Equilibrium on International Assets and Goods Markets *

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Abstract

Most of the international asset pricing models are developed in the second situation where purchasing power parity (PPP) is not respected.Investors of different countries do not agree on expected security returns. However, in this case, an equilibrium on the international assets market may exist but not on the international goods market. Our purpose in this paper is to give conditions under which we have equilibrium, not only on the international assets markets but also on the international good market. More precisely, we focus on the link between no-arbitrage, equilibrium and PPP. At equilibrium, assets markets must clear and international goods market balance. In particular, equilibrium goods prices respect the PPP.

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1 Introduction

Very often, the international asset pricing models are developed around two considerations. The first one is to take into account of the differences of taxes between countries or of the presence of barriers to the international exchange of assets. The second one is to claim that real returns on assets differ between the nations. Nations are defined as geographical zones where agents use the same currency in order to deflate prices. The first type of considerations is not very easy to improve. So, most of the international asset pricing models are developed in the second situation where purchasing power parity (PPP) is not respected. These models are partial equilibrium asset pricing models and exchange rates are exogenous.

Since the PPP is not respected, investors of different countries do not agree on expected security returns. However, in this case, an equilibrium on the international assets market may exist but not on the international goods market.

Our purpose in this paper is to give conditions under which we have equilibrium, not only on the international assets markets but also on the international good market. More precisely, we focus on the link between no-arbitrage, equilibrium and PPP. For that, as in Hart [8], we consider a two-period international model. In period 0 agents buy or sell financial assets. In period 1, they buy or sell goods with their initial endowments and the gains of their financial investments in period 0. In our model, contrarily to Solnik [14], investors are not constrained to exchange goods only on their domestic markets. In period 0, they optimally choose their portfolios by using expected utility functions. In the second period, they consume with their initial endowments and the gains yielded by their investments in period 0. Security returns and goods are valued in domestic currencies. At equilibrium, assets markets must clear and international goods market balance. In particular, equilibrium goods prices respect the PPP.

Using no-arbitrage conditions we obtain equilibrium on the international asset market. We differ from Solnik [14] who assumes equilibrium already exists and PPP does not hold. Under a condition on the security returns, we get as in Ross and Walsh [13] that PPP holds for consumption good prices. We also obtain, as in Dumas [6], the result that equilibrium does not exist on the international good market if PPP is not respected, under risk neutrality. Actually, our result is stronger. When the agents are risk neutral, an equilibrium on the international good market exists if, and only if, PPP holds.

The paper is organized as follows. Section 2 presents the general model with its assumptions. In particular, we introduce no-arbitrage conditions and a condition on the security returns. In Section 3, we provide existence of equilibrium theorems for two models: consumption good model and wealth model. In Section 4 we study the link between PPP and equilibrium. Comments are given in Section 5. We give an example where the condition on the security returns does not hold. However, there exists an equilibrium on the asset market but since PPP is not satisfied, no equilibrium can exists for the international good market. We also link our results to the general expression of assets pricing in international assets pricing models (see e.g. Fontaine [7]). We also show that when the agents are risk neutral, an equilibrium on the international good market exists if, and only if, PPP holds. Section 6 concludes.

2 The Model

We consider a two-period economy with $L+1$ countries and K assets. We suppose there exists one consumption good which may be traded between the $L+1$ countries. In each country there is only one consumer. In period 0, agent i, $(i = 0, \ldots, L)$ purchases assets and consumes in period 1. There are S states of nature in period 1. If state s occurs, in period 1, the consumer in country i will consume c^i_s:

$$c^i_s = \omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta i_k$$

where θ^i is the portfolio she purchased in period 0, ω^i_s is the initial endowment of consumption good, $R^i_k(s) \geq 0$ is the return of asset k in country i. The initial endowment ω^i_s and the return $R^i_k(s)$ are valued in currency of country i.

We consider two cases: the two-period consumption model, and the two-period wealth model.

In the first case: for any i, any s:

$$c_s^i = \omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^i \geq 0$$

The consumption set X^i is

$$X^i = \left\{ \theta \in \mathbb{R}^K : \text{for any } s,\ \omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k \geq 0 \right\}$$

In the second case:

$$c_s^i = \omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^i \in \mathbb{R}$$

The consumption set X^i is $\mathbb{R}^K$

Let $(\pi_s^i \geq 00)$ in the S-unit simplex be the belief of agent i. If q is the asset price, agent i will solve:

$$(\mathcal{P}) \qquad \begin{aligned} & \max \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^i) \\ & \theta^i \in X^i, \\ & \sum_{k=1}^{K} q_k \theta_k^i \leq 0. \end{aligned}$$

We suppose that for any i, agent i has no initial endowment for the assets. (We actually consider the net purchases of the agents).
The return of a security is to be interpreted as the total value of one unit of security in the second period, including received dividends payments (therefore, returns should not be confused with rates of returns). Returns are unknown in the first period, but investors are assumed to have probabilistic beliefs about them.

We make the following assumptions:
A1: For any i, any s, $\sum_{k=1}^{K} R_k^i(s) > 0$
A2: For any i, any k, $\sum_{s=1}^{S} R_k^i(s) > 0$
These assumptions are not very stringent. If **A1** is not satisfied for some

i, some s, in this case, country i will not make any exchange on the asset market in state s. If **A2** is not satisfied for some i, some k, country i will never purchase asset k.

A3: For any i, there exists no non-null $(\theta_1, \dots, \theta_K)$ which satisfies

$$\forall s, \sum_{k=1}^{K} R_k^i(s)\theta_k = 0$$

This assumption means that, for any country i, the K assets are not redundant.

