Chapter 4. Strong Bisimulation

Goals

- Introduce the notion of strong bisimulation.
- Techniques for establishing strong bisimulation.
- Properties of strong bisimulation.

Equivalence relations on agents

Two agents P and Q are equivalent if they cannot be distinguished by any agent interacting with each of them.

There are several such relations: (behavioural) equality '$=$', strong bisimilarity '$\sim$', weak bisimilarity $\approx$, and observation congruence $\approx_o$:

- $\sim$ and $\approx$ are defined in terms of patterns of actions agents can perform.
- $\sim$ and $\approx$ are distinguished by their treatment of τ, we have $\sim \subset \approx$.
- Observation congruence is a proper subset of weak bisimulation, we have $\sim \subset \approx_o \subset \approx$.
- $=$ is sitting between $\sim$ and $\approx$; in fact $=$ is the same as $\approx_o$.
- We first study $\sim$, which treats τ exactly like any other actions.

Reasons for studying $\sim$ first

- It is relatively simpler.
- It shares many common properties with the other two equivalences.
- Many equational laws used in practice hold for strong bisimulation.
- The technique for establishing this equivalence is more or less the same as the techniques for the other two equivalences.

0. Basic Notions and Definitions

- **Sets:** A set S is a collection of objects (elements). A set S is often denoted as:
  - for a finite set: $S = \{e_1, e_2, \ldots, e_n\}$
  - for an infinite enumerable set: $S = \{e_1, e_2, \ldots, \}$
  - in general $S = \{e \ : \ P(e)\}$, where P is a predicate
  - empty set: $\emptyset = \{e : P(e) \wedge \neg P(e)\}$

$$\{1, 2, 3\} = \{2, 1, 3\}$$

- Repetition of elements in a set is insignificant:

$$\{2, 2, 3\} = \{2, 3\}$$

Subsets:

A set S_1 is a subset of S_2, $S_1 \subseteq S_2$, if each element of S_1 is an element of S_2.

Set operations: Given sets S_1 and S_2

- intersection: $S_1 \cap S_2$, $\bigcap\{S_i : i \in I\}$
- union: $S_1 \cup S_2$, $\bigcup\{S_i : i \in I\}$
- complement: $S_1 - S_2$ is the complement of S_2 *w.r.t.* S_1
- product: $S_1 \times S_2 = \{(e_1, e_2) : e_1 \in S_1, e_2 \in S_2\}$, $\Pi\{S_i : i \in I\}$
- power set $\mathcal{P}(S_1) = \{S \ : \ S \subseteq S_1\}$

What is $\mathcal{P}(\{1,2,3\})$?

$\mathcal{P}(\{1,2,3\}) \ = \ \{$

Relations

A binary relation $\mathcal{R}$ on a set S is a subset of $S \times S$, i.e. the elements of $\mathcal{R}$ are the pairs (e_1, e_2), where $e_1, e_2 \in S$.

If elements e_1 and e_2 are related by $\mathcal{R}$, we often write this as $e_1 \mathcal{R} e_2$.

Relation operations: Given S and relations $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_i$ for $i \in I$, the following are also relations on S:

- the identity: $Id_S = \{(e, e) : e \in S\}$
- composition $\mathcal{R}_1\mathcal{R}_2 = \{(e_1, e_2) : \exists m \in S.(e_1, m) \in \mathcal{R}_1, (m, e_2) \in \mathcal{R}_2\}$
- converse: $\mathcal{R}^{-1} = \{(y, x) : (x, y) \in \mathcal{R}\}$
- union: $R_1 \cup R_2$, $\bigcup_{i \in I} \mathcal{R}_i$
- intersection: $R_1 \cap R_2$, $\bigcap_{i \in I} \mathcal{R}_i$

Important properties of relations

Let $R \subseteq S \times S$.

- $\mathcal{R}$ is reflexive if $e\mathcal{R}e$ for all $e \in S$, i.e. $Id_s \subseteq R$
- $\mathcal{R}$ is symmetric if whenever $e_1\mathcal{R}e_2$ then $e_2\mathcal{R}e_1$: i.e. $R^{-1} = R$
- $\mathcal{R}$ is transitive if whenever $e_1\mathcal{R}e_2$ and $e_2\mathcal{R}e_3$ then $e_1\mathcal{R}e_3$, i.e. $RR \subseteq R$
- $\mathcal{R}$ is an equivalence relation if it is reflexive, symmetric and transitive.

