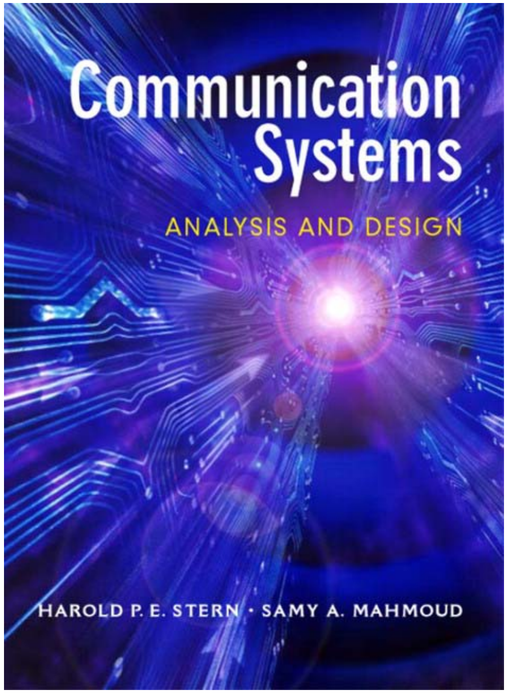
Chapter 2

Frequency Domain Analysis

Chapter 2

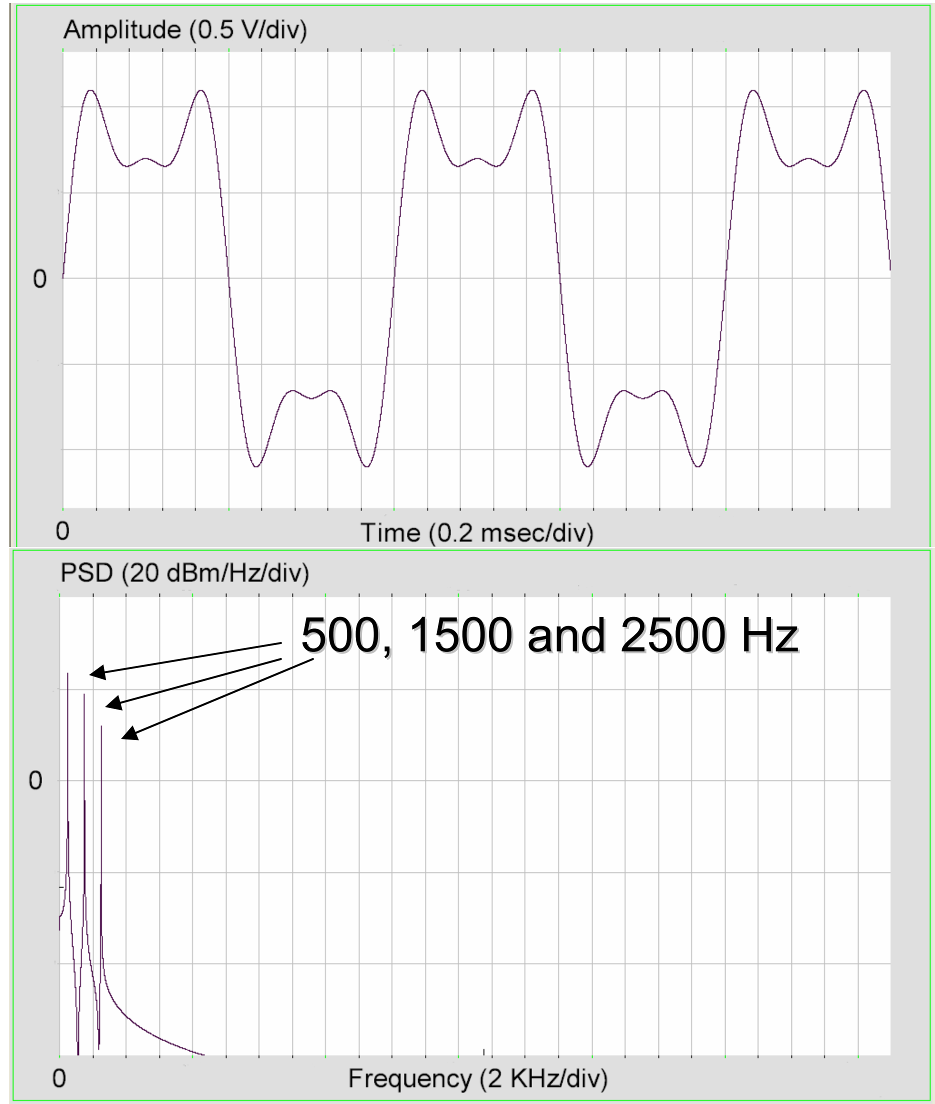
Frequency Domain Analysis

- *Why Study Frequency Domain Analysis?*
- Pages 6-13



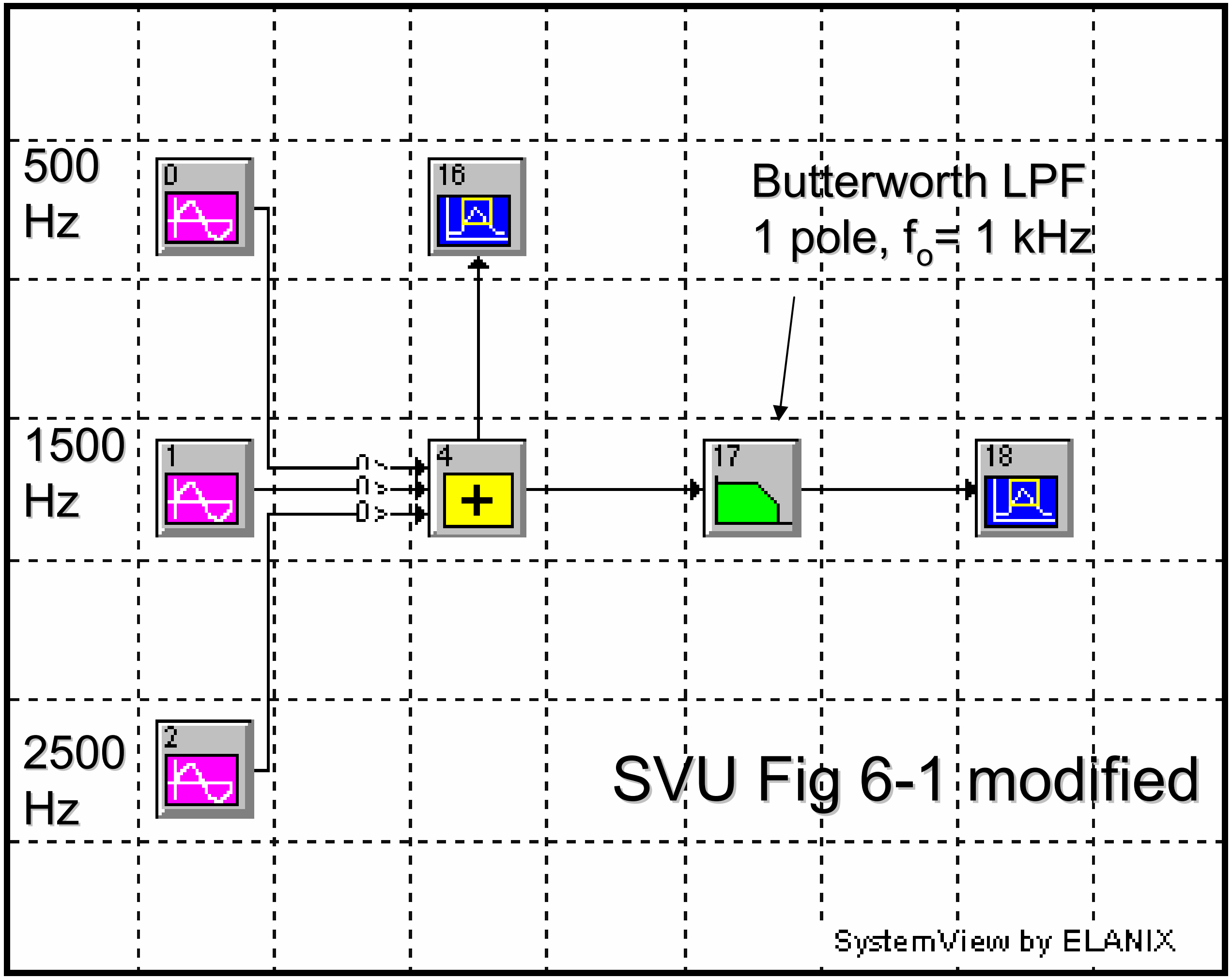
Why frequency domain analysis?

- Allows simple algebra rather than time-domain differential equations to be used
- Transfer functions can be applied to transmitter, communication channel and receiver
- Channel bandwidth, noise and power are easier to evaluate

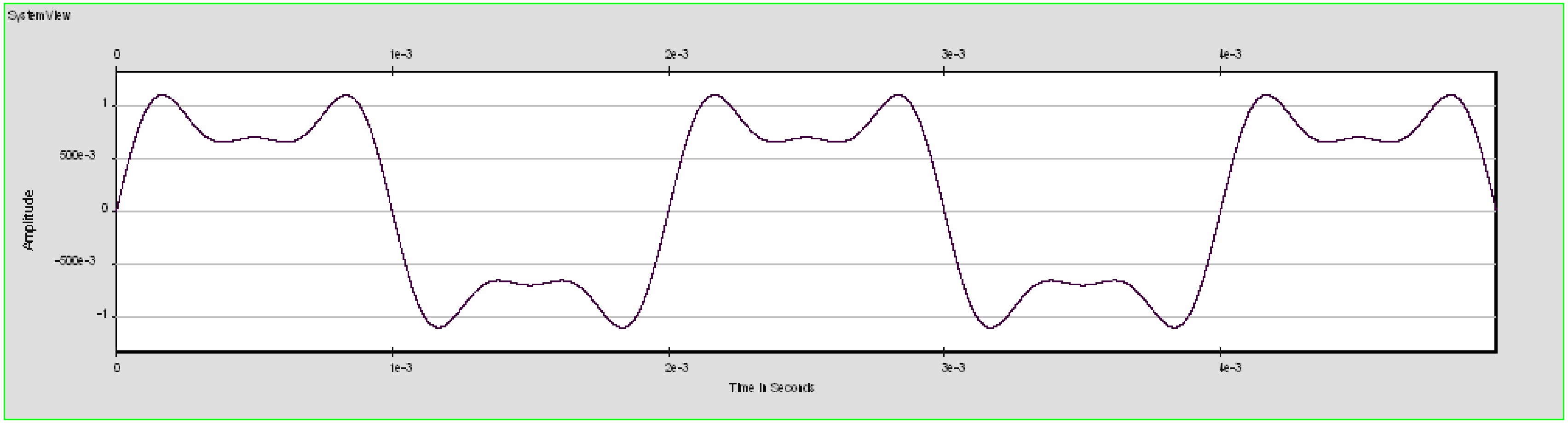


SVU Figure 6.2 and Figure 6.3

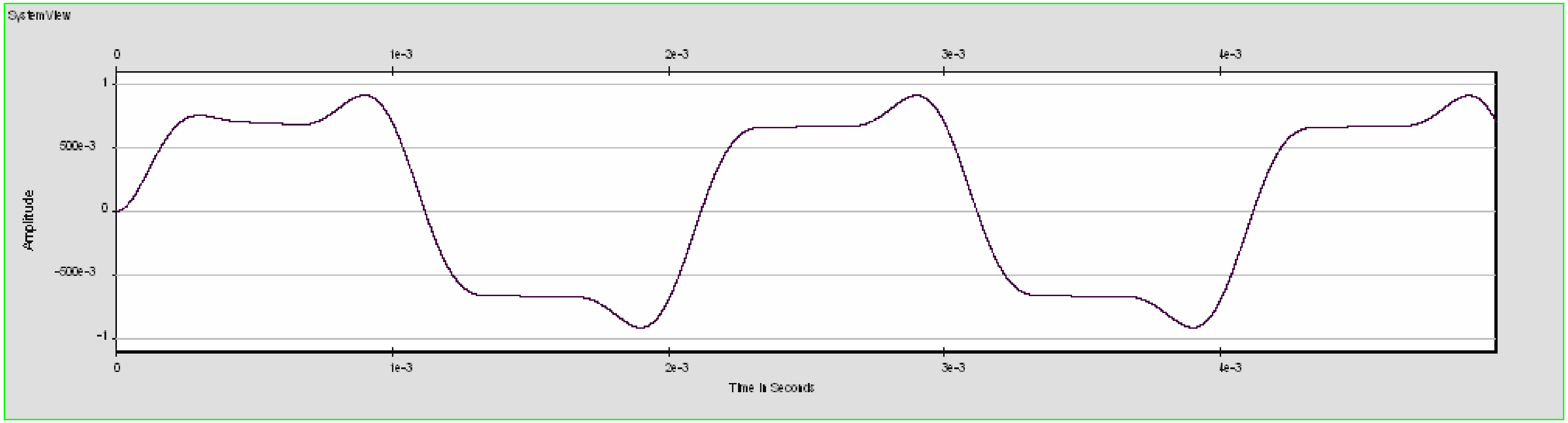
- Example 2.1 Input sum of three sinusoids



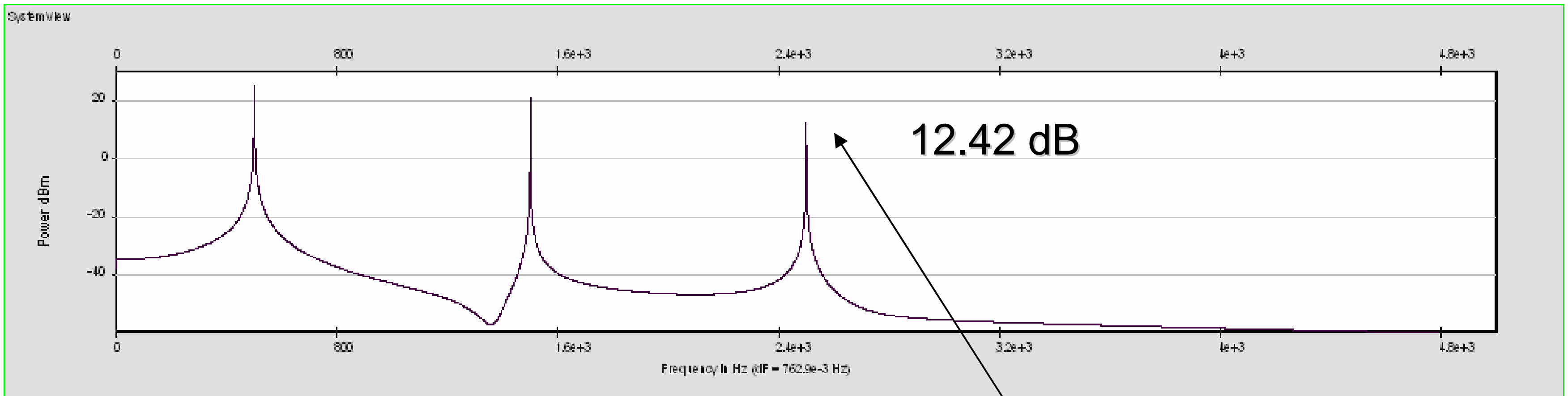
- Example 2.1 Input sum of three sinusoids



