Chapter 3

Mixed strategy equilibria

This chapter studies mixed strategy equilibria, extending the considerations of chapter 2.

3.1 Learning objectives

After studying this chapter, you should be able to:

- explain what mixed strategies and mixed equilibria are;
- state the main theorems about mixed strategies: the best response condition, Nash's theorem on the existence of mixed equilibria, and the minimax theorem;
- find mixed equilibria of small bimatrix games, using geometric tools like the upper envelope method where necessary;
- give the definition of a degenerate game, and find all equilibria of small degenerate games;
- explain the concept of max-min strategies and their importance for zero-sum games, and apply this to given games.

3.2 Further reading

The strong emphasis on geometric concepts for understanding mixed strategies is not found to this extent in other books. For zero-sum games, which are the subject of section 3.14, a detailed treatment is given in the following book:

- Mendelson, Elliot *Introducing Game Theory and Its Applications.* (Chapman & Hall / CRC, 2004) [ISBN 1584883006].

Later sections and appendices in that book treat, more briefly, also the non-zero-sum games studied here.

In section 3.5, an "inspection game" is used to motivate mixed strategies. A book on inspection games is

- Avenhaus, Rudolf, and Morton J. Canty *Compliance Quantified.* (Cambridge University Press, 1996) [ISBN 0521019192].

Nash's central paper is

- Nash, John F. "Non-cooperative games." *Annals of Mathematics*, Vol. 54 (1951), pp. 286–295.

This paper is easy to obtain. It should be relatively accessible after a study of this chapter.

3.3 Introduction

A game in strategic form does not always have a Nash equilibrium in which each player deterministically chooses one of his strategies. However, players may instead randomly select from among these *pure* strategies with certain probabilities. Randomising one's own choice in this way is called a *mixed* strategy. A profile of mixed strategies is called a *mixed equilibrium* if no player can gain on average by unilateral deviation. Nash showed in 1951 that any finite strategic-form game has a mixed equilibrium.

In this chapter, we first discuss how a player's payoffs represent his preference for random outcomes via an *expected-utility* function. This is illustrated first with a single-player decision problem and then with a game whether to comply or not with a legal requirement. The game is known as an *inspection game* between a player II called inspectee and an inspector as player I. This game has no equilibrium using pure (that is, deterministic) strategies. However, active randomisation comes up naturally in this game, because the inspector can and will choose to inspect only some of the time. The mixed equilibrium in this game demonstrates many aspects of general mixed equilibria, in particular that a player's randomisation probabilities depend on the payoffs of the *other* player.

We then turn to general two-player games in strategic form, called bimatrix games. The payoff matrices for the two players are convenient to represent expected payoffs via multiplication with vectors of probabilities which are the players' mixed strategies. These mixed strategies, as vectors of probabilities, have a geometric interpretation, which we study in detail. Mixing with probabilities is geometrically equivalent to taking *convex combinations*.

The *best response condition* gives a convenient, finite condition when a mixed strategy is a best response to the mixed strategy of the other player (or against the mixed strategies of all other players in a game with more than two players, which we do not consider). Namely, only the *pure* strategies that are best responses are allowed to have positive probability. These pure best responses must have maximal, and hence equal, expected payoff. This is stated as an "obvious fact" in Nash's paper, and it is not hard to prove. However, the best response condition is central for understanding and computing mixed-strategy equilibria.

In section 3.10, we state and prove Nash's theorem that every game has an equilibrium. We follow Nash's original proof, and state and briefly motivate *Brouwer's fixed point theorem* used in that proof, but do not prove the fixed point theorem itself.

Sections 3.11–3.13 show how to find equilibria in bimatrix games, in particular when one player has only two strategies.

The special and interesting case of zero-sum games is treated in section 3.14.

3.4 Expected-utility payoffs

The payoffs in a game represent a player's preferences, in that higher payoffs correspond to more desirable outcomes for the player. In this sense, the payoffs have the role of an "ordinal" utility function, meaning that only the order of preference is important. For example, the order among payoffs 0, 8, and 10 can equally be represented by the numbers 0, 2, and 10.

As soon as the outcome is random, the payoffs also represent a "cardinal" utility, which means that the relative size of the numbers also matters. The reason is that average (that is, *expected*) payoffs are considered to represent the preference for the random outcomes. For example, the payoffs 0, 8, and 10 do not represent the same preferences as 0, 2 and 10 when the player has to decide between a coin-toss that gives 0 or 10 with probability $1/2$ each, which is a risky choice, or taking the "middle" payoff. The expected payoff of the coin-toss is 5, which is larger than 2 but smaller than 8, so it matters whether the middle payoff is 2 or 8. In a game, a player's payoffs always represent his "expected utility" function in the sense that the payoffs can be weighted with probabilities in order to represent the player's preference for a random outcome.

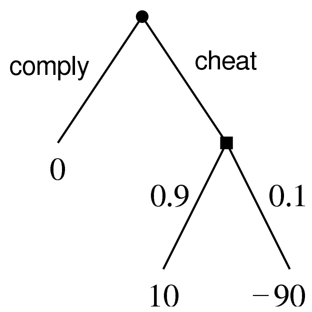

Figure 3.1 One-player decision problem to decide between "comply" and "cheat". This problem demonstrates expected-utility payoffs. With these payoffs, the player is indifferent.

As a concrete example, figure 3.1 shows a game with a single player who can decide to comply with a regulation, like to buy a parking permit, or to cheat otherwise. The payoff when she chooses to comply is 0. Cheating involves a 10 percent chance of getting caught and having to pay a penalty, stated as the negative payoff -90, and otherwise a 90 percent chance of gaining a payoff of 10. With these numbers, cheating leads to a random outcome with an expected payoff of $0.9 \times 10 + 0.1 \times (-90)$, which is zero, so that the player is exactly *indifferent* (prefers neither one nor the other) between her two available moves.

If the payoffs are monetary amounts, each payoff unit standing for a dollar, say, one would not necessarily assume such a *risk neutrality* on the part of the player. In practice, decision-makers are typically *risk averse*, meaning they prefer the safe payoff of 0 to the gamble with an expectation of 0.

In a game-theoretic model with random outcomes, as in the game above, the payoff is not necessarily to be interpreted as money. Rather, the player's attitude towards risk is incorporated into the payoff figure as well. To take our example, the player faces a punishment or reward when cheating, depending on whether she is caught or not. Suppose that the player's decision only depends on the probability of being caught, which is 0.1 in figure 3.1, so that she would cheat if that probability was zero. Moreover, set the reward for cheating arbitrarily to 10 units, as in the figure above, and suppose that being caught has clearly defined consequences for the player, like regret and losing money and time. Then there must be a certain probability of getting caught where the player in the above game is indifferent, say 4 percent. This determines the utility $-u$, say, for "getting caught" by the equation

$$0 = 0.96 \times 10 + 0.04 \times (-u)$$

which states equal expected utility for the choices "comply" and "cheat". This equation is equivalent to $u = 9.6/0.04 = 240$. That is, in the above game, the negative utility -90 would have to be replaced by -240 to reflect the player's attitude towards the risk of getting caught. With that payoff, she will now *prefer* to comply if the probability of getting caught stays at 0.1.

The point of this consideration is to show that payoffs exist, and can be constructed, that represent a player's preference for a risky outcome, as measured by the resulting *expected payoff*. These payoffs do not have to represent money. The existence of such expected-utility payoffs depends on a certain consistency of the player when facing choices with random outcomes. This can be formalised, but the respective theory, known as the *von Neumann–Morgenstern* axioms for expected utility, is omitted here for brevity.

In practice, the risk attitude of a player may not be known. A game-theoretic analysis should be carried out for different choices of the payoff parameters in order to test how much they influence the results. Often, these parameters represent the "political" features of a game-theoretic model, those most sensitive to subjective judgement, compared to the more "technical" part of a solution. In particular, there are more involved variants of the inspection game discussed in the next section. In those more complicated models, the technical part often concerns the optimal usage of limited inspection resources, like maximising the probability of catching a player who wants to cheat. A separate step afterwards is the "political decision" when to declare that the inspectee has actually cheated. Such models and practical issues are discussed in the book by Avenhaus and Canty, *Compliance Quantified* (see section 3.2).

3.5 Example: Compliance inspections

Suppose a player, whom we simply call "inspectee", must fulfil some legal obligation (such as buying a transport ticket, or paying tax). The inspectee has an incentive to violate this obligation. Another player, the "inspector", would like to verify that the inspectee is abiding by the rules, but doing so requires inspections which are costly. If the inspector does inspect and catches the inspectee cheating, the inspector can demand a large penalty payment for the noncompliance.

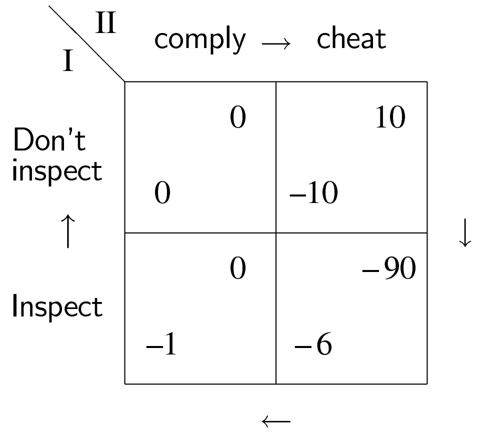

Figure 3.2 Inspection game between an inspector (player I) and inspectee (player II).

Figure 3.2 shows possible payoffs for such an inspection game. The standard outcome, which defines the reference payoff zero to both inspector (player I) and inspectee (player II), is that the inspector chooses "Don't inspect" and the inspectee chooses to comply. Without inspection, the inspectee prefers to cheat because that gives her payoff 10, with resulting negative payoff -10 to the inspector. The inspector may also decide to inspect. If the inspectee complies, inspection leaves her payoff 0 unchanged, while the inspector incurs a cost resulting in a negative payoff -1. If the inspectee cheats, however, inspection will result in a heavy penalty (payoff -90 for player II) and still create a certain amount of hassle for player I (payoff -6).

In all cases, player I would strongly prefer if player II complied, but this is outside of player I's control. However, the inspector prefers to inspect if the inspectee cheats (because -6 is better than -10), indicated by the downward arrow on the right in figure 3.2. If the inspector always preferred "Don't inspect", then this would be a dominating strategy and be part of a (unique) equilibrium where the inspectee cheats.

The circular arrow structure in figure 3.2 shows that this game has no equilibrium in pure strategies. If any of the players settles on a deterministic choice (like "Don't inspect" by player I), the best reponse of the other player would be unique (here to cheat by player II), to which the original choice would not be a best reponse (player I prefers to inspect when the other player cheats, against which player II in turn prefers to comply). The strategies in a Nash equilibrium must be best responses to each other, so in this game this fails to hold for any pure strategy profile.

What should the players do in the game of figure 3.2? One possibility is that they prepare for the worst, that is, choose a max-min strategy. A max-min strategy maximises the player's worst payoff against all possible choices of the opponent. The max-min strategy (as a pure strategy) for player I is to inspect (where the inspector guarantees himself payoff -6), and for player II it is to comply (which guarantees her payoff 0). However, this is not a Nash equilibrium and hence not a stable recommendation to the two players, because player I could switch his strategy and improve his payoff.

A *mixed strategy* of player I in this game is to inspect only with a certain probability. In the context of inspections, randomising is also a practical approach that reduces costs. Even if an inspection is not certain, a sufficiently high chance of being caught should deter from cheating, at least to some extent.

The following considerations show how to find the probability of inspection that will lead to an equilibrium. If the probability of inspection is very low, for example one percent, then player II receives (irrespective of that probability) payoff 0 for complying, and payoff $0.99 \times 10 + 0.01 \times (-90) = 9$, which is bigger than zero, for cheating. Hence, player II will still cheat, just as in the absence of inspection.

If the probability of inspection is much higher, for example 0.2, then the expected payoff for "cheat" is $0.8 \times 10 + 0.2 \times (-90) = -10$, which is less than zero, so that player II prefers to comply. If the inspection probability is either too low or too high, then player II has a unique best response. As shown above, such a pure strategy cannot be part of an equilibrium.

Hence, the only case where player II herself could possibly randomise between her strategies is if both strategies give her the same payoff, that is, if she is *indifferent*. As stated and proved formally in theorem 3.1 below, it is never optimal for a player to assign a positive probability to a pure strategy that is inferior to other pure strategies, given what the other players are doing. It is not hard to see that player II is indifferent if and only if player I inspects with probability 0.1, because then the expected payoff for cheating is $0.9 \times 10 + 0.1 \times (-90) = 0$, which is then the same as the payoff for complying.

