Chapter 2

Petri Nets

2.1 Petri Nets Boundedness

2.1.1 Boundedness of Finite Petri Nets

We prove that a Petri net with an initial marking is bounded if and only if it has a finite number of reachable markings. Consider a Petri net N and a marking μ. We say that marking μ' is reachable from μ if there is some sequence of transitions $t_0, \ldots, t_n$ and markings $\mu_0, \ldots, \mu_{n+1}$ such that $\mu_0 = \mu$, $\mu' = \mu_{n+1}$, and for every $0 \leq i \leq n$ we have μ_{i+1} is the t_i-successor of μ_i.

Theorem 2.1 *A Petri net with initial marking is bounded if and only if it has a finite number of reachable markings.*

Proof Consider an unbounded Petri net N. We have to show that there are infinitely many markings reachable from the initial marking. According to the definition of unbounded Petri net, there is some place p such that for every $k \geq 0$, there is a reachable marking μ_k such that $\mu_k(p) > k$. We construct by induction an infinite sequence of reachable markings $\nu_1, \nu_2, \ldots$ such that for every $i \geq 1$ we have $\nu_{i+1}(p) > \nu_i(p)$. Let ν_1 be the marking such that $\nu_1(p) > 1$. Suppose that we have constructed the sequence $\nu_1, \ldots, \nu_n$. Let $k = \nu_n(p)$. By unboundedness, there is a marking $\mu_{\nu_n(p)}$ such that $\mu_{\nu_n(p)}(p) > \nu_n(p)$. Set $\nu_{n+1} = \mu_{\nu_n(p)}$.

In the other direction, suppose that there are infinitely many different reachable markings. Consider the place p_1. If the markings assign infinitely many different values to p_1 then we are done. Otherwise, suppose that p_1 is bounded by k. There has to be some value $v_1 \leq k$ such that infinitely many different markings assign p_1 the value v_1. Restrict our attention to the infinitely many markings that mark p_1 with v_1. Continuing by induction we either find an unbounded place or find infinitely many markings that are fixed on an increasing subset of the locations. As the set of locations is finite, at the end we get a place that is assigned infinitely many different values.

2.1.2 The Dining Philosophers

Consider the following Petri net for the *dining philosophers* problem.

Let μ_0 be the marking depicted in the figure, where all philosophers are thinking and all chopsticks are free. That is for $1 \leq i \leq 4$ we have $\mu_0(c_i) = 1$, $\mu_0(e_i) = 0$, and $\mu_0(t_i) = 1$.

For a marking μ let $E_\mu = \sum_{i=1}^{4} \mu(e_i)$, $C_\mu = \sum_{i=1}^{4} \mu(c_i)$, and $T_\mu = \sum_{i=1}^{4} \mu(t_i)$. That is, E_μ is the number of tokens on locations $e_1, \ldots, e_4$, etc. Let $i \oplus 1$ denote $(i \bmod 4) + 1$. Thus, $4 \oplus 1 = 1$ and for $i < 4$, we have $i \oplus 1 = i + 1$. Dually, $i \ominus 1 = (i + 2 \bmod 4) + 1$. Thus, $1 \ominus 1 = 4$ and for $i > 1$, we have $i \ominus 1 = i - 1$.

Lemma 2.2 *No two adjacent philosophers can eat simulateneously.*

Proof We prove the following invariants for every marking μ reachable from μ_0.

$$\begin{array}{lcl}
I_1 & : & 2E_\mu + C_\mu = 4 \\
I_{2,i} & : & \mu(e_i) + \mu(t_i) = 1 \\
I_{3,i} & : & \mu(e_i) + \mu(c_i) \leq 1 \\
I_{4,i} & : & \mu(e_i) + \mu(c_{i\oplus 1}) \leq 1 \\
I_{5,i} & : & \mu(e_i) + \mu(e_{i\oplus 1}) \leq 1
\end{array}$$

Clearly, all invariants hold for the marking μ_0. Suppose that all invariants hold for μ. Let μ' be the t successor of μ.

If $t = eat_j$, then $\mu'(t_j) = \mu(t_j) - 1$, $\mu'(e_j) = \mu(e_j) + 1$, $\mu'(c_j) = \mu(c_j) - 1$, and $\mu'(c_{j\oplus 1}) = \mu(c_{j\oplus 1}) - 1$. Thus, I_1 and $I_{2,i}$ hold forall i. In order to enable eat_j, it must be the case that $\mu(c_j) = 1$ and $\mu(c_{j\oplus 1}) = 1$. Thus, it must be that $\mu(e_j) = 0$, $\mu(e_{j\oplus 1}) = 0$, and $\mu(e_{j\ominus 1}) = 0$. It follows that $I_{3,i}$, $I_{4,i}$, and $I_{5,i}$ hold as well forall i.

If $t = think_j$, then $\mu'(t_j) = \mu(t_j){+}1$, $\mu'(e_j) = \mu(e_j){-}1$, $\mu'(c_j) = \mu(c_j){+}1$, and $\mu'(c_{j\oplus 1}) = \mu(c_{j\oplus 1}) + 1$. Thus, I_1 and $I_{2,i}$ hold forall i. As $\mu(e_i) = 1$ it must be that $\mu(c_i) = 0$, $\mu(c_{i\oplus 1}) = 0$, $\mu(e_{i\oplus 1}) = 0$, and $\mu(e_{i\ominus 1}) = 0$. It follows that $I_{3,i}$, $I_{4,i}$, and $I_{5,i}$ hold as well forall i.

Invariant $I_{5,i}$ is exactly the mutual exclusion property we want to prove.

It also follows from these invariants that the Petri net is safe, i.e., every place has at most one token in every reachable marking.

2.1.3 The Readers Writers Problem

Consider the following Petri net for the *readers writers* problem.

Let μ_0 be the marking such that $\mu_0(idle_w) = m$, $\mu_0(idle_r) = n$, $\mu_0(capacity) = k$, and $\mu_0(reading) = \mu_0(writing) = 0$.

Lemma 2.3 *The following are true: (a) at most k readers can read simultaneously, (b) at most 1 writer can write simultaneously, and (c) readers and writers cannot read and write simultaneously.*

Proof We prove the following invariants for every marking μ reachable from μ_0.

$$\begin{aligned} I_1 &: \mu(idle_r) + \mu(idle_w) + \mu(capacity) + (k+1)\mu(writing) + \\ &\qquad\qquad 2\mu(reading) = m + n + k \\ I_2 &: \mu(reading) + \mu(capacity) + k\mu(writing) = k \end{aligned}$$

The two invariants hold for the initial marking. Suppose that I_1 and I_2 hold for μ. Let μ' be the t successor of μ.

If $t = read$, then $\mu(capacity) > 0$, $\mu(idle_r) > 0$, and $\mu'(capacity) = \mu(capacity) - 1$, $\mu'(idle_r) = \mu(idle_r) -!$, and $\mu'(reading) = \mu(reading) + 1$.

If $t = write$, then $\mu(capacity) \geq k$, $\mu(idle_w) > 0$ and $\mu'(capacity) = \mu(capacity) - k$, $\mu'(idle_w) = \mu(idle_w) - 1$, and $\mu'(writing) = \mu(writing) + 1$.

The cases where $t = finish\_read$ and $t = finish\_write$ are similar.

It follows that $\mu(capacity) \leq k$, $\mu(reading) \leq k$, $\mu(writing) \leq 1$ and that $\mu(reading) + \mu(writing) \leq 1$.

The following invariants follow.

$$\begin{aligned} I_3 &: \mu(idle_w) + \mu(writing) = m \\ I_4 &: \mu(idle_r) + \mu(reading) = n \\ I_5 &: \mu(reading) \leq k \end{aligned}$$

So in addition every place in the Petri net is bounded.

2.2 Unboundedness of Finite Petri Nets

We prove that the reachability tree algorithm is complete. That is, whenever a Markov chain is unbounded the reachability tree algorithm finds a marking with a *.

Theorem 2.4 *A Petri net with initial marking is unbounded if and only if there exists a reachable marking μ and a sequence of transitions σ such that $\mu \vec{\sigma} \mu'$ for some marking μ' such that $\mu' > \mu$.*

Proof Consider an unbounded Petri net N. We have to show that there are reachable markings μ and μ' as above. According to the definition of unbounded Petri net, there is some place p such that for every $k \geq 0$, there is a reachable marking μ_k such that $\mu_k(p) > k$. We have already established that there is an infinite sequence of reachable markings $\nu_1, \nu_2, \ldots$ and a place p such that for every $i \geq 1$ we have $\nu_{i+1}(p) > \nu_i(p)$. Consider the *infinite* reachability tree that includes all the markings $\nu_1, \ldots$. This tree has a finitely branching degree: if the number of transitions in the Petri net is m, every node in the tree has at most m successors. König's lemma proves that there is an infinite path in the tree that visits infinitely many of these markings[1]. Consider the place p_1. If p_1 is unbounded along the infinite path, then there is a subsequence of $\nu_1, \ldots$ such that the markings of p_1 are strictly increasing in the subsequence. If p_1 is bounded along the infinite path, then there is some value v_1 such that infinitely many markings in the sequence assign p_1 with v_1. We then take the subsequence that assigns p_1 the value v. Proceed in the same way to consider all the places in the net. At the end, we have an infinite subsequence of $\nu_1, \ldots$ such that for every place the markings are not decreasing and are increasing for at least the original place p. It follows that we can find μ and μ' as required.