P: For every state s, every country i, $\pi_s^i > 0$

We also asume

U1: For any i, the utility function u^i is concave, strictly increasing, differentiable in $\mathbb{R}_{++}$ for the consumption model and in $\mathbb{R}$ for the wealth model . For the wealth model, we denote $a^i = u^{i\prime}(+\infty),\ b^i = u^{i\prime}(-\infty),\ i = 0, \dots, L$.

Definition 1

We say that $\{\theta_k^i\}_{i,k}$ is a net trade if for any k, $\sum_i \theta_k^i = 0$

We introduce an assumption called Consistency Condition:

($\mathcal{C}$) There exist $[(\tau_s^{*i} > 0); i = 0, \dots, L; s = 1, \dots, S]$, such that

For any net trade $\{\theta_k^i\}$, one has: $\forall s, \sum_i \tau_s^{*i} c_s^i = \sum_i \tau_s^{*i} \omega_s^i$

We say, in this case, that the sequence of prices $(\tau_s^{*i})_{i,s}$ satisfies the Consistency Condition ($\mathcal{C}$). Observe that using the prices (τ_s^{*i}), the international goods trade balance.

If we normalize by taking $\tau_s^{*0} = 1, \quad \forall s$, then $(\tau_s^{*i})_{s=1,\dots,S}$ is the exchange rate between country i and 0 in state s.

Definition 2

An equilibrium is a list $[(\theta^{*i}, (c_s^{*i}; p_s^{*i})_{s=1,\dots,S})_{i=0,\dots,L}, q^* \neq 0]$ such that

(1) $\forall i,\ \theta^{*i}$ will solve problem ($\mathcal{P}$) given q^*

(2) $\sum_{i=0}^{L} \theta^{*i} = 0$

(3) $\forall i,\ \forall s,\ c_s^{*i} = \omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^{*i}$

(4) The sequence of prices $((p_s^{*i})_{s=1,\dots,S})_{i=0,\dots,L}$ satisfies

$$\forall i, \sum_s p_s^{*i} c_s^{*i} = \sum_s p_s^{*i} \omega_s^{*i}$$

and the Consistency Condition $(\mathcal{C})$

Relation (2) is the market clearing on the asset market while condition $(\mathcal{C})$ implies $\forall s,\ \sum_{i=0}^{L} p_s^{*i} c_s^{*i} = \sum_{i=0}^{L} p_s^{*i} \omega_s^i$.

This condition is the balance on the consumption goods market in currency of country 0. At an equilibrium, we allow investors to hold portfolios which yield negative rates of return with positive probability. But in the second period, the value of the returns obtained from the net purchases of assets traded in period 0 will be zero.

We first have

Proposition 1 *Assume* $omega_s^i > 0, foralli,\ \forall s$ *if we consider the consumption model. Then Condition* $\mathcal{C}$ *is equivalent to*

$$(\mathcal{E})\ \forall i \neq 0,\ \forall s, \forall k,\ \tau_s^{*i} R_k^i(s) = R_k^0(s).$$

Proof: Let (θ_k^i) a net trade defined as follows:

Fix some country i and some asset k. Take $\theta_k^0 = \epsilon,\ \theta_k^i = -\epsilon,\ \theta_{k'}^0 = \theta_{k'}^i = 0, \forall k' \neq k,\ \theta^j = 0, \forall j \neq 0, k$. For $\epsilon > 0$ small enough, $(\theta^i) \in X^i, \forall i$. Then

$$\forall s,\ \sum_j \tau_s^{*j} c_s^j = \sum_j \tau_s^{*j} \omega_s^j + (R_k^0(s) - \tau_s^{*i} R_k^i(s))\epsilon$$

If $(\mathcal{C})$ holds, then $R_k^0(s) = \tau_s^{*i} R_k^i(s)$. The converse is obvious. ■

We now introduce **No arbitrage conditions**

Definition 3

w is a useful [1] assets purchase for agent i if for any $\lambda \geq 0$, for any $\theta \in X^i$, one has:

$$(a) \qquad \theta + \lambda w \in X^i$$

$$(b) \qquad \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k)) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k)$$

Let W^i denote the set of useful vectors for agent i.

[1] For a definition of useful and useless purchases, see e.g. Werner [15]

Proposition 2 *For the consumption model, we have*

$$W^i = \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^i(s) w_k \geq 0, \ \forall s \right\}$$

Proof: Consider (a) in the previous definition. Divide the LHS by λ and let it go to infinity. We obtain $\sum_{k=1}^{K} R_k^i(s) w_k \geq 0$.
Conversely, assume $\forall s, \sum_{k=1}^{K} R_k^i(s) w_k \geq 0$. Then obviously, for any $\theta \in X^i$, any $\lambda \geq 0$, one has (a). From the increasingness of u^i, we have

$$\sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k)$$

We obtain (b). ■

Proposition 3 *Consider the wealth model. A vector w is useful for i if and only if:*

$$\forall \theta \in \mathbb{R}^K, \ \sum_{k'=1}^{K} w_{k'} \sum_{s=1}^{S} \pi_s^i u^{i\prime}(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k) R_{k'}^i(s) \geq 0 \tag{1}$$

Proof: It is very similar to those given in Dana and Le Van [3], [4] by using the concavity and the differentiability of the u^i. ■

We can have another characterization of W^i for the wealth model. The proof of the following proposition is adapted from Dana and Le Van [4].