Examples of equivalence relations:

- Equality of numbers
- CCS's $=$

Algebra

An algebra $\mathcal{G}$ is a structure which consists of a set S, called the carrier set, and a set of operations on the elements of S:

$$\mathcal{G} =< S, op_1, \ldots, op_n > \qquad op_i(x_{i1}, \ldots, x_{ik_i}) \in S$$

Congruence

A relation $\mathcal{R}$ on S of $\mathcal{G}$ is a congruence w.r.t $\mathcal{G}$ if

(a) $\mathcal{R}$ is an equivalence relation, and

(b) for each op_i and any closed terms over S $e_{i1}, \ldots, e_{ik_i}$ and $e'_{i1}, \ldots, e'_{ik_i}$,

$$\begin{array}{lll} \text{if} & e_{ij} \; \mathcal{R} \; e'_{ij} & \text{for } j = 1, \ldots, k_i \\ \text{then} & op_i(e_{i1}, \ldots, e_{ik_i}) \; \mathcal{R} \; op_i(e'_{i1}, \ldots, e'_{ik_i}). & \end{array}$$

1. Bisimulation Games

A game for a pair of agents (P_0, Q_0) is a finite or infinite sequence of the form

$$(P_0, Q_0), \ldots, (P_i, Q_i), \ldots$$

- played by two participants or observers, player I and player II.
- Player I attempts to show that P_i and Q_i have a different behaviour, and player II tries to prevent this.
- For each j the pair (P_{j+1}, Q_{j+1}) is determined as the result of a next step from the previous pair (P_j, Q_j) by the following rules.

Rules of the game

- First player I chooses P_j (or Q_j) and a valid transition

$$P_j \xrightarrow{\alpha} P_{j+1} \quad or \quad Q_j \xrightarrow{\alpha} Q_{j+1}$$

- Then player II chooses a corresponding valid transition from the other agent

$$Q_j \xrightarrow{\alpha} Q_{j+1} \qquad or \qquad P_j \xrightarrow{\alpha} P_{j+1}$$

Winner of the game

If at any point a player is unable to make a move then the other player wins the game:

- Player I is stuck, and she loses, if both agents are deadlocked.
- Player II loses if she can find no matching transition.
- If the game continues forever (is infinite) or if there is a repeated configuration, i.e. there is a pair (P_{j+1}, Q_{j+1}) which appears twice in the game sequence, then Player II wins.

Winning strategy

A player has a wining strategy from (P_0, Q_0) if she is able to win any game from this pair.

Game equivalence

Two agents P_0 and Q_0 are game equivalent if Player II has a winning strategy from (P_0, Q_0). In other words, P_0 and Q_0 are game equivalent if whatever moves Player I makes, they can be matched by Player II.

Example 1:

Consider the following agents.

$$Cl_1 \stackrel{def}{=} tick.Cl_1 \qquad\qquad Cl_2 \stackrel{def}{=} tick.tick.Cl_2$$

Any game from (Cl_1, Cl_2) must progress through the steps

$$(Cl_1, CL_2), (Cl_1, tick.Cl_2), (Cl_1, Cl_2), \ldots$$

Thus, player II always wins.

Example 2:

Consider the following agents

$$V \stackrel{def}{=} 1p.(little.collect.V + 1p.big.collect.V)$$
$$V' \stackrel{def}{=} 1p.little.collect.V' + 1p.1p.big.collect.V'$$

Player I has a winning strategy from (V, V'):

1. Player I chooses agent V' and the transition
$$V' \stackrel{1p}{\rightarrow} 1p.big.collect.V'$$
2. Play II has to take V and makes
$$V \stackrel{1p}{\rightarrow} little.collect.V + 1p.big.collect.V$$
3. Player I opts for $little.collect.V + 1p.big.collect.V$ and chooses
$$little.collect.V + 1p.big.collect.V \stackrel{little}{\rightarrow} collect.V$$
4. Player II cannot make the corresponding transition with the agent $1p.big.collect.V'$.
5. Thus, Player I wins, and V and V' are not equivalent.

