Chapter 5. Weak Bisimulation and Observation Congruence

Goals for Chapter 5:

- Introduce the notion of *weak bisimulation*
- Properties of weak bisimulation
- Techniques for establishing weak bisimulation
- Formal definition of observation congruence
- Properties of observation congruence
- Differences and relationships between the three bisimulation equivalences

1 Introduction

In the last chapter we introduced strong bisimulation, in which every α action of one agent must be matched by an α action of the other agent. This even holds for silent actions, for example

$$a.\tau.b.\mathbf{0} \not\sim a.b.\mathbf{0}$$

Now, we shall relax this requirement, only as far as the silent actions τ are concerned. The notion of *weak bisimulation* (WB) will be introduced which treats τ actions as unobservable, i.e.

- it merely requires that each τ action of one agent is matched by *zero* or *more* τ actions of the other.
- and that each external (observable) action l of one agent is matched by an l action accompanied by *zero* or *more* τ actions of the other agent.

In such a setting we shall have that $a.\tau.b.\mathbf{0}$ and $a.b.\mathbf{0}$ are *weakly bisimilar*.

As in the last chapter, we shall motivate weak bisimulation by a variation of interaction games.

2 Weak Bisimulation Games

A *WB game* of interaction from a pair of agents (P_0, Q_0) is a finite or infinite sequence of the form

$$(P_0, Q_0), \ldots, (P_i, Q_i), \ldots$$

played by two participants or observers, player I and player II, such that

- Player I attempts to show that an *observable* difference in behaviour of P_i and Q_i is detectable, whereas player II tries to prevent this.

- For each j the pair (P_{j+1}, Q_{j+1}) is determined as the result of a next step from the previous pair (P_j, Q_j) as follows.
  - Firstly, player I chooses P_j (or Q_j) and a transition $P_j \xrightarrow{\alpha} P_{j+1}$ (or $Q_j \xrightarrow{\alpha} Q_{j+1}$).
  - Then, player II has to choose the other agent Q_j (or P_j) and responds as follows:
    * if $\alpha = \tau$, she can either choose Q_j (or P_j) as Q_{j+1} (or P_{j+1}) without making a transition, or she can make one or more τ transitions
      $Q_j \xrightarrow{\tau} \ldots \xrightarrow{\tau} Q_{j+1} \quad (\text{or} \quad P_j \xrightarrow{\tau} \ldots \xrightarrow{\tau} P_{j+1})$
    * otherwise, if $\alpha \neq \tau$, she chooses a corresponding transition from the other agent accompanied, before and/or after, by zero or more τ transitions
      $Q_j(\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* Q_{j+1} \quad (\text{or} \quad P_j(\xrightarrow{\tau})^* \xrightarrow{\alpha} (\xrightarrow{\tau})^* P_{j+1}),$

    where $(\xrightarrow{\tau})^*$ means zero, one or more $\xrightarrow{\tau}$.
- If at any point a player is unable to make a move then the other player *wins* the game:
  - Player I is stuck if both agents are deadlocked.
  - Player II is at a loss if no corresponding transition is available.
  - If the game continues forever (is infinite) or if there is a repeated configuration, the pair (P_{j+1}, Q_{j+1}) has already occurred previously, then player II also wins.
- A player has a *wining strategy* from (P_0, Q_0) if she is able to win any game from this pair.
- Two agents P_0 and Q_0 are *WB game equivalent* if player II has a winning strategy from (P_0, Q_0): whatever moves player I makes can always be matched by player II.

Remark. Obviously, P and P are WB game equivalent for all agents P.

Example 1: Consider $(P, \tau.P)$. Whenever $P \xrightarrow{\alpha} P'$ by player I, player II can respond with

$$\tau.P \xrightarrow{\tau} \xrightarrow{\alpha} P'$$

And, if player I chooses $\tau.P \xrightarrow{\tau} P$, then player II can respond by simply not making any transition on P. Thus, player II always wins, which implies that P and $\tau.P$ are WB game equivalent.

