A Relativistic Study of Diffusion Using Local Random Walks

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May 2020

1 Introduction

There have been numerous attempts to derive the behaviour of diffusing gases on lorentzian manifolds, flat or otherwise, with various degrees of success. Notably, there has not yet been a derivation which is uncontentiously fully relativistic while simultaneously preserving the second law of thermodynamics, the most agreed upon so far being that of the telegrapher equation.

In the following we will find that by modifying the Markovian process to rely on local proper time we will be able to analyze the diffusion of a gas in terms of a series of random walks to retrieve an equation of motion which does not suffer from any of the above mentioned issues, and which is relatively easily generalized onto a curved spacetime. Throughout we will also draw analogies between this theory and the quantum mechanical theory of statistics, and find they bare a resemblance similar to that found in a previous writing.

Throughout this study we use $\hbar = k = c = 1$ and the $(+ - - -)$ signature.

2 Local Random Walks

The typical derivation of the random walk in euclidean space treats the motion of the particles of the gas as a Markovian process, where after a certain amount of invariant global time, all the particles step a fixed distance in a random direction in a way that is uncorrelated with their previous state. It has been shown, however, that Markovian processes are impossible in relativity, as they give rise to first order differential equations in the timelike coordinate. It is not surprising that Markovian processses are relativistically invalid; as was expressed in their here wording, they rely on invariant global time, which is unphysical.

Let us instead adapt the Markovian process to the local frame of a random walker, where normally they rely on a global time, here they will wait some proper time before rerandomizing their motion. In a sense, this proper time will behave like a decay time before the particle becomes completely uncorrelated with its past state.

Consider randomly choosing a single particle among a gas of particles and observing it in some coordinate frame until it interacts with the surrounding system and changes rapidity and direction. In general, we expect the particle to move some proper length δs before changing to a new spacetime orientation (η, ζ, ψ) with probability densities $q(\delta s)$ and $\rho(\eta, \zeta, \psi)$. If we take the system to the infinite limit, the former distribution will converge to a delta function around 0, and so the change in state will be entirely determined by the orientation distribution.

We can translate the particles behaviour into the language of manifolds by saying that after each system interaction, the particle will randomly choose from the set of unit tangent vectors in its local frame $\{\xi(\eta, \zeta, \psi)\}$ and move a proper length δs along the geodesic corresponding to that tangent vector and intersecting the position of the particle. The master equation for the position of the particle is then

$$\phi(\boldsymbol{x}) = \int d\eta d\zeta d\psi \; \phi(\boldsymbol{x} - \delta s \boldsymbol{\xi}) \rho(\eta, \zeta, \psi) \tag{1}$$

where $\phi(\boldsymbol{x})dV$ is the probability of finding the particle in the volume element dV. Taylor expanding around the proper length along the geodesic corresponding to $\boldsymbol{\xi}$

$$\phi(\boldsymbol{x}) = \phi(\boldsymbol{x}) - \int d\eta d\zeta d\psi \; \delta s \rho(\eta, \zeta, \psi) \xi^\mu \nabla_\mu \phi(\boldsymbol{x}) - \frac{1}{2} \delta s^2 \rho(\eta, \zeta, \psi) \xi^\nu \nabla_\nu (\xi^\mu \nabla_\mu \phi(\boldsymbol{x})) + \cdots \tag{2}$$

where we have used the normalization of $\rho(\eta, \zeta, \psi)$ to take $\phi(\boldsymbol{x})$ outside of the integral. Now, because the $\boldsymbol{\xi}$ obey the geodesic equation, we can commute them outside the covariant derivatives, which with some algebra allows us to reach the final expression

$$\alpha^\mu \nabla_\mu \phi = \kappa^{\mu\nu} \nabla_\mu \nabla_\nu \phi \tag{3}$$

where we have

$$\alpha^\mu = \int d\eta d\zeta d\psi \; \rho(\eta, \zeta, \psi) \xi^\mu \tag{4}$$

and

$$\kappa^{\mu\nu} = \frac{1}{2} \delta s \int d\eta d\zeta d\psi \; \rho(\eta, \zeta, \psi) \xi^\mu \xi^\nu \tag{5}$$

Because $\int d\eta \rho(\eta, \zeta, \psi) = 1$ is Lorentz invariant, and the $\boldsymbol{\xi}$ are simply chosen from the local tangent space, the α^μ and $\kappa^{\mu\nu}$ are tensorial, and so Eqn. (3) is fully tensorial

To see that the results here satisfy all the relevant conditions, let us study the equation in Minkowski space.

