A Simple Model of Brownian Motion in Spacetime

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June 2021

Abstract

The simplest possible random walk model in Minwkoski spacetime is constructed and analyzed. It is shown that the model leads to a non-hyperbolic heat equation with the unique property that its coefficients depend on the boundary conditions of the distribution, which allows one to safely break Lorentz symmetry at the level of system initialisation. The properties of this PDE are analyzed, and it is found that the PDE has a Klein-Gordon like Lagrangian description under the appropriate coordinate transformation, which with a Wick rotation becomes the Klein-Gordon equation exactly; a conserved current similar to Fourier's heat law; and a Riemannian metric with a Cartesian elliptic symmetry.

1 Introduction

Relativistic diffusion has remained an unsolved problem for about 50 years. Many frameworks exist, but none are well agreed upon. The most agreed relativistic heat equation takes the form of the telegrapher equation; the telegrapher equation admits worrying properties, such as fixed speed wavefronts, and resonances which allow for violations of the second law of thermodynamics. More sophisticated approaches invoke SDEs, but fail to provide simple analytical models or big picture solutions to the problem. In this article, we will construct a relativistic random walk model based on local past-future decorrelation, then derive a non-hyperbolic PDE describing the random walk. We will show that the exact coefficients in the PDE are dependent on the systems boundary conditions, allowing us to safely break Lorentz invariance such as to do away with heat resonances and shock waves. We will also show that under a coordinate transformation, the PDE can be put into the form of a Klein-Gordon equation with a latent symmetry group. Wick rotation and Lorentz invariant boundary conditions turns it into the KGE exactly (in analogy to the relationship between the heat and Shrodinger equations). In this article we take $c = \hbar = 1$ and $(+---)$.

2 Random Walk Model

The Model To be relativistic, the equation must be at least second order in all coordinates, and thus cannot be described by a Markovian process in spacetime. If it is to be described by a Markovian process, that process must take place in phase space. The internal state of our randomly walking particle is therefore given by (at least) its 4-velocity. If we imagine the decorrelation of the particles current state from its past state behaves like a decay process (in which the particle decays information about its past self), then the likelihood the particle has adopted a completely new state between observations depends on the proper time experienced by the particle between those observations. We can then approximate the motion of the particle by saying it chooses some 4-velocity X at random, then maintains that 4-velocity until it becomes completely decorrelated from its past, at which point it chooses a new 4-velocity at random. On average that decorrelation will take ξ units of proper time, such that the particle will have moved from x to $x + \xi X$

The PDE If we parameterize the 4-velocity by spacetime orientation, Ω, then we can denote the probability that the particle chooses the 4-velocity $X(\Omega)$ by $\rho(\Omega)d^{d-1}\Omega$ at a given step. Under Lorentz symmetry, obviously $\rho(\Omega) = \rho(\Omega + \delta\Omega) = \rho_0$ is constant; however, for now we will keep $\rho(\Omega)$ general for reasons which will become clear later. Then, if we denote the probability the particle is found at point x by $\phi(x)d^dx$, the master equation for this random walk becomes

$$\phi(x) = \int d^{d-1}\Omega\rho(\Omega)\phi(x - \xi X) =: \int \mu(\Omega)\phi(x - \xi X) \tag{1}$$

We would like to Taylor expand the right hand side for small ξ; this is non-trivial, since the components of X can become arbitrarily large given any choice of Ω. This can be fixed by demanding that $\rho(\Omega)$ vanish sufficiently quickly for high rapidities, such that the contributions from low accuracy ϕ expansions to the value of the integral are suppressed. Performing the expansion to second order

$$\phi = \int \mu(\Omega)\Big(\phi - \xi X^\mu \partial_\mu \phi + \frac{1}{2}\xi^2 X^\mu X^\nu \partial_\mu \partial_\nu \phi + \cdots\Big) \tag{2}$$

The ϕ are independent of Ω, and the integration measure is normalized, so this reduces to

$$H^{\mu\nu}\partial_\mu \partial_\nu \phi = Q^\mu \partial_\mu \phi \tag{3}$$

where

$$Q^\mu = \xi \int \mu(\Omega) X^\mu = \xi \langle X^\mu \rangle \tag{4}$$

$$H^{\mu\nu} = \frac{1}{2}\xi^2 \int \mu(\Omega) X^\mu X^\nu = \frac{1}{2}\xi^2 \langle X^\mu X^\nu \rangle \tag{5}$$