Example 2: Consider the following agents

$$V \stackrel{def}{=} 1p.(little.collect.V + 1p.big.collect.V)$$

$$V' \stackrel{def}{=} 1p.little.collect.V' + 1p.1p.big.collect.V'$$

Player I has a winning strategy from (V, V') as follows

1. Player I chooses:

$$V' \xrightarrow{1p} 1p.big.collect.V'$$

2. Play II has to make:

$$V \xrightarrow{1p} little.collect.V + 1p.big.collect.V$$

3. Player I opts:

$$little.collect.V + 1p.big.collect.V \overset{little}{\rightarrow} collect.V$$

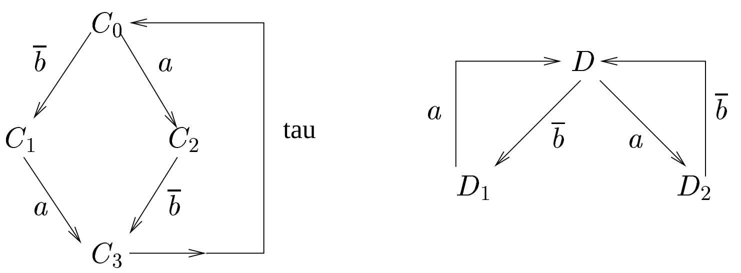
4. Player II cannot make the corresponding transition.

5. Thus, V and V' are not WB game equivalent. Recall, that in the last chapter we have shown that V and V' are not SB game equivalent.

Example 3: Let

$$\begin{array}{ll} C_0 \overset{def}{=} \overline{b}.C_1 + a.C_2 & C_1 \overset{def}{=} a.C_3 \\ C_2 \overset{def}{=} \overline{b}.C_3 & C_3 \overset{def}{=} \tau.C_0 \end{array}$$

$$\begin{array}{ll} D \overset{def}{=} a.D_2 + \overline{b}.D_1 & \\ D_1 \overset{def}{=} a.D & D_2 \overset{def}{=} \overline{b}.D \end{array}$$



Then, C_0 and D are WB game equivalent as any game will go through the following pairs of states

$$(C_0, D), (C_1, D_1), (C_2, D_2), (C_3, D), (C_0, D), \ldots$$

Example: This a bigger example taken from 2001 exam paper.

Consider a system with a specification $Spec \overset{def}{=} a.(b.c.\mathbf{0} + c.b.\mathbf{0})$. By playing weak bisimulation games decide which of the following implementations are correct. Define the appropriate winning strategies to support your answers.

1. $(a.s.\mathbf{0} | \overline{s}.b.\overline{u}.\mathbf{0} | u.c.\mathbf{0}) \setminus \{s, u\}$

2. $(a.\overline{s}.\mathbf{0} | s.b.\mathbf{0} | s.c.\mathbf{0}) \setminus \{s, u\}$

3. $(a.\overline{s}.\overline{s}.\mathbf{0} | s.b.\mathbf{0} | s.c.\mathbf{0}) \setminus \{s, u\}$

4. $(a.\overline{s}.\overline{u}.\mathbf{0} | s.b.\mathbf{0} | u.c.\mathbf{0}) \setminus \{s, u\}$

Try to construct answers before you look at them below.

1. Not correct. Player I chooses *Spec* and action a followed by c. Player II cannot match these actions with the implementation agent: action a is possible but then c is guarded by b.
2. Not correct. Player I chooses *Spec* and action a followed by c (or by b) and gets to $b.\mathbf{0}$. Player II can match these actions with the implementation agent:

$$(a.\overline{s}.\mathbf{0}|s.b.\mathbf{0}|s.c.\mathbf{0}) \setminus \{s,u\} \xrightarrow{a}\xrightarrow{\tau}\xrightarrow{c} (\mathbf{0}|s.b.\mathbf{0}|\mathbf{0}) \setminus \{s,u\}$$

