Chapter 4. Strong Bisimulation

Goals for Chapter 4:

- Introduce the notion of strong bisimulation
- Techniques for establishing strong bisimulation
- Properties of strong bisimulation

1 Introduction

In the last chapter we have defined one notion of equivalence: agents P and Q are equivalent if $P = Q$, namely if P can be rewritten into Q using the laws and rules of equational reasoning for CCS (as given in Chapter 3). In this chapter we define a notion of equivalence based on the idea of that we only wish to distinguish between agents P and Q if the distinction can be detected by an external agent interacting with each of the agents. Intuitively, this equivalence will be defined in terms of the transitions of agents which describe precisely the interaction capabilities of agents.

In this chapter we shall treat τ, the silent action, exactly like any other action, namely as visible. So, our equivalence that we shall call strong bisimulation will even distinguish between $a.\mathbf{0}$ and $a.\tau.\mathbf{0}$. In the next chapter we shall take account of our overall intention that τ cannot be observed; this will yield a weaker notion of equivalence called observation congruence.

Equivalence relations between agents

We want to treat two agents P and Q equivalent

- If no distinction can be detected by an external agent (observer) interacting with each of the agents.
- This agent may be ourselves, or another agent from the calculus.
- There are three such relations: strong bisimulation '$\sim$', and weak bisimulation $\approx$, and observation congruence '$\approx_o$' (which, as we will show later, coincides with '$=$').
- $\sim$ and $\approx$ are defined in terms of agents' patterns of transitions.
- $\sim$ and $\approx$ differ in the way they treat silent actions τ.
- $\approx_o$ is sitting between $\sim$ and $\approx$, and is more closely related to $\approx$ than to $\sim$.

Firstly, we shall study $\sim$, which treats τ exactly like any other action.

Some reason for studying $\sim$ first

- It is relatively simpler.
- It shares many common properties with the other two equivalences.
- Many of the equational laws studied so far hold for strong bisimulation.
- The techniques for establishing strong bisimulation are similar to the techniques for establishing the other two equivalences.

Basic notions and definitions

- **Sets:** A set S is a collection of objects (elements). A set S is often denoted as:
  - for a finite set: $S = \{e_1, e_2, \ldots, e_n\}$
  - for an infinite enumerable set: $S = \{e_1, e_2, \ldots, \}$
  - in general $S = \{e \,:\, P(e)\}$, where P is a predicate
  - empty set: $\emptyset = \{e : P(e) \wedge \neg P(e)\}$
- The order of the elements in a set is insignificant:

  $$\{1, 2, 3\} = \{2, 1, 3\}$$

- Repetitions of elements in a set are insignificant:

  $$\{2, 2, 3\} = \{2, 3\}$$

Subsets

A set S_1 is a subset of S_2, written as $S_1 \subseteq S_2$, if each element of S_1 is an element of S_2.

Set operations

Given sets S_1 and S_2, we have the following set operations.

- intersection: $S_1 \cap S_2$, $\bigcap\{S_i : i \in I\}$
- union: $S_1 \cup S_2$, $\bigcup\{S_i : i \in I\}$
- complement: $S_1 - S_2$ is the complement of S_2 with respect to S_1
- product: $S_1 \times S_2 = \{(e_1, e_2) : e_1 \in S_1, e_2 \in S_2\}$, $\Pi\{S_i : i \in I\}$
- power set $\mathcal{P}(S_1) = \{S \,:\, S \subseteq S_1\}$

Note that $S_1 \in \mathcal{P}(S_1)$ and $\emptyset \in \mathcal{P}(S_1)$.

The power set of $\{1, 2, 3\}$ is

$$\mathcal{P}(\{1, 2, 3\}) = \{$$

Relations

A relation $\mathcal{R}$ on a set S is a subset of $S \times S$, i.e. $\mathcal{R} = \{(e_1, e_2) : e_1, e_2 \in S\}$. We shall write $e_1 \mathcal{R} e_2$ if and only if $(e_1, e_2) \in \mathcal{R}$.

Relation operations

Consider S and binary relations $\mathcal{R}_1, \mathcal{R}_2$, $\mathcal{R}_i$, for $i \in I$, on S. The following are binary relations on S.

- identity: $Id_S = \{(e, e) : e \in S\}$
- composition $\mathcal{R}_1 \mathcal{R}_2 = \{(e_1, e_2) : \exists m, (e_1, m) \in \mathcal{R}_1, (m, e_2) \in \mathcal{R}_2\}$
- converse: $\mathcal{R}^{-1} = \{(y, x) : (x, y) \in \mathcal{R}$
- union: $R_1 \cup R_2$, $\bigcup_{i \in I} \mathcal{R}_i$
- intersection: $R_1 \cap R_2$, $\bigcap_{i \in I} \mathcal{R}_i$

Consider $\mathcal{R} \subseteq S \times S$. The following are some of the important properties of relations:

- $\mathcal{R}$ is reflexive if $e \mathcal{R} e$ for all $e \in S$, i.e. $Id_s \subseteq R$
- $\mathcal{R}$ is symmetric if whenever $e_1 \mathcal{R} e_2$ then $e_2 \mathcal{R} e_1$: i.e. $R^{-1} = R$
- $\mathcal{R}$ is transitive if whenever $e_1 \mathcal{R} e_2$ and $e_2 \mathcal{R} e_3$ then $e_1 \mathcal{R} e_3$, i.e. $RR \subseteq R$
- $\mathcal{R}$ is an equivalence relation if it is reflexive, symmetric and transitive.

