ECE 4213/5213

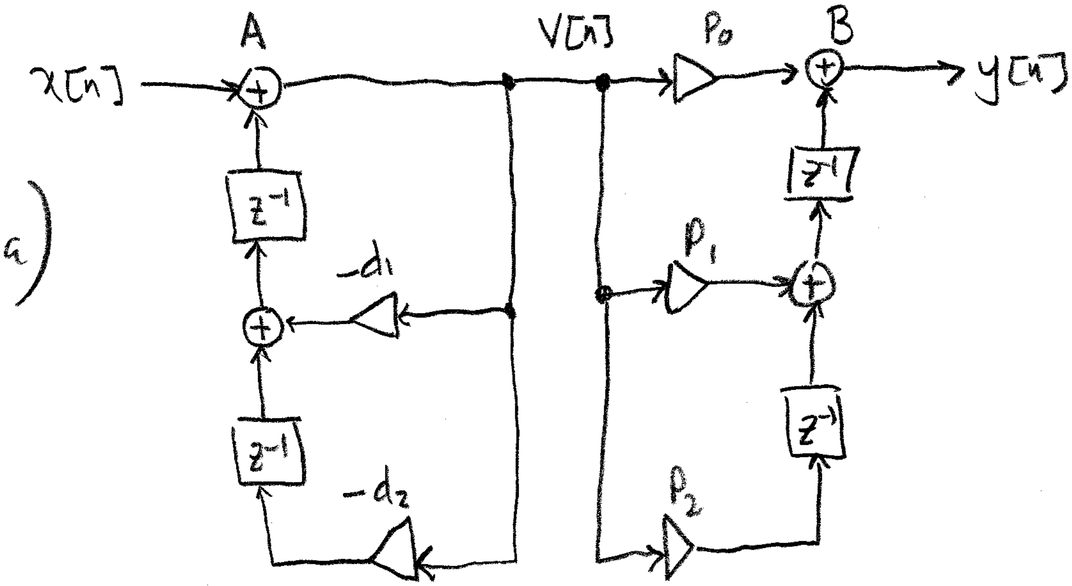
DSP

HW 4 SOLUTION

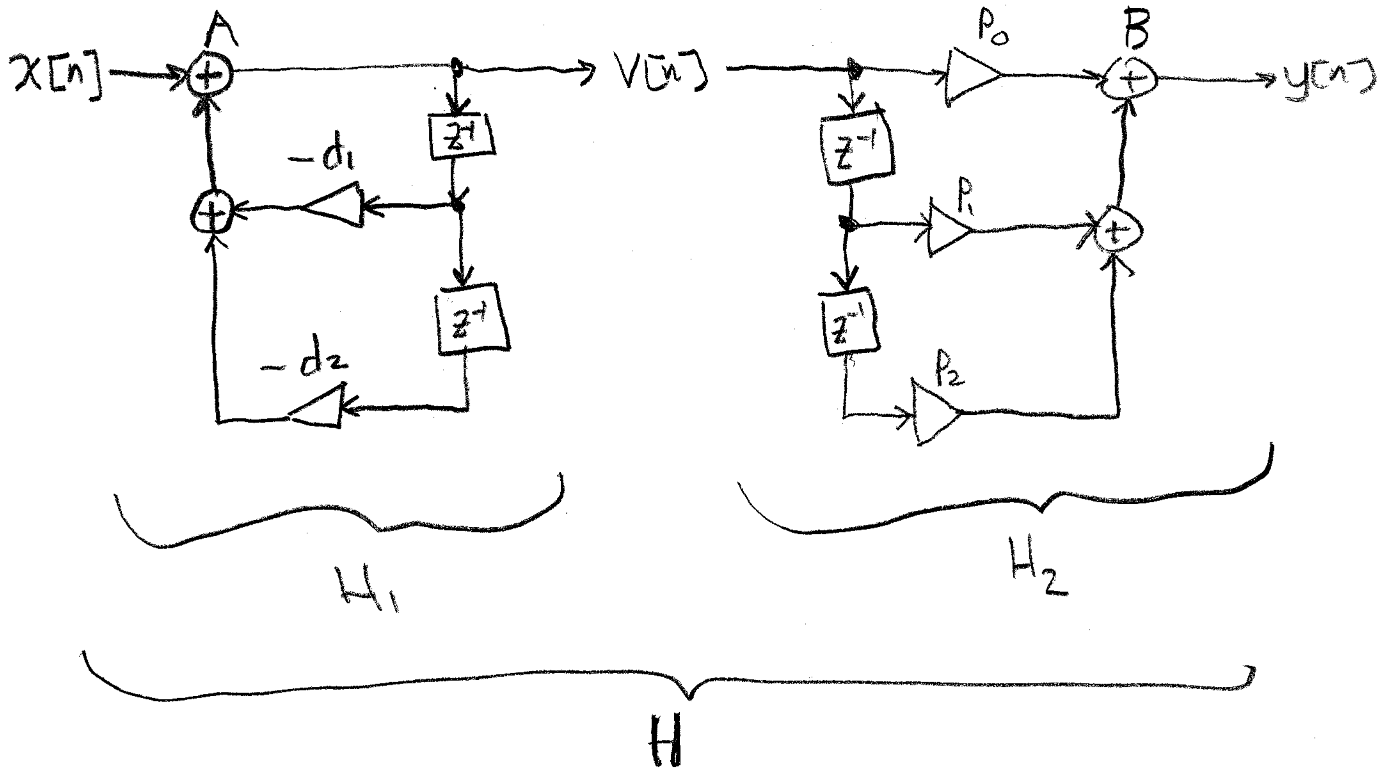
HAVLICEK

$\boxed{2.4a)-1}$

2.4a)



Move the delays and break into two systems:



$\longrightarrow$

$\boxed{2.4a)\text{-}2}$

- for system H_1, the input is $x[n]$ and the output is $v[n]$.

At node A, we have:

$$v[n] = x[n] - d_1 v[n-1] - d_2 v[n-2]$$

$$v[n] + d_1 v[n-1] + d_2 v[n-2] = x[n]$$

z-xform: $V(z) + d_1 z^{-1} V(z) + d_2 z^{-2} V(z) = X(z)$

$$V(z)\left[1 + d_1 z^{-1} + d_2 z^{-2}\right] = X(z)$$

$$H_1(z) = \frac{V(z)}{X(z)} = \frac{1}{1 + d_1 z^{-1} + d_2 z^{-2}}$$

- for system H_2, the input is $v[n]$ and the output is $y[n]$.

At node B, we have:

$$y[n] = p_0 v[n] + p_1 v[n-1] + p_2 v[n-2]$$

z-transform: $Y(z) = p_0 V(z) + p_1 z^{-1} V(z) + p_2 z^{-2} V(z)$

$$= \left[p_0 + p_1 z^{-1} + p_2 z^{-2}\right] V(z)$$

$$H_2(z) = \frac{Y(z)}{V(z)} = p_0 + p_1 z^{-1} + p_2 z^{-2}$$

$\longrightarrow$

The overall system H is a series connection of H_1 and H_2, so

$\boxed{2.4a)-3}$

$$H(z) = H_1(z)H_2(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2}}{1 + d_1 z^{-1} + d_2 z^{-2}} = \frac{Y(z)}{X(z)}$$

cross Multiply:

$$Y(z)\left[1 + d_1 z^{-1} + d_2 z^{-2}\right] = X(z)\left[p_0 + p_1 z^{-1} + p_2 z^{-2}\right]$$

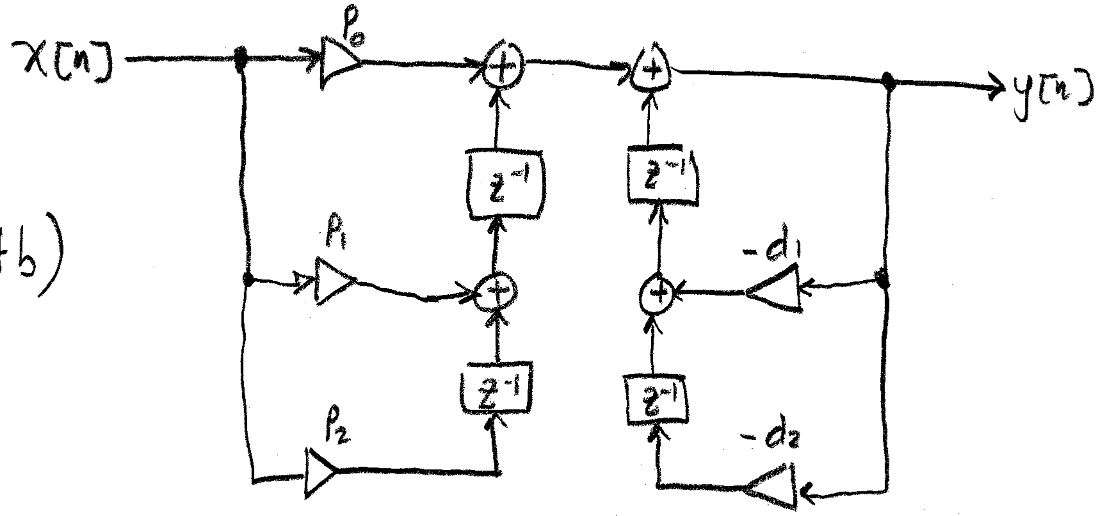
$$Y(z) + d_1 z^{-1} Y(z) + d_2 z^{-2} Y(z) = p_0 X(z) + p_1 z^{-1} X(z) + p_2 z^{-2} X(z)$$

inverse z transform:

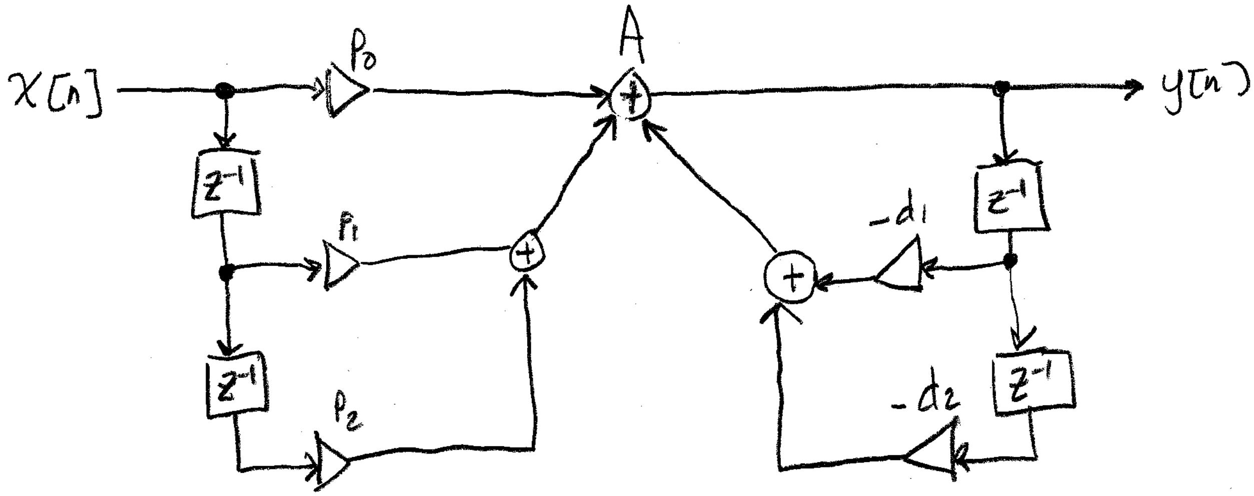
$$\underline{\underline{y[n] + d_1 y[n-1] + d_2 y[n-2] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2]}}$$

2.4b)-1

2.4b)



Commute the delays and multipliers and combine the two adders in the middle:



