitment games of section 2.14 from the economic perspective of "Stackelberg leadership".

2.3 Introduction

In this chapter, we introduce several main concepts of non-cooperative game theory: *game trees* (with perfect information), which describe explicitly how a game evolves over time, and *strategies*, which describe a player's possible "plans of action". A game can be described in terms of strategies alone, which defines a game in *strategic form.*

Game trees can be solved by *backward induction*, where one finds optimal moves for each player, given that all future moves have already been determined. The central concept for games in strategic form is the *Nash equilibrium*, where each player chooses a strategy that is optimal given what the other players are doing. We will show that backward induction always produces a Nash equilibrium, also called "subgame perfect Nash equilibrium" or SPNE.

The main difference between game trees and games in strategic form is that in a game tree, players act sequentially, being aware of the previous moves of the other players. In contrast, players in a game in strategic form move *simultaneously*. The difference between these two descriptions becomes striking when *changing* a two-player game in strategic form to a game with *commitment*, described by a game tree where one player moves first and the other second, but which otherwise has the same payoffs.

Every game tree can be converted to strategic form, which, however, is often much larger than the tree. A general game in strategic form can only be represented by a tree by modelling "imperfect information" where a player is not aware of another player's action. Game trees with imperfect information are treated in chapter 4.

2.4 Definition of game trees

Figure 2.1 shows an example of a game tree. We always draw trees downwards, with the *root* at the top. (Conventions on drawing game trees vary. Sometimes trees are drawn from the bottom upwards, sometimes from left to right, and sometimes from the center with edges in any direction.)

The *nodes* of the tree denote game states. (In a combinatorial game, game states are called "positions".) Nodes are connected by lines, called *edges*. An edge from a node u

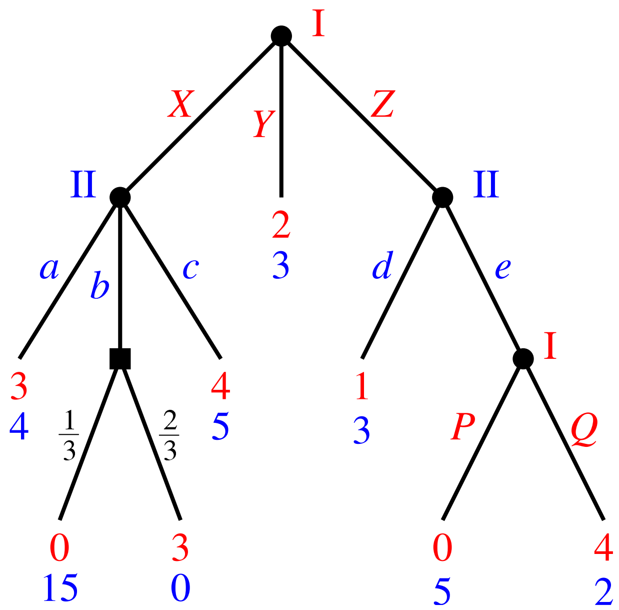


Figure 2.1 Example of a game tree. The square node indicates a chance move. At a leaf of the tree, the top payoff is to player I, the bottom payoff to player II.

to a successor node v (where v is drawn below u) indicates a possible *move* in the game. This may be a move of a "personal" player, for example move X in figure 2.1 of player I. Then u is also called a *decision node*. Alternatively, u is a *chance node*. A chance node is drawn here as a *square*, like the node u that follows move b of player II in figure 2.1. The next node v is then determined by a random choice according to the probability associated with the edge leading from u to v. In figure 2.1, these probabilities are $\frac{1}{3}$ for the left move and $\frac{2}{3}$ for the right move.

Nodes without successors in the tree are called terminal nodes or *leaves*. At such a node, every player gets a *payoff*, which is a real number (in our examples often an integer). In figure 2.1, leaves are not explicitly drawn, but the payoffs given instead, with the top payoff to player I and the bottom payoff to player II.

The game tree, with its decision nodes, moves, chance probabilities and payoffs, is known to the players, and defines the game completely. The game is played by starting at the root. At a decision node, the respective player chooses a move which determines the next node. At a chance node, the move is made randomly according to the given probabilities. Play ends when a leaf is reached, where all players receive their payoffs.

Players are interested in maximising their own payoff. If the outcome of the game is random, then the players are assumed to be interested in maximising their *expected payoffs*. In figure 2.1, the expected payoff to player I after the chance move is $\frac{1}{3}\times 0+\frac{2}{3}\times 3=2$, and to player II it is $\frac{1}{3}\times 15+\frac{2}{3}\times 0=5$. In this game, the chance node could therefore be replaced by a leaf with payoff 2 for player I and payoff 5 for player II. These expected payoffs reflect the players' preferences for the random outcomes, including their attitude to the risk involved when facing such uncertain outcomes. This "risk-neutrality" of payoffs does not necessarily hold for actual monetary payoffs. For example, if the payoffs to player II in figure 2.1 were millions of dollars, then in reality that player would probably prefer receiving 5 million dollars for certain to a lottery that gave her 15 million dollars with probability $1/3$ and otherwise nothing. However, payoffs can be adjusted to reflect the player's attitude to risk, as well as representing the "utility" of an outcome like money so that one can take just the expectation. In the example, player II may consider a lottery that gives her 15 million dollars with probability $1/3$ and otherwise nothing only as a valuable as getting 2 million dollars for certain. In that case, "15 million dollars" may be represented by a payoff of 6 so that the said lottery has an expected payoff 2, assuming that "2 million dollars" are represented by a payoff of 2 and "getting nothing" by a payoff of 0. This is discussed in greater detail in section 3.4.

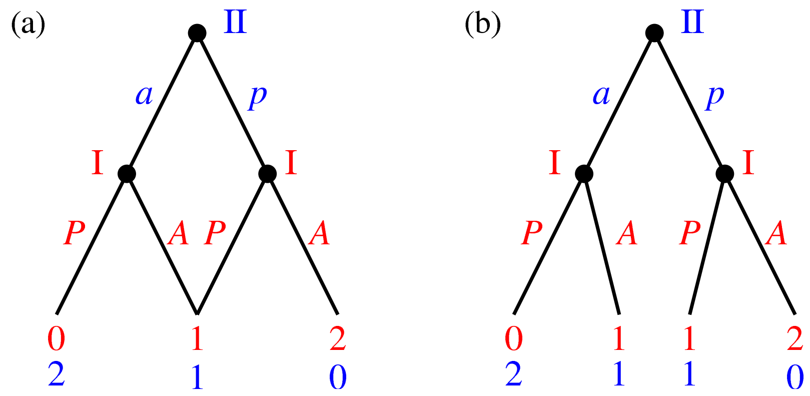


Figure 2.2 The left picture (a) is not a tree because the leaf where both players receive payoff 1 is reachable by two different paths. The correct tree representation is shown on the right in (b).

For the game tree, it does not matter how it is drawn, but only its "combinatorial" structure, that is, how its nodes are connected by edges. A tree can be considered as a special "directed graph", given by a set of nodes, and a set of edges, which are pairs of nodes (u,v). A *path* in such a graph from node u to node v is a sequence of nodes $u_0u_1\ldots u_k$ so that $u_0 = u$, $u_k = v$, and so that (u_i,u_{i+1}) is an edge for $0 \le i < k$. The graph is a tree if it has a distinguished node, the root, so that to each node of the graph there is a unique path from the root.

Figure 2.2 demonstrates the tree property that every node is reached by a unique path from the root. This fails to hold in the left picture (a) for the middle leaf. Such a structure may make sense in a game. For example, the figure could represent two people in a pub where first player II chooses to pay for the drinks (move p) or to accept (move a) that the first round is paid by player I. In the second round, player I may then decide to either accept not to pay (move A) or to pay (P). Then the players may only care how often, but not when, they have paid a round, with the middle payoff pair $(1,1)$ for two possible "histories" of moves. The tree property prohibits game states with more than one history, because the history is represented by the unique path from the root. Even if certain differences in the game history do not matter, these game states are distinguished, mostly as a matter of mathematical convenience. The correct representation of the above game is shown in the right picture, figure 2.2(b).

Game trees are also called *extensive games* with *perfect information*. Perfect information means that a player always knows the game state and therefore the complete history of the game up to then. Game trees can be enhanced with an additional structure that represents "imperfect information", which is the topic of chapter 4.

Note how game trees differ from the combinatorial games studied in chapter 1:

(a) A combinatorial game can be described very compactly, in particular when it is given as a sum of games. For general game trees, such sums are typically not considered.

(b) The "rules" in a game tree are much more flexible: more than two players and chance moves are allowed, players do not have to alternate, and payoffs do not just stand for "win" or "lose".

(c) The flexibility of game trees comes at a price, though: The game description is much longer than a combinatorial game. For example, a simple instance of nim may require a huge game tree. Regularities like the mex rule do not apply to general game trees.

2.5 Backward induction

Which moves should the players choose in a game tree? "Optimal" play should maximise a player's payoff. This can be decided irrespective of other players' actions when the player is the *last* player to move. In figure 2.2(b), player I maximises his payoff by move A at both his decision nodes, because at the left node he receives 1 rather than 0 with that move, and at the right node payoff 2 rather than 1. Going backwards in time, player II has to make her move a or p at the root of the game tree, where she will receive either 1 or 0, assuming the described future behaviour of player I. Consequently, she will choose a.