2. Strong Bisimulation

Definition 1 A relation $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ is a strong bisimulation (SB) if, whenever $P\mathcal{S}Q$ and $\alpha \in Act$, then

1. if $P \stackrel{\alpha}{\rightarrow} P'$, then, for some Q', $Q \stackrel{\alpha}{\rightarrow} Q'$ and $P'\mathcal{S}Q'$, and
2. if $Q \stackrel{\alpha}{\rightarrow} Q'$, then, for some P', $P \stackrel{\alpha}{\rightarrow} P'$ and $P'\mathcal{S}Q'$.

Agents P and Q are strongly bisimilar, or simply bisimilar, written $P \sim Q$, if there is a SB $\mathcal{S}$ such that $P\mathcal{S}Q$.

Theorem Agents P and Q are strongly bisimilar if and only if they are game equivalent.

In other words

Agents P and Q are strongly bisimilar if and only if Player II has a winning strategy from (P, Q).

Example 3

For Cl_1 and Cl_2 in Example 1, $Cl_1 \sim Cl_2$ since the following is a bisimulation

$$\{(Cl_1, Cl_2), (Cl_1, tick.Cl_2)\}$$

Example 4 Consider the agents

$$\begin{aligned} A_0 &\stackrel{def}{=} a.A_1 \\ A_1 &\stackrel{def}{=} a.A_2 + b.A_0 \\ A_2 &\stackrel{def}{=} b.A_1 \end{aligned}$$

and

$$\begin{aligned} B &\stackrel{def}{=} a.b.B \\ C &\stackrel{def}{=} B|B \end{aligned}$$

Then $A_0 \sim C$ since the following is a bisimulation

$$\{(A_0, C), (A_1, b.B|B), (A_1, B|b.B), (A_2, b.B|b.B)\}.$$

Proposition 1

Assume that each $\mathcal{S}_i$ $(i \in I = \{1, 2, \ldots\})$ is a SB. Then the following relations are SBs

$$\begin{array}{llll} (1) & Id_{\mathcal{P}} & (3) & \mathcal{S}_1\mathcal{S}_2 \\ (2) & \mathcal{S}_i^{-1} & (4) & \bigcup_{i \in I} \mathcal{S}_i \end{array}$$

Proof of (3):

Suppose that

$$P\mathcal{S}_1\mathcal{S}_2R$$

Then for some Q

$$P\mathcal{S}_1Q \text{ and } Q\mathcal{S}_2R$$

Now let $P \xrightarrow{\alpha} P'$. Then for some Q' we have since $P\mathcal{S}_1Q$

$$Q \xrightarrow{\alpha} Q' \text{ and } P'\mathcal{S}_1Q'$$

Also since $Q\mathcal{S}_2R$ we have, for some R',

$$R \xrightarrow{\alpha} R' \text{ and } Q'\mathcal{S}_2R'$$

Hence $P'\mathcal{S}_1\mathcal{S}_2R'$. Similarly if $R \xrightarrow{\alpha} R'$ then we can find P' such that

$$P \xrightarrow{\alpha} P' \text{ and } P'\mathcal{S}_1\mathcal{S}_2R'$$

3. Properties of $\sim$

Proposition 2

1. $\sim$ is the largest SB.
2. $\sim$ is an equivalence relation.
3. $P \sim Q$ iff, for all $\alpha \in Act$
   (a) Whenever $P \xrightarrow{\alpha} P'$ then, for some Q',
   $$Q \xrightarrow{\alpha} Q' \text{ and } P' \sim Q'$$
   (b) Whenever $Q \xrightarrow{\alpha} Q'$ then, for some P'
   $$P \xrightarrow{\alpha} P' \text{ and } P' \sim Q'$$

Proof:

(1). It is easy to prove that

$$\sim = \bigcup \{\mathcal{S} : \mathcal{S} \text{ is a SB}\}$$

(2). Reflexivity: For any P, $P \sim P$ by **P**.1(1).

Symmetry: If $P \sim Q$, then $P\mathcal{S}Q$ for some SB $\mathcal{S}$. Hence $Q\mathcal{S}^{-1}P$, and so $Q \sim P$ by **P**.1(2).

Transitivity: If $P \sim Q$ and $Q \sim R$, then for some SB $\mathcal{S}_1$ and $\mathcal{S}_2$

$$P\mathcal{S}_1Q \text{ and } Q\mathcal{S}_2R$$

Hence $P\mathcal{S}_1\mathcal{S}_2R$, so $P \sim R$ by **P**.1(3).