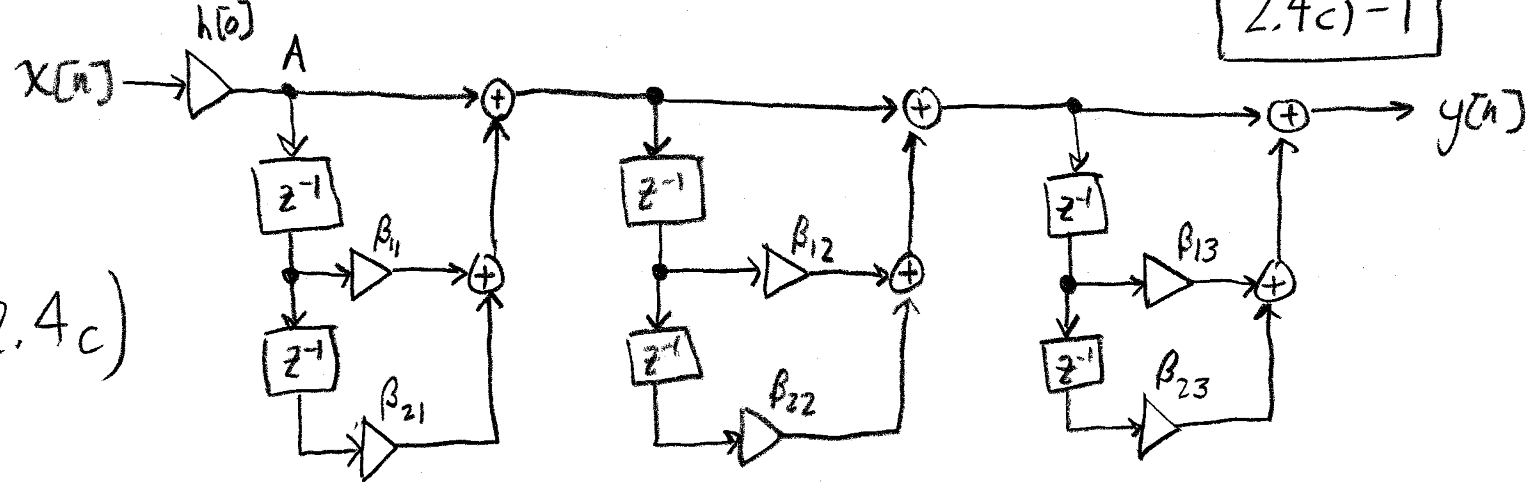
at node A, we have:

$$y[n] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2] - d_1 y[n-1] - d_2 y[n-2]$$

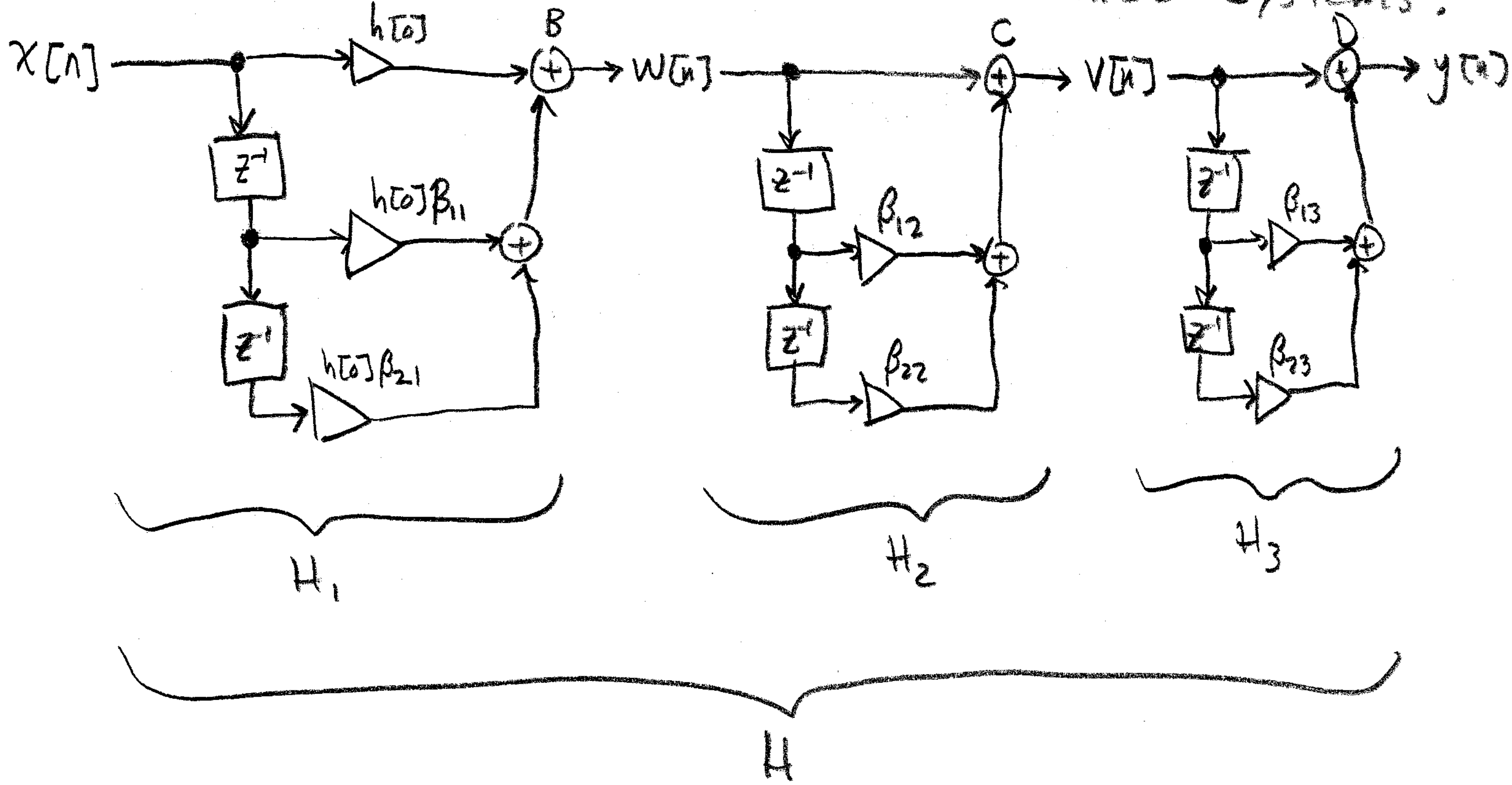
$$y[n] + d_1 y[n-1] + d_2 y[n-2] = p_0 x[n] + p_1 x[n-1] + p_2 x[n-2]$$

2.4c)-1

2.4c)



- Push the multiplication by h[0] through node A and consider as a series connection of three systems:



For system H_1 the input is $x[n]$ and the output is $w[n]$.

At node B, we have

$$w[n] = h[0]x[n] + h[0]\beta_{11}x[n-1] + h[0]\beta_{21}x[n-2]$$

Apply z-transform:

$\boxed{2.4c)-2}$

$$W(z) = h[0]\,X(z) + h[0]\beta_{11} z^{-1} X(z) + h[0]\beta_{21} z^{-2} X(z)$$

$$W(z) = \left[h[0] + h[0]\beta_{11} z^{-1} + h[0]\beta_{21} z^{-2}\right] X(z)$$

$$H_1(z) = \frac{W(z)}{X(z)} = h[0] + h[0]\beta_{11} z^{-1} + h[0]\beta_{21} z^{-2}$$

For system H_2, the input is $w[n]$ and the output is $v[n]$.

At node C_5 we have

$$v[n] = w[n] + \beta_{12} w[n-1] + \beta_{22} w[n-2]$$

z-transform: $V(z) = W(z) + \beta_{12} z^{-1} W(z) + \beta_{22} z^{-2} W(z)$

$$V(z) = \left[1 + \beta_{12} z^{-1} + \beta_{22} z^{-2}\right] W(z)$$

$$H_2(z) = \frac{V(z)}{W(z)} = 1 + \beta_{12} z^{-1} + \beta_{22} z^{-2}$$

For system H_3, the input is $v[n]$ and the output is $y[n]$.

- the diagram for H_3 is the same as the one for H_2. Using the above result for H_2, we have

$$H_3(z) = \frac{Y(z)}{V(z)} = 1 + \beta_{13} z^{-1} + \beta_{23} z^{-2}.$$

The overall system H is the series connection of H_1, H_2, and H_3. So, we have

$\longrightarrow$

$$H(z) = H_1(z)\,H_2(z)\,H_3(z)$$

$\boxed{2.4c)-3}$

$$= \left[h[0] + h[0]\beta_{11}z^{-1} + h[0]\beta_{21}z^{-2}\right]\left[1+\beta_{12}z^{-1} + \beta_{22}z^{-2}\right]\left[1+\beta_{13}z^{-1}+\beta_{23}z^{-2}\right]$$

$$= \Big[h[0] + h[0]\beta_{12}z^{-1} + h[0]\beta_{22}z^{-2} + h[0]\beta_{11}z^{-1} + h[0]\beta_{11}\beta_{12}z^{-2} + h[0]\beta_{11}\beta_{22}z^{-3}$$
$$+ h[0]\beta_{21}z^{-2} + h[0]\beta_{21}\beta_{12}z^{-3} + h[0]\beta_{21}\beta_{22}z^{-4}\Big]$$
$$\times\left[1+\beta_{13}z^{-1} + \beta_{23}z^{-2}\right]$$

$$= \Big\{h[0] + (h[0]\beta_{12} + h[0]\beta_{11})z^{-1} + (h[0]\beta_{22} + h[0]\beta_{11}\beta_{12} + h[0]\beta_{21})z^{-2}$$
$$+ (h[0]\beta_{11}\beta_{22} + h[0]\beta_{21}\beta_{12})z^{-3} + h[0]\beta_{21}\beta_{22}z^{-4}\Big\}\left[1+\beta_{13}z^{-1}+\beta_{23}z^{-2}\right]$$

$$= h[0] + (h[0]\beta_{12} + h[0]\beta_{11})z^{-1} + (h[0]\beta_{22} + h[0]\beta_{11}\beta_{12} + h[0]\beta_{21})z^{-2}$$
$$+ (h[0]\beta_{11}\beta_{22} + h[0]\beta_{21}\beta_{12})z^{-3} + h[0]\beta_{21}\beta_{22}z^{-4}$$
$$+ h[0]\beta_{13}z^{-1} + (h[0]\beta_{12}\beta_{13} + h[0]\beta_{11}\beta_{13})z^{-2}$$
$$+ (h[0]\beta_{22}\beta_{13} + h[0]\beta_{11}\beta_{12}\beta_{13} + h[0]\beta_{21}\beta_{13})z^{-3}$$
$$+ (h[0]\beta_{11}\beta_{22}\beta_{13} + h[0]\beta_{21}\beta_{12}\beta_{13})z^{-4} + h[0]\beta_{21}\beta_{22}\beta_{13}z^{-5}$$
$$+ h[0]\beta_{23}z^{-2} + (h[0]\beta_{12}\beta_{23} + h[0]\beta_{11}\beta_{23})z^{-3}$$
$$+ (h[0]\beta_{22}\beta_{23} + h[0]\beta_{11}\beta_{12}\beta_{23} + h[0]\beta_{21}\beta_{23})z^{-4}$$
$$+ (h[0]\beta_{11}\beta_{22}\beta_{23} + h[0]\beta_{21}\beta_{12}\beta_{23})z^{-5} + h[0]\beta_{21}\beta_{22}\beta_{23}z^{-6}$$

