Term paper Presentation

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Optical Flow

- Optical flow is used to compute the motion of the pixels of an image sequence. It provides a dense (point to point) pixel correspondence.
- *Correspondence problem:* determine where the pixels of an image at time **t** are in the image at time **t+1**.

Assumptions: **BC (Grey value constancy assumption)**

❖It is assumed that the grey value of a pixel is not changed by the displacement.

$$I(x,y,\mathrm{t}) = I(x+u, y+v, \mathrm{t}+1) \qquad , I:\Omega \subset \mathbb{R}^3 \to \mathbb{R} \qquad (1)$$

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + ... \qquad \text{General Taylor series expansion}$$

$$I(x+u, y+v, \mathrm{t}+1) = I(x,y,\mathrm{t}) + \frac{\partial I(x,y,\mathrm{t})}{\partial x}u + \frac{\partial I(x,y,\mathrm{t})}{\partial y}v + \frac{\partial I(x,y,\mathrm{t})}{\partial t} + ...$$

$$I(x+u, y+v, \mathrm{t}+1) = I(x,y,\mathrm{t}) + \nabla I_x u + \nabla I_y v + I_t + ...$$

$$I_x u_x + I_y v_y + I_t = 0$$

$$\nabla I.\,\mathbf{u} + I_t = 0$$

❖Displacement vector $\mathbf{w} := (u, v, 1)$

Assumptions: **GC (Gradient constancy assumption.)**

❖It is assumed that the gradient value of a pixel is not changed by the displacement.

$$\nabla I(x,y,\mathrm{t}) = \nabla I(x+u, y+v, \mathrm{t}+1) \qquad (2)$$

$$\nabla = (\partial_x, \partial_y)'$$

❖**Smoothness assumption:** Outliers in estimation, vanishing gradient

How to solve optical flow constraints ?

$$I_x u_x + I_y v_y + I_t = 0$$

$$\nabla I(x,y,t) = \nabla I(x+u, y+v, t+1)$$

- local methods:
  - Lucas–Kanade technique: One equation two unknowns

$$\underset{25\times 2}{\underbrace{\begin{bmatrix} I_x(\mathbf{p}_1) & I_y(\mathbf{p}_1) \\ I_x(\mathbf{p}_2) & I_y(\mathbf{p}_2) \\ \vdots & \vdots \\ I_x(\mathbf{p}_{25}) & I_y(\mathbf{p}_{25}) \end{bmatrix}}_{A}} \underset{2\times 1}{\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{d}} = - \underset{25\times 1}{\underbrace{\begin{bmatrix} I_t(\mathbf{p}_1) \\ I_t(\mathbf{p}_2) \\ \vdots \\ I_t(\mathbf{p}_{25}) \end{bmatrix}}_{b}}$$

  - Bigun's structure tensor method
- Global methods:
  - Horn/Schunck
  - <u>Total variations</u>

Multiscale approach

- In the case of displacements that are larger than one pixel per frame, the cost functional in a variational formulation must be expected to be multi-modal,
  - i.e. a minimization algorithm could easily be trapped in a local minimum. In order to find the global minimum, it can be useful to apply multiscale ideas:

Horn/Schunck

$$E_{\text{Horn\_Schunck}}(\mathbf{u}) = \int_\Omega \left(\nabla I \cdot \mathbf{u} + \frac{\partial}{\partial t} I \right)^2 + \alpha \left(|\nabla u_1|^2 + |\nabla u_2|^2 \right).$$