(3) Fairly direct from the definition.

Proposition 3

$\sim$ is a congruence relation (called strong congruence): if $P_1 \sim P_2$ then

1. $\alpha.P_1 \sim \alpha.P_2$
2. $P_1 + Q \sim P_2 + Q$
3. $P_1|Q \sim P_2|Q$
4. $P_1 \backslash L \sim P_2 \backslash L$
5. $P_1[f] \sim P_2[f]$
6. If $\tilde{A} \stackrel{def}{=} \tilde{P}$, then $\tilde{A} \sim \tilde{P}$

Proof:

For (3), we show $\mathcal{S}$ is a SB, where

$$\mathcal{S} = \{(P_1|Q, P_2|Q) : P_1 \sim P_2\}$$

For any $(P_1|Q, P_2|Q) \in \mathcal{S}$. Let

$$P_1|Q \stackrel{\alpha}{\to} R$$

There are three cases to consider

1. $P_1 \stackrel{\alpha}{\to} P_1'$, and $R \equiv P_1'|Q$. Then since $P_1 \sim P_2$, we have

$$P_2 \stackrel{\alpha}{\to} P_2' \text{ and } P_1' \sim P_2'$$

Hence

$$P_2|Q \stackrel{\alpha}{\to} P_2'|Q \text{ and } (P_1'|Q, P_2'|Q) \in \mathcal{S}$$

2. $Q \stackrel{\alpha}{\to} Q'$ and $R \equiv P_1|Q'$. Then also

$$P_2|Q \stackrel{\alpha}{\to} P_2|Q' \text{ and } (P_1|Q', P_2|Q') \in \mathcal{S}$$

3. $\alpha = \tau$, $P_1 \stackrel{l}{\to} P_1'$ and $Q \stackrel{\bar{l}}{\to} Q'$, and $R \equiv P_1'|Q'$. Then since $P_1 \sim P_2$, we have

$$P_2 \stackrel{l}{\to} P_2' \text{ and } P_2|Q \stackrel{\tau}{\to} P_2'|Q'$$

with $P_1' \sim P_2'$. Thus

$$(P_1'|Q', P_2'|Q') \in \mathcal{S}$$

Proof of (6). We consider the single defining equation

$$A \stackrel{def}{=} P$$

By rule **Con**, $A \stackrel{\alpha}{\to} A'$ iff $P \stackrel{\alpha}{\to} A'$. So by **P.**2(3), $A \sim P$.

Proposition 4 $\sim$ is preserved by recursion

Let $\tilde{E}$ and $\tilde{F}$ be agent expressions and contain variables $\tilde{X}$ at most. If for any agents $\tilde{R}$,

$$\tilde{E}\{\tilde{R}/\tilde{X}\} \sim \tilde{F}\{\tilde{R}/\tilde{X}\}$$

and if

$$\tilde{A} \stackrel{def}{=} \tilde{E}\{\tilde{A}/\tilde{X}\} \text{ and } \tilde{B} \stackrel{def}{=} \tilde{F}\{\tilde{B}/\tilde{X}\}$$

then

$$\tilde{A} \sim \tilde{B}$$

Proposition 5 (implies **P**.3.1: monoid laws)

1. $P + Q \sim Q + P$ — commutativity
2. $P + (Q + R) \sim (P + Q) + R$ — associativity
3. $P + P \sim P$ — Idempotence
4. $P + \mathbf{0} \sim P$ — $\mathbf{0}$ is the zero element of $+$

Proof of (2):

Suppose that

$$P + (Q + R) \xrightarrow{\alpha} P'$$

Then by the semantic rules $\mathbf{Sum}_j$,

- either $P \xrightarrow{\alpha} P'$,
- or $Q \xrightarrow{\alpha} P'$
- or $R \xrightarrow{\alpha} P'$

In each case we can easily infer by $\mathbf{Sum}_j$ that

$$(P + Q) + R \xrightarrow{\alpha} P'$$

and we know $P' \sim P'$ This establish (a) of **P.**2(3), and (b) is similar.