$\longrightarrow$

$\boxed{2.4c)-4}$

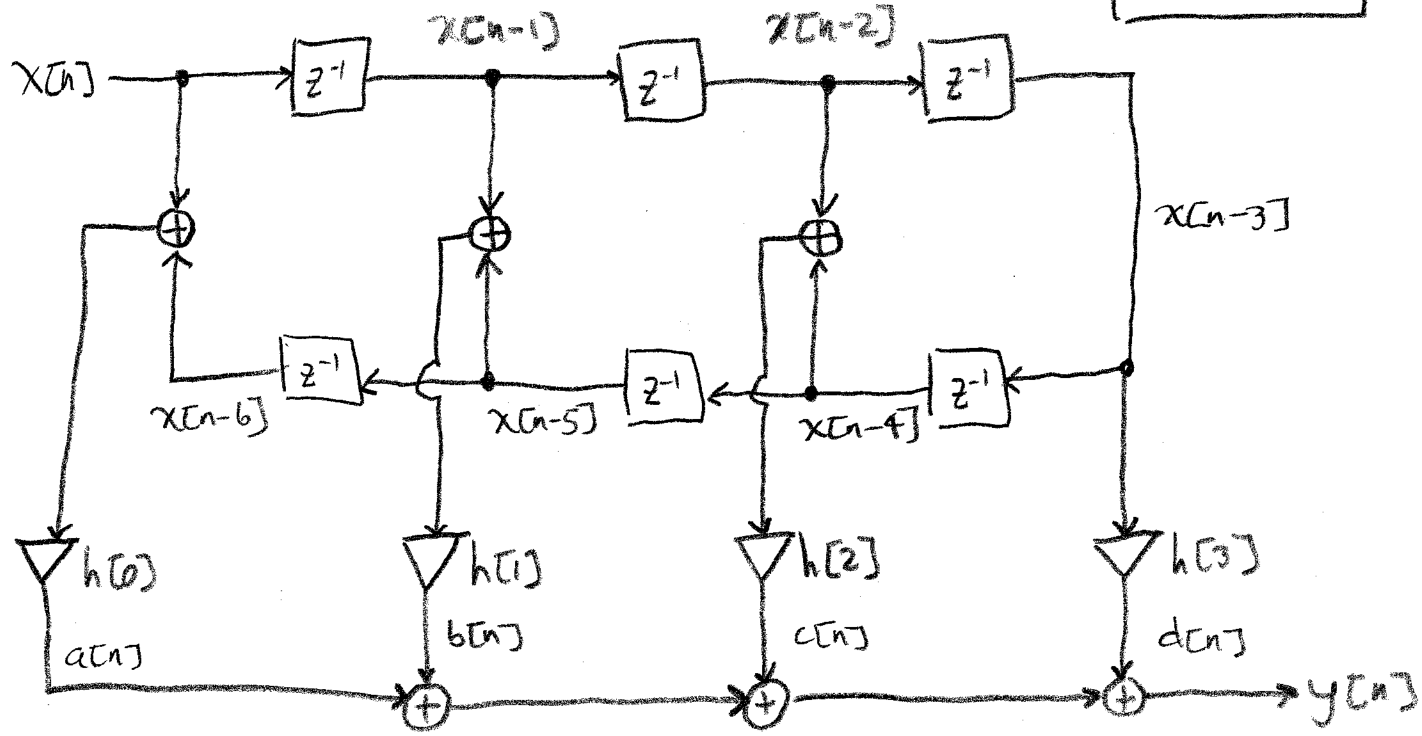
$$\ldots\ H(z) = h[0] + (h[0]\beta_{12} + h[0]\beta_{11} + h[0]\beta_{13})z^{-1}$$
$$+ (h[0]\beta_{22} + h[0]\beta_{11}\beta_{12} + h[0]\beta_{21} + h[0]\beta_{12}\beta_{13} + h[0]\beta_{11}\beta_{13} + h[0]\beta_{23})z^{-2}$$
$$+ (h[0]\beta_{11}\beta_{22} + h[0]\beta_{21}\beta_{12} + h[0]\beta_{22}\beta_{13} + h[0]\beta_{11}\beta_{12}\beta_{13} + h[0]\beta_{21}\beta_{13} + h[0]\beta_{12}\beta_{23} + h[0]\beta_{11}\beta_{23})z^{-3}$$
$$+ (h[0]\beta_{21}\beta_{22} + h[0]\beta_{11}\beta_{22}\beta_{13} + h[0]\beta_{21}\beta_{12}\beta_{13} + h[0]\beta_{22}\beta_{23} + h[0]\beta_{11}\beta_{12}\beta_{23} + h[0]\beta_{21}\beta_{23})z^{-4}$$
$$+ (h[0]\beta_{21}\beta_{22}\beta_{13} + h[0]\beta_{11}\beta_{22}\beta_{23} + h[0]\beta_{21}\beta_{12}\beta_{23})z^{-5}$$
$$+ h[0]\beta_{21}\beta_{22}\beta_{23}z^{-6}$$

Note that $H(z) = \frac{Y(z)}{X(z)}$. Cross multiplying with the above expression for $H(z)$ and taking inverse z-transform, we have:

$$y[n] = h[0]x[n] + h[0](\beta_{12} + \beta_{11} + \beta_{13})x[n-1]$$
$$+ h[0](\beta_{22} + \beta_{11}\beta_{12} + \beta_{21} + \beta_{12}\beta_{13} + \beta_{11}\beta_{13} + \beta_{23})x[n-2]$$
$$+ h[0](\beta_{11}\beta_{22} + \beta_{21}\beta_{12} + \beta_{22}\beta_{13} + \beta_{11}\beta_{12}\beta_{13} + \beta_{21}\beta_{13} + \beta_{12}\beta_{23} + \beta_{11}\beta_{23})x[n-3]$$
$$+ h[0](\beta_{21}\beta_{22} + \beta_{11}\beta_{22}\beta_{13} + \beta_{21}\beta_{12}\beta_{13} + \beta_{22}\beta_{23} + \beta_{11}\beta_{12}\beta_{23} + \beta_{21}\beta_{23})x[n-4]$$
$$+ h[0](\beta_{21}\beta_{22}\beta_{13} + \beta_{11}\beta_{22}\beta_{23} + \beta_{21}\beta_{12}\beta_{23})x[n-5]$$
$$+ h[0]\beta_{21}\beta_{22}\beta_{23}x[n-6]$$

2.4d)

2.4d)-1



$$a[n] = h[0]\left(x[n] + x[n-6]\right)$$

$$b[n] = h[1]\left(x[n-1] + x[n-5]\right)$$

$$c[n] = h[2]\left(x[n-2] + x[n-4]\right)$$

$$d[n] = h[3]\,x[n-3]$$

$$y[n] = a[n] + b[n] + c[n] + d[n]$$

$$y[n] = h[0]\left(x[n] + x[n-6]\right) + h[1]\left(x[n-1] + x[n-5]\right) + h[2]\left(x[n-2] + x[n-4]\right) + h[3]\,x[n-3]$$

2.21a)

n	<-2	-2	-1	0	1	2	>2
$x_1[n]$	0	$-1+j3$	$2-j7$	$4-j5$	$3+j5$	$-2-j$	0
$x_1^*[-n]$	0	$-2+j$	$3-j5$	$4+j5$	$2+j7$	$-1-j3$	0

$$x_{1cs}[n] = \frac{1}{2}\left(x_1[n] + x_1^*[-n]\right)$$

$$= \frac{1}{2}\left\{-3+j4 \quad 5-j12 \quad 8 \quad 5+j12 \quad -3-j4\right\}, \quad -2 \le n \le 2$$

$$= \left\{-\frac{3}{2}+j2 \quad \frac{5}{2}-j6 \quad 4 \quad \frac{5}{2}+j6 \quad -\frac{3}{2}-j2\right\}, \quad -2 \le n \le 2$$

$$x_{1ca}[n] = \frac{1}{2}\left(x_1[n] - x_1^*[-n]\right)$$

$$= \frac{1}{2}\left\{1+j2 \quad -1-j2 \quad -j10 \quad 1-j2 \quad -1+j2\right\}, \quad -2 \le n \le 2$$

$$= \left\{\frac{1}{2}+j \quad -\frac{1}{2}-j \quad -j5 \quad \frac{1}{2}-j \quad -\frac{1}{2}+j\right\}, \quad -2 \le n \le 2$$

2.21b) $x_2[n] = e^{j2\pi n/5} + e^{j\pi n/3}$

$x_2^*[n] = e^{-j2\pi n/5} + e^{-j\pi n/3}$

$x_2^*[-n] = e^{j2\pi n/5} + e^{j2\pi n/3}$

$$x_{2cs}[n] = \tfrac{1}{2}\left(x_2[n] + x_2^*[-n]\right)$$
$$= \tfrac{1}{2}\left(e^{j2\pi n/5} + e^{j\pi n/3} + e^{j2\pi n/5} + e^{j\pi n/3}\right)$$
$$= \tfrac{1}{2}\left(2e^{j2\pi n/5} + 2e^{j\pi n/3}\right)$$
$$= \underline{\underline{e^{j2\pi n/5} + e^{j\pi n/3}}}$$

NOTE: in this case, $x_2[n] = x_{2cs}[n]$, since $x_2[n]$ was already conjugate symmetric to start with.

$$x_{2ca}[n] = \tfrac{1}{2}\left(x_2[n] - x_2^*[-n]\right)$$
$$= \tfrac{1}{2}\left(e^{j2\pi n/5} + e^{j\pi n/3} - e^{j2\pi n/5} - e^{j\pi n/3}\right)$$
$$= \underline{\underline{0}}$$

NOTE: in this case, $x_{2ca}[n]$ is zero because $x_2[n]$ was already conjugate symmetric to start with; e.g., the conjugate antisymmetric part is identically zero.

2.21c) $x_3[n] = j\cos(2\pi n/7) - \sin(2\pi n/4)$

$$x_3^*[n] = -j\cos(2\pi n/7) - \sin(2\pi n/4)$$

$$x_3^*[-n] = -j\cos(-2\pi n/7) - \sin(-2\pi n/4)$$

$$= -j\cos(2\pi n/7) + \sin(2\pi n/4)$$

$$x_{3cs}[n] = \frac{1}{2}\{x_3[n] + x_3^*[-n]\}$$

$$= \frac{1}{2}\{j\cos(2\pi n/7) - \sin(2\pi n/4) - j\cos(2\pi n/7) + \sin(2\pi n/4)\}$$

$$= \underline{\underline{0}}$$

Note: $x_3[n]$ was conjugate antisymmetric to start with, so the conjugate symmetric part is identically zero.

$$x_{3ca}[n] = \frac{1}{2}\{x_3[n] - x_3^*[-n]\}$$

$$= \frac{1}{2}\{j\cos(2\pi n/7) - \sin(2\pi n/4) + j\cos(2\pi n/7) - \sin(2\pi n/4)\}$$

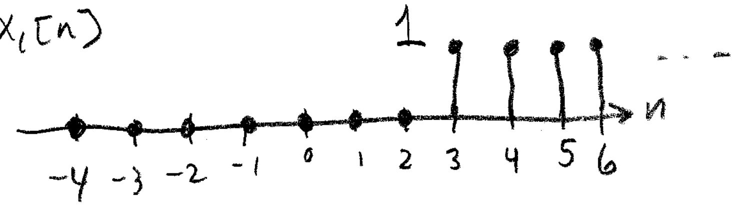
$$= \frac{1}{2}\{2j\cos(2\pi n/7) - 2\sin(2\pi n/4)\}$$

$$= \underline{\underline{j\cos(2\pi n/7) - \sin(2\pi n/4)}}$$

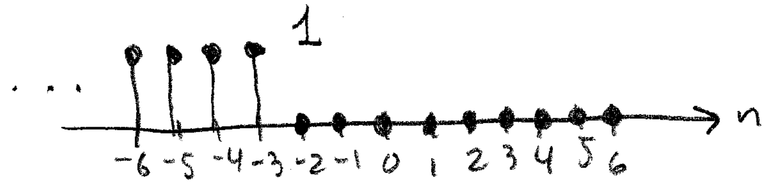
Note: $x_3[n]$ was conjugate antisymmetric to start with, so the conjugate antisymmetric part is just $x_3[n]$ itself.

2.27a) $x_1[n] = u[n-3]$

$x_1[n]$

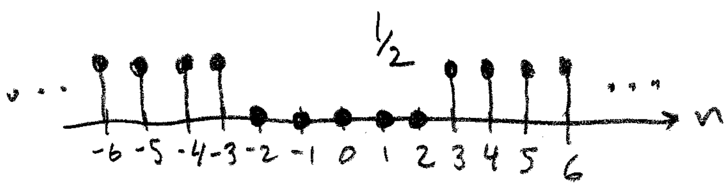


$x_1[-n]$



$$x_{1ev}[n] = \frac{1}{2}\left(x_1[n] + x_1[-n]\right)$$

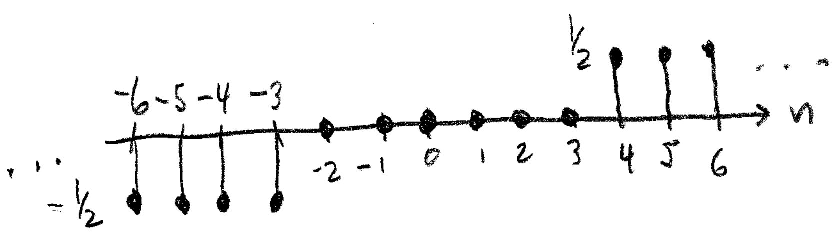
$=$



$$= \frac{1}{2}u[-n-3] + \frac{1}{2}u[n-3] = \frac{1}{2}u\left[|n|-3\right]$$

$$x_{1od}[n] = \frac{1}{2}\left(x_1[n] - x_1[-n]\right)$$

$=$

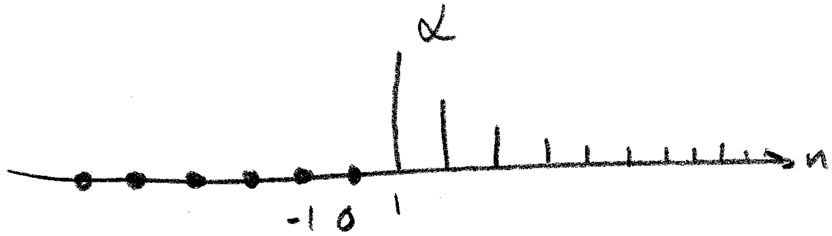


$$= -\frac{1}{2}u[-n-3] + \frac{1}{2}u[n-3] = \frac{1}{2}\operatorname{sgn}(n)\, u[|n|-3]$$