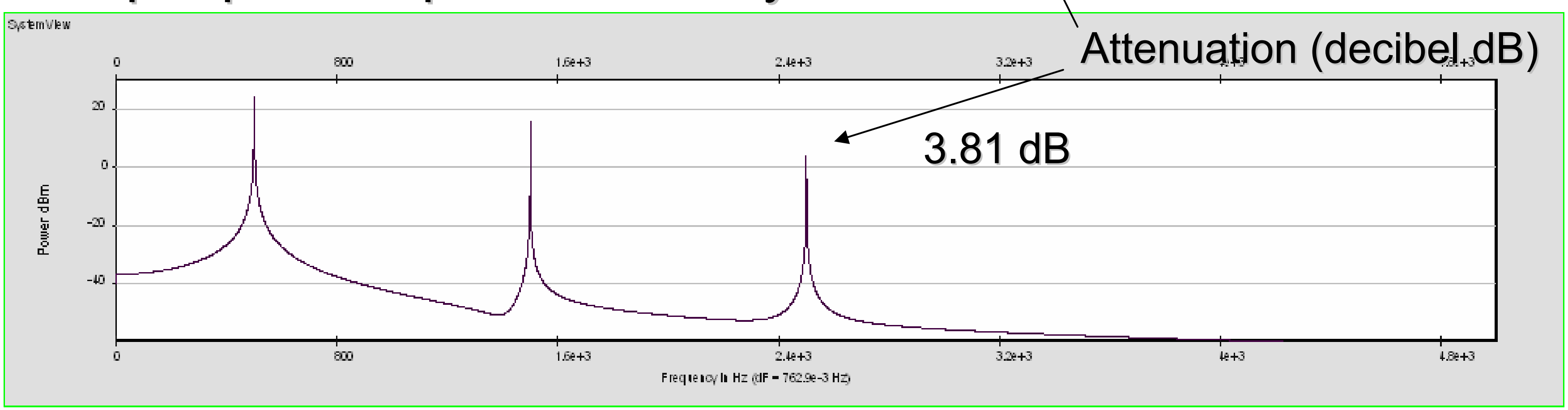
- Output after Butterworth LPF



- Input power spectral density of the sum of three sinusoids



- Output power spectral density after Butterworth LPF



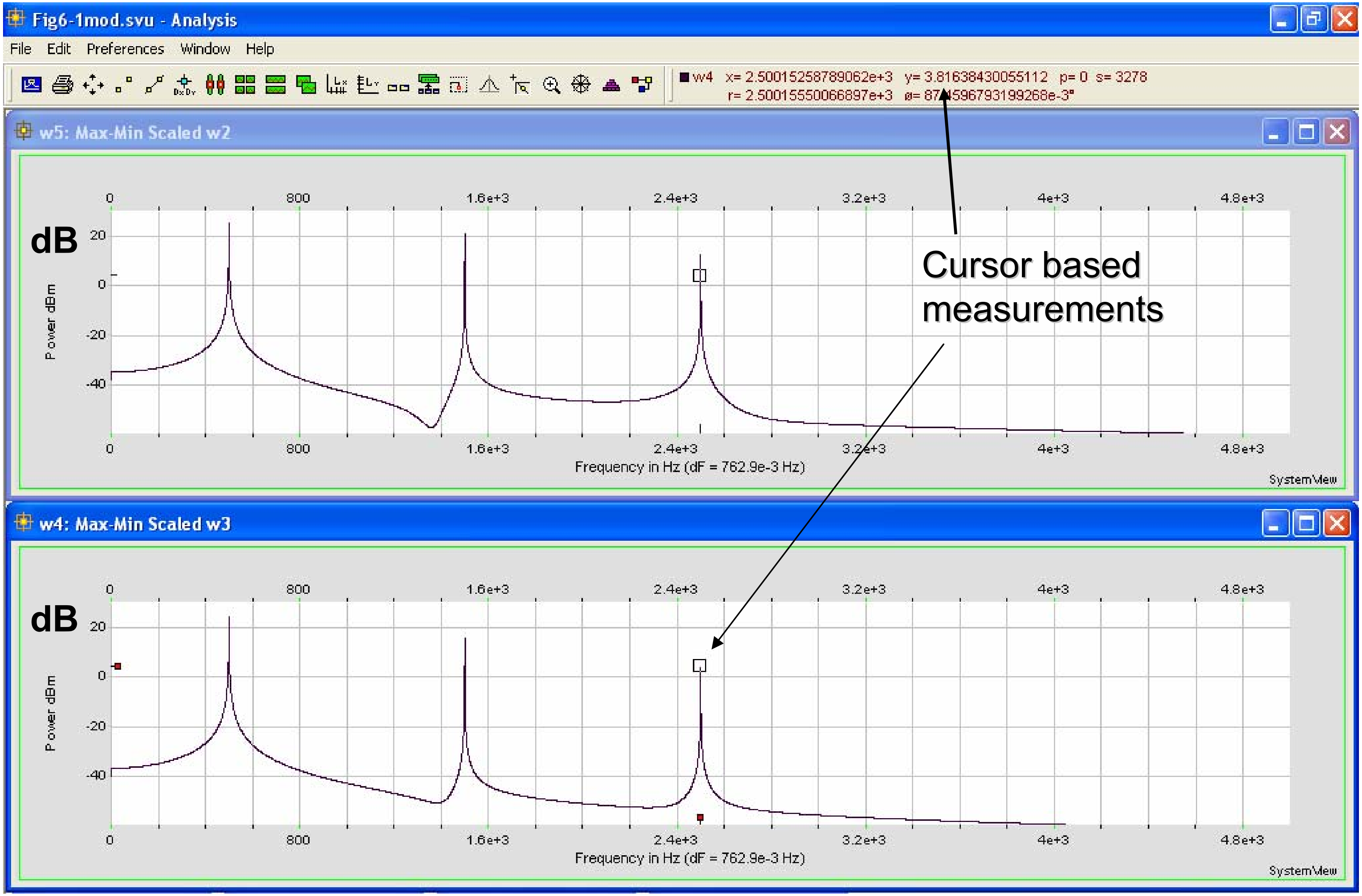
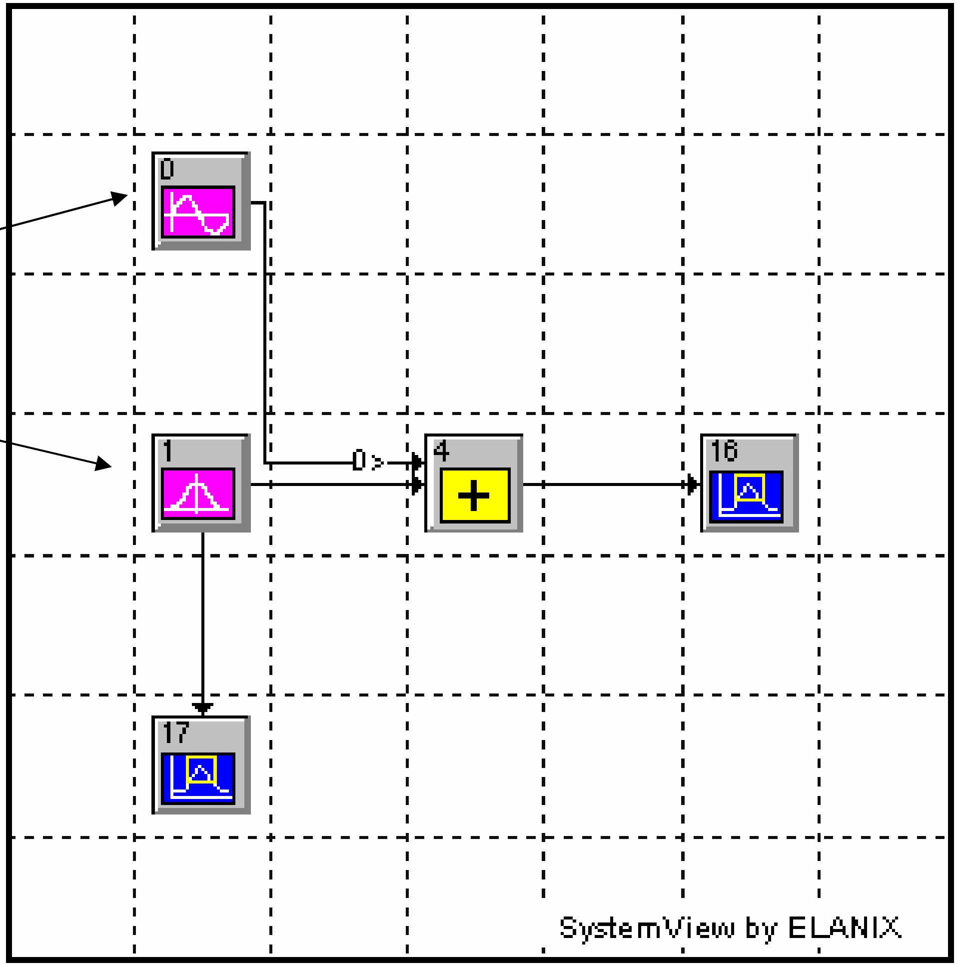


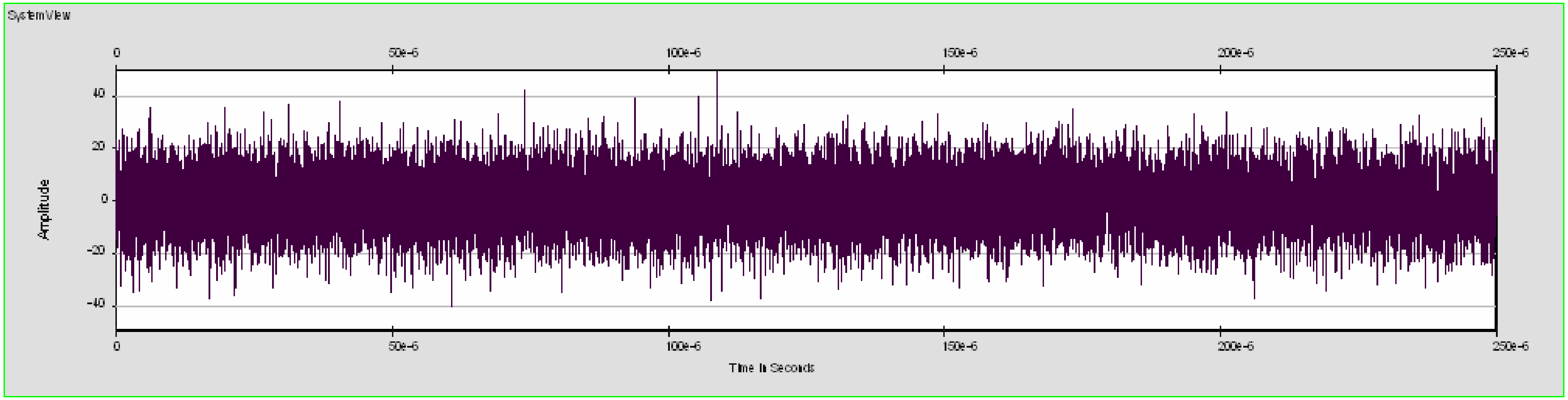
Fig6-1mod.svu - Analysis
File Edit Preferences Window Help
w4 x= 2.500152587890 62e+3 y= 3.81638430055112 p= 0 s= 3278
r= 2.50015550066897e+3
w5: Max-Min Scaled w2
dB
Power dBm
Frequency in Hz (dF = 762.9e-3 Hz)
SystemView
Cursor based
measurements
w4: Max-Min Scaled w3
dB
Power dBm
Frequency in Hz (dF = 762.9e-3 Hz)
SystemView

- Example 2.2

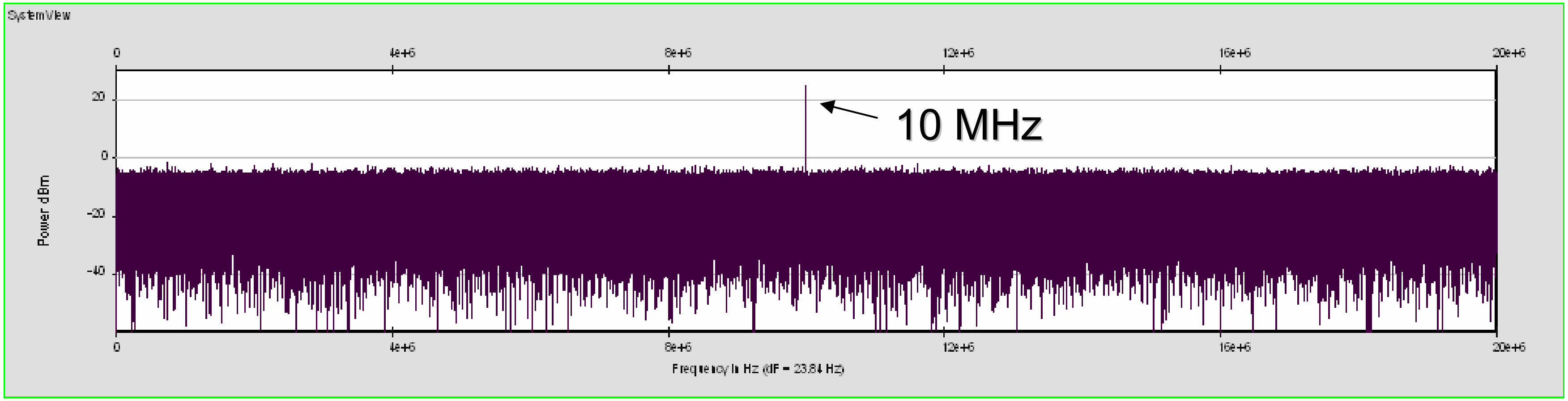
10 MHz sinusoid with additive white Gaussian noise (AWGN)



- Example 2.2 10 MHz sinusoid with AWGN



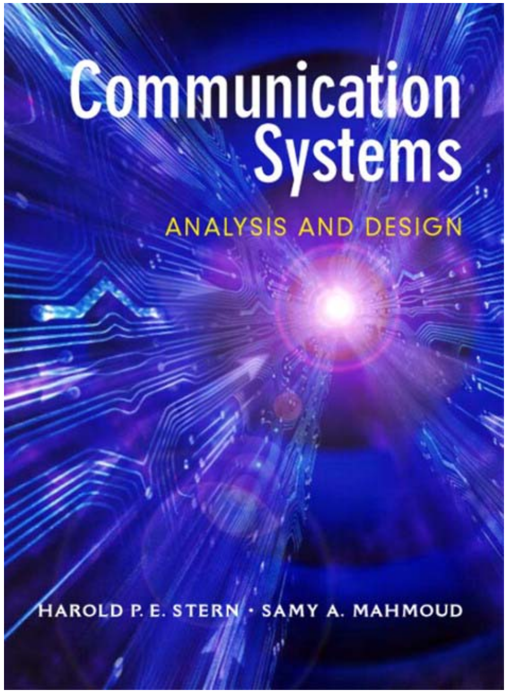
- Power spectral density of 10 MHz sinusoid with AWGN



Chapter 2

Frequency Domain Analysis

- *The Fourier Series*
- Pages 13-38



- Fourier Series

Jean Baptiste Joseph Fourier was a French mathematician and physicist who is best known for initiating the investigation of Fourier Series and its application to problems of heat flow. The Fourier transform is also named in his honor.

1768-1830

$$s(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi n f_o t + \varphi_n)$$

- Fourier series coefficients:

trignometric	a_n b_n
polar	X_n
complex	c_n

$$X_0 = a_0 \qquad X_n = \sqrt{a_n^2 + b_n^2}$$

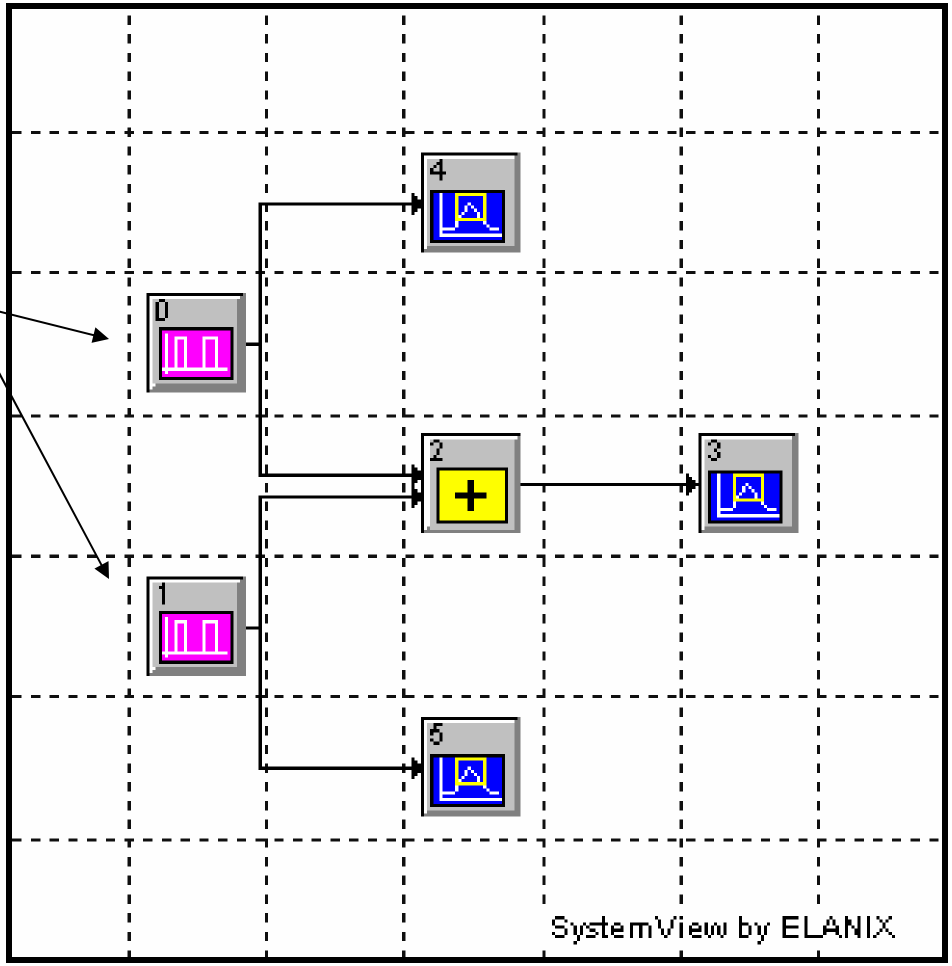
$$|\,c_n\,| = X_n \,/\, 2 \qquad X_n = |\,2\,c_n\,|$$

Agilent EEsof EDA
SystemVue
System Design Software

SystemVue simulation can provide the magnitude of the complex Fourier series coefficients for any periodic waveform.

