Possible Error in The Discrete alembertian

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Abstract

The discrete alembertian is an important pathway to the scalar curvature of complex directed acyclic systems. In particular, and with regards to current causal set investigations, this allows one to start studying the scalar curvature of causets. I argue here that an error has been made in the currently proposed discrete alembertian operators. I then propose a new operator to fix this problem. In doing so, the value of graph matrix notation becomes apparent.

1 Notation

Let us start with graph notation. Because these are meant to be brief notes I wont go over DAG/causet notation. Let each node in an N node DAG be represented by a basis vector x_n. x_n is a vector of zeros, except at the nth position, where it is one. Then, the matrix F is defined such that Fx_n is a vector representing the entire future of x_n. Similarly, F_1x_n is the immediate future of x_n, while F_mx_n is the set of nodes m hops into the future from x_n. The past matrix P is defined similarly. The past and future are related simply by $F = P^T$.

Let N be a vector of ones. Then the size of the future of x will simply be $Fx \cdot N$. Similarly, the size of the causal diamond $D(x, y)$ is just

$$V(x, y) = Fx \cdot Py = x^T F^T Py = x \cdot P^2 y \tag{1}$$

If we want the entire causal diamond, we have to define the selector. The selector is defined as

$$S(v) = \sum_n x_n x_n^T (v \cdot x_n) \tag{2}$$

where v is some vector. This looks complicated, but is really just the map:

$$S : \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \longrightarrow \begin{bmatrix} v_1 & 0 & 0 & & & \\ 0 & v_2 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & v_3 & & & \\ & \cdot & & & & \\ & \cdot & & & & \\ & \cdot & & & & \end{bmatrix} \tag{3}$$

ie. a diagonalization of v. Similarly, S^{-1} is one over the diagonalization. Note, S is not a matrix, but has the same linearity as the dot product. With this, the causal diamond is

$$D(x,y) = S(Fx) \cdot Py = Fx \cdot S(Py) \tag{4}$$

The selector is an interesting bit of notation, which can be used to derive some diffusion equations (which I will put here later).

Finally a scalar field is defined by a vector ϕ such that $\phi \cdot x_n = \phi_n$. The scalar field is typically used to find/understand the alembertian operator B.

2 Current Definition

The current definition is based on 4 principles (this is lifted from arxiv:1403.1622v1 [hep-th]).

- Linearity: B should be linear in ϕ
- Retardness: $B\phi(x)$ should depend only on $\phi(x)$ in the past of x
- Label Invariance: B should be invariant under change of labels of nodes
- Neighbourly Democracy: All nth neighbours of x should contribute equally to $B\phi(x)$

This leads to the discrete alembertian, in the notation here,

$$\xi^2 B\phi(x) = a\phi \cdot x + \sum_{n=1} b_n P_n x \cdot \phi \tag{5}$$

which implies

$$\begin{aligned} \xi^2 B &= a\delta + \sum_{n=1} b_n P_n x \\ \partial^2 \phi &= B^T \phi \end{aligned} \tag{6}$$

where δ is the identity, ξ the length scale, and $\partial^2\phi \cdot x$ is the alembertian at x. The fourth principle, however, is not quite correct. Each node in $P_n x$ cannot contribute equally because not all of the branches of the past nodes go to x. For one node, only one of many branches may reach x, while for another all its branches may reach x. Limiting isotropy then implies the two nodes contribute differently. A branching future can be understood in the context of the alembertian as wavelet propogation at a wavefront (cf Huygens principle). Futures which branch out of the past lightcone of x are waves propagating too far away to influence the event in question.