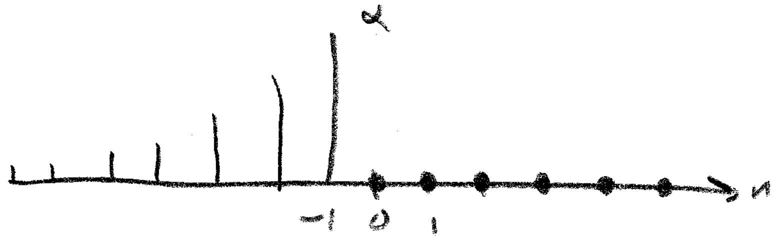
2.27b)

$\boxed{2.27b)-1}$

$$x_2[n] = \alpha^n u[n-1]$$

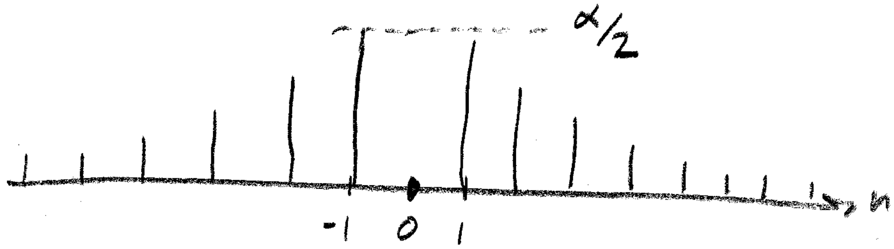


$$x_2[-n] = \alpha^{-n} u[-n-1]$$



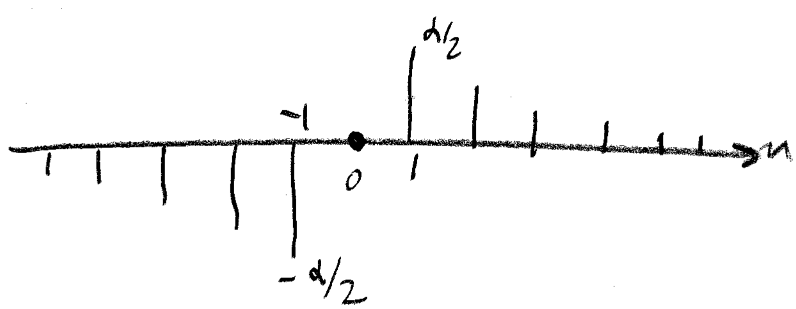
$$x_{2ev}[n] = \frac{1}{2}\left(x_2[n] + x_2[-n]\right) = \frac{1}{2}\alpha^n u[n-1] + \frac{1}{2}\alpha^{-n} u[-n-1]$$

$$= \frac{1}{2}\alpha^{|n|} u[|n|-1]$$



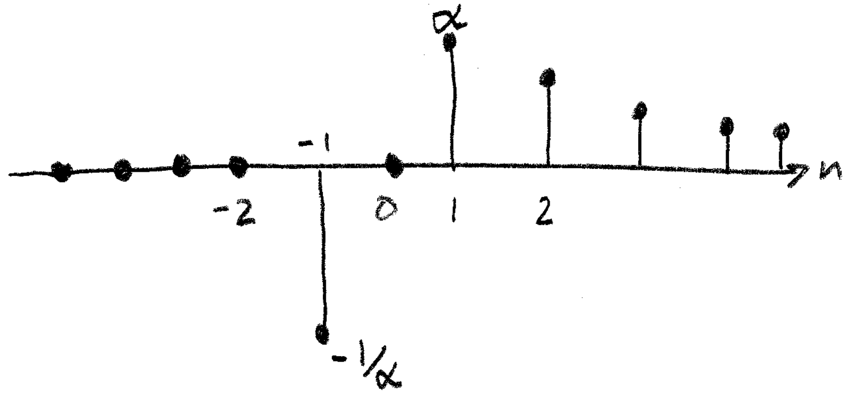
$$x_{2od}[n] = \frac{1}{2}\left(x_2[n] - x_2[-n]\right) = \frac{1}{2}\alpha^n u[n-1] - \frac{1}{2}\alpha^{-n} u[-n-1]$$

$$= \frac{1}{2}\operatorname{sgn}(n)\, \alpha^{|n|} u[|n|-1]$$

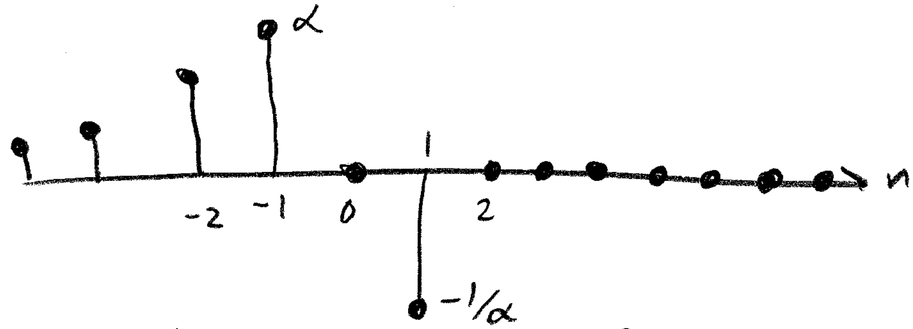


2.27c) $x_3[n] = n\alpha^n u[n+1]$

$\boxed{2.27c)-1}$



$x_3[-n] = -n\alpha^{-n} u[-n+1]$



From these graphs, we have

$$x_{3ev}[n] = \frac{1}{2}\left(x_3[n] + x_3[-n]\right) = \begin{cases} -\frac{n}{2}\alpha^{-n}, & n \le -2 \\ \frac{\alpha}{2} - \frac{1}{2\alpha}, & n = -1 \\ 0, & n = 0 \\ \frac{\alpha}{2} - \frac{1}{2\alpha}, & n = 1 \\ \frac{n}{2}\alpha^{n}, & n \geq 2 \end{cases}$$

$$= \frac{|n|}{2}\alpha^{|n|} u[|n|-2] + \frac{\alpha^2 - 1}{2\alpha}\delta[|n|-1].$$

$\boxed{2.27c)-2}$

$$x_{3od}[n] = \frac{1}{2}\left(x_3[n] - x_3[-n]\right)$$

$$= \begin{cases} \frac{n}{2}\alpha^n, & n \le -2 \\ -\frac{\alpha}{2} - \frac{1}{2\alpha}, & n = -1 \\ 0, & n = 0 \\ \frac{\alpha}{2} + \frac{1}{2\alpha}, & n = 1 \\ \frac{n}{2}\alpha^n, & n \geq 2 \end{cases}$$

$$= \frac{n}{2}\alpha^{|n|}u[|n|-2] + \mathrm{sgn}(n)\frac{\alpha^2+1}{2\alpha}\delta[|n|-1]$$

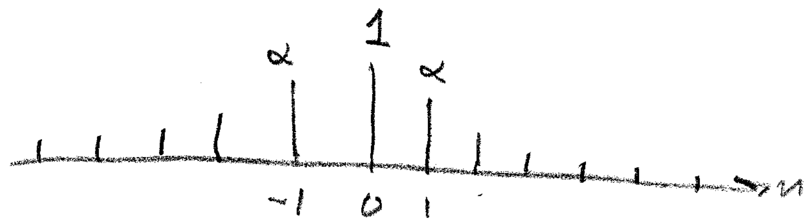
2.27d) $x_4[n] = \alpha^{|n|}$ $\boxed{2.27d)-1}$

Since $x_4[n]$ is <u>even</u>, it follows immediately that

$x_{4ev}[n] = x_4[n] = \alpha^{|n|}$ and $x_{4od}[n] = 0$.

But let's work it out and show this:

$$x_4[n] = \alpha^{|n|}$$



$$x_4[-n] = \alpha^{|-n|} = \alpha^{|n|} = x_4[n]$$

$$x_{4ev}[n] = \frac{1}{2}\left(x_4[n] + x_4[-n]\right) = \frac{1}{2}\left(x_4[n] + x_4[n]\right) = x_4[n]$$

$$x_{4od}[n] = \frac{1}{2}\left(x_4[n] - x_4[-n]\right) = \frac{1}{2}\left(x_4[n] - x_4[n]\right) = 0.$$

2.39a) $\tilde{x}_a[n] = e^{j\pi n/4}$

$$\omega_0 = \frac{\pi}{4}$$

$$\frac{\omega_0}{2\pi} = \frac{\pi/4}{2\pi} = \frac{1}{8}$$

Fundamental Period $= \underline{\underline{8}}$

One period of $\tilde{x}_a[n]$ looks like one period of $\tilde{x}_a(t)$.

2.39b) $\tilde{x}_b[n] = \cos(0.6\pi n + 0.3\pi) = \cos\left(\frac{6\pi}{10}n + \frac{3\pi}{10}\right)$

$$\omega_0 = \frac{6\pi}{10} = \frac{3\pi}{5}$$

$$\frac{\omega_0}{2\pi} = \frac{\frac{3\pi}{5}}{2\pi} = \frac{3}{10}$$

Fundamental Period $= \underline{\underline{10}}$

One period of $\tilde{x}_b[n]$ looks like <u>three</u> periods of $\tilde{x}_b(t)$.

2.39 c) $\tilde{x}_c[n] = \mathrm{Re}\{e^{j\pi n/8}\} + \mathrm{Im}\{e^{j\pi n/5}\}$ $\boxed{2.39c\text{-}1}$

$$= \cos\left(\frac{\pi}{8}n\right) + \sin\left(\frac{\pi}{5}n\right)$$

For the cosine, $\omega_0 = \frac{\pi}{8}$ and $\frac{\omega_0}{2\pi} = \frac{\pi}{16\pi} = \frac{1}{16}$

For the sine, $\omega_0 = \frac{\pi}{5}$ and $\frac{\omega_0}{2\pi} = \frac{\pi}{10\pi} = \frac{1}{10}$

So the cosine has fundamental period 16 and the sine has fundamental period 10.

The fundamental period of $\tilde{x}_c[n]$ is the <u>lowest</u> <u>common</u> <u>multiple</u> of these.

$$N_0 = \mathrm{lcm}(16, 10) = \mathrm{lcm}(2^4, 2\cdot 5)$$

we need four 2's and a 5.

$$= 2\cdot 2\cdot 2\cdot 2\cdot 5 = \underline{\underline{80}}$$

2.39d) $\tilde{x}_d[n] = 6\sin\left(\frac{15\pi}{100}n\right) - \cos\left(\frac{12\pi}{100}n + \frac{\pi}{10}\right)$

For the sine, $\omega_0 = \frac{15\pi}{100} = \frac{3\pi}{20}$ and $\frac{\omega_0}{2\pi} = \frac{3\pi}{40\pi} = \frac{3}{40}$.

So the fundamental period of the sine is 40.

For the cosine, $\omega_0 = \frac{12\pi}{100} = \frac{3\pi}{25}$ and $\frac{\omega_0}{2\pi} = \frac{3\pi}{50\pi} = \frac{3}{50}$.

So the fundamental period of the cosine is 50.