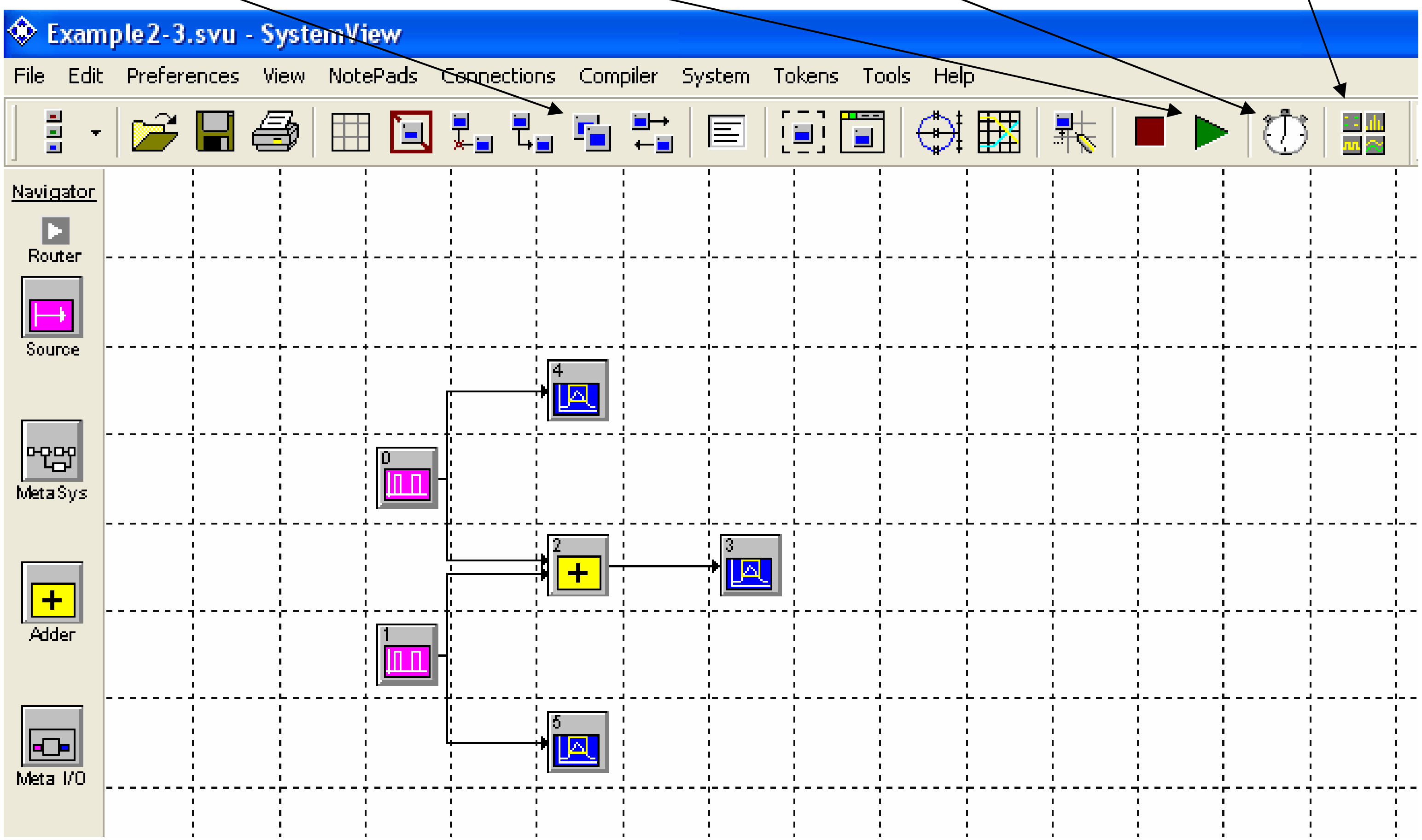
- Example 2.3

Complex pulse
as the addition of
two periodic pulses



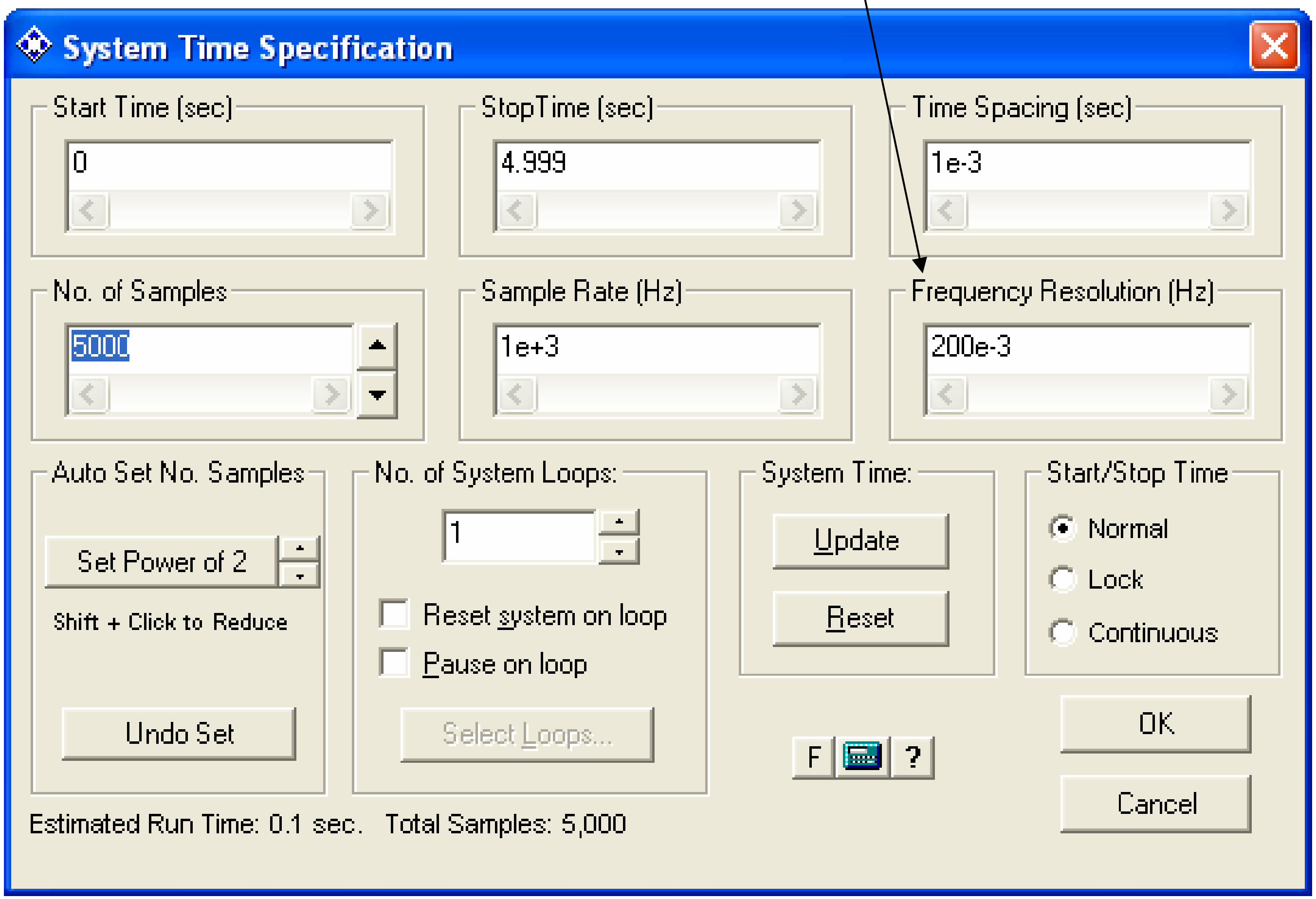
- Example 2.3 SystemVue Design Window

Editing Simulate System Time Analysis Window



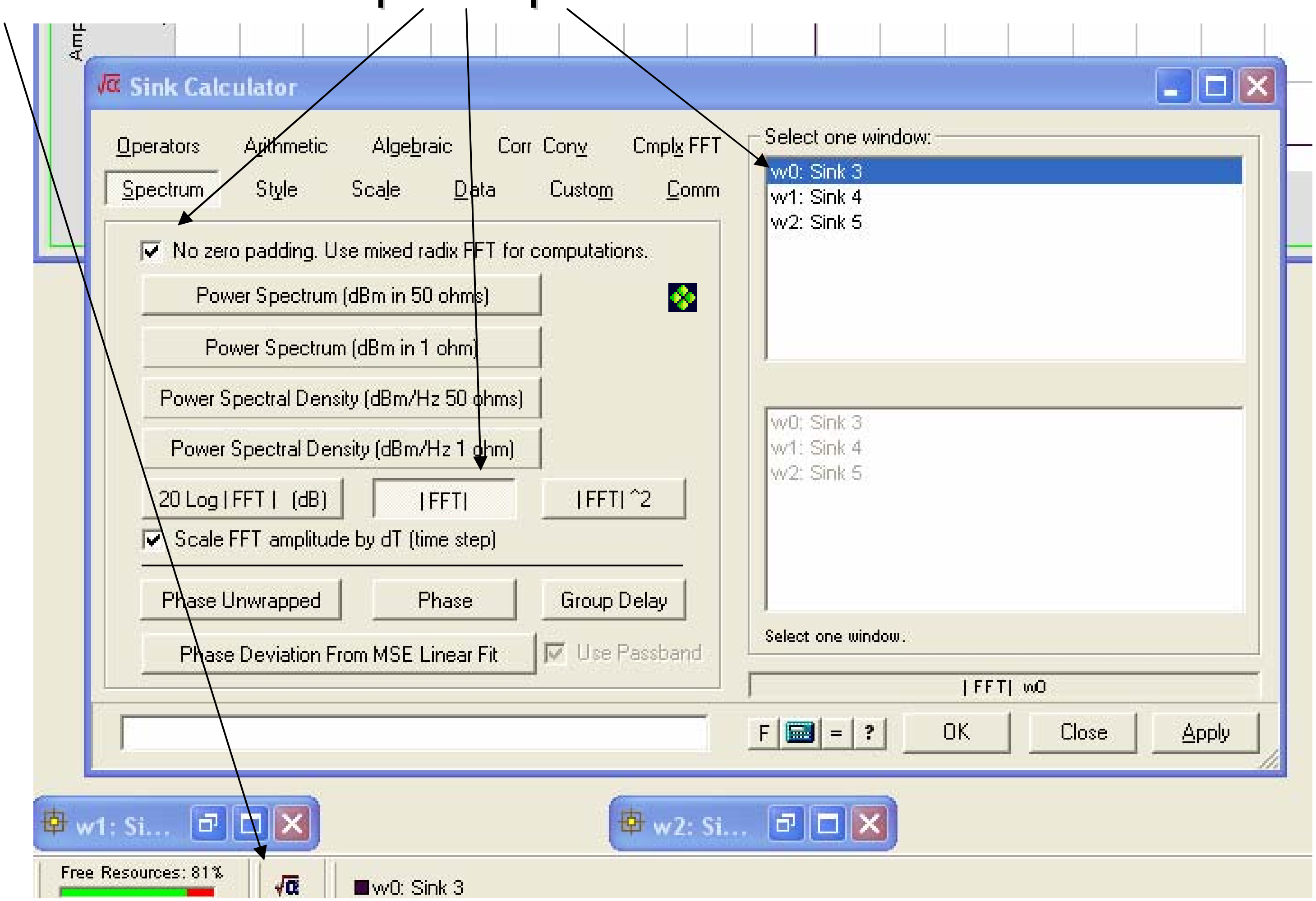
- Example 2.3 SystemVue System Time

Fundamental frequency $f_o = 0.2$ Hz, $T_o = 5$ sec



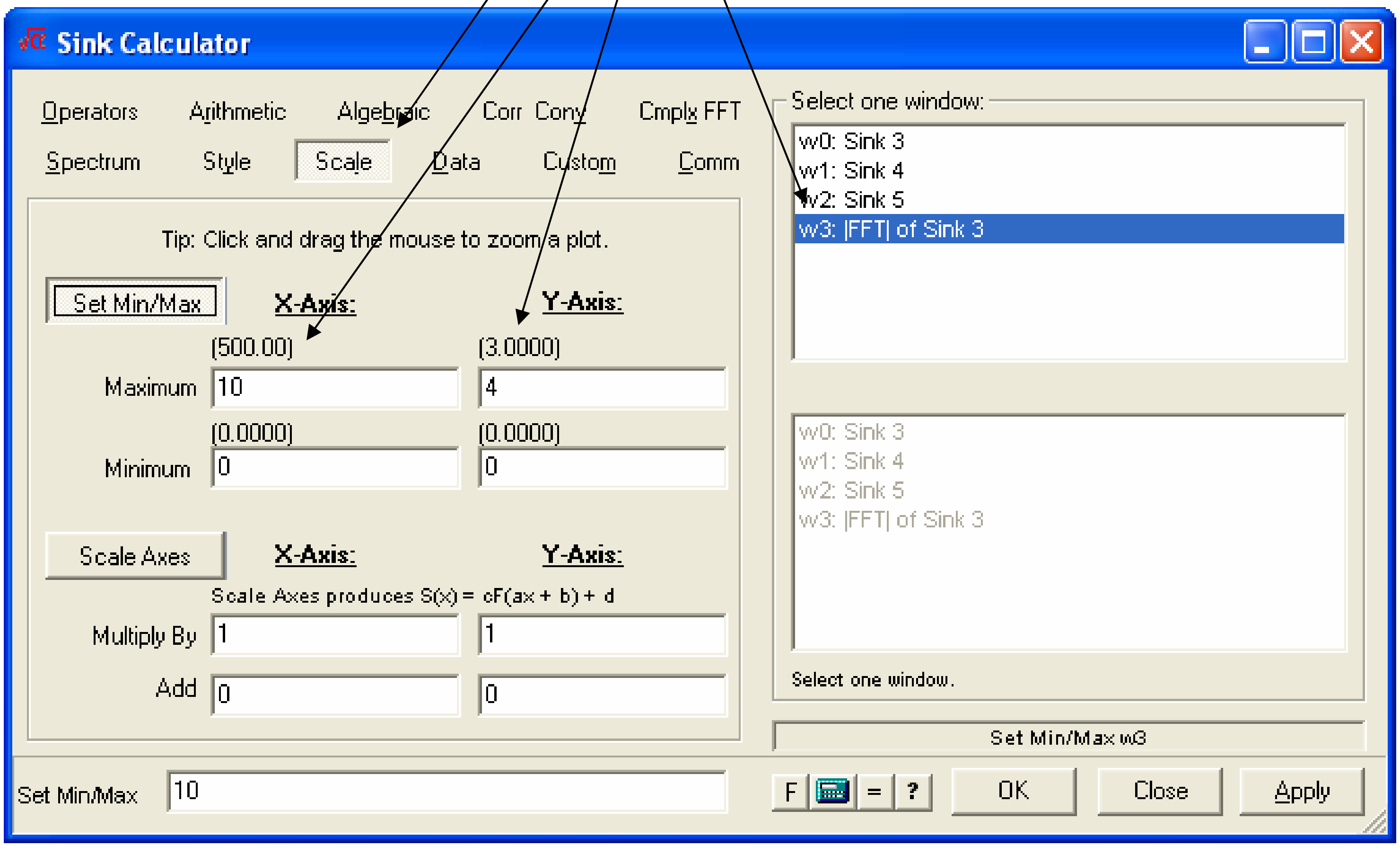
- Example 2.3 SystemVue Analysis Window

Sink calculator | FFT |