We can use wave propogation in the limiting DAG to get the correct alembertian. Following a point on the wavefront down some path, we know its

amplitude will diminish. In the context of Huygens principle, this is because wavelets are propagating off the path of the main wavefront-point. We also know by time symmetry and isotropy, excepting interference with other waves, all wavefronts will diminish as the same function of proper time. Thus, branches of the same length have the same contribution. Fortunately, we can write the number of branches of length n between y and x as $y \cdot P_1^n x$ (where y precedes x). The operator can then be simplified to

$$\xi^2 B = \sum_{n=0} a_n P_1^n = f(P_1) \tag{7}$$

It is known that the normally used alembertian has fluctuations that diverge as the length scale vanishes. This can be fixed in an ad-hoc way by introducing another length scale, but this is undesirable. I suspect that the alembertian proposed here does not have this fluctuation problem.

Time symmetry means each node should have about the same degree. Thus the number of branches along a path is proportional to its length. The waves should also have an oscillatory property, and so the coefficients might be something like

$$\xi^2 B = 1 + \sum_{n=1} \frac{(-1)^n}{n} P_1^n \tag{8}$$

which makes B

$$\xi^2 B = 1 - S^{-1} \log(\Lambda + 1) S \tag{9}$$

where Λ is the eigenvalue matrix of P_1 and S the diagonalizing matrices.

3 Measuring Coefficients

We need to find the coefficients. We can try to find them mathematically, but finding the expectation value of such a structure would be complicated (aside - perhaps try Campbell's theorems). Instead we'll find them computationally. For smaller ξ the $\bar{a}_n$ should converge. Can find them with a curve fit. We know that for $\phi_n = e^{ip \cdot X_n}$ we get $(B^T \phi)_n = -p^2 \phi_n$. Here, X_n is the position of x_n in the PPP.

The procedure: generate a graph of size N. Do a curve fit between the functions above for different values of p. Get the $\bar{a}_n$ and the associated variances. Graph the $\bar{a}_n$ and the variances against N.

This should work, but I'm a bit worried. It'll be a lot of work if it doesn't work properly.

There is another...

A different way to do this that I think will work better.
1. create graph.
2. put a point down somewhere in the graph, call this A.
3. calculate the value of each of the $P_1^n \phi$ at A. Here, ϕ_n is again $e^{ip \cdot X_n}$.

4. repeat this on the same graph for many different A. Then repeat this for many graphs.
5. take the averages of the values of the alembertians at A (from the calculations of $P_1^n\phi$ across the graphs times those coefficients.
6. fit these points to $-p^2 e^{ip\cdot X}$
Good, except how do we deal with p? What is the probability that it works for one p but not others?

4 Diffusion

I will model diffusion as a flow of a scalar field from the immediate past to the present to the immediate future. The flow should be isotropic, so if $\phi \cdot y$ has n neihbours in its immediate future, then each neighbour should receive $\phi \cdot y/n$. If we let x_i be the set of nodes to the immediate past of x, then

$$\phi \cdot x = \sum_i \frac{\phi \cdot x_i}{N \cdot F_1 x_i} \tag{10}$$

for any x. Noting that $F_1^T N$ is a vector containing the sizes of the immediate futures of the nodes, we can write the terms in the sum as $S^{-1}(F_1^T N)P_1 x$, and so

$$\phi \cdot x = S^{-1}(F_1^T N)P_1 x \cdot \phi \tag{11}$$

By linearity, and the completeness of x, we can do

$$\begin{aligned} \phi \cdot (x_1 \oplus x_2 \oplus x_3 \oplus \cdots) &= S^{-1}(F_1^T N)P_1(x_1 \oplus x_2 \oplus x_3 \oplus \cdots) \cdot \phi \\ \phi \cdot \delta &= S^{-1}(F_1^T N)P_1 \delta\phi \\ \phi &= S^{-1}(F_1^T N)P_1 \phi \end{aligned} \tag{12}$$

which is a simple eigenvalue problem. Again, the number of paths to the past of a node is $F_1^n x$. In the alembertian, it is the paths of each past which have equivalence. Perhaps then we can write the solution as

$$\phi = \sum_{n=0} S^{-1}[(F_1^n)^T N]P_1^n \phi \tag{13}$$

and avoid the coefficients altogether. We see now that this notation lends itself nicely to wrting down transport phenomena on DAGs (as is normally the case in graph theory).