With this mixed strategy of player I (don't inspect with probability 0.9 and inspect with probability 0.1), player II is indifferent between her strategies. Hence, she can *mix* them (that is, play them randomly) without losing payoff. The only case where, in turn, the original mixed strategy of player I is a best response is if player I is indifferent. According to the payoffs in figure 3.2, this requires player II to comply with probability 0.8 and to cheat with probability 0.2. The expected payoffs to player I are then for "Don't inspect" $0.8 \times 0 + 0.2 \times (-10) = -2$, and for "Inspect" $0.8 \times (-1) + 0.2 \times (-6) = -2$, so that player I is indeed indifferent, and his mixed strategy is a best response to the mixed strategy of player II.

This defines the only Nash equilibrium of the game. It uses mixed strategies and is therefore called a *mixed* equilibrium. The resulting expected payoffs are -2 for player I and 0 for player II.

The preceding analysis shows that the game in figure 3.2 has a mixed equilibrium, where the players choose their pure strategies according to certain probabilities. These probabilities have several noteworthy features.

First, the equilibrium probability of 0.1 for inspecting makes player II indifferent between complying and cheating. As explained in section 3.4 above, this requires payoffs to be *expected utilities*.

Secondly, mixing seems paradoxical when the player is indifferent in equilibrium. If player II, for example, can equally well comply or cheat, why should she gamble? In particular, she could comply and get payoff zero for certain, which is simpler and safer. The answer is that precisely because there is no incentive to choose one strategy over the other, a player can mix, and only in that case there can be an equilibrium. If player II would comply for certain, then the only optimal choice of player I is not to inspect, making the choice of complying not optimal, so this is not an equilibrium.

The least intuitive aspect of mixed equilibrium is that the probabilities depend on the *opponent's payoffs* and not on the player's own payoffs (as long as the qualitative preference structure, represented by the arrows, remains intact). For example, one would expect that raising the penalty -90 in figure 3.2 for being caught lowers the probability of cheating in equilibrium. In fact, it does not. What does change is the probability of inspection, which is reduced until the inspectee is indifferent.

3.6 Bimatrix games

In the following, we discuss mixed equilibria for general games in strategic form. We always assume that each player has only a finite number of given pure strategies. In order to simplify notation, we consider the case of two players. Many definitions and results carry over without difficulty to the case of more than two players.

Recall that a game in strategic form is specified by a finite set of "pure" strategies for each player, and a payoff for each player for each *strategy profile*, which is a tuple of strategies, one for each player. The game is played by each player independently and simultaneously choosing one strategy, whereupon the players receive their respective payoffs.

For two players, a game in strategic form is also called a *bimatrix game* (A,B). Here, A and B are two payoff matrices. By definition, they have equal dimensions, that is, they are both $m \times n$ matrices, having m rows and n columns. The m rows are the pure strategies i of player I and the n columns are the pure strategies j of player II. For a row i, where $1 \leq i \leq m$, and column j, where $1 \leq j \leq n$, the matrix entry of A is a_{ij} as payoff to player I, and the matrix entry of B is b_{ij} as payoff to player II.

Usually, we depict such a game as a table with m rows and n columns, so that each cell of the table corresponds to a pure strategy pair (i, j), and we enter both payoffs a_{ij} and b_{ij} in that cell, a_{ij} in the lower-left corner, preferably written in red if we have colours at hand, and b_{ij} in the upper-right corner of the cell, displayed in blue. The "red" numbers are then the entries of the matrix A, the "blue" numbers those of the matrix B. It does not matter if we take two matrices A and B, or a single table where each cell has two entries (the respective components of A and B).

A *mixed strategy* is a randomised strategy of a player. It is defined as a probability distribution on the set of pure strategies of that player. This is played as an "active randomisation": Using a lottery device with the given probabilities, the player picks each pure strategy according to its probability. When a player plays according to a mixed strategy, the other player is not supposed to know the outcome of the lottery. Rather, it is assumed that the opponent knows that the strategy chosen by the player is a random event, and bases his or her decision on the resulting distribution of payoffs. The payoffs are then "weighted with their probabilities" to determine the *expected payoff*, which represents the player's preference, as explained in section 3.4.

A pure strategy is a special mixed strategy. Namely, consider a pure strategy i of player I. Then the mixed strategy x that selects i with probability one and any other pure strategy with probability zero is effectively the same as the pure strategy i, because x chooses i with certainty. The resulting expected payoff is the same as the pure strategy payoff, because any unplayed strategy has probability zero and hence does not affect the expected payoff, and the pure strategy i is weighted with probability one.

3.7 Matrix notation for expected payoffs

Unless specified otherwise, we assume that in the two-player game under consideration, player I has m strategies and player II has n strategies. The pure strategies of player I, which are the m rows of the bimatrix game, are denoted by $i = 1,\ldots,m$, and the pure strategies of player II, which are the n columns of the bimatrix game, are denoted by $j = 1,\ldots,n$.

A mixed strategy is determined by the probabilities that it assigns to the player's pure strategies. For player I, a mixed strategy x can therefore be identified with the m-tuple of probabilities $(x_1,x_2,\ldots,x_m)$ that it assigns to the pure strategies $1,2,\ldots,m$ of player I. We can therefore consider x as an element of m-space (written $\mathbb{R}^m$). We assume that the vector x with m components is a *row vector*, that is, a $1\times m$ matrix with a single row and m columns. This will allow us to write expected payoffs in a short way.

A mixed strategy y of player II is an n-tuple of probabilities y_j for playing the pure strategies $j = 1,\ldots,n$. That is, y is an element of $\mathbb{R}^n$. We write y as a *column vector*, as $(y_1,y_2,\ldots,y_n)^\top$, that is, the row vector $(y_1,y_2,\ldots,y_n)$ *transposed*. Transposition in general applies to any matrix. The transpose $B^\top$ of the payoff matrix B, for example, is the $n\times m$ matrix where the entry in row j and column i is b_{ij}, because transposition means exchanging rows and columns. A column vector with n components is therefore considered as an $n\times 1$ matrix; transposition gives a row vector, a $1\times n$ matrix.

Normally, all vectors are considered as column vectors, so $\mathbb{R}^n$ is equal to $\mathbb{R}^{n\times 1}$, the set of all $n\times 1$ matrices with n rows and one column. We have made an exception in defining a mixed strategy x of player I as a row vector. Whether we mean row or column vectors will be clear from the context.

Suppose that player I uses the mixed strategy x and that player II uses the mixed strategy y. With these conventions, we can now succinctly express the expected payoff to player I as xAy, and the expected payoff to player II as xBy.

In order to see this, recall that the *matrix product* CD of two matrices C and D is defined when the number of columns of C is equal to the number of rows of D. That is, C is a $p\times q$ matrix, and D is a $q\times r$ matrix. The product CD is then a $p\times r$ matrix with entry $\sum_{k=1}^{q} c_{ik}d_{kj}$ in row i and column j, where c_{ik} and d_{kj} are the respective entries of C and D. Matrix multiplication is associative, that is, for another $r \times s$ matrix E the matrix product CDE is a $p \times s$ matrix, which can be computed either as $(CD)E$ or as $C(DE)$.

For mixed strategies x and y, we read xAy and xBy as matrix products. This works because x, considered as a matrix, is of dimension $1 \times m$, both A and B are of dimension $m \times n$, and y is of dimension $n \times 1$. The result is a 1×1 matrix, that is, a single real number.

It is best to think of xAy being computed as $x(Ay)$, that is, as the product of a row vector x that has m components with a column vector Ay that has m components. (The matrix product of two such vectors is also known as the *scalar product* of these two vectors.) The column vector Ay has m rows. We denote the entry of Ay in row i by $(Ay)_i$ for each row i. It is given by

$$(Ay)_i = \sum_{j=1}^{n} a_{ij} y_j \quad \text{for } 1 \le i \le m. \tag{3.1}$$

That is, the entries a_{ij} of row i of player I's payoff matrix A are multiplied with the probabilities y_j of their columns, so *$(Ay)_i$ is the expected payoff to player I when playing row i.* One can also think of y_j as a linear coefficient of the jth column of the matrix A. That is, Ay is the linear combination of the column vectors of A, each multiplied with its probability under y. This linear combination Ay is a vector of expected payoffs, with one expected payoff $(Ay)_i$ for each row i.

Furthermore, xAy is the expected payoff to player I when the players use x and y, because

$$x(Ay) = \sum_{i=1}^{m} x_i (Ay)_i = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} a_{ij} y_j = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i y_j) a_{ij}. \tag{3.2}$$

Because the players choose their pure strategies i and j independently, the probability that they choose the pure strategy pair (i, j) is the product $x_i y_j$ of these probabilities, which is the coefficient of the payoff a_{ij} in (3.2).

Analogously, xBy is the expected payoff to player II when the players use the mixed strategies x and y. Here, it is best to read this as $(xB)y$. The vector xB, as the product of a $1 \times m$ with an $m \times n$ matrix, is a $1 \times n$ matrix, that is, a row vector. Each column of that row corresponds to a strategy j of player II, for $1 \le j \le n$. We denote the respective column entry by $(xB)_j$. It is given by $\sum_{i=1}^{m} x_i b_{ij}$, which is the scalar product of x with the jth column of B. That is, $(xB)_j$ is the expected payoff to player II when player I plays x and player II plays the pure strategy j. If these numbers are multiplied with the column probabilities y_j and added up, then the result is the expected payoff to player II, which in analogy to (3.2) is given by

$$(xB)y = \sum_{j=1}^{n} (xB)_j y_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_i b_{ij} \right) y_j = \sum_{j=1}^{n} \sum_{i=1}^{m} (x_i y_j) b_{ij}. \tag{3.3}$$

3.8 Convex combinations and mixed strategy sets

It is useful to regard mixed strategy vectors as geometric objects. A mixed strategy x of player I assigns probabilities x_i to the pure strategies i. The pure strategies, in turn, are special mixed strategies, namely the unit vectors in $\mathbb{R}^m$, for example $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ if $m=3$. The mixed strategy (x_1,x_2,x_3) is then a linear combination of the pure strategies, namely $x_1\cdot(1,0,0)+x_2\cdot(0,1,0)+x_3\cdot(0,0,1)$, where the linear coefficients are just the probabilities. Such a linear combination is called a *convex* combination because the coefficients sum to one and are non-negative.

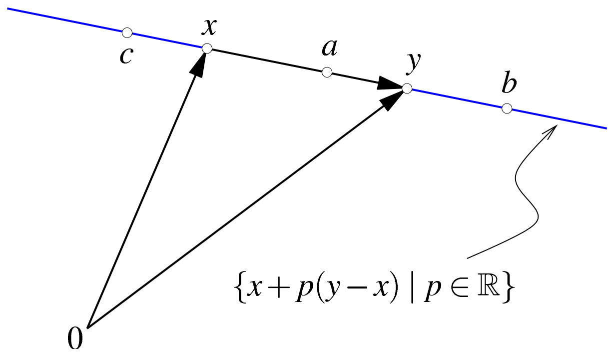

Figure 3.3 The line through the points x and y is given by the points $x+p(y-x)$ where $p\in\mathbb{R}$. Examples are point a for $p=0.6$, point b for $p=1.5$, and point c when $p=-0.4$. The line segment connecting x and y results when p is restricted to $0\le p\le 1$.

Figure 3.3 shows two points x and y, here in the plane, but the picture may also be regarded as a suitable view of the situation in a higher-dimensional space. The line that goes through the points x and y is obtained by adding to the point x, regarded as a vector, any multiple of the difference $y-x$. The resulting vector $x+p\cdot(y-x)$, for $p\in\mathbb{R}$, gives x when $p=0$, and y when $p=1$. Figure 3.3 gives some examples a, b, c of other points. When $0\le p\le 1$, as for point a, the resulting points give the *line segment* joining x and y. If $p>1$, then one obtains points on the line through x and y on the other side of y relative to x, like the point b in figure 3.3. For $p<0$, the corresponding point, like c in figure 3.3, is on that line but on the other side of x relative to y.

The expression $x+p(y-x)$ can be re-written as $(1-p)x+py$, where the given points x and y appear only once. This expression (with $1-p$ as the coefficient of the first vector and p of the second) shows how the line segment joining x to y corresponds to the real interval $[0,1]$ for the possible values of p, with the endpoints 0 and 1 of the interval corresponding to the endpoints x and y, respectively, of the line segment.

In general, a *convex combination* of points $z_1,z_2,\ldots,z_k$ in some space is given as any linear combination $p_1\cdot z_1+p_2\cdot z_2+\cdots+p_k\cdot z_k$ where the linear coefficients $p_1,\ldots,p_k$ are non-negative and sum to one. The previously discussed case corresponds to $z_1=x$, $z_2=y$, $p_1=1-p$, and $p_2=p\in[0,1]$.