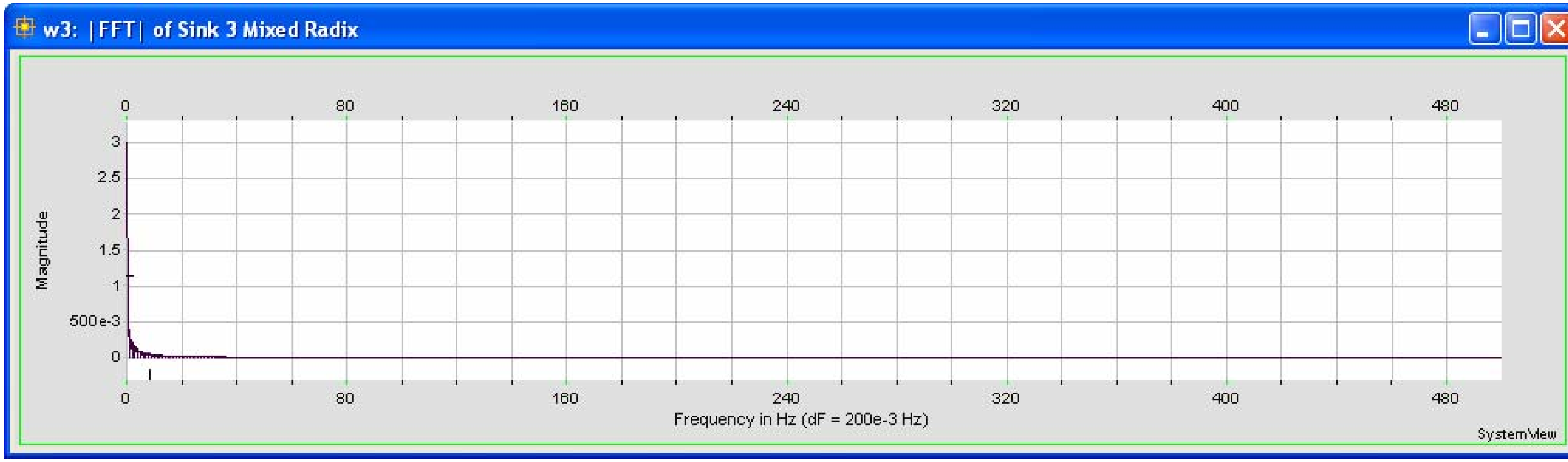


- Example 2.3 SystemVue Analysis Window

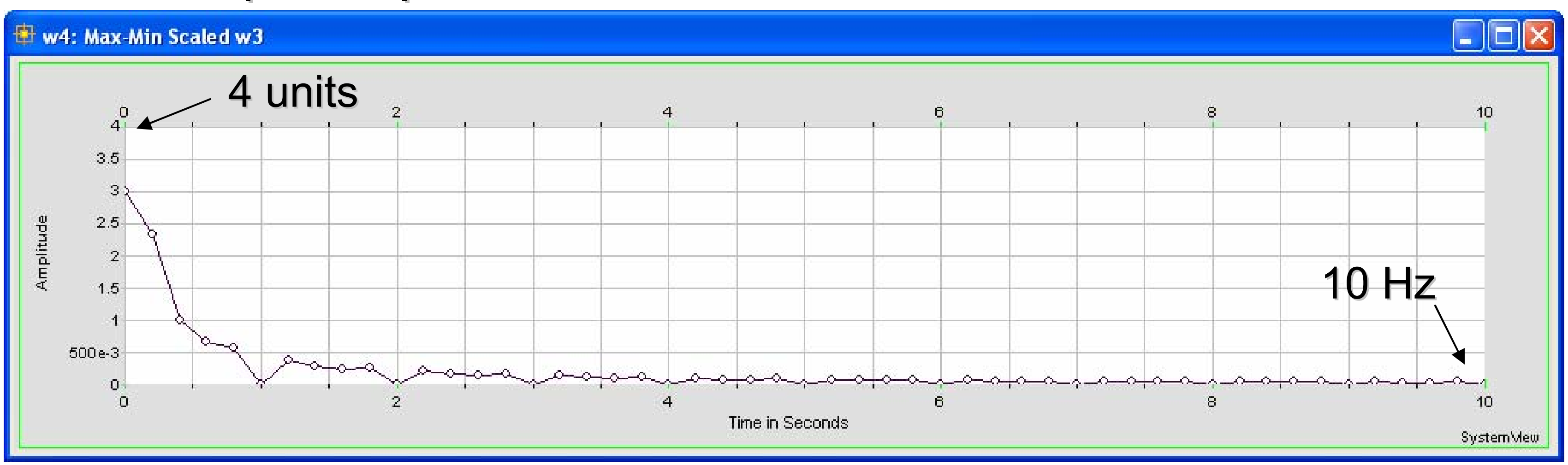
Sink calculator Scale Display



- Example 2.3 Unscaled | FFT |

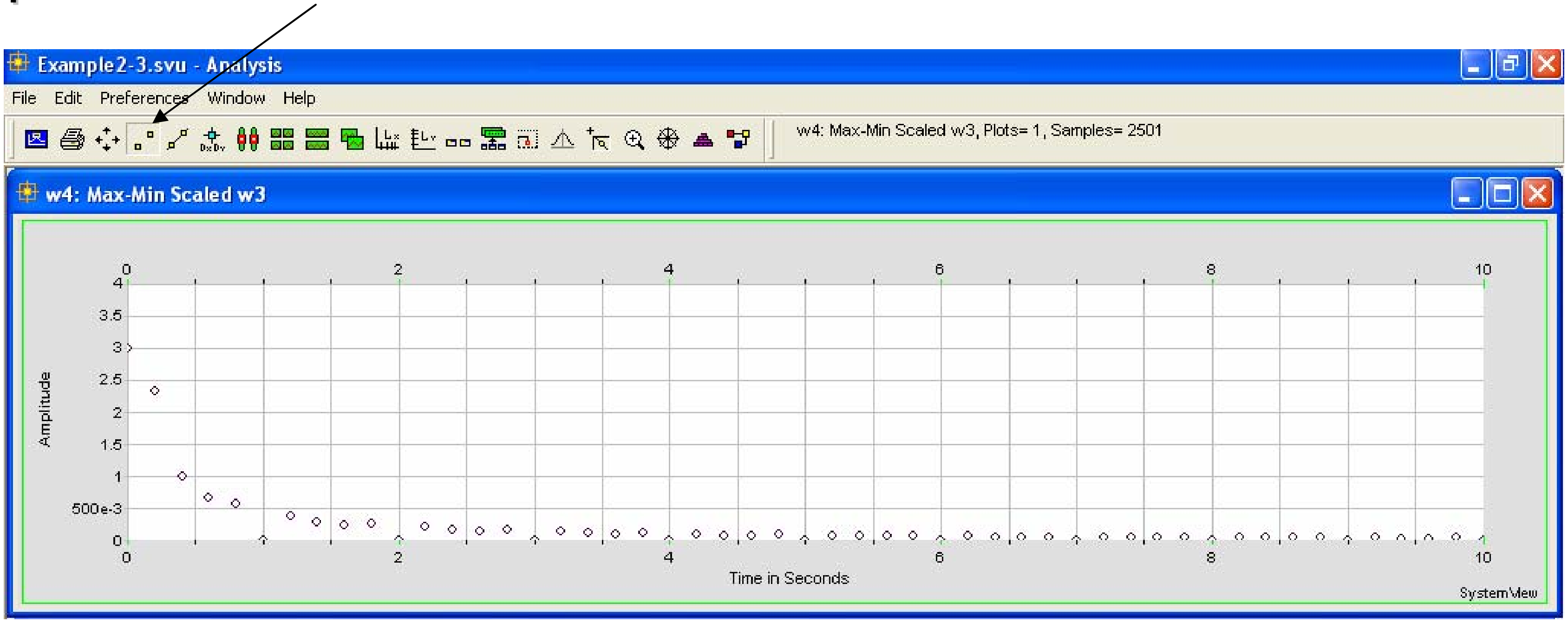


- Scaled | FFT |

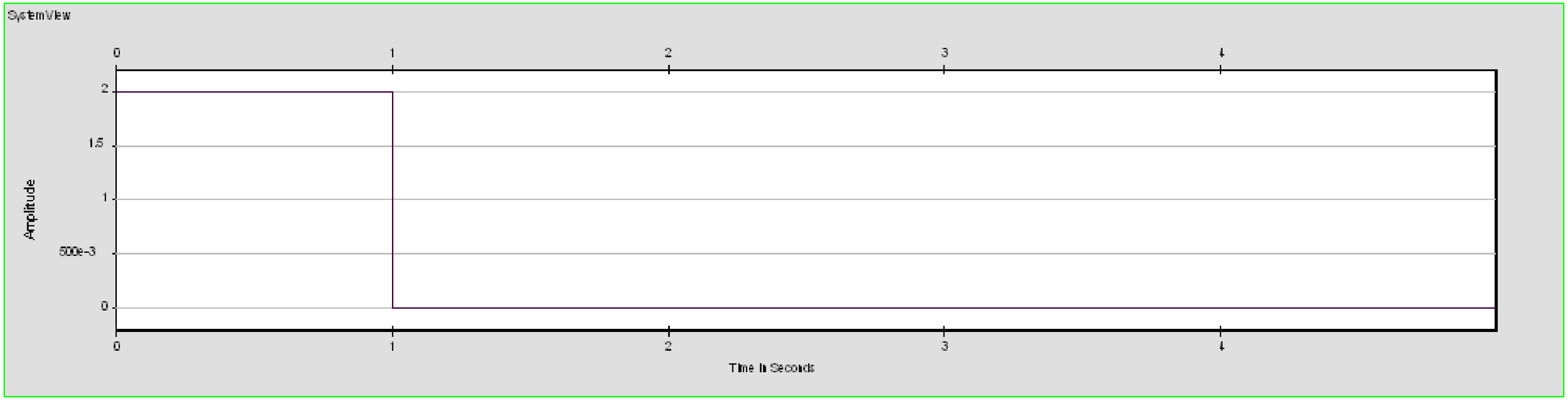


- Example 2.3 Scaled | FFT |

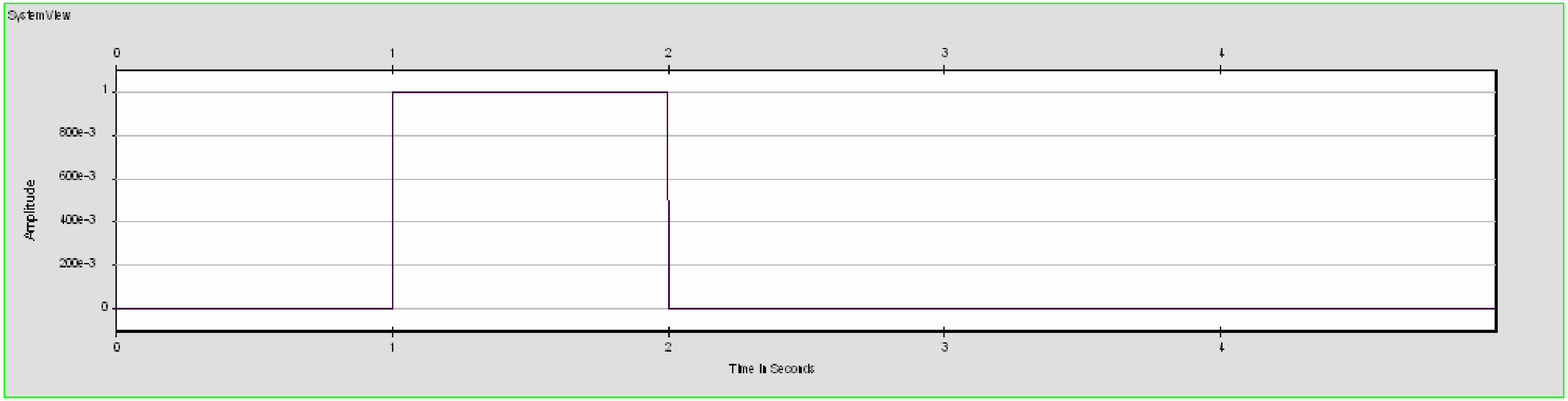
  The Fourier series components are *discrete.* In the *SystemVue* Analysis Window the connection between data points can be eliminated if warranted.