In the other direction, suppose that there are such markings μ and μ' and a sequence of transitions σ. Let $\sigma = t_1 \cdots t_n$ and let $\mu = \mu_0$, μ_1, $\ldots$, μ_{n-1}, $\mu_n = \mu'$ be the sequence of markings that result from firing these transitions. We construct an infinite sequence of markings starting from μ_0. We prove by induction that $\mu_m > \mu_{m-n} \geq \mu_{m \bmod n}$ for every $m \geq n$. The claim holds trivially for μ_n and μ_0. Suppose that the claim holds for m

[1] König's lemma constructs by induction a path in the tree where each node in the path has an infinite number of the interesting markings below it. This is definitely true for the root. Consider a path from the root to a node with an infinite number of interesting markings below it. Then, as this node has finitely many immediate successors. One of these successors has infinitely many interesting markings below it. It follows that we can find a path longer by one. By induction, we construct an infinite path in the tree.

and prove for $m+1$. By assumption $\mu_m \geq \mu_{m \bmod n}$. Hence, the transition $m \bmod n+1$ is enabled in μ_m. Consider a place that that $\mu_m(p) > \mu_{m-n}(p)$. Clearly, after firing $t_{m \bmod n+1}$ we have $\mu_{m+1}(p) > \mu_{m+1-n}$. Similarly, for every place we have $\mu_{m+1}(p') \geq \mu_{m+1 \bmod n}$. In particular, there is some place p such that $\mu_0(p) < \mu_n(p) < \mu_{2n}(p) < \ldots$. It follows that the marking of p is unbounded.

2.3 Exercises

Exercises marked with ^{s} have solutions in Section 2.4.

^{s}1. Consider the Petri Net $N = (P, T, F, w)$, where

- $P = \{p_1, p_2, p_3\}$;
- $T = \{t_1, t_2, t_3\}$;
- $F = \{(p_1, t_1), (p_1, t_3), (p_2, t_1), (p_2, t_2), (p_3, t_3), (t_1, p_2), (t_1, p_3), (t_2, p_3), (t_3, p_1), (t_3, p_2)\}$;
- $w(x, y) = \begin{cases} 2 & (x, y) = (p_1, t_1) \\ 1 & (x, y) \in F \setminus \{(p_1, t_1)\} \end{cases}$.

(a) Draw the graph of the Petri net.

(b) Determine the presets and the postsets of p_2 and t_2.

(c) Determine the set of enabled transition under each of the markings μ_1 and μ_2 where:

- $\mu_1(p_1) = 1$; $\mu_1(p_2) = 0$; $\mu_1(p_3) = 1$;
- $\mu_2(p_1) = 2$; $\mu_2(p_2) = 1$; $\mu_2(p_3) = 1$.

In each case, for each enabled transition, draw a diagram showing the marking that results from firing that transition.

2. Consider the Petri net $N = (P, T, F, w)$ where:

$$T = \{t_1, t_2, t_3, t_4\}; \quad F = \{(t_1, p_1), (t_2, p_2), (t_3, p_3), (p_1, t_2), (p_1, t_3), (p_2, t_4), (p_3, t_4)\};$$
$$P = \{p_1, p_2, p_3\}; \quad w(x, y) = \begin{cases} 2 & \text{if } (x, y) = (p_1, t_2) \\ 1 & \text{otherwise} \end{cases}.$$

(a) Draw the graph of the Petri net N.

(b) Write down the preset and the postset of p_1.

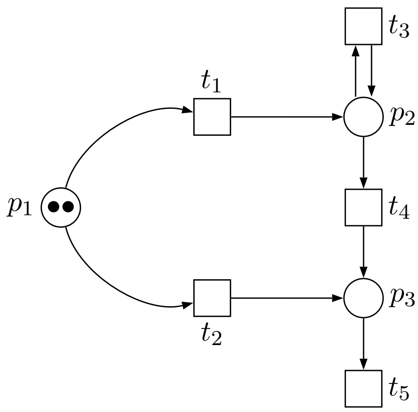
(c) Suppose we have a marking μ with $\mu(p_1) = 1$, $\mu(p_2) = 0$ and $\mu(p_3) = 2$.

i. Draw the graph of the N showing this marking μ (i.e. the graph of N with suitable tokens added).

ii. Which transitions are enabled under the marking μ? Justify your answer briefly.

iii. For each transition t that is enabled under μ, draw the graph of the Petri net showing the marking that results from firing t.

[s]3. Consider the Petri net below with the marking as shown:



(a) Calculate the sequence of markings corresponding to the transition sequence t_1, t_2, t_3, t_4, t_5.

(b) Find a transition sequence that generates the marking sequence $[1,0,0], [0,0,1], [0,0,0]$, where a marking $[n_1, n_2, n_3]$ signifies that there are n_i tokens in place p_i $(i = 1, 2, 3)$.

[s]4. Change the workshop example from Lectures 11 and 12 such that:

- There are two types of items that may arrive to the workshop.
- The workshop produces two types of products:
  - A product composed from one item of type 1 and one item of type 2.
  - A product compoased from two items of type 2.
- The workshop can produce two products of the first type simultaneously but only one of the second type.

[s]5. Consider a vending machine that sells one type of chocolate bar which costs 50p. The machine accepts only 50p and £1 coins (other coins need not be considered). As soon as a valid coin has been put into the machine, the machine will automatically dispense a chocolate bar if one is available; the machine can also provide the correct change.

The machine has a "coin return" button, which prompts the machine to return any coin entered that has not been used in buying a bar of chocolate. The machine has 100 storage slots for bars of chocolate. At any time, a serviceman can refill some or all of these empty slots. In addition, a serviceman can remove some or all of the coins which the machine has taken for dispensed chocolate bars or can insert some more 50p coins (so that change is available).

Model the above system as a Petri net and draw a diagram of the Petri net (labelling the transitions and places appropriately). An initial marking should be indicated that corresponds to a machine currently containing four bars of chocolate and containing two 50p's (and no £1 coins) with no order pending (i.e. all previous orders have been processed with the chocolate delivered and change given as appropriate).

Note: You may make any reasonable assumptions you like in modelling the system and creating the Petri net. For example, you could model the machine as "holding" a 50p or £1 coin before dispensing a chocolate bar (while it checks for availability and change, even though a bar will be dispensed automatically if one is available and the correct change is present).

6. We are going to develop a variant of the readers-writers problem shown in class. There are three types of processes: readers, writers, and editors. All processes are accessing a shared resource. There is mutual exclusion between the different types of processes. That is, at every given time, at most one type of processes is accessing the resource. Then, there are at most k_r readers that access the resource simultaneously, there are at most k_w writers that access the resource simultaneously, and at most k_e editors that access the resouce simultaneously. Devise a Petri net that will suppurt this kind of mutual exclusion. A drawing of the Petri net is sufficient.

7. We wish to model a workshop that has the following properties:

   - Pieces of wood arrive at the workshop. There is storage place for at most ten pieces of untreated wood in the workshop's entrance.
   - There are three painting stations. Each station can handle at most one piece of wood at a time.
   - Untreated wood can move to a free painting station.
   - Once a piece of wood is painted it it put on a queue of wood cutting.
   - There is room for at most seven pieces of painted wood to await cutting.
   - There is one cutter and it can handle one piece of wood at a time.

- A cut piece of wood leaves the workshop.

Model this situation by means of a Petri net. A diagram of the Petri net is quite sufficient here. In the diagram, show the marking that represents the situation where there are four pieces of untreated wood in the workshop's entrance, one piece of wood in two of the painting stations (and none at the third), two pieces of wood awaiting cutting, and one piece of wood that is already cut awaiting to leave the workshop.

8. In this exercise you will create a Petri net model of a teleportation system (cf. *The Stars My Destination* by Alfred Bester). The teleportation system is built between the north pole and the south pole. However, current teleportation technology does not enable direct teleport between such remote locations. So an intermediate station is built in the core of the earth to break the journey. Due to obvious reasons, the intermediate station has to be quite small and can contain the atoms of at most one person simultaneously.

9. We are going to develop a model for the production line of a car. The parts of the car that are going to be considered are the chassis, wheels, engine, and roof. The model should include input trays for all these parts. Input trays should contain at most 6 of each element. The production line starts with taking the chasis and putting the engine on it. Then, four wheels are added, and finally the roof is attached. There is an additional switch that changes the work of the production line to produce cars without roofs.

   Model this production line by means of a Petri net. A diagram of the Petri net is quite sufficient here. In the diagram, show the marking that represents the situation where there are no objects at all in the production line. Label transitions and places carefully to make your solution understandable.