Proposition 4 *Consider the wealth model. Let $w \in X^i$ and let $\zeta_s = \sum_k R_k^i(s) w_k, \ \forall s, \ S^+ = \{s : \zeta_s \geq 0\}, \ S^- = \{s : \zeta_s < 0\}$. The vector w is useful is for i if and only if,*

$$a^i \sum_{s \in S^+} \pi_s^i \zeta_s + b^i \sum_{s \in S^-} \pi_s^i \zeta_s \geq 0 \tag{2}$$

Proof: From Proposition 3, w is useful, if and only if, for any $\theta \in \mathbb{R}^K$, we have

$$\sum_{s=1}^{S} \pi_s^i u^i \left(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k + \lambda w_k) \right) \geq \sum_{s=1}^{S} \pi_s^i u^i (\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k), \ \forall \lambda \geq 0.$$

Take $\theta = 0$. We then have

$$\sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \lambda \zeta_s) \geq \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i), \ \forall \lambda \geq 0.$$

Thus, ζ is useful for the function $(c_s)_s \to \sum \pi_s^i u^i(c_s)$. We then have for any $(c_s)_s$

$$0 \geq \sum_{s=1}^{S} \pi_s^i u^i(c_s) - \sum_{s=1}^{S} \pi_s^i u^i(c_s + \zeta_s) \geq - \sum_{s=1}^{S} \pi_s^i u^{i\prime}(c_s)\zeta_s.$$

This implies $\sum_{s=1}^{S} \pi_s^i u^{i\prime}(c_s)\zeta_s \geq 0$. For any $s \in S^+$ let c_s go to $+\infty$, and for $s \in S^-$, let c_s go $-\infty$. We then obtain (2).

The converse is obvious since $u^{i\prime}$ is non-increasing. ■

Remark 1

The set of useful vectors is larger for the wealth model. It includes the set of useful vectors of the consumption model. But when $a^i = 0$, or $b^i = +\infty$, they coincide.

Corollary 1 *Consider the wealth model. If $a^i = 0$ or $b^i = +\infty$ then* $W^i = \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^i(s) w_k \geq 0, \ \forall s \right\}$

Proof: It is obvious. ■

Definition 4

A vector q is a no-arbitrage price for agent i if $q \cdot w > 0$, for all $w \in W^i$.

Let S^i denote the cone of no-arbitrage prices for agent i. Then, obviously, $S^i = -\operatorname{int}(W^i)^0$. Under assumption **A3**, the sets W^i do not contain lines and the sets S^i are non empty (see e.g. Dana, Le Van and Magnien [5]).

In finance, there is another concept of no-arbitrage. We call it NA1. A vector q is a $NA1$ price, or more simply $NA1$, if for any country i, for any portfolio θ which satisfies $R^i_k(s) \cdot \theta \geq 0,\ \forall s$, and $R^i_k(s') \cdot \theta > 0$ for some s', then we have $q \cdot \theta > 0$.

Proposition 5 *Under $(\mathcal{C})$, a vector q is NA1 if and only if:*
$\forall s,\ R^0_k(s) \cdot \theta \geq 0$ *and* $R^0_k(s') \cdot \theta > 0$ *for some* s', *then* $q \cdot \theta > 0$.

Proof: Obvious. ■

Proposition 6 *Consider the consumption model. Assume A3. Then q is NA1 if and only if it is a no-arbitrage price.*

Proof: Let q be no-arbitrage. Given i, let w satisfy $R^i_k(s) \cdot w \geq 0,\ \forall s$ and $R^i_k(s') \cdot w > 0$ for some s'. In this case $w \in W^i \setminus \{0\}$. Hence $q \cdot w > 0$. That means q is NA1.
Conversely, let q be NA1. Given i, let $w \in W^i \setminus \{0\}$. then we have $R^i_k(s) \cdot w \geq 0,\ \forall s$ and $R^i_k(s') \cdot w > 0$ for some s'. If not, $R^i_k(s) \cdot w = 0,\ \forall s$ and from A3, $w = 0$: a contradiction. Since q is NA1, we have $q \cdot w > 0$, i.e. q is no-arbitrage. ■

Proposition 7 *Consider the consumption model. (a) If q^* is an equilibrium price then it is NA1.*
(b) Assume A3. If q^* is an equilibrium price then it is both NA1 and no-arbitrage.

Proof: (a) Given i, let ψ satisfy $R^i_k(s) \cdot \psi \geq 0,\ \forall s$ and $R^i_k(s') \cdot \psi > 0$ for some s'. Let θ^{*i} denote the associated equilibrium portfolio. Since u^i is strictly increasing, and $\pi^i_s > 0,\ \forall s$, we have

$$\sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)(\theta^{*i}_k + \psi_k)) > \sum_{s=1}^{S} \pi^i_s u^i(\omega^i_s + \sum_{k=1}^{K} R^i_k(s)\theta^{*i}_k)$$

That implies $q \cdot \psi > 0$.
(b) The result follows from (a) and Proposition 6. ■

Proposition 8 *Consider the wealth model. (a) If q is no-arbitrage, then it is NA1. If q^* is an equilibrium price, then it is NA1.*
(b) Assume A3. If u^i is strictly concave then

$$\sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)(\theta_k + w_k)) > \sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k)$$

for any θ, any $w \in W^i \setminus \{0\}$. And any equilibrium price is no-arbitrage.