A set of points is called *convex* if it contains with any points $z_1,z_2,\ldots,z_k$ also every convex combination of these points. Equivalently, one can show that a set is convex if it contains with any two points also the line segment joining these two points; one can then obtain combinations of k points for $k > 2$ by iterating convex combinations of only two points.

The coefficients in a convex combination can also be regarded as probabilities, and conversely, a probability distribution on a finite set can be seen as a convex combination of the unit vectors.

In a two-player game with m pure strategies for player I and n pure strategies for player II, we denote the sets of mixed strategies of the two players by X and Y, respectively:

$$\begin{aligned} X &= \{(x_1,\ldots,x_m) \mid x_i \ge 0 \text{ for } 1 \le i \le m,\ \sum_{i=1}^{m} x_i = 1\}, \\ Y &= \{(y_1,\ldots,y_n)^\top \mid y_j \ge 0 \text{ for } 1 \le j \le n,\ \sum_{j=1}^{n} y_j = 1\}. \end{aligned} \tag{3.4}$$

For consistency with section 3.7, we assume that X contains row vectors and Y column vectors, but this is not an important concern.

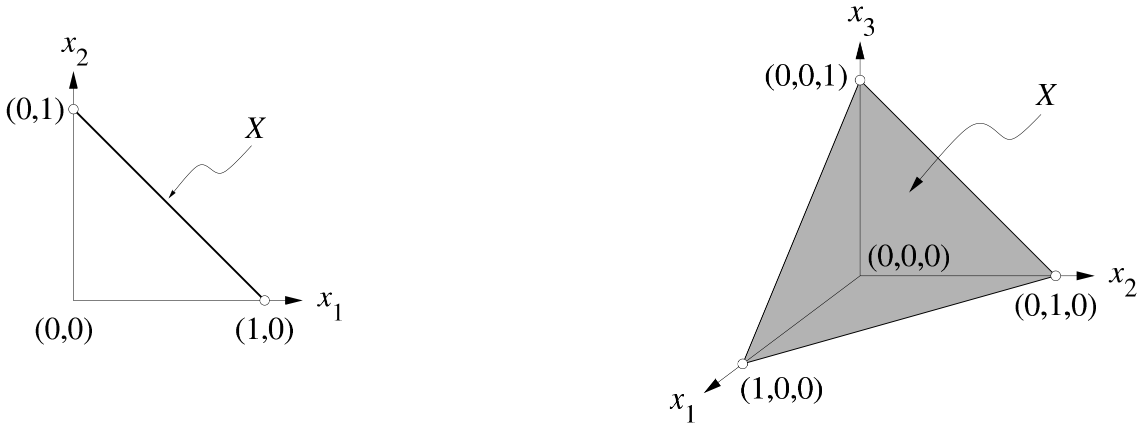

Figure 3.4 Examples of player I's mixed strategy set X when $m = 2$ (left) and $m = 3$ (right), as the set of convex combinations of the unit vectors.

Examples of X are shown in figure 3.4. When $m = 2$, then X is just the line segment joining $(1,0)$ to $(0,1)$. If $m = 3$, then X is a triangle, given as the set of convex combinations of the unit vectors, which are the vertices of the triangle.

It is easily verified that in general X and Y are convex sets.

3.9 The best response condition

A mixed strategy equilibrium is a profile of mixed strategies such that no player can improve his expected payoff by unilaterally changing his own strategy. In a two-player game, an equilibrium is a pair (x,y) of mixed strategies such that x is a best response to y and vice versa. That is, player I cannot get a better expected payoff than xAy by choosing any other strategy than x, and player II cannot improve her expected payoff xBy by changing y.

It does not seem easy to decide if x is a best response to y among all possible mixed strategies, that is, if x maximises xAy for all x in X, because X is an infinite set. However, the following theorem, known as the *best response condition*, shows how to recognise this. This theorem is not difficult but it is important to understand. We discuss it afterwards.

Theorem 3.1 (Best response condition) *Let x and y be mixed strategies of player I and II, respectively. Then x is a best response to y if and only if for all pure strategies i of player I,*

$$x_i > 0 \implies (Ay)_i = \max\{(Ay)_k \mid 1 \le k \le m\}. \tag{3.5}$$

Proof. Recall that $(Ay)_i$ is the ith component of Ay, which is the expected payoff to player I when playing row i, according to (3.1). Let $u = \max\{(Ay)_k \mid 1 \le k \le m\}$, which is the maximum of these expected payoffs for the *pure* strategies of player I. Then

$$\begin{aligned} xAy = \sum_{i=1}^m x_i\,(Ay)_i &= \sum_{i=1}^m x_i\,(u-(u-(Ay)_i) = \sum_{i=1}^m x_i\,u - \sum_{i=1}^m x_i\,(u-(Ay)_i) \\ &= u - \sum_{i=1}^m x_i\,(u-(Ay)_i). \end{aligned} \tag{3.6}$$

Because, for any pure strategy i, both x_i and the difference of the maximum payoff u and the payoff $(Ay)_i$ for row i is non-negative, the sum $\sum_{i=1}^m x_i(u-(Ay)_i)$ is also non-negative, so that $xAy \le u$. The expected payoff xAy achieves the maximum u if and only if that sum is zero, that is, if $x_i > 0$ implies $(Ay)_i = u$, as claimed. $\square$

Consider the phrase "x is a best response to y" in the preceding theorem. This means that among all mixed strategies in X of player I, x gives maximum expected payoff to player I. However, the pure best responses to y in (3.5) only deal with the pure strategies of player I. Each such pure strategy corresponds to a row i of the payoff matrix. In that row, the payoffs a_{ij} are multiplied with the column probabilities y_j, and the sum over all columns gives the expected payoff $(Ay)i$ for the pure strategy i according to (3.1). This pure strategy is a best response if and only if no other row gives a higher payoff.

The first point of the theorem is that the condition whether a pure strategy is a best response is very easy to check, as one only has to compute the m expected payoffs $(Ay)_i$ for $i = 1,\ldots,m$. For example, if player I has three pure strategies ($m = 3$), and the expected payoffs in (3.1) are $(Ay)_1 = 4$, $(Ay)_2 = 4$, and $(Ay)_3 = 3$, then only the first two strategies are pure best responses. If these expected payoffs are 3, 5, and 3, then only the second strategy is a best response. Clearly, at least one pure best response exists, because the numbers $(Ay)_k$ in (3.5) have their maximum u for at least one k. The theorem states that only pure best responses i may have positive probability x_i if x is to be a best response to y.

A second consequence of theorem 3.1, used also in its proof, is that a mixed strategy can never give a higher payoff than the best pure strategy. This is intuitive because "mixing" amounts to *averaging*, which is an average weighted with the probabilities, in the way that the overall expected payoff xAy in (3.6) is obtained from those in (3.1) by multiplying (weighting) each row i with weight x_i and summing over all rows i, as shown in (3.2). Consequently, any pure best response i to y is also a mixed best response, so the maximum of xAy for $x \in X$ is the same as when x is restricted to the unit vectors in $\mathbb{R}^m$ that represent the pure strategies of player I.

$\Rightarrow$ Try now exercise 3.1 on page 91, with the help of theorem 3.1. This will help you appreciate methods for finding mixed equilibria that you will learn in later sections. You can also answer exercise 3.2 on page 92, which concerns a game with three players.

3.10 Existence of mixed equilibria

In this section, we give the original proof of John Nash from 1951 that shows that any game with a finite number of players, and finitely many strategies per player, has a mixed equilibrium. This proof uses the following theorem about continuous functions. The theorem concerns compact sets; here, a set is compact if it is closed (containing all points near the set) and bounded.

Theorem 3.2 (Brouwer's fixed point theorem) *Let S be a subset of $\mathbb{R}^N$ that is convex and compact, and let f be a continuous function from S to S. Then f has at least one* fixed point, *that is, a point s in S so that $f(s) = s$.*

We do not prove theorem 3.2, but instead demonstrate its assumptions with examples where not all assumptions hold and the conclusion fails. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$. This is clearly a continuous function, and the set $\mathbb{R}$ of all real numbers is closed and convex, but f has no fixed point. Some assumption of the fixed point theorem must be violated, and in this case it is compactness, because the set $\mathbb{R}$ is not bounded. Consider another function $f\colon [0,1] \to [0,1]$, given by $f(x) = x^2$. The fixed points of this function are 0 and 1. If we consider the function $f(x) = x^2$ as a function on the open interval $(0,1)$ (which is $[0,1]$ without its endpoints), then this function has no longer any fixed points. In this case, the missing condition is that the function is not defined on a closed set, which is therefore not compact. Another function is $f(x) = 1 - x$ for $x \in \{0,1\}$, where the domain of this function has just two elements, so this is a compact set. This function has no fixed points which is possible because its domain is not convex. Finally, the function on $[0,1]$ defined by $f(x) = 1$ for $0 \le x \le 1/2$ and by $f(x) = 0$ for $1/2 < x \le 1$ has no fixed point, which is possible in this case because the function is not continuous. These simple examples demonstrate why the assumptions of theorem 3.2 are necessary.

Theorem 3.3 (Nash [1951]) *Every finite game has at least one equilibrium in mixed strategies.*

Proof. We will give the proof for two players, to simplify notation. It extends in the same manner to any finite number of players. The set S that is used in the present context is the product of the sets of mixed strategies of the players. Let X and Y be the sets of mixed strategies of player I and player II as in (3.4), and let $S = X \times Y$.

Then the function $f{:}S \to S$ that we are going to construct maps a pair of mixed strategies (x,y) to another pair $f(x,y) = (\bar{x},\bar{y})$. Intuitively, a mixed strategy probability x_i (of player I, similarly y_j of player II) is changed to $\bar{x}_i$, such that it will decrease if the pure strategy i does worse than the average of all pure strategies. In equilibrium, all pure strategies of a player that have positive probability do equally well, so no sub-optimal pure strategy can have a probability that is reduced further. This means that the mixed strategies do not change, so this is indeed equivalent to the fixed point property $(x,y) = (\bar{x},\bar{y}) = f(x,y)$.

In order to define f as described, consider the following functions $\chi{:}X \times Y \to \mathbb{R}^m$ and $\psi{:}X \times Y \to \mathbb{R}^n$ (we do not worry whether these vectors are row or column vectors; it suffices that $\mathbb{R}^m$ contains m-tuples of real numbers, and similarly $\mathbb{R}^n$ contains n-tuples). For each pure strategy i of player I, let $\chi_i(x,y)$ be the ith component of $\chi(x,y)$, and for each pure strategy j of player II, let $\psi_j(x,y)$ be the jth component of $\psi(x,y)$. The functions χ and ψ are defined by

$$\chi_i(x,y) = \max\{0,(Ay)_i - xAy\}, \qquad \psi_j(x,y) = \max\{0,(xB)_j - xBy\},$$

for $1 \le i \le m$ and $1 \le j \le n$. Recall that $(Ay)_i$ is the expected payoff to player I against y when he uses the pure strategy i, and that $(xB)_j$ is the expected payoff to player II against x when she uses the pure strategy j. Moreover, xAy and xBy are the overall expected payoffs to player I and player II, respectively. So the difference $(Ay)_i - xAy$ is positive if the pure strategy i gives more than the average xAy against y, zero if it gives the same payoff, and negative if it gives less. The term $\chi_i(x,y)$ is this difference, except that it is replaced by zero if the difference is negative. The term $\psi_j(x,y)$ is defined analogously. Thus, $\chi(x,y)$ is a non-negative vector in $\mathbb{R}^m$, and $\psi(x,y)$ is a non-negative vector in $\mathbb{R}^n$. The functions χ and ψ are continuous.

The pair of vectors (x,y) is now changed by replacing x by $x+\chi(x,y)$ in order to get $\bar{x}$, and y by $y+\psi(x,y)$ to get $\bar{y}$. Both sums are non-negative. The only problem is that in general these new vectors are no longer probabilities because their components do not sum to one. For that purpose, they are "re-normalised" by the following functions $r: \mathbb{R}^m \to \mathbb{R}^m$ and $s: \mathbb{R}^n \to \mathbb{R}^n$, defined by their components r_i and s_j, that is, $r(x) = (r_1(x),\ldots,r_m(x))$, and $s(y) = (s_1(y),\ldots,s_n(y))$:

$$r_i(x_1,\ldots,x_m) = \frac{x_i}{\sum_{k=1}^m x_k}, \qquad s_j(y_1,\ldots,y_n) = \frac{y_j}{\sum_{k=1}^n y_k}.$$

Clearly, if $x_i \ge 0$ for $1 \le i \le m$ and $\sum_{k=1}^m x_k > 0$, then $r(x)$ is defined and is a probability distribution, that is, an element of the mixed strategy set X. Analogously, $s(y) \in Y$.