10. We are going to develop a model for an assembly line that works as follows.

    - Items of two types arrive to the assembly line. There is a storage place for at most ten items of each type in the beginning of the assembly line.
    - There are two assembly stations. Station 1 takes two items of type 1 and one item of type 2 and station 2 takes two items of

type 2 and one item of type 1 and assemble them together to a product. Each assembly station can assemble one product at a time.

- Products need to be packed in packages of 2. There are at most 3 products awaiting packing at any given time. There is one packing machine that can handle at most two (pairs of) packages at a time.
- Packed products need to be checked.
- A checked package leaves the assembly line.

Model this situation by means of a Petri net. A diagram of the Petri net is quite sufficient here. In the diagram, show the marking that represents the situation where there are four pieces of each item in the beginning of the assembly line, one product being assembled in one assembly station (and none in the other), two products awaiting packing, and one package being checked. There should be no products being packed, no packages awaiting checking, and no packages awaiting to leave the assembly line.

11. We are going to develop a model for an assembly line that works as follows.

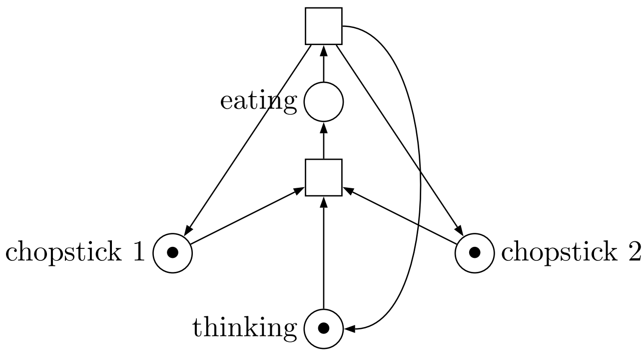
    - Items of three types arrive to the assembly line. There is a storage place for at most ten items of each type in the beginning of the assembly line.
    - There are three assembly stations. Each station takes two items of different types and assembles them together to a product. Each assembly station can assemble one pair at a time.
    - Products need to be checked.
    - One a product is checked it is put in a queue for packing.
    - There is room for at most 5 products to await packing.
    - There is one packing machine that can handle at most two products at a time.
    - A packed product leaves the assembly line.

    Model this situation by means of a Petri net. A diagram of the Petri net is quite sufficient here. In the diagram, show the marking that represents the situation where there are four pieces of each item in the beginning of the assembly line, two items awaiting assembly in one assembly station (and none in the two others), one product being checked, and two awaiting packing. There should be no products being packed and no products awaiting to leave the assembly line.

12. We are going to develop a model for a workshop with a drill, a press, and an oven. The workshop processes two types of objects (A and B) that arrive to their respective input trays. There can be at most five items of each type in each input tray. There is a robot arm that can lift an object and then put it down in a different location. The robot arm needs o move the objects from the input trays to the tools and then to the output tray. Products of type A require processing by the oven, then the drill, and finally the press. Products of type B require processing by the drill, then the press, and finally the oven. In addition, the drill and the press cannot be used simultaneously.

    Model this situation by means of a Petri net. A diagram of the Petri net is quite sufficient here. In the diagram, show the marking that represents the situation where there are no objects at all in the workshop. Label transitions and places carefully to make your solution understandable.

[s] 13. We have seen the following model for the dining philosophers:



- Change the model of a philosopher such that a philosopher first picks up the fork to its left and then picks up the fork to its right.
- Make two philosophers sit around a round table. How do you choose to connect the chopsticks to the other philosopher?

[s] 14. In a computer system, three processes run concurrently and each of the three processes requires various resources. There are three resources: memory, disc and coprocessor. The processes work as follows:

- Process 1 starts in an idle state. It can then request the memory and the coprocessor together and, after using them, it releases both resources. After releasing the resources, the process returns to the idle state; it can now start the procedure all over again.
- Process 2 starts in an idle state. It can then request all three resources at once; after using them, it releases all of them and enters an idle state again. The same procedure can now be repeated.
- Process 3 starts in an idle state. It can request the coprocessor. After using it, Process 3 releases the coprocessor and requests disk access; it then releases disc access and goes back to an idle state before the same procedure can be repeated.

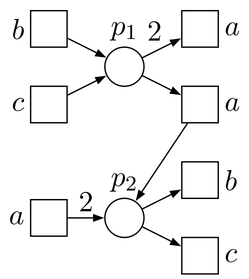
At no time should more than one process have access to the same resource.

(a) Draw a Petri net that models the system described above. Show a suitable initial marking reflecting the situation where all the processes are idle (and all the resources are available).

(b) Is the Petri net safe? (You can just explain this in words - you do not need to construct the reachability tree.)

[s] 15. Use a Petri net to model a 4 bit counter. For simplicity, the update of multiple bits can be done by a chain of transitions and not a single transition. Start with a single bit and increase the number of bits gradually.

[s] 16. Consider the following Petri net.

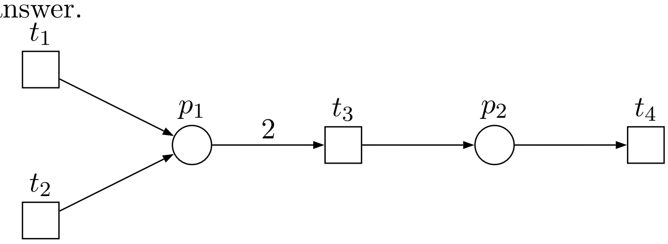


What is the language of this Petri net if the initial marking is $[0, 0]$ and the set of accepting markings is $\{[0, 0]\}$? Is it regular?

[s] 17. What is the language accepted by the labelled Petri net shown below where:

- $\Sigma = \{a, b, c, d\}$;
- a, b, c and d are the labels of the transitions t_1, t_2, t_3 and t_4 respectively;
- the initial marking μ_0 is given by $\mu_0(p_1) = 2$, $\mu_0(p_2) = 0$;
- the set of accepting markings is $\{\mu_1\}$ where $\mu_1(p_1) = \mu_1(p_2) = 0$.

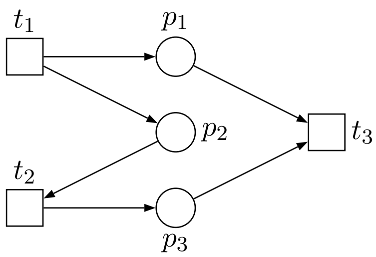
Justify your answer.



18. What is the language accepted by the labelled Petri net shown below where:

    - $\Sigma = \{a, b, c\}$;
    - a, b, and c are the labels of the transitions t_1, t_2, and t_3 respectively;
    - the initial marking μ_0 is given by $\mu_0(p_1) = \mu_0(p_2) = \mu_0(p_3) = 0$;
    - the set of accepting markings is $\{\mu_0\}$.

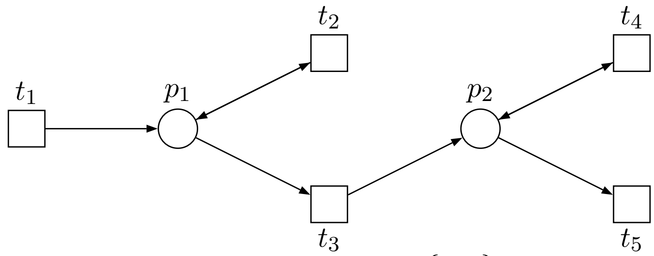
    Justify your answer.



19. What is the language accepted by the labelled Petri net shown below, where:

    - $\Sigma = \{a, b, c\}$;
    - the label of t_1 is a, the label of t_2 and t_3 is b, and the label of t_4 and t_5 is c;
    - the initial marking μ_0 is given by $\mu_0(p_1) = \mu_0(p_2) = 0$;
    - the set of accepting markings is $\{\mu_0\}$.

Justify your answer.

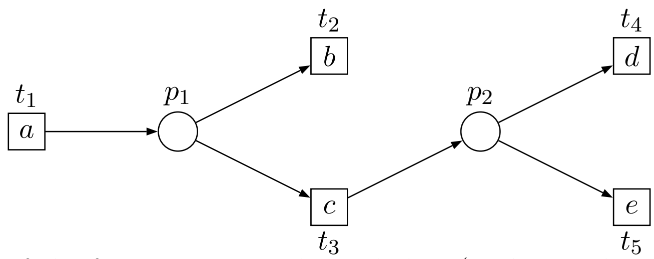


20. Devise a Petri net whose alphabet is $\Sigma = \{a, b\}$ and whose languages is $\{w \mid \sharp_a(w) = \sharp_b(w)\}$.

21. What is the language accepted by the labelled Petri net shown below where:

    - $\Sigma = \{a, b, c, d, e\}$;
    - the labels are as depicted in the diagram;
    - the initial marking μ_0 is given by $\mu_0(p_1) = \mu_0(p_2) = 0$;
    - the set of accepting markings is $\{\mu_0\}$.