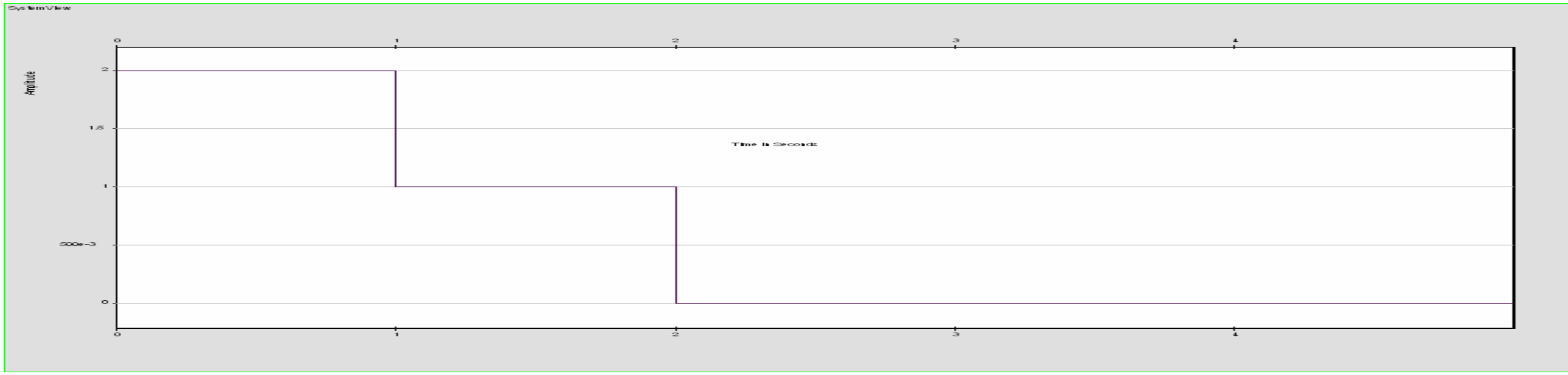
- Example 2.3 First periodic pulse



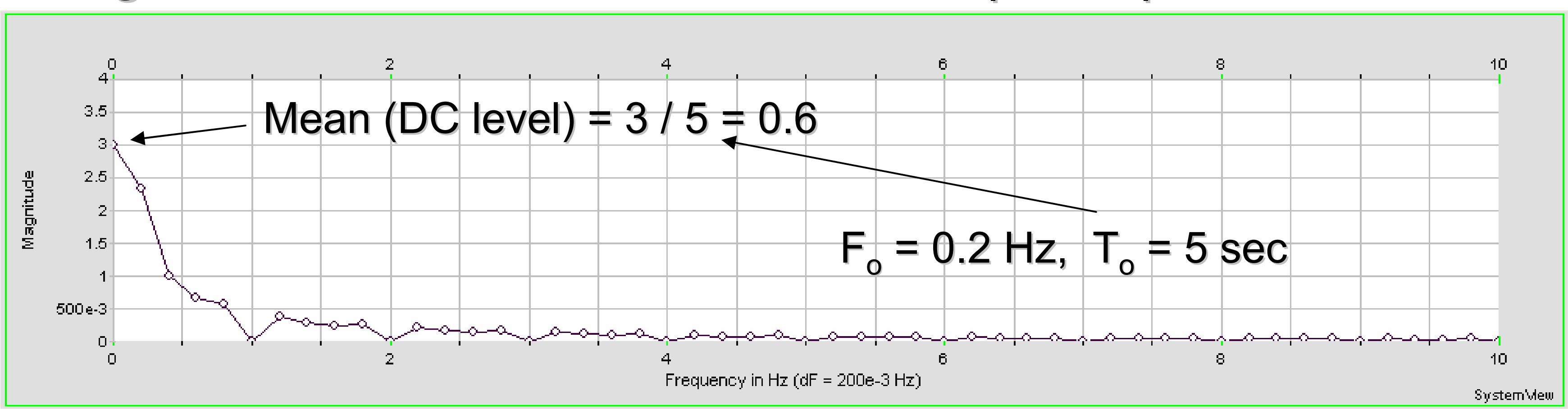
- Second periodic pulse



- Example 2.3 Sum of first and second periodic pulses

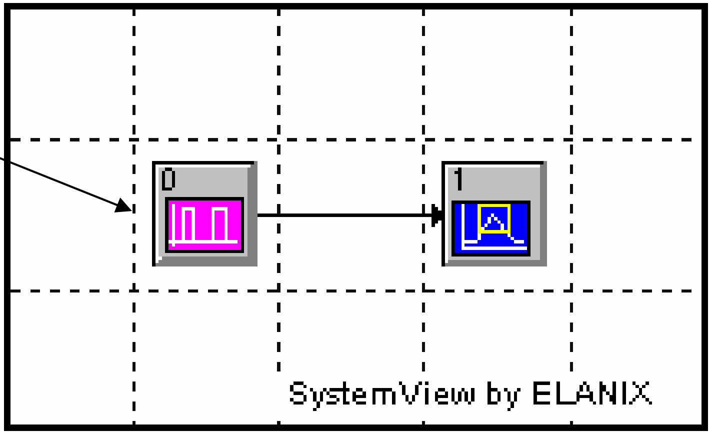


- Magnitude of the Fourier Transform | FFT |



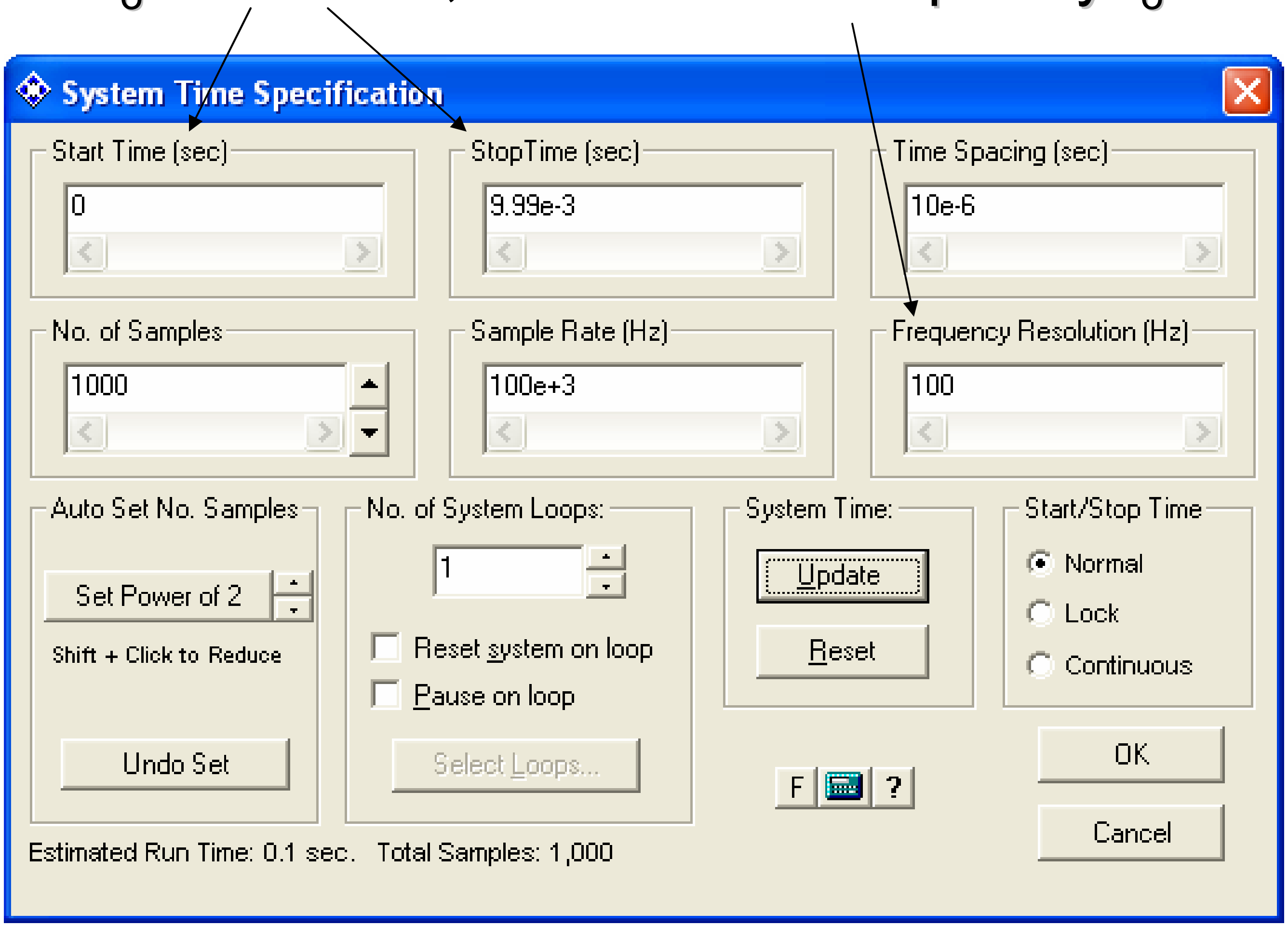
- Example 2.7

Rectangular pulse train

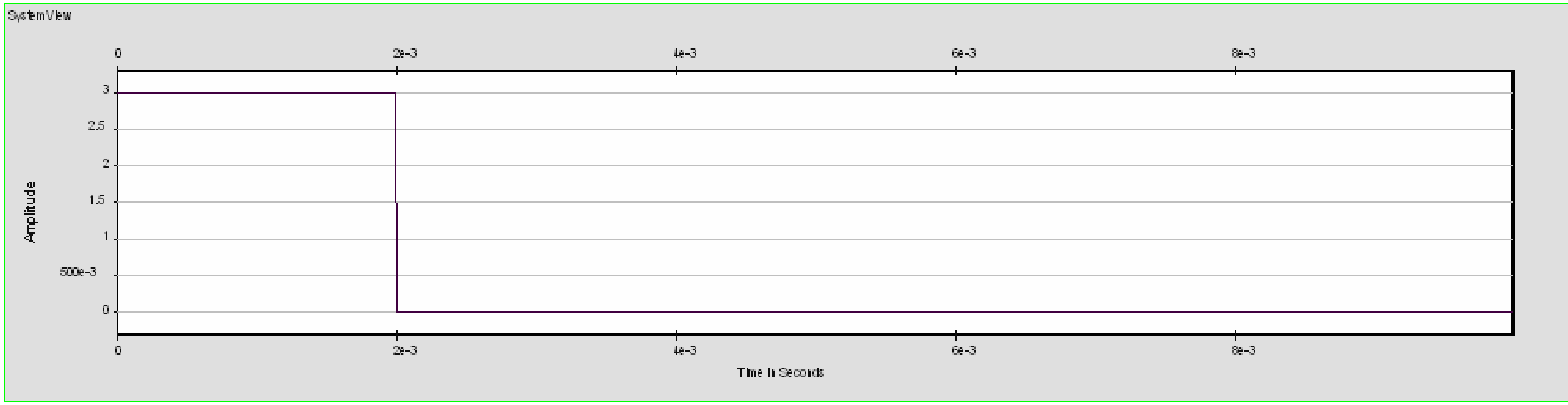


- Example 2.7 SystemVue System Time

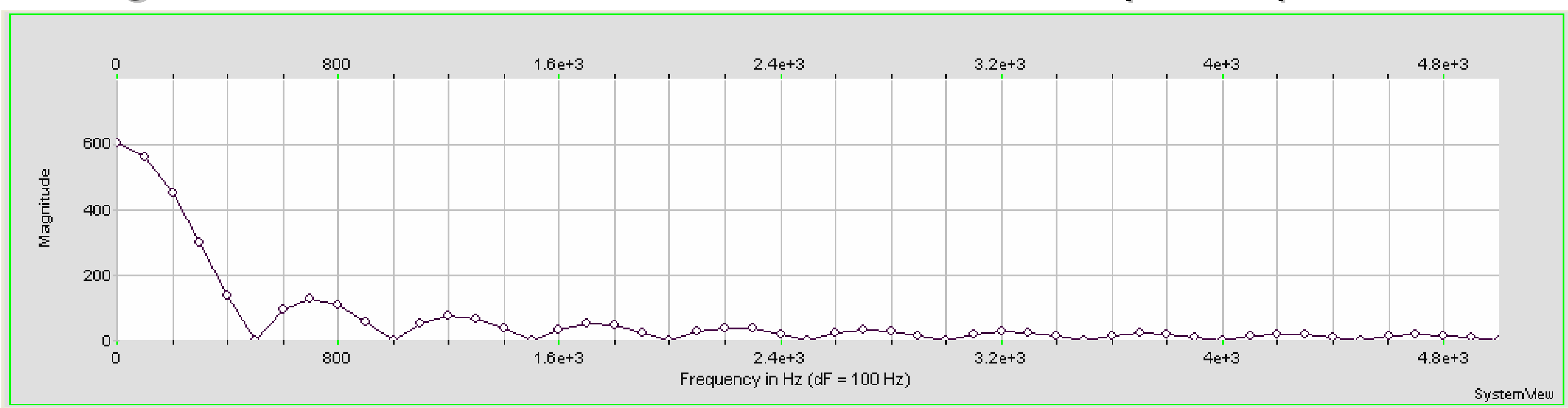
Period $T_o \approx 10$ msec, fundamental frequency $f_o = 100$ Hz



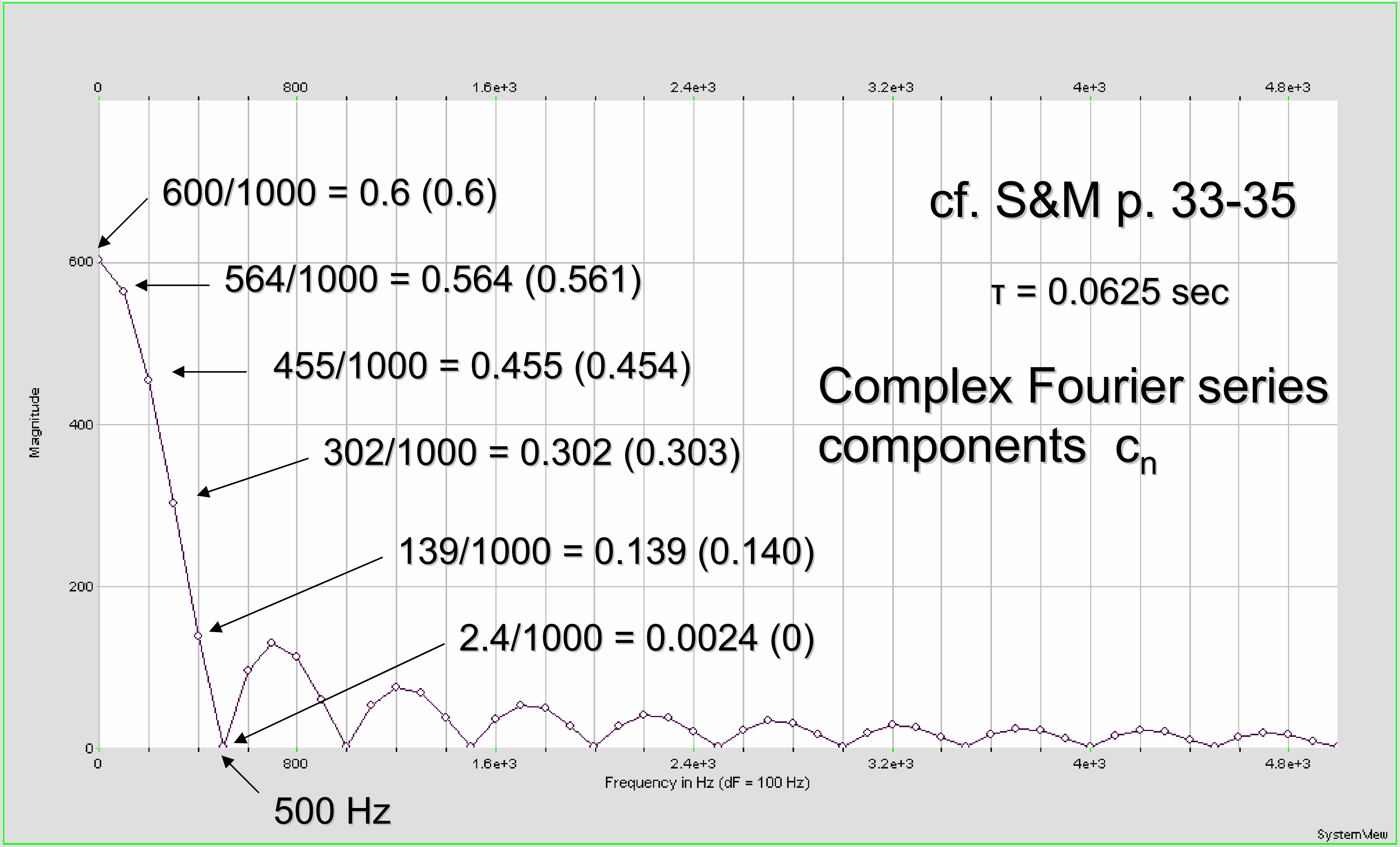
- Example 2.7 One cycle of periodic pulse



- Magnitude of the Fast Fourier Transform | FFT |



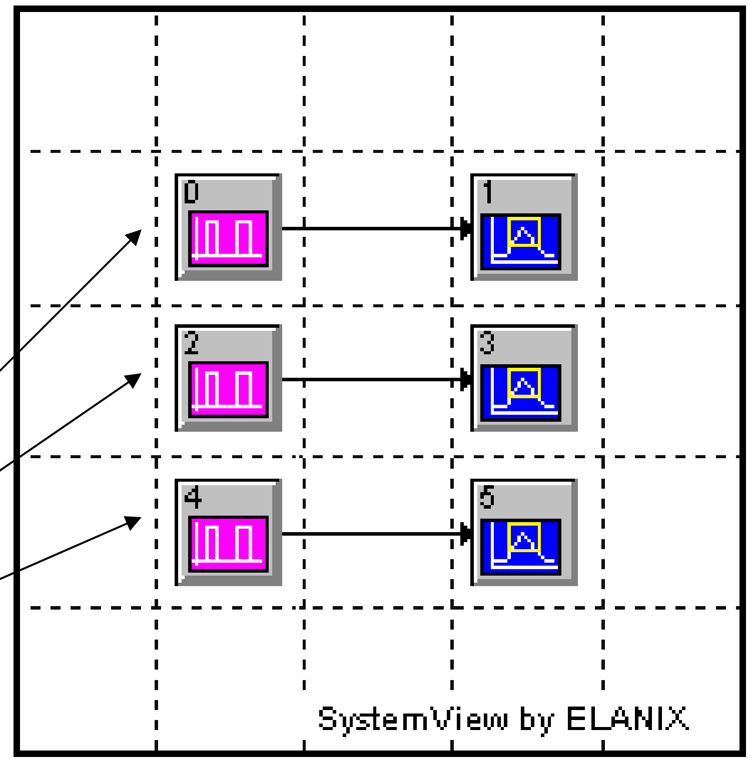
- Example 2.7 | FFT |



- Example 2.8

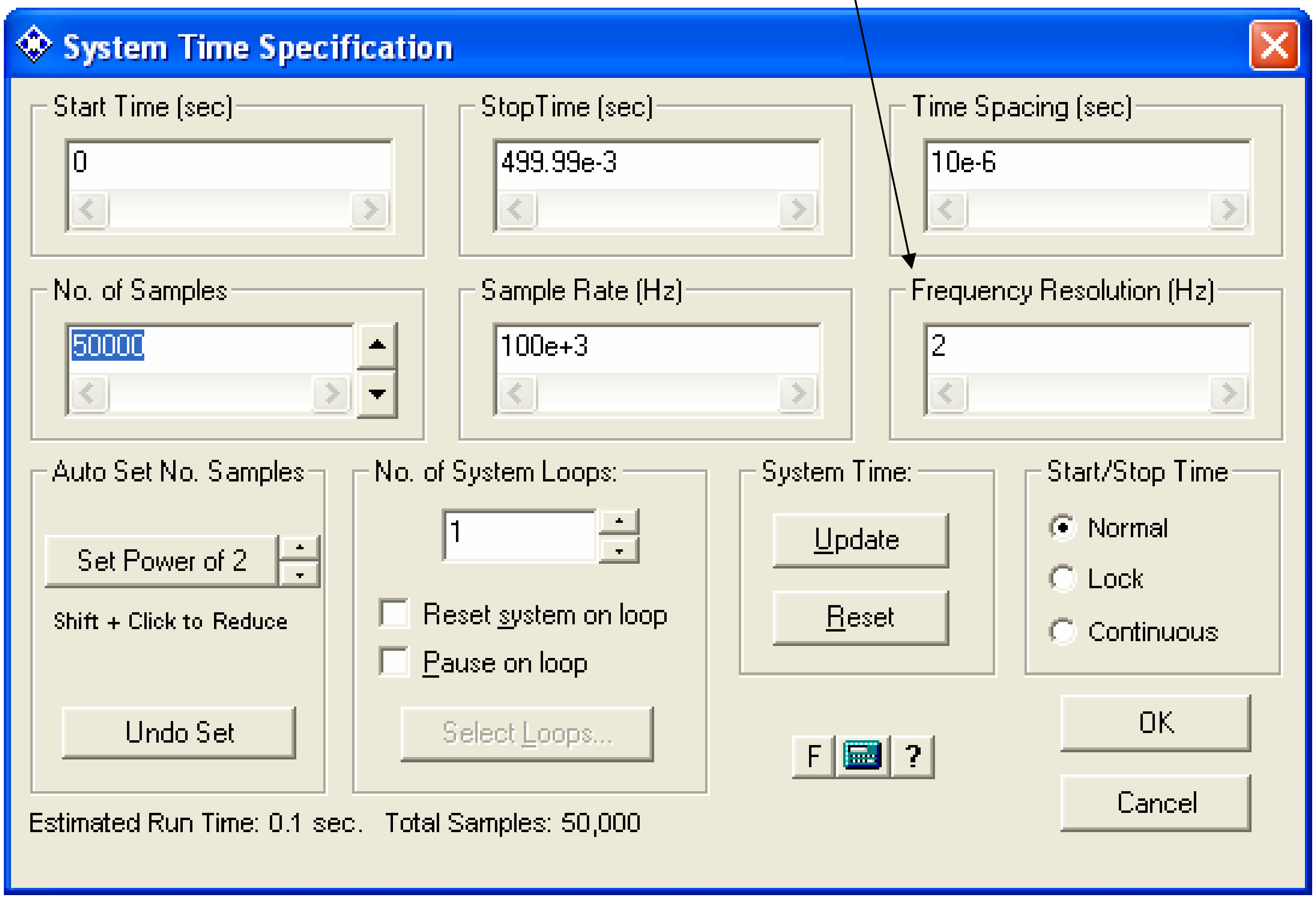
  Rectangular pulse trains with pulse period of 0.5 sec and pulse widths of