Proposition 6 (implies **P.**3.8-3.10)

1. $P|Q \sim Q|P$ – commutativity
2. $P|(Q|R) \sim (P|Q)|R$ – associativity
3. $P|\mathbf{0} \sim P$ – $\mathbf{0}$ is an unit
4. $P\backslash L \sim P$ if $\mathcal{L}(P) \cap (L \cup \overline{L}) = \emptyset$
5. $P\backslash K\backslash L \sim P\backslash(K \cup L)$
6. $P[f]\backslash L \sim (P\backslash f^{-1}(L))[f]$
7. $(P|Q)\backslash L \sim P\backslash L|Q\backslash L$ if
$$\mathcal{L}(P) \cap \overline{\mathcal{L}(Q)} \cap (L \cup \overline{L}) = \emptyset$$
8. $P[Id] \sim P$
9. $P[f] \sim P[f']$ if $f \restriction \mathcal{L}(P) = f' \restriction \mathcal{L}(P)$
10. $P[f][f'] \sim P[f' \circ f]$
11. $(P|Q)[f] \sim P[f]|Q[f]$ if $f \restriction (L \cup \overline{L})$ is one to one, where $L = \mathcal{L}(P|Q)$

Proof of (1):

We show that $\mathcal{S}$ is a SB, where

$$\mathcal{S} = \{(P|Q, Q|P) : P, Q \in \mathcal{P}\}$$

Suppose that $(P|Q) \xrightarrow{\alpha} P'$. There are three main cases, with sub-cases:

1. $P \xrightarrow{\alpha} P_1$, and $P' \equiv P_1|Q$ Then we also have
$Q|P \xrightarrow{\alpha} Q|P_1$ and $(P_1|Q, Q|P_1) \in \mathcal{S}$

2. $Q \xrightarrow{\alpha} Q_1$ and $P' \equiv P|Q_1$. Then have: $Q|P \xrightarrow{\alpha} Q_1|P$ and $(P|Q_1, Q_1|P) \in \mathcal{S}$

3. $\alpha = \tau$, $P \xrightarrow{l} P_1$, $Q \xrightarrow{\bar{l}} Q_1$ and $P' \equiv P_1|Q_1$. Then
$Q|P \xrightarrow{\tau} Q_1|P_1$ and $(P_1|Q_1, Q_1|P_1) \in \mathcal{S}$.

This proves condition (1) of **Def**.1. Condition (2) follows by a symmetric argument.

Proposition 7 (implies the expansion law)

Let $P \equiv (P_1[f_1] | \ldots | P_n[f_n]) \backslash L$. Then

$$
\begin{array}{rcl}
P & \sim & \sum\{ f_i(\alpha).(P_1[f_1] | \ldots | P_i'[f_i] | \ldots | P_n[f_n]) \backslash L : \\
 & & P_i \stackrel{\alpha}{\to} P_i', f_i(\alpha) \notin L \cup \overline{L} \} \\
 & + & \sum\{ \tau.(P_1[f_1] | \ldots | P_i'[f_i] | \ldots | P_j'[f_j] | \ldots | P_n[f_n]) \backslash L : \\
 & & P_i \stackrel{l_1}{\to} P_i', P_j \stackrel{l_2}{\to} P_j', f_i(l_1) = \overline{f_j(l_2)}, i < j \}
\end{array}
$$

Proposition 8 (implies **P**.3.4(2))

Let E_i $(i \in I)$ contain at most the variables X_j $(j \in I)$, and let these variables are weakly guarded in each E_i, i.e. every occurrence of X_i in E_j is within some subexpression $\alpha.F$ of E_j with $\alpha \in Act$. Then,

If $\tilde{P} \sim \tilde{E}\{\tilde{P}/\tilde{X}\}$ and $\tilde{Q} \sim \tilde{E}\{\tilde{Q}/\tilde{X}\}$ then $\tilde{P} \sim \tilde{Q}$

4. Summary

- We introduced the notion of strong bisimulation. Strong bisimulation requires that, if $P \sim Q$, then every α action, including the silent action τ, of P or Q must be matched by an α action of the other.
- To prove that two agents are strongly equivalent, we either
  - establish (define) a strong bisimulation which contains the pair of the two agents, or
  - define a strong bisimulation game starting from the pair of the two agents such that Player II has a winning strategy.
- We can also prove that two agents are strongly equivalent by using the properties of the strong bisimulation relation $\sim$.