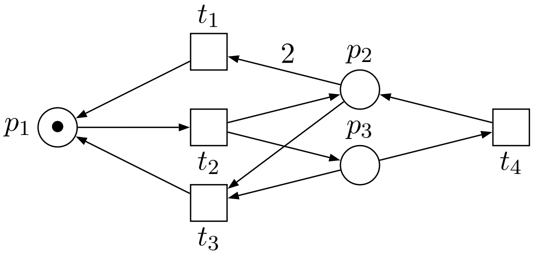
    Justify your answer.

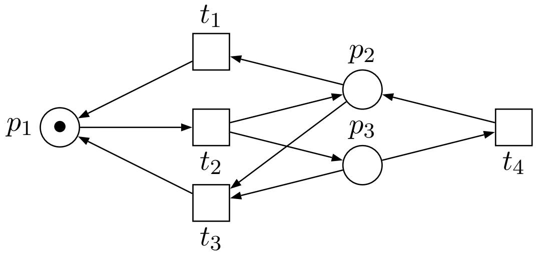


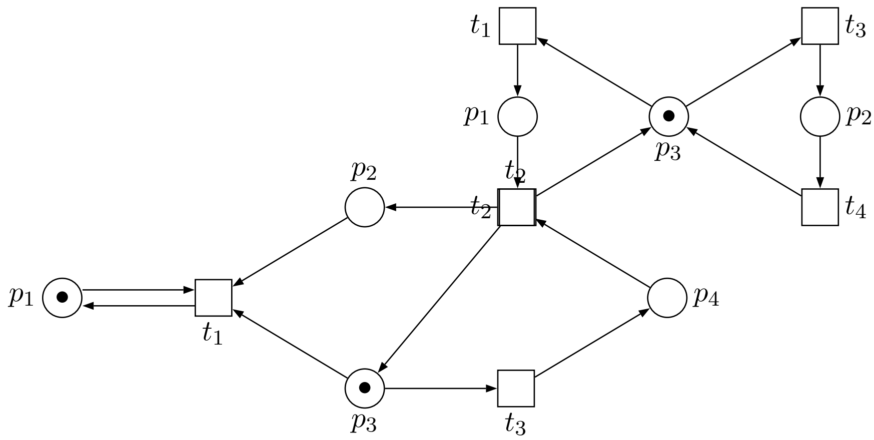
[s] 22. For each of the four Petri nets shown below (with initial markings as indicated):

    (a) Construct the reachability tree for the Petri net (using extended markings if necessary).

    (b) Determine if the Petri net is:
    - safe;
    - k-safe for some k;
    - unbounded.





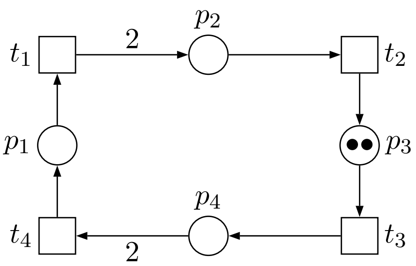


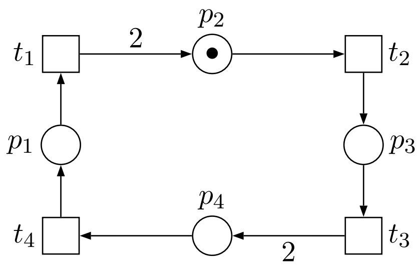
[s]23. For each of the two Petri nets shown below (with initial markings as indicated):

(a) Construct the reachability tree for the Petri net (using extended markings if necessary).

(b) Determine if the Petri net is:

- safe;
- k-safe for some k;
- unbounded.

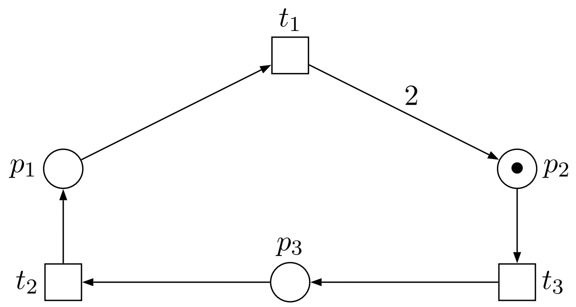


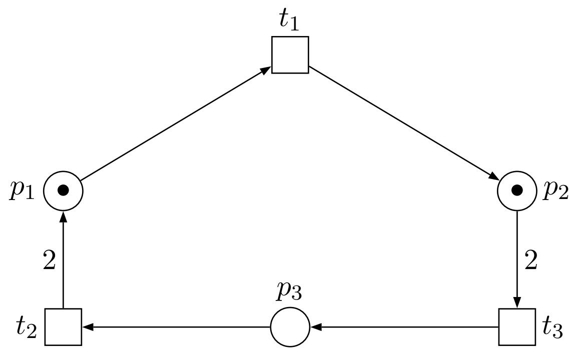


24. For each of the two Petri nets shown below (with initial markings as indicated):

    (a) construct the full reachability tree for the Petri net (using extended markings if necessary);

    (b) determine if the Petri net is: (i) safe; (ii) k-safe for some k; (iii) unbounded.

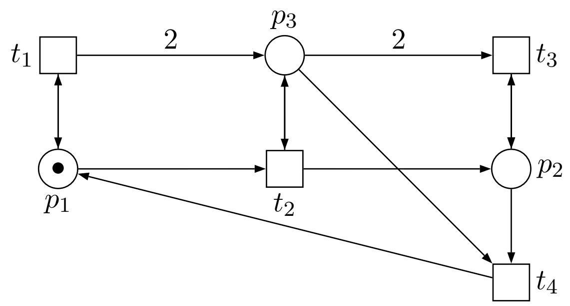


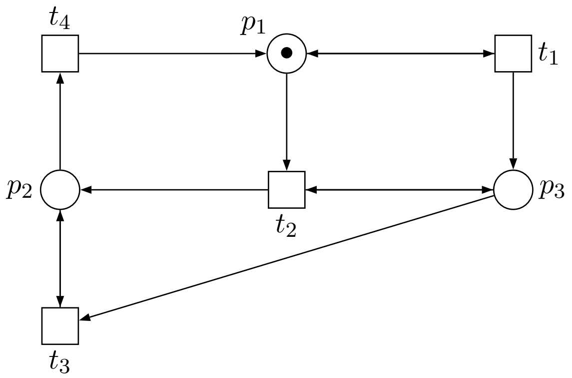


25. For each of the two Petri nets shown below (with initial markings as indicated):

    (a) construct the full reachability tree for the Petri net (using extended markings if necessary);

(b) determine if the Petri net is: (i) safe; (ii) k-safe for some k; (iii) unbounded.

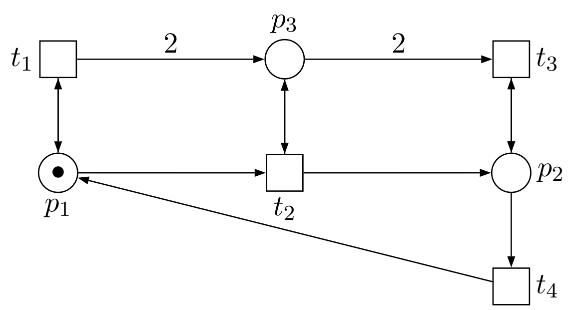


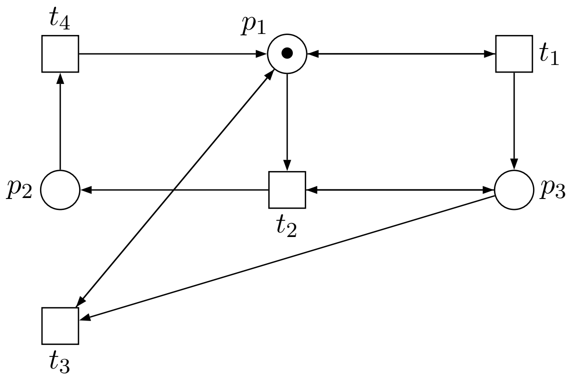


26. For each of the two Petri nets shown below (with initial markings as indicated):

    (a) construct the full reachability tree for the Petri net (using extended markings if necessary);

    (b) determine if the Petri net is: (i) safe; (ii) k-safe for some k; (iii) unbounded.





[s] 27. Revisit the dining philosophers network from the previous Surgery. Show that a network with three philosophers all (one of them inversed) is live. That is, from every reachable marking there is a way to fire every transition.

[s] 28. We are going to model a single-lane bridge. The model includes a single-lane bridge and sensors sensing whether cars are coming from the left and the right. They operate as follows:

- Each sensor can sense whenever a car arrives from its side.
- Each sensor can sense when a car enters the bridge (i.e., leaves the queue on its side).
- The bridge can sense when a car enters it from either side and when a car leaves it.
- At any given time there are at most two cars on the bridge travelling in the same direction.

(a) Construct a Petri-Net model of this system.