The function $f{:}X \to Y$ is now defined by

$$f(x,y) = \bigl(r(x+\chi(x,y)),\ s(y+\psi(x,y))\bigr).$$

What is a fixed point (x,y) of that function, so that $f(x,y) = (x,y)$? Among all strategies i where $x_i > 0$, consider the smallest pure strategy payoff $(Ay)_i$ against y, that is, $(Ay)_i = \min\{(Ay)_k \mid x_k > 0\}$. Then $(Ay)_i \le xAy$, which is proved analogously to (3.6), so the component $\chi_i(x,y)$ of $\chi(x,y)$ is zero. This means that the respective term $x_i + \chi_i(x,y)$

is equal to x_i. Conversely, consider any other pure strategy l of player I that gets the maximum payoff $(Ay)_l = \max_k (Ay)_k$. If that payoff is better than the average xAy, then clearly $\chi_l(x,y) > 0$, so that $x_l + \chi_l(x,y) > x_l$. Because $\chi_k(x,y) \geq 0$ for all k, this implies $\sum_{k=1}^m (x_k + \chi_k(x,y)) > 1$, which is the denominator in the re-normalisation with r in $r(x + \chi(x,y))$. This re-normalisation will now *decrease* the value of x_i for the pure strategy i with $(Ay)_i \leq xAy$, so the relative weight x_i of the pure strategy i decreases. (It would stay unchanged, that is, not decrease, if $x_i = 0$.) But $(Ay)_l > xAy$ can only occur if there is some sub-optimal strategy i (with $(Ay)_i \leq xAy < (Ay)_l$) that has positive probability x_i. In that case, $r(x + \chi(x,y))$ is *not equal* to x, so that $f(x,y) \neq (x,y)$.

Analogously, if $\psi(x,y)$ has some component that is positive, then the respective pure strategy of player II has a better payoff than xBy, so $y \neq s(y + \psi(x,y))$ and (x,y) is not a fixed point of f. In that case, y is also not a best response to x.

Hence, the function f has a fixed point (x,y) if and only if both $\chi(x,y)$ and $\psi(x,y)$ are zero in all components. But that means that xAy is the maximum possible payoff $\max_i (Ay)_i$ against y, and xBy is the maximum possible payoff $\max_j (xB)_j$ against x, that is, x and y are mutual best responses. The fixed points (x,y) are therefore exactly the Nash equilibria of the game. □

3.11 Finding mixed equilibria

How can we find all Nash equilibria of a two-player game in strategic form? It is easy to determine the Nash equilibria where both players use a pure strategy, because these are the cells of the payoff table where both payoffs are best-response payoffs, shown by a box around that payoff. We now describe how to find mixed strategy equilibria, using the best response condition theorem 3.1.

We first consider 2×2 games. Our first example above, the inspection game in figure 3.2, has no pure-strategy equilibrium. There we have determined the mixed strategy probabilities of a player so as to make the other player indifferent between his or her pure strategies, because only then that player will mix between these strategies. This is a consequence of theorem 3.1: Only pure strategies that get maximum, and hence equal, expected payoff can be played with positive probability in equilibrium.

A mixed strategy equilibrium can exist also in 2×2 games that have pure-strategy equilibria. As an example, consider the battle of sexes game, shown on the left in figure 3.5, which has the pure strategy equilibria (C,c) and (S,s). As indicated by the boxes, the best response of the other player depends on the player's own strategy: If player I plays C then player II's best response is c, and the best response to S is s. Suppose now that player I plays a mixed strategy, playing C with probability $1-p$ and S with probability p. Clearly, when p is close to zero so that player I almost certainly chooses C, the best response will still be c, whereas if p is close to one the best response will be s. Consequently, there is some probability so that player II is indifferent. This probability p is found as follows. Given p, which defines the mixed strategy of player I, the expected payoff to player II when she plays c is $2(1-p)$, and when she plays s that expected payoff

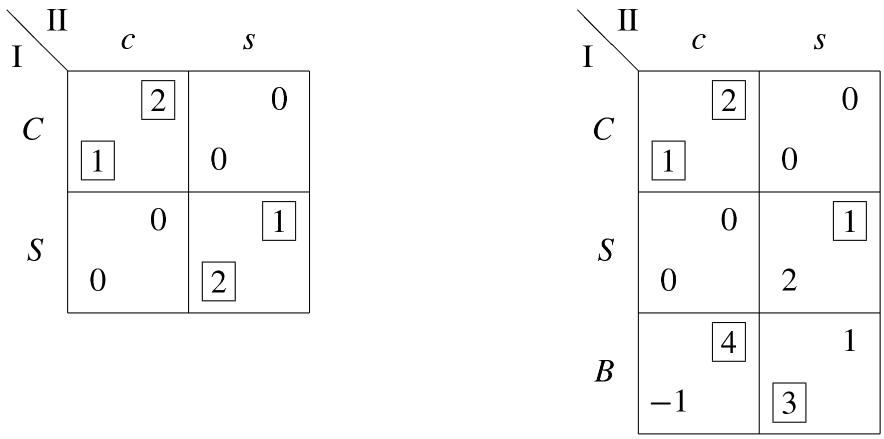

Figure 3.5 Battle of sexes game (left) and a 3×2 game (right) which is the same game with an extra strategy B for player I.

is simply p. She is indifferent between c and s if and only if these expected payoffs are equal, that is, $2(1-p) = p$ or $2 = 3p$, that is, $p = 2/3$. This mixed strategy of player I, where he plays C and S with probabilities $1/3$ and $2/3$, respectively, gives player II for both her strategies, the two columns, the same expected payoff $2/3$. Then player II can mix between c and s, and a similar calculation shows that player I is indifferent between his two strategies if player II uses the mixed strategy $(2/3, 1/3)$, which is the vector of probabilities for her two strategies c, s. Then and only then player I gets the same expected payoff for both rows, which is $2/3$. This describes the mixed equilibrium of the game, which we write as $((1/3, 2/3), (2/3, 1/3))$, as a pair (x, y) of mixed strategies x and y, which are probability vectors.

In short, the rule to find a mixed strategy equilibrium in a 2×2 game is to make the other player indifferent, because only in that case can the other player mix. This indifference means the expected payoffs for the two opponent strategies have to be equal, and the resulting equation determines the player's own probability.

The mixed strategy probabilities in the battle of sexes game can be seen relatively quickly by looking at the payoff matrices: For example, player II must give twice as much weight to strategy c compared to s, because player I's strategy C only gets one payoff unit from c whereas his strategy S gets two payoff units when player II chooses s. Because the strategy pairs (C, s) and (S, c) give payoff zero, the two rows C and S give the same expected payoff only when c has probability $2/3$ and s has probability $1/3$.

We now explain a quick method, the "difference trick", to find the probabilities in a mixed strategy equilibrium of a general 2×2 game. Consider the game on the left in figure 3.6. When player II plays T, then l is a best response, and when player II plays B, then r is a best response. Consequently, there must be a way to mix T and B so that player II is indifferent between l and r. We consider now the difference Δ in payoffs to the other player for both rows: When player I plays T, then the difference between the two payoffs to player II is $\Delta = 2 - 1 = 1$, and when player I plays B, then that difference, in absolute value, is $\Delta = 7 - 3 = 4$, as shown on the side of the game. (Note that these differences, when considered as differences between the payoffs for l

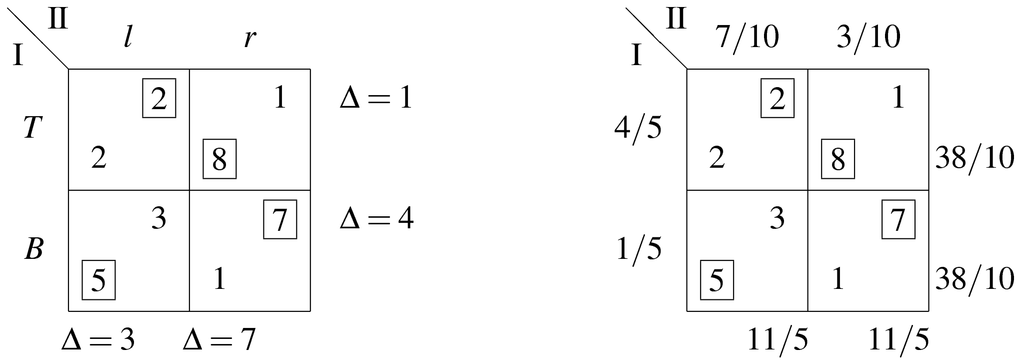

Figure 3.6 The "difference trick" to find equilibrium mixed strategy probabilities in a 2×2 game. The left figure shows the game and the difference in payoffs to the other player for each strategy. As shown on the right, these differences are assigned to the respective other own strategy and are re-normalised to become probabilities. The fractions $11/5$ and $38/10$ are the resulting equal expected payoffs to player II and player I, respectively.

versus r, have opposite sign because l is preferred to r against T, and the other way around against B; otherwise, player II would always prefer the same strategy and could not be made indifferent.) Now the payoff difference Δ is assigned as a probability weight to the respective *other* strategy of the row player, meaning T is given weight 4 (which is the Δ computed for B), and B is given weight 1 (which is the Δ for T). The probabilities for T and B are then chosen proportional to these weights, so one has to divide each weight by 5 (which is the sum of the weights) in order to obtain probabilities. This is shown on the right in figure 3.6. The expected payoffs to player II are also shown there, at the bottom, and are for l given by $(4 \cdot 2 + 1 \cdot 3)/5 = 11/5$, and for r by $(4 \cdot 1 + 1 \cdot 7)/5 = 11/5$, so they are indeed equal as claimed. Similarly, the two payoff differences for player I in columns l and r are 3 and 7, respectively, so l and r should be played with probabilities that are proportional to 7 and 3, respectively. With the resulting probabilities $7/10$ and $3/10$, the two rows T and B get the same expected payoff $38/10$.

$\Rightarrow$ A proof that this "difference trick" always works is the subject of exercise 3.3 on page 92.

Next, we consider games where one player has two and the other more than two strategies. The 3×2 game on the right in figure 3.5 is like the battle of sexes game, except that player I has an additional strategy B. Essentially, such a game can be analysed by considering the 2×2 games obtained by restricting both players to two strategies only, and checking if the resulting equilibria carry over to the whole game. First, the pure-strategy equilibrium (C, c) of the original battle of sexes game is also an equilibrium of the larger game, but (S, s) is not because S is no longer a best response to s because B gives a larger payoff to player I. Second, consider the mixed strategy equilibrium of the smaller game where player I chooses C and S with probabilities $1/3$ and $2/3$ (and B with probability zero), so that player II is indifferent between c and s. Hence, the mixed strategy $(2/3, 1/3)$ of player II is still a best response, against which C and S both give the same expected payoff $2/3$. However, this is not enough to guarantee an equilibrium, because C and S have to be best responses, that is, their payoff must be at least as large as that for the additional strategy B. That payoff is $(-1)\cdot 2/3+3\cdot 1/3=1/3$, so it is indeed not larger than the payoff for C and S. In other words, the mixed equilibrium of the smaller game is also a mixed equilibrium of the larger game, given by the pair of mixed strategies $((1/3,2/3,0),(2/3,1/3))$.

Other "restricted" 2×2 games are obtained by letting player I play only C and B, or only S and B. In the first case, the difference trick gives the mixed strategy $(3/5,2/5)$ of player II (for playing c,s) so that player I is indifferent between C and B, where both strategies receive expected payoff $3/5$. However, the expected payoff to the third strategy S is then $4/5$, which is higher, so that C and B are not best responses, which means we cannot have an equilibrium where only C and B are played with positive probability by player I. Another reason why no equilibrium strategy of player II mixes C and B is that against both C and B, player II's best response is always c, so that player I could not make player II indifferent by playing in that way.

Finally, consider S and B as pure strategies that player I mixes in a possible equilibrium. This requires, via the difference trick, the mixed strategy $(0,3/4,1/4)$ of player I (as probability vector for playing his three pure strategies C,S,B) so that player II is indifferent between c and s, both strategies receiving payoff 1 in that case. Then if player II uses the mixed strategy $(1/2,1/2)$ (in order to make player I indifferent between S and B), the expected payoffs for the three rows C,S,B are $1/2$, 1, and 1, respectively, so that indeed player I only uses best responses with positive probability, which gives a third Nash equilibrium of the game.

Why is there no mixed equilibrium of the 3×2 game in figure 3.5 where player I mixes between his three pure strategies? The reason is that player II has only a single probability at her disposal (which determines the complementary probability for the other pure strategy), which does not give enough freedom to satisfy two equations, namely indifference between C and S as well as between S and B (the third indifference between C and B would then hold automatically). We have already computed the probabilities $(2/3,1/3)$ for c,s that are needed to give the same expected payoff for C and B, against which the payoff for row B is different, namely $1/3$. This alone suffices to show that it is not possible to make player I indifferent between all three pure strategies, so there is no equilibrium where player I mixes between all of them. In certain games, which are called "degenerate" and which are treated in section 3.13, it is indeed the case that player II, say, mixes only between two pure strategies, but where "by accident" three strategies of player I have the same optimal payoff. This is normally not the case, and leads to complications when determining equilibria, which will be discussed in the later section on degenerate games.