Proof: (a)Let q be no-arbitrage. Given i, let w satisfy $R_k^i(s) \cdot w \geq 0,\ \forall s$ and $R_k^i(s') \cdot w > 0$ for some s'. In this case $w \in W^i \setminus \{0\}$. Hence $q \cdot w > 0$. That means q is NA1.
Given i, let ψ satisfy $R_k^i(s) \cdot \psi \geq 0,\ \forall s$ and $R_k^i(s') \cdot \psi > 0$ for some s'. Let θ^{*i} denote the associated equilibrium portfolio. Since u^i is strictly increasing, and $\pi_s^i > 0,\ \forall s$, we have

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^{*i} + \psi_k)) > \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)\theta_k^{*i})$$

That implies $q \cdot \psi > 0$.
(b) Let $w \in W^i \setminus \{0\}$. Then from A3, $\sum_s R_k^i(s) w_k \neq 0$. If

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + w_k)) = \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)\theta_k^i) \qquad (3)$$

then, by strict concavity of the u^i:

$$\begin{aligned}
\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + \frac{1}{2}w_k)) &> \frac{1}{2}\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)\theta_k^i) \\
&+ \frac{1}{2}\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + w_k)) \\
&= \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + w_k)) \text{ by (3)}
\end{aligned}$$

which is a contradiction since

$$\sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + w_k)) \geq \sum_{s=1}^S \pi_s^i u^i(\omega_s^i + \sum_{k=1}^K R_k^i(s)(\theta_k^i + \frac{1}{2}w_k))$$

Let $[(\theta^{*i}), q^*]$ be an equilibrium. Then for any $w \in W^i \setminus \{0\}$ we have

$$\sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)(\theta_k^{*i} + w_k)) > \sum_{s=1}^{S} \pi_s^i u^i(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^{*i})$$

This implies $q^* \cdot w > 0$. ■

Remark 2 For the wealth model, excepted the cases $a^i = 0$ or $b^i = +\infty$, we do not have the equivalence between NA1 prices and no-arbitrage prices as in the consumption model.

3 Existence of equilibrium

Proposition 9 *Assume **A1, A2, P, U1** and the following no-arbitrage condition*

$$(\mathcal{NA}) \ \cap_{i=0}^{L} S^i \neq \emptyset$$

Then there exist $[(\theta^{*i})_{i=0,\dots,L}; q^* >> 0]$ such that
(a) $\forall i$, θ^{*i} solves problem $(\mathcal{P})$
(b) $\sum_{i=0}^{L} \theta^{*i} = 0$

Proof: The proof may be found in several papers, e.g., Werner [15], Page and Wooders [9], Dana, Le Van, Magnien [5]. The strict positivity of q^* comes from the strict increasingness of the u^i and assumptions **A1, A2**. ■

Proposition 10 *Consider the consumption model. Assume A1, A2, A3, P, U1 and $\mathcal{C}$. Assume that for any i, $\omega_s^i > 0, \forall s$. Then there exists an equilibrium. The equilibrium prices satisfy PPP*

$$\forall i, \ \forall s, \ p_s^{*i} = \tau_s^{*i} p_s^{*0}$$

Proof: See Appendix. ■

Proposition 11 *Consider the wealth model. Assume A1, A2, A3, P, U1, condition* $(\mathcal{C})$*, and for any* i*, either* $a^i = 0$ *or* $b^i = +\infty$*. Then there exists an equilibrium. The prices* (p_s^{*i}) *satisfy PPP.*

Proof: In this case, from Proposition 4, for any i, $W^i = \{w \in \mathbb{R}^K : \sum_s R_k^i(s) w_k \geq 0, \ \forall s\}$. The proof is therefore the same as for Proposition 10. ■

More generally,

Proposition 12 *Consider the wealth model. Assume A1, A2, A3, P, U1, condition* $(\mathcal{C})$*, and for any* i*,* $a^i < u^{i\prime}(\omega_s^i + \sum_{k=1}^K R_k^i(s)\theta_k) < b^i$*,* $\forall \theta$*. Then there exists an equilibrium if, and only if, there exists a no-arbitrage price, i.e. there exist* $[(\theta^i, \lambda^i > 0)_{i=0,\dots,L}]$ *such that*

$$\forall i, \ \forall j, \ \forall k', \ \lambda^i \sum_s \pi_s^i u^{i\prime}(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) R_{k'}^i = \lambda^j \sum_s \pi_s^j u^{j\prime}(\omega_s^j + \sum_k R_k^j(s)\theta_k^j) R_{k'}^j$$

The prices (p_s^{*i}) *satisfy PPP.*

Proof: See Appendix. ■

4 Equilibrium and PPP

In this section we emphasize the role of condition $(\mathcal{C})$ or equivalently $(\mathcal{E})$ and the existence of PPP through the following proposition.

Proposition 13 *Assume A1, A2, A3, P, U1 and* $\mathcal{C}$*. Let* $[\theta^{*i}, q^*]$ *solve* $\mathcal{P}$ *for any* i *and* $\sum_i \theta^{*i} = 0$*. Let*

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i}, \ \forall i, \ \forall s$$

Then there exists a price system $(\tilde{p}_s^{*i})_{i,s}$ *such that* $[(c_s^{*i}), (\tilde{p}_s^{*i})]$ *is an equilibrium for the model where*

(a) each agent i *solves:*

$$\max \sum_s \pi_s^i u^i(c_s^i)$$

under the constraints:

$$c^i \in X^i = \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K,\ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \cap \mathbb{R}_+^S \text{ for the consumption model}$$

$$c^i \in X^i = \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K,\ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \text{ for the wealth model}$$

and the budget constraint

$$\sum_s \tilde{p}_s^{*i} c_s^i \leq \sum_s \tilde{p}_s^{*i} \omega_s^i$$

and

(b) $\forall s,\ \forall i,\ \tilde{p}_s^{*i} = \tau_s^{*i} \tilde{p}_s^{*0}$

In other words, the prices system $(\tilde{p}_s^{*i})_{i,s}$ *satisfies the PPP.*

Conversely, under A3, if $[(c_s^{*i}); (\tilde{p}_s^{*i}); i = 0, \ldots, L; s = 1, \ldots, S]$ *is an equilibrium for the model given just above with* $\tilde{p}_s^{*i} = \tau_s^{*i} \tilde{p}_s^{*0},\ \forall i,\ \forall s$ *then* $[\theta^{*i}, q^*]$ *solve* $\mathcal{P}$ *for any* i*, where*

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s) \theta_k^{*i},\ \forall i,\ \forall s$$

and

$$q^* = \sum_s \tilde{p}_s^{*0} R^0(s)$$

and $\sum_i \theta^{*i} = 0$.