   This is the only way she can match these actions. Then, Player I continues with $b.\mathbf{0}$and performs b. We easily see that $(\mathbf{0}|s.b.\mathbf{0}|\mathbf{0}) \setminus \{s,u\} \stackrel{b}{\not\rightarrow}$.
3. Correct. Easy winning strategy by Player II. After the initial a, Player II waits for Player I to perform the next visible action, which she can clearly match.
4. Correct. Easy winning strategy by Player II similar to the one in part 3. Player II can always match actions Player I can do, possibly first performing some silent actions. Notice that the implementation agent has the derivation tree equivalent to $a.\tau.(b.\tau.c.\mathbf{0}+\tau.(b.c.\mathbf{0}+c.b.\mathbf{0}))$. This is equivalent to $a.(b.c.\mathbf{0}+\tau.(b.c.\mathbf{0}+c.b.\mathbf{0}))$ by simplifying using 1st τ law. Using Corollary 3, Chapter 3 (page 54), we further simplify to $a.\tau.(b.c.\mathbf{0}+c.b.\mathbf{0})$, and again using 1st τ law we obtain $a.(b.c.\mathbf{0}+c.b.\mathbf{0})$, i.e. the specification.

3 Weak Bisimulation

Definition 1 Act^* is the set of all finite sequences of actions in Act; $\varepsilon \in Act^*$ is the empty sequence; α^n is the sequence of n αs.

Definition 2 For $t \in Act^*$, $\widehat{t}$ is the sequence gained by deleting all occurrences of τ from t.
Note: $\widehat{\tau^n} = \varepsilon$.

Definition 3 If $t = \alpha_1 \ldots \alpha_n \in Act^*$, then $E \xrightarrow{t} E'$ if

$$E \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} E'$$

Definition 4 If $t = \alpha_1 \ldots \alpha_n \in Act^*$, then $E \stackrel{t}{\Rightarrow} E'$ if

$$E(\xrightarrow{\tau})^* \xrightarrow{\alpha_1} (\xrightarrow{\tau})^* \cdots (\xrightarrow{\tau})^* \xrightarrow{\alpha_n} (\xrightarrow{\tau})^* E'$$

For example, $E \stackrel{ab}{\Rightarrow} E'$ means that there exists $p,q,r \geq 0$ such that $E \xrightarrow{\tau^p} \xrightarrow{a} \xrightarrow{\tau^q} \xrightarrow{b} \xrightarrow{\tau^r} E'$.

About $\xrightarrow{t}$, $\stackrel{t}{\Rightarrow}$, and $\stackrel{\widehat{t}}{\Rightarrow}$

Each specifies an action sequence with *exactly the same observable actions* as t, but they are different with respect to τ actions:

- $\xrightarrow{t}$ specifies *exactly* the τ actions occurring in t.
- $\stackrel{t}{\Rightarrow}$ specifies *at least* the τ actions occurring in t.

- $\stackrel{\hat{t}}{\Rightarrow}$ specifies *nothing* about τ actions.

We have the following property: $P \stackrel{t}{\rightarrow} P'$ implies $P \stackrel{t}{\Rightarrow} P'$ and $P \stackrel{t}{\Rightarrow} P'$ implies $P \stackrel{\hat{t}}{\Rightarrow} P'$.

Definition 5 A relation $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ is a *weak bisimulation* (WB) if, whenever $P\mathcal{S}Q$ and $\alpha \in Act$, then

1. if $P \stackrel{\alpha}{\rightarrow} P'$, then, for some Q', $Q \stackrel{\hat{\alpha}}{\Rightarrow} Q'$ and $P'\mathcal{S}Q'$, and
2. if $Q \stackrel{\alpha}{\rightarrow} Q'$, then, for some P', $P \stackrel{\hat{\alpha}}{\Rightarrow} P'$ and $P'\mathcal{S}Q'$.

Agents P and Q are *weakly bisimilar* or *observation equivalent*, written $P \approx Q$, if there is a WB $\mathcal{S}$ such that $P\mathcal{S}Q$.

