

# Variational view on image segmentation and the Mumford-Shah model

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# Outline

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# Segmentation problem

$$I : \Omega \rightarrow \mathbb{R}$$

## Segmentation

Partition the domain of an image into regions and/or contours that correspond to the imaged objects and their boundaries

$$\Omega = \bigcup_{i=1}^n \Omega_i \text{ or } \Omega = \left( \bigcup_{i=1}^n \Omega_i \right) \cup \Gamma$$

- Variants of thresholding
- Clustering
- Spatial context ("what the neighbors say")
- Region-based methods (e.g. region growing, region merging)
- Edge-based segmentation (e.g. watershed transform)
- Methods focusing on mathematically transparent optimization criteria

# The Mumford-Shah functional

## Piecewise-smooth Mumford-Shah functional

$$E(u, \Gamma) = \iint_{\Omega} [I(\mathbf{x}) - u(\mathbf{x})]^2 dA + \lambda \iint_{\Omega \setminus \Gamma} |\nabla u(\mathbf{x})| dA + \nu |\Gamma|$$

$$\Omega \subset \mathbb{R}^2$$

$$I : \Omega \rightarrow \mathbb{R}$$

$$u : \Omega \rightarrow \mathbb{R}$$

$$\Gamma \subset \Omega$$

# Data and regularity terms

## Data term

$$\iint_{\Omega} [I(\mathbf{x}) - u(\mathbf{x})]^2 dA$$

Approximation  $u$  matches the original image  $I$  in the least-squares sense.

## Regularity term 1: Dirichlet energy

$$\lambda \iint_{\Omega \setminus \Gamma} |\nabla u(\mathbf{x})| dA$$

Penalize gradient outside boundary  $\Gamma$  (to obtain piecewise smooth  $u$ ).

## Regularity term 2: discontinuity penalty

$$\nu |\Gamma|$$

Allow discontinuities, but not too many.

# Piecewise-constant case of Mumford-Shah

Regions  $\Omega_1, \dots, \Omega_n$ , separated by boundary  $\Gamma$  ( $\Omega = (\bigcup_{i=1}^n \Omega_i) \cup \Gamma$ ).

$\lambda \rightarrow \infty$ . Each region  $\Omega_i$  contains pixels with constant intensity  $u_i$ .

## Piecewise-constant Mumford-Shah functional

$$E(\{u_1, \dots, u_n\}, \Gamma) = \sum_{i=1}^n \iint_{\Omega_i} [I(\mathbf{x}) - u_i]^2 dA + \nu |\Gamma|$$

(also known as Ising model)

Can be solved in polynomial time for  $n = 2$ . NP-hard problem for  $n > 2$ .

# Towards the canonical form of functional

Want:

$$E = \iint_{\Omega} f(\text{functions}, \text{derivatives}, \text{parameters}) dA$$

# Green theorem with a sample functional (1)

Consider a functional:

$$E(\Gamma) = \iint_{int(\Gamma)} f(x_1, x_2) dx_1 dx_2$$

Green's theorem:

$$\iint_{int(\Gamma)} (\nabla \times \mathbf{v}) dA = \int_{\Gamma} \mathbf{v} ds$$

where:

$\mathbf{v} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  – a vector field,  $\mathbf{v}(s) = \begin{bmatrix} a(x_1, x_2) \\ b(x_1, x_2) \end{bmatrix}$ , chosen so that

$$f = \frac{\partial b}{\partial x_1} - \frac{\partial a}{\partial x_2}$$

$\Gamma : [0, 1] \rightarrow \mathbb{R}^2$  – boundary as a parametric curve.

## Green theorem with a sample functional (2)

$$\iint_{int(\Gamma)} (\nabla \times \mathbf{v}) dA = \int_{\Gamma} \mathbf{v} ds$$

$$E(C) = \int_{\Gamma} (a dx_1 + b dx_2) = \int_0^1 (a x_1 + b x_2) ds = \int_0^1 L(x_1, x_2, \frac{dx_1}{ds}, \frac{dx_2}{ds})$$

Euler-Lagrange equation 1:

$$\frac{\partial L}{\partial x_1} - \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{x}_1} \right) = \frac{\partial a}{\partial x_1} \dot{x}_1 + \frac{\partial b}{\partial x_1} \dot{x}_2 - \frac{da}{ds} = \left( \frac{\partial b}{\partial x_1} - \frac{\partial a}{\partial x_2} \right) \dot{x}_2$$

Euler-Lagrange equation 2:

$$\frac{\partial L}{\partial x_2} - \frac{d}{ds} \left( \frac{\partial L}{\partial \dot{x}_2} \right) = \frac{\partial a}{\partial x_2} \dot{x}_1 + \frac{\partial b}{\partial x_2} \dot{x}_2 - \frac{db}{ds} = \left( -\frac{\partial b}{\partial x_1} + \frac{\partial a}{\partial x_2} \right) \dot{x}_1$$

## Green theorem with a sample functional (3)

$$\frac{\partial L}{\partial \Gamma} = f(x_1, x_2) \begin{bmatrix} \dot{x}_2 \\ -\dot{x}_1 \end{bmatrix}$$

Tangent vector:  $\mathbf{t} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ . Normal vector:  $\mathbf{n} = \begin{bmatrix} \dot{x}_2 \\ -\dot{x}_1 \end{bmatrix}$ .  $\langle \mathbf{n}, \mathbf{t} \rangle = 0$

$$\frac{\partial L}{\partial \Gamma} = f(x_1, x_2) \mathbf{n}$$

Interpretation: change in energy occurs when contour  $\Gamma$  evolves in the normal direction.

# Piecewise-constant M.-S. with $n = 2$ regions

$$E(\Gamma) = \iint_{int(\Gamma)} [I(\mathbf{x}) - u_{int}]^2 dA + \iint_{ext(\Gamma)} [I(\mathbf{x}) - u_{ext}]^2 dA + \nu |\Gamma|$$

$$\frac{dE}{d\Gamma} = [(I(\mathbf{x}) - u_{int})^2 - (I(\mathbf{x}) - u_{ext})^2 - \nu \kappa(\mathbf{x})] \mathbf{n}$$

where  $\kappa(\mathbf{x})$  is local curvature of  $\Gamma$

Gradient descent:

$$\frac{\partial \Gamma(s, t)}{\partial t} = -\frac{dE}{d\Gamma} = [(I(\mathbf{x}) - u_{ext})^2 - (I(\mathbf{x}) - u_{int})^2 + \nu\kappa]\mathbf{n}$$

At each boundary point  $\mathbf{x}$ , displace the curve:

- outwards if  $|I(\mathbf{x}) - u_{int}| < |I(\mathbf{x}) - u_{ext}|$
- inwards if  $|I(\mathbf{x}) - u_{int}| > |I(\mathbf{x}) - u_{ext}|$