Proof: See Appendix. ■

5 Comments

5.1 Comment 1

Condition $(\mathcal{E})$ means that for any portfolio $\theta_1, \ldots, \theta_k$, the return it yields will be the same for any country i if it is valued in currency 0. This condition is very important. We give an example where it is not satisfied and we have no equilibrium.

We consider a consumption model with two countries, 0 and 1, two states of nature and two assets. We assume

$$R^o = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad R^1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

In this economy, condition $(\mathcal{E})$ is not satisfied. We have

$$\begin{array}{rcl} W^0 & = & \{(\theta_1, \theta_2) : \theta_1 \geq 0,\ \theta_1 + 2\theta_2 \geq 0\} \\ W^1 & = & \{(\theta_1, \theta_2) : \theta_2 \geq 0,\ 2\theta_1 + \theta_2 \geq 0\} \\ S^0 & = & \{(p_1, p_2) : p_1 > 0,\ p_2 > 0,\ 2p_1 - p_2 > 0\} \\ S^1 & = & \{(p_1, p_2) : p_1 > 0,\ p_2 > 0,\ 2p_2 - p_1 > 0\} \end{array}$$

One can check that $(1,1) \in S^0 \cap S^1$. From Proposition 9, there exist $[(\theta^{*i})_{i=0,1}; (q^*(1), q^*(2))]$ such that
(a) $\forall i,\ \theta^{*i}$ solves problem $(\mathcal{P})$
(b) $\sum_{i=0}^{2} \theta^{*i} = 0$
(c) $(q^*(1), q^*(2)) >> 0$
If we have an equilibrium then $\forall s,\ \sum_i p_s^* c_s^{i*} = \sum_i p_s^* \omega_s^i$ which implies

$$\sum_i \sum_k R_k^i(s) p_s^{*i} \theta_k^{*i} = 0,\ \forall s$$

In our case, we have in particular

$$R_1^0(1)p_1^{*0}\theta_1^{*0} + R_2^0(1)p_1^{*0}\theta_2^{*0} + R_1^1(1)p_1^{*1}\theta_1^{*1} + R_2^1(1)p_1^{*1}\theta_2^{*1} = 0$$

Since $\theta_1^{*0} + \theta_1^{*1} = 0$, we get

$$\left[R_1^0(1)p_1^{*0} - R_1^1(1)p_1^{*1}\right]\theta_1^{*0} + \left[R_2^0(1)p_1^{*0} - R_2^1(1)p_1^{*1}\right]\theta_2^{*0} = 0$$

Replacing $R_1^0, R_1^1, R_2^0, R_2^1$ by their values, we finally obtain

$$p_1^{*0}\theta_1^{*0} - p_1^{*1}\theta_2^{*0} = 0$$

which is a contradiction since $q_1^*\theta_1^{*0} + q_2^*\theta_2^{*0} = 0$.

5.2 Comment 2

Consider condition $(\mathcal{E})$. We assume that for any country i, the asset i is riskless. The returns $R_i^i(s)$ will not depend on s and are assumed to be constant. Condition $(\mathcal{E})$ may be written as

$$Log\tau_s^{*i} = LogR_i^0(s) - LogR_i^i$$

Let $E_i^i = LogR_i^i$. Assume that the returns are given, as in Fontaine [7], relation (5)

$$LogR_i^0(s) = E_i^0(s) + \sum_{m=1}^{M} b_{im}^0 \tilde{f}_m^0(s)$$

where $\tilde{f}_m^0$ are the common factors, we then obtain

$$Log\tau_s^{*i} = E_i^0(s) - E_i^i + \sum_{m=1}^{M} b_{im}^0 \tilde{f}_m^0(s) \tag{4}$$

which is relation (9) in Fontaine [7].
More generally, assume that

$$LogR_k^j(s) = E_k^j(s) + \sum_{m=1}^{M} b_{km}^j \tilde{f}_m^j(s)$$

Let $r_{jk}^0(s) = Log(\tau_s^{*j} R_k^j(s))$. r_{jk}^0 is the return of asset k in country j valued in currency 0. We get:

$$r_{jk}^0(s) = E_k^j(s) + \sum_{m=1}^{M} b_{km}^j \tilde{f}_m^j(s) + E_i^0(s) - E_i^i + \sum_{m=1}^{M} b_{im}^0 \tilde{f}_m^0(s) \tag{5}$$

which corresponds to relation (11) in Fontaine [7]. If Relation (5) holds for any country j, for any asset k, we then have an equilibrium in the two-period consumption model. However, this condition is not sufficient for the wealth model. Actually, to get Relation (4), Fontaine [7] considers a wealth model and supposes there exists no arbitrage opportunity. In this case, we have also an equilibrium for his two-period wealth model

5.3 Comment 3

An equilibrium price is given by

$$\begin{aligned}
\forall i,\ \forall k,\ q_k^* &= \lambda^i \sum_{s=1}^{S} \pi_s^i u^{i\prime}(\omega_s^i + \sum_{k=1}^{K} R_k^i(s)\theta_k^{*i}) R_k^i(s) \\
&= \lambda^i \sum_{s=1}^{S} \pi_s^i u^{i\prime} \left(\omega_s^i + \sum_{k=1}^{K} \frac{R_k^0(s)\theta_k^{*i}}{\tau_s^{*i}} \right) \frac{R_k^0(s)}{\tau_s^{*i}} \qquad (6)
\end{aligned}$$

