Abstract

I try to get a fractal vector calculus, but the efforts are not very successful. From these considerations, however, I do get an interesting sort of path integral. In this path integral, I take the dimension of the scalar field at each point in space to be infinitesimal, allowing it to be evaluated in a well defined way.

1 Fractal Vector Calculus

The main property we want from a fractal calculus is that our mass scales with exponent Δ. That is

$$\int dx^{\Delta} f(\alpha x) = \alpha^{-\Delta} \int dx^{\Delta} f(x) \tag{1}$$

for arbitrary constant α, and arbitrary function f. Interestingly, there is a nice way to define fractal integrals on functions with Laurent series. Consider

$$\frac{d^{\Delta}}{dx^{\Delta}} x^n = \frac{n!}{(n-\Delta)!} x^{n-\Delta} \tag{2}$$

The continuous limit is just

$$\frac{d^{\Delta}}{dx^{\Delta}} x^n = \frac{\Gamma(n+1)}{\Gamma(n-\Delta+1)} x^{n-\Delta} \tag{3}$$

and so the integral

$$\int dx^{\Delta} x^n = \frac{\Gamma(n+1)}{\Gamma(n+\Delta+1)} x^{n+\Delta} \tag{4}$$

for $n, \Delta \in \mathbb{R}$ (except perhaps where we hit negative integers of the gamma function, where I suspect we get a logarithm), which obeys (1). But of course, we don't have to use just one measure. In fact, we can use infinitely many:

$$\begin{aligned} \int \prod_i dx_i^{\mu_i} f(\alpha x) &= \alpha^{-\sum_i \mu_i} \int \prod_i dx_i^{\mu_i} f(x) \\ &= \alpha^{-\Delta} \int \prod_i dx_i^{\mu_i} f(x) \end{aligned} \tag{5}$$

where

$$\sum_i \mu_i = \Delta \tag{6}$$

and so vectors will take the form

$$x = \sum_i \mu_i x^i e_i^{\mu_i} \tag{7}$$

with basis vectors $e_i^{\mu_i}$, and the factor of μ_i is just so that a vector of ones will return the dimension. In this sense, we have many different axes, but the axes themselves have fractional dimensions, rather than dimensions of unity. What this all means for these vectors I honestly have no idea, and will have to think about some more.

This can all be made even more general by considering functionals, where instead of (6) we have

$$\Delta = \int \mu(dQ) = \int dQ\rho(Q) \tag{8}$$

where $\mu(dQ)(Q)$ is the dimensional measure, and instead of (7)

$$x = \int \mu(dQ)x(Q)e^{\mu}(Q) \tag{9}$$

Now if we do $\rho = \sum_i \mu_i \delta(Q - Q_i)$ we will get the discrete equations (making it clear why we included the coefficients on (7)).

Let us define the integral measure $\mathcal{D}x = \prod_Q dx^{\mu(dQ)}$ and consider doing the integral of a functional of the form

$$F[x] = \prod_Q f_Q(x_Q) \tag{10}$$

where f_Q can be expanded as

$$f_Q(x_Q) = \sum_n \alpha_{n,Q} x_Q^n \tag{11}$$

where we are now writing the Qs as subscripts, eg. $x(Q) = x_Q$, for convenience. Then

$$\begin{aligned} \int \mathcal{D}xF[x] &= \int \prod_Q dx_Q^{\mu(dQ)} \prod_Q f_Q(x_Q) \\ &= \int \prod_Q dx_Q^{\mu(dQ)} \sum_n \alpha_{n,Q} x_Q^n \\ &= \prod_Q \sum_n \alpha_{n,Q} \frac{\Gamma(n+1)}{\Gamma(n+1+\mu(dQ))} x_Q^{n+\mu(dQ)} \\ &= \prod_Q x_Q^{\mu(dQ)} \prod_Q \sum_n \alpha_{n,Q} \frac{\Gamma(n+1)}{\Gamma(n+1+\mu(dQ))} x_Q^n \end{aligned} \tag{12}$$

Now, we will taylor expand the gamma function

$$\frac{\Gamma(n+1)}{\Gamma(n+1+\mu(dQ))} = 1 - \frac{\Gamma'(n+1)}{\Gamma(n+1)}\mu(dQ) \tag{13}$$

and integrate the outer coefficient to get

$$
\begin{aligned}
\int \mathcal{D}xF[x] &= e^{\int \mu(dQ)\ln x_Q} \prod_Q \sum_n \alpha_{n,Q}\left(1 - \frac{\Gamma'(n+1)}{\Gamma(n+1)}\mu(dQ)\right) x_Q^n \\
&= e^{\int \mu(dQ)\ln x_Q} \prod_Q \left(f_Q - \sum_n \alpha_{n,Q}\frac{\Gamma'(n+1)}{\Gamma(n+1)}\mu(dQ)\cdot x_Q^n\right) \\
&= e^{\int \mu(dQ)\ln x_Q} \prod_Q f_Q \prod_Q \left(1 - \frac{1}{f_Q}\sum_n \alpha_{n,Q}\frac{\Gamma'(n+1)}{\Gamma(n+1)}\mu(dQ)\cdot x_Q^n\right) \\
&= e^{\int \mu(dQ)\ln x_Q} F[x] \prod_Q \exp\left(-\frac{1}{f_Q}\sum_n \alpha_{n,Q}\frac{\Gamma'(n+1)}{\Gamma(n+1)}\mu(dQ)\cdot x_Q^n\right) \\
&= F[x]\exp\int \mu(dQ)\left(\ln x_Q - \frac{1}{f_Q}\sum_n \alpha_{n,Q}\frac{\Gamma'(n+1)}{\Gamma(n+1)}x_Q^n\right) \\
&= F[x]\exp\int \mu(dQ)\left(\ln x_Q - \frac{1}{f_Q}\sum_n \alpha_{n,Q}(H_n-\gamma)x_Q^n\right) \\
&= F[x]\exp\int \mu(dQ)\left(\ln x_Q - \frac{1}{f_Q}\sum_n \alpha_{n,Q}H_n x_Q^n + \gamma\right) \\
&= F[x]\exp\int \mu(dQ)\left(\ln x_Q - \frac{\hat{h}_Q f_Q}{f_Q} + \gamma\right)
\end{aligned}
\tag{14}
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where we have used $\frac{\Gamma'(n+1)}{\Gamma(n+1)} = H_n - \gamma$ where H_n are the harmonic numbers, and γ is the Euler-Mascheroni constant. We then defined the linear operator $\hat{h}_Q$ which has the behaviour $\hat{h}_Q x_Q^n = H_n$. We have now written a tractable expression for a path integral without discretizing space! Thats quite cool.