3 Diffusion in Minkowski Space

In flat spacetime, Eqn. (3) can be written

$$\alpha^\mu \partial_\mu \phi = \kappa^{\mu\nu} \partial_\mu \partial_\nu \phi \tag{6}$$

Because space is isotropic, $\rho(\eta, \zeta, \psi)$ should be symmetric, meaning $\alpha^i = \kappa^{\mu\nu} = \kappa^{\nu\mu} = 0$ in the rest frame of the system. If time were isotropic as well, then $\alpha^\mu = 0$, and Eqns. (6) and (3) reduce to simple wave equations. This implies that the first order "damping term" arises due to the arrow of time, and can in fact be interpreted as an entropy contribution. Writing Eqn. (6) in this coordinate system

$$\alpha^0 \frac{\partial \phi}{\partial t} = \kappa^{00} \frac{\partial^2 \phi}{\partial t^2} + \kappa^{11} \frac{\partial^2 \phi}{\partial x^2} + \kappa^{22} \frac{\partial^2 \phi}{\partial y^2} + \kappa^{33} \frac{\partial^2 \phi}{\partial z^2} \tag{7}$$

where α^0 and the $\kappa^{\mu\mu}$ are positive. This equation is hyperbolic, and so does not have wavelike solutions, but in fact steadily decaying solutions. Fourier transforming and using the standard ansatz, we find

$$\phi = \int d^3k e^{\gamma t} (A e^{\Omega t} + B e^{-\Omega t}) e^{ik^i x_i} \tag{8}$$

where

$$\begin{aligned} \Omega &= \frac{1}{2\kappa^{00}} \sqrt{(\alpha^0)^2 + 4\kappa^{00}\kappa^{ij} k_i k_j} \\ \gamma &= \frac{\alpha^0}{2\kappa^{00}} \end{aligned} \tag{9}$$

and $\boldsymbol{k}$ is the relevant wavevector. One can see that with $\kappa^{\mu\nu} \geq 0$, normalizable solutions ($A = 0$) are indeed diffusive. Furthermore, because $\tilde{\phi}$ is even, we know ϕ will be real, and can therefore be interpreted as a probability. If the solutions could be complex, then we might have had to interpret ϕ as a probability density or a statistical field. The only disadvantage to not having to do this is that ϕ will not be automatically normalized in other function bases. This is purely a calculational inconvenience, as ϕ is still interpretable as a density in those bases and can simply be re-normalized.

4 Temperature

The naive assumption would be that, just as in the classical case, the temperature and probability distributions are proportional. As was shown in a previous writing, however, temperature is not so simple relativistically, and in general thermodynamic systems obey

$$dS = g_{\mu\nu}\beta^\mu dp^\nu \tag{10}$$

where S is the entropy, $\boldsymbol{g}$ the metric, $\boldsymbol{p}$ the four momentum, and $\boldsymbol{\beta}$ the inverse four temperature. If we wish to extend this, we might then assert that $\phi^{-1} \propto \sqrt{\beta^\mu\beta_\mu}$. There is, however, a more physically intuitive (rather than simply mathematically intuitive) and general statement we can make about the nature of temperature which leads us to a relationship between ϕ and $\boldsymbol{\beta}$ which we can reach by considering the Lagrangian associated with Eqn. (6).

In nonequilibrium systems, the effective definition of temperature concerns the tendency of heat to flow away from a system, and is expressed classically in terms of Fourier's law.

$$\begin{aligned} \boldsymbol{J} &= -\boldsymbol{A}\cdot\nabla u \\ &= \frac{\boldsymbol{A}}{\beta_0^2}\cdot\nabla\beta_0 \end{aligned}$$