Consider the case where all the countries risk-neutral ($u^i(x) = x$). Assume A1, A2, A3, P, U1 and ($\mathcal{E}$). From our existence of equilibrium results, if an equilibrium exists we then have PPP. Let us prove the converse. From (6), if an equilibrium exists with risk-neutral agents then, up to a scalar, asset prices are

$$\forall k = 1, \ldots, K,\ q_k^* = \sum_s \pi_s^i R_k^i(s)$$

and consumption prices are therefore (π_s^i). Assume they satisfy PPP:

$$\forall i,\ \forall s,\ \pi_s^i = \tau_s^{*i} \pi_s^0$$

Then

$$\forall k = 1, \ldots, K,\ q_k^* = \sum_s \pi_s^0 R_k^0(s)$$

Let (ψ_k^i) be a portfolio net trade, i.e. $\sum_{i=0}^{L} \psi_k^i = 0$. We claim that $[(\psi^i, (c_s^{*i}; \pi_s^i)_{s=1,\ldots,S})_{i=0,\ldots,L}, q^*]$ is an equilibrium where

$$\forall i,\ \forall s,\ c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\psi_k^i$$

Indeed, consider some country i and let (θ_k^i) satisfy

$$\sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) > \sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\psi_k^i)$$

Equivalently, since $u^i(c) = c$:

$$
\begin{aligned}
\sum_k \sum_s \pi_s^i R_k^i(s)\theta_k^i &> \sum_k \sum_s \pi_s^i R_k^i(s)\psi_k^i \\
\sum_k \sum_s \pi_s^0 \tau_s^{*i} R_k^i(s)\theta_k^i &> \sum_k \sum_s \pi_s^0 \tau_s^{*i} R_k^i(s)\psi_k^i \\
\sum_k \sum_s \pi_s^0 R_k^0(s)\theta_k^i &> \sum_k \sum_s \pi_s^0 R_k^0(s)\psi_k^i \\
q^* \cdot \theta^i &> q^* \cdot \psi^i
\end{aligned}
$$

Hence $[(\psi^i), q^*]$ solve $(\mathcal{P})$. It is easy to check that

$$
\begin{aligned}
\forall i,\ \sum_s \pi_s^i c_s^{*i} &= \sum_s \pi_s^i \omega_s^i \\
\forall s,\ \sum_i \pi_s^i c_s^{*i} &= \sum_i \pi_s^i \omega_s^i
\end{aligned}
$$

Our claim is true.

6 Conclusion

Our paper attempts to link, when we are in presence of international markets, the General Equilibrium and the Finance frameworks. It emphasizes the role of exchange rates and the respect vs the non-respect of the Purchasing Power Parity. If PPP is not respected, we cannot have an equilibrium on the international goods markets but we may have an equilibrium on the international financial assets. In the usual literature, for instance Rogoff [12], the common feeling is that PPP is not respected, even in the long run and that testing PPP will introduce a lot of problems. The implication of these considerations is that we have a discrepancy between these two international markets. Our paper may therefore open to future empirical research testing the coherency between international financial markets and international goods markets. It might be interesting to see, during the recent financial crisis, (i) whether the discrepancies between the two markets were widened or not, and (ii) if the deviations from PPP were bigger or not, compare to the situations before the crises.

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7 Appendix

Proof of Proposition 10 We know that condition $\mathcal{C}$ is equivalent to condition $\mathcal{E}$. Under $(\mathcal{E})$, the set W^i

$$\begin{aligned} W^i &= \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^i(s) w_k \geq 0, \ \forall s \right\} \\ &= \left\{ w \in \mathbb{R}^K : \frac{1}{\tau_s^{*i}} \sum_{k=1}^{K} R_k^0(s) w_k \geq 0, \ \forall s \right\} \\ &= \left\{ w \in \mathbb{R}^K : \sum_{k=1}^{K} R_k^0(s) w_k \geq 0, \ \forall s \right\} \end{aligned}$$

is independent of i and hence S^i is the same for all i. We will show that S^0 is non-empty. Indeed, let $w \in W^0 \setminus \{0\}$. Then there exists s' such that $\sum_{k=1,\ldots,K} R_k^0(s') w_k > 0$. If not, we have: $\forall s, \ \sum_{k=1,\ldots,K} R_k^0(s) w_k = 0$. From **A3**, $w = 0$ which is a contradiction. Now, let $q \in \mathbb{R}^K$ be defined by $\forall k, \ q_k = \sum_{s=1,\ldots,S} R_k^0(s)$. Then $q \cdot w > 0$ for any $w \in W^0 \setminus \{0\}$. That means $q \in S^0$. The No-Arbitrage condition $(\mathcal{NA})$ is therefore satisfied.

From Proposition 9, there exist $[(\theta^{*i})_{i=0,\ldots,L}; q^* \neq 0]$ such that

$$\begin{array}{ll} (a) & \forall i,\ \theta^{*i} \text{ solves problem } (\mathcal{P}) \\ (b) & \displaystyle\sum_{i=0}^{L} \theta^{*i} = 0. \end{array}$$

Let

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i},\ \forall i,\ \forall s$$

and let q^* be an equilibrium price. We know that q^* is NA1. From Dana and Jeanblanc-Piqué [2], there exists $\big((\beta_s^i > 0); i = 0,\ldots,L; s = 1,\ldots,S\big)$ such that $\forall i,\ q^* = \sum_s \beta_s^i R^i(s)$. Define $\tilde{p}_s^{*i} = \beta_s^i,\ s = 1,\ldots,S; i = 0,\ldots,L$. We have

$$\forall i,\ q_k^* = \sum_s \tilde{p}_s^{*i} R_k^i(s) = \sum_s \frac{\tilde{p}_s^{*i}}{\tau_s^{*i}} R_k^0(s) = \sum_s \tilde{p}_s^{*0} R_k^0(s) \tag{7}$$