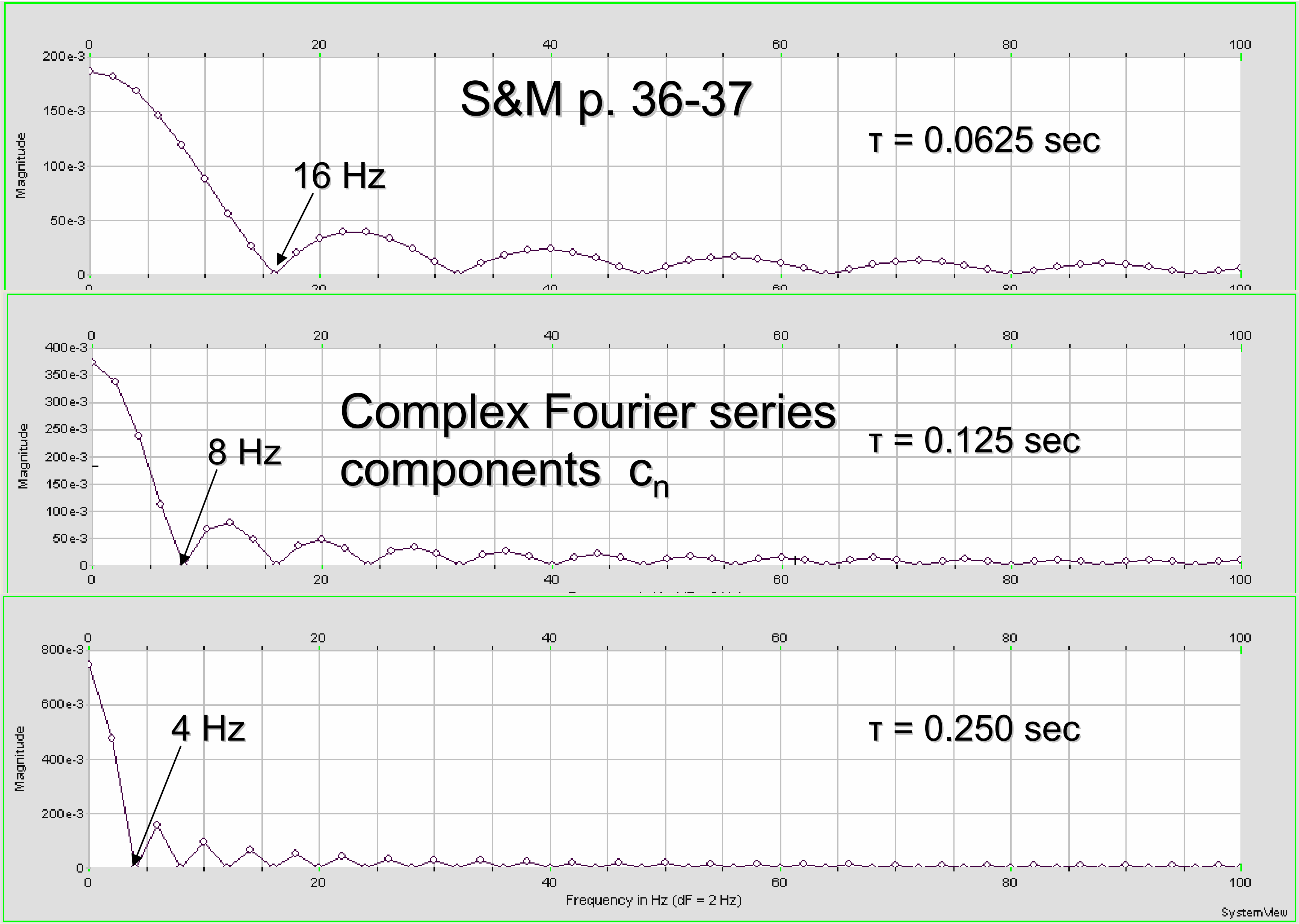
  0.0625 sec
  0.125 sec
  0.250 sec



- Example 2.8 SystemVue System Time

Fundamental frequency f_o= 2 Hz, T_o = 0.5 sec

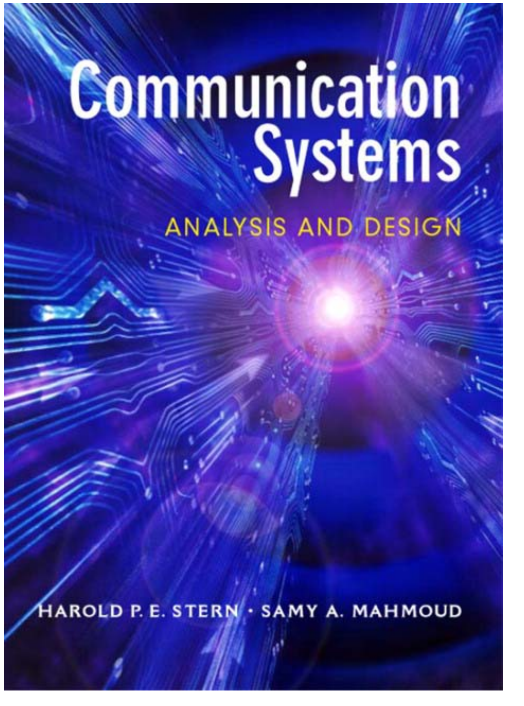



S&M p. 36-37
τ = 0.0625 sec
16 Hz
Complex Fourier series components c_n
τ = 0.125 sec
8 Hz
τ = 0.250 sec
4 Hz
Magnitude
Frequency in Hz (dF = 2 Hz)
SystemView

Chapter 2

Frequency Domain Analysis

- *Power in the Frequency Domain*
- Pages 38-52



- Average Normalized (R = 1Ω) Power

Periodic signal as a frequency domain representation

$$s(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi n f_o t + \varphi_n)$$

Average normalized power in the signal as a time domain or frequency domain representation

$$P_S = \frac{1}{T}\int_{t_o}^{t_o+T} s^2(t)\,dt = X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2}$$

Parseval's Theorem

- *Parseval's Theorem*

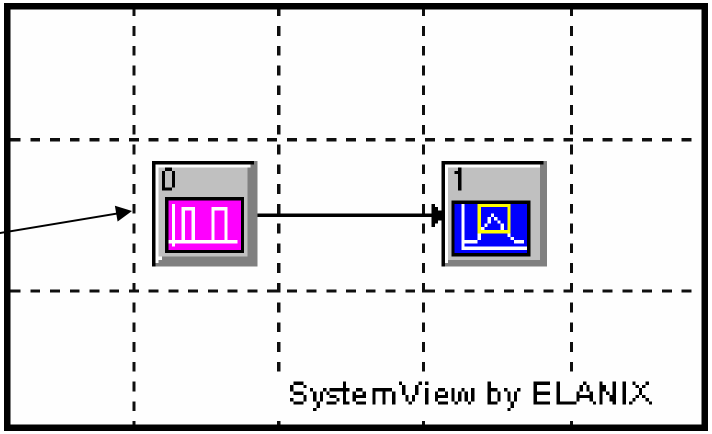
1755-1836

Marc-Antoine Parseval des Chênes was a French mathematician, most famous for what is now known as Parseval's Theorem, which presaged the equivalence of the Fourier Transform. A monarchist opposed to the French Revolution, Parseval fled the country after being imprisoned in 1792 by Napoleon for publishing tracts critical of the government.

$$P_S = \frac{1}{T}\int_{t_o}^{t_o+T} s^2(t)\,dt = X_0^2 + \sum_{n=1}^{\infty}\frac{X_n^2}{2}$$