Overall, the fundamental period of $\tilde{x}_d[n]$ is

$$N_0 = \mathrm{lcm}(40, 50) = \mathrm{lcm}(2\cdot2\cdot2\cdot5,\ 2\cdot5\cdot5)$$

$$= 2\cdot2\cdot2\cdot5\cdot5 = \underline{\underline{200}}$$

2.39e) $\tilde{x}_e[n] = \underbrace{\sin\left(\frac{\pi}{10}n + \frac{75\pi}{100}\right)} \underbrace{-3\cos\left(\frac{8\pi}{10}n + \frac{2\pi}{10}\right)} + \underbrace{\cos\left(\frac{13\pi}{10}n\right)}$

$\omega_0 = \frac{\pi}{10}$

$\frac{\omega_0}{2\pi} = \frac{\pi}{20\pi} = \frac{1}{20}$

$N_0 = 20$

$\omega_0 = \frac{8\pi}{10} = \frac{4\pi}{5}$

$\frac{\omega_0}{2\pi} = \frac{4\pi}{10\pi} = \frac{4}{10} = \frac{2}{5}$

$N_0 = 5$

$\omega_0 = \frac{13\pi}{10}$

$\frac{\omega_0}{2\pi} = \frac{13\pi}{20\pi} = \frac{13}{20}$

$N_0 = 20$

Overall, $N_0 = \mathrm{lcm}(20, 5, 20) = \mathrm{lcm}(2\cdot2\cdot5,\ 5,\ 2\cdot2\cdot5)$

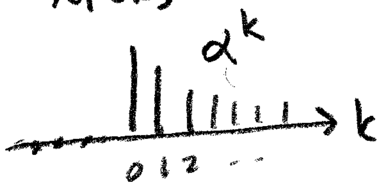
$$= 2\cdot2\cdot5 = \underline{\underline{20}}.$$

2.46a) $x_1[n] = \alpha^n u[n]$

$x_2[n] = u[n]$

$\boxed{2.46a-1}$

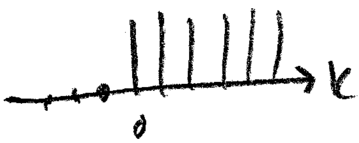
$x_1[k]$



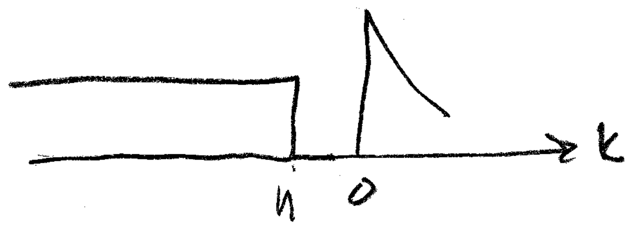
$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x_1[n-k]\, x_2[k]$$

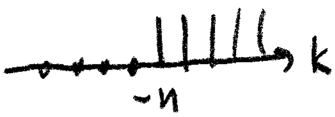
$x_2[k]$



I) $n<0$:



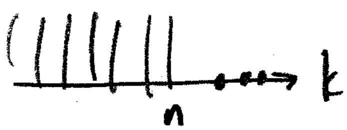
$x_2[k - -n] = x_2[n+k]$



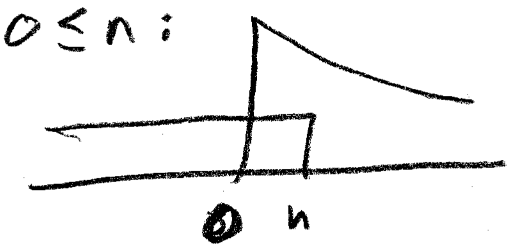
$x_1[k]\, x_2[n-k] = 0 \quad \forall k$

$x_2[n-k]$

$\Rightarrow y[n] = 0.$



II) $0 \leq n$:



you get nonzero terms from $k=0$ to $k=n$:

$$y[n] = \sum_{k=0}^{n} x_1[k]\, x_2[n-k] = \sum_{k=0}^{n} \alpha^k \cdot 1$$

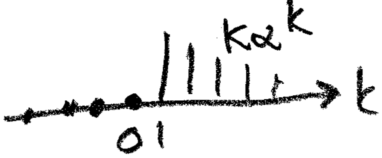
$$= \sum_{k=0}^{n} \alpha^k = \frac{\alpha^0 - \alpha^{n+1}}{1-\alpha} = \frac{1-\alpha^{n+1}}{1-\alpha}$$

All together : $y[n] = \begin{cases} 0, & n<0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & n \geq 0 \end{cases} = \dfrac{1-\alpha^{n+1}}{1-\alpha} u[n].$

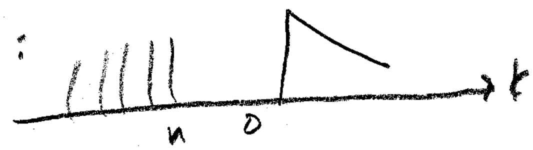
2.46b) $x_1[n] = n\alpha^n u[n]$ $\quad x_2[n] = u[n]$ $\quad \boxed{2.46b\text{-}1}$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k].$$

$x_1[k]$

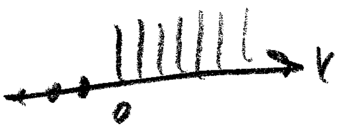


I) $n < 0$:

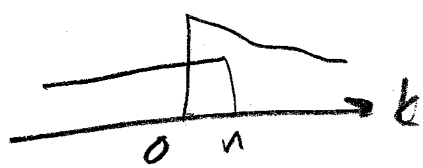


$$\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] = \sum_{k=-\infty}^{\infty} 0 = 0.$$

$x_2[k]$



II) $n \geq 0$:



$x_2[k - -n]$

$= x_2[n+k]$

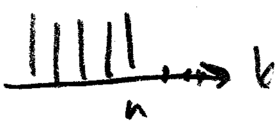
$$y[n] = \sum_{k=0}^{n} x_1[k]\, x_2[n-k]$$

$$= \sum_{k=0}^{n} k\alpha^k \cdot 1$$

$$= \frac{\alpha(1-(n+1)\alpha^n + n\alpha^{n+1})}{(1-\alpha)^2}.$$

This formula is given on the formula sheet

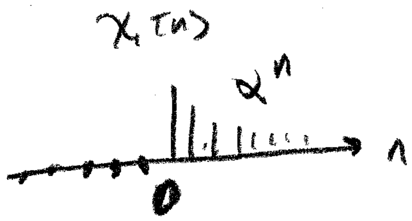
$x_2[n-k]$



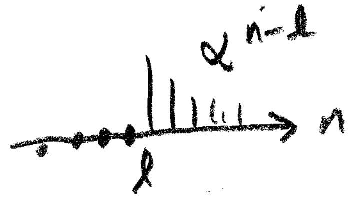
All together:

$$y[n] = \left\{ \frac{\alpha[1-(n+1)\alpha^n + n\alpha^{n+1}]}{(1-\alpha)^2} \right\} u[n]$$

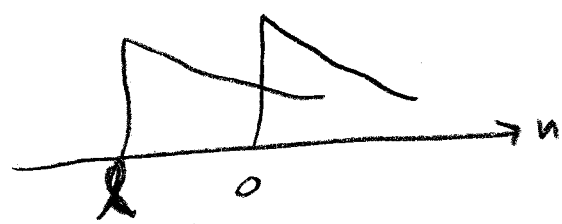
2.53a) $x_1[n] = \alpha^n u[n]$



$$r_{x_1x_1}[l] = \sum_{n\in\mathbb{Z}} x_1[n]\,x_1[n-l]$$



Case I) $l < 0$:



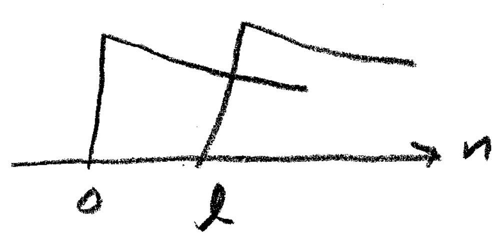
$$r_{xx}[l] = \sum_{n=0}^{\infty} x_1[n]\,x_1[n-l] = \sum_{n=0}^{\infty} \alpha^n \alpha^{n-l}$$

$$= \sum_{n=0}^{\infty} \alpha^n \alpha^n \alpha^{-l} = \alpha^{-l}\sum_{n=0}^{\infty} \alpha^{2n}$$

$$= \alpha^{-l}\sum_{n=0}^{\infty} (\alpha^2)^n = \alpha^{-l}\,\frac{1}{1-\alpha^2}$$

$$= \frac{\alpha^{-l}}{1-\alpha^2} \quad \text{provided } |\alpha^2| < 1.$$

Case II) $l \geq 0$



$$r_{xx}[l] = \sum_{n=l}^{\infty} x_1[n]\,x_1[n-l]$$

$$= \sum_{n=l}^{\infty} \alpha^n \alpha^{n-l} = \sum_{n=l}^{\infty} \alpha^n \alpha^n \alpha^{-l}$$

$$= \alpha^{-l} \sum_{n=l}^{\infty} \alpha^{2n} = \alpha^{-l} \sum_{n=l}^{\infty} (\alpha^2)^n$$

$$= \alpha^{-l} \lim_{A \to \infty} \sum_{n=l}^{A} (\alpha^2)^n = \alpha^{-l} \lim_{A \to \infty} \frac{\alpha^{2l} - \alpha^{2A+2}}{1-\alpha^2}$$

$$= (\text{provided } |\alpha^2| < 1) \quad \alpha^{-l} \frac{\alpha^{2l} - 0}{1-\alpha^2}$$

$$= \frac{\alpha^l}{1-\alpha^2}$$

All Together: $r_{xx}[l] = \begin{cases} \dfrac{\alpha^{-l}}{1-\alpha^2}, & l < 0 \\ \dfrac{\alpha^l}{1-\alpha^2}, & l \geq 0 \end{cases}$

or: $r_{xx}[l] = \dfrac{\alpha^{|l|}}{1-\alpha^2}$.

→ This function is clearly even in "l" because "l" appears only in absolute value.

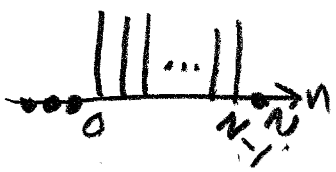
→ Since $|\alpha| < 1$ is required for convergence, $\alpha^{|l|}$ is monotonically decreasing in $|l|$ and the peak occurs at $l = 0$.

$\boxed{2.53b\text{-}1}$

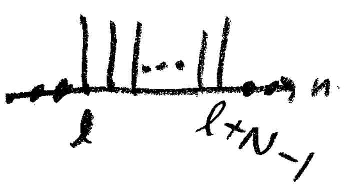
2.53b) $x_2[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{other} \end{cases}$

$= u[n] - u[n-N].$

$x_2[n]$

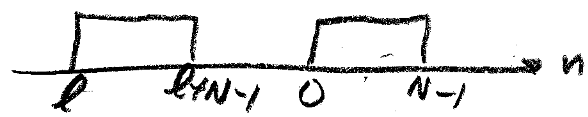


$x_2[n-\ell]$



Case I) $\ell + N - 1 < 0$:

$\ell < 1 - N = -(N-1)$:

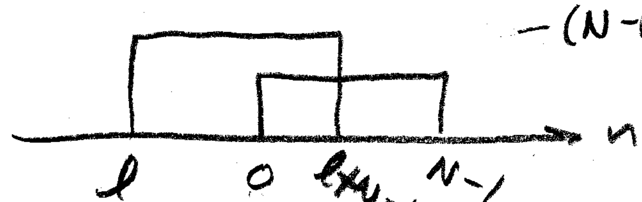


$r_{xx}[\ell] = 0.$

Case II) $\ell + N - 1 \ge 0$ and $\ell + N - 1 < N - 1$:

$\ell \ge 1 - N = -(N-1)$ and $\ell < 0$:

$-(N-1) \le \ell < 0$:



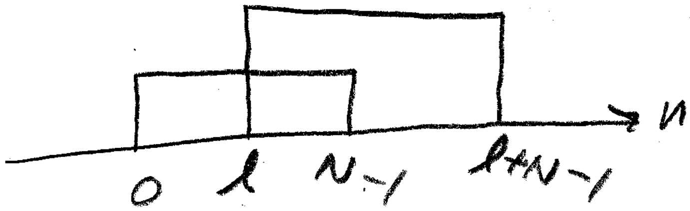
$$r_{xx}[\ell] = \sum_{n=0}^{\ell+N-1} x_2[n]\, x_2[n-\ell] = \sum_{n=0}^{\ell+N-1} 1 = N + \ell$$

Case III) $\ell + N - 1 \ge N - 1$ and $\ell < N$:

$\ell \ge 0$ and $\ell < N$:

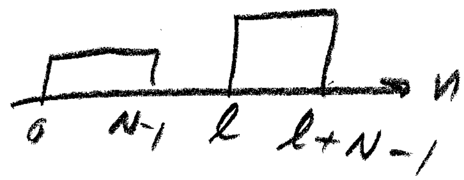
$0 \le \ell < N$:

$\longrightarrow$



$$r_{xx}[l] = \sum_{n=l}^{N-1} x_2[n]\, x_2[n-l] = \sum_{n=l}^{N-1} 1 = N-l.$$

case III) $l \geq N$;



$$r_{xx}[l] = 0.$$

all Together: $r_{xx}[l] = \begin{cases} 0, & l < -(N-1) \\ N+l, & -(N-1) \leq l < 0 \\ N-l, & 0 \leq l < N \\ 0, & l \geq N \end{cases}$

$$= \begin{cases} N-|l|, & |l| < N \\ 0, & |l| \geq N \end{cases}$$

→ Even in "l" because only $|l|$ appears.

→ For $l \in \mathbb{Z}$, $N-|l|$ is decreasing in $|l|$, so the max occurs when $l=0$.

4.3a) $y[n] = x[n+3]$

$\boxed{4.3a-1}$

(1) Linear: Let the input be $x_1[n]$. Then the output is $y_1[n] = H\{x_1[n]\} = x_1[n+3]$.

Now let the input be $x_2[n]$. Then the output is
$$y_2[n] = H\{x_2[n]\} = x_2[n+3].$$

Let $a, b \in \mathbb{C}$ be constants and let $x_3[n] = a x_1[n] + b x_2[n]$.