This process is called *backward induction*: Starting with the decision nodes closest to the leaves, a player's move is chosen which maximises that player's payoff at the node. In general, a move is chosen in this way for each decision node provided all subsequent moves have already been decided. Eventually, this will determine a move for every decision node, and hence for the entire game. Backward induction is also known as *Zermelo's algorithm*. (This is attributed to an article by Zermelo (1913) on chess. Later, people decided that Zermelo proved something different, in fact a more complicated property, so that Zermelo's algorithm is sometimes called "Kuhn's algorithm", according to a paper by Kuhn (1953), which we cite on page 96.)

The move selected by backward induction is not necessarily unique, if there is more than one move giving maximal payoff to the player. In the game in figure 2.1, backward induction chooses either move b or move c for player II, both of which give her payoff 5 (which is an expected payoff for move b) that exceeds her payoff 4 for move a. At the right-most node, player I chooses Q. This determines the preceding move d by player II which gives her the higher payoff 3 as opposed to 2 (via move Q). In turn, this means that player I, when choosing between X, Y, or Z at the root of the game tree, will get payoff 2 for Y and payoff 1 for Z; the payoff when he chooses X depends on the choice of player II: if that is b, then player I gets 2, and can choose either X or Y, both of which give him maximal payoff 2. If player II chooses c, however, then the payoff to player I is 4 when choosing X, so this is the unique optimal choice. To summarise, the possible combinations of moves that can arise in figure 2.1 by backward induction are, simply listing the moves for each player: (XQ, bd), (XQ, cd), and (YQ, bd). Note that Q and d are always chosen, but that Y can only be chosen in combination with the move b by player II.

The moves determined by backward induction are therefore, in general, not unique, and possibly interdependent.

Backward induction gives a *unique* recommendation to each player if there is always only one move that gives maximal payoff. This applies to *generic* games. A generic game is a game where the payoff parameters are real numbers that are in no special dependency of each other (like two payoffs being equal). In particular, it should be allowed to replace them with values nearby. For example, the payoffs may be given in some practical scenario which has some "noise" that effects the precise value of each payoff. Then two such payoffs are equal with probability zero, and so this case can be disregarded. In generic games, the optimal move is always unique, so that backward induction gives a unique result.

2.6 Strategies and strategy profiles

Definition 2.1 In a game tree, a *strategy* of a player specifies a move for every decision node of that player. A *strategy profile* is a tuple of strategies, with one strategy for each player of the game.

If the game has only two players, a strategy profile is therefore a pair of strategies, with one strategy for player I and one strategy for player II.

In the game tree in figure 2.1, the possible strategies for player I are XP, XQ, YP, YQ, ZP, ZQ. The strategies for player II are ad, ae, bd, be, cd, ce. For simplicity of notation, we have thereby specified a strategy simply as a list of moves, one for each decision node of the player. When specifying a strategy in that way, this must be done with respect to a fixed order of the decision nodes in the tree, in order to identify each move uniquely. This matters when a move name appears more than once, for example in figure 2.2(b). In that tree, the strategies of player I are AA, AP, PA, PP, with the understanding that the first move refers to the left decision node and the second move to the right decision node of player I.

Backward induction defines a move for every decision node of the game tree, and therefore for every decision node of each player, which in turn gives a strategy for each player. The result of backward induction is therefore a strategy profile.

⇒ Exercise 2.1 on page 52 studies a game tree which is not unlike the game in figure 2.1. You can already answer part (a) of this exercise, and apply backward induction, which answers (e).

2.7 Games in strategic form

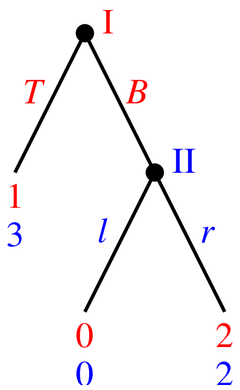
Assume a game tree is given. Consider a strategy profile, and assume that players move according to their strategies. If there are no chance moves, their play leads to a unique leaf. If there are chance moves, a strategy profile may lead to a probability distribution on the leaves of the game tree, with resulting *expected* payoffs. In general, any strategy profile defines an expected payoff to each player (which also applies to a payoff that is obtained deterministically, where the expectation is computed from a probability distribution that assigns probability one to a single leaf of the game tree).

Definition 2.2 The *strategic form* of a game is defined by specifying for each player the set of strategies, and the payoff to each player for each strategy profile.

For two players, the strategic form is conveniently represented by a table. The rows of the table represent the strategies of player I, and the columns the strategies of player II. A strategy profile is a strategy pair, that is, a row and a column, with a corresponding cell of the table that contains two payoffs, one for player I and the other for player II.

If m and n are positive integers, then an $m \times n$ game is a two-player game in strategic form with m strategies for player I (the rows of the table) and n strategies for player II (the columns of the table).

(a)



(b)

I \ II	l	r
T	1, 3	1, 3
B	0, 0	2, 2

Figure 2.3 Extensive game (a) and its strategic form (b). In a cell of the strategic form, player I receives the bottom left payoff, and player II the top right payoff.

Figure 2.3 shows an extensive game in (a), and its strategic form in (b). In the strategic form, T and B are the strategies of player I given by the top and bottom row of the table, and l and r are the strategies of player II, corresponding to the left and right column of the table. The strategic form for the game in figure 2.1 is shown in figure 2.4.

A game in strategic form can also be given directly according to definition 2.2, without any game tree that it is derived from. Given the strategic form, the game is played as follows: Each player chooses a strategy, independently from and simultaneously with the other players, and then the players receive their payoffs as given for the resulting strategy profile.

I \ II	*ad*	*ae*	*bd*	*be*	*cd*	*ce*
XP	4 3	4 3	5 2	5 2	5 4	5 4
XQ	4 3	4 3	5 2	5 2	5 4	5 4
YP	3 2	3 2	3 2	3 2	3 2	3 2
YQ	3 2	3 2	3 2	3 2	3 2	3 2
ZP	3 1	5 0	3 1	5 0	3 1	5 0
ZQ	3 1	2 4	3 1	2 4	3 1	2 4

Figure 2.4 Strategic form of the extensive game in figure 2.1. The less redundant reduced strategic form is shown in figure 2.13 below.

2.8 Symmetric games

Many game-theoretic concepts are based on the strategic form alone. In this section, we discuss the possible *symmetries* of such a game. We do this by presenting a number of standard games from the literature, mostly 2×2 games, which are the smallest games which are not just one-player decision problems.

(a)

I \ II	*c*	*d*
C	2 2	3 0
D	0 3	1 1

(b)

I \ II	*c*	*d*
C	2 2	3 0
D	0 3	1 1

Figure 2.5 Prisoner's dilemma game (a), its symmetry shown by reflection along the dotted diagonal line in (b).

Figure 2.5(a) shows the well-known "prisoner's dilemma" game. Each player has two strategies, called C and D for player I and c and d for player II. (We use upper case letters for player I and lower case letters for player II for easier identification of strategies and moves.) These letters stand for "cooperate" and "defect", respectively. The story behind the name "prisoner's dilemma" is that of two prisoners held suspect of a serious crime. There is no judicial evidence for this crime except if one of the prisoners testifies against the other. If one of them testifies, he will be rewarded with immunity from prosecution (payoff 3), whereas the other will serve a long prison sentence (payoff 0). If both testify, their punishment will be less severe (payoff 1 for each). However, if they both "cooperate" with each other by not testifying at all, they will only be imprisoned briefly for some minor charge that can be held against them (payoff 2 for each). The "defection" from that mutually beneficial outcome is to testify, which gives a higher payoff no matter what the other prisoner does (which makes "defect" a dominating strategy, as discussed in the next section). However, the resulting payoff is lower to both. This constitutes their "dilemma".

The prisoner's dilemma game is *symmetric* in the sense that if one exchanges player I with player II, and strategy C with c, and D with d, then the game is unchanged. This is demonstrated in figure 2.5(b) with the dotted line that connects the top left of the table with the bottom right: the payoffs remain the same when reflected along that dotted line. In order to illustrate the symmetry in this visual way, payoffs have to be written in diagonally opposite corners in a cell of the table.

This representation of payoffs in different corners of cells of the table is due to Thomas Schelling, which he used, for example, in his book *The Strategy of Conflict* (1961). Schelling modestly (and not quite seriously) calls the staggered payoff matrices his most important contribution to game theory, despite that fact that this book was most influential in applications of game theory to social science. Schelling received the 2005 Nobel prize in economics for his contribution to the understanding of conflict and cooperation using game theory.

The payoff to player I has to appear in the bottom left corner, the payoff to player II at the top right. These are also the natural positions for the payoffs that leave no ambiguity about which player each payoff belongs to, because the table is read from top to bottom and from left to right.

For this symmetry, the order of the strategies matters (which it does not for other aspects of the game), so that, when exchanging the two players, the first row is exchanged with the first column, the second row with the second column, and so on. A non-symmetric game can sometimes be made symmetric when re-ordering the strategies of one player, as illustrated in figure 2.7 below. Obviously, in a symmetric game, both players must have the same number of strategies.