Proposition 1 A relation $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ is a WB iff whenever $P\mathcal{S}Q$ then

1. if $P \stackrel{l}{\rightarrow} P'$ then for some Q', $Q \stackrel{l}{\Rightarrow} Q'$ and $P'\mathcal{S}Q'$,
2. if $P \stackrel{\tau}{\rightarrow} P'$ then for some Q', $Q(\stackrel{\tau}{\rightarrow})^*Q'$ and $P'\mathcal{S}Q'$,
3. if $Q \stackrel{l}{\rightarrow} Q'$ then for some P', $P \stackrel{l}{\Rightarrow} P'$ and $P'\mathcal{S}Q'$,
4. if $Q \stackrel{\tau}{\rightarrow} Q'$ then for some P', $P(\stackrel{\tau}{\rightarrow})^*P'$ and $P'\mathcal{S}Q'$,

Corollary 2 $P \sim Q$ implies $P \approx Q$.
Thus, all the **equational** laws for $\sim$ hold for $\approx$.

Example 4: $P \approx \tau.P$ since the following is a WB

$$\{(P, \tau.P), (P, P) : P \in \mathcal{P}\}$$

Recall that $P \neq \tau.P$.
Example 5: For agents as in Example 3 in the last section, $C_0 \approx D$ since the following is a WB

$$\{(C_0, D), (C_1, D_1), (C_2, D_2), (C_3, D)\}.$$

Example 6: $a.\tau.b.\mathbf{0} \approx a.b.\mathbf{0}$. The following is a WB

$$\{(a.\tau.b.\mathbf{0}, a.b.\mathbf{0}), (\tau.b.\mathbf{0}, b.\mathbf{0}), (b.\mathbf{0}, b.\mathbf{0}), (\mathbf{0}, \mathbf{0})\}$$

Example 7: Although $b.\mathbf{0} \approx \tau.b.\mathbf{0}$ but if $a \neq b$, then

$$a.\mathbf{0} + b.\mathbf{0} \not\approx a.\mathbf{0} + \tau.b.\mathbf{0},$$

meaning that $\approx$ is not a congruence for CCS.

Proof: If there is a WB $\mathcal{S}$ such that $(LHS, RHS) \in \mathcal{S}$, then

Since	$RHS \xrightarrow{\tau} b.\mathbf{0}$	
We require	$LHS \stackrel{\widehat{\tau}\equiv\varepsilon}{\Rightarrow} P'$	for some P'
and	$(P', b.\mathbf{0}) \in \mathcal{S}$.	
But	$P' \equiv LHS$	
and	$(LHS, b.\mathbf{0}) \notin \mathcal{S}$	

Thus, $\approx$ is not a congruence relation: i.e. although $b.\mathbf{0} \approx \tau.b.\mathbf{0}$, but we do not have, as shown above,

$$a.\mathbf{0} + b.\mathbf{0} \approx a.\mathbf{0} + \tau.b.\mathbf{0}.$$

4 Properties of Weak Bisimulation

Weak bisimilarity shares many properties with strong bisimilarity.

Proposition 3 Assume that each $\mathcal{S}_i$ $(i = 1, 2, \ldots)$ is a WB. Then the following relations are WBs

(1) $Id_{\mathcal{P}}$ (3) $\mathcal{S}_1\mathcal{S}_2$

(2) $\mathcal{S}_i^{-1}$ (4) $\bigcup_{i\in I} \mathcal{S}_i$

Proposition 4

1. $\approx$ is the largest WB.
2. $\approx$ is an equivalence relation.
3. $P \approx Q$ iff, for all $\alpha \in Act$
   (a) Whenever $P \xrightarrow{\alpha} P'$ then, for some Q',
   $$Q \stackrel{\widehat{\alpha}}{\Rightarrow} Q' \text{ and } P' \approx Q'$$
   (b) Whenever $Q \xrightarrow{\alpha} Q'$ then, for some P'
   $$P \stackrel{\widehat{\alpha}}{\Rightarrow} P' \text{ and } P' \approx Q'$$

$\approx$ **is not** a congruence relation since it does not preserve Summation, i.e. it is not in general the case that if $P \approx Q$ then $P + R \approx Q + R$, as shown in Example 7. However we have the following proposition.