Examples of equivalence relations are

- equality over numbers, sets, etc.
- CCS's $=$.
- (strong, or weak) bisimulation.
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Algebra

An algebra $\mathcal{G}$ is a structure which consists of a set S, called the carrier set, and a set of operators on the elements of S:

$$\mathcal{G} =< S, op_1, \ldots, op_n > \quad \text{where} \quad op_i(x_{i1}, \ldots, x_{ik_i}) \in S \quad \text{for } x_{i1}, \ldots, x_{ik_i} \in S$$

Congruence

A relation $\mathcal{R}$ on S of $\mathcal{G}$ is a congruence w.r.t. $\mathcal{G}$ if

(a) $\mathcal{R}$ is an equivalence relation, and

(b) for each operator op_i and any elements $e_{i1}, \ldots, e_{ik_i}$ and $e'_{i1}, \ldots, e'_{ik_i}$ of S

$$\text{if} \quad e_{ij} \ \mathcal{R} \ e'_{ij}, \text{ for all } i \text{ and } j = 1, \ldots, k_i, \quad \text{then} \quad op_i(e_{i1}, \ldots, e_{ik_i}) \ \mathcal{R} \ op_i(e'_{i1}, \ldots, e'_{ik_i}).$$

In this module we are interested in equivalences on CCS agents that are congruences with respect to operators of CCS. We say that a relation $\mathcal{R}$ on agents is a congruence with respect to CCS whenever the following condition holds:

$$\forall P_i, Q_i.(\text{ if } P_i \ \mathcal{R}\ Q_i, \quad \text{then} \quad \forall op_i \in \text{CCS}.\ op_i(P_i, \ldots, P_k)\ \mathcal{R}\ op_i(Q_i, \ldots, Q_k))$$

We shall see (Proposition 3) that strong bisimulation $\sim$ is a congruence for CCS.

2 Bisimulation Games

It may be easier to understand the notion of strong bisimulation by considering first the interaction games played on a pair of agents. The games are between two players, and the existence of a winning strategy for one of the players means, as we shall explain below, that the pair of agents is strongly bisimilar.

A game of interaction from a pair of agents (P_0, Q_0) is a finite or infinite sequence of the form

$$(P_0, Q_0), \ldots, (P_i, Q_i), \ldots$$

that is played by two participants called player I and player II.

- Player I attempts to show that the behaviour of P_i and Q_i are different, whereas player II tries to prevent this.
- For each j the pair (P_{j+1}, Q_{j+1}) is determined as the result of a next step from the previous pair (P_j, Q_j) as follows:
  - Firstly, player I chooses P_j (or Q_j) and a valid transition (according the transition rules given in Chapter 2) $P_j \stackrel{\alpha}{\to} P_{j+1}$ (or $Q_j \stackrel{\alpha}{\to} Q_{j+1}$).
  - Then, player II chooses a corresponding valid transition from the other agent, i.e. $Q_j \stackrel{\alpha}{\to} Q_{j+1}$ (or $P_j \stackrel{\alpha}{\to} P_{j+1}$).
- If at any point a player is unable to make a move then the other player wins the game:
  - Player I is stuck if both agents are deadlocked, and she loses the game as she has failed to show a difference between the behaviour of the two agent.
  - Player II is at a loss if no corresponding transition is available, as then player I has found a difference between the behaviour of the two agents, i.e. it is when player I makes a transition for one agent while player II cannot make a corresponding transition for the other agent.
  - If the game continues forever (i.e. is infinite) or if there is a repeated configuration (i.e. the pair (P_{j+1}, Q_{j+1}) has occurred previously in the sequence of pairs), then player II also wins. For, similarly as in the first case, player I can never find a transition in one agent such that player II cannot find the corresponding transition in the other , and in the later case the game can be repeated again and again from the repeated configuration.
- A player has a wining strategy from (P_0, Q_0) if she is able to win any game from this pair.

- Two agents P_0 and Q_0 are game equivalent if player II has a winning strategy from (P_0, Q_0): whatever moves player I makes they can always be matched by player II. Therefore, P_0 and Q_0 are game equivalent (i.e. player II is able to win any game) iff every possible transition of one agent can be matched by a corresponding transition of the other agent.