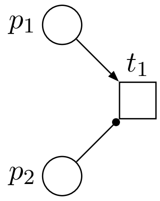
(b) Show that your model is live and deadlock free.

29. Show that for every labelled Petri net, if the Petri net is bounded, then its language is regular.

30. Give an intersection construction for Petri nets. That is, consider two Petri nets $N_i = (P_i, T_i, F_i, \omega_i, \Sigma, l_i, \mu_0^i, A_i)$, for $i \in \{1, 2\}$, over the same alphabet Σ. Show how to construct a Petri net $N = (P, T, F, \omega, \Sigma, l, \mu_0, A)$ such that $L(N) = L(N_1) \cap L(N_2)$.

31. A more advanced form of Petri nets includes *negation arcs*, that is, transitions can depend on there being *no* tokens in some places.

For example, consider the transition below.



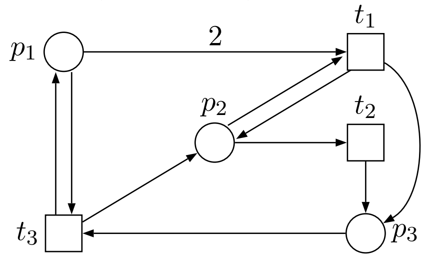
In this new semantics transition t_1 can fire if there is at least one token in p_1 and there are no tokens in p_2.

Show that checking what is the bound on the number of tokens even in one particular place in such a Petri net is undercidable.

You can use a reduction from the emptiness problem of a two-counter machine. That is, whether a two-counter machine accepts some input word or its language is the empty language. Choose your favorite definition of a two-counter machine.

2.4 Solutions to Selected Exercises

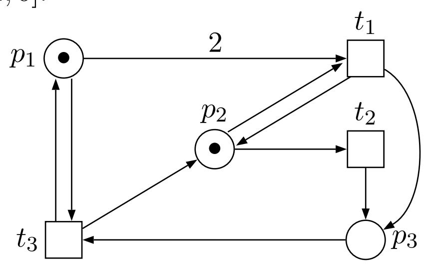
1. (a) We obtain the following Petri net graph:



(b) For the presets and postsets we have:

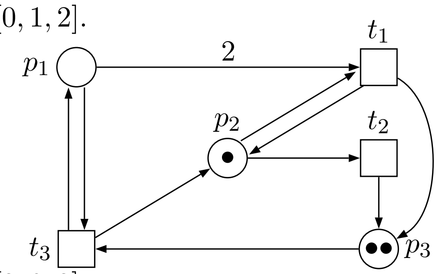
$$\bullet p_2 = \{t_1, t_3\}; \qquad \bullet t_2 = \{p_2\};$$
$$p_2\bullet = \{t_1, t_2\}; \qquad t_2\bullet = \{p_3\}.$$

(c) Marking $\mu_1 = [1, 0, 1]$: only transition t_3 is enabled and we have $[1, 0, 1] \xrightarrow{t_3} [1, 1, 0]$.

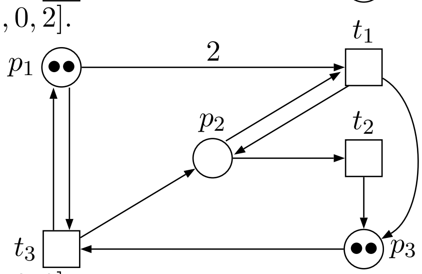


Marking $\mu = [2, 1, 1]$: All the transitions are enabled, and we have:

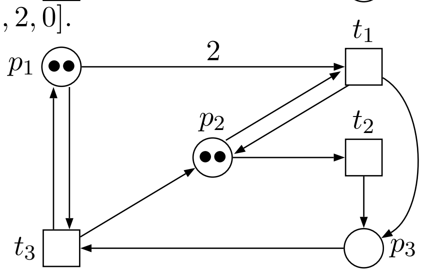
- $[2, 1, 1] \xrightarrow{t_1} [0, 1, 2]$.



- $[2, 1, 1] \xrightarrow{t_2} [2, 0, 2]$.



- $[2, 1, 1] \xrightarrow{t_3} [2, 2, 0]$.



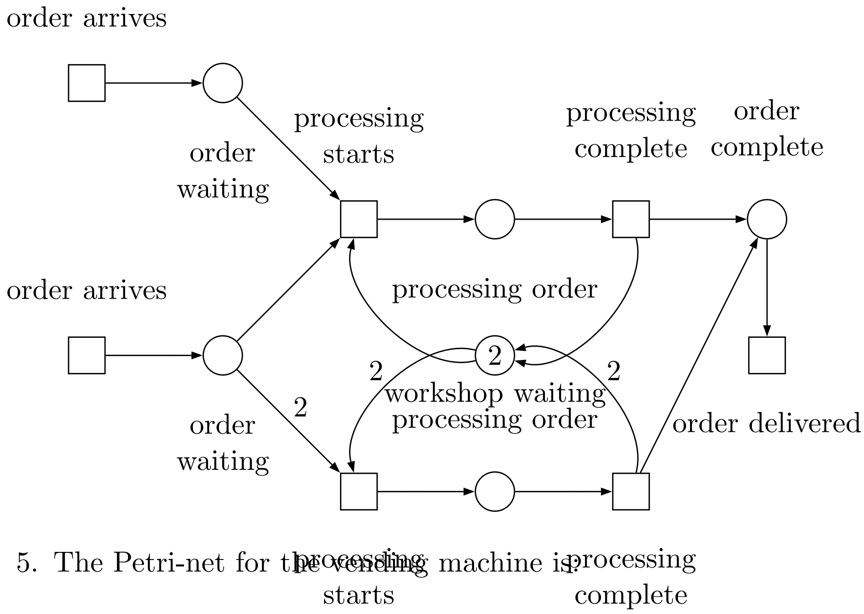
3. (a) We obtain the following sequence of markings:

$$[2, 0, 0] \xrightarrow{t_1} [1, 1, 0] \xrightarrow{t_2} [0, 1, 1] \xrightarrow{t_3} [0, 1, 1] \xrightarrow{t_4} [0, 0, 2] \xrightarrow{t_5} [0, 0, 1].$$

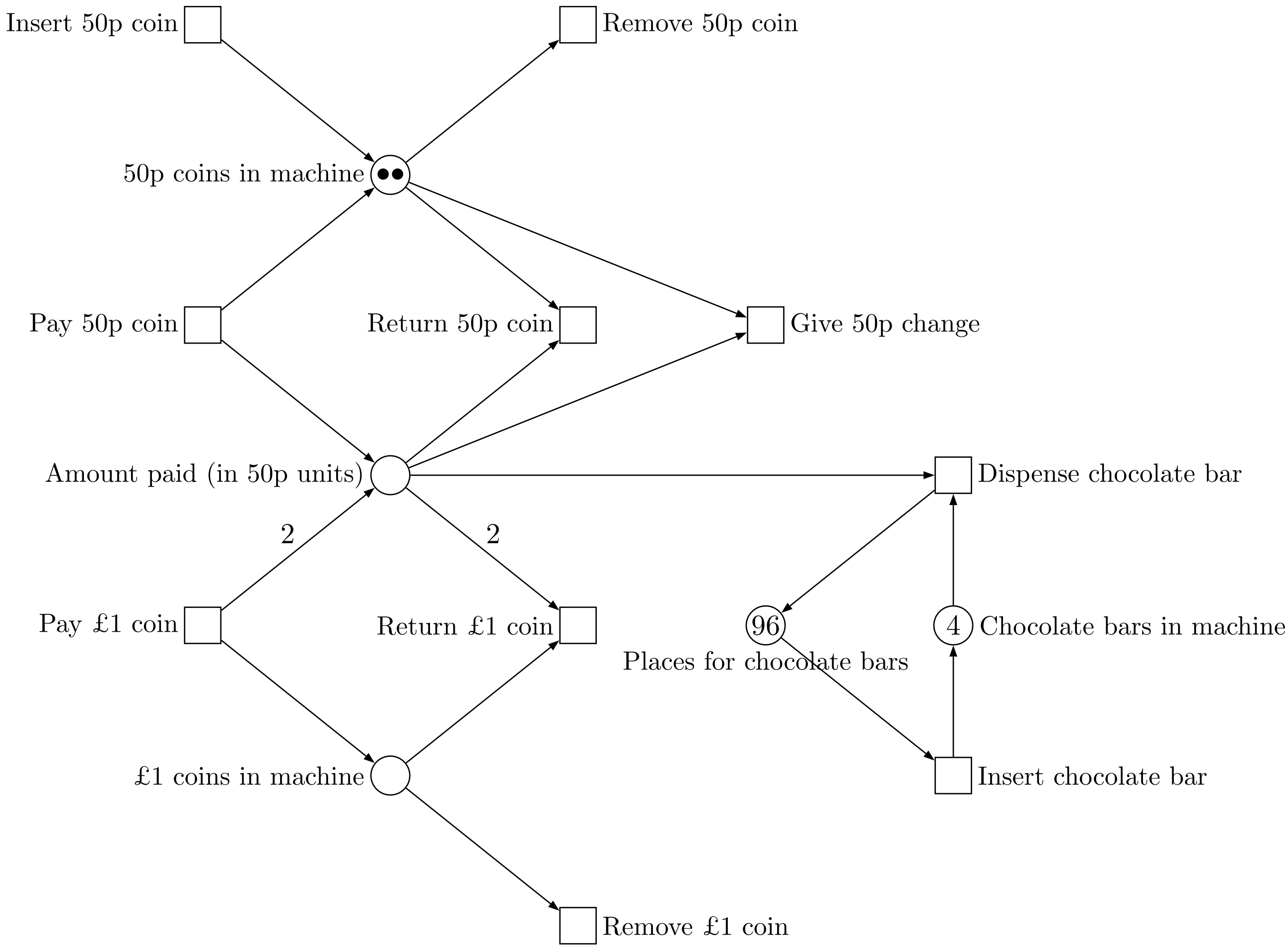
   (b) We could have the following sequence of transitions:

$$[1, 0, 0] \xrightarrow{t_2} [0, 0, 1] \xrightarrow{t_5} [0, 0, 0].$$

4. My modified workshop is:

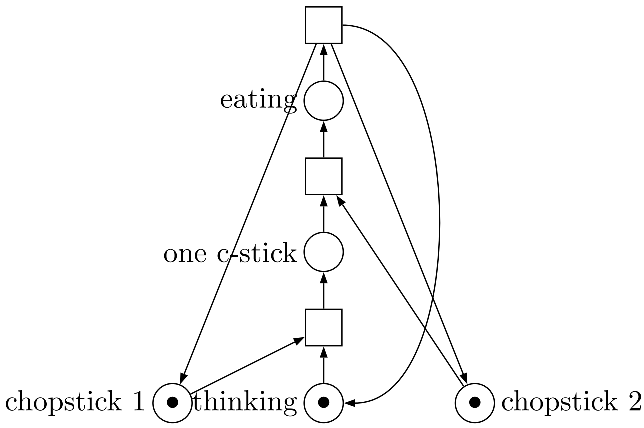


5. The Petri-net for the vending machine is:

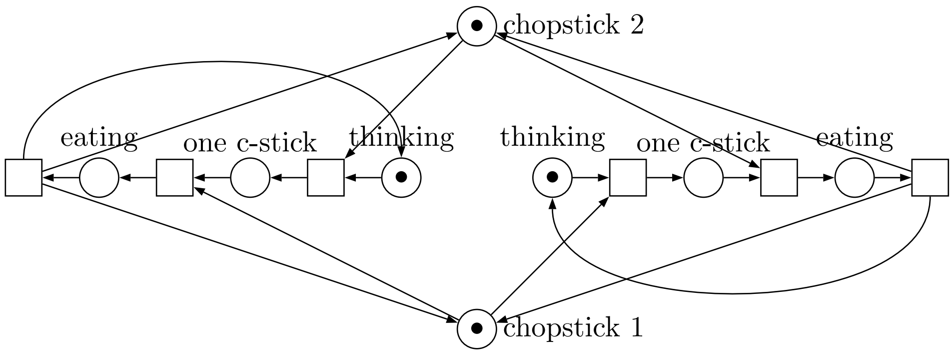


Note. The operations "Insert 50p coin", "Remove 50p coin", "Remove £1 coin" and "Insert chocolate bar" are performed by the serviceman; the operations "Pay 50p coin" and "Pay £1 coin" are performed by a customer; the operations "Give 50p change", "Return 50p coin" and "Return £1 coin" are prompted by requests from the customer (as, of course, is the operation "Dispense chocolate bar").

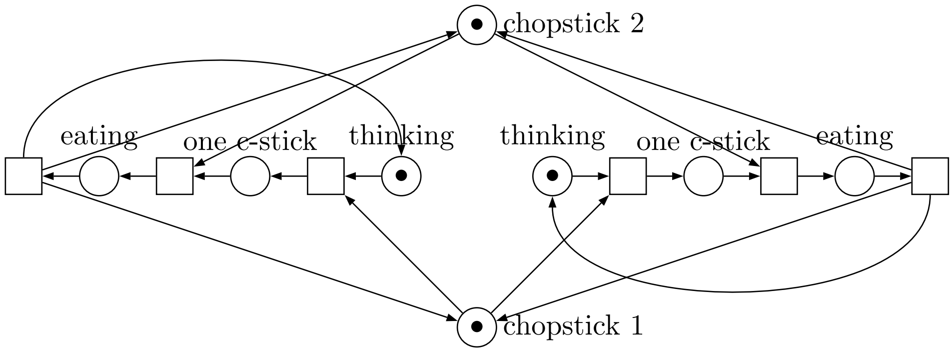
13. We add another intermediate place between thinking and eating. The first transition takes the first chopstick and the second transition takes the second.



Consider the following connection of the two philosophers. It is obtained by rotating the philosopher around the table making it treat the chopstick to its left and to its right just like the first philosopher.

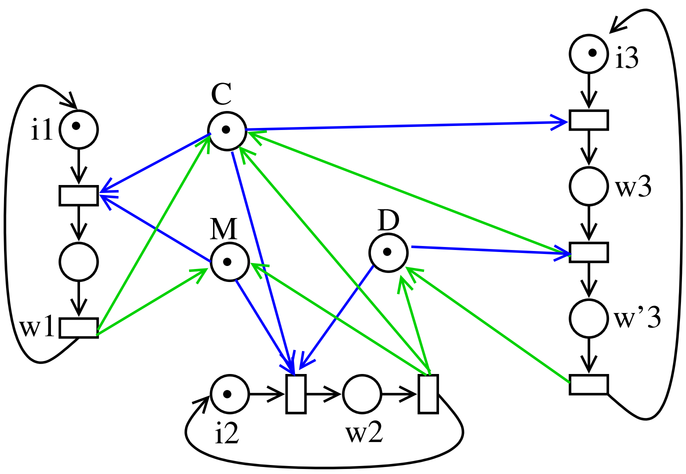


If both philosophers pick up chopstick then both are stuck. Indeed, both chopsticks are taken but neither philosopher can progress to the eating place. The solution is to turn around one philosopher:



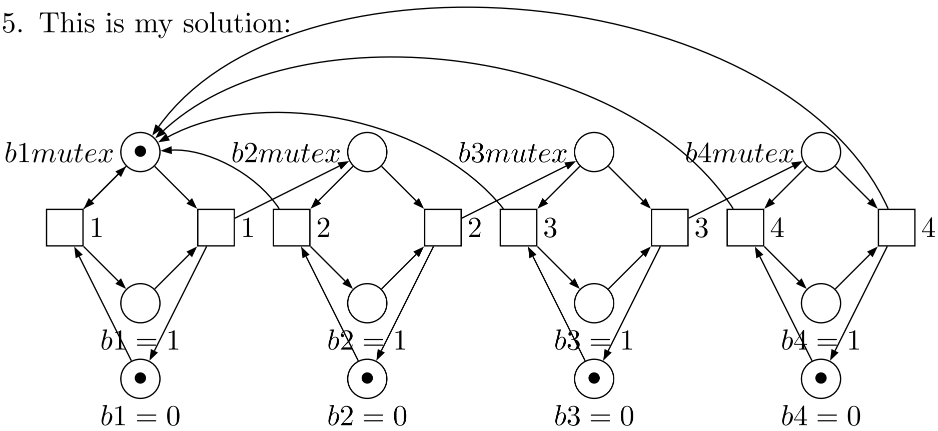
When one philosopher is inverted (with respect to left and right) then this protocol works for an arbitrary number of philosophers.

14. We get the following Petri net, where i_1, i_2, i_3 represent the idle states of the processes, w_1, w_2, w_3, w_3' are the working states and C, D, M stand for coprocessor, disk and memory.



As there is at most one token in any place, this Petri net is safe.

15. This is my solution:

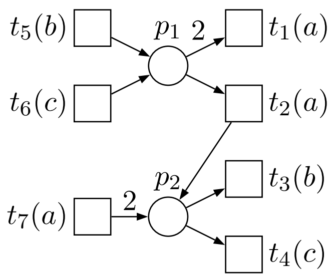


16. The language of the Petri net is:

$$L = \{w \in \{a, b, c\} \mid \sharp_a(w) = \frac{1}{3}|w|\}$$

In order to make sure that the number of as is third the number of letters we have to make sure that every a is matched by two other letters (b or c). For that we have two places, intuitively, counting whether we "owe" as or "owe" bs and cs. If we "owe" bs and cs and another a arrives then the "debt" increases by 2. If we "owe" bs and cs and an a or a b arrives the "debt" decreases by 1. If we "owe" as and an a arrives then either the "debt" decreases by 2 or the "debt" decreases by 1 and the bs and cs "debt" increases by 1. If we "owe" as and a b or a c arrives then the "debt" increases by 1.

This can be made more formal by stating the following invariant. First, give names to all the transitions.