Proposition 5 If $P \approx Q$, $P_1 \approx P_2$ and $P_i \approx Q_i$ for $i \in I$, then

1. $\alpha.P \approx \alpha.Q$
2. $\sum_{i\in I} \alpha_i.P_i \approx \sum_{i\in I} \alpha_i.Q_i$
3. $P_1|Q \approx P_2|Q$
4. $P_1\backslash L \approx P_2\backslash L$

5. $P_1[f] \approx P_2[f]$

Proposition 6 all the equational laws for $=$ in Chapter 3 are valid for $\approx$, if $E = F$ is a law, then $E\rho \approx F\rho$, where ρ is any ground substitution over CCS terms.

In other words, if $P = Q$ can be proved by the laws from Chapter 3, then $P \approx Q$. The converse is not true: although $\tau.a.\mathbf{0} \approx a.\mathbf{0}$ but not $\tau.a.\mathbf{0} = a.\mathbf{0}$! Hence, $= \subseteq \approx$.

5 Observation Congruence

Observation congruence is very similar to weak bisimulation:

Definition 7 Agents P and Q are *observation congruent*, denoted by $P \approx_o Q$, if for all $\alpha \in Act$,

1. if $P \stackrel{\alpha}{\to} P'$, then, for some Q', $Q \stackrel{\alpha}{\Rightarrow} Q'$ and $P' \approx Q'$, and
2. if $Q \stackrel{\alpha}{\to} Q'$, then, for some P', $P \stackrel{\alpha}{\Rightarrow} P'$ and $P' \approx Q'$

Remarks.

- $\approx_o$ differs from $\approx$ only in one respect: $\stackrel{\alpha}{\Rightarrow}$ takes the place of $\stackrel{\hat{\alpha}}{\Rightarrow}$ for the first actions of P and Q.
- Thus, each action of P or Q must be matched by *at least* one action of the other agent, but this only applies to the first actions of P and Q, for the subsequent actions the matching has to be as for WB.

This becomes even clearer from the following proposition.

Proposition 7 $P \approx_o Q$ iff

1. if $P \stackrel{l}{\to} P'$, then, for some Q', $Q \stackrel{l}{\Rightarrow} Q'$ and $P' \approx Q'$,
2. if $P \stackrel{\tau}{\to} P'$, then, for some Q', $Q \stackrel{\tau}{\Rightarrow} Q'$ and $P' \approx Q'$,
3. if $Q \stackrel{l}{\to} Q'$, then, for some P', $P \stackrel{l}{\Rightarrow} P'$ and $P' \approx Q'$,
4. if $Q \stackrel{\tau}{\to} Q'$, then, for some P', $P \stackrel{\tau}{\Rightarrow} P'$ and $P' \approx Q'$,

Proposition 8

1. $\approx_o$ is a congruence relation, and thus it is an equivalence relation.
2. All the equational laws for $=$ in Chapter 3 are valid for $\approx_o$.

Part 2 means that if $P = Q$ can be proved using the laws from Chapter 3, then $P \approx_o Q$. Importantly, the converse is also true, i.e. if $P \approx_o Q$, then $P = Q$ can be proved using the laws from Chapter 3. Hence,

$$P \approx_o Q \;\text{ iff }\; P = Q$$

6 Relationship between $\sim$, $\approx$ and $\approx_o$

Proposition 9

1. If $P \sim Q$ then $P \approx Q$, i.e. $\sim \subseteq \approx$
2. if $P \sim Q$ then $P \approx_o Q$, i.e. $\sim \subseteq \approx_o$
3. if $P \approx_o Q$ then $P \approx Q$, i.e. $\approx_o \subseteq \approx$
4. Thus, all the **equational** laws for $\sim$ hold for $\approx_o$, and all the **equational** laws for $\approx_o$ hold for $\approx$.