Example 1: Consider the following agents.

$$Cl_1 \stackrel{def}{=} tick.Cl_1$$

$$Cl_2 \stackrel{def}{=} tick.tick.Cl_2$$

Any game from (Cl_1, Cl_2) must progress through the steps

$$(Cl_1, CL_2), (Cl_1, tick.Cl_2), (Cl_1, Cl_2), \cdots$$

Thus, player II always wins.

Example 2: Consider the following agents

$$V \stackrel{def}{=} 1p.(little.collect.V + 1p.big.collect.V)$$

$$V' \stackrel{def}{=} 1p.little.collect.V' + 1p.1p.big.collect.V'$$

Player I has a winning strategy from (V, V') as follows

1. Player I chooses

$$V' \stackrel{1p}{\rightarrow} 1p.big.collect.V'$$

2. Play II has to make the move

$$V \stackrel{1p}{\rightarrow} little.collect.V + 1p.big.collect.V$$

3. Player I opts for

$$little.collect.V + 1p.big.collect.V \stackrel{little}{\rightarrow} collect.V$$

4. Now, player II cannot make the corresponding transition, namely $1p.big.collect.V \stackrel{little}{\nrightarrow}$.

5. Thus, V and V' are not equivalent.

3 Strong Bisimulation

Definition 1 A relation $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ is a strong bisimulation (SB) if, whenever $P\mathcal{S}Q$ and $\alpha \in Act$, then the following conditions hold:

1. if $P \stackrel{\alpha}{\to} P'$, then, for some Q', $Q \stackrel{\alpha}{\to} Q'$ and $P'\mathcal{S}Q'$
2. if $Q \stackrel{\alpha}{\to} Q'$, then, for some P', $P \stackrel{\alpha}{\to} P'$ and $P'\mathcal{S}Q'$

Agents P and Q are strongly bisimilar or strongly equivalent, written $P \sim Q$, if there is a SB $\mathcal{S}$ such that $P\mathcal{S}Q$.

In terms of interaction games, condition 1 in the definition of SB says that for every possible transition for the first agent that player I is able to make, player II can make a corresponding transition for the second agent, and after this, the game can continue until player I cannot make any transition. Condition 2 says that for every possible transition for the second agent that player I is able to make, play II can make a corresponding transition for the first agent, and after this, the game can continue until player I cannot make any transition. Thus, the two conditions together say that for any possible transition that player I can make for one of the two agents, player II can make a corresponding transition for the other agent, and after this the game can continue until player I cannot make another transition. Therefore, any pair of agents that are strongly bisimilar are game equivalent. This gives us the following result.

Theorem. Two agents are game equivalent if and only if they are strongly bisimilar.

Example 3 For Cl_1 and Cl_2 in Example 1, $Cl_1 \sim Cl_2$ since the following is a strong bisimulation.

$$\mathcal{S}_1 = \{(Cl_1, Cl_2), (Cl_1, tick.Cl_2)\}$$

Example 4 Consider the agents

$$A_0 \stackrel{def}{=} a.A_1 \quad A_1 \stackrel{def}{=} a.A_2 + b.A_0 \quad A_2 \stackrel{def}{=} b.A_1$$

$$B \stackrel{def}{=} a.b.B \quad C \stackrel{def}{=} B|B$$

Then $A_0 \sim C$ since the following is a bisimulation

$$\mathcal{S}_2 = \{(A_0, C), (A_1, b.B|B), (A_1, B|b.B), (A_2, b.B|b.B)\}$$

These two examples illustrate the methods for proving two agents strongly bisimilar by constructing a SB which contains the pair of the two agents.

It is easy to check $\mathcal{S}_1$ and $\mathcal{S}_2$ are SBs. However, how are these SBs constructed when we are asked to prove two agents strongly bisimilar? We will describe this process below.

Proving two agents strongly bisimilar by constructing a SB is often very difficult and tedious. We need some rules or laws (i.e. theorems) which allow us to prove strong bisimilarity between two agents from known (already proven) strongly bisimilar agents.

Proposition 1 Assume that each $\mathcal{S}_i$ ($i \in I = \{1, 2, \ldots\}$) is a SB. Then the following relations are SBs

(1) $Id_{\mathcal{P}}$ (3) $\mathcal{S}_1\mathcal{S}_2$

(2) $\mathcal{S}_i^{-1}$ (4) $\bigcup_{i \in I} \mathcal{S}_i$

Proof of (3): Suppose that

$$P\mathcal{S}_1\mathcal{S}_2R$$

Then for some Q

$$P\mathcal{S}_1Q \text{ and } Q\mathcal{S}_2R$$

Now let $P \xrightarrow{\alpha} P'$. Then for some Q' we have since $P\mathcal{S}_1Q$

$$Q \xrightarrow{\alpha} Q' \text{ and } P'\mathcal{S}_1Q'$$

Also since $Q\mathcal{S}_2R$ we have, for some R',

$$R \xrightarrow{\alpha} R' \text{ and } Q'\mathcal{S}_2R'$$

Hence $P'\mathcal{S}_1\mathcal{S}_2R'$. Similarly if $R \xrightarrow{\alpha} R'$ then we can find P' such that

$$P \xrightarrow{\alpha} P' \text{ and } P'\mathcal{S}_1\mathcal{S}_2R'.$$

4 Constructing Strong Bisimulations

In this section you will practice verifying whether or not two agents are strongly bisimilar.