The game of "chicken" is another symmetric game, shown in figure 2.6. The two strategies are A and C for player I and a and c for player II, which may be termed "aggressive" and "cautious" behaviour, respectively. The aggressive strategy is only advantageous (with payoff 2 rather than 0) if the other player is cautious, whereas a cautious strategy always gives payoff 1 to the player using it.

The game known as the "battle of sexes" is shown in figure 2.7(a). In this scenario, player I and player II are a couple each deciding whether to go to a concert (strategies C

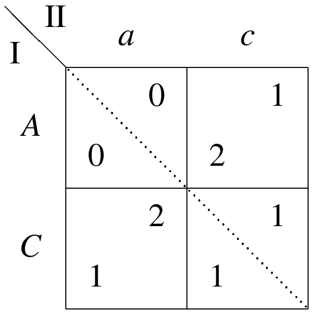


Figure 2.6 The game “chicken”, its symmetry indicated by the diagonal dotted line.

Figure 2.7 The “battle of sexes” game (a), which is symmetric if the strategies of one player are exchanged, as shown in (b).

and c) or to a sports event (strategies S and s). The players have different payoffs arising from which event they go to, but that payoff is zero if they have to attend the event alone.

This game is not symmetric when written as in figure 2.7(a), where strategy C of player I would be exchanged with strategy c of player II, and correspondingly S with s, because the payoffs for the strategy pairs (C,c) and (S,s) on the diagonal are not the same for both players, which is clearly necessary for symmetry. However, changing the order of the strategies of one player, for example of player II as shown in figure 2.7(b), makes this a symmetric game.

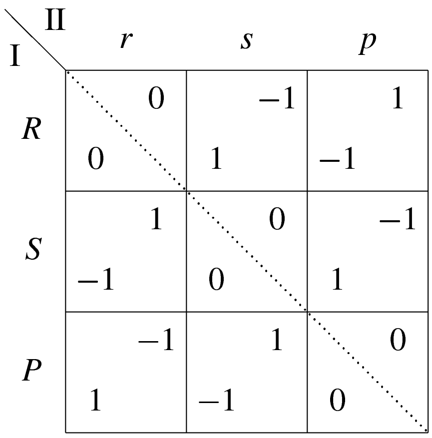


Figure 2.8 The “rock-scissors-paper” game.

Figure 2.8 shows a 3×3 game known as "rock-scissors-paper". Both players choose simultaneously one of their three strategies, where rock beats scissors, scissors beat paper, and paper beats rock, and it is a draw otherwise. This is a *zero-sum* game because the payoffs in any cell of the table sum to zero. The game is symmetric, and because it is zero-sum, the pairs of strategies on the diagonal, here (R,r), (S,s), and (P,p), must give payoff zero to both players: otherwise the payoffs for these strategy pairs would not be the same for both players.

2.9 Symmetries involving strategies*

In this section,[1] we give a general definition of symmetry that may apply to any game in strategic form, with possibly more than two players. This definition allows also for symmetries among strategies of a player, or across players. As an example, the rock-scissors-paper game also has a symmetry with respect to its three *strategies*: When cyclically replacing R by S, S by P, and P by R, and r by s, s by p, and p by r, which amounts to moving the first row and column in figure 2.8 into last place, respectively, then the game remains the same. (In addition, the players may also be exchanged as described.) The formal definition of a symmetry of the game is as follows.

Definition 2.3 Consider a game in strategic form with N as the finite set of players, and strategy set Σ_i for player $i \in N$, where $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$. The payoff to player i is given by $a_i((s_j)_{j\in N})$ for each strategy profile $(s_j)_{j\in N}$, where $s_j \in \Sigma_j$. Then a *symmetry* of the game is given by a bijection $\phi{:}N \to N$ and a bijection $\psi_i : \Sigma_i \to \Sigma_{\phi(i)}$ for each $i \in N$ so that

$$a_i((s_j)_{j\in N}) = a_{\phi(i)}((\psi_j(s_j))_{j\in N}) \tag{2.1}$$

for each player $i \in N$ and each strategy profile $(s_j)_{j\in N}$.

Essentially, a symmetry is a way of re-naming players and strategies that leaves the game unchanged. In definition 2.3, the strategy sets Σ_i of the players are assumed to be pairwise disjoint. This assumption removes any ambiguity when looking at a strategy profile $(s_j)_{j\in N}$, which defines one strategy s_j in Σ_j for each player j, because then the particular order of these strategies does not matter, because every strategy belongs to exactly one player. In our examples, strategy sets have been disjoint because we used upper-case letters for player I and lower-case letters for player II.

A symmetry as in definition 2.3 is, first, given by a re-naming of the players with the bijection ϕ, where player i is re-named as $\phi(i)$, which is either another player in the game or the same player. Secondly, a bijection ψ_i re-names the strategies s_i of player i as strategies $\psi_i(s_i)$ of player $\phi(i)$. Any strategy profile $(s_j)_{j\in N}$ thereby becomes a re-named strategy profile $(\psi_j(s_j))_{j\in N}$. With this re-naming, player $\phi(i)$ has taken the role of player i, so one should evaluate his payoff, normally given by the payoff function a_i, as $a_{\phi(i)}((\psi_j(s_j))_{j\in N})$ when applied to the re-named strategy profile. If this produces in

[1]This is an additional section which can be omitted at first reading, and has non-examinable material, as indicated by the star sign * following the section heading.

all cases the same payoffs, as stated in (2.1), then the re-naming has produced the same game in strategic form, and is therefore called a symmetry of the game.

For two players, the reflection along the diagonal is obviously simpler to state and to observe. In this symmetry among the two players, with player set $N = \{\mathsf{I}, \mathsf{II}\}$ and with strategy sets Σ_{I} and Σ_{II}, we used $\phi(\mathsf{I}) = \mathsf{II}$ and thus $\phi(\mathsf{II}) = \mathsf{I}$, and a *single* bijection $\psi_{\mathsf{I}} : \Sigma_{\mathsf{I}} \to \Sigma_{\mathsf{II}}$ for re-naming the strategies, where ψ_{II} is the inverse of ψ_{I}. This last condition does not have to apply in definition 2.3, which makes this definition more general.

I \ II	h	t
H	1, -1	-1, 1
T	-1, 1	1, -1

Figure 2.9 The "matching pennies" game.

The game of "matching pennies" in figure 2.9 illustrates the greater generality of definition 2.3. This zero-sum game is played with two players that each have a penny, which the player can choose to show as heads or tails, which is the strategy H or T for player I, and h or t for player II. If the pennies match, for the strategy pairs (H,h) and (T,t), then player II has to give her penny to player I; if they do not, for the strategy pairs (H,t) and (T,h), then player I loses his penny to player II. This game has an obvious symmetry in strategies where heads are exchanged with tails for both players, but where the players are not re-named, so that ϕ is the identity function on the player set. However, the game cannot be written so that it remains the same when reflected along the diagonal, because the diagonal payoffs would have to be zero, as in the rock-scissors-paper game, so it is not symmetric among players in the sense of any of the earlier examples. Definition 2.3 does capture the symmetry between the two players as follows (for example): ϕ exchanges I and II, and $\psi_{\mathsf{I}}(H) = h$, $\psi_{\mathsf{I}}(T) = t$, but $\psi_{\mathsf{II}}(h) = T$, $\psi_{\mathsf{II}}(t) = H$. That is, the sides of the penny keep their meaning for player I when he is re-named as player II, but heads and tails change their role for player II. In effect, this exchanges the players' preference for matching versus non-matching pennies, as required for the symmetry.

Please note: In the following discussion, when we call a game "symmetric", we always mean the simpler symmetry of a two-player game described in the previous section 2.8 that can be seen by reflecting the game along the diagonal.

2.10 Dominance and elimination of dominated strategies

This section discusses the concepts of strict and weak dominance, which apply to strategies and therefore to games in strategic form.

Definition 2.4 Consider a game in strategic form, and let s and t be two strategies of some player P of that game. Then s *dominates* (or strictly dominates) t if for any fixed strategies of the other players, the payoff to player P when using s is higher than his payoff when using t. Strategy s *weakly dominates* t if for any fixed strategies of the other players, the payoff to player P when using s is at least as high as when using t, and in at least one case strictly higher.

In a two-player game, player P in definition 2.4 is either player I or player II. For player I, his strategy s dominates t if row s of the payoffs to player I is in each component larger than row t. If one denotes the payoff to player I for row i and column j by $a(i,j)$, then s dominates t if $a(s,j) > a(t,j)$ for each strategy j of player II. Similarly, if the payoff to player II is denoted by $b(i,j)$, and s and t are two strategies of player II, which are columns of the payoff table, then s dominates t if $b(i,s) > b(i,t)$ for each strategy i of player I.

In the prisoner's dilemma game in figure 2.5, strategy D of player I dominates strategy C, because, with the notation of the preceding paragraph, $a(D,c) = 3 > 2 = a(C,c)$ and $a(D,d) = 1 > 0 = a(C,d)$. Because the game is symmetric, strategy d of player II also dominates c.