Then
$$\begin{aligned} y_3[n] &= H\{x_3[n]\} \\ &= x_3[n+3] \\ &= a x_1[n+3] + b x_2[n+3] \\ &= a y_1[n] + b y_2[n] \ \checkmark \end{aligned}$$

Therefore, the system <u>is</u> <u>linear</u>.

(2) Causal: Let the input be $x[n] = n$.

Then the output is $y[n] = H\{x[n]\} = x[n+3] = n+3$.

When $n=0$, we have $y[0] = x[3] = 3$, which depends on the <u>future</u> input $x[3]$.

Therefore, the system is <u>not</u> <u>causal</u>.

$\boxed{4.3a\text{-}2}$

(3) Stable: Suppose $x[n]$ is a bounded input. Then $\exists B \in \mathbb{R}$, $B>0$, s.t. $|x[n]| \leq B \;\forall n \in \mathbb{Z}$.

Now, when $x[n]$ is the input, the output is given by

$$y[n] = H\{x[n]\} = x[n+3].$$

So $|y[n]| = |x[n+3]| \leq B$.

Therefore, $y[n]$ is bounded by B.

Since every bounded input produces a bounded output, the system <u>is</u> <u>stable</u>.

(4) Shift Invariant: Let the input be $x_1[n]$. Then the output is $y_1[n] = H\{x_1[n]\} = x_1[n+3]$.

Then $y_1[n-n_0] = x_1[n-n_0+3] \;\forall n_0 \in \mathbb{Z}$.

Now let $x_2[n] = x_1[n-n_0]$, where $n_0 \in \mathbb{Z}$.

Then $y_2[n] = H\{x_2[n]\} = x_2[n+3] = x_1[n-n_0+3]$

$= y_1[n-n_0]$ ✓

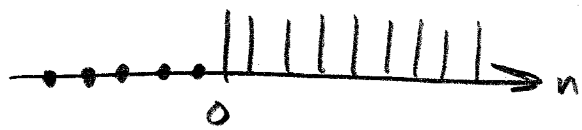
Therefore the system <u>is</u> <u>shift</u> <u>invariant</u>.

4.3b) $y[n] = x[2-n] + \alpha, \quad \alpha \in \mathbb{C}, \ \alpha \neq 0.$

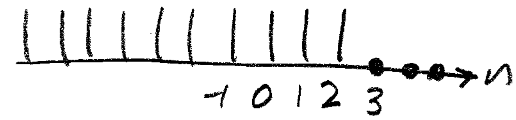
$\boxed{4.3b-1}$

(1) Linear: Let $x_1[n] = u[n]$

$u[n]$



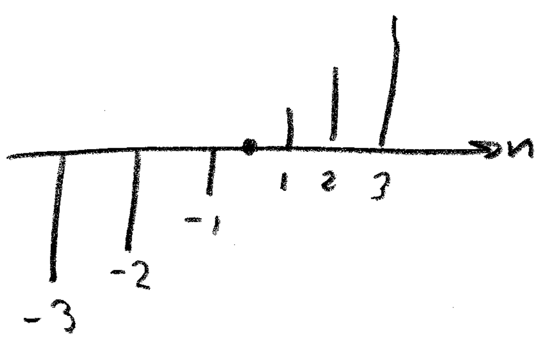
$u[2-n] = u[-n - -2]$



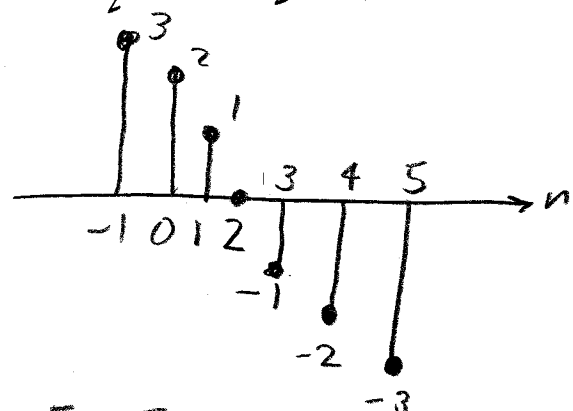
Then $y_1[n] = H\{x_1[n]\} = u[2-n] + \alpha.$

Let $x_2[n] = n.$

$x_2[n] = n$



$x_2[2-n] = 2-n$



Then $y_2[n] = H\{x_2[n]\} = x_2[2-n] + \alpha$

$= 2 - n + \alpha$

Now let $x_3[n] = x_1[n] + x_2[n] = u[n] + n$

Then $y_3[n] = H\{x_3[n]\} = x_3[2-n] + \alpha = u[2-n] + 2 - n + \alpha.$

But $y_1[n] + y_2[n] = u[2-n] + 2 - n + \underline{\underline{2\alpha}} \neq y_3[n].$

So the system is <u>not</u> <u>linear</u>.

(2) Causal : Let the input be $x[n] = n$. $\boxed{4.3b\text{-}2}$

Then, as in part (1), the output is given by

$$y[n] = H\{x[n]\} = x[2-n] + \alpha = 2-n+\alpha.$$

When $n=0$, we have $y[0] = x[2]+\alpha = 2+\alpha$.

Since $y[0]$ depends on the <u>future</u> input $x[2]$, the system is <u>not causal</u>.

(3) Stable : Let $x[n]$ be a bounded input. Then $\exists B \in \mathbb{R}$, $B>0$, s.t. $|x[n]| \leq B \quad \forall n \in \mathbb{Z}$.

The output is given by

$$y[n] = H\{x[n]\} = x[2-n] + \alpha,$$

so
$$\begin{aligned} |y[n]| &= |x[2-n] + \alpha| \\ &\leq |x[2-n]| + |\alpha| \\ &\leq B + |\alpha|. \end{aligned}$$

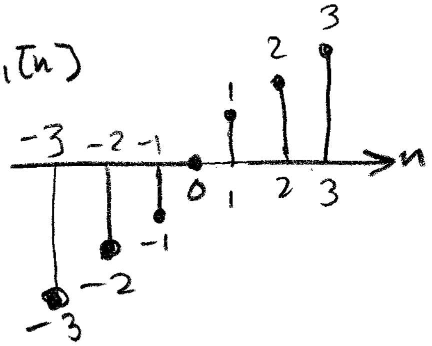
So $y[n]$ is bounded by $B+|\alpha|$.

Since every bounded input produces a bounded output, the system <u>is stable</u>.

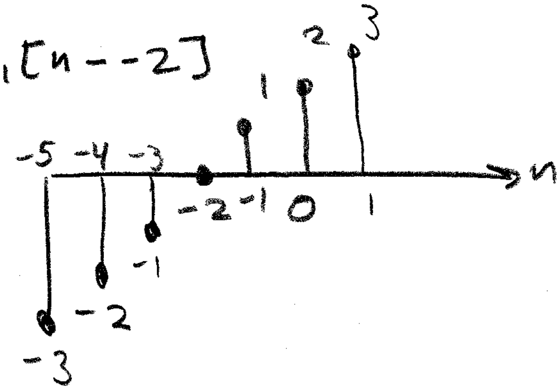
(4) Shift invariant:

Let $x_1[n] = n$.

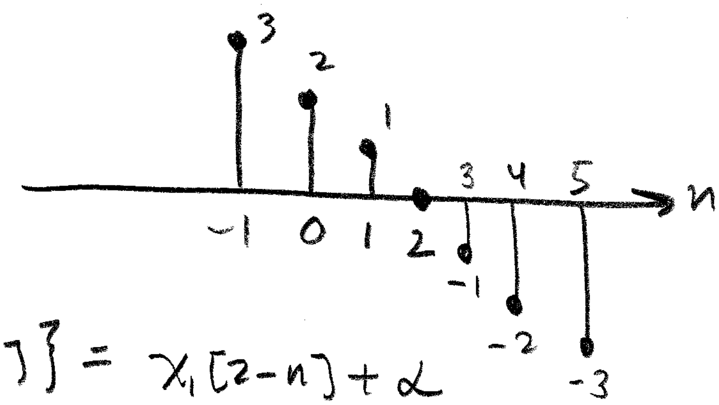
$x_1[n]$



$x_1[n--2]$

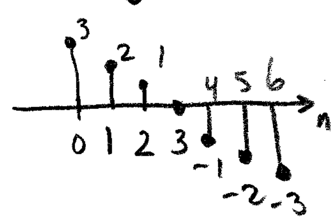


$x_1[2-n] = x_1[-n--2] = 2-n$



Then $y_1[n] = H\{x_1[n]\} = x_1[2-n] + \alpha$
$= 2-n+\alpha.$

Let $n_0 = 1$. Then $y_1[n-n_0] = y_1[n-1]$

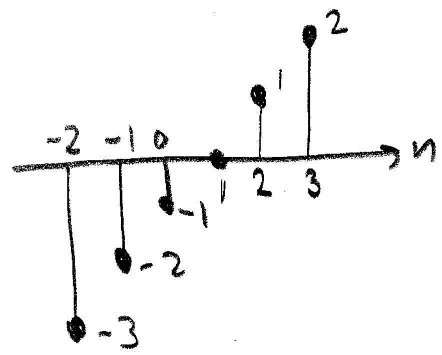
$=$  $+\alpha$

$= 3-n+\alpha.$

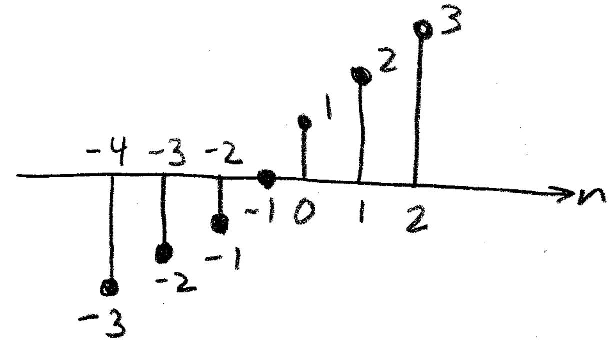
$\longrightarrow$

Now let $x_2[n] = x_1[n-n_0] = x_1[n-1] = n-1$: $\boxed{4.3b-4}$

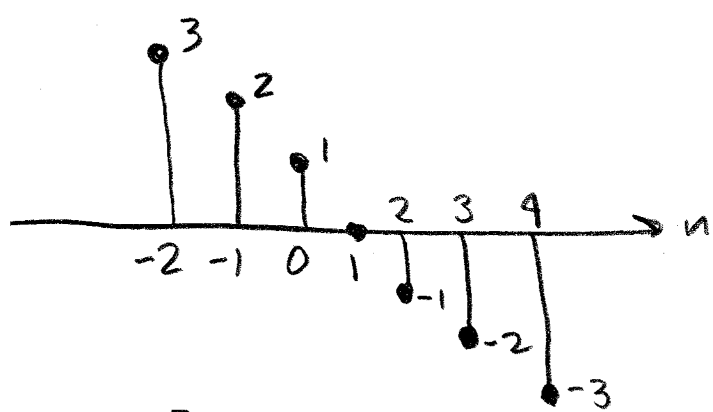
$x_2[n] = n-1$



$x_2[n--2] = x_2[2+n] = n+1$



$x_2[-n--2] = x_2[2-n] = 1-n$



Then $y_2[n] = H\{x_2[n]\} = x_2[2-n]+\alpha = 1-n+\alpha$.

Since $y_2[n] \neq y_1[n-1]$, the system is

<u>not</u> <u>shift</u> <u>invariant</u>.

4.3c) $y[n] = \ln(1 + |x[n]|)$

(1) Linear: Let the input be $x_1[n]$. Then the output is $y_1[n] = \ln(1 + |x_1[n]|)$.

Let the input be $x_2[n]$. Then the output is $y_2[n] = \ln(1 + |x_2[n]|)$.

Let $a, b \in \mathbb{C}$ be constants.

Then $a y_1[n] + b y_2[n] = a \ln(1 + |x_1[n]|) + b \ln(1 + |x_2[n]|)$.

Now let $x_3[n] = a x_1[n] + b x_2[n]$

Then
$$\begin{aligned} y_3[n] &= \ln(1 + |x_3[n]|) \\ &= \ln(1 + |a x_1[n] + b x_2[n]|) \\ &\neq a y_1[n] + b y_2[n] \text{ in general.} \end{aligned}$$

Therefore the system is <u>not linear</u>.

(2) Causal: At time $n = n_0$, the value of the output $y[n_0]$ depends on the current value of the input $x[n_0]$, but not on past values of the input $x[n]$ for $n < n_0$ and not on future values of the input $x[n]$ for $n > n_0$.

So the system <u>is causal</u> (and also memoryless).

(3) Stable: Let $x[n]$ be a bounded input.