If μ is a marking that results from reading the word w the following invariant holds:

$$I \ : \ (2 \cdot \sharp_a(w) - \sharp_b(w) - \sharp_c(w)) + (\mu(p_1) - \mu(p_2)) = 0$$

Clearly, for ϵ the invariant holds. Suppose that it holds for a word w and consider an extension $w \cdot \sigma$. If t_1 fires, then the number of as increases by 1 but the marking on p_1 decreases by 2. If t_2 fires, then the number of as increases by 1 but the marking on p_1 decreases by 1 and the marking on p_2 increases by 1. If t_3 fires, then the number of bs increases by 1 but the marking on p_2 decreases by 1. The rest of the transitions are similar.

So an accepted word w we know that $2\sharp_a(w) = \sharp_b(w) + \sharp_c(w)$. But $|w| = \sharp_a(w) + \sharp_b(w) + \sharp_c(w)$ leading to $3\sharp_a(w) = |w|$ as required.

This language is not regular as well.

17. Every time we fire t_1 (with label a) or t_2 (with label b) we put a token on p_1; every time we fire t_3 (with label c) we consume two tokens from p_1. In the initial marking μ_0 there are two tokens on p_1. So, for any prefix p of an accepted string, we must have:

$$2\sharp_c(p) \quad \leqslant \quad \sharp_a(p) + \sharp_b(p) + 2. \tag{2.1}$$

Every time we fire t_3 (with label c) we put a token on p_2; every time we fire t_4 (with label d) we consume a token from p_2. In the initial marking μ_0 there are no tokens on p_2. So, for any prefix p of an accepted string, we must have:

$$\sharp_d(p) \quad \leqslant \quad \sharp_c(p). \tag{2.2}$$

When we finish there must be no tokens present; so, for any accepted string s, we must have:

$$2\sharp_c(s) \quad = \quad \sharp_a(s) + \sharp + b(s) + 2; \tag{2.3}$$

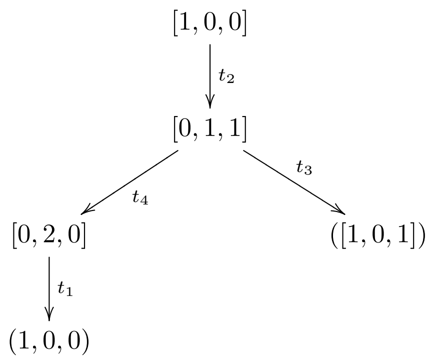
$$\sharp_d(s) \quad = \quad \sharp_c(s). \tag{2.4}$$

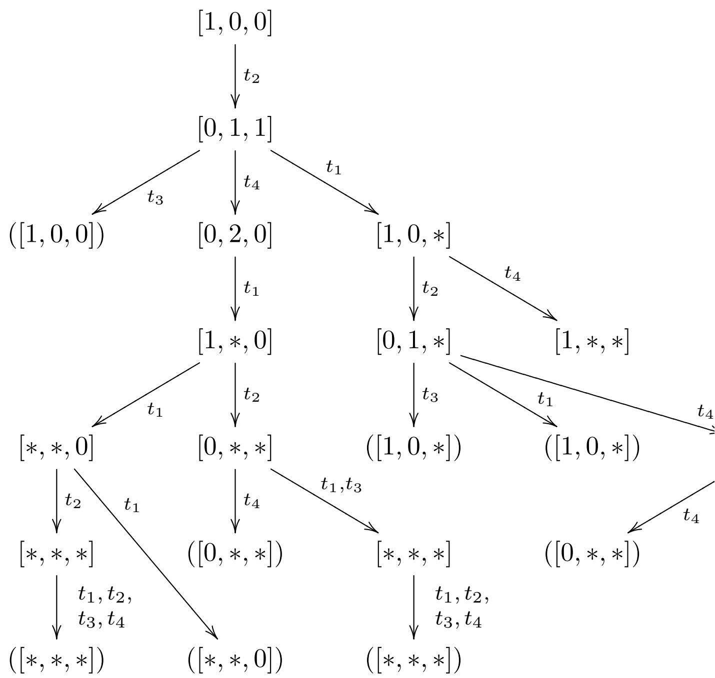
Putting conditions (2.1), (2.2), (2.3) and (2.4) together, we get that the language accepted by the labelled Petri net is:

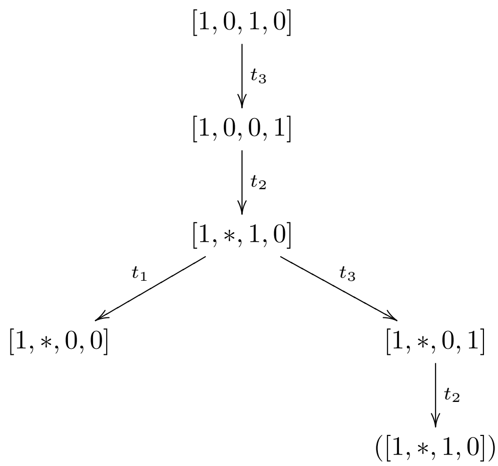
$$\begin{aligned}\{s \in \{a,b,c,d\}^* : \ & 2\sharp_c(s) = \sharp_a(s) + \sharp_b(s) + 2, \quad \sharp_d(s) = \sharp_c(s), \\ & 2\sharp_c(p) \leqslant \sharp_a(p) + \sharp_b(p) + 2 \text{ for any prefix } p \text{ of } s, \\ & \sharp_d(p) \leqslant \sharp_c(p) \text{ for any prefix } p \text{ of } s\}.\end{aligned}$$

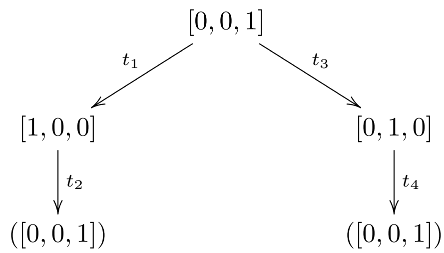
22. Reachability trees for the four Petri nets (where we have used extended

markings):









Inspecting the trees, we see that the left Petri net is unbounded and the right Petri net is safe (so that $k = 1$ is the least value of k for which it is k-safe).

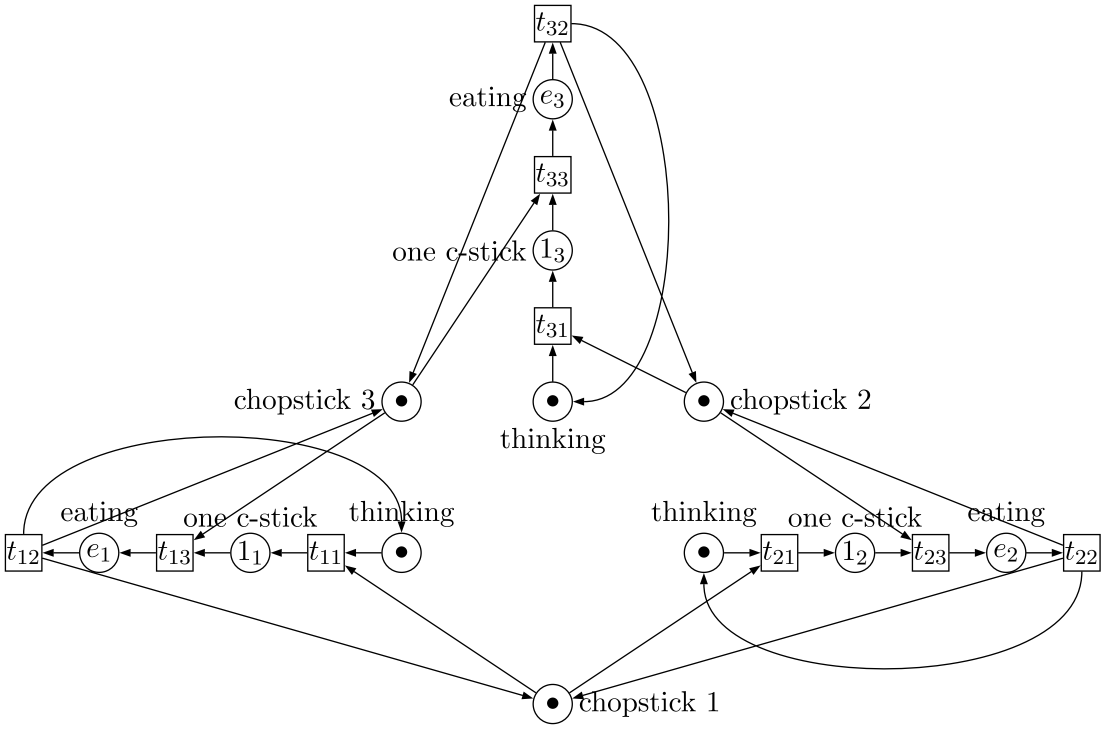
23. The first Petri net is 2-safe (and hence bounded) but not safe:

$$
\begin{tikzcd}[column sep=small]
& {[0,0,2,0]} \arrow[d, "t_3"'] & \\
& {[0,0,1,1]} \arrow[d, "t_3"'] & \\
& {[0,0,0,2]} \arrow[d, "t_4"'] & \\
& {[1,0,0,0]} \arrow[d, "t_1"'] & \\
& {[0,2,0,0]} \arrow[d, "t_2"'] & \\
& {[0,1,1,0]} \arrow[dl, "t_2"] \arrow[dr, "t_3"'] & \\
{([0,0,2,0])} & & {[0,1,0,1]} \arrow[d, "t_2"'] \\
& & {([0,0,1,1])}
\end{tikzcd}
$$