It would be inaccurate to define dominance by saying that a strategy s dominates strategy t if s is "always" better than t. This is only correct if "always" means "given the *same* strategies of the other players". Even if s dominates t, it may happen that *some* payoff when playing s is worse than some other payoff when playing t. For example, $a(C,c) = 2 > 1 = a(D,d)$ in the prisoner's dilemma game, so this is a case where the dominating strategy D gives a lower payoff than C. However, the strategy used by the other player is necessarily different in that case. When s dominates t, then strategy s is better than t when considering any arbitrary (but same) fixed strategies of the other players.

Dominance is sometimes called strict dominance in order to distinguish it from weak dominance. Consider a two-player game with payoffs $a(i,j)$ to player I and $b(i,j)$ to player II when they play row i and column j. According to definition 2.4, strategy s of player I weakly dominates t if $a(s,j) \geq a(t,j)$ for all j and $a(s,j) > a(t,j)$ for at least one column j. The latter condition ensures that if the two rows s and t of payoffs to player I are equal, s is not said to weakly dominate t (because for the same reason, t could also be said to dominate s). Similarly, if s and t are strategies of player II, then s weakly dominates t if the column s of payoffs $b(i,s)$ to player II is in each component i at least as large as column t, and strictly larger, with $b(i,s) > b(i,t)$, in at least one row i. An example of such a strategy is l in figure 2.3(b), which is weakly dominated by r.

When a strategy s of player P dominates his strategy t, player P can always improve his payoff by playing s rather than t. This follows from the way a game in strategic form is played, where the players choose their strategies simultaneously, and the game is played only once.[2] Then player P may consider the strategies of the other players as fixed, and

[2]The game in strategic form is considered as a "one-shot" game. Many studies concern the possible emergence of cooperation in the prisoner's dilemma when the game is repeated, which is a different context.

his own strategy choice cannot be observed and should not influence the choice of the others, so he is better off playing s rather than t.

In consequence, one may study the game where all dominated strategies are *eliminated*. If some strategies are eliminated in this way, one then obtains a game that is simpler because some players have fewer strategies. In the prisoner's dilemma game, elimination of the dominated strategies C and c results in a game that has only one strategy per player, D for player I and d for player II. This strategy profile (D,d) may therefore be considered as a "solution" of the game, a recommendation of a strategy for each player.

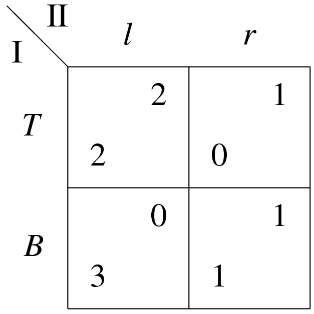


Figure 2.10 The "quality game", with T and B as good or bad quality offered by player I, and l and r as buying or refusing to buy the product as strategies of player II.

If one accepts that a player will never play a dominated strategy, one may eliminate it from the game and continue eliminating strategies that are dominated in the resulting game. This is called *iterated elimination of dominated strategies*. If this process ends in a unique strategy profile, the game is said to be *dominance solvable*, with the final strategy profile as its solution.

The "quality game" in figure 2.10 is a game that is dominance solvable in this way. The game is nearly identical to the prisoner's dilemma game in figure 2.5, except for the payoff to player II for the top right cell of the table, which is changed from 3 to 1. The game may describe the situation of, say, player I as a restaurant owner, who can provide food of good quality (strategy T) or bad quality (B), and a potential customer, player II, who may decide to eat there (l) or not (r). The customer prefers l to r only if the quality is good. However, whatever player II does, player I is better off by choosing B, which therefore dominates T. After eliminating T from the game, player II's two strategies l and r remain, but in this smaller game r dominates l, and l is therefore eliminated in a second iteration. The resulting strategy pair is (B,r).

When eliminating dominated strategies, the order of elimination does not matter, because if s dominates t, then s still dominates t in the game where some strategies (other than t) are eliminated. In contrast, when eliminating weakly dominated strategies, the order of elimination may matter. Moreover, a weakly dominated strategy, such as strategy l of player II in figure 2.3, may be just as good as the strategy that weakly dominates it, if the other player chooses some strategy, like T in figure 2.3, where the two strategies have equal payoff. Hence, there are no strong reasons for eliminating a weakly dominated strategy in the first place.

⇒ Exercise 2.5 on page 54 studies weakly dominated strategies, and what can happen when these are eliminated. You can answer this exercise except for (d) which you should do after having understood the concept of Nash equilibrium, treated in the next section.

2.11 Nash equilibrium

Not every game is dominance solvable, as the games of chicken and the battle of sexes demonstrate. The central concept of non-cooperative game theory is that of *equilibrium*, often called *Nash equilibrium* after John Nash, who introduced this concept in 1950 for general games in strategic form (the equivalent concept for zero-sum games was considered earlier). An equilibrium is a strategy profile where each player's strategy is a "best response" to the remaining strategies.

Definition 2.5 Consider a game in strategic form and a strategy profile given by a strategy s_j for each player j. Then for player i, his strategy s_i is a *best response* to the strategies of the remaining players if no other strategy gives player i a higher payoff, when the strategies of the other players are unchanged. An *equilibrium* of the game, also called *Nash equilibrium*, is a strategy profile where the strategy of each player is a best response to the other strategies.

In other words, a Nash equilibrium is a strategy profile so that no player can gain by changing his strategy *unilaterally*, that is, with the remaining strategies kept fixed.

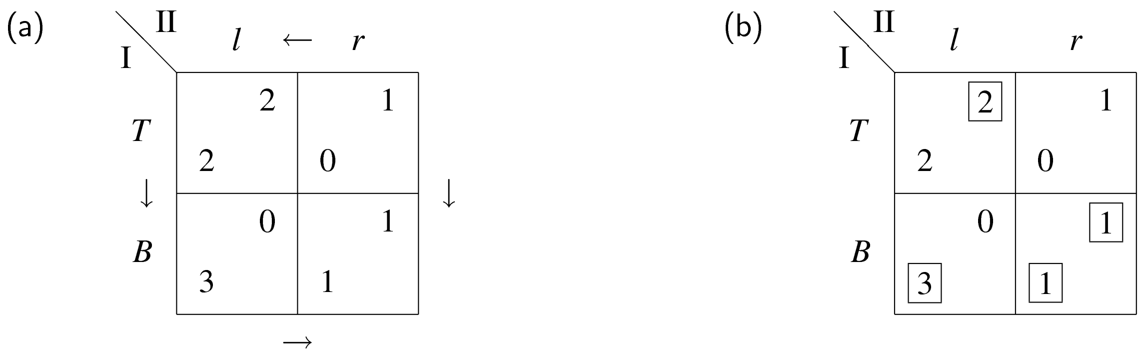


Figure 2.11 Indicating best responses in the quality game with arrows (a), or by putting best response payoffs in boxes (b).

Figure 2.11(a) shows the quality game of figure 2.10 where the best responses are indicated with arrows: The downward arrow on the left shows that if player II plays her left strategy l, then player I has best response B; the downward arrow on the right shows that the best response to r is also B; the leftward arrow at the top shows that player II's best response to T is l, and the rightward arrow at the bottom that the best response to B is r.

This works well for 2×2 games, and shows that a Nash equilibrium is where the arrows "flow together". For the quality game, player I's arrows both point downwards, which shows that B dominates T, and the bottom arrow points right, so that starting from any cell and following the arrows, one arrives at the equilibrium (B,r).

In figure 2.11(b), best responses are indicated by boxes drawn around the payoffs for each strategy that is a best response. For player I, a best response is considered against a strategy of player II. That strategy is a column of the game, so the best response payoff is the maximum of each column (which may occur more than once) of the payoffs of player I. Similarly, a best response payoff for player II is the maximum in each row of the payoffs to player II. Unlike arrows, putting best response payoffs into boxes can be done easily even when a player has more than two strategies. A Nash equilibrium is then given by a cell of the table where both payoffs are boxed.

Figure 2.12 Best responses and Nash equilibria in the game of chicken (a), and in the battle of sexes game (b).

The game of chicken in figure 2.12(a) has two equilibria, (A,c) and (a,C). If a game is symmetric, like in this case, then an equilibrium is symmetric if it does not change under the symmetry (that is, when exchanging the two players). Neither of the two equilibria (A,c) and (a,C) is symmetric, but they map to each other under the symmetry (any non-symmetric equilibrium must have a symmetric counterpart; only symmetric equilibria map to themselves).

As figure 2.12(b) shows, the battle of sexes game has two Nash equilibria, (C,c) and (S,s). (When writing the battle of sexes game symmetrically as in figure 2.7(b), its equilibria are not symmetric either.) The prisoner's dilemma game has one equilibrium (D,d), which is symmetric.

It is clear that a dominated strategy is never a best response, and hence cannot be part of an equilibrium. Consequently, one can eliminate any dominated strategy from a game, and not lose any equilibrium. Moreover, this elimination cannot create additional equilibria because any best response in the absence of a dominated strategy remains a best response when adding the dominated strategy back to the game. By repeating this argument when considering the iterated elimination of dominated strategies, we obtain the following proposition.