Let

$$Z = \{z \in \mathbb{R}^S : \sum_s z_s R_k^0(s) = 0,\ \forall k\}$$

$Z = \{0\}$ if the market is complete. From (7), we get

$$\forall i, \tilde{p}_s^{*i} = \tau_s^{*i}(\tilde{p}_s^{*0} + z_s^i)$$

with $(z^i) \in Z$. Define

$$\begin{aligned} \forall i \neq 0,\ \forall s,\ p_s^{*i} &= \tilde{p}_s^{*i} - \tau_s^{*i} z_s^i = \tau_s^{*i}\tilde{p}_s^{*0} && (8) \\ p_s^{*0} &= \tilde{p}_s^{*0} && (9) \end{aligned}$$

Now, let

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i},\ \forall i,\ \forall s$$

We have

$$\sum_s p_s^{*i} c_s^{*i} = \sum_s p_s^{*i}\omega_s^i + \sum_k q_k^*\theta_k^{*i} = \sum_s p_s^{*i}\omega_s^i$$

since $\sum_k q_k^*\theta_k^{*i} = 0$.

Observe that, for any s, any i, we have

$$p_s^{*i} R_k^i(s) = p_s^{*0} R_k^0(s)$$

Hence

$$\begin{aligned} p_s^{*i} c_s^{*i} &= p_s^{*i} \omega_s^i + \sum_k p_s^{*i} R_k^i(s) \theta_k^{*i} \\ &= p_s^{*i} \omega_s^i + \sum_k p_s^{*0} R_k^0(s) \theta_k^{*i} \end{aligned}$$

Summing over i we get

$$\sum_i p_s^{*i} c_s^{*i} = \sum_i p_s^{*i} \omega_s^i$$

i.e. the prices (p_s^{*i}) satisfy the Consistency Condition. Obviously, they also satisfy PPP. We end the proof.

Proof of Proposition 12 (1) Assume there exist $[(\theta^i, \lambda^i > 0)_{i=0,\ldots,L}]$ such that

$$\forall i,\ \forall j,\ \forall k',\ \lambda^i \sum_s \pi_s^i u^{i\prime}(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) R_{k'}^i = \lambda^j \sum_s \pi_s^j u^{j\prime}(\omega_s^j + \sum_k R_k^j(s)\theta_k^j) R_{k'}^j$$

Let

$$q_{k'} = \lambda^i \sum_s \pi_s^i u^{i\prime}(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) R_{k'}^i,\ \forall k'$$

We will show that q is no-arbitrage. Indeed, let $w \in W^i \setminus \{0\}$. Let $\zeta_s = \sum_k R_k^i(s) w_k,\ \forall s$. We will show

$$q \cdot w = \lambda^i \sum_s \pi_s^i u^{i\prime}(\omega_s^i + \sum_k R_k^i(s)\theta_k^i)\zeta_s > 0$$

From A3, $\zeta \neq 0$. Let $S^+ = \{s : \zeta_s \geq 0\}$, $S^- = \{s : \zeta_s < 0\}$. We have

$$\lambda^i \sum_s \pi_s^i u^{i\prime}(\omega_s^i + \sum_k R_k^i(s)\theta_k^i)\zeta_s > \lambda^i \left(a^i \sum_{s \in S^+} \pi_s^i \zeta_s + b^i \sum_{s \in S^-} \pi_s^i \zeta_s \right) \geq 0$$

That means q is no-arbitrage for any agent i. Under A1, A2, P, U1, if there exists a no-arbitrage price then (see e.g. Werner [15], Page and Wooders [9], Dana, Le Van, Magnien [5]) there exist $[(\theta^{*i})_{i=0,\ldots,L}; q^* >> 0]$ such that

(a) $\forall i,\ \theta^{*i}$ solves problem $(\mathcal{P})$

(b) $\sum_{i=0}^L \theta^{*i} = 0$

The proof of the existence of (p_s^{*i}) which satisfy condition (4) of an equilibrium is the same as in the proof of Proposition 10.

Conversely, if $[(\theta^{*i})_{i=0,\dots,L};\ q^* \neq 0]$ are the equilibrium port-folio and equilibrium assets prices, then

$$\forall k',\ q^*_{k'} = \lambda^{*i} \sum_{s=1}^{S} \pi^i_s u^{i\prime} \left(\omega^i_s + \sum_{k=1}^{K} R^i_k(s) \theta^{*i}_k \right) R^i_{k'}(s),\ \lambda^{*i} > 0$$

and

$$\forall i, a^i < u^{i\prime} \left(\omega^i_s + \sum_{k=1}^{K} R^i_k(s) \theta^{*i}_k \right) < b^i$$

One can show as just above that $q^* \in \cap_i S^i$, i.e. a no-arbitrage price.