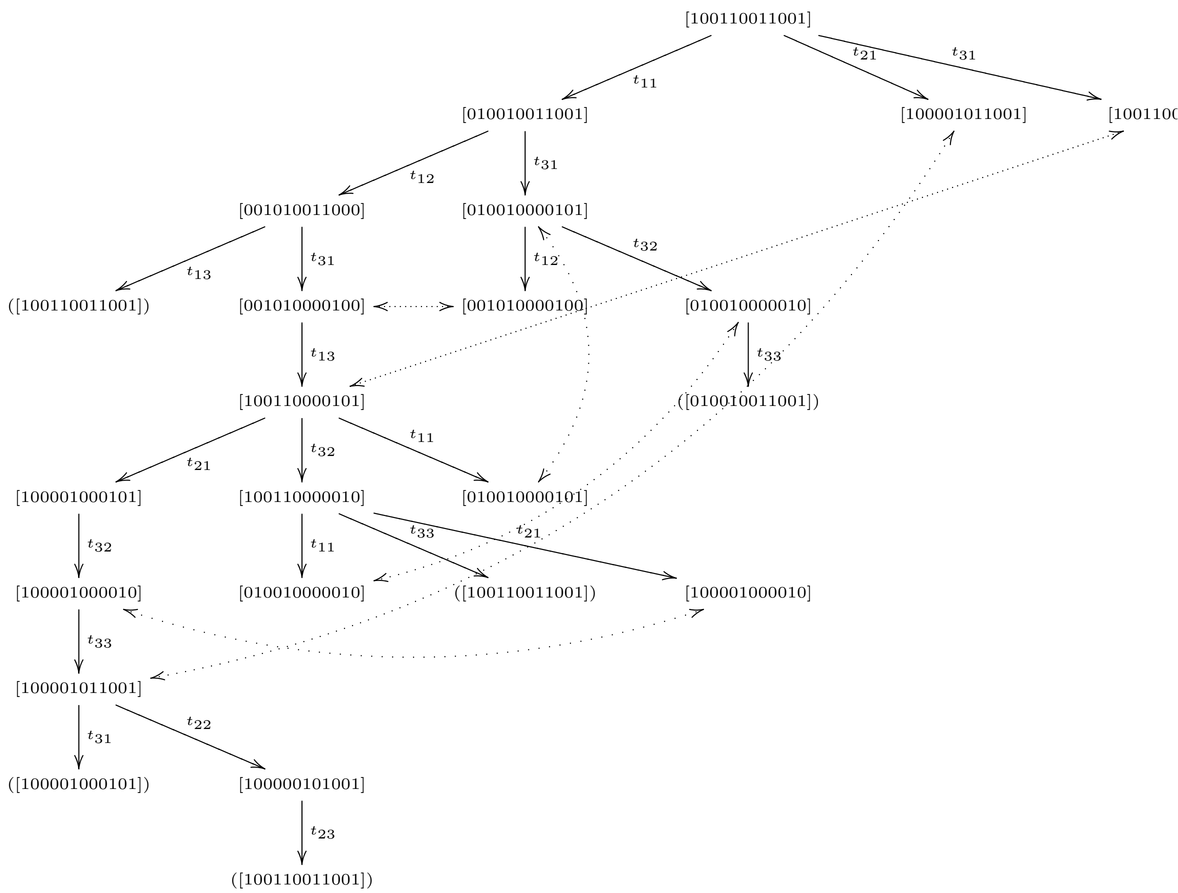
The second Petri net is unbounded (and hence not k-safe for any k):

$$
\begin{tikzcd}[column sep=tiny]
& & & {[0,1,0,0]} \arrow[d, "t_2"'] & & & & \\
& & & {[0,0,1,0]} \arrow[d, "t_3"'] & & & & \\
& & & {[0,0,0,2]} \arrow[d, "t_4"'] & & & & \\
& & & {[1,0,0,1]} \arrow[dl, "t_4"] \arrow[dr, "t_1"] & & & & \\
& & {[0,\ast,0,\ast]} \arrow[dl, "t_2"] \arrow[dr, "t_4"'] & & {[2,0,0,0]} \arrow[dr, "t_1"] & & & \\
& {[0,\ast,\ast,\ast]} \arrow[dl, "{t_2,t_3}"'] \arrow[d, "t_4"] & & {[\ast,\ast,0,\ast]} \arrow[dl, "{t_1,t_4}"'] \arrow[d, "t_2"] & & {[\ast,\ast,0,0]} \arrow[dl, "t_1"'] \arrow[dr, "t_2"] & & \\
{([0,\ast,\ast,\ast])} & {[\ast,\ast,\ast,\ast]} \arrow[d, "{t_1,t_2,t_3,t_4}"] & {([\ast,\ast,0,\ast])} & {([\ast,\ast,\ast,\ast])} & {([\ast,\ast,0,0])} & & {[\ast,\ast,\ast,0]} \arrow[dl, "{t_1,t_2}"] \arrow[dr, "t_3"] & \\
& {([\ast,\ast,\ast,\ast])} & & & & {([\ast,\ast,\ast,0])} & & {([\ast,\ast,\ast,\ast])}
\end{tikzcd}
$$

27. This is the Petri net for three dining philosophers. Philosopher 1 (on the left) takes his left chopstick first and then his right. Philosophers 2 and 3 take their right chopstick first and then their left.

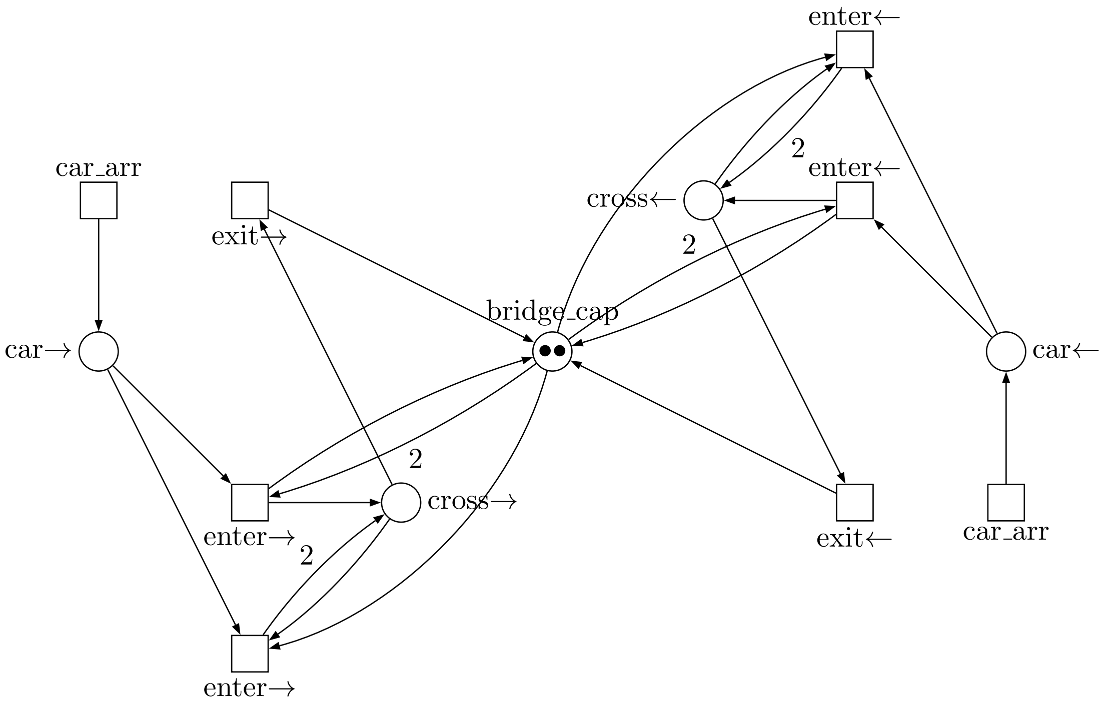


The vector of locations is $[t_1, 1_1, e_1, c_1, t_2, 1_2, e_2, c_2, t_3, 1_3, e_3, c_3]$. Then, the reachability tree is:



We notice that it is possible to get to the initial marking from every marking in the reachability tree. Then, as every transition appears at least once in the reachability tree liveness follows.

28. I give two possible solutions. The first, allocates different places to the cars moving in different directions.



The second solution includes places signalling the direction of movement on the bridge. Thicker edges have weight 2.