Then $\exists B \in \mathbb{R}$, $B > 0$, s.t. $|x[n]| \leq B$ $\forall n \in \mathbb{Z}$.

The output is given by

$$y[n] = \ln(1 + |x[n]|),$$

so

$$|y[n]| = |\ln(1 + |x[n]|)|$$

$$\leq |\ln(1+B)| \quad \text{(since } \ln x \text{ is monotonically increasing for } x \geq 1.)$$

$$= \ln(1+B)$$

Then $y[n]$ is bounded by $\ln(1+B)$.

Since every bounded input signal produces a bounded output signal, the system <u>is stable</u>.

(4) Shift Invariant: Let the input be $x_1[n]$. Then the output is $y_1[n] = H\{x_1[n]\} = \ln(1 + |x_1[n]|)$.

$\forall n_0 \in \mathbb{Z}$, we have $y_1[n - n_0] = \ln(1 + |x_1[n-n_0]|)$.

Now let $n_0 \in \mathbb{Z}$ and let $x_2[n] = x_1[n - n_0]$. The output is $y_2[n] = H\{x_2[n]\} = H\{x_1[n-n_0]\} = \ln(1 + |x_1[n-n_0]|)$.

Since $y_2[n] = y_1[n - n_0]$, the system <u>is shift invariant</u>.

4.3d) $$y[n] = \beta + \sum_{\ell=-1}^{3} x[n-\ell]$$ $\boxed{4.3d\text{-}1}$

$$= \beta + x[n-3] + x[n-2] + x[n-1] + x[n] + x[n+1], \quad \beta \in \mathbb{C}, \ \beta \neq 0.$$

(1) Linear: Let $x_1[n] = 0$, e.g., $x_1[n]$ is the signal that is everywhere equal to zero.

Then $$y_1[n] = H\{x_1[n]\} = \beta + \sum_{\ell=-1}^{3} x_1[n-\ell]$$
$$= \beta + 0 = \beta.$$

Let $x_2[n] = 1$ (constant signal everywhere equal to one). Then
$$y_2[n] = H\{x_2[n]\} = \beta + \sum_{\ell=-1}^{3} x_2[n-\ell]$$
$$= \beta + \sum_{\ell=-1}^{3} 1 = \beta + 5.$$

Now let $x_3[n] = x_1[n] + x_2[n] = x_2[n] = 1$.
We have immediately from above that
$$y_3[n] = H\{x_3[n]\} = H\{x_2[n]\} = y_2[n] = \beta + 5.$$

But $y_1[n] + y_2[n] = \beta + \beta + 5 = 2\beta + 5 \neq y_3[n]$.

So the system is <u>not</u> <u>linear</u>.

(2) causal: when $n=0$, we have

$$y[0] = \beta + \sum_{l=-1}^{3} x[n-l] = \beta + x[-3] + x[-2] + x[-1] + x[0] + x[1],$$

which depends on the future value $x[1]$ of the input $x[n]$ at $n=1$.

Therefore, the system is <u>not causal</u>.

(3) Stable: Let $x[n]$ be a bounded input. Then $\exists B \in \mathbb{R}$, $B>0$, s.t. $|x[n]| \leq B \;\; \forall n \in \mathbb{Z}$.

The output is given by

$$y[n] = \beta + x[n-3] + x[n-2] + x[n-1] + x[n] + x[n+1],$$

So $|y[n]| = |\beta + x[n-3] + x[n-2] + x[n-1] + x[n] + x[n+1]|$

$$\leq |\beta| + |x[n-3]| + |x[n-2]| + |x[n-1]| + |x[n]| + |x[n+1]|$$

$$\leq |\beta| + 5B.$$

Therefore, $y[n]$ is bounded by $|\beta| + 5B$ when $x[n]$ is bounded by B.

Since every bounded input signal produces a bounded output signal, the system <u>is stable</u>.

(4) Shift Invariant:

Let $x_1[n]$ be the input and let $n_0 \in \mathbb{Z}$.
The output is given by

$$y_1[n] = H\{x_1[n]\}$$
$$= \beta + x_1[n-3] + x_1[n-2] + x_1[n-1] + x_1[n] + x_1[n+1].$$

So $y_1[n-n_0] = \beta + x_1[n-n_0-3] + x_1[n-n_0-2] + x_1[n-n_0-1]$
$+ x_1[n-n_0] + x_1[n-n_0+1].$

Now let $x_2[n] = x_1[n-n_0]$.
Then $y_2[n] = H\{x_2[n]\}$

$$= \beta + x_2[n-3] + x_2[n-2] + x_2[n-1] + x_2[n] + x_2[n+1]$$

$$= \beta + x_1[n-n_0-3] + x_1[n-n_0-2] + x_1[n-n_0-1]$$
$$+ x_1[n-n_0] + x_1[n-n_0+1]$$
$$= y_1[n-n_0]. \checkmark$$

Therefore the system is shift invariant.

4.8) The system I/O relation is: $\boxed{4.8-1}$

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n]$$

$$y[n] = x[n+1] - 2x[n] + x[n-1]$$

Let $x_1[n]$ be the input. Then we have

$$y_1[n] = H\{x_1[n]\} = x_1[n+1] - 2x_1[n] + x_1[n-1].$$

Let $x_2[n]$ be the input. Then

$$y_2[n] = H\{x_2[n]\} = x_2[n+1] - 2x_2[n] + x_2[n-1].$$

Let $c_1, c_2 \in \mathbb{C}$ be arbitrary constants.

Then
$$\begin{aligned} c_1 y_1[n] + c_2 y_2[n] = \; & c_1 x_1[n+1] - 2c_1 x_1[n] \\ & + c_1 x_1[n-1] \\ & + c_2 x_2[n+1] - 2c_2 x_2[n] \\ & + c_2 x_2[n-1]. \end{aligned}$$

Let $x_3[n] = c_1 x_1[n] + c_2 x_2[n]$.

Then
$$\begin{aligned} y_3[n] = H\{x_3[n]\} &= x_3[n+1] - 2x_3[n] + x_3[n-1] \\ &= \left(c_1 x_1[n+1] + c_2 x_2[n+1]\right) \\ &\quad -2\left(c_1 x_1[n] + c_2 x_2[n]\right) \\ &\quad +\left(c_1 x_1[n-1] + c_2 x_2[n-1]\right) \end{aligned}$$

$$= c_1 x_1[n+1] + c_2 x_2[n+1]$$
$$- 2c_1 x_1[n] - 2c_2 x_2[n]$$
$$+ c_1 x_1[n-1] + c_2 x_2[n-1]$$

$$= c_1 x_1[n+1] - 2c_1 x_1[n] + c_1 x_1[n-1]$$
$$+ c_2 x_2[n+1] - 2c_2 x_2[n] + c_2 x_2[n-1]$$

$$= c_1 \left(x_1[n+1] - 2x_1[n] + x_1[n-1] \right)$$
$$+ c_2 \left(x_2[n+1] - 2x_2[n] + x_2[n-1] \right)$$

$$= c_1 y_1[n] + c_2 y_2[n] \checkmark$$

$\Rightarrow$ The system is linear.

Let $x_1[n]$ be the input. Then

$$y_1[n] = H\{x_1[n]\}$$
$$= x_1[n+1] - 2x_1[n] + x_1[n-1].$$

Let $n_0 \in \mathbb{Z}$. Then

$$y_1[n-n_0] = x_1[n+1-n_0] - 2x_1[n-n_0] + x_1[n-1-n_0].$$

$\longrightarrow$

4.8-3

Now let $x_2[n] = x_1[n-n_0]$.

Then $y_2[n] = H\{x_2[n]\}$

$$= x_2[n+1] - 2x_2[n] + x_2[n-1]$$

$$= x_1[n-n_0+1] - 2x_1[n-n_0] + x_1[n-n_0-1]$$

$$= x_1[n+1-n_0] - 2x_1[n-n_0] + x_1[n-1-n_0]$$

$$= y_1[n-n_0] \checkmark$$

The system <u>is</u> time invariant.

Consider the case $n=2$. At $n=2$,

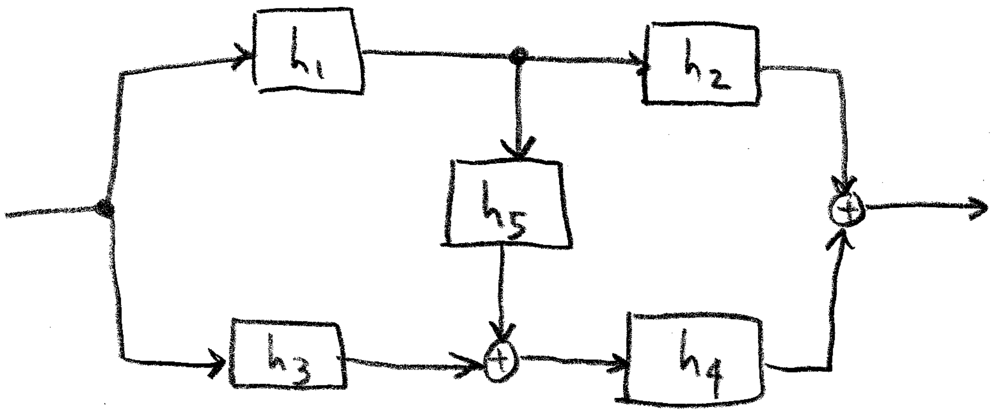
$$y[2] = x[3] - 2x[2] + x[1].$$

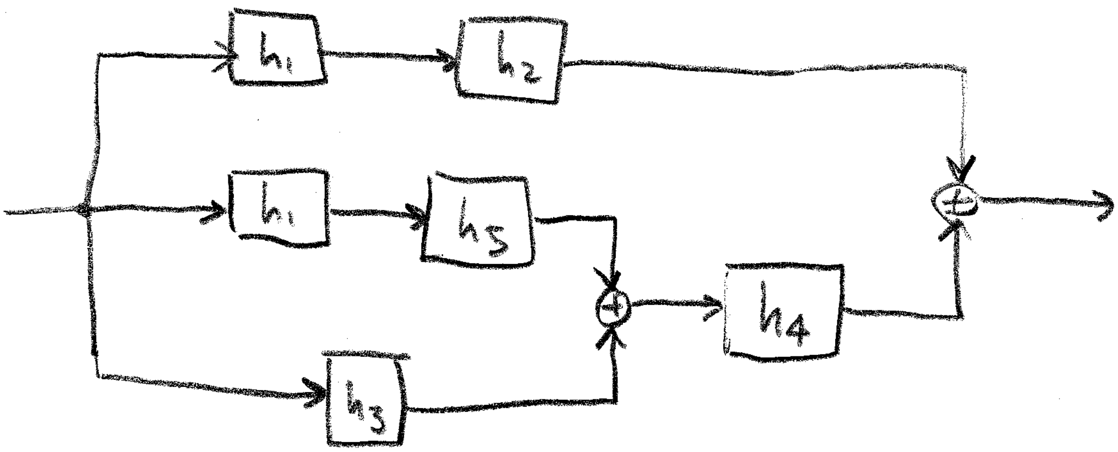
$\Rightarrow$ calculation of the output $y[2]$ requires the <u>future</u> input $x[3]$.