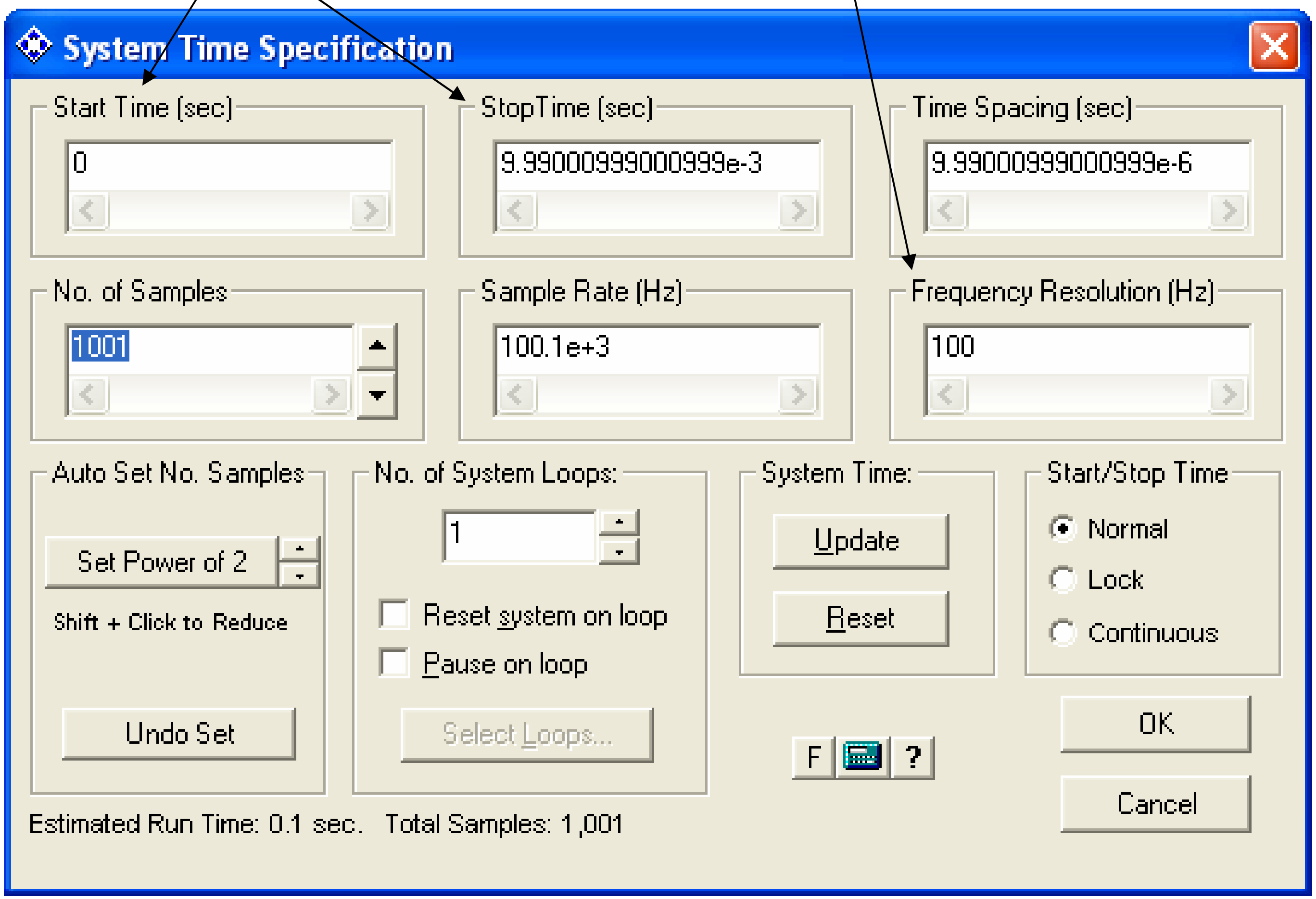
- Example 2.9

Normalized power spectrum of a periodic rectangular pulse train



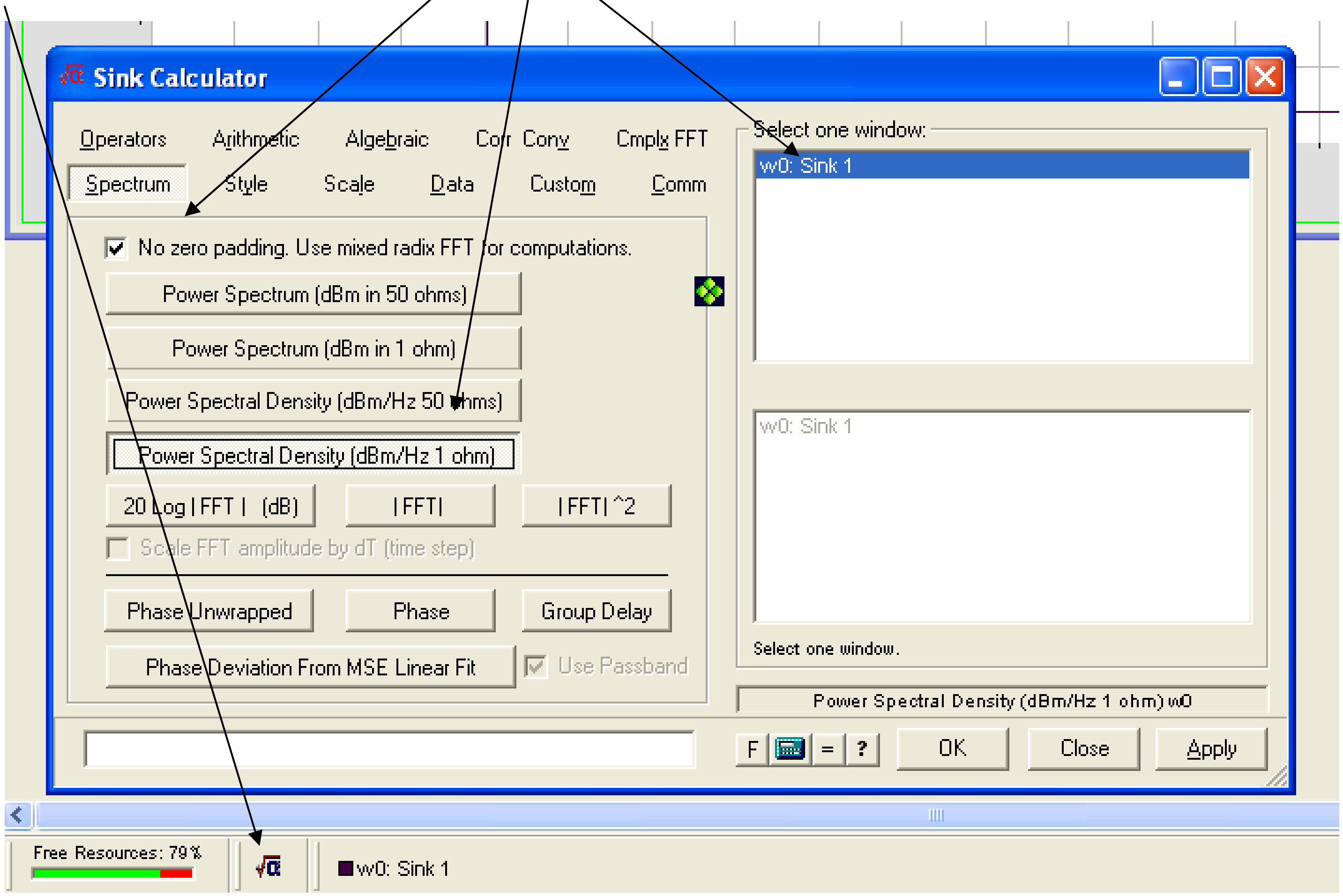
- Example 2.8 SystemVue System Time

Period $T_o \approx 10$ msec, frequency resolution = 100 Hz

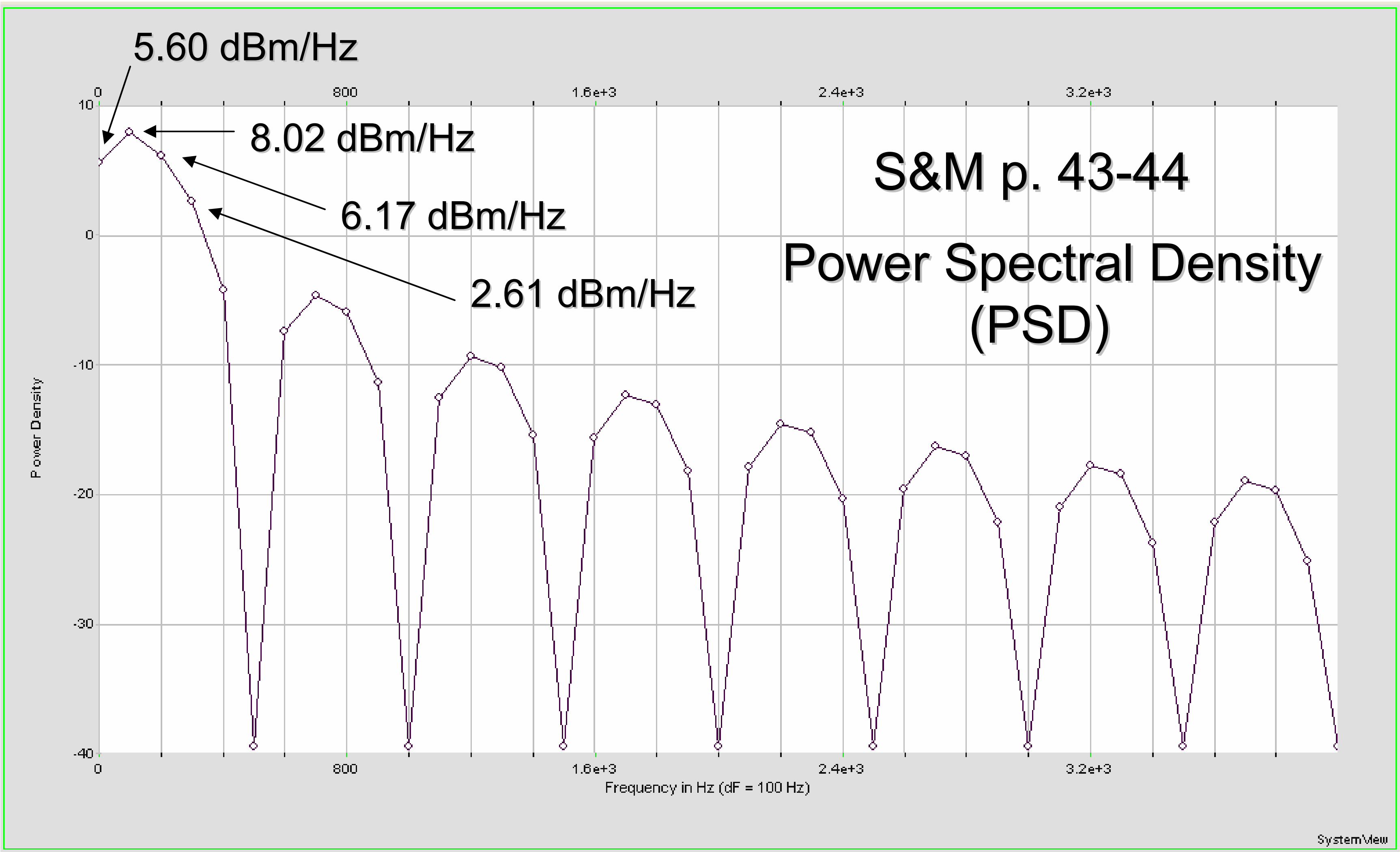


- Example 2.9 SystemVue Analysis Window

Sink calculator Power Spectral Density dBm/Hz 1Ω

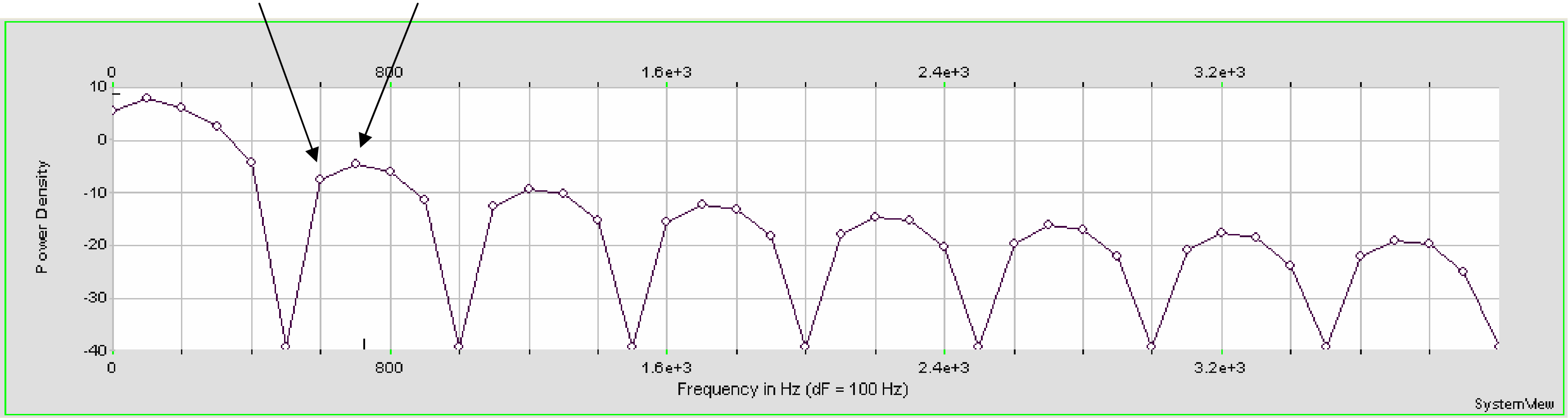


- Example 2.9 PSD dBm/Hz 1Ω



- Power Spectral Density dBm/Hz 1Ω

The power spectral density (PSD) for periodic signals is *discrete* because of the fundamental frequency $f_o = 1/T_o$ = 100 Hz here.



However, for aperiodic signals the PSD is conceptually *continuous*. Periodic signals contain *no information* and only aperiodic signals are, in fact, communicated.

- Power Spectral Density dBm/Hz 1Ω

dBm is decibel (dB) referenced to 1 normalized milliwatt (mW = 10^{-3} W, normalized V^2/R, R = 1Ω)

dBm = 10 log (Power/ 10^{-3} V^2) normalized R = 1Ω

$$5.6\ \text{dBm/Hz} \quad f_o = 100\ \text{Hz}$$

$$5.6 = 10 \log (\text{Power/Hz} / 10^{-3})$$

$$\text{Power/Hz} = 10^{5.6/10}(10^{-3}) = 3.63 \text{ x } 10^{-3}\ V^2/\text{Hz}$$

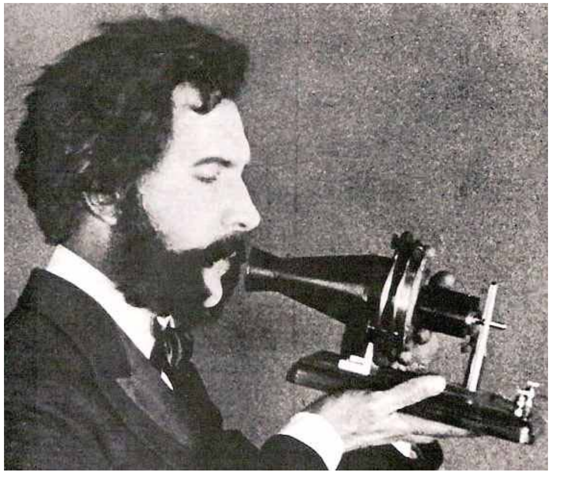
$$\text{Power} = (\text{Power/Hz})(f_o\ \text{Hz})$$

$$\text{Power} = (3.63 \text{ x } 10^{-3}\ V^2/\text{Hz})(100\ \text{Hz})$$

$$\text{Power} = 0.36\ V^2 \quad (0.36\ V^2, \text{S \& M p. 43})$$

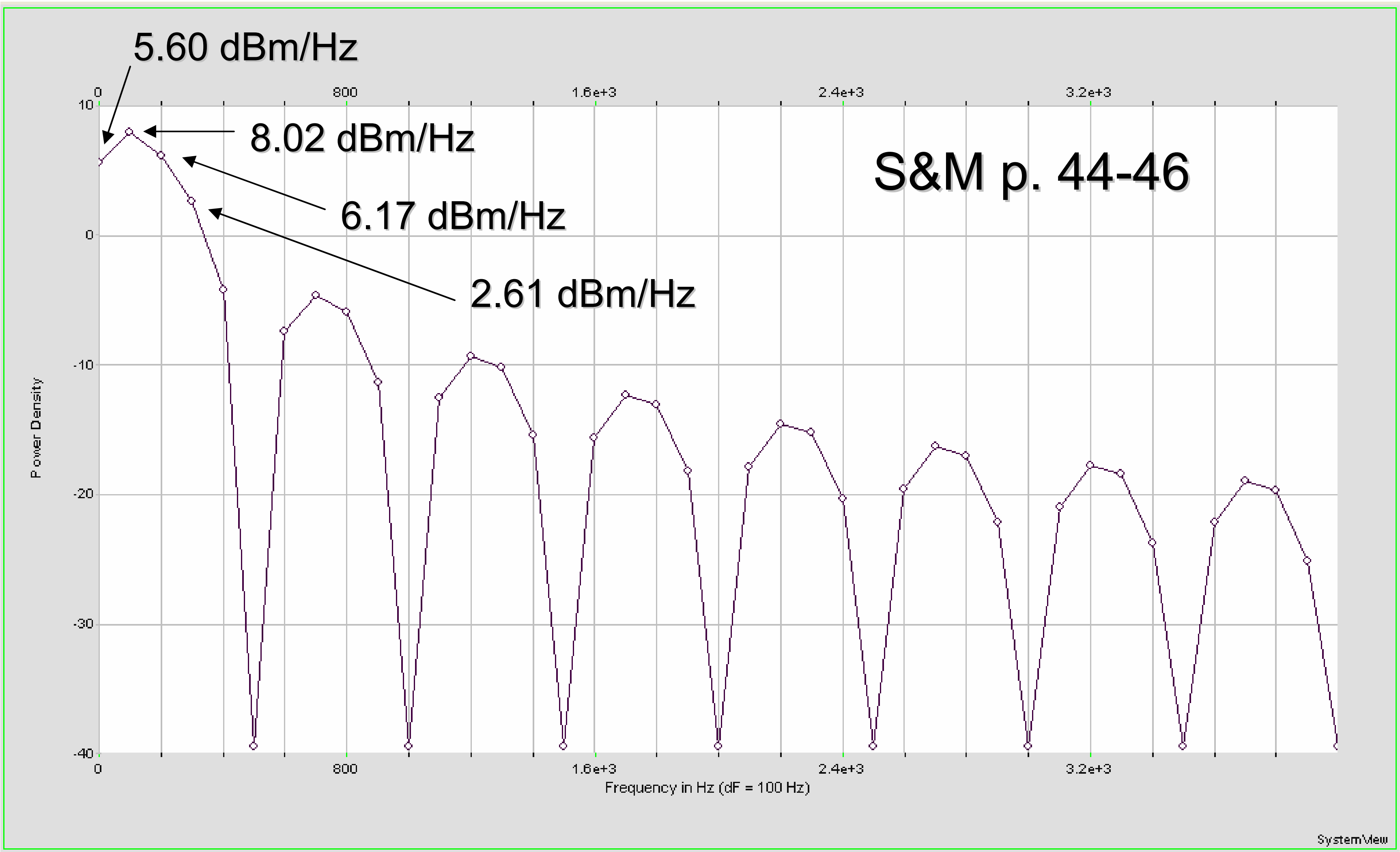
- de·ci·bel (dĕs'ə-bəl, -bĕl')
  n. (*Abbr.* dB)
  A unit used to express relative difference in power or intensity, usually between two acoustic or electric signals, equal to ten times the common logarithm of the ratio of the two levels.

  The *bel* (B) as a unit of measurement was originally proposed in 1929 by W. H. Martin of Bell Labs. The bel was too large for everyday use, so the decibel (dB), equal to 0.1 B, became more commonly used.

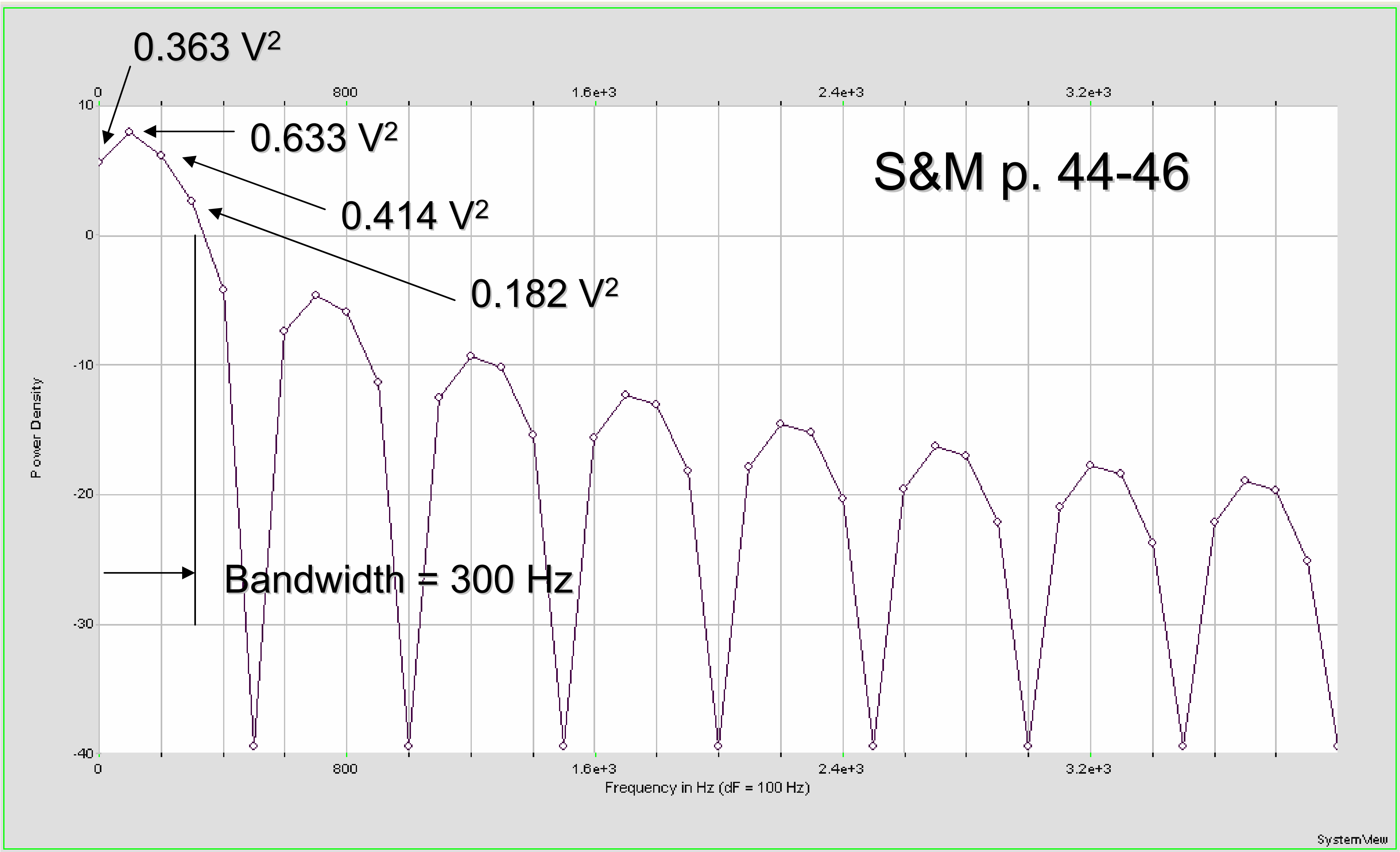


Alexander Graham Bell
1847-1922

- Example 2.10 PSD dBm/Hz 1Ω



• Example 2.10 PSD dBm/Hz 1Ω Converted to V^2



- Bandwidth

The *bandwidth* of a signal is the width of the frequency band in Hertz that contains a sufficient number of the signal's frequency components to reproduce the signal with an acceptable amount of distortion.

Bandwidth is a nebulous term and communication engineers must always define what if meant by "bandwidth" in the context of use.

- Total Power in the Signal

Parseval's Theorem allows us to determine the *total normalized power* in the signal without the infinite sum of Fourier series components by integrating in the temporal domain:

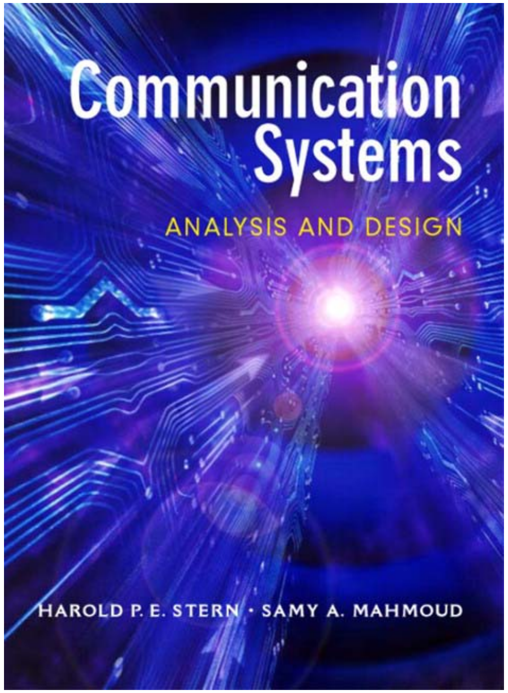
$$P_S = \frac{1}{T}\int_{t_o}^{t_o+T} s^2(t)\,dt = X_0^2 + \sum_{n=1}^{\infty}\frac{X_n^2}{2}$$

The total power in the signal then is 1.8 V^2 and the percentage of the total power in the signal in a bandwidth of 300 Hz then is approximately 88% (S&M p. 45) .