$\Rightarrow$ You can now attempt exercise 3.1 on page 91, if you have not done this earlier. Compare your solution with the approach for the 3×2 game in figure 3.5.

3.12 The upper envelope method

The topic of this section is the "upper-envelope" diagram that simplifies finding mixed equilibria, in particular of games where one player has only two strategies.

The left graph in figure 3.7 shows the upper-envelope diagram for the left game in figure 3.5. This diagram is a plot of the expected payoff of one player against the mixed strategy of the other player. We use this typically when that mixed strategy is determined by a single probability, that is, we consider a player who has only two pure strategies, in this example player II. The horizontal axis represents the probability for playing the second pure strategy s of player II, so this is $q = \text{prob}(s)$. In the vertical direction, we plot the resulting expected payoffs to the other player, here player I, for his pure strategies. In this game, this is the expected payoff $1-q$ when he plays row C, and the expected payoff $2q$ when he plays row S. The plots are lines because expected payoffs are linear functions of the mixed strategy probabilities.

These lines are particularly easy to obtain graphically. To see this, consider row C of player I. When player II plays her left strategy c (where $q = 0$), row C gives payoff 1 (according to the table of payoffs to player I). When player II plays her right strategy s (where $q = 1$), row C gives payoff 0. The probabilities $q = 0$ or $q = 1$ and corresponding payoffs define the two endpoints with co-ordinates $(0,1)$ and $(1,0)$ of a line that describes the expected payoff $1-q$ as a function of q, which consists of the points $(q, 1-q)$ for $q \in [0,1]$. As explained in section 3.8 above, this line is just the set of convex combinations $(1-q)\cdot(0,1)+q\cdot(1,0)$ of the two endpoints. The first component of such a convex combination is always q and second component $(1-q)\cdot 1+q\cdot 0$ is the expected payoff. For row S, this expected payoff is $(1-q)\cdot 0+q\cdot 2$, so the line of expected payoffs for S connects the left endpoint $(0,0)$ with the right endpoint $(1,2)$.

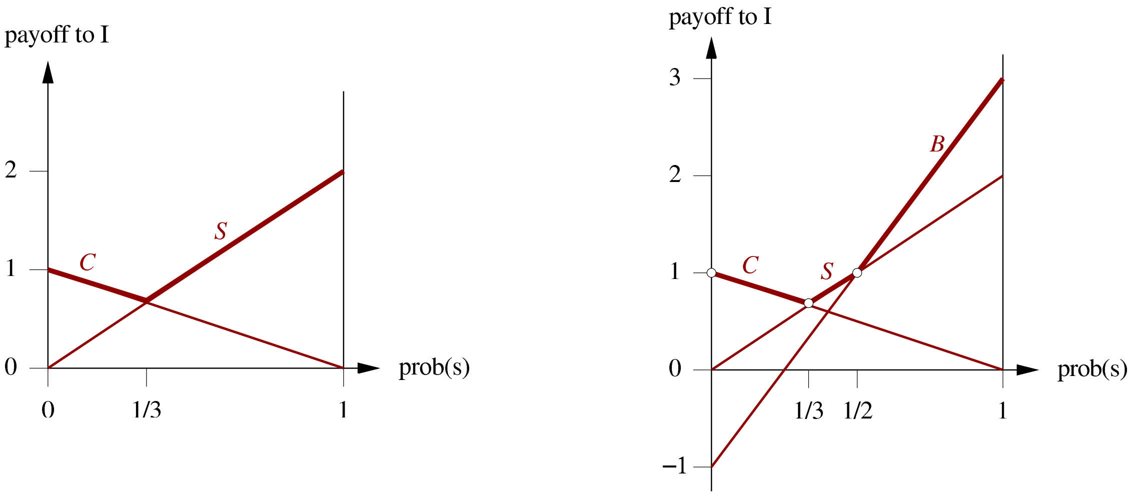

Figure 3.7 Upper envelope of expected payoffs to player I for the two games in figure 3.5.

The upper-envelope diagram can also be described as the "goalpost" method. First, identify a player who has only two pure strategies (here player II). Second, plot the probability (here q) for the second of these pure strategies horizontally, along the interval $[0,1]$. Third, erect a vertical line (a "goalpost") at the left endpoint 0 and at the right endpoint 1 of that interval. Fourth, do the following for each pure strategy of the other player (which are here rows C and S for player I): Mark the payoff (as given in the game description) against the left strategy (of player II, that is, against $q = 0$) as a "height" on the left goalpost, and against the right strategy ($q = 1$) as a height on the right goalpost, and connect the two heights by a line, which gives the expected payoff as a function of q for $0 \leq q \leq 1$.

Note that for these goalposts it is very useful to consider player II's mixed strategy as the vector of probabilities $(1-q,q)$ (according to the second step above), because then the left goalpost corresponds to the left strategy of player II, and the right goalpost to the right strategy of player II. (This would not be the case if q was taken as the probability for the left strategy, writing player II's mixed strategy as $(q,1-q)$, because then $q = 0$ would be the left goalpost but represent the right pure strategy of player II, creating an unnecessary source of confusion.)

In the left diagram in figure 3.7, the two lines of expected payoffs for rows C and S obviously intersect exactly where player I is indifferent between these two rows, which happens when $q = 1/3$. However, the diagram not only gives information about this indifference, but about player I's preference in general: Any pure strategy that is a *best response* of player I is obviously the *topmost* line segment of all the lines describing the expected payoffs. Here, row C is a best response whenever $0 \leq q \leq 1/3$, and row S is a best response whenever $1/3 \leq q \leq 1$, with indifference between C and S exactly when $q = 1/3$. The *upper envelope* of expected payoffs is the *maximum* of the lines that describe the expected payoffs for the different pure strategies, and it is indicated by the bold line segments in the diagram. Marking this upper envelope in bold is therefore the fifth step of the "goalpost" method.

The upper envelope is of particular use when a player has more than two pure strategies, because mixed equilibria of a 2×2 are anyhow quickly determined with the "difference trick". Consider the upper-envelope diagram on the right of figure 3.7, for the 3×2 game in figure 3.5. The extra strategy B of player I gives the line connecting height -1 on the left goalpost to height 3 on the right goalpost, which shows that B is a best response for all q so that $1/2 \leq q \leq 1$, with indifference between S and B when $q = 1/2$. The upper envelope consists of three line segments for the three pure strategies C, S, and B of player I. Moreover, there are only two points where player I is indifferent between any two such strategies, namely between C and S when $q = 1/3$, and between S and B when $q = 1/2$. These two points, indicated by small circles on the upper envelope, give the only two mixed strategy probabilities of player II where player I can mix between two pure strategies (the small circle for $q = 0$ corresponds to the pure equilibrium strategy c). There is a third indifference between C and B, for $q = 2/5$, where the two lines for C and B intersect, but this intersection point is below the line for S and hence not on the upper envelope. Consequently, it does not have to be investigated as a possible mixed strategy equilibrium strategy where player I mixes between the two rows C and B because they are not best responses.

Given these candidates of mixtures for player I, we can quickly find if and how the two respective pure strategies can be mixed in order to make player II indifferent using the difference trick, as described in the previous section.

So the upper envelope restricts the pairs of strategies that need to be tested as possible strategies that are mixed in equilibrium. Exercise 3.4, for example, gives a 2×5 game, where the upper envelope may consist of up to five line segments (there may be fewer line segments if some pure strategy of player II is never a best response). Consequently, there are up to four points where these line segments join up, so that player II is indifferent between two strategies and can mix. Without the upper envelope, there would not be four but ten (which are the different ways to pick two out of five) pure strategy pairs to be checked as possible pure strategies that player II mixes in equilibrium, which would be much more laborious to investigate.

$\Rightarrow$ The familiar exercise 3.1 on page 91 can also be solved with the upper envelope method. This method is particular help for larger games as in exercise 3.4 on page 93.

In principle, the upper-envelope diagram is also defined for the expected payoffs against a mixed strategy that mixes between more than two pure strategies. This diagram is much harder to draw and visualise in this case because it requires higher dimensions. We demonstrate this for the game on the right in figure 3.5, although we do not suggest using this as a method to find all equilibria of such a game.

Consider the set of mixed strategies of player I for the 3×2 game of figure 3.5, which is a triangle. Then the payoff to the other player II needs an extra "vertical" dimension. A perspective drawing of the resulting three-dimensional picture is shown in figure 3.8, in several stages. First (top left), the mixed strategy triangle is shown with goalposts erected on its corners $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Next (top right), we mark the heights 2, 0, and 4 on these posts which are the payoffs to player II for her first strategy c. The resulting expected payoffs define a plane above the triangle through these height markings. The third picture (bottom left) does the same for the second strategy s of player II, which has payoffs 0, 1, and 1 as heights in the three corners of the triangle. The final picture (bottom right) gives the upper envelope as the maximum of the two expected-payoff planes (which in the lower dimension as in figure 3.7 is drawn in bold). The small circles indicate the three mixed strategies of player I which are his strategies, respectively, in the three Nash equilibria $((1,0,0),(1,0))$ $((1/3,2/3,0),(2/3,1/3))$, and $((0,3/4,1/4),(1/2,1/2))$. The first of these three equilibria is the pure strategy equilibrium (C,c), and the last two involve mixing both columns c and s of player II. Hence, player II must be indifferent between both c and s, so player I must choose a mixed strategy where the planes of expected payoffs for these two strategies c and s of player II meet. (These two planes meet in a line, which is given by any convex combination of the two mixed strategies $(1/3,2/3,0)$ and $(0,3/4,1/4)$ of player I; however, player I must choose one of these two endpoints because he can only mix either between C and S or between S and B as described earlier, depending on the mixed strategy of player II.)

$\Rightarrow$ The simple exercise 3.5 on page 93 helps to understand dominated strategies with the upper-envelope diagram.

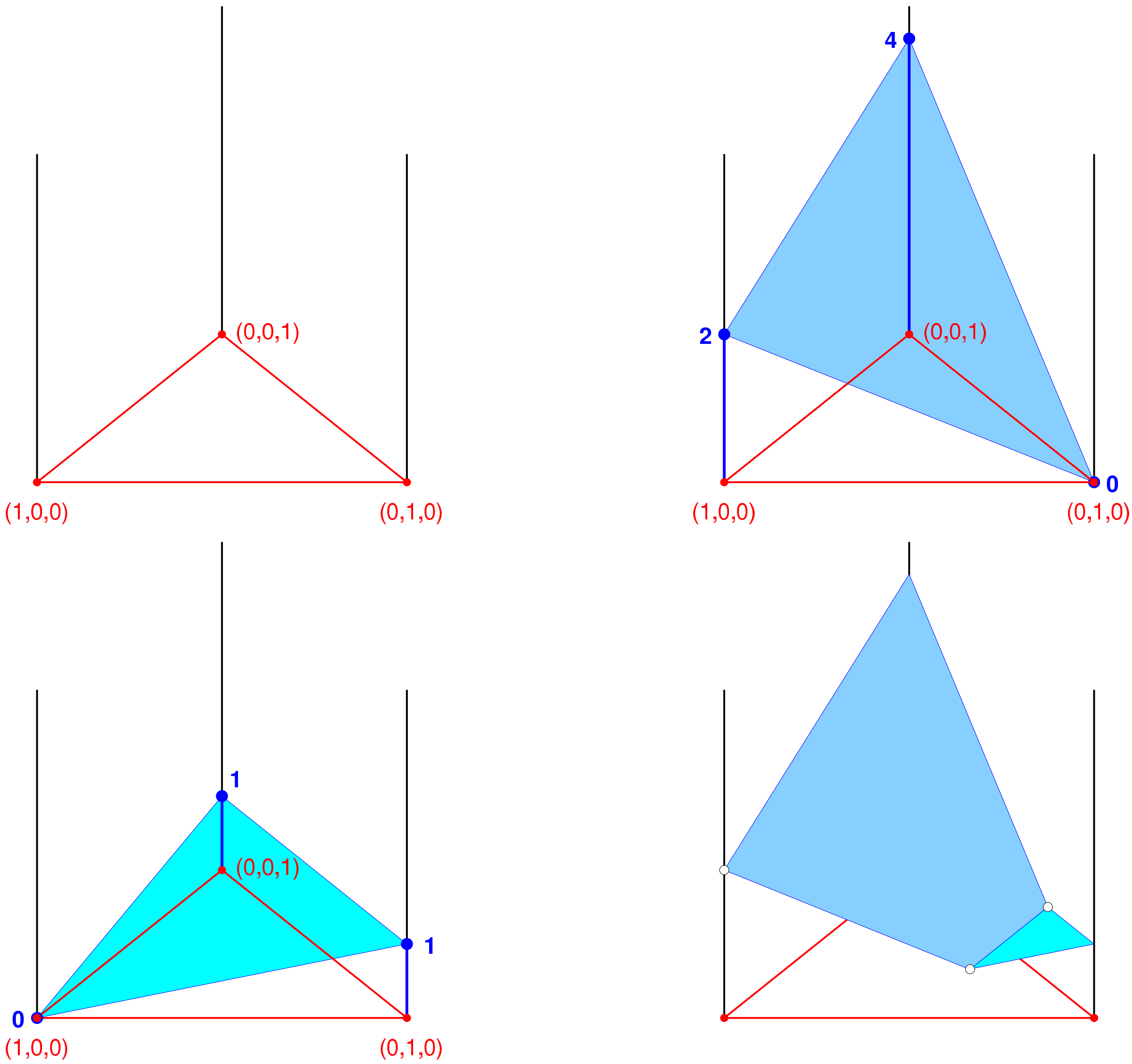

Figure 3.8 Expected payoffs to player II as a function of the mixed strategy (x_1,x_2,x_3) of player I for the 3×2 game in figure 3.5. Top left: goalposts for the three pure strategies $C=(1,0,0)$, $S=(0,1,0)$, $B=(0,0,1)$; top right: plane of expected payoffs for the left column c, with payoffs $2,0,4$ against C,S,B; bottom left: plane of expected payoffs for the right column s, with payoffs $0,1,1$; bottom right: *upper envelope* (maximum) of expected payoffs, with small white circles indicating equilibrium strategies, which match the right picture of figure 3.7.