Proof of Proposition 13 Let $[\theta^{*i}, q^*]$ solve $\mathcal{P}$ for any i and $\sum_i \theta^{*i} = 0$. In this case, q^* is NA1. From Dana and Jeanblanc-Piqué [2], there exists $\big((\beta^i_s > 0); i = 0, \dots, L; s = 1, \dots, S\big)$ such that $\forall i,\ q^* = \sum_s \beta^i_s R^i(s)$. Define $p^{*i}_s = \beta^i_s,\ s = 1, \dots, S; i = 0, \dots, L$. We have

$$\forall i,\ q^*_k = \sum_s p^{*i}_s R^i_k(s) = \sum_s \frac{p^{*i}_s}{\tau^{*i}_s} R^0_k(s) = \sum_s p^{*0}_s R^0_k(s) \tag{10}$$

Let

$$Z = \{z \in \mathbb{R}^S : \sum_s z_s R^0_k(s) = 0,\ \forall k\}$$

$Z = \{0\}$ if the market is complete. From (10), we get

$$\forall i, p^{*i}_s = \tau^{*i}_s (p^{*0}_s + z^i_s)$$

with $(z^i) \in Z$. Define

$$\forall i \neq 0,\ \forall s,\ \tilde{p}^{*i}_s = p^{*i}_s - \tau^{*i}_s z^i_s = \tau^{*i}_s p^{*0}_s \tag{11}$$

$$\tilde{p}^{*0}_s = p^{*0}_s \tag{12}$$

Now, let

$$c^{*i}_s = \omega^i_s + \sum_k R^i_k(s) \theta^{*i}_k,\ \forall i,\ \forall s$$

We have

$$\sum_s \tilde{p}^{*i}_s c^{*i}_s = \sum_s \tilde{p}^{*i}_s \omega^i_s + \sum_k q^*_k \theta^{*i}_k = \sum_s \tilde{p}^{*i}_s \omega^i_s$$

since $\sum_k q_k^* \theta_k^{*i} = 0$.
Observe that, for any portfolio of country i, θ^i,

$$q^* \cdot \theta^i = \sum_s \tilde{p}_s^{*i} (\sum_k R_k^i(s) \theta_k^i)$$

Now, let

$$\sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) > \sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^{*i})$$

This implies $q^* \cdot \theta^i > q^* \cdot \theta^{*i}$ or equivalently

$$\sum_s \tilde{p}_s^{*i} (\sum_k R_k^i(s)\theta_k^i) > \sum_s \tilde{p}_s^{*i} (\sum_k R_k^i(s)\theta_k^{*i})$$

And if we define

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i}, \ \forall i, \ \forall s$$

$$c_s^i = \omega_s^i + \sum_k R_k^i(s)\theta_k^i, \ \forall i, \ \forall s$$

we obtain

$$\sum_s \tilde{p}_s^{*i} c_s^{*i} > \sum_s \tilde{p}_s^{*i} c_s^i$$

That means $[(c_s^{*i}); (\tilde{p}_s^{*i}); i = 0, \ldots, L; s = 1, \ldots, S]$ is an equilibrium for the model where

(a) each agent i solves:

$$\max \sum_s \pi_s^i u^i(c_s^i)$$

under the constraints:

$$c^i \in X^i \ = \ \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, \ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \cap \mathbb{R}_+^S \text{ for the consumption model}$$

$$c^i \in X^i \ = \ \{c \in \mathbb{R}^S \mid \exists \theta \in \mathbb{R}^K, \ c_s = \omega_s^i + \sum_k R_k^i \theta_k\} \text{ for the wealth model}$$

and the budget constraint

$$\sum_s p_s^{*i} c_s^i \ \leq \ \sum_s p_s^{*i} \omega_s^i$$

and

(b) $\forall s,\ \forall i,\ \ \tilde{p}_s^{*i} = \tau_s^{*i}\tilde{p}_s^{*0}$

Relation (b) implies the balance on the consumption good market valued in currency 0, i.e. $\sum_i \tilde{p}_s^{*i} c_s^{*i} = \sum_i \tilde{p}_s^{*i}\omega_s^i,\ \forall s$.

Conversely, under A3, one can check that if $[(c_s^{*i});(\tilde{p}_s^{*i}); i = 0,\dots,L; s = 1,\dots,S]$ is an equilibrium for the model given just above with $\tilde{p}_s^{*i} = \tau_s^{*i}\tilde{p}_s^{*0},\ \forall i,\ \forall s$ then $[\theta^{*i}, q^*]$ solve where $\mathcal{P}$ for any i and $\sum_i \theta^{*i} = 0$, where

$$c_s^{*i} = \omega_s^i + \sum_k R_k^i(s)\theta_k^{*i},\ \forall i,\ \forall s$$

and

$$q^* = \sum_s \tilde{p}_s^{*0} R^0(s).$$

Indeed, let

$$\sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) > \sum_s \pi_s^i u^i(\omega_s^i + \sum_k R_k^i(s)\theta_k^{*i})$$

That implies

$$\sum_s \tilde{p}_s^{*i}(\omega_s^i + \sum_k R_k^i(s)\theta_k^i) > \sum_s \tilde{p}_s^{*i}(\omega_s^i + \sum_k R_k^i(s)\theta_k^{*i})$$

or equivalently

$$\sum_k (\sum_s \tilde{p}_s^{*i} R_k^i(s))\theta_k^i > \sum_k (\sum_s \tilde{p}_s^{*i} R_k^i(s))\theta_k^{*i}$$

Under $(\mathcal{E})$, we get

$$q^* \cdot \theta^i > q^* \cdot \theta^{*i}$$

It remains to show that the asset market clears. Since

$$\tilde{p}_s^{*i} = \tau_s^{*i}\tilde{p}_s^{*0},\ \forall i,\ \forall s$$

and

$$\sum_s p_s^{*i} c_s^i = \sum_s p_s^{*i}\omega_s^i,\ \forall i$$

we have

$$\sum_i \sum_k \tau_s^{*i} R_k^i(s) \theta_k^{*i} = 0$$

or equivalently

$$\sum_k R_k^0(s) (\sum_i \theta_k^{*i}) = 0$$

Assumption A3 implies $\sum_i \theta_k^{*i} = 0, \ \forall k$. The proof is complete.