- Penalizes high gradients of **u**
- Disallows discontinuities

Total variation: L1

- TV L1: $E(\mathbf{u}) = \lambda \int_\Omega |I_1(\mathbf{x}+\mathbf{u},t) - I_0(x,t)| dx + \int_\Omega |\nabla u_1| + |\nabla u_2|$

$$I_1(\mathbf{x}+\mathbf{u}) - I_0(\mathbf{x}) = I_1(\mathbf{x}+\mathbf{u}_0) - I_0(\mathbf{x}) + \nabla I_1(\mathbf{x}+\mathbf{u}_0)(\mathbf{u}-\mathbf{u}_0)$$
$$\rho(\mathbf{u}) = I_1(\mathbf{x}+\mathbf{u}_0) - I_0(\mathbf{x}) + \nabla I_1(\mathbf{x}+\mathbf{u}_0)(\mathbf{u}-\mathbf{u}_0)$$

Hence we can Write the above equation as

$$E(\mathbf{u}) = \int_\Omega |\nabla u_1| + |\nabla u_2| + \lambda |\rho(\mathbf{u})|.$$

Convex relaxation

$$E_\theta(\mathbf{u}, \mathbf{v}) = \int_\Omega |\nabla u_1| + |\nabla u_2| + \frac{1}{2\theta}|\mathbf{u} - \mathbf{v}|^2 + \lambda |\rho(\mathbf{v})| .$$

How to solve ?

1. Fixed $\mathbf{v}$, solve

$$\min_{\mathbf{u}} \int_\Omega |\nabla u_1| + |\nabla u_2| + \frac{1}{2\theta}|\mathbf{u} - \mathbf{v}|^2 .$$

2. Fixed $\mathbf{u}$, solve

$$\min_{\mathbf{v}} \int_\Omega \frac{1}{2\theta}|\mathbf{u} - \mathbf{v}|^2 + \lambda |\rho(\mathbf{v})| .$$

How to solve 1:In a nutshell

- The idea is to replace the optimization of the flow field *u* by the optimization of a vector field *P* that is related to *u* by $u = v - \lambda div(p)$ .The vector field is the one that minimizes

$$\min\left\{\|\lambda div\ p - u\|^2 \quad |P_{i,j}|^2 - 1 \le 0\ \forall\ i, j\ = 1, \ldots, N\right\},$$

- Solution 1: do Projected gradient descent as

$$p^{l+1} = \Pr oj\left(u^l - \tau\nabla\left(\lambda div\left(p^l\right) - \frac{u}{\lambda}\right)\right)$$

How to solve 1:Chambolle approach

- The total variation of *u* is defined by (In discrete setting)

$$J(u) = \sum_{1 \le i, j \le N} |(\nabla u)_{i,j}|, \qquad (1)$$

- The continues formulation of the TV norm is given by

$$J(u) = \sup\left\{ \int_\Omega u(x)\,\mathrm{div}\,\xi(x)\,dx : \ \xi \in C_c^1(\Omega; \mathbb{R}^2), |\xi(x)| \le 1\ \forall x \in \Omega \right\} \qquad (2)$$

Compact support, zero at the boundary

$$|\nabla u(x)| = \sup_{|\xi(x)| \le 1} \xi . \nabla u$$

$$\int u(x) div(\xi) dx = \cancel{\int (u(x)\xi(x))' dx} + \int \xi(x) \nabla u(x)$$

$$\int u(x) div(\xi) dx = \int \xi(x) \nabla u(x), \quad |\xi(x)| \le 1, \xi = \frac{\nabla u}{|\nabla u|}$$

$$= \int \frac{(\nabla u)^2}{|\nabla u|} = \int |\nabla u(x)|$$

$$(f g)' = f' g + f g'$$

How to solve 1: cont'

- Legendre transform : is a transformation from a convex differentiable function f(x) to a function that depends on the family of tangents s=∇_xf(x) is gives as

$$J^*(\mathbf{s}) = \mathbf{s}^T \mathbf{x}(\mathbf{s}) - J(\mathbf{x}(\mathbf{s}))$$

- Legendre-Fenchel transform: is a generalization of Legendre transform to non convex and non differentiable functions is defined as

$$J^*(v) = \sup_u \langle u, v \rangle_X - J(u)$$

With $\langle u, v \rangle_X = \sum_{i,j} u_{i,j} v_{i,j}$ "Characteristic function " function of closed convex set *K*

$$J^*(v) = \chi_K(v) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{otherwise.} \end{cases} \qquad (3)$$