3.13 Degenerate games

In this section, we treat games that have to be analysed more carefully in order to find all their Nash equilibria. We completely analyse an example, and then give the definition of a *degenerate game* that applies to 2×2 games like in this example. We then discuss larger games, and give the general definition 3.5 when an $m\times n$ game is degenerate.

Figure 3.9 is the strategic form, with best response payoffs, of the "threat game" of figure 2.3. It is similar to the battle of sexes game in that it has two pure Nash equilibria, here (T,l) and (B,r). As in the battle of sexes game, we should expect an additional mixed strategy equilibrium, which indeed exists. Assume this equilibrium is given by the mixed strategy $(1-p,p)$ for player I and $(1-q,q)$ for player II. The difference trick gives $q=1/2$ so that player I is indifferent between T and B and can mix, and for player I the difference trick gives $p=0$ so that player II is indifferent between l and r.

In more detail, if player I plays T and B with probabilities $1-p$ and p, then the expected payoff to player II is $3(1-p)$ for her strategy l, and $3(1-p)+2p$ for r, so that she is indifferent between l and r only when $p=0$. Indeed, because l and r give the same payoff when player I plays T, but the payoff for r is larger when player I plays B, any positive probability for B would make r the unique best response of player II, against which B is the unique best response of player I. Hence, the only Nash equilibrium where B is played with positive probability is the pure-strategy equilibrium (B,r).

I \ II	l	r
T	$\boxed{1}$, $\boxed{3}$	1, $\boxed{3}$
B	0, 0	$\boxed{2}$, $\boxed{2}$

Figure 3.9 Strategic form of the threat game, which is a degenerate game.

Consequently, a mixed-strategy equilibrium of the game is $((1,0),(1/2,1/2))$. However, only player II uses a properly mixed strategy in this equilibrium, whereas the strategy of player I is the pure strategy T. The best response condition requires that player II is indifferent between her pure strategies l and r because both are played with positive probability, and this implies that T is played with probability one, as just described. For player I, however, the best response condition requires that T has maximal expected payoff, which does *not* have to be the same as the expected payoff for B because B does not have positive probability! That is to say, player I's indifference between T and B, which we have used to determine player II's mixed strategy $(1-q,q)$ as $q=1/2$, is too strong a requirement. All we need is that T is a best response to this mixed strategy, so the expected payoff 1 to player I when he plays T has to be *at least as large* as his expected payoff $2q$ when he plays B, that is, $1 \geq 2q$. In other words, the equation $1=2q$ has to be replaced by an *inequality*.

The inequality $1 \geq 2q$ is equivalent to $0 \leq q \leq 1/2$. With these possible values for q, we obtain an infinite set of equilibria $((1,0),(1-q,q))$. The two extreme cases for q, namely $q=0$ and $q=1/2$, give the pure-strategy equilibrium (T,l) and the mixed equilibrium $((1,0),(1/2,1/2))$ that we found earlier.

Note: An equilibrium where at least one player uses a mixed strategy that is not a pure strategy is always called a mixed equilibrium. Also, when we want to find "all mixed equilibria” of a game, we usually include the pure-strategy equilibria among them (which are at any rate easy to find).

The described set of equilibria (where player II uses a mixed strategy probability q from some interval) has an intuition in terms of the possible “threat” in this game that player II plays l (with an undesirable outcome after player I has chosen B, in the game tree in figure 2.3(a)). First, r weakly dominates l, so if there is any positive probability that player I ignores the threat and plays B, then the threat of playing l at all is not sustainable as a best response, so any equilibrium where player II does play l requires that player I’s probability p for playing B is zero. Second, if player I plays T, then player II is indifferent between l and r and she can in principle play anything, but in order to maintain the threat T must be a best response, which requires that the probability $1-q$ for l is sufficiently high. Player I is indifferent between T and B (and therefore T still optimal) when $1-q=1/2$, but whenever $1-q \geq 1/2$ the threat works as well. In other words, “threats” can work even when the threatened action with the undesirable outcome is uncertain, as long as its probability is sufficiently high.

The complications in this game arise because the game in figure 3.9 is “degenerate” according to the following definition.

Definition 3.4 A 2×2 game is called *degenerate* if some player has a pure strategy with two pure best responses of the other player.

A degenerate game is complicated to analyse because one player can use a pure strategy (like T in the threat game) so that the other player can mix between her two best responses, but the mixed strategy probabilities are not constrained by the equation that the other player has to be indifferent between his strategies. Instead of that equation, it suffices to fulfil the *inequality* that the first player’s pure strategy is a best response. Note that the mentioned equation, as well as the inequality, is a consequence of the best response condition, theorem 3.1.

⇒ Answer exercise 3.6 on page 93.

I \ II	l	c	r
T	0, $\boxed{2}$	$\boxed{3}$, 1	$\boxed{2}$, 0
B	$\boxed{1}$, 0	2, 1	0, $\boxed{2}$

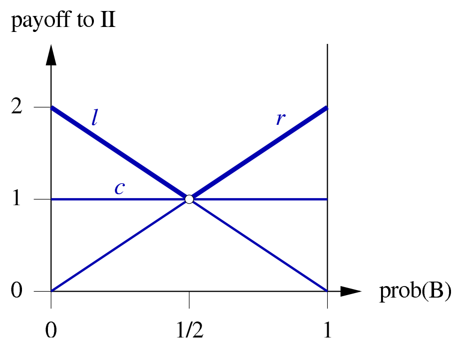

Figure 3.10 Degenerate 2×3 game and upper envelope of expected payoffs to player II.

Degeneracy occurs for larger games as well, which requires a more general definition. Consider the 2×3 game in figure 3.10. Each pure strategy has a unique best response, so the condition in definition 3.4 does not hold. This game does not have a pure-strategy equilibrium, so both players have to use mixed strategies. We apply the upper envelope method, plotting the payoff to player II as a function of the probability that player I plays B, shown on the right in figure 3.10. Here, the lines of expected payoffs cross in a single point, and all three pure strategies of player II are best responses when player I uses the mixed strategy $(1/2, 1/2)$.

How should we analyse this game? Clearly, we only have an equilibrium if both players mix, which requires the mixed strategy $(1/2, 1/2)$ for player I. Then player I has to be made indifferent, and in order to achieve this, player II can mix between any of her three best responses l, c, and r. We now have the same problem as earlier in the threat game, namely too much freedom: Three probabilities for player II, call them y_l, y_c, and y_r, have to be chosen so that $y_l + y_c + y_r = 1$, and so that player I is indifferent between his pure strategies T and B, which gives the equation $3y_c + 2y_r = y_l + 2y_c$. These are three unknowns subject to two equations, with an underdetermined solution.

We give a method to find out all possible probabilities, which can be generalised to the situation that three probabilities are subject to only two linear equations, like here. First, sort the coefficients of the probabilities in the equation $3y_c + 2y_r = y_l + 2y_c$, which describes the indifference of the expected payoffs to the other player, so that each probability appears only once and so that its coefficient is *positive*, which here gives the equation $y_c + 2y_r = y_l$. Of the three probabilities, normally two appear on one side of the equation and one on the other side. The "extreme solutions" of this equation are obtained by setting either of the probabilities on the side of the equation with two probabilities to zero, here either $y_c = 0$ or $y_r = 0$. In the former case, this gives the solution $(2/3, 0, 1/3)$ for (y_l, y_c, y_r), in the latter case the solution $(1/2, 1/2, 0)$. In general, we have $0 \le y_c \le 1/2$, which determines the remaining probabilities as $y_l = y_c + 2y_r = y_c + 2(1 - y_l - y_c) = 2 - 2y_l - y_c$ or $y_l = 2/3 - y_c/3$, and $y_r = 1 - y_l - y_c = 1 - (2/3 - y_c/3) - y_c = 1/3 - 2y_c/3$.

Another way to find the extreme solutions to the underdetermined mixed strategy probabilities of player II is to simply ignore one of the three best responses and apply the difference trick: Assume that player II uses only his best responses l and c. Then the difference trick, to make player I indifferent, gives probabilities $(1/2, 1/2, 0)$ for l, c, r. If player II uses only his best responses l and r, the difference trick gives the mixed strategy $(2/3, 0, 1/3)$. These are exactly the two mixed strategies just described. If player II uses only his best responses c and r, then the difference trick does not work because player I's best response to both c and r is always T.

The following is a definition of degeneracy for general $m \times n$ games.

Definition 3.5 (Degenerate game) A two-player game is called *degenerate* if some player has a mixed strategy that assigns positive probability to exactly k pure strategies so that the other player has more than k pure best responses to that mixed strategy.

In a degenerate game, a mixed strategy with "too many" best responses creates the difficulty that we have described with the above examples. Consider, in contrast, the "normal" situation that the game is non-degenerate.

Proposition 3.6 *In any equilibrium of a two-player game that is not degenerate, both players use mixed strategies that mix the same number of pure strategies.*

$\Rightarrow$ You are asked to give the easy and instructive proof of proposition 3.6 in exercise 3.7 on page 93.

In other words, in a non-degenerate game we have pure-strategy equilibria (both players use exactly one pure strategy), or mixed-strategy equilibria where both players mix between exactly two strategies, or equilibria where both player mix between exactly three strategies, and so on. In general, both players mix between exactly k pure strategies. Each of these strategies has a probability, and these probabilities are subject to k equations in order to make the other player indifferent between his or her pure strategies: One of these equations states that the probabilities sum to one, and the other $k-1$ equations state the equality between the first and second, second and third, etc., up to the $(k-1)$st and kth strategy of the other player. These equations have unique solutions. (It can be shown that non-unique solutions can occur only in a degenerate game.) These solutions have to be checked for the equilibrium property: The resulting probabilities have to be non-negative, and the unused pure strategies of the other player must not have a higher payoff. Finding mixed strategy probabilities by equating the expected payoffs to the other player no longer gives unique solutions in degenerate games, as we have demonstrated.

Should we or should we not care about degenerate games? In a certain sense, degenerate games can be ignored when considering games that are given in strategic form, and where each cell of the payoff table arises from a different circumstance of the interactive situation that is modelled by the game. In that case, it is "unlikely" that two payoffs are identical, which is necessary to get two pure best responses to a pure strategy, which is the only case where a 2×2 game can be degenerate. A small change of the payoff will result in a unique preference of the other player. Similarly, it is unlikely that more than two lines defining the upper envelope of expected payoffs cross in one point, or that more than three planes of expected payoffs against a mixed strategy that mixes three pure strategies (like in the two planes in figure 3.8) cross in one point, and so on. In other words, "generic" (or "almost all") games in strategic form are not degenerate. On the other hand, degeneracy is very likely when looking at a game tree, because a payoff of the game tree will occur repeatedly in the strategic form, as demonstrated by the threat game.