Knowing the Expansion Law (or indeed Proposition 8 below), we have no doubt that the agents $(a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$ and $a.\tau.d.\mathbf{0}$ are strongly bisimilar. In what follows we show how to construct a strong bisimulation S such that our pair of agents $((a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ a.\tau.d.\mathbf{0})$ belongs to S. We shall construct the relation using the definition of strong bisimulation, Definition 1 from the last section.

The first step in exhibiting whether or not a pair of agents are strongly bisimilar is to assume that they are strongly bisimilar, namely there is a strong bisimulation relation S to which the pair belongs, and use the definition of strong bisimulation to verify your assumption. If the pair is indeed strongly bisimilar, we will construct the relation S in the process of verifying this. Otherwise, if the pair is not strongly bisimilar, we will get stuck when using Definition 1 while verifying strong bisimulation.

Lets assume strong bisimulation S such that $((a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ a.\tau.d.\mathbf{0}) \in S$. The definition of strong bisimulation has two conditions. For our pair of agents condition 1 says that whatever transitions the agent $(a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$ can do the agent $a.\tau.d.\mathbf{0}$ must match them. Condition 2 says that whatever transitions $a.\tau.d.\mathbf{0}$ can do $(a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$ must match them. Both conditions also require that after respective transitions the pairs of resulting agents are in S. Lets examine transitions of our agents. It is usual that agents perform several transitions in some of their states, then we need to examine all these transitions using Definition 1.

We start with condition 1 and the transitions of $(a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$. We see that

$$(a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b \xrightarrow{a} (b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$$

with the agent on the right of $\rightarrow$ being P'. In fact this is the only transition of $(a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$. By condition 1, we need to find an agent Q' such that $a.\tau.d.\mathbf{0} \xrightarrow{a} Q'$ and $P'SQ'$. Clearly $a.\tau.d.\mathbf{0}$

has a unique a derivative $\tau.d.\mathbf{0}$ (sometimes there may be several such derivatives, then you would need to choose the 'best one'). In our case $P'SQ'$ means

$$((b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ \tau.d.\mathbf{0}) \in S.$$

Now, we check condition 2 for our original pair. Since $a.\tau.d.\mathbf{0}$ has only one transition we immediately obtain

$$\text{if} \quad a.\tau.d.\mathbf{0} \xrightarrow{a} \tau.d.\mathbf{0}, \quad \text{then} \quad (a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b \xrightarrow{a} (b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$$

and again we require that $((b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ \tau.d.\mathbf{0}) \in S$. After checking conditions 1 and 2 for our original pair of agents, we know that if S exists, it contains the pairs

$$((a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ a.\tau.d.\mathbf{0})$$

$$((b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ \tau.d.\mathbf{0})$$

We continue verifying conditions 1 and 2 with the agents of the second pair above. We notice that again both agents have single transitions. As action b is restricted and $\overline{c}$ is renamed to $\overline{b}$ in $(b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b$, we have

$$(b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b \xrightarrow{\tau} (\mathbf{0} \mid d.\mathbf{0}[b/c])\backslash b.$$

This is matched by $\tau.d.\mathbf{0} \xrightarrow{\tau} d.\mathbf{0}$. Both conditions require that the pair of the τ derivatives belong to S, i.e. $((\mathbf{0} \mid d.\mathbf{0}[b/c])\backslash b,\ d.\mathbf{0}) \in S$. So, if S exists, then $((\mathbf{0} \mid d.\mathbf{0}[b/c])\backslash b,\ d.\mathbf{0})$ is its third pair.

So far we have checked conditions 1 and 2 for the first two pairs. Now, we continue with the third pair $((\mathbf{0} \mid d.\mathbf{0}[b/c])\backslash b,\ d.\mathbf{0})$. Luckily, again, both agents have single matching transitions:

$$(\mathbf{0} \mid d.\mathbf{0}[b/c])\backslash b \xrightarrow{d} (\mathbf{0} \mid \mathbf{0}[b/c])\backslash b \quad \text{and} \quad d.\mathbf{0} \xrightarrow{d} \mathbf{0}$$