How to solve 1: cont'

- Note: $J(u) = \sup_{v \in K} \langle u, v \rangle_X.$

And the closure set K $\left\{ \operatorname{div} \xi : \xi \in C_c^1(\Omega; \mathbb{R}^2),\, |\xi(x)| \le 1\ \forall x \in \Omega \right\}.$

How to solve 1: cont'

- Objective: $$\min_{u \in X} \frac{\|u - g\|^2}{2\lambda} + J(u), \qquad (6)$$

- Euler equation for (6) is

$$u - g + \lambda \partial J(u) \ni 0.$$

$$\Rightarrow (g - u)/\lambda \in \partial J(u)$$

$$u \in \partial J^*((g - u)/\lambda)$$

$$\frac{g}{\lambda} \in \frac{g - u}{\lambda} + \frac{1}{\lambda} \partial J^*\left(\frac{g - u}{\lambda}\right),$$

How to solve 1: cont'

$$w = (g - u)/\lambda$$

Indicator function

$$\frac{\|w - (g/\lambda)\|^2}{2} + \frac{1}{\lambda} J^*(w).$$

$$\Rightarrow w = \pi_K(g/\lambda).$$

Hence the solution of (6) is given by

$$u = g - \pi_{\lambda K}(g). \qquad (7)$$

In the discrete case, setting $\pi_{\lambda K}(g) = div(p)$

$$\Rightarrow \min\{\|\lambda \operatorname{div} p - g\|^2 : p \in Y,\ |p_{i,j}|^2 - 1 \le 0\ \forall i, j = 1, \ldots, N\}. \qquad (8)$$

Chambolle approach

Euler Lagrange formulation

$$-(\nabla(\lambda \operatorname{div} p - g))_{i,j} + \alpha_{i,j} p_{i,j} = 0 \quad \text{Lagrange multiplier } \alpha_{i,j} \ge 0,$$

$$\Rightarrow \alpha_{i,j} = |(\nabla(\lambda \operatorname{div} p - g))_{i,j}|.$$

semi-implicit gradient descent

We choose $\tau > 0$, let $p^0 = 0$ and for any $n \ge 0$,

$$p_{i,j}^{n+1} = p_{i,j}^n + \tau\big((\nabla(\operatorname{div} p^n - g/\lambda))_{i,j} - |(\nabla(\operatorname{div} p^n - g/\lambda))_{i,j}|\, p_{i,j}^{n+1}\big),$$

Chambolle approach: cont'

$$p_{i,j}^{n+1} = \frac{p_{i,j}^n + \tau(\nabla(\mathrm{div}\, p^n - g/\lambda))_{i,j}}{1 + \tau|(\nabla(\mathrm{div}\, p^n - g/\lambda))_{i,j}|}. \qquad (9)$$

How to solve equation 2:

2. Fixed $\mathbf{u}$, solve

$$\min_{\mathbf{v}} \int_\Omega \frac{1}{2\theta}|\mathbf{u}-\mathbf{v}|^2 + \lambda|\rho(\mathbf{v})|.$$

$$\rho(v)(x) = a^T v + b,\ a \in \mathbb{R}^d \text{ and } b \in \mathbb{R}$$

$$v(x) = u(x) - \pi_{\lambda\theta[-a,a]}\left(u + \frac{b}{\|a\|^2}a\right)$$

$$\pi_{\lambda\theta[-a,a]}\left(u + \frac{b}{\|a\|^2}a\right) = \begin{cases} -\lambda\theta a & \text{if} \quad a^T u + b < -\lambda\theta\|a\|^2 \\ \lambda\theta a & \text{if} \quad a^T u + b > \lambda\theta\|a\|^2 \\ \dfrac{a^T u + b}{\|a\|^2}a & \text{if} \quad |a^T u + b| \le \lambda\|a\|^2 \end{cases}$$