We conclude with a useful proposition that can be used when looking for all equilibria of a degenerate game. In the threat game in figure 3.9, there are two equilibria $((1,0),(1,0))$ and $((1,0),(1/2,1/2))$ that are obtained, essentially, by "ignoring" the degeneracy. These two equilibria have the same strategy $(1,0)$ of player I. Similarly, the game in figure 3.10 has two mixed equilibria $((1/2,1/2),(1/2,1/2,0))$ and $((1/2,1/2),(2/3,0,1/3))$, which again share the same strategy of player I. The following proposition states that two mixed equilibria that have the same strategy of one player (here stated for player I, but it holds similarly for the other player) can be combined by convex combinations to obtain further equilibria, as it is the case in the above examples.

Proposition 3.7 *Consider a bimatrix game (A,B) with X as the set of mixed strategies of player I, and Y as the set of mixed strategies of player II. Suppose that (x,y) and (x',y)*

are equilibria of the game, where $x, x' \in X$ and $y \in Y$. Then $((1-p)x + px'), y)$ is also an equilibrium of the game, for any $p \in [0,1]$.

Proof. Let $\bar{x} = (1-p)x + px'$. Clearly, $\bar{x} \in X$ because $\bar{x}_i \geq 0$ for all pure strategies i of player I, and

$$\sum_{i=1}^{m} \bar{x}_i = \sum_{i=1}^{m} ((1-p)x_i + px'_i) = (1-p)\sum_{i=1}^{m} x_i + p\sum_{i=1}^{m} x'_i = (1-p) + p = 1.$$

The pair $(\bar{x}, y)$ is an equilibrium if $\bar{x}$ is a best response to y and vice versa. For any mixed strategy $\tilde{x}$ of player I, we have $xAy \geq \tilde{x}Ay$, and $x'Ay \geq \tilde{x}Ay$, and consequently $\bar{x}Ay = ((1-p)x + px')Ay = (1-p)xAy + px'Ay \geq (1-p)\tilde{x}Ay + p\tilde{x}Ay = \tilde{x}Ay$, which shows that $\bar{x}$ is a best response to y. In other words, these inequalities hold because they are preserved under convex combinations. In a similar way, y is a best response to x and x', that is, for any $\tilde{y} \in Y$ we have $xBy \geq xB\tilde{y}$ and $x'By \geq x'B\tilde{y}$. Again, taking convex combinations shows $\bar{x}By \geq \bar{x}B\tilde{y}$. □

We come back to the example of figure 3.10 to illustrate proposition 3.7: This game has two equilibria (x, y) and (x, y') where $x = (1/2, 1/2)$, $y = (1/2, 1/2, 0)$, and $y' = (2/3, 0, 1/3)$, so this is the situation of proposition 3.7, except that the two equilibria have same mixed strategy of player I rather than player II. Consequently, for any $\bar{y}$ which is a convex combination of y and y' we have a Nash equilibrium $(x, \bar{y})$. The equilibria (x, y) and (x, y') are "extreme" in the sense that y and y' have as many zero probabilities as possible, which means that y and y' are the endpoints of the line segment consisting of equilibrium strategies $\bar{y}$ (that is, $(x, \bar{y})$ is an equilibrium), where $\bar{y} = (1-p)y + py'$.

⇒ Express the set of Nash equilibria for the threat game in figure 3.9 with the help of proposition 3.7.

3.14 Zero-sum games

Zero-sum games are games of two players where the interests of the two players are directly opposed: One player's loss is the other player's gain. Competitions between two players in sports or in parlor games can be thought of as zero-sum games. The combinatorial games studied in chapter 1 are also zero-sum games.

Consider a zero-sum game in strategic form. If the row player chooses row i and the column player chooses column j, then the payoff $a(i, j)$ to the row player can be considered as a cost to the column player which she tries to minimise. That is, her payoff is given by $b(i, j) = -a(i, j)$, that is, $a(i, j) + b(i, j) = 0$, which explains the term "zero-sum". The zero-sum game is completely specified by the matrix A with entries $a(i, j)$ of payoffs to the row player, and the game is therefore often called a *matrix game*. In principle, such a matrix game is a special case of a bimatrix game (A, B), where the column player's payoff matrix is $B = -A$. Consequently, we can analyse matrix games just like bimatrix games. However, zero-sum games have additional strong properties, which are the subject of this section, that do not hold for general bimatrix games.

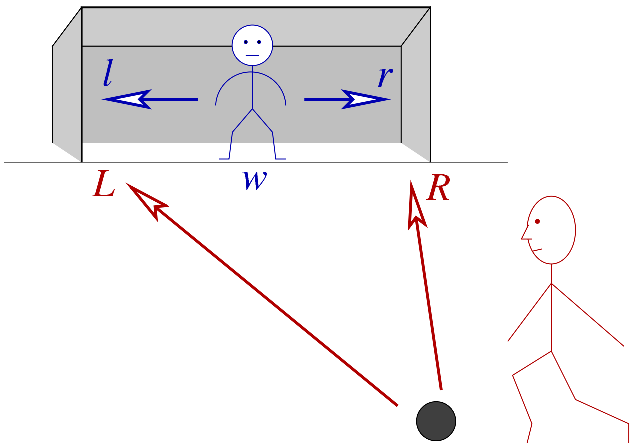

Figure 3.11 Strategic options for striker and goalkeeper in a football penalty.

Figure 3.11 shows the football penalty kick as a zero-sum game between striker and goalkeeper. For simplicity, assume that the striker's possible strategies are L and R, that is, to kick into the left or right corner, and that the goalkeeper's strategies are l, w, r, where l and r mean that he jumps immediately into the left or right corner, respectively, and w that he waits and sees where the ball goes and jumps afterwards. On the left in figure 3.12, we give an example of resulting probabilities that the striker scores a goal, depending on the choices of the two players. Clearly, the striker (player I) tries to maximise and the goalkeeper (player II) to minimise that probability, and we only need this payoff to player I to specify the game. The two players in a zero-sum game are therefore often called "Max" and "min", respectively (where we keep our tradition of using lower case letters for the column player). In reality, one would expect higher scoring probabilities when the goalkeeper chooses w, but the given numbers lead to a more interesting equilibrium. In this case, the striker does not score particularly well, especially when he kicks into the right corner.

Max \ min	l	w	r
L	0.5	0.7	1.0
R	1.0	0.6	0.4

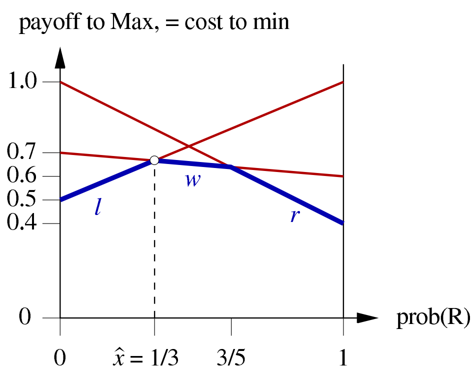

Figure 3.12 Left: the football penalty as a zero-sum game with scoring probabilities as payoffs to the striker (row player), which the goalkeeper (column player) tries to minimise. Right: lower envelope of best response costs to the minimiser.

In the payoff table in figure 3.12, we have indicated the pure best responses with little boxes, in the lower left for the row player and in the upper right for the column player, similar to the boxes put around the payoffs in a bimatrix game when each cell has two numbers. (Other conventions are to underline a best response cost for the minimiser, and to put a line above the best response payoff for the maximiser, or to put a circle or box, respectively, around the number.) When considering these best responses, remember that the column player minimises.

The best responses show that the game has no pure-strategy equilibrium. This is immediate from the sports situation: obviously, the goalkeeper's best response to a kick into the right or left corner is to jump into that corner, whereas the striker would rather kick into the opposite corner chosen by the goalkeeper. We therefore have to find a mixed equilibrium, and active randomisation is obviously advantageous for both players. Conveniently, the resulting expected scoring probability is just a probability, so the players are risk-neutral with respect to the numbers in the game matrix, which therefore represent the expected-utility function (as discussed in section 3.4) for player I, and cost for player II.

We first find the equilibria of the game as we would do this for a 2×3 game, except that we only use the payoffs to player I. Of course, this game must be solved with the goalpost method, because it is about football! The right picture in figure 3.12 gives the expected payoffs to player I as a function of his probability, say x, of choosing R, depending on the reponses l, w, or r of player II. These payoffs to player I are costs to player II, so the best-response costs to player II are given by the minimum of these lines, which defines the *lower envelope* shown by bold line segments. There are two intersection points on the lower envelope, of the lines for l and w when $x = 1/3$, and of the lines for w and r when $x = 3/5$, which are easily found with the difference trick (where it does not matter if one considers costs or payoffs, because the differences are the same in absolute value). So player II's best response to the mixed strategy $(1-x,x)$ is l for $0 \leq x \leq 1/3$, and w for $1/3 \leq x \leq 3/5$, and r for $3/5 \leq x \leq 1$. Only for $x = 1/3$ and for $x = 3/5$ can player II mix, and only in the first case, when l and w are best responses, can player I, in turn, be made indifferent. This gives the unique mixed equilibrium of the game, $((2/3, 1/3), (1/6, 5/6, 0))$. The resulting expected costs for the three columns l, w, r are $2/3$, $2/3$, $4/5$, where indeed player II assigns positive probability only to the smallest-cost columns l and w. The expected payoffs for both rows are $2/3$.

So far, nothing is new. Now, we consider the lower envelope of expected payoffs to player I (as a function of his own mixed strategy defined by x, which for simplicity we call "the mixed strategy x") from that player's own perspective: It represents the worst possible expected payoff to player I, given the possible responses of player II. If player I wants to "secure" the maximum of this worst possible payoff, he should maximise over the lower envelope with a suitable choice of x. This maximum is easily found for $\hat{x} = 1/3$, shown by a dashed line in figure 3.12.

The strategy $\hat{x}$ is called a *max-min strategy*, and the resulting payoff is called his *max-min payoff*. In general, let A be the matrix of payoffs to player I and let X and Y be the sets of mixed strategies of player I and player II, respectively. Then a max-min strategy of player I is a mixed strategy $\hat{x}$ with the property

$$\min_{y \in Y} \hat{x}Ay = \max_{x \in X} \min_{y \in Y} xAy, \tag{3.7}$$

which means the following: The left-hand side of (3.7) shows the smallest payoff that player I gets when he plays strategy $\hat{x}$, assuming that player II chooses a strategy y that minimises player I's payoff. The right-hand side of (3.7) gives the largest such "worst-case" payoff over all possible choices of x. The equality in (3.7) states that $\hat{x}$ is the strategy of player I that achieves this best possible worst-case payoff. It is therefore also sometimes called a *security* strategy of player I, because it defines what player I can secure to get (in terms of expected payoffs).

As a side observation, it is easy to see that the payoff $\min_{y\in Y} xAy$ which describes the "worst case" payoff to player I when he plays x can be obtained by only looking at the *pure* strategies, the columns j, of player II. In other words, $\min_{y\in Y} xAy = \min_j (xA)_j$. The reason is that with a mixed strategy y, player II cannot produce a worse expected payoff to player I than with a pure strategy j, because the expectation represents a "weighted average" of the pure-strategy payoffs $(xA)_j$ with the probabilities y_j as weights, so it is never worse than the smallest of these pure-strategy payoffs (which is proved in the same way as the best-response condition theorem 3.1). However, we keep the minimisation over all mixed strategies y in (3.7) because we will later exchange taking the maximum and the minimum. For player I, the max-min strategy $\hat{x}$ is better if he can use a mixed strategy, as the football penalty game shows: If he was restricted to playing pure strategies only, his (pure-strategy) max-min payoff $\max_i \min_j a(i,j)$ would be 0.5, with pure max-min strategy L, rather than the max-min payoff $2/3$ when he is allowed to use a mixed strategy.

The max-min payoff in (3.7) is well defined and does not depend on what player II does, because we always take the minimum over player II's choices y. The function $\min_{y\in Y} xAy$ is a continuous function of x, and x is taken from the compact set X, so there is some $\hat{x}$ where the maximum $\max_{x\in X} \min_{y\in Y} xAy$ of that function is achieved. The maximum (and thus the max-min payoff) is unique, but there may be more than one choice of $\hat{x}$, for example if in the game in figure 3.12, the payoff 0.7 for the strategy pair (L,w) was replaced by 0.6, in which case the lower envelope would have its maximum 0.6 along an interval of probabilities that player I chooses R, with any $\hat{x}$ in that interval defining a max-min strategy.

Recall how the max-min strategy $\hat{x}$ is found graphically over the lower envelope of expected payoffs. The max-min strategy $\hat{x}$ identifies the "peak" of that lower envelope. The "hat" accent $\hat{x}$ may be thought of as representing that peak.

$\Rightarrow$ Try exercise 3.8 on page 93, which is a variant of the football penalty game.

