Local Diff Struct On Graphs Notes

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1 Motivation

As far as I can tell, there are two levels of structure to manifolds: topological structure, and differential structure. In the emerging field of network geometry people seem to take typical manifold properties and find their equivalent in graphs. Obviously this isnt always fruitful, because networks will have less information than manifolds, making it unclear when doing something like, say, defining a local curvature, will even make sense. Globally, and in the limit of many nodes, these efforts are more fruitful, but I think less powerful. I think it would be of interest to try to generalize ideas used in the study of local behaviour on manifolds such that they become applicable in networks. ie. to take a "ground up" approach to network geometry.

We should then consider which level of structure to try to generalize to networks. Topology is on a lower level than differential structure, so maybe that would have been a more sensible thing to try to look into? Ok, well maybe thats another possible direction we can take things. Either way this is the basic idea of both of them:

Topological structure: we try to define what it means to be in a "local neighbourhood" of a point, then look at what sort of symmetries that set of points has, or even, if there are enough points there, what sort of continuous space that set of points approximates.

Differential structure: we try to define what it means to be in a "local neighbourhood" of a point, then try to identify a tangent space in that local neighbourhood. It might seem as though it is necessary to get the local topological space first, but we will see that this depends on how we define the local neighbourhoord.

Whatever we choose, we still need to get a grasp on the notion of local neighbourhood. I think there are two good ways to do this, which might be ultimately identical (?), but for these notes I will use the first definition in a way that makes that choice important.

Definition 1 (with motivation): Lets imagine an hRGG on a Riemannian manifold with node density ρ and connection radius r. We might expect that with $\rho \to \infty$ our network will approximate the manifold, but this is impossible, because if r stays fixed, then we can form discontinuous paths between different points on the "manifold", and our shortest paths will be discontinuous as well. Instead, what we need is that $\rho \to \infty$ AND $r \to 0$. In this limit points will only connect to eachother if they share a local neighbourhood on the manifold, and so all geodesics will be defined properly. This then motivates us to define the local neighbourhood of a point as the point and all its neighbours. This is rather intuitive. For DAGs this doesn't work because light cones are infinite, but with DAGs we have transitive reduction.

Definition 2 (with motivation): when we perform PPP, what we're essentially doing is cutting the manifold up into sections and representing each section with a point. Which sections are adjacent gives us information about the topology of the manifold, and if we do our node connecting properly our network will capture this. In this sense we are doing a real space renormalization (RSR) of the manifold. If our network is large enough, we might then imagine that there exists an RSR which doesn't affect the basic properties of the network, and so we can say that the partitioning step of any valid RSR defines a local neighbourhood. Naturally we would then identify each of those partitions as a single point in the renormalized network, and so this definition in some sense coincides with the first.

In these notes I'll try to get the differential structure, mainly because with differential structure we will basically be able to take any concept from differential geometry and, if not generalize it, identify why its not generalizable. I'll also use the first definition of local neighbourhood because its easier to work with mathematically and creates an interesting discussion about symmetry.

2 Differential Structure

On a d-dimensional manifold, at each point there is a d-dimensional tangent space. For most physical manifolds, that tangent space has the property that its invariant under some Lie Group. More precisely, it is a vector space invariant under the action of a linear representation of the Lie symmetry of the manifolds local topology (I think thats right). On networks, using definition 1, there is an obvious permutation symmetry S_k at each degree k node. This is different from manifolds in that 1. the symmetry is different at each point in the network and 2. the symmetry is discontinuous, meaning we do not have a nice tangent vector space with exactly the local symmetry and nothing more. Point 1. means that parallel transporting on networks might require some additional thought, but point 2. indicates that perhaps it will be necessary to search for some not-quite vector space in which we can form faithful representations of S_k.

Another difference between manifolds and networks is that networks have a local *fractal* dimension, while on manifolds the dimension is just the dimension of the tangent spaces. Defining the local neighbourhoods as we have makes calculating the fractal dimension quite easy. If we say the scale of our neighbourhoods is δs, and the local fractal dimension Δ, then we expect the volume of a $\Delta - ball$ encompassing the the neighbourhood of a k degree node to be $k\delta s^{\Delta}$. Thus

$$\begin{aligned} k\delta s^{\Delta} &= \frac{\pi^{\frac{\Delta}{2}}}{\Gamma\left(1+\frac{\Delta}{2}\right)}\delta s^{\Delta} \\ k &= \frac{\pi^{\frac{\Delta}{2}}}{\Gamma\left(1+\frac{\Delta}{2}\right)} \end{aligned} \tag{1}$$

Note that curvature vanishes in the local neighbourhood of any point on a manifold, and so this analogy is still natural. Interestingly, we then have locally varying symmetries leading to some sort of strange maybe-vector spaces, each associated with a similarly varying fractal dimension. It has been suggested to use fractal vector spaces to study porous media, and so this all makes me think that the local vector space that we want is a fractal vector space. But what is a fractal vector space? Nobody seems to know! Most basically, they should be a vectorization of a fractal dimensioned space, which is a space with mass scaling exponent Δ. I sorta took a crack at it here, but I'm not too confident about it. It does give some nice math though.