General form of TVL1 with data attachment term

$$E_\theta(\mathbf{u},\mathbf{v}) = \int_\Omega |\nabla u_1| + |\nabla u_2| + \frac{1}{2\theta}|\mathbf{u}-\mathbf{v}|^2 + \lambda|\rho(\mathbf{v})|$$

$$\min_u \int_\Omega \left\{|\nabla u| + \frac{1}{2\theta}(u-v)^2\right\} d\boldsymbol{x}. \qquad (6)$$

Proposition 1 *The solution of Eq. (6) is given by*

$$u = v - \theta\,\mathbf{div}\,\boldsymbol{p}, \qquad (8)$$

where $\mathbf{p} = (p^1, p^2)$ *fulfills*

$$\nabla(\theta\,\mathbf{div}\,\mathbf{p} - v) = |\nabla(\theta\,\mathbf{div}\,\mathbf{p} - v)|\,\mathbf{p}, \qquad (9)$$

which can be solved by the following iterative fixed-point scheme:

$$\mathbf{p}^{k+1} = \frac{\mathbf{p}^k + \tau\nabla(\mathbf{div}\,\mathbf{p}^k - v/\theta)}{1 + \tau|\nabla(\mathbf{div}\,\mathbf{p}^k - v/\theta)|}, \qquad (10)$$

where $\mathbf{p}^0 = \mathbf{0}$ *and the time step* $\tau \le 1/8$.

Chambolles Projection algorithm

2. Fixed $\mathbf{u}$, solve

$$\min_{\mathbf{v}} \int_\Omega \frac{1}{2\theta}|\mathbf{u}-\mathbf{v}|^2 + \lambda|\rho(\mathbf{v})|.$$

$$\rho(v)(x) = a^T v + b,\ a \in \mathbb{R}^d \text{ and } b \in \mathbb{R}$$

$$v(x) = u(x) - \pi_{\lambda\theta[-a,a]}\left(u + \frac{b}{\|a\|^2}a\right)$$

$$\pi_{\lambda\theta[-a,a]}\left(u + \frac{b}{\|a\|^2}a\right) = \begin{cases} -\lambda\theta a & \text{if} \quad a^T u + b < -\lambda\theta\|a\|^2 \\ \lambda\theta a & \text{if} \quad a^T u + b > \lambda\theta\|a\|^2 \\ \dfrac{a^T u + b}{\|a\|^2}a & \text{if} \quad |a^T u + b| \le \lambda\|a\|^2 \end{cases}$$

Point wise solution

Numerical Details

- To compute the gradient of the image I1, we use central differences along each direction, with Neumann boundary conditions.

$$\frac{\partial}{\partial x}I_1(i,j) = \begin{cases} \frac{I_1(i+1,j) - I_1(i-1,j)}{2} & \text{if} 1 < i < N_x \\ 0 & \text{otherwise} \end{cases},$$

$$\frac{\partial}{\partial y}I_1(i,j) = \begin{cases} \frac{I_1(i,j+1) - I_1(i,j-1)}{2} & \text{if} 1 < j < N_y \\ 0 & \text{otherwise} \end{cases}.$$

Numerical Details

- To compute the gradient of each component of the flow u, we use forward differences withNeumann boundary conditions.

$$\frac{\partial}{\partial x}u(i,j) = \begin{cases} u(i+1,j) - u(i,j) & \text{if} 1 \leq i < N_x \\ 0 & \text{if} i = N_x \end{cases},$$
$$\frac{\partial}{\partial y}u(i,j) = \begin{cases} u(i,j+1) - u(i,j) & \text{if} 1 \leq j < N_y \\ 0 & \text{if} j = N_y \end{cases}.$$