Lorentz Symmetry Under Lorentz transformations, $X(\Omega) \to \Lambda(\delta\Omega)X(\Omega) = X(\Omega + \delta\Omega)$, and thus H and Q both transform as tensors. If $\rho = \rho_0$ is constant, then $\mu(\Omega) = \mu(\Omega + \delta\Omega)$, and

$$\begin{aligned}\Lambda(\delta\Omega)H\Lambda(\delta\Omega)^T &= \frac{1}{2}\xi^2 \int \mu(\Omega) X(\Omega+\delta\Omega) \otimes X(\Omega+\delta\Omega) \\ &= \frac{1}{2}\xi^2 \int \mu(\Omega+\delta\Omega) X(\Omega+\delta\Omega) \otimes X(\Omega+\delta\Omega) = H\end{aligned} \tag{6}$$

so $\Lambda H \Lambda^T = H$, implying H is proportional to the Minkowski metric. By similar arguments, $\Lambda Q = Q$, implying $Q = 0$. Therefore (3) reduces to a simple wave equation, which is clearly non-physical. In general, any PDE in the form (3) must maintain these symmetries on its coefficients, such that a non-resonant second order PDE cannot be fully relativistic. Normally, this would mean we either have to give up Lorentz symmetry, or the second law of thermodynamics. The derived PDE, however, has the special property that its coefficients are partially dependent on the observers knowledge of the distribution $\rho(\Omega)$, which itself is partially encoded in $\partial_\mu \phi$. Since the underlying mechanics governing the behaviour of the system maintains the full Poincare symmetry, the average momentum $\langle p \rangle \propto Q$, is conserved, such that the boundary conditions for $\partial_\mu \phi$ alone would specify $\rho(\Omega)$ with sufficient accuracy to pick out a preferred frame for the PDE.

Consider, for example, a box containing a gas of particles in Minkowski spacetime described by this model. The box will be moving at some 4-velocity proportional to both the average 4-velocity $Q = \xi\langle X \rangle$ of the particles and the average $\partial^\mu \phi$ across the distribution. Since there are no external forces on the box, its 4-velocity never changes, and so the initial conditions for $\partial^\mu \phi$ specify Q up to a constant for all points in time. The average 4-velocity also specifies an average orientation in spacetime, therefore constraining $\rho(\Omega)$ and, by extension, H. We therefore do not need to choose $\rho = \rho_0$, and do not need to maintain Lorentz invarviance at the level of H and Q, as it is safely being broken at the level of system initialization.

Ellipticity By spatial isotropy, it is possible to choose a frame such that H is diagonal. From (5) it is clear that H is positive semi-definite, and therefore elliptic in this frame. While normally ellipticity implies instantaneous propagation of boundary condition perturbations, here, perturbation of the boundary conditions implies perturbation of the coefficients H and Q, and therefore non-trivial, not-necessarily-superluminal propagations. Furthermore, this ellipticity ultimately arises from the Lorentz symmetry breaking described above, which itself does not occur at the physical level. We will therefore presume that there is no superluminal propagation in spite of the ellipticity.

3 Symmetry and Conservation

Latent Geometry While we don't necessarily have Lorentz invariance, we can still identify the symmetry group consisting of matrices L such that $LHL^T = H$. We can then identify H as a non-Euclidean, non-Lorentzian metric tensor acting to raise and lower capital Roman indices, ie. $u^A =: H^{AB}u_B$. Physically, we define contravariant vectors and coordinates as corresponding between the capital Roman and lower case Greek indices, ie. $u^A := u^\alpha$ and $x^A := x^\alpha$ (implying $\partial_A := \partial_\alpha$). Although these manipulations are not strictly necessary for the following calculations, it will allow us to omit H in formulae, and provide some insight later on.

Conservation Laws The non-relativistic heat equation can be derived from Fourier's law, $j^\mu = (\phi, -q_0\partial^i\phi)$, and conservation, $\partial_\mu j^\mu = 0$. An exact analogy can be made here, where Fourier's law becomes

$$j^A = (\partial^A - Q^A)\phi \tag{7}$$

such that $\partial_A j^A = 0$ returns (3). If we wish to associate a symmetry to this conserved current we must first find the Lagrangian of the system. The presence of a first order damping term in the PDE implies the Lagrangian should depend on spacetime. The solution is

$$\mathcal{L} = \frac{1}{2}e^{-Q\cdot x}\partial_A\phi\partial^A\phi \tag{8}$$

The infinitesimal symmetry generated by j^A is then $\phi \to \phi + \frac{1}{2}\epsilon e^{Q\cdot x}$, and corresponds to a position weighted amplitude translation in the finite limit. This leads to $\mathcal{L} \to \mathcal{L} + \epsilon\partial_A(Q^A\phi)$, through which one can easily verify the conservation of j^A using Noether's theorem.