Both conditions require that the pair $((\mathbf{0} \mid \mathbf{0}[b/c])\backslash b,\ \mathbf{0})$ belong to S. We need to check this using Definition 1. We notice that neither of the agents have any transitions, so conditions 1 and 2 are vacuously satisfied. Hence, there is nothing left to check: we have verified that the agents of the original pair are strongly bisimilar, and in the process of doing so we have constructed a particular strong bisimulation S:

$$\begin{aligned} S \quad = \quad & \{((a.b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ a.\tau.d.\mathbf{0}),\ ((b.\mathbf{0} \mid \overline{c}.d.\mathbf{0}[b/c])\backslash b,\ \tau.d.\mathbf{0}),\ ((\mathbf{0} \mid d.\mathbf{0}[b/c])\backslash b,\ d.\mathbf{0}), \\ & ((\mathbf{0} \mid \mathbf{0}[b/c])\backslash b,\ \mathbf{0})\} \end{aligned}$$

Next, we consider a pair of agents that are not strongly bisimilar. We show that $a.\mathbf{0} \mid b.c.\mathbf{0}$ and $a.b.c.\mathbf{0} + b.a.c.\mathbf{0}$ are not strongly bisimilar. We assume that $(a.\mathbf{0} \mid b.c.\mathbf{0}, a.b.c.\mathbf{0} + b.a.c.\mathbf{0}) \in S$, S is some strong bisimulation and we will show that S cannot exist.

We begin by trying to verify conditions 1 and 2 of Definition 1. We notice that this time both original agents have two transitions each. So, we must check condition 1 for the two transitions of $a.\mathbf{0} \mid b.c.\mathbf{0}$ and then check condition 2 for the two transitions of $a.b.c.\mathbf{0}+b.a.c.\mathbf{0}$. Using our knowledge of the derivation trees and interaction games for the two agents, we firstly check condition 1 with the transition

$$a.\mathbf{0} \mid b.c.\mathbf{0} \xrightarrow{b} a.\mathbf{0} \mid c.\mathbf{0}.$$

This is matched in a unique way by

$$a.b.c.\mathbf{0} + b.a.c.\mathbf{0} \xrightarrow{b} a.c.\mathbf{0}.$$

So far, so good. We need to verify that $a.\mathbf{0} \mid c.\mathbf{0}$ and $a.c.\mathbf{0}$ are strongly bisimilar, i.e.

$$(a.\mathbf{0} \mid c.\mathbf{0}, a.c.\mathbf{0}) \in S$$

for some strong bisimulation S. But, checking condition 1 we have

$$a.\mathbf{0} \mid c.\mathbf{0} \xrightarrow{c} a.\mathbf{0} \mid \mathbf{0}$$

and this transition cannot be matched: $a.c.\mathbf{0} \stackrel{c}{\nrightarrow}$. So, condition 1 is not satisfied by S, and we do not need to check anything more (included the remaining three transitions of the original pair of agents). Hence, we cannot construct S according to Definition 1. So, $a.\mathbf{0} \mid b.c.\mathbf{0}$ and $a.b.c.\mathbf{0} + b.a.c.\mathbf{0}$ are not strongly bisimilar as there is no strong bisimulation S such that $(a.\mathbf{0} \mid b.c.\mathbf{0},\ a.b.c.\mathbf{0} + b.a.c.\mathbf{0}) \in S$.

5 Properties of $\sim$

Proposition 2

1. $\sim$ is the largest SB.
2. $\sim$ is an equivalence relation.
3. $P \sim Q$ iff, for all $\alpha \in Act$
   (a) Whenever $P \xrightarrow{\alpha} P'$ then, for some Q',
   $$Q \xrightarrow{\alpha} Q' \text{ and } P' \sim Q'$$
   (b) Whenever $Q \xrightarrow{\alpha} Q'$ then, for some P'
   $$P \xrightarrow{\alpha} P' \text{ and } P' \sim Q'$$

Proof:

(1). It is easy to prove that

$$\sim = \bigcup \{\mathcal{S} : \mathcal{S} \text{ is a SB}\}$$

(2). Reflexivity: For any P, $P \sim P$ by $\mathbf{P}.1(1)$.
Symmetry: If $P \sim Q$, then $P\mathcal{S}Q$ for some SB $\mathcal{S}$. Hence $Q\mathcal{S}^{-1}P$, and so $Q \sim P$ by $\mathbf{P}.1(2)$.
Transitivity: If $P \sim Q$ and $Q \sim R$, then for some SB $\mathcal{S}_1$ and $\mathcal{S}_2$

$$P\mathcal{S}_1Q \text{ and } Q\mathcal{S}_2R$$

Hence $P\mathcal{S}_1\mathcal{S}_2R$, so $P \sim R$ by $\mathbf{P}.1(3)$.

(3) Fairly direct using Definition 1.

Proposition 3 $\sim$ is a congruence relation (called strong congruence) with respect to CCS: for all CCS agents P_1 and P_2 if $P_1 \sim P_2$, then the following statements hold.