Numerical Details

- For computing the divergences of the dual variables p, we use the adjoint of the gradient of u, which corresponds to using backward differences:

$$\mathrm{div}(\mathbf{p})(i,j) = \begin{cases} p_1(i,j) - p_1(i-1,j) & \text{if} 1 < i < N_x \\ p_1(i,j) & \text{if} i = 1 \\ -p_1(i-1,j) & \text{if} i = N_x \end{cases}$$
$$+ \begin{cases} p_2(i,j) - p_2(i,j-1) & \text{if} 1 < j < N_y \\ p_2(i,j) & \text{if} j = 1 \\ -p_2(i,j-1) & \text{if} j = N_y \end{cases}.$$

Extensions:

1. Warping
2. Illumination change approximation

$$\rho(\mathrm{w}, v) = I_t + (\nabla I)^T{}_{i,j}\left(v_{i,j} - v^0_{i,j}\right) + \beta w$$

3. Adding gradient consistency term

Algorithm

Algorithm 1: Pyramidal structure management

Input: $I_0, I_1, \tau, \lambda, \theta, \varepsilon, \eta, N_{maxiter}, N_{warps}, N_{scales}$
Output: $\mathbf{u}$

Normalize images between 0 and 255
Convolve the images with a Gaussian of $\sigma = 0.8$
Create the pyramid of images I^s using η (with $s = 0, \ldots, N_{scales} - 1$)
$\mathbf{u}^{N_{scales}-1} \leftarrow (0,0)$
for $s \leftarrow N_{scales} - 1$ **to** 0 **do**
    `TV-L`^{1}`_optical_flow`$(I_0, I_1, u^0, \tau, \lambda, \theta, \varepsilon, N_{maxiter}, N_{warps})$
    if $s > 0$ **then**
        $\mathbf{u}^{s-1}(\mathbf{x}) := 2\mathbf{u}^s(\mathbf{x}/\eta)$
    end
end

time step (τ), data attachment weight (λ), tightness (θ), stopping criterion threshold (ε), downsampling factor (η), number of scales (Nscales), number of warps (Nwarps).

Implementation:

```
pyramid_levels = 100; % as much as possible
pyramid_factor = 0.90;
```

```
width_Pyramid{i} = pyramid_factor*width_Pyramid{i-1};
height_Pyramid{i} = pyramid_factor*height_Pyramid{i-1};
```

Source code available: http://www.ipol.im/pub/art/2013/26/?utm_source=doi

Algorithm

Procedure `TV-L¹_optical_flow`$(I_0, I_1, u^0, \tau, \lambda, \theta, \varepsilon, N_{maxiter}, N_{warps})$

$\mathbf{p}_1 \leftarrow (0,0)$
$\mathbf{p}_2 \leftarrow (0,0)$
for $w \leftarrow 1$ **to** N_{warps} **do**
    Compute $I_1(\mathbf{x}+\mathbf{u}^0(\mathbf{x})), \nabla I_1(\mathbf{x}+\mathbf{u}^0(\mathbf{x}))$ using bicubic interpolation
    $n \leftarrow 0$
    while $n < N_{maxiter}$ **and** *stopping_criterion* $> \varepsilon$ **do**
        $\mathbf{v} \leftarrow TH(\mathbf{u}, \mathbf{u}^0)$
        $\mathbf{u} \leftarrow \mathbf{v} + \theta \mathrm{div}(\mathbf{p})$
        $\mathbf{p} \leftarrow \frac{\mathbf{p}+\tau/\theta\nabla\mathbf{u}}{1+\tau/\theta|\nabla\mathbf{u}|}$
        $n \leftarrow n+1$
    end
end

time step (τ), data attachment weight (λ), tightness (θ), stopping criterion threshold (ε), downsampling factor (η), number of scales (Nscales), number of warps (Nwarps).

Reference:

- J. Pérez, E. Meinhardt-Llopis, and G. Facciolo, "TV-L1 Optical Flow Estimation," Image Process. Line, vol. 1, pp. 137–150, 2013.
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