We also expect conserved currents associated with translation invariance, as well as L transformation invariance. While this can be achieved directly through (8), it is easier to first change variables to $\psi = e^{-Q\cdot x/2}\phi$. Then, up to a total derivative, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_A\psi\partial^A\psi + \frac{1}{2}m^2\psi^2 \tag{9}$$

which gives

$$(\partial_A\partial^A - m^2)\psi = 0 \tag{10}$$

where

$$m^2 := \frac{1}{4}Q_A Q^A \tag{11}$$

We thus have something like the Klein Gordon equation, but with an imaginary mass and corresponding momentum $Q^A/2$. This makes the metric interpretation of H clearer, while also allowing us to read off the conserved stress energy and angular momentum. If we then define the covariant derivative $D_A := \partial_A - Q_A/2$, we can multiply (10) by $e^{Q\cdot x/2}$ to get

$$(D_A D^A - m^2)\phi = 0 \tag{12}$$

We might then wish to think of the current $J^A = D^A\phi$, which is not conserved, but covariantly sourced by the mass of the system. We therefore have four currents: a conserved current describing the transport of heat, a standard stress energy, an angular momentum associated with the $LHL^T = H$ symmetry, and a covariant heat current sourced by the systems characteristic mass m.

Wick Rotation and Quantum Mechanics It is well known that the Schrodinger equation is simply the non-relativistic heat equation with an imaginary time. That is, the heat and Schrodinger equations are related by a Wick rotation. Something similar can be done with the above calculations; if we let $p^A = iQ^A/2$ and $p_A p^A = M^2$, then (10) becomes the Klein Gordon equation with mass M and Minkowski metric replaced by H. If we also let $\rho = \rho_0$ such that H becomes (proportional to) the Minkowski metric, then (10) becomes the Klein Gordon equation identically. Interestingly, this also means the change of variables to $\phi \to \psi$ corresponds to an evolution factor $e^{ip\cdot x}$, indicating that $Q/2$ is like a global arrow of time for the system. This is further supported by the fact that allowing space*time* isotropy in $\rho(\Omega)$ leads to $Q = 0$ (but not $H = 0$) by symmetry over the measure.

4 1+1 Dimensions

Hyperbolic and Elliptic Cases Let us take the $1 + 1$d case as an example. Then X is given by $X^T = (\cosh\eta, \sinh\eta)$ and the correlation function H is

$$\begin{aligned} H &= \frac{1}{2}\xi^2 \int d\eta\rho(\eta) \begin{bmatrix} \cosh^2\eta & \cosh\eta\sinh\eta \\ \cosh\eta\sinh\eta & \sinh^2\eta \end{bmatrix} \\ &= \frac{1}{4}\xi^2 \int d\eta\rho(\eta) \begin{bmatrix} \cosh(2\eta)+1 & \sinh(2\eta) \\ \sinh(2\eta) & \cosh(2\eta)-1 \end{bmatrix} \\ &= \frac{1}{4}\xi^2 \begin{bmatrix} c+1 & s \\ s & c-1 \end{bmatrix} = \frac{1}{4}\xi^2\left(\tilde{\eta} + \begin{bmatrix} c & s \\ s & c \end{bmatrix}\right) \end{aligned} \tag{13}$$

where

$$c = \int d\eta\rho(\eta)\cosh(2\eta) \tag{14}$$

$$s = \int d\eta \rho(\eta) \sinh(2\eta) \tag{15}$$

and $\tilde{\eta}$ is the Minkowski metric. Under Lorentz invariance, $\rho = \rho_0$ is constant and, for normalization, zero. In such a case, $c = s = 0$, and so $H \propto \tilde{\eta}$ as expected. Otherwise, using the first line of (13), the discriminant of the PDE is

$$\begin{aligned} \frac{D}{4} &= \left(\int \mu(\eta) \cosh\eta \sinh\eta \right)^2 - \int \mu(\eta) \cosh^2\eta \int \mu(\eta) \sinh^2\eta \\ &= \langle \cosh\eta, \sinh\eta \rangle^2 - \langle \cosh\eta, \cosh\eta \rangle \langle \sinh\eta, \sinh\eta \rangle \end{aligned} \tag{16}$$

By Cauchy-Schwarz, $D < 0$, so the equation is elliptic. We have therefore retrieved the two cases from above. For interest and generality we will proceed with the elliptic case.

