Matrix Basics

(Appendix B)

Matrices

- A matrix is a 2D array of numbers.
- A matrix can be *rectangular* or *square*.
- Example:

$$D = \begin{pmatrix} 5 & 10 & 3 \\ 2 & -4 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

  - D is rectangular, A is square.
  - D is a 2×3 matrix (2 rows and 3 columns).
  - A is a 2×2 matrix.

- The value in the ith row and jth column (numbering starts from zero) of a matrix X is usually denoted by $x_{i,j}$ (i.e. using a lowercase version of the name of the matrix). For example:

$$d_{0,1} = 10, a_{1,1} = 8.$$

Operations on Matrices

Transposition

The *transpose* of X is written as X^T and is the matrix which has X's rows as its columns. For example:

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \quad A^T = \begin{pmatrix} 2 & 6 \\ 4 & 8 \end{pmatrix}$$

$$D = \begin{pmatrix} 5 & 10 & 3 \\ 2 & -4 & 0 \end{pmatrix} \quad D^T = \begin{pmatrix} 5 & 2 \\ 10 & -4 \\ 3 & 0 \end{pmatrix}$$

Scalar Multiplication

A matrix can be multiplied by a *scalar* (a single value) by multipying all of its entries by the scalar. For example:

$$2 \cdot D = D \cdot 2 = \begin{pmatrix} 10 & 20 & 6 \\ 4 & -8 & 0 \end{pmatrix}.$$

Matrix Multiplication

To multiply two 2×2 matrices:

$$Y = \begin{pmatrix} y_{0,0} & y_{0,1} \\ y_{1,0} & y_{1,1} \end{pmatrix}, Z = \begin{pmatrix} z_{0,0} & z_{0,1} \\ z_{1,0} & z_{1,1} \end{pmatrix}$$

$$Y \cdot Z = \begin{pmatrix} y_{0,0} \cdot z_{0,0} + y_{0,1} \cdot z_{1,0} & y_{0,0} \cdot z_{0,1} + y_{0,1} \cdot z_{1,1} \\ y_{1,0} \cdot z_{0,0} + y_{1,1} \cdot z_{1,0} & y_{1,0} \cdot z_{0,1} + y_{1,1} \cdot z_{1,1} \end{pmatrix}$$

For example:

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 2 \cdot 1 + 4 \cdot 5 & 2 \cdot 3 + 4 \cdot 7 \\ 6 \cdot 1 + 8 \cdot 5 & 6 \cdot 3 + 8 \cdot 7 \end{pmatrix} = \begin{pmatrix} 22 & 34 \\ 46 & 74 \end{pmatrix}$$

Matrix Multiplication

- Matrix multiplication is *associative*. This means $A \cdot B \cdot C$ can be calculated as $(A \cdot B) \cdot C$ or $A \cdot (B \cdot C)$.
- Matrix multiplication is **not commutative**. This means $A \cdot B$ and $B \cdot A$ need not be equal. For example:

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}, B = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, A \cdot B = \begin{pmatrix} 22 & 34 \\ 46 & 74 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 1 \cdot 2 + 3 \cdot 6 & 1 \cdot 4 + 3 \cdot 8 \\ 5 \cdot 2 + 7 \cdot 6 & 5 \cdot 4 + 7 \cdot 8 \end{pmatrix} = \begin{pmatrix} 20 & 28 \\ 52 & 76 \end{pmatrix}$$

Matrix Multiplication for 2×2 DCT

You need to calculate the product DAD^T where A is some 2×2 matrix, e.g. $A = \begin{pmatrix} 10 & 30 \\ 20 & 40 \end{pmatrix}$, and $D = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

$$
\begin{aligned}
DAD^T &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 10 & 30 \\ 20 & 40 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 10 & 30 \\ 20 & 40 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 10 & 30 \\ 20 & 40 \end{pmatrix}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2} \underbrace{\begin{pmatrix} 30 & 70 \\ -10 & -10 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}} \\
&= \frac{1}{2} \begin{pmatrix} 100 & -40 \\ -20 & 0 \end{pmatrix} = \begin{pmatrix} 50 & -20 \\ -10 & 0 \end{pmatrix}
\end{aligned}
$$

Matrix Multiplication for 2×2 DCT (MISTAKE)

Calculate the product DAD^T where A is some 2×2 matrix, e.g. $A = \begin{pmatrix} 10 & 30 \\ 20 & 40 \end{pmatrix}$, and $D = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

$$DAD^T = \frac{1}{2} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 10 & 30 \\ 20 & 40 \end{pmatrix}} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 30 & 70 \\ -10 & -10 \end{pmatrix}}$$

WRONG: matrix mult. not commutative

$$= \frac{1}{2} \begin{pmatrix} 20 & 60 \\ 40 & 80 \end{pmatrix} = \begin{pmatrix} 10 & 30 \\ 20 & 40 \end{pmatrix}$$