Proposition 2.6 *If a game in strategic form is dominance solvable, its solution is the only Nash equilibrium of the game.*

⇒ In order to understand the concept of Nash equilibrium in games with more than two players, exercise 2.6 on page 55 is very instructive.

We have seen that a Nash equilibrium may not be unique. Another drawback is illustrated by the rock-scissors-paper game in figure 2.8, and the game of matching pennies in figure 2.9, namely that the game may have no equilibrium "in pure strategies", that is, when the players may only use exactly one of their given strategies in a deterministic way. This drawback is overcome by allowing each player to use a "mixed strategy", which means that the player chooses his strategy randomly according to a certain probability distribution. Mixed strategies are the topic of the next chapter.

In the following section, we return to game trees, which do have equilibria even when considering only non-randomised or "pure" strategies.

2.12 Reduced strategies

The remainder of this chapter is concerned with the connection of game trees and their strategic form, and the Nash equilibria of the game.

We first consider a simplification of the strategic form. Recall figure 2.4, which shows the strategic form of the extensive game in figure 2.1. The two rows of the strategies XP and XQ in that table have identical payoffs for both players, as do the two rows YP and YQ. This is not surprising, because after move X or Y of player I at the root of the game tree, the decision node where player I can decide between P or Q cannot be reached, because that node is preceded by move Z, which is excluded by choosing X or Y. Because player I makes that move himself, it makes sense to replace both strategies XP and XQ by a less specific "plan of action" $X*$ that precribes only move X. We *always* use the star "$*$" as a placeholder. It stands for an unspecified move at the respective unreachable decision node, to identify the node with its unspecified move in case the game has many decision nodes.

Leaving moves at unreachable nodes unspecified in this manner defines a *reduced strategy* according to the following definition. Because the resulting expected payoffs remain uniquely defined, tabulating these reduced strategies and the payoff for the resulting reduced strategy profiles gives the *reduced strategic form* of the game. The reduced strategic form of the game tree of figure 2.1 is shown in figure 2.13.

Definition 2.7 In a game tree, a *reduced strategy* of a player specifies a move for every decision node of that player, except for those decision nodes that are unreachable due to an earlier own move. A *reduced strategy profile* is a tuple of reduced strategies, one for each player of the game. The *reduced strategic form* of a game tree lists all reduced strategies for each player, and tabulates the expected payoff to each player for each reduced strategy profile.

The preceding definition generalises definitions 2.1 and 2.2. It is important that the only moves that are left unspecified are at decision nodes which are unreachable due to an earlier *own* move of the player. A reduced strategy must not disregard a move because

I \ II	*ad*	*ae*	*bd*	*be*	*cd*	*ce*
$X*$	$\boxed{3}$, 4	3, 4	$\boxed{2}$, $\boxed{5}$	2, $\boxed{5}$	$\boxed{4}$, $\boxed{5}$	$\boxed{4}$, $\boxed{5}$
$Y*$	2, $\boxed{3}$	2, $\boxed{3}$	$\boxed{2}$, $\boxed{3}$	2, $\boxed{3}$	2, $\boxed{3}$	2, $\boxed{3}$
ZP	1, 3	0, $\boxed{5}$	1, 3	0, $\boxed{5}$	1, 3	0, $\boxed{5}$
ZQ	1, $\boxed{3}$	$\boxed{4}$, 2	1, $\boxed{3}$	$\boxed{4}$, 2	1, $\boxed{3}$	$\boxed{4}$, 2

Figure 2.13 Reduced strategic form of the extensive game in figure 2.1. The star $*$ stands for an arbitrary move at the second decision node of player I, which is not reachable after move X or Y.

another player may not move there, because that possibility cannot be excluded by looking only at the player's own moves. In the game tree in figure 2.1, for example, no reduction is possible for the strategies of player II, because neither of her moves at one decision node precludes a move at her other decision node. Therefore, the reduced strategies of player II in that game are the same as her (unreduced, original) strategies.

In definition 2.7, reduced strategies are defined without any reference to payoffs, only by looking at the structure of the game tree. One could derive the reduced strategic form in figure 2.13 from the strategic form in figure 2.4 by identifying the strategies that have identical payoffs for both players and replacing them by a single reduced strategy. Some authors define the reduced strategic form in terms of this elimination of duplicate strategies in the strategic form. (Sometimes, dominated strategies are also eliminated when defining the reduced strategic form.) We define the reduced strategic form without any reference to payoffs because strategies may have identical payoffs by accident, due to special payoffs at the leaves of the game (this does not occur in generic games). Moreover, we prefer that reduced strategies refer to the structure of the game tree, not to some relationship of the payoffs, which is a different aspect of the game.

Strategies are combinations of moves, so for every additional decision node of a player, each move at that node can be combined with move combinations already considered. The number of move combinations therefore grows exponentially with the number of decision nodes of the player, because it is the product of the numbers of moves at each node. In the game in figure 2.14, for example, where m possible initial moves of player I are followed each time by two possible moves of player II, the number of strategies of player II is 2^m. Moreover, this is also the number of reduced strategies of player II because no reduction is possible. This shows that even the reduced strategic form can be exponentially large in the size of the game tree. If a player's move is preceded by an own

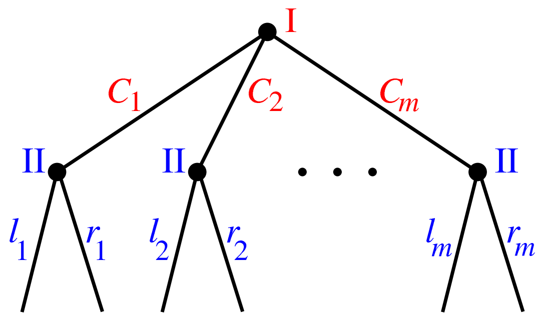


Figure 2.14 Extensive game with m moves $C_1, \ldots, C_m$ of player I at the root of the game tree. To each move C_i, player II may respond with two possible moves l_i or r_i. Player II has 2^m strategies.

earlier move, the reduced strategic form is smaller because then that move can be left unspecified if the preceding own move is not made.

⇒ Exercise 2.4 on page 54 tests your understanding of reduced strategies.

In the reduced strategic form, a Nash equilibrium is defined, as in definition 2.5, as a profile of reduced strategies, each of which is a best response to the others.

In this context, we will for brevity sometimes refer to reduced strategies simply as "strategies". This is justified because, when looking at the reduced strategic form, the concepts of dominance and equilibrium can be applied directly to the reduced strategic form, for example to the table defining a two-player game. Then "strategy" means simply a row or column of that table. The term "reduced strategy" is only relevant when referring to the extensive form.

The Nash equilibria in figure 2.13 are identified as those pairs of (reduced) strategies that are best responses to each other, with both payoffs surrounded by a box in the respective cell of the table. These Nash equilibria in reduced strategies are $(X*, bd)$, $(X*, cd)$, $(X*, ce)$, and $(Y*, bd)$.

⇒ You are now in a position answer (b)–(e) of exercise 2.1 on page 52.

2.13 Subgame perfect Nash equilibrium (SPNE)

In this section, we consider the relationship between Nash equilibria of games in extensive form and backward induction.

The 2×2 game in figure 2.3(b) has two Nash equilibria, (T, l) and (B, r). The strategy pair (B, r) is also obtained, uniquely, by backward induction. We will prove shortly that backward induction always defines a Nash equilibrium.

The equilibrium (T, l) in figure 2.3(b) is not obtained by backward induction, because it prescribes the non-optimal move l for player II at her only decision node. Moreover, l is a weakly dominated strategy. Nevertheless, this is a Nash equilibrium because T is the best response to l, and, moreover, l is a best response to T because against T, the payoff to player II is no worse than when choosing r. This can also be seen when considering the game tree in figure 2.3(a): Move l is a best response to T because player II never has to make that move when player I chooses T, so player II's move does not affect her payoff, and she is indifferent between l and r. On the other hand, only when she makes move l is it optimal for player I to respond by choosing T, because against r player I would get more by choosing B.

The game in figure 2.3(a) is also called a "threat game" because it has a Nash equilibrium (T,l) where player II "threatens" to make the move l that is bad for both players, against which player I chooses T, which is then advantageous for player II compared to the backward induction outcome when the players choose (B,r). The threat works only because player II never has to execute it, given that player I acts rationally and chooses T.

The concept of Nash equilibrium is based on the strategic form because it applies to a strategy profile. When applied to a game tree, the strategies in that profile are assumed to be chosen by the players *before* the game starts, and the concept of best response applies to this given expectation of what the other players will do.

With the game tree as the given specification of the game, it is often desirable to keep its sequential interpretation. The strategies chosen by the players should therefore also express some "sequential rationality" as expressed by backward induction. That is, the moves in a strategy profile should be optimal for any part of the game, including subtrees that cannot be reached due to earlier moves, possibly of other players, like the tree starting at the decision node of player II in the threat game in figure 2.3(a).