Remark. None of the inverses in Proposition 9 is valid in general, namely

$b.\mathbf{0} \approx \tau.b.\mathbf{0}$, but $b.\mathbf{0} \not\approx_o \tau.b.\mathbf{0}$

$\tau.b.\mathbf{0} \approx_o \tau.\tau.b.\mathbf{0}$, but $\tau.b.\mathbf{0} \not\sim \tau.\tau.b.\mathbf{0}$

Proposition 10

1. If $P \approx Q$ and P and Q are *stable*, i.e. have no immediate τ-transitions, then $P \approx_o Q$.
2. If $P \approx Q$, then $\alpha.P \approx_o \alpha.Q$

Further reading: Milner's book *Communication and Concurrency*, chapters 4, 5 and 7.

7 Exercises

1. Choose one or more of your fellow CO3007 students and play the weak bisimulation (WB) games to determine whether or not the following pairs of CCS agents are WB game equivalent. Remember, that compared to strong bisimulation games, that Player II has now more flexibility to make her moves. To an α move by Player I, Player II can respond as follows.

   - if $\alpha = \tau$, then she can make zero, one or more τ transitions.
   - if $\alpha \neq \tau$, then she needs to perform α accompanied, before and/or after, by zero or more τ transitions.

   Moreover, remember that it is Player I who has to show the difference between the agents' behaviour. Hence, if Player I plays resolutely and is always able to show the difference in behaviour, then the agents are *not* WB game equivalent—in other words Player I has a winning strategy. If Player II plays resolutely and Player I cannot show the difference between the agents' behaviour, then the agents are WB game equivalent—in other words Player II has a winning strategy.

   (a) $a.(b.\mathbf{0} + c.\mathbf{0})$ and $a.(b.\mathbf{0} + c.\mathbf{0}) + a.b.\mathbf{0} + a.c.\mathbf{0}$
   (b) $\tau.P$ and P
   (c) $\alpha.\tau.P$ and $\alpha.P$

(d) $\tau.P$ and $P + \tau.P$

(e) $\alpha.(P + \tau.Q))$ and $\alpha.(P + \tau.Q)) + \alpha.Q$

(f) $\tau.(P + \alpha.(Q + \tau.R))$ and $\tau.(P + \alpha.(Q + \tau.R)) + \alpha.R$

(g) $a.b.\mathbf{0} + a.c.\mathbf{0}$ and $\tau.(a.b.\mathbf{0} + a.c.\mathbf{0}) + a.c.\mathbf{0}$

(h) $a.(b.\mathbf{0} + c.\mathbf{0})$ and $\tau.(a.b.\mathbf{0} + a.c.\mathbf{0}) + a.c.\mathbf{0}$

(i) $a.(b.\mathbf{0} + c.d.\mathbf{0})[e/b, e/c]$ and $a.e.d.\mathbf{0}$

2. Prove whether or not the following pairs of agents are weakly bisimilar. If they are weakly bisimilar, then construct a weak bisimulation relation for the pair. Otherwise, show why the pair fails the definition of weak bisimulation.

   (a) $\alpha.(P + \tau.\tau.P + \tau.P)$ and $\alpha.P$

   (b) $(\overline{c}.\mathbf{0}|a.b.d.\mathbf{0}|\overline{b}.c.e.\mathbf{0}) \setminus \{b, c\}$ and $a.\tau.(d.e.\mathbf{0} + \tau.(d.e.\mathbf{0} + e.d.\mathbf{0}))$

   (c) $\tau.(P + \alpha.(Q + \tau.R))$ and $\tau.(P + \alpha.(Q + \tau.R)) + \alpha.R$

   (d) $a.\tau.b.\mathbf{0} + a.(b.\mathbf{0} + c.\mathbf{0} + \tau.b.\mathbf{0})$ and $a.(c.\mathbf{0} + \tau.b.\mathbf{0}) + a.(b.\mathbf{0} + c.\mathbf{0} + \tau.b.\mathbf{0})$

   (e) $a.B$ and A, where $A \stackrel{def}{=} a.b.A$ and $B \stackrel{def}{=} b.a.B$

   Additionally, support your answers by writing down the winning strategies for Player I or Player II, as appropriate.

