Homework 4

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Exercises

Prove (or disprove) the following results, showing all steps of your argument.

1. The sum of two odd integers is even.
   Let a = 2n+1, b = 2k+1

$$\begin{aligned} a + b &= (2n + 1) + (2k + 1) \\ &= 2n + 2k + 2 \\ &= 2(n + k + 1) \end{aligned}$$

   Therefore the result holds

2. The sum of two even integers is even.
   Let a = 2n, b = 2k

$$\begin{aligned} a + b &= 2n + 2k \\ &= 2(n + k) \end{aligned}$$

   Therefore the result holds

3. The square of an even number is even.
   Let a = 2n

$$\begin{aligned} a^2 &= (2n)^2 \\ &= 4n^2 \\ &= 2(2n)^2 \end{aligned}$$

   Therefore the result holds

4. The product of two odd integers is odd.
Let a = 2n+1, b = 2k+1

$$
\begin{aligned}
ab &= (2n+1)(2k+1) \\
&= 4kn + 2n + 2k + 1 \\
&= 2(2kn + n + k) + 1
\end{aligned}
$$

Therefore the result holds

5. If n^3 + 5 is odd then n is even, for any n ∈ Z.
We can solve this by using contraposition, so let n = 2k+1

$$
\begin{aligned}
n^3 + 5 &= (2k+1)^3 + 5 \\
&= 8k^3 + 12k^2 + 6k + 1 + 5 \\
&= 8k^3 + 12k^2 + 6k + 6 \\
&= 2(4k^3 + 6k^2 + 3k + 3)
\end{aligned}
$$

Therefore the original result holds

6. If 3n + 2 is even then n is even, for any n ∈ Z.
We can solve this using contraposition, so let n = 2k+1

$$
\begin{aligned}
3n + 2 &= 3(2k+1) + 2 \\
&= 6k + 3 + 2 \\
&= 6k + 4 + 1 \\
&= 2(3k+2) + 1
\end{aligned}
$$

Therefore the original result holds

7. The sum of a rational number and an irrational number is irrational.
We can prove this via contradiction, Let I be a irrational number and let n = p/q

$$
\begin{aligned}
n + I &= a/b \\
p/q + I &= a/b \\
I &= a/b - p/q \\
&= (aq - bp)/bq
\end{aligned}
$$

This means that I is rational, a contradiction to our original statement

8. The product of two irrational numbers is irrational.
Since we know that $\sqrt{2}$ is irrational, we can do the following:

$$\begin{aligned}\sqrt{2}\sqrt{2} &= 2^{\frac{1}{2}}2^{\frac{1}{2}} \\ &= 2^1 \\ &= 2\end{aligned}$$

So a product of two irrationals is rational, therefore result is false.

9. Use mathematical induction to prove that the first n even integers add up to n(n + 1)
Base case: Set n = 1

$$2 = 1(1 + 1) = 2$$

Therefore the base case holds

Induction Hypothesis: Assume the property holds when n = $k - 1$

$$2 + 4 + ... + 2(k - 1) = k(k - 1)$$

Inductive Step: Show that if property holds for k-1 then it must hold for k

$$\begin{aligned}2 + 4 + ... + 2(k - 1) &= k(k - 1) \\ &= k^2 - k + 2k \\ &= k^2 + k \\ &= k(k + 1)\end{aligned}$$

Therefore the property holds for all positive integers.

10. Use mathematical induction to prove that n^3 + 2n is divisible by 3.
Base case: Set n = 1

$$1^3 + 2(1)/3 = 3/3 = 1$$

Therefore the base case holds

Induction Hypothesis: Assume if it works for n = k then k^3 + 2k is divisible by 3

Inductive Step: Show that if it works for k, then it will work for k+1

$$
\begin{aligned}
&(k+1)^3 + 2(k+1) \\
&= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= k^3 + 2k + 3k^2 + 3k + 3 \\
&= k^3 + 2k + 3(k^2 + k + 1)
\end{aligned}
$$

Therefore result is true for all n.