1. $\alpha.P_1 \sim \alpha.P_2$
2. $P_1 + Q \sim P_2 + Q$
3. $P_1|Q \sim P_2|Q$
4. $P_1\backslash L \sim P_2\backslash L$
5. $P_1[f] \sim P_2[f]$
6. If $\tilde{A} \stackrel{def}{=} \tilde{P}$, then $\tilde{A} \sim \tilde{P}$

Proof: For statement (3) we show $\mathcal{S}$ is a SB, where

$$\mathcal{S} = \{(P_1|Q, P_2|Q) : P_1 \sim P_2\}$$

Suppose $(P_1|Q, P_2|Q) \in \mathcal{S}$. Let

$$P_1|Q \stackrel{\alpha}{\rightarrow} R$$

There are three cases to consider

1. $P_1 \stackrel{\alpha}{\rightarrow} P_1'$, and $R \equiv P_1'|Q$. Then since $P_1 \sim P_2$, we have

$$P_2 \stackrel{\alpha}{\rightarrow} P_2' \text{ and } P_1' \sim P_2'$$

Hence

$$P_2|Q \stackrel{\alpha}{\rightarrow} P_2'|Q \text{ and } (P_1'|Q, P_2'|Q) \in \mathcal{S}$$

2. $Q \stackrel{\alpha}{\rightarrow} Q'$ and $R \equiv P_1|Q'$. Then also

$$P_2|Q \stackrel{\alpha}{\rightarrow} P_2|Q' \text{ and } (P_1|Q', P_2|Q') \in \mathcal{S}$$

3. $\alpha = \tau$, $P_1 \stackrel{l}{\rightarrow} P_1'$ and $Q \stackrel{\bar{l}}{\rightarrow} Q'$, and $R \equiv P_1'|Q'$. Then since $P_1 \sim P_2$, we have

$$P_2 \stackrel{l}{\rightarrow} P_2' \text{ and } P_2|Q \stackrel{\tau}{\rightarrow} P_2'|Q'$$

with $P_1' \sim P_2'$. Thus

$$(P_1'|Q', P_2'|Q') \in \mathcal{S}$$

For statement (6) we consider the single defining equation

$$A \stackrel{def}{=} P$$

By rule **Con**, $A \stackrel{\alpha}{\to} A'$ iff $P \stackrel{\alpha}{\to} A'$. So by **P.**2(3), $A \sim P$.

Before we show that $\sim$ is preserved by recursion, we have to prepare the ground. In a simple case we will show that, if $A \stackrel{def}{=} E\{A/X\}$, $B \stackrel{def}{=} F\{B/X\}$ and $E \sim F$, then $A \sim B$. Consider

$$A \stackrel{def}{=} b.\mathbf{0} + a.(A \mid c.\mathbf{0}) \qquad \text{and} \qquad B \stackrel{def}{=} a.(c.\mathbf{0} \mid B) + b.\mathbf{0}$$

Because the definitions of A and B are equivalent according to presented in this chapter rules of strong bisimulation (Propositions 3, 5 and 6) we expect A and B to be equivalent. We obtain this as follows: We can write the above two definitions as

$$\begin{array}{lll} A & \stackrel{def}{=} & E\{A/X\} \quad \text{where } E \equiv b.\mathbf{0} + a.(X \mid c.\mathbf{0}) \\ B & \stackrel{def}{=} & F\{B/X\} \quad \text{where } F \equiv a.(c.\mathbf{0} \mid X) + b.\mathbf{0} \end{array}$$

and our laws for $\sim$ above ensure that $E \sim F$, and the proposition below does the rest.

Proposition 4 ($\sim$ is preserved by recursion)
Let $\tilde{E}$ and $\tilde{F}$ be agent expressions and assume that they contain at most variables $\tilde{X}$.

$$\text{If} \quad \tilde{E} \sim \tilde{F}, \quad \text{namely for any agents } \tilde{R}, \tilde{E}\{\tilde{R}/\tilde{X}\} \sim \tilde{F}\{\tilde{R}/\tilde{X}\},$$

and

$$\tilde{A} \stackrel{def}{=} \tilde{E}\{\tilde{A}/\tilde{X}\} \text{ and } \tilde{B} \stackrel{def}{=} \tilde{F}\{\tilde{B}/\tilde{X}\}, \quad \text{then} \quad \tilde{A} \sim \tilde{B}.$$

Proposition 5 (implies **P.**3(1))

1. $P + Q \sim Q + P$ — commutativity
2. $P + (Q + R) \sim (P + Q) + R$ — associativity
3. $P + P \sim P$ — Idempotence
4. $P + \mathbf{0} \sim P$ — $\mathbf{0}$ is the zero element of $+$

Proof of (2): Suppose that

$$P + (Q + R) \stackrel{\alpha}{\to} P'$$

Then by the semantic rules $\mathbf{Sum}_j$, either $P \stackrel{\alpha}{\to} P'$, or $Q \stackrel{\alpha}{\to} P'$ or $R \stackrel{\alpha}{\to} P'$. In each case we can easily infer by $\mathbf{Sum}_j$ that

$$(P + Q) + R \stackrel{\alpha}{\to} P'$$

and we of course know $P' \sim P'$. This establishes (a) of **P.**2(3), and (b) is shown similarly.