(11)

where $\boldsymbol{J}$ is the heat current, u is the classical temperature density, and $\boldsymbol{A}$ is the heat conduction matrix. If we wish to translate this into the formalism given above, then one might wish to do

$$\begin{aligned} j^\mu &= \frac{1}{(\beta^0)^2}A^{\mu\sigma}\partial_\sigma\beta^0 \\ T^{\mu\nu} &= \frac{1}{(\beta^\nu)^2}A^{\mu\sigma}\partial_\sigma\beta^\nu \end{aligned} \tag{12}$$

where $\boldsymbol{T}$ is the stress tensor. This, however, would not be tensorial, as one can see from the reciprocal beta squared term. Instead, because $\beta^i = 0$ in the classical limit, we can do

$$T^{\mu\nu} = \frac{1}{\boldsymbol{\beta}^2}A^{\mu\sigma}\partial_\sigma\beta^\nu \tag{13}$$

which is both tensorial and agrees with the classical limit. For homogoneous systems this reads

$$T^{\mu\nu} = \frac{1}{\boldsymbol{\beta}^2}\partial_\sigma(A^{\mu\sigma}\beta^\nu) \tag{14}$$

and isotropic systems

$$T^{\mu\nu} = \frac{A}{\boldsymbol{\beta}^2}\partial^\mu\beta^\nu \tag{15}$$

Interestingly, Eqn. (14) implies that if a single component $T^{\mu'\nu'}$ is zero, then the quantity $A^{\mu'0}\beta^{\nu'}$ is locally conserved in a homogeneous spacetime. If we let $\mu'\nu' = i0$ then we have a vanishing momentum implying a local conservation. Normally vanishing momentum in a homogeneous spacetime implies conservation of relative positions, creating an odd analogy between position and this temperature like object.

Let us now try to form a stress tensor from the equations of motion to relate Eqn. (13) to ϕ. Consider the Lagrangian for the Minkowski equations of motion

$$\mathcal{L} = \frac{1}{2}e^{-\gamma_\rho x^\rho}\kappa^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{16}$$

where $\boldsymbol{\gamma}$ satisfies $\gamma_\mu\kappa^{\mu\nu} = \alpha^\nu$. We note again that for a spacetime-symmetric system $\boldsymbol{\alpha} = 0$, which would kill the exponential term. Therefore the introduction of entropy and the arrow of time produces a spacetime dependent decay term in the Lagrangian, which manifests in the equations of motion as something similar to a damping.

If we wish to find a stress tensor from Eqn. (14) we must be wary of the explicit spatial dependence of $\mathcal{L}$ on $\boldsymbol{x}$. While the spatial transformation will still have $\delta\phi = \partial_\mu\phi\delta x^\mu$ implying $\delta\mathcal{L} = \partial_\mu\mathcal{L}\delta x^\mu$, $\delta\mathcal{L}$ will now be

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial_\mu\phi}\delta\phi + \partial_\mu\mathcal{L}\delta x^\mu \tag{17}$$

which in following the typical prescription set out by Noether's theorem eventually returns

$$\begin{aligned}\partial_\mu T^{\mu\nu} = \partial_\mu\left(e^{-\gamma_\rho x^\rho}\kappa^{\mu\sigma}\partial_\sigma\phi\partial^\nu\phi\right) = 0\\ \implies T^{\mu\nu} = e^{-\gamma_\rho x^\rho}\kappa^{\mu\sigma}\partial_\sigma\phi\partial^\nu\phi\end{aligned} \tag{18}$$

which when we combine with Eqn. (13) gives

$$e^{-\gamma_\rho x^\rho}\kappa^{\mu\sigma}\partial_\sigma\phi\partial^\nu\phi = \frac{1}{\boldsymbol{\beta}^2}A^{\mu\sigma}\partial_\sigma\beta^\nu \tag{19}$$

For an isotropic system this is just

$$e^{-\gamma_\rho x^\rho}\kappa^{\mu\sigma}\partial_\sigma\phi\partial^\nu\phi = \frac{A}{\boldsymbol{\beta}^2}\partial^\mu\beta^\nu \tag{20}$$

There are a few pieces of information we can take from Eqn. (18) and (19). Firstly, the stress tensor is not necessarily symmetric, implying a non-conserved angular momentum. This arises due to the dissipation term in the Lagrangian, implying vorticities arising from dissipation, a familiar concept from fluids. Secondly, Eqn. (20) tells us that a purely rotating ϕ distribution, and therefore a purely antisymmetric stress tensor, will have an antisymmetric $\partial^\mu\beta^\nu$ term, therefore identifying the inverse four temperature field with a killing field in this case. Finally, the form of Eqn. (19) correctly identifies a kinetic term with the flow of heat, however in such a way that dismisses the assumption $\phi^{-1} \propto \sqrt{\beta^\mu \beta_\mu}$

Before moving on, it is important to note that this section is in general based on so far unresolved assumptions regarding relativistic temperature and heat flow, and is built essentially off nice analogies. This demands a criticallity concerning, in particular, Eqns. (12)-(15), (19), and (20). Crucially, the other sections do not depend on these relationships.