In a game tree, a *subgame* is any subtree of the game, given by a node of the tree as the root of the subtree and all its descendants. (Note: This definition of a subgame applies only to games with perfect information, which are the game trees considered so far. In extensive games with imperfect information, which we consider later, the subtree is a subgame only if all players *know* that they are in that subtree.) A strategy profile that defines a Nash equilibrium for every subgame is called a *subgame perfect Nash equilibrium* or SPNE.

> ⇒ You can now answer the final question (e) of exercise 2.1 on page 52, and exercise 2.2 on page 53.

Theorem 2.8 *Backward induction defines an SPNE.*

Proof. Recall the process of backward induction: Starting with the nodes closest to the leaves, consider a decision node u, say, with the assumption that all moves after u have already been selected. Among the moves at u, select a move that *maximises* the expected payoff to the player that moves at u. (Expected payoffs must be regarded if there are chance moves after u.) In that manner, a move is selected for every decision node, which determines an entire strategy profile.

We prove the theorem inductively: Consider a non-terminal node u of the game tree, which may be a decision node (as in the backward induction process), or a chance node. Suppose that the moves at u are $c_1, c_2, \ldots, c_k$, which lead to subtrees $T_1, T_2, \ldots, T_k$ of the game tree, and assume, as inductive hypothesis, that the moves selected so far define an

SPNE in each of these trees. (As the "base case" of the induction, this is certainly true if each of $T_1, T_2, \ldots, T_k$ is just a leaf of the tree, so that u is a "last" decision node considered first in the backward induction process.) The induction step is completed if one shows that, by selecting the move at u, one obtains an SPNE for the subgame with root u.

First, suppose that u is a chance node, so that the next node is chosen according to the fixed probabilities specified in the game tree. Then backward induction does not select a move for u, and the inductive step holds trivially: For every player, the payoff in the subgame starting at u is the expectation of the payoffs for each subgame T_i (weighted with the probability for move c_i), and if a player could improve on that payoff, she would have to do so by changing her moves within at least one subtree T_i, which, by inductive hypothesis, she cannot.

Secondly, suppose that u is a decision node, and consider a player *other* than the player to move at u. Again, for that player, the moves in the subgame starting at u are completely specified, and, irrespective of what move is selected for u, she cannot improve her payoff because that would mean she could improve her payoff already in some subtree T_i.

Finally, consider the player to move at u. Backward induction selects a move for u that is best for that player, given the remaining moves. Suppose the player could improve his payoff by choosing some move c_i and additionally change his moves in the subtree T_i. But the resulting improved payoff would *only* be the improved payoff in T_i, that is, the player could already get a better payoff in T_i itself, contradicting the inductive assumption that the moves selected so far defined an SPNE for T_i. This completes the induction. $\square$

This theorem has two important consequences. In backward induction, each move can be chosen deterministically, so that backward induction determines a profile of pure strategies. Theorem 2.8 therefore implies that game trees have Nash equilibria, and it is not necessary to consider randomised strategies. Secondly, subgame perfect Nash equilibria exist. For game trees, we can use "SPNE" synonymously with "strategy profile obtained by backward induction".

Corollary 2.9 *Every game tree with perfect information has a pure-strategy Nash equilibrium.*

Corollary 2.10 *Every game tree has an SPNE.*

In order to describe an SPNE, it is important to consider *unreduced* (that is, fully specified) strategies. Recall that in the game in figure 2.1, the SPNE are (XQ, bd), (XQ, cd), and (YQ, bd). The reduced strategic form of the game in figure 2.13 shows the Nash equilibria $(X{*}, bd)$, $(X{*}, cd)$, and $(Y{*}, bd)$. However, we cannot call any of these an SPNE because they leave the second move of player I unspecified, as indicated by the $*$ symbol. In this case, replacing $*$ by Q results in all cases in an SPNE. The full strategies are necessary to determine if they define a Nash equilibrium in every subgame. As seen in figure 2.13, the game has, in addition, the Nash equilibrium $(X{*}, ce)$. This is not subgame perfect because it prescribes the move e, which is not part of a Nash equilibrium in the subgame starting with the decision node where player II chooses between

d and e: When replacing $*$ by P, this would not be the best response by player I to e in that subgame, and when replacing $*$ by Q, player II's best response would not be e. This property can only be seen in the game tree, not even in the unreduced strategic form in figure 2.4, because it concerns an unreached part of the game tree due to the earlier move X of player I.

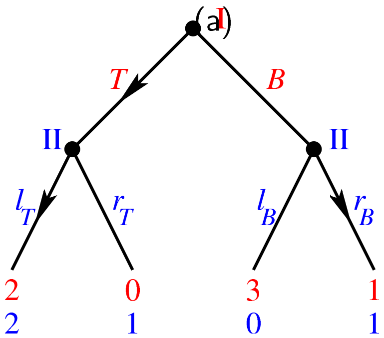
> ⇒ For understanding subgame perfection, exercise 2.3 on page 54 is very instructive. Exercise 2.7 on page 56 studies a three-player game.

2.14 Commitment games

The main difference between game trees and games in strategic form is that in a game tree, players act sequentially, being aware of the previous moves of the other players. In contrast, players in a game in strategic form move *simultaneously*. The difference between these two descriptions becomes striking when *changing* a two-player game in strategic form to a *commitment game*, as described in this section.

We start with a two-player game in strategic form, say an $m \times n$ game. The corresponding commitment game is an extensive game with perfect information. One of the players (we always choose player I) moves *first*, by choosing one of his m strategies. Then, unlike in the strategic form, player II is *informed* about the move of player I. For each move of player I, the possible moves of player II are the n strategies of the original strategic form game, with resulting payoffs to both players as specified in that game.

(a)



(b)

I \ II	$l_T l_B$	$l_T r_B$	$r_T l_B$	$r_T r_B$
T	2, $\boxed{2}$	$\boxed{2}$, $\boxed{2}$	0, 1	0, 1
B	$\boxed{3}$, 0	1, $\boxed{1}$	$\boxed{3}$, 0	$\boxed{1}$, $\boxed{1}$

Figure 2.15 The extensive game (a) is the *commitment* game of the quality game in figure 2.10. Player I moves first, and player II can *react* to that choice. The arrows show the backward induction outcome. Its strategic form with best-response payoffs is shown in (b).

An example is figure 2.15, which is the commitment game for the quality game in figure 2.10. The original strategies T and B of player I become his moves T and B. In general, if player I has m strategies in the given game, he also has m strategies in the commitment game. In the commitment game, each strategy l or r of player II becomes a move, which depends on the preceding move of player I, so we call them l_T and r_T when following move T, and l_B and r_B when following move B. In general, when player I has m moves in the commitment game and player II has n responses, then each combination of moves defines a pure strategy of player II, so she has n^m strategies in the commitment game. Figure 2.15(b) shows the resulting strategic form of the commitment game in the example. It shows that the given game and the commitment game are very different.

We then look for subgame perfect equilibria of the commitment game, that is, apply backward induction, which we can do because the game has perfect information. Figure 2.15(a) shows that the resulting unique SPNE is $(T, l_T r_B)$, which means that (going backwards) player II, when offered high quality T, chooses l_T, and r_B when offered low quality. In consequence, player I will then choose T because that gives him the higher payoff 2 rather than just 1 with B. Note that it is important that player I has the power to *commit* to T and cannot change later back to B after seeing that player II chose l; without that commitment power, the game would not be accurately reflected by the extensive game as described.

The commitment game in figure 2.15 also demonstrates the difference between an equilibrium, which is a strategy profile, and the *equilibrium path*, which is described by the moves that are actually played in the game when players use their equilibrium strategies. If the game has no chance moves, the equilibrium path is just a certain path in the game tree. In contrast, a strategy profile defines a move in every part of the game tree. Here, the SPNE $(T, l_T r_B)$ specifies moves l_T and r_B for both decision points of player II. Player I chooses T, and then only l_T is played. The equilibrium path is then given by move T followed by move l_T. However, it is *not sufficient* to simply call this equilibrium (T, l_T) or (T, l), because player II's move r_B must be specified to know that T is player I's best response.

We only consider SPNE when analysing commitment games. For example, the Nash equilibrium (B, r) of the original quality game in figure 2.10 can also be found in figure 2.15, where player II *always* chooses the move r corresponding to her equilibrium strategy in the original game, which is the strategy $r_T r_B$ in the commitment game. This defines the Nash equilibrium $(B, r_T r_B)$ in the commitment game, because B is a best response to $r_T r_B$, and $r_T r_B$ is a best response to B (which clearly holds as a general argument, starting with any Nash equilibrium of the original game). However, this Nash equilibrium of the commitment game is not subgame perfect, because it prescribes the suboptimal move r_T off the equilibrium path; this does not affect the Nash equilibrium property because T is not chosen by player I. In order to compare a strategic-form game with its commitment game in an interesting way, we consider only SPNE of the commitment game.

A practical application of a game-theoretic analysis may be to reveal the potential effects of changing the "rules" of the game. This is illustrated by changing the quality game to its commitment version.

$\Rightarrow$ This is a good point to try exercise 2.8 on page 56.

Games in strategic form, when converted to a commitment game, typically have a *first-mover advantage*. A player in a game becomes a first mover or "leader" when he can commit to a strategy as described, that is, choose a strategy irrevocably and inform the other players about it; this is a change of the "rules of the game". The first-mover advantage states that a player who can become a leader is not worse off than in the original game where the players act simultaneously. In other words, if one of the players has the power to commit, he or she should do so.