Proposition 6 (implies **P.**3.8-3.10)

1. $P|Q \sim Q|P$ — commutativity
2. $P|(Q|R) \sim (P|Q)|R$ — associativity
3. $P|\mathbf{0} \sim P$ — $\mathbf{0}$ is an unit
4. $P\backslash L \sim P$ if $\mathcal{L}(P) \cap (L \cup \overline{L}) = \emptyset$
5. $P\backslash K\backslash L \sim P\backslash(K \cup L)$
6. $P[f]\backslash L \sim (P\backslash f^{-1}(L))[f]$
7. $(P|Q)\backslash L \sim P\backslash L|Q\backslash L \quad$ if $\quad \mathcal{L}(P) \cap \overline{\mathcal{L}(Q)} \cap (L \cup \overline{L}) = \emptyset$
8. $P[Id] \sim P$
9. $P[f] \sim P[f']$ if $f \restriction \mathcal{L}(P) = f' \restriction \mathcal{L}(P)$
10. $P[f][f'] \sim P[f' \circ f]$
11. $(P|Q)[f] \sim P[f]|Q[f]$ if $f \restriction (L \cup \overline{L})$ is one to one, where $L = \mathcal{L}(P|Q)$

Proof of (2): We show that $\mathcal{S}$ is a SB, where $\mathcal{S} = \{(P_1|(P_2|P_3), (P_1|P_2)|P_3) : P_1, P_2, P_3 \in \mathcal{P}\}$. Suppose that $P_1|(P_2|P_3) \stackrel{\alpha}{\to} P'$. There are three main cases, with sub-cases:

1. $P_1 \stackrel{\alpha}{\to} P_1'$, and $P' \equiv P_1'|(P_2|P_3)$ Then we must have

$$(P_1|P_2)|P_3 \stackrel{\alpha}{\to} (P_1'|P_2)|P_3$$

and $(P_1'|(P_2|P_3), (P_1'|P_2)|P_3) \in \mathcal{S}$

2. $P_2|P_3 \stackrel{\alpha}{\to} P_{23}'$ and $P' \equiv P_1|P_{23}'$. Then there are three sub-cases:

2.1. $P_2 \stackrel{\alpha}{\to} P_2'$ and $P_{23}' \equiv P_2'|P_3$.
Then $P' \equiv P_1|(P_2'|P_3)$, and it is easy to show that

$$(P_1|P_2)|P_3 \stackrel{\alpha}{\to} (P_1|P_2')|P_3$$

and

$$(P_1|(P_2'|P_3), (P_1|P_2')|P_3) \in \mathcal{S}$$

2.2. $P_3 \stackrel{\alpha}{\to} P_3'$ and $P_{23}' \equiv P_2|P_3'$. This is proved similarly as for **2.1**.

2.3. $\alpha = \tau$, $P_2 \stackrel{l}{\to} P_2'$, $P_3 \stackrel{\overline{l}}{\to} P_3'$ and $P_{23}' \equiv P_2'|P_3'$
Then $P' \equiv P_1|(P_2'|P_3')$, and it is easy to show that

$$(P_1|P_2)|P_3 \stackrel{\tau}{\to} (P_1|P_2')|P_3'$$

and clearly

$$(P_1|(P_2'|P_3'), (P_1|P_2')|P_3') \in \mathcal{S}$$

3. $\alpha = \tau$, $P_1 \stackrel{l}{\to} P_1'$, $P_2|P_3 \stackrel{\overline{l}}{\to} P_{23}'$ and $P' \equiv P_1'|P_{23}'$. There are two sub-cases:

3.1. $P_2 \xrightarrow{\bar{l}} P_2'$ and $P_{23}' \equiv P_2'|P_3$.
Then $P' \equiv P_1'|(P_2'|P_3)$, and it is easy to show that

$$(P_1|P_2)|P_3 \xrightarrow{\tau} (P_1'|P_2')|P_3$$

and clearly

$$(P_1'|(P_2'|P_3), (P_1'|P_2')|P_3) \in \mathcal{S}$$

3.2. $P_3 \xrightarrow{\bar{l}} P_3'$ and $P_{23}' \equiv P_2|P_3'$. The proof is similar to **3.1**.

This proves condition (1) of **Def.**1. Condition (2) follows by a symmetric argument.