A max-min strategy, and the max-min payoff, only depends on the player's *own* payoffs. It can therefore also be defined for a game with general payoffs which are not necessarily zero-sum. In that case, a max-min strategy represents a rather pessimistic view of the world: Player I chooses a strategy $\hat{x}$ that maximises his potential loss against an opponent who has nothing else on her mind than harming player I. In a *zero-sum* game, however, that view is eminently rational: Player II is not trying to harm player I, but she is merely trying to maximise her own utility. This is because her payoffs are directly opposed those of player I, her payoff matrix being given by $B = -A$. So the expression $\min_{y\in Y} xAy$ in (3.7) can also be written as $-\max_{y\in Y} xBy$ which is the (negative of the) best response payoff to player II. This is exactly what player II tries to do when maximising her payoff against the strategy x of player I. As described above, the lower envelope of payoffs to player I is just the negative of the upper envelope of payoffs to player II, which defines her best responses.

So a max-min strategy is a natural strategy to look at when studying a zero-sum game. Our goal is now to show that in a zero-sum game, a max-min strategy is the same as an equilibrium strategy. This has some striking consequences, for example the uniqueness of the equilibrium payoff in any equilibrium of a zero-sum game, which we will discuss.

We consider such a max-min strategy not only for player I but also for player II, whose payoff matrix is $B = -A$. According to (3.7), this is a mixed strategy $\hat{y}$ of player II with the property

$$\min_{x\in X} xB\hat{y} = \max_{y\in Y} \min_{x\in X} xBy,$$

which we re-write as

$$-\max_{x\in X} xA\hat{y} = -\min_{y\in Y} \max_{x\in X} xAy$$

or, omitting the minus signs,

$$\max_{x\in X} xA\hat{y} = \min_{y\in Y} \max_{x\in X} xAy. \tag{3.8}$$

Equation (3.8) expresses everything in terms of a single matrix A of payoffs to player I, which are costs to player II that she tries to minimise. In order to simplify the discussion of these strategies, a max-min strategy of player II (in terms of her own payoffs) is therefore also called a *min-max* strategy (with the min-max understood in terms of her own costs, which are the payoffs to her opponent). So a min-max strategy of player II is a strategy $\hat{y}$ that minimises the worst-case cost $\max_{x\in X} xA\hat{y}$ that she has to pay to player I, as stated in (3.8).

The first simple relationship between the max-min payoff to player I in a zero-sum game and the min-max cost to player II is

$$\max_{x\in X} \min_{y\in Y} xAy \le \min_{y\in Y} \max_{x\in X} xAy. \tag{3.9}$$

The left-hand side is the "secure" payoff that player I can guarantee to himself (on average) when using a max-min strategy $\hat{x}$, the right-hand side the "secure" cost that player II can guarantee to herself to pay at most (on average) when using a min-max strategy $\hat{y}$. It is clear that this is the only direction in which the inequality can hold if that "guarantee" is to be meaningful. For example, if player I could guarantee to get a payoff of 10, and player II could guarantee to pay at most 5, something would be wrong. The proof of (3.9) is similarly simple:

$$\max_{x\in X} \min_{y\in Y} xAy = \min_{y\in Y} \hat{x}Ay \le \hat{x}A\hat{y} \le \max_{x\in X} xA\hat{y} = \min_{y\in Y} \max_{x\in X} xAy.$$

The main theorem on zero-sum games states the *equation* "max min = min max", to be understood for the payoffs, that is, equality instead of an inequality in (3.9). This is also called the "minimax" theorem, and it is due to von Neumann.

The next theorem clarifies the connection between max-min strategies (which for player II we called min-max strategies) and the concept of Nash equilibrium strategies. Namely, for zero-sum games they are the same. Furthermore, the existence of a Nash equilibrium strategy pair (x^*,y^*) implies the minimax theorem, that is, equality in (3.9).

Theorem 3.8 (von Neumann's minimax theorem) *Consider a zero-sum game with payoff matrix A to player I. Let X and Y be the sets of mixed strategies of player I and II, respectively, and let $(x^*,y^*) \in X \times Y$. Then (x^*,y^*) is a Nash equilibrium if and only if x^* is a max-min strategy and y^* is a min-max strategy, that is,*

$$\min_{y\in Y} x^*Ay = \max_{x\in X}\ \min_{y\in Y} xAy, \qquad \max_{x\in X} xAy^* = \min_{y\in Y}\ \max_{x\in X} xAy. \tag{3.10}$$

Furthermore,

$$\max_{x\in X}\ \min_{y\in Y} xAy = \min_{y\in Y}\ \max_{x\in X} xAy. \tag{3.11}$$

Proof. Consider a max-min strategy $\hat{x}$ in X and a min-max strategy $\hat{y}$ in Y, which fulfil (3.10) in place of x^* and y^*, respectively. Such strategies exist because X and Y are compact sets. Let (x^*,y^*) be any Nash equilibrium of the game, which exists by Nash's theorem 3.3. The equilibrium property is equivalent to the "saddle point" property

$$\max_{x\in X} xAy^* = x^*Ay^* = \min_{y\in Y} x^*Ay, \tag{3.12}$$

where the left equation states that x^* is a best response to y^* and the right equation states that y^* is a best response to x^* (recall that $B = -A$, so player II tries to minimise her expected costs xAy). Then

$$\hat{x}A\hat{y} \le \max_{x\in X} xA\hat{y} = \min_{y\in Y}\ \max_{x\in X} xAy \le \max_{x\in X} xAy^* = x^*Ay^* \tag{3.13}$$

and

$$\hat{x}A\hat{y} \ge \min_{y\in Y} \hat{x}Ay = \max_{x\in X}\ \min_{y\in Y} xAy \ge \min_{y\in Y} x^*Ay = x^*Ay^*, \tag{3.14}$$

so all these inequalities hold as equalities. With the second inequality in both (3.13) and (3.14) as an equality we get (3.10), that is, y^* is a min-max and x^* is a max-min strategy. Having the first inequality in both (3.13) and (3.14) as an equality gives

$$\max_{x\in X} xA\hat{y} = \hat{x}A\hat{y} = \min_{y\in Y} \hat{x}Ay,$$

which says that $(\hat{x},\hat{y})$ is a Nash equilibrium. Furthermore, we obtain (3.11). □

One consequence of theorem 3.8 is that in any Nash equilibrium of a zero-sum game, a player receives the same payoff, namely the max-min payoff to player I in (3.11), which is the min-max cost to player II. This payoff is also called the *value* of the zero-sum game.

Another important consequence of the theorem 3.8 is that the concept of "Nash equilibrium strategy" in a zero-sum game is *independent* of what the opponent does, because it is the same as a max-min strategy (when considered in terms of the player's own payoffs). This is far from being the case in general-payoff games, as, for example, the game of "chicken" demonstrates. In particular, if player I's strategies x^* and $\bar{x}$ are both part of a Nash equilibrium in a zero-sum game, they are part of *any* Nash equilibrium. So Nash equilibrium strategies are interchangeable: if (x^*, y^*) and $(\bar{x}, \bar{y})$ are Nash equilibria, so are $(x^*, \bar{y})$ and $(\bar{x}, y^*)$. The reason is that x^* and $\bar{x}$ are then max-min strategies, and y^* and $\bar{y}$ are min-max strategies, which has nothing to do with what the opponent does.

A further consequence is that if (x^*, y^*) is a Nash equilibrium and y^* is the *unique* best response (for example, this is often the case when y^* is a pure strategy), then y^* is also the unique min-max strategy. The reason is that if there was another min-max strategy $\hat{y}$ that is different from y^*, then $(x^*, \hat{y})$ would be a Nash equilibrium, and consequently $\hat{y}$ would be a best response to x^*, contradicting the uniqueness of the best response y^*. Similarly, if, say, $\hat{x}$ is the unique max-min strategy of player I (for example, obtained by plotting the lower envelope against x as in figure 3.12), it is also the only Nash equilibrium strategy of player I.

These properties of zero-sum games make equilibrium strategies, or equivalently max-min/min-max strategies, a particularly convincing solution concept. They are also called "optimal" strategies for the players.

3.15 Exercises for chapter 3

Exercise 3.1 shows how to find all mixed equilibria of a small bimatrix game, which is however not a 2×2 game. It is useful to attempt this exercise as early as possible, because this will help you appreciate the methods developed later. The given solution to this exercise does not use the upper envelope method, but you should try using that method once you have learned it; it will help you solve the exercise faster. Exercise 3.2 is a (not easy) continuation of exercise 2.7, and asks about mixed equilibria in a three-player game tree. In exercise 3.3, you are asked to derive the "difference trick". Exercise 3.4 demonstrates the power of the upper envelope method for $2 \times n$ games. In exercise 3.5, you should understand the concept of dominated strategies with the help of of the upper-envelope diagram, which provides a useful geometric insight. A degenerate game is treated in exercise 3.6. Exercise 3.7 shows that in non-degenerate games, Nash equilibria are always played on square games that are part of the original game, in the sense that both players mix between the same number of pure strategies. Exercise 3.8 is a variant of the football penalty.

Exercise 3.1 Find all Nash equilibria of the following 3×2 game, including equilibria consisting of mixed strategies.

I \ II	l	r
T	0, 1	6, 0
M	2, 0	5, 2
B	3, 3	3, 4

Exercise 3.2 Consider the three-player game tree in figure 2.22 on page 56, already considered in exercise 2.7.

(a) Is the following statement true or false? Justify your answer.

For each of players I, II, or III, the game has a Nash equilibrium in which that player plays a mixed strategy that is not a pure strategy.

(b) Is the following statement true or false? Justify your answer.

In every Nash equilibrium of the game, at least one player I, II, or III plays a pure strategy.

Exercise 3.3 In the following 2×2 game, A,B,C,D are the payoffs to player I, which are real numbers, no two of which are equal. Similarly, a,b,c,d are the payoffs to player II, which are real numbers, also no two of which are equal.

I \ II	left	right
Top	A, a	B, b
Bottom	C, c	D, d

(a) Under which conditions does this game have a mixed equilibrium which is not a pure-strategy equilibrium?
[*Hint: Think in terms of the arrows, and express this using relationships between the numbers, as, for example,* $A > C$.]

(b) Under which conditions in (a) is this the only equilibrium of the game?

(c) Consider one of the situations in (b) and compute the probabilities $1-p$ and p for playing "Top" and "Bottom", respectively, and $1-q$ and q for playing "left" and "right", respectively, that hold in equilibrium.

(d) For the solution in (c), give a simple formula for the quotients $(1-p)/p$ and $(1-q)/q$ in terms of the payoff parameters. Try to write this formula such that denominator and numerator are both positive.

(e) Show that the mixed equilibrium strategies do not change if player I's payoffs A and C are replaced by $A+E$ and $C+E$, respectively, for some constant E, and similarly if player II's payoffs a and b are replaced by $a+e$ and $b+e$, respectively, for some constant e.

Exercise 3.4 Consider the following 2×5 game:

I \ II	a	b	c	d	e
T	2	4	3	5	0
	0	2	1	0	3
B	7	4	5	0	8
	1	0	4	1	0

(a) Draw the expected payoffs to player II for all her strategies a, b, c, d, e, in terms of the probability p, say, that player I plays strategy B. Indicate the best responses of player II, depending on that probability p.

(b) Using the diagram in (a), find all pure or mixed equilibria of the game.

Exercise 3.5 Consider the quality game in figure 2.10.

(a) Use the "goalpost" method twice, to draw the upper envelope of best-response payoffs to player I against the mixed strategy of player II, and vice versa.

(b) Explain which of these diagrams shows that a pure strategy dominates another pure strategy.

(c) Explain how that diagram shows that if a strategy s dominates a strategy t, then t is never a best response, even against a *mixed* strategy of the other player.

Exercise 3.6 Find all equilibria of the following degenerate game, which is a variant of the inspection game in figure 3.2. The difference is that player II, the inspectee, derives no gain from acting illegally (playing r) even if she is not inspected (player I choosing T).

I \ II	l	r
T	0	0
	0	-10
B	0	-90
	-1	-6

Exercise 3.7 Prove proposition 3.6.

Exercise 3.8 Consider the duel in a football penalty kick. The following table describes this as a zero-sum game. It gives the probability of scoring a goal when the row player (the striker) adopts one of the strategies L (shoot left), R (shoot right), and the column player (the goalkeeper) uses the strategies l (jump left), w (wait then jump), r (jump right). The row player is interested in maximising and the column player in minimising the probability of scoring a goal.

Max \ min	l	w	r
L	0.6	0.7	1.0
R	1.0	0.8	0.7

(a) Find an equilibrium of this game in pure or mixed strategies, and the equilibrium payoff. Why is the payoff unique?

(b) Now suppose that player I has an additional strategy M (shoot in the middle), so that the payoff matrix is

Max \ min	l	w	r
L	0.6	0.7	1.0
R	1.0	0.8	0.7
M	1.0	0.0	1.0

Find an equilibrium of this game, and the equilibrium payoff.
[*Hint: The result from (a) will be useful.*]