5 Relationship to Quantum Mechanics and Gravity

It is well known that thermodynamics has an important relationship to quantum gravity, although it is unclear exactly what this relationship is. There is clearly also some vague sense in which Eqn. (6) resembles the Klein Gordon Equation, differing in the metric, the entropy term, and the quantity ϕ. We will now show that changing the variable ϕ to something more similar to Dirac's postulate $\psi = A(x)e^{iS}$ returns a hyperbolic Klein Gordon equation. After this we will show that coupling the orientational distribution to spacetime essentially works to couple that hyperbolic equation of motion to a Riemannian manifold although in a way that causes the mass term to vary.

Let $\varphi = \phi e^{-\frac{1}{2}\gamma_\rho x^\rho}$ and consider the following Lagrangian,

$$\begin{aligned}\mathcal{L}_\varphi &= \frac{1}{2}\kappa^{\mu\nu}\Big(\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{4}\gamma_\mu\gamma_\nu\varphi^2\Big)\\ &= \frac{1}{2}e^{-\gamma_\rho x^\rho}\kappa^{\mu\nu}\Big(\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\gamma_\mu\gamma_\nu\phi^2 - \gamma_\mu\phi\partial_\nu\phi\Big)\end{aligned} \tag{21}$$

The variation of the last two terms is zero:

$$\begin{aligned}\frac{\delta}{\delta\phi}e^{-\gamma_\rho x^\rho}\kappa^{\mu\nu}\Big(\gamma_\mu\gamma_\nu\phi^2 - \gamma_\mu\partial_\nu\phi\Big) &= e^{-\gamma_\rho x^\rho}\kappa^{\mu\nu}\Big(- \gamma_\mu\gamma_\nu\phi + \gamma_\mu\partial_\nu\phi - \gamma_\mu\partial_\nu\phi + \gamma_\mu\gamma_\nu\phi\Big)\\ &= 0\end{aligned} \tag{22}$$

and therefore we have that

$$\frac{\delta\mathcal{L}_\varphi}{\delta\phi} = \frac{\delta\mathcal{L}_\phi}{\delta\phi} \tag{23}$$

return the same equations of motion. We can now write the time evolved Lagrangian as

$$\mathcal{L}_\varphi = \frac{1}{2}\kappa^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}m^2\varphi^2 \tag{24}$$

where

$$m^2 = \frac{1}{4}\kappa^{\mu\nu}\gamma_\mu\gamma_\nu \tag{25}$$

thus identifying $\boldsymbol{\gamma}$ with a sort of momentum. Varying this with respect to φ then gives the equation of motion

$$(\kappa^{\mu\nu}\partial_\mu\partial_\nu - m^2)\varphi = 0 \tag{26}$$

This bears some resemblance to the Klein Gordon equation, except with an imaginary mass, and this $\boldsymbol{\kappa}$ diffusion term (with a $(++++)$ signature) instead of the flat spacetime metric $\boldsymbol{\eta}$. Interestingly, the metric like quantity $\boldsymbol{\kappa}$ arises from diffusive processes on a Minkowski spacetime, rather than any preconceived notion of a curved spacetime. One might then posit that if we were to allow $\rho(\eta, \zeta, \psi)$ to vary in $\boldsymbol{x}$, then follow a similar procedure for finding the $\mathcal{L}_\varphi$ from $\mathcal{L}_\phi$, we would get a similar equation coupled to a curved manifold. Because $\boldsymbol{\kappa}$ has an all plusses signature, this manifold we expect to be Riemannian.