Chapter 2

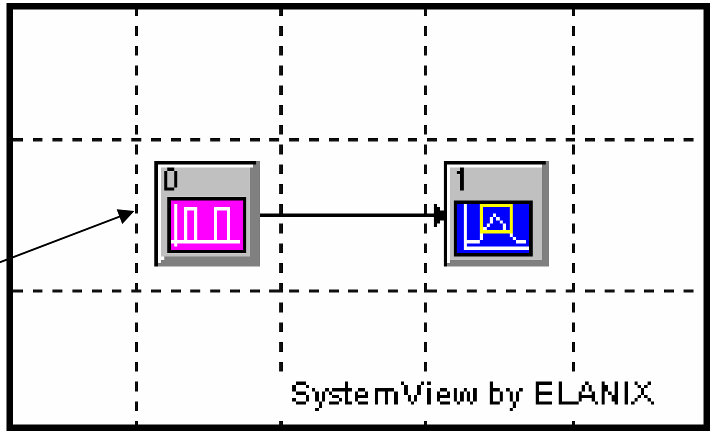
Frequency Domain Analysis

- *The Fourier Transform*
- Pages 52-69



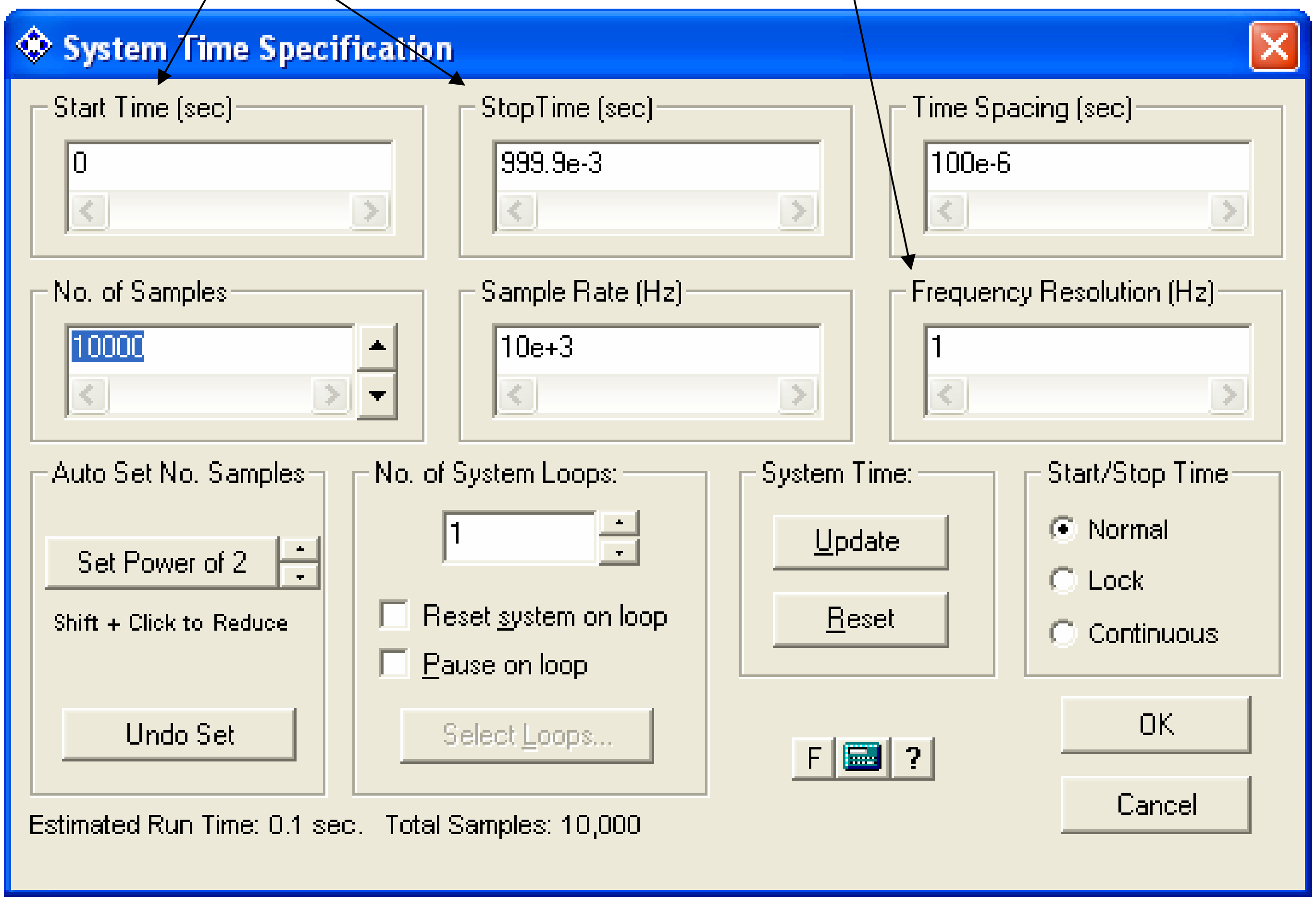
- Example 2.12

Spectrum of a simulated single pulse from a very low duty cycle rectangular pulse train

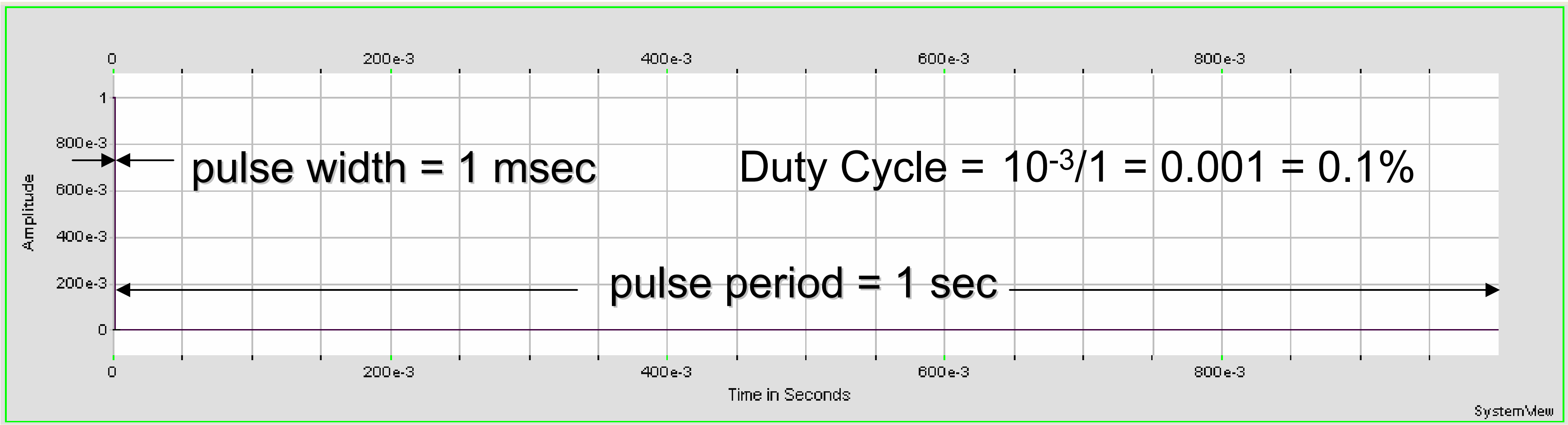


- Example 2.7 SystemVue System Time

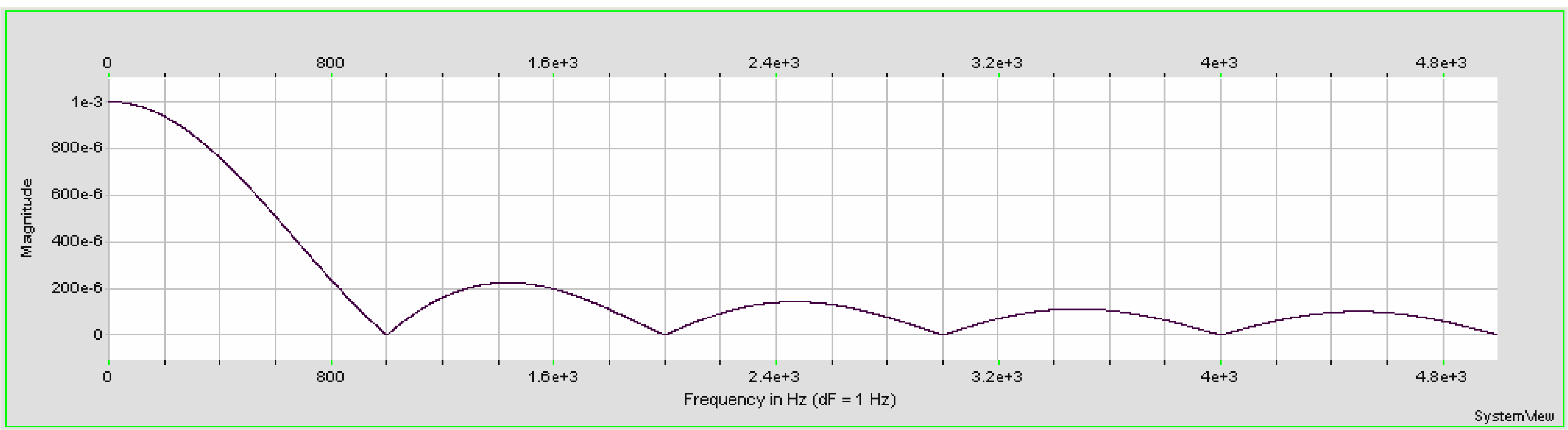
Period $T_o \approx 1$ sec, fundamental frequency $f_o = 1$ Hz



- Example 2.12 Simulated single pulse

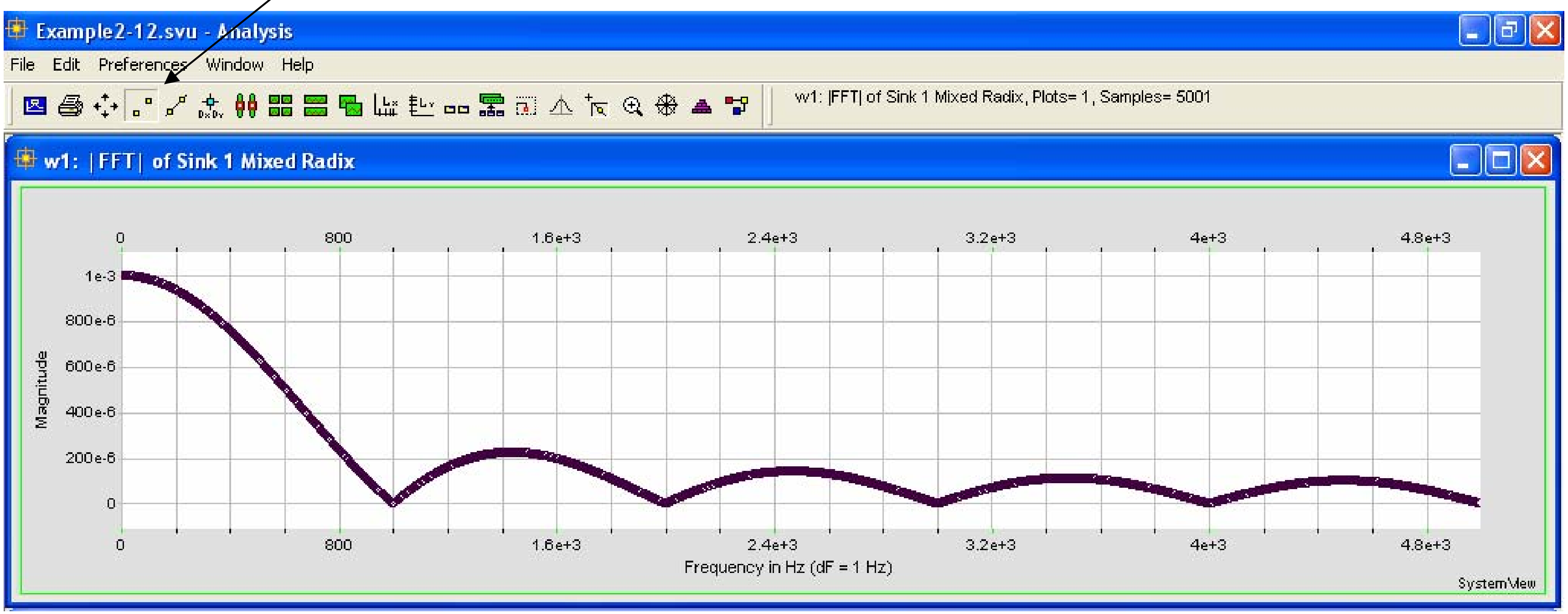


- Magnitude of the Fourier Transform | FFT |

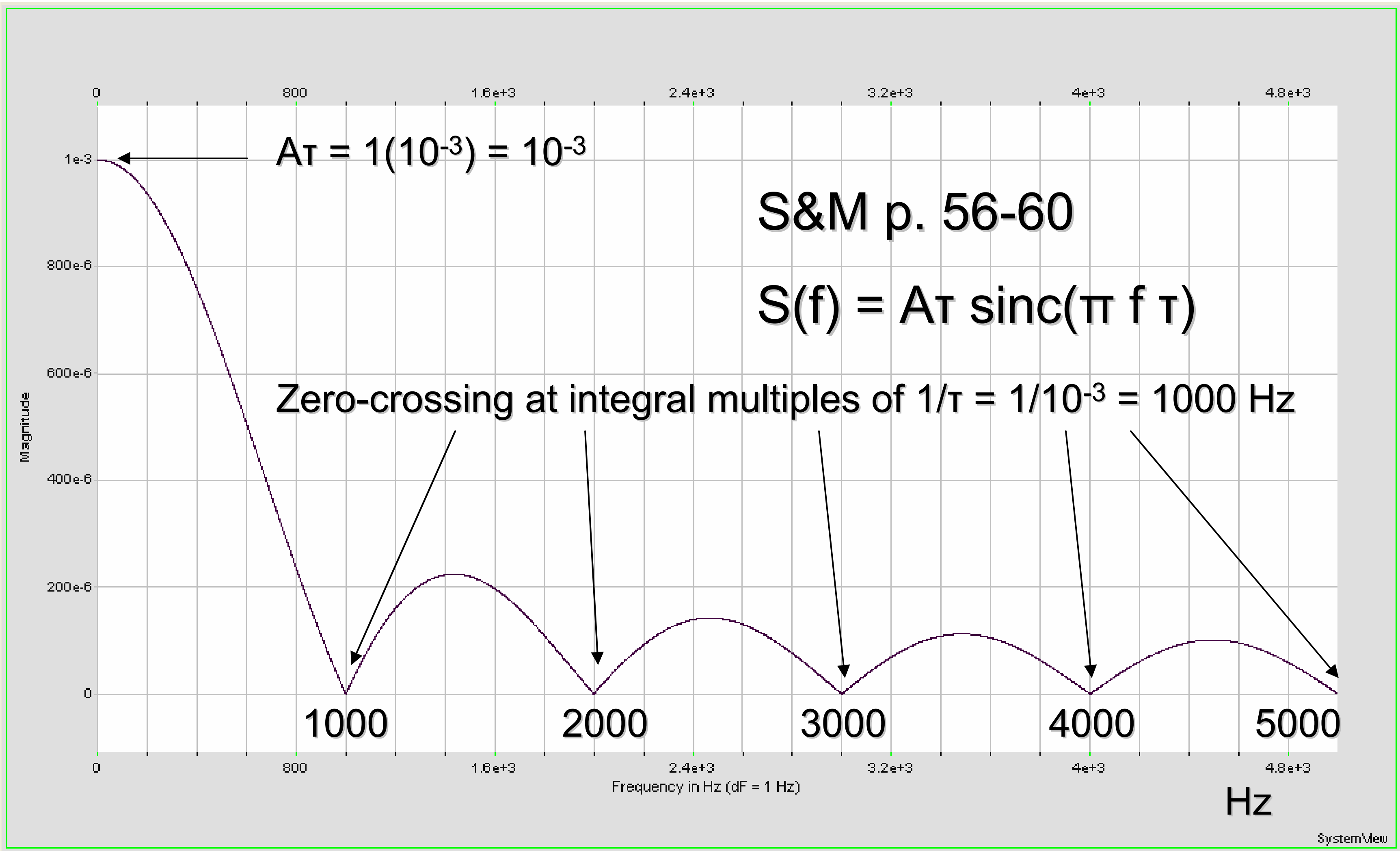


- Example 2.12 Simulated single pulse