This statement must be interpreted carefully. For example, if more than one player has the power to commit, then it is not necessarily best to go first. For example, consider changing the game in figure 2.10 so that player II can commit to her strategy, and player I moves second. Then player I will always respond by choosing B because this is his dominant strategy in figure 2.10. Backward induction would then amount to player II choosing l, and player I choosing $B_l B_r$, with the low payoff 1 to both. Then player II is not worse off than in the simultaneous-choice game, as asserted by the first-mover advantage, but does not gain anything either. In contrast, making player I the first mover as in figure 2.15 is beneficial to both.

The first-mover advantage is also known as *Stackelberg leadership*, after the economist Heinrich von Stackelberg, who formulated this concept for the structure of markets in 1934. The classic application is to the duopoly model by Cournot, which dates back to 1838.

I \ II	h	m	l	n
H	0, 0	12, 8	18, $\boxed{9}$	$\boxed{36}$, 0
M	8, 12	$\boxed{16}$, $\boxed{16}$	$\boxed{20}$, 15	32, 0
L	$\boxed{9}$, 18	15, $\boxed{20}$	18, 18	27, 0
N	0, $\boxed{36}$	0, 32	0, 27	0, 0

Figure 2.16 Duopoly game between two chip manufacturers who can decide between high, medium, low, or no production, denoted by H,M,L,N for player I and h,m,l,n for player II. Payoffs denote profits in millions of dollars.

As an example, suppose that the market for a certain type of memory chip is dominated by two producers. The players can choose to produce a certain quantity of chips, say either high, medium, low, or none at all, denoted by H,M,L,N for player I and h,m,l,n for player II. The market price of the memory chips decreases with increasing total quantity produced by both companies. In particular, if both choose a high quantity of production, the price collapses so that profits drop to zero. The players know how increased production lowers the chip price and their profits. Figure 2.16 shows the game in strategic form, where both players choose their output level simultaneously. The symmetric payoffs are derived from Cournot's model, explained below.

This game is dominance solvable (see section 2.10 above). Clearly, no production is dominated by low or medium production, so that row N and column n in figure 2.16 can be eliminated. Then, high production is dominated by medium production, so that row H and column h can be omitted. At this point, only medium and low production remain. Then, regardless of whether the opponent produces medium or low, it is always better for each player to produce a medium quantity, eliminating L and l. Only the strategy pair (M,m) remains. Therefore, the Nash equilibrium of the game is (M,m), where both players make a profit of \$16 million.

Consider now the commitment version of the game, with a game tree corresponding to figure 2.16 just as figure 2.15 is obtained from figure 2.10. We omit this tree to save space, but backward induction is easily done using the strategic form in figure 2.16: For each row H, M, L, or N, representing a possible commitment of player I, consider the best response of player II, as shown by the boxed payoff entry for player II *only* (the best responses of player I are irrelevant). The respective best responses are unique, defining the backward induction strategy $l_H m_M m_L h_N$ of player II, with corresponding payoffs 18, 16, 15, 0 to player I when choosing H, M, L, or N, respectively. Player I gets the maximum of these payoffs when he chooses H, to which player II will respond with l_H. That is, among the anticipated responses by player II, player I does best by announcing H, a high level of production. The backward induction outcome is thus that player I makes a profit \$18 million, as opposed to only \$16 million in the simultaneous-choice game. When player II must play the role of the follower, her profits fall from \$16 million to \$9 million.

The first-mover advantage comes from the ability of player I to credibly commit himself. After player I has chosen H, and player II replies with l, player I would like to be able switch to M, improving profits even further from \$18 million to \$20 million. However, once player I is producing M, player II would change to m. This logic demonstrates why, when the players choose their quantities simultaneously, the strategy combination (H,l) is not an equilibrium. The commitment power of player I, and player II's appreciation of this fact, is crucial.

The payoffs in figure 2.16 are derived from the following simple model due to Cournot. The high, medium, low, and zero production numbers are 6, 4, 3, and 0 million memory chips, respectively. The profit per chip is $12 - Q$ dollars, where Q is the total quantity (in millions of chips) on the market. The entire production is sold. As an example, the strategy combination (H,l) yields $Q = 6+3 = 9$, with a profit of \$3 per chip. This yields the payoffs of \$18 million and \$9 million for players I and II in the (H,l) cell in figure 2.16. Another example is player I acting as a monopolist (player II choosing n), with a high production level H of 6 million chips sold at a profit of \$6 each.

In this model, a monopolist would produce a quantity of 6 million even if numbers other than 6, 4, 3, or 0 were allowed, which gives the maximum profit of \$36 million. The two players could cooperate and split that amount by producing 3 million each, corresponding to the strategy combination (L,l) in figure 2.16. The equilibrium quantities, however, are 4 million for each player, where both players receive less. The central four cells in figure 2.16, with low and medium production in place of "cooperate" and "defect", have the structure of a prisoner's dilemma game (figure 2.5), which arises here in a natural economic context. The optimal commitment of a first mover is to produce a quantity of 6 million, with the follower choosing 3 million.

These numbers, and the equilibrium ("Cournot") quantity of 4 million, apply even when arbitrary quantities are allowed. That is, suppose x and y are the quantities of production for player I and player II, respectively. The payoffs $a(x,y)$ to player I and $b(x,y)$ to player II are defined as

$$\begin{aligned} a(x,y) &= x\cdot(12-x-y), \\ b(x,y) &= y\cdot(12-x-y). \end{aligned} \tag{2.2}$$

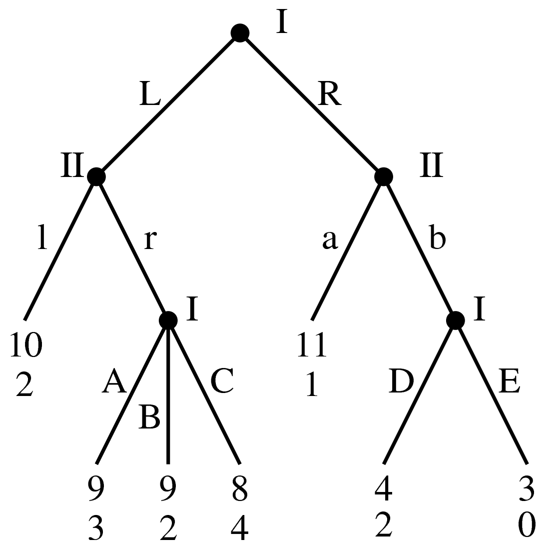
Then it is easy to see that player I's best response $x(y)$ to y is given by $6-y/2$, and player II's best response $y(x)$ to x is given by $6-x/2$. The pair of linear equations $x(y)=x$ and $y(x)=y$ has the solution $x=y=4$, which is the above Nash equilibrium (M,m) of figure 2.16. In the commitment game, player I maximises his payoff, assuming the unique best response $y(x)$ of player II in the SPNE, by maximising $a(x,y(x))$, which is $x\cdot(12-x-6+x/2)=x\cdot(12-x)/2$. That maximum is given for $x=6$, which happens to be the strategy H (high production) in figure 2.16. The best response of player II to that commitment is $6-6/2=3$, which we have named strategy l for low production.

$\Rightarrow$ Exercise 2.9 on page 57 studies a game with infinite strategy sets like (2.2), and makes interesting observations, in particular in (d), concerning the payoffs to the two players in the commitment game compared to the original simultaneous game.

2.15 Exercises for chapter 2

Exercises 2.1 and 2.2 ask you to apply the main concepts of this chapter to a simple game tree. A more detailed study of the concept of SPNE is exercise 2.3. Exercise 2.4 is on counting strategies and reduced strategies. Exercise 2.5 shows that iterated elimination of *weakly* dominated strategies is a problematic concept. Exercises 2.6 and 2.7 concern three-player games, which are very important to understand because the concepts of dominance and equilibrium require to keep the strategies of *all other* players fixed; this is not like just having another player in a two-player game. Exercise 2.8 is an instructive exercise on commitment games. Exercise 2.9 studies commitment games where in the original game both players have infinitely many strategies.

Exercise 2.1 Consider the following game tree. At a leaf, the top payoffs are for player I, the bottom payoffs are for player II.



(a) What is the number of strategies of player I and of player II?

(b) How many reduced strategies do they have?

(c) Give the reduced strategic form of the game.

(d) What are the Nash equilibria of the game in reduced strategies?

(e) What are the subgame perfect Nash equilibria of the game?

Exercise 2.2 Consider the game tree in figure 2.17. At a leaf, the top payoff is for player I, the bottom payoff for player II.