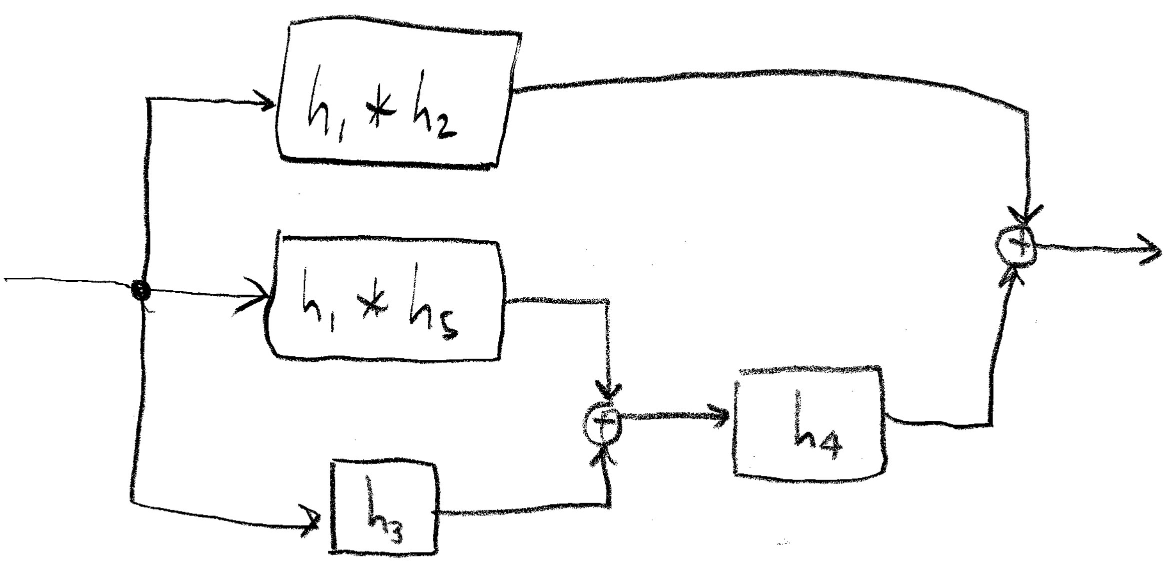
Therefore, the system is <u>not</u> <u>causal</u>

4.30a)

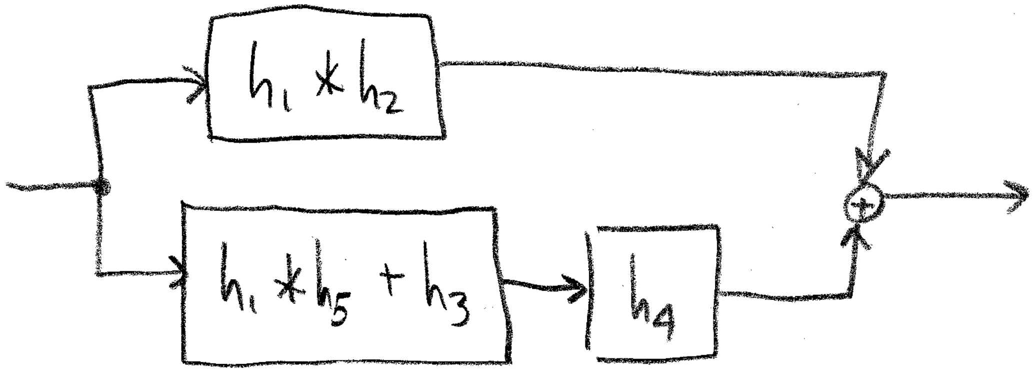
$\boxed{4.30a-1}$

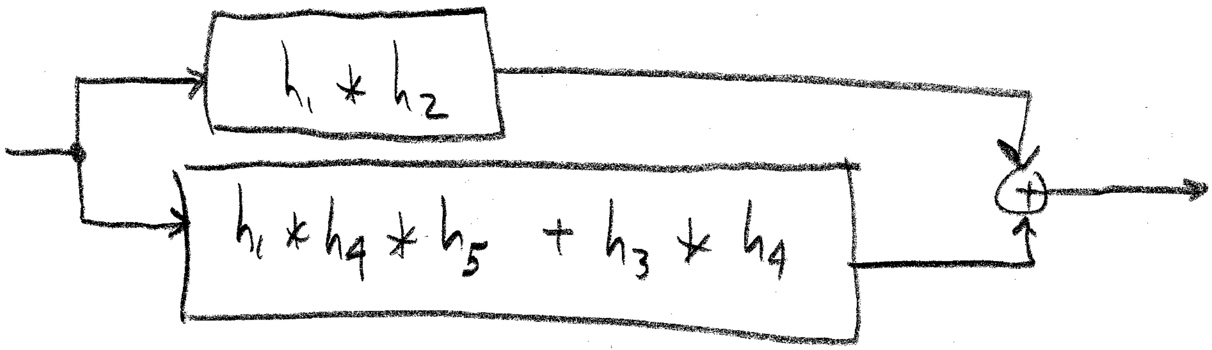


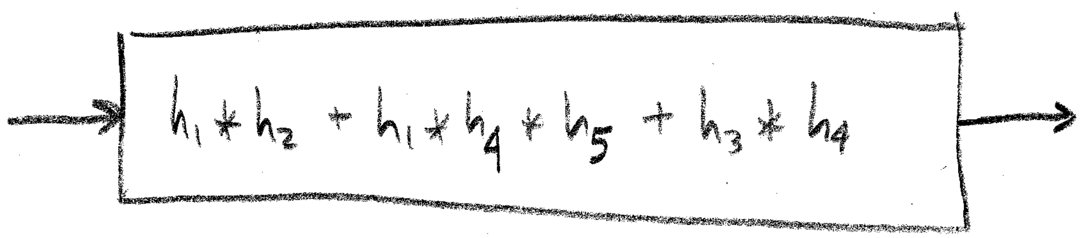




4.30a-2



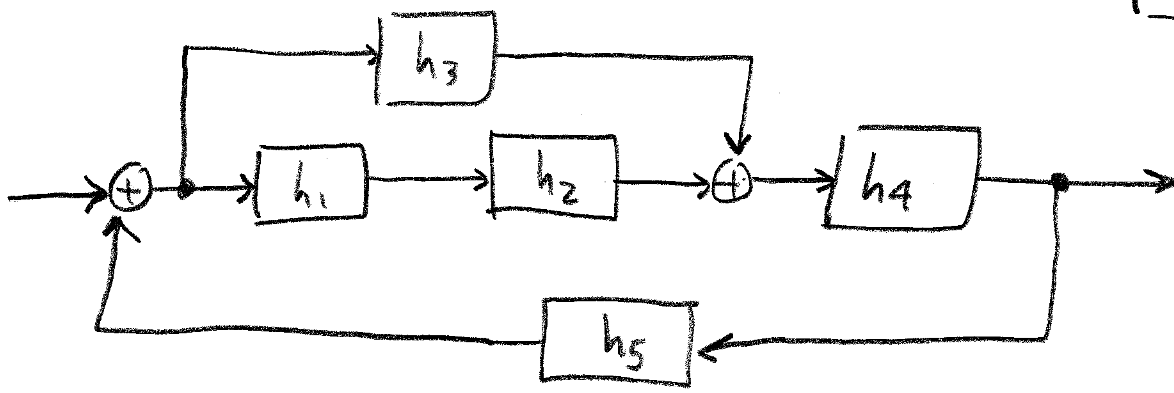


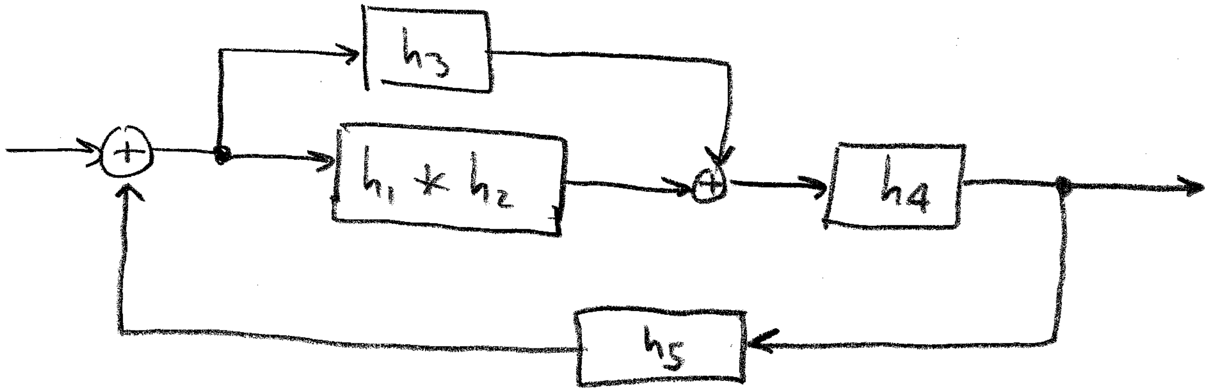


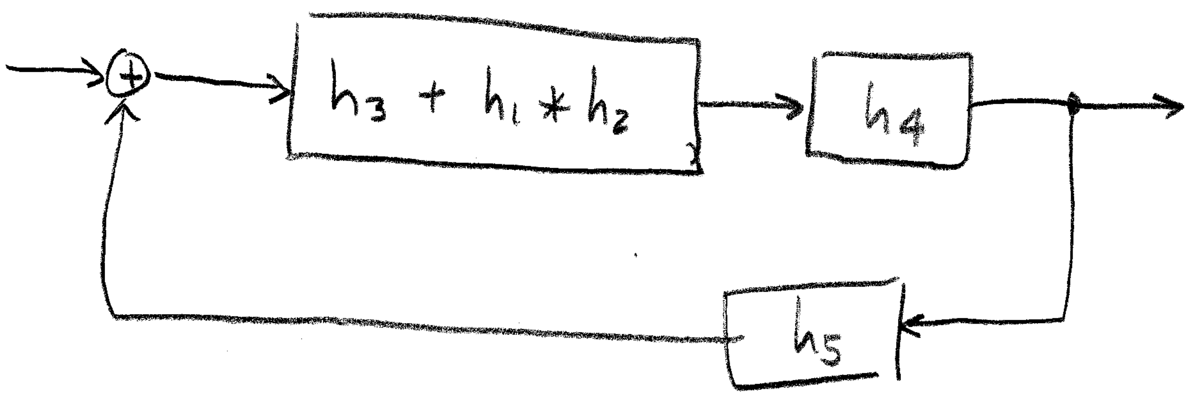
$$h[n] = h_1[n] * h_2[n] + h_1[n] * h_4[n] * h_5[n] + h_3[n] * h_4[n]$$

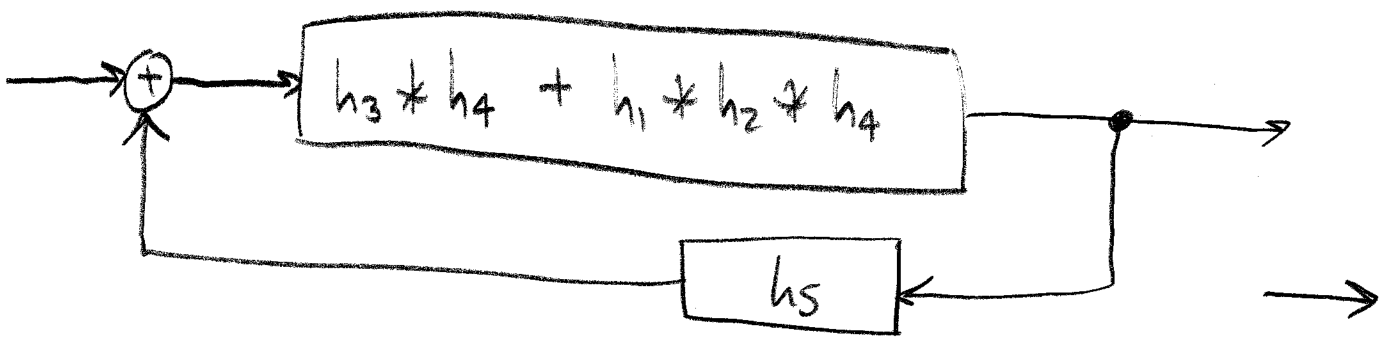
4.30b)

4.30b-1

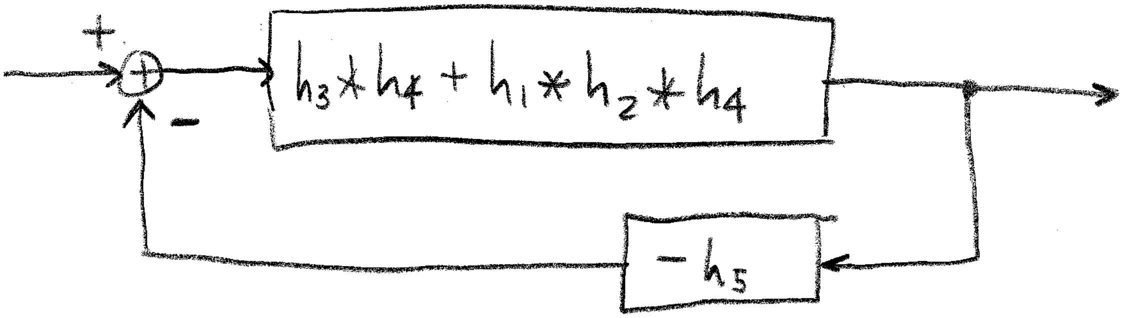








4.30b-2



Forward path gain: $H_3(z)H_4(z) + H_1(z)H_2(z)H_4(z)$

Reverse path gain: $-H_5(z)$

$$H(z) = \frac{\text{Forward Gain}}{1 + (\text{Forward Gain})(\text{Reverse Gain})}$$

$$= \frac{H_3(z)H_4(z) + H_1(z)H_2(z)H_4(z)}{1 - H_3(z)H_4(z)H_5(z) - H_1(z)H_2(z)H_4(z)H_5(z)}$$

$h[n] = \mathcal{Z}^{-1}\{H(z)\}$ (no general closed form solution in time domain)