Coupling $\rho(\eta, \zeta, \psi)$ to spacetime gives the diffusion equation

$$\alpha^\mu(\boldsymbol{x})\partial_\mu\phi = \kappa^{\mu\nu}(\boldsymbol{x})\partial_\mu\partial_\nu\phi \tag{27}$$

The Lagrangian for this is now

$$\mathcal{L}_\phi = \frac{1}{2}e^{-G}\sqrt{\kappa}\kappa^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{28}$$

where

$$\kappa^{\mu\nu}\partial_\mu G - \partial_\mu\kappa^{\mu\nu} - \partial_\mu \ln\sqrt{\kappa}\kappa^{\mu\nu} = \alpha^\nu \tag{29}$$

We can now see that $\partial_\mu G$ is essentially the generalization of γ_μ to probabilistically curved space, such that G might be identified as an action, or perhaps entropy. This implies that φ should become $\varphi = \phi e^{-\frac{1}{2}G}$. Consider now the φ Lagrangian

$$\mathcal{L}_\varphi = \frac{1}{2}\sqrt{\kappa}\kappa^{\mu\nu}\Big(\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{4}\partial_\mu G\partial_\nu G\varphi^2\Big) \tag{30}$$

Expanding in terms of ϕ

$$\mathcal{L}_\varphi = \frac{1}{2}\sqrt{\kappa}\kappa^{\mu\nu}\Big(\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\phi^2\partial_\mu G\partial_\nu G - \phi\partial_\mu\phi\partial_\nu G\Big)e^{-G} \tag{31}$$

Varying with respect to ϕ

$$\frac{\delta\mathcal{L}_\varphi}{\delta\phi} = \frac{\delta\mathcal{L}_\phi}{\delta\phi} - \frac{1}{2}\partial_\mu(\sqrt{\kappa}\kappa^{\mu\nu}\partial_\nu G)\varphi \tag{32}$$

and so we can fix the φ Lagrangian so that it is equivalent to the ϕ Lagrangian by simply adjusting the mass term so that it cancels with this auxiliary term in Eqn. (32). Such a term would essentially turn out as a lot of garbage.

It will then be interesting and useful to make the following identifications:

$$\Pi^{\rho}_{\mu\nu} = \frac{1}{2}\kappa^{\rho\sigma}\Big(\partial_\mu(\kappa^{-1})_{\nu\sigma} + \partial_\nu(\kappa^{-1})_{\sigma\mu} - \partial_\sigma(\kappa^{-1})_{\mu\nu}\Big) \tag{33}$$

and now

$$D_\mu v_\nu = \partial_\mu v^\nu + \Pi^{\nu}_{\mu\sigma} v^\sigma \tag{34}$$

$$D_\mu v^\nu = \partial_\mu v_\nu - \Pi^{\sigma}_{\mu\nu} v_\sigma \tag{35}$$

where we note that the indices are still raised and lowered under the Lorentzian manifold background. We can now use the known identity

$$\kappa^{\mu\nu}\Pi^{\rho}_{\mu\nu} = \frac{-1}{\sqrt{\kappa}}\partial_\mu(\sqrt{\kappa}\kappa^{\mu\rho}) \tag{36}$$

to write the contracted covariant derivative as

$$\kappa^{\mu\nu} D_\mu v_\nu = \frac{1}{\sqrt{\kappa}}\partial_\mu(\sqrt{\kappa}\kappa^{\mu\rho} v_\rho) \tag{37}$$

where after applying the product rule one must relabel indices on the Christoffel term to retrieve the LHS from the right. Using this we can rewrite Eqn. (32) as

$$\frac{\delta\mathcal{L}_\varphi}{\delta\phi} = \frac{\delta\mathcal{L}_\phi}{\delta\phi} - \frac{1}{2}\sqrt{\kappa}\kappa^{\mu\nu} D_\mu(\partial_\nu G)\varphi \tag{38}$$

therefore demanding $\mathcal{L}_\varphi$ be written as

$$\mathcal{L} = \frac{1}{2}\sqrt{\kappa}\Big(\kappa^{\mu\nu} D_\mu\varphi D_\nu\varphi + M^2\varphi^2\Big) \tag{39}$$

leading to the equation of motion

$$\frac{\delta\mathcal{L}_\varphi}{\delta\varphi} = \kappa^{\mu\nu} D_\mu D_\nu\varphi - M^2\varphi = 0 \tag{40}$$

where

$$M^2 = \kappa^{\mu\nu}\Big(\frac{1}{4}D_\mu G D_\nu G - \frac{1}{2}D_\mu D_\nu G\Big) \tag{41}$$

and we have written many of the partial derivatives on scalars as covariant derivatives for the sake of symmetry.