3. For the pairs of agents as in question 1 work out whether or not they are strongly bisimilar. If you claim that agents in a pair are not strongly bisimilar, then prove this formally. Hint: use the definition of strong bisimulation.

4. Prove that the following agents are equal. Mention any laws you use.

$$a.(c.\mathbf{0} + \tau.(\tau.b.\mathbf{0} + c.\mathbf{0})) + a.(b.\tau.\mathbf{0} + \tau.(b.\mathbf{0} + c.\mathbf{0}))$$
$$a.(b.\mathbf{0} + c.\tau.\mathbf{0}) + a.b.\mathbf{0} + a.(\tau.b.\mathbf{0} + b.\mathbf{0} + \tau.(\tau.b.\mathbf{0} + c.\mathbf{0}))$$

5. Consider a system with a specification $Spec \stackrel{def}{=} a.b.c.\mathbf{0}+b.a.c.\mathbf{0}$. By playing weak bisimulation games decide which of the following implementations are correct. Define the appropriate winning strategies to support your answers.

   (a) $(a.n.\mathbf{0}|\overline{n}.b.\overline{m}.\mathbf{0}|m.c.\mathbf{0}) \setminus \{n, m\}$

   (b) $(a.\overline{n}.\mathbf{0}|b.n.\mathbf{0}|n.\overline{n}.c.\mathbf{0}) \setminus \{n, m\}$

   (c) $(n.a.\overline{n}.\mathbf{0}|\overline{n}.b.n.\mathbf{0}|n.\overline{n}.c.\mathbf{0}) \setminus \{n, m\}$

   (d) $(n.a.\overline{m}.\mathbf{0}|\overline{n}.b.n.\mathbf{0}|m.\overline{n}.c.\mathbf{0}) \setminus \{n, m\}$

6. (a) Draw the transition graph for $(P|Q|R) \setminus \{b\}$, where

$$\begin{aligned} P &\stackrel{def}{=} a.(\overline{b}.\mathbf{0} + c.S) \\ Q &\stackrel{def}{=} P[b/a] \\ R &\stackrel{def}{=} b.c.S \\ S &\stackrel{def}{=} o.S \end{aligned}$$

(b) Write the defining equations for an agent A, using only constant, prefixing, sum and $\mathbf{0}$ combinators, such that the transition graph of A is identical (up to a change of agent names that appear in the nodes) to the transition graph of the agent $(P|Q|R) \setminus \{b\}$.

(c) Using the laws of equational reasoning for CCS, in particular the τ-laws and the laws for the sum combinator, construct an agent B that has the same observable behaviour as the agent A, but does not use prefixing with silent actions. Explain your construction and state the CCS laws you used.

7. Consider CCS extended with a new unary combinator f which is defined operationally as follows:

$$\frac{E \xrightarrow{a} E' \quad E \xrightarrow{b} E''}{f(E) \xrightarrow{ok} \mathbf{0}} \qquad\qquad \frac{E \xrightarrow{\tau} E'}{f(E) \xrightarrow{\tau} f(E')}$$

(a) Draw transition graphs for $f(\tau.(a.\mathbf{0}+b.\mathbf{0}))$, $f(\tau.(a.\mathbf{0}+b.\mathbf{0}+\tau.b.\mathbf{0}))$ and $f(\tau.(a.\mathbf{0}+\tau.b.\mathbf{0}))$.

(b) Which of the agents $\tau.(a.\mathbf{0}+b.\mathbf{0})$, $\tau.(a.\mathbf{0}+b.\mathbf{0}+\tau.b.\mathbf{0})$ and $\tau.(a.\mathbf{0}+\tau.b.\mathbf{0})$ are observationally congruent?

(c) Which of the agents $f(\tau.(a.\mathbf{0} + b.\mathbf{0}))$, $f(\tau.(a.\mathbf{0} + b.\mathbf{0} + \tau.b.\mathbf{0}))$ and $f(\tau.(a.\mathbf{0} + \tau.b.\mathbf{0}))$ are observationally congruent?