Symmetry group Let us now find the set of matrices satisfying $LHL^T = H$. To do this we will first diagonalise H using the fundamental representation of $O(\theta) \in SO(2)$. Define $\bar{H} = O^T H O$ and $\bar{L} = O^T L O$. If we choose θ such that $-\tan(2\theta) = s$, then

$$\bar{H} = \begin{bmatrix} c + \sqrt{s^2+1} & 0 \\ 0 & c - \sqrt{s^2+1} \end{bmatrix} =: \begin{bmatrix} b_+^2 & 0 \\ 0 & b_-^2 \end{bmatrix} \tag{17}$$

where we have defined $b_\pm = \sqrt{c \pm \sqrt{s^2+1}}$. The matrices $\bar{L}$ satisfying $O^T LHL^T O = \bar{L}\bar{H}\bar{L}^T = \bar{H}$ can easily be shown to be

$$\bar{L} = \begin{bmatrix} \cos\beta & \frac{b_+}{b_-} \sin\beta \\ -\frac{b_-}{b_+} \sin\beta & \cos\beta \end{bmatrix} \tag{18}$$

for some arbitrary parameter $\beta \in \mathbb{R}$. From this we can determine that L is

$$Ł = \frac{1}{b_+ b_-} \begin{bmatrix} b_+ b_- \cos\beta - s \sin\beta & (1+c)\sin\beta \\ (1-c)\sin\beta & b_+ b_- \cos\beta + s \sin\beta \end{bmatrix} \tag{19}$$

Again, if $\rho = \rho_0$ then $c = s = 0$, $b_\pm = \sqrt{\pm 1}$, and L is a boost. Physically, $L(\beta)$ rotates a point along the ellipse with major axis somehow oriented according to θ. Note, c and s are dependent on the choice of frame, and so the exact structure of these matrices is also frame dependent.

Specific Case Let us consider the specific $1+1$ dimensional case of a gas of equally massive particles moving with average 4-velocity ∂_t, meaning the systems arrow of time is $Q = Q^0 \partial_t$. Assuming spatial isotropy implies $H^{01} = H^{10} = 0$ by symmetry. We therefore have

$$H^{00} \partial_t^2 \phi + H^{11} \partial_x^2 \phi = Q^0 \partial_t \phi \tag{20}$$

If we further take the distribution to be a Gaussian with standard deviation σ then the equation becomes

$$\frac{1}{2} \xi e^{2\sigma^2} (\partial_t^2 + \partial_x^2)\phi + \frac{1}{2} \xi (\partial_t^2 - \partial_x^2)\phi = e^{\sigma^2/2} \partial_t \phi \tag{21}$$

Note that this does not actually vanish for $\sigma \to \infty$. This is because for any allowed value of σ there is a preferred frame, so it is not possible to limit to the Lorentz invariant case. To achieve the Lorentz invariant case, one must set ρ exactly to zero.

5 Discussion and Conclusion

The purpose of this article was to show that Brownian motion in phase space can be achieved with a relatively simple model and still give sensical results. The most important feature of the model was the dependence of the coefficients of the derived PDE on the initial conditions of the distribution. This meant that the PDE could be both second order in time and non-hyperbolic without breaking Lorentz invariance. This, however, also meant that sourcing the equations of motion is non-trivial. The models inability to handle such external perturbations is perhaps related to the fact that it assumes local equilibrium everwhere; if this is the case, then one might be able to safely include adiabatic sources without creating superluminal propagations.

We also showed that under a simple coordinate change, the PDE could be put into a Klein-Gordon like form. This led to the interpretation of the tensor coefficients H and Q as a Riemannian metric and characteristic momentum respectively. Wick rotating the distribution and re-establishing Lorentz symmetry returned the

Klein-Gordon equation exactly, demonstrating that the derived PDE is to the Klein-Gordon equation what the heat equation is to the Schrodinger equation.

Finally it was shown that, aside from the conserved quantities associated with its Klein-Gordon form, the PDE admitted a conserved quantity corresponding to Fourier's classical heat current. Interestingly, this current is distinct from the stress energy. Naively, its zeroth component corresponds to a density of particles, however the density of particles is already given by ϕ as ensured by the formulation of the master equation. Perhaps then this points to an as of yet poorly understood relationship between densities of information. flows of information through time, and probability.