With Eqn. (40) we have, again, almost retrieved the KGE in a curved spacetime, except the metric has an all plusses signature, the mass is imagniary, and the φ is not a probability amplitude. The mass also features a second order derivative action term which does not naturally vanish with a vanishing diffusion constant α. If we take the identification of G as an action seriously, then we expect $D_\mu G = T_{\mu 0}$ to be the conjugate momentum, such that the aforementioned additional term does vanish. This would then give a much closer analogy to the curved KGE.

With this, the results of this section are to give a nice dichotomy between this statistical formulation of diffusion, and the quantum mechanical formulation of motion. In fact, the two theories can be nicely wrapped together by the nature of their phase transformations: while wavefunctions are comprised of circular phases with the evolution term e^{iS}, statistical distributions are comprised of hyperbolic phases with the evolution term $e^{-\frac{G}{2}}$. In fact, if we substitute Eqns. (4) and (5) into the restriction $\gamma_\mu \kappa^{\mu\nu} = \alpha^\nu$ we have

$$\gamma_\mu \int d\eta d\zeta d\psi \ \rho(\eta, \zeta, \psi) \xi^\mu \xi^\nu \propto \int d\eta d\zeta d\psi \ \rho(\eta, \zeta, \psi) \xi^\nu \tag{42}$$

which given $\xi_\mu \xi^\mu = 1$ implies that $\gamma^\mu \sim \xi^\mu \sim p^\mu$ is similar to the momentum, and so the evolution term is $\sim e^{-p_\mu x^\mu} \sim e^{-S}$. By these considerations one might even posit that the evolution term is precisely e^{-S}. Interestingly, this relationship we have between the classical statistical and the quantum statistical theories is the same relationship we have between the two types of Lorentz rotation. A gauge transformation on a combination of the quantum and this classical theory might then return a gravitational theory. Something similar was noted in a previous paper on relativistic thermodynamics, and made more exact by introducing the hyperbolic complex number σ defined by $\sigma^2 = 1$. It will likely be valuable to apply the same technique to the framework built here.

6 Conclusion

The basis of this writing was in the derivation of a diffusion equation on Lorentzian Manifolds via the consideration of a series of local random walks, where the decorellation of a particles present state from its past state is modelled as a decay with a decay time. The result of this was a second order hyperbolic differential equation with an asymmetric sense of the future manifesting as a first order damping/entropy term, whose presence then led to an explicit spacetime dependence in the Lagrangian density associated with the equations of motion.

This Lagrangian density was then used to find the systems stress energy tensor, which allowed us to relate the probability distribution to the temperature distribution in a non-trivial way via a (contentious) relativistic extension of Fourier's law.

It was then shown that the explicit spacetime-dependence of the Lagrangian could be absorbed into the probability distribution as a spacetime evolution term in a way analogous to Dirac's postulate of quantum mechanics such as to return a hyperbolic, Klein Gordon like equation of motion. From this it was noted that quantum statistical motion and classical statistical motion are essentially equivalent up to whether the basic phase transformation of the theory is circular or hyperbolic, similar to the differentiation between spacelike and timelike rotations.

Inspired by these methods, the orientational distribution of the particle was then coupled to spacetime and the probability distribution transformed into the spacetime evolved form in a way similar to the Minkowski case in hopes that it would return the same differential equation on a Riemannian manifold. The result was indeed such an equation, although with a mass term coupled to spacetime beyond what is expected from the equivalent KGE on a curved background.

The actual solutions of these equations were not properly explored here, and most of the studies performed were in flat space. Thus, it would be interesting to study the behaviour of classical gasses near black holes using these formulae, and to in general look further into their more pragmatic physical consequences such as to verify or dismiss different parts of this framework. It would also be interesting to look further into these equations relationship to quantum mechanics and gravity, and see if perhaps there is some deeper reason for the hyperbolic/circular transformation symmetry between the two theories, as well as if the extra term from Eqn. (41) is somehow recognizable physically, or otherwise vanishes.