(d) Draw some conclusions concerning f and observational congruence.

8. Consider a sorting machine *Sort* which repeatedly accepts three natural numbers on the ports in_1, in_2 and in_3, and then outputs them in ascending order on the ports $\overline{out_1}$, $\overline{out_2}$ and $\overline{out_3}$. Assume for simplicity that the first number is input on in_1, the second one on in_2 and the last number on in_3; similarly the smallest number is output first on $\overline{out_1}$ and the largest number is output last on $\overline{out_3}$.

(a) In the value passing CCS, write a specification of the agent *Sort* without using parallel composition, relabelling or restriction combinators.

(b) In the value passing CCS, write a specification of an agent S which repeatedly inputs two natural numbers—the first number on the port in_1 and the second on the port in_2—and then outputs them in ascending order on the ports $\overline{out_1}$ and $\overline{out_2}$. Assume that the smaller of the two numbers is output first on $\overline{out_1}$. Do not use parallel composition, restriction or relabelling combinators in S.

(c) Construct an agent *SORT* from several copies of S such that *SORT* implements *Sort*. Explain carefully your construction, for example, via a suitable picture that shows how the ports of copies of S are renamed and connected. In your construction, you will need to use relabelling, restriction and parallel composition combinators. Give an informal justification of your construction.

9. The following family of agent expressions encodes a counter storing a natural number:

$$\begin{array}{lcl} C_0 & \stackrel{def}{=} & up.C_1 \\ C_1 & \stackrel{def}{=} & up.C_2 + down.C_0 \\ \dots & & \\ C_n & \stackrel{def}{=} & up.C_{n+1} + down.C_{n-1} \\ \dots & & \end{array}$$

The agent C below is meant to be an implementation of the counter storing a natural number.

$$\begin{array}{rcl} C & \stackrel{def}{=} & up.(C|D) \\ D & \stackrel{def}{=} & down.\mathbf{0} \end{array}$$

Verify that C is an implementation of the counter, i.e. prove $C_0 \approx C$.

10. A binary relation over CCS agents is a *simulation* if, whenever $P\mathcal{S}Q$, for any CCS agents P and Q, and $\alpha \in Act$, then

$$\text{if } P \stackrel{\alpha}{\to} P', \text{ then, for some } Q',\ Q \stackrel{\widehat{\alpha}}{\Rightarrow} Q' \text{ and } P'\mathcal{S}Q'.$$

Agents P and Q are *similar*, written $P \sim_S Q$, if there is a simulation $\mathcal{S}$ such that $P\mathcal{S}Q$.

Is $\sim_S$ an equivalence relation? If it is, then prove it formally. If it is not, then explain carefully why not.

Explain the relationship between $\approx_o$ and $\sim_S$.

For each of the following three pairs of agents, state whether they are observation congruent, and whether they are similar. In each case, if they are observation congruent, then prove this by defining a suitable observation congruence, and if they are similar, then define a suitable simulation relation. Moreover, in each case, if the agents are *not* observation congruent or *not* similar, then explain carefully why not.

(a) $1p.(small.\overline{smallch}.\mathbf{0} + 1p.big.\overline{bigch}.\mathbf{0})$ and $1p.small.\overline{smallch}.\mathbf{0} + 1p.1p.big.\overline{bigch}.\mathbf{0}$

(b) $1p.small.\overline{smallch}.\mathbf{0} + 1p.1p.big.\overline{bigch}.\mathbf{0}$ and $1p.(small.\overline{smallch}.\mathbf{0} + 1p.big.\overline{bigch}.\mathbf{0})$

(c) $a.b.c.\mathbf{0} + a.b.\mathbf{0}$ and $a.b.c.\mathbf{0}$

Propose two laws for CCS agents that are valid for simulation and that only involve choice combinator, prefixing combinators and agent constants. Your laws will have the form $P = Q$, where P and Q are CCS agent expressions, $P \sim_S Q$ and both P and Q are written using only choice combinator, prefixing combinators and agent constants. Explain why these laws are valid for simulation.