Proposition 7 (implies the expansion law): Let $P \equiv (P_1[f_1]| \dots |P_n[f_n])\backslash L$. Then,

$$\begin{aligned} P \quad \sim \quad & \sum\{f_i(\alpha).(P_1[f_1]| \dots |P_i'[f_i]| \dots |P_n[f_n])\backslash L : \quad P_i \xrightarrow{\alpha} P_i', f_i(\alpha) \notin L \cup \overline{L}\} \\ + \quad & \sum\{\tau.(P_1[f_1]| \dots |P_i'[f_i]| \dots |P_j'[f_j]| \dots |P_n[f_n])\backslash L : \\ & P_i \xrightarrow{l_1} P_i', P_j \xrightarrow{l_2} P_j', f_i(l_1) = \overline{f_j(l_2)}, i < j\} \end{aligned}$$

Proposition 8 (implies **P**.3.4(2)):
Let E_i $(i \in I)$ contain at most the variables X_j $(j \in I)$, and let these variables be weakly guarded in each E_i, i.e. every occurrence of X_i in E_j is within some subexpression $\alpha.F$ of E_j with $\alpha \in Act$. Then

If $\tilde{P} \sim \tilde{E}\{\tilde{P}/\tilde{X}\}$ and $\tilde{Q} \sim \tilde{E}\{\tilde{Q}/\tilde{X}\}$, then $\tilde{P} \sim \tilde{Q}$.

Now, we can see that all the equational laws listed in Chapter 3, except for the τ-laws, hold for $\sim$. The validity of these laws for $\sim$ can be formally proven using mainly the definition of $\sim$.

Question: Why the τ-laws from Chapter 3 do not hold for $\sim$?
This will be answered in the next chapter.

6 Summary

- We have introduced the notion of strong bisimulation. The strong equivalence (and congruence) $P \sim Q$ requires that every α action, including the silent action τ, of P or Q must be matched by an α action of the other agent.

- To prove that two agents are strongly equivalent, we need to establish (define) a strong bisimulation relation which contains the pair of these two agents. Or, we need to construct a strong bisimulation game and show that Player II has a winning strategy.

- Sometimes it is also possible to prove that two agents are strongly equivalent by using the properties of strong bisimulation $\sim$.

7 Exercises

1. Play the strong bisimulation games to determine whether or not the following pairs CCS agents are game equivalent.

   (a) $a.(b.\mathbf{0} + c.\mathbf{0})$ and $a.(b.\mathbf{0} + c.\mathbf{0}) + a.b.\mathbf{0} + a.c.\mathbf{0}$

   (b) $\tau.(P + \alpha.(Q + \tau.R))$ and $\tau.(P + \alpha.(Q + \tau.R)) + \alpha.R$

   (c) Let $A \stackrel{def}{=} a.b.A$ and $B \stackrel{def}{=} b.a.B$. Are $b.A$ and B game equivalent?

   (d) $a.(b.\mathbf{0}|c.\mathbf{0})$ and $a.(b.c.\mathbf{0} + c.b.\mathbf{0})$

   (e) $(a.b.(c.\mathbf{0} + d.\mathbf{0}) + a.c.b.\mathbf{0}) \setminus c$ and $a.b.d.\mathbf{0} + a.\mathbf{0}$

   (f) $a.(b.\mathbf{0} + c.d.\mathbf{0})[e/b, e/c]$ and $a.e.d.\mathbf{0}$

   (g) $a.\overline{b}.\mathbf{0}[e/b]|c.\overline{d}.\mathbf{0})[e/c]) \setminus e$ and $a.\tau.d.\mathbf{0}$

2. Find a strong bisimulation which contains the pair

   $$(Sem_3(0), Sem|Sem|Sem)$$

   where Sem and $Sem_n(0)$ are the semaphores defined in Chapter 1.

3. Consider the agents

   $$\begin{aligned} A_0 &\stackrel{def}{=} a.A_1 \\ A_1 &\stackrel{def}{=} a.A_2 + b.A_0 \\ A_2 &\stackrel{def}{=} b.A_1 \end{aligned}$$

   $$\begin{aligned} B &\stackrel{def}{=} a.b.B \\ C &\stackrel{def}{=} B|B \end{aligned}$$

   Prove $A_0 \sim C$.

4. Prove that $P + (Q + R) \sim (P + Q) + R$.

5. **Cooks**. Ann and Mary are good friends and good cooks; they share a pot and a pan, which they acquire, use, release, to finish cooking, and then to enjoy eating the meal. Write a specification of the system consisting of Ann and Mary, and the pot and the pan. Are you sure that your system may not deadlock, i.e. are you sure that it is not possible that none of Ann and Mary may be starved? If not, modify your specification so that it may not deadlock.

   This is one of the typical questions of two or more agents (or processes) sharing one or more resources, such as processes sharing memories, processors sharing devices, processes sharing communication channels, etc. In such a case, nondeterminism may cause the danger of deadlocks if mutual exclusion is required for accessing the shared resource, i.e. no more then one agents can access the resource at the same time, which is often the case. Thus, it is important to avoid deadlocks. This can be done by adding new agent(s) plays the role as a coordinator, or by eliminating some non-determinism.