3 Fractal Vector Calculus

The main property we want is that our mass scales with exponent Δ. That is

$$\int dx^{\Delta} f(\alpha x) = \alpha^{-\Delta} \int dx^{\Delta} f(x) \tag{2}$$

for arbitrary constant α, and arbitrary function f. Interestingly, there is a nice way to define fractal integrals on functions with Laurent series. Consider

$$\frac{d^{\Delta}}{dx^{\Delta}} x^n = \frac{n!}{(n-\Delta)!} x^{n-\Delta} \tag{3}$$

The continuous limit is just

$$\frac{d^{\Delta}}{dx^{\Delta}} x^n = \frac{\Gamma(n+1)}{\Gamma(n-\Delta+1)} x^{n-\Delta} \tag{4}$$

and so the integral

$$\int dx^{\Delta} x^n = \frac{\Gamma(n+1)}{\Gamma(n+\Delta+1)} x^{n+\Delta} \tag{5}$$

for $n, \Delta \in \mathbb{R}$ (except perhaps where we hit negative integers of the gamma function, where I suspect we get a logarithm), which obeys (2). But of course, we don't have to use just one measure. In fact, we can use infinitely many:

$$
\begin{aligned}
\int \prod_i dx_i^{\mu_i} f(\alpha x) &= \alpha^{-\sum_i \mu_i} \int \prod_i dx_i^{\mu_i} f(x) \\
&= \alpha^{-\Delta} \int \prod_i dx_i^{\mu_i} f(x)
\end{aligned} \tag{6}
$$

where

$$
\sum_i \mu_i = \Delta \tag{7}
$$

and so vectors will take the form

$$
x = \sum_i \mu_i x^i e_i^{\mu_i} \tag{8}
$$

with basis vectors $e_i^{\mu_i}$, and the factor of μ_i is just so that a vector of ones will return the dimension. In this sense, we have many different axes, but the axes themselves have fractional dimensions, rather than dimensions of unity. What this all means for these vectors I honestly have no idea, and will have to think about some more.

This can all be made even more general by considering functionals, where instead of (7) we have

$$
\Delta = \int \mu(dQ) = \int dQ \rho(Q) \tag{9}
$$

where $\mu(dQ)(Q)$ is the dimensional measure, and instead of (8)

$$
x = \int \mu(dQ) x(Q) e^{\mu}(Q) \tag{10}
$$

Now if we do $\rho = \sum_i \mu_i \delta(Q - Q_i)$ we will get the discrete equations (making it clear why we included the coefficients on (8)).

Let us define the integral measure $\mathcal{D}x = \prod_Q dx^{\mu(dQ)}$ and consider doing the integral of a functional of the form

$$
F[x] = \prod_Q f_Q(x_Q) \tag{11}
$$

where f_Q can be expanded as

$$
f_Q(x_Q) = \sum_n \alpha_{n,Q} x_Q^n \tag{12}
$$

where we are now writing the Qs as subscripts, eg. $x(Q) = x_Q$, for convenience. Then

$$
\begin{aligned}
\int \mathcal{D}xF[x] &= \int \prod_Q dx_Q^{\mu(dQ)} \prod_Q f_Q(x_Q) \\
&= \int \prod_Q dx_Q^{\mu(dQ)} \sum_n \alpha_{n,Q} x_Q^n \\
&= \prod_Q \sum_n \alpha_{n,Q} \frac{\Gamma(n+1)}{\Gamma(n+1+\mu(dQ))} x_Q^{n+\mu(dQ)} \\
&= \prod_Q x_Q^{\mu(dQ)} \prod_Q \sum_n \alpha_{n,Q} \frac{\Gamma(n+1)}{\Gamma(n+1+\mu(dQ))} x_Q^n
\end{aligned} \tag{13}
$$

Now, we will taylor expand the gamma function

$$
\frac{\Gamma(n+1)}{\Gamma(n+1+\mu(dQ))} = 1 - \frac{\Gamma'(n+1)}{\Gamma(n+1)} \mu(dQ) \tag{14}
$$

and integrate the outer coefficient to get

$$
\begin{aligned}
\int \mathcal{D}xF[x] &= e^{\int \mu(dQ) \ln x_Q} \prod_Q \sum_n \alpha_{n,Q} \left(1 - \frac{\Gamma'(n+1)}{\Gamma(n+1)} \mu(dQ)\right) x_Q^n \\
&= e^{\int \mu(dQ) \ln x_Q} \prod_Q \left(f_Q - \sum_n \alpha_{n,Q} \frac{\Gamma'(n+1)}{\Gamma(n+1)} \mu(dQ) \cdot x_Q^n \right) \\
&= e^{\int \mu(dQ) \ln x_Q} \prod_Q f_Q \prod_Q \left(1 - \frac{1}{f_Q} \sum_n \alpha_{n,Q} \frac{\Gamma'(n+1)}{\Gamma(n+1)} \mu(dQ) \cdot x_Q^n \right) \\
&= e^{\int \mu(dQ) \ln x_Q} F[x] \prod_Q \exp\left(- \frac{1}{f_Q} \sum_n \alpha_{n,Q} \frac{\Gamma'(n+1)}{\Gamma(n+1)} \mu(dQ) \cdot x_Q^n \right) \\
&= F[x] \exp \int \mu(dQ) \left(\ln x_Q - \frac{1}{f_Q} \sum_n \alpha_{n,Q} \frac{\Gamma'(n+1)}{\Gamma(n+1)} x_Q^n \right)
\end{aligned} \tag{15}
$$

which is a monster, but its quite a bit further than I though would be possible to get. We have also written a tractable expression for a path integral without discretizing space! Thats quite cool.

I dont have the rest of the notes organized in my head, but theres not much more. Essentially, I'm not sure if this is exactly what we want with regards to finding a local tangent space. Where would the S_k symmetry come from? We have to somehow use the fractal scaling property to restrict more properties of the vectorization/parameterization of the space (or perhaps even get more properties?). Maybe to do this we should scale vectors in a particular way? If what we've done is right, and we cant add any more restrictions, then it has a really big partition symmetry, in that we can choose many different measures $\mu(dQ)$ to get the dimensionality Δ.