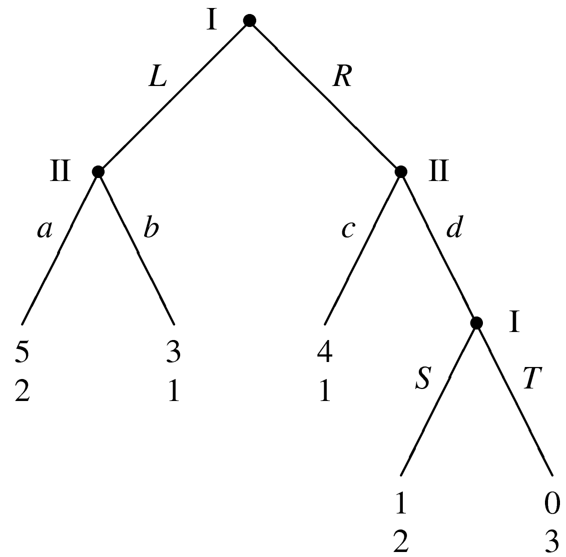


Figure 2.17 Game tree for exercise 2.2.

(a) What is the number of strategies of player I and of player II? How many reduced strategies does each of the players have?

(b) Give the reduced strategic form of the game.

(c) What are the Nash equilibria of the game in reduced strategies? What are the subgame perfect equilibria of the game?

(d) Identify every pair of reduced strategies where one strategy weakly or strictly dominates the other, and indicate if the dominance is weak or strict.

Exercise 2.3 Consider the game trees in figure 2.18.

(a) For the game tree in figure 2.18(a), find all Nash equilibria (in pure strategies). Which of these are subgame perfect?

(b) In the game tree in figure 2.18(b), the payoffs a,b,c,d are positive real numbers. For each of the following statements (i), (ii), (iii), decide if it is true or false, justifying your answer with an argument or counterexample; you may refer to any standard results. For any $a,b,c,d > 0$,

  (i) the game always has a subgame perfect Nash equilibrium (SPNE);

  (ii) the payoff to player II in any SPNE is always at least as high as her payoff in any Nash equilibrium;

  (iii) the payoff to player I in any SPNE is always at least as high as his payoff in any Nash equilibrium.

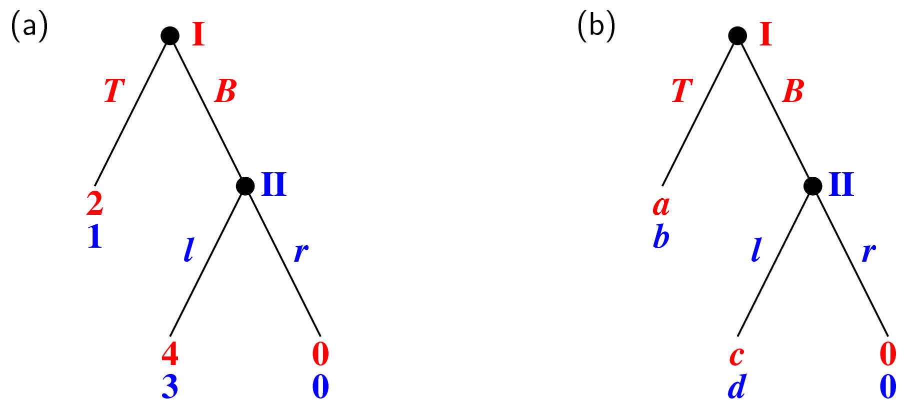


Figure 2.18 Game trees for exercise 2.3

Exercise 2.4 Consider the two game trees (a) and (b) in figure 2.19. In each case, how many strategies does each player have? How many reduced strategies?

Exercise 2.5 Consider the 3×3 game in figure 2.20.

(a) Identify all pairs of strategies where one strategy weakly dominates the other.

(b) Assume you are allowed to remove a weakly dominated strategy of some player. Do so, and repeat this process (of iterated elimination of weakly dominated strategies) until you find a single strategy pair of the original game. (Remember that two strategies with identical payoffs do *not* weakly dominate each other!)

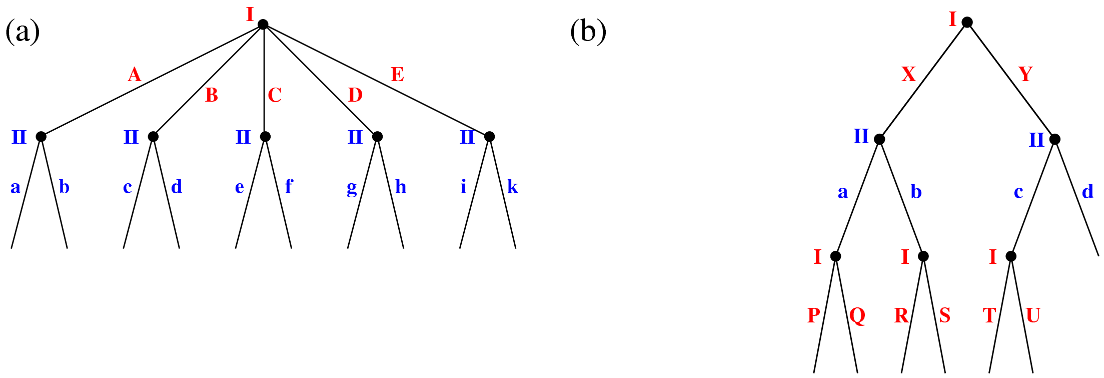


Figure 2.19 Game trees for exercise 2.4. Payoffs have been omitted because they are not relevant for the question.

I \ II	l	c	r
T	0 1	1 3	1 1
M	1 1	0 3	1 0
B	2 2	3 3	2 0

Figure 2.20 3×3 game for exercise 2.5.

(c) Find such an iterated elimination of weakly dominated strategies that results in a strategy pair other than the one found in (b), where *both* strategies, and the payoffs to the players, are different.

(d) What are the Nash equilibria (in pure strategies) of the game?

Exercise 2.6 Consider the three-player game in strategic form in figure 2.21. Each player has two strategies: Player I has the strategies T and B (the top and bottom row below), player II has the strategies l and r (left or right column in each 2×2 panel), and player III has the strategies L and R (the right or left panel). The payoffs to the players in each cell are given as triples of numbers to players I, II, III.

(a) Identify all pairs of strategies where one strategy strictly, or weakly, dominates the other.
[*Hint: Make sure you understand what dominance means for more than two players. Be careful to consider the correct payoffs.*]

(b) Apply iterated elimination of strictly dominated strategies to this game. What are the Nash equilibria of the game?

I \ II	l	r
T	3,4,4	1,3,3
B	8,1,4	2,0,6

III: L

I \ II	l	r
T	4,0,5	0,1,6
B	5,1,3	1,2,5

III: R

Figure 2.21 Three-player game for exercise 2.6.

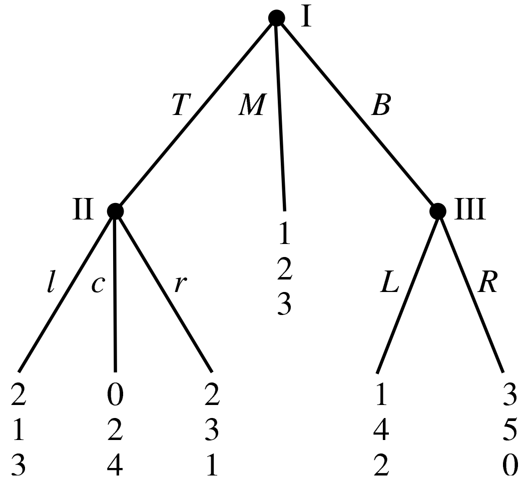


Figure 2.22 Game tree with three players for exercise 2.7. At a leaf, the topmost payoff is to player I, the middle payoff is to player II, and the bottom payoff is to player III.

Exercise 2.7 Consider the three-player game tree in figure 2.22.

(a) How many strategy *profiles* does this game have?

(b) Identify all pairs of strategies where one strategy strictly, or weakly, dominates the other.

(c) Find all Nash equilibria in pure strategies. Which of these are subgame perfect?

Exercise 2.8 Consider a game G in strategic form. Recall that the *commitment game* derived from G is defined by letting player I choose one of his pure strategies x which is then *announced* to player II, who can then in each case choose one of her strategies in G as a response to x. The resulting payoffs are as in the original game G.

(a) If G is an $m \times n$ game, how many strategies do player I and player II have, respectively, in the commitment game?

For each of the following statements, determine whether they are true or false; justify your answer by a short argument or counterexample.

(b) In an SPNE of the commitment game, player I never commits to a strategy that is strictly dominated in G.

(c) In an SPNE of the commitment game, player II never chooses a move that is a strictly dominated strategy in G.

Exercise 2.9 Let G be the following game: player I chooses a (not necessarily integer) non-negative number x, and player II in the same way a non-negative number y. The resulting (symmetric) payoffs are

$$\begin{aligned} &x \cdot (4 + y - x) \quad \text{for player I,} \\ &y \cdot (4 + x - y) \quad \text{for player II.} \end{aligned}$$

(a) Given x, determine player II's best response $y(x)$ (which is a function of x), and player I's best response $x(y)$ to y. Find a Nash equilibrium, and give the payoffs to the two players.

(b) Find an SPNE of the commitment game, and give the payoffs to the two players.

(c) Are the equilibria in (a) and (b) unique?

(d) Let G be a game where the best response $y(x)$ of player II to any strategy x of player I is always unique. Show that in any SPNE of the commitment game, the payoff to player I is at least as large as his payoff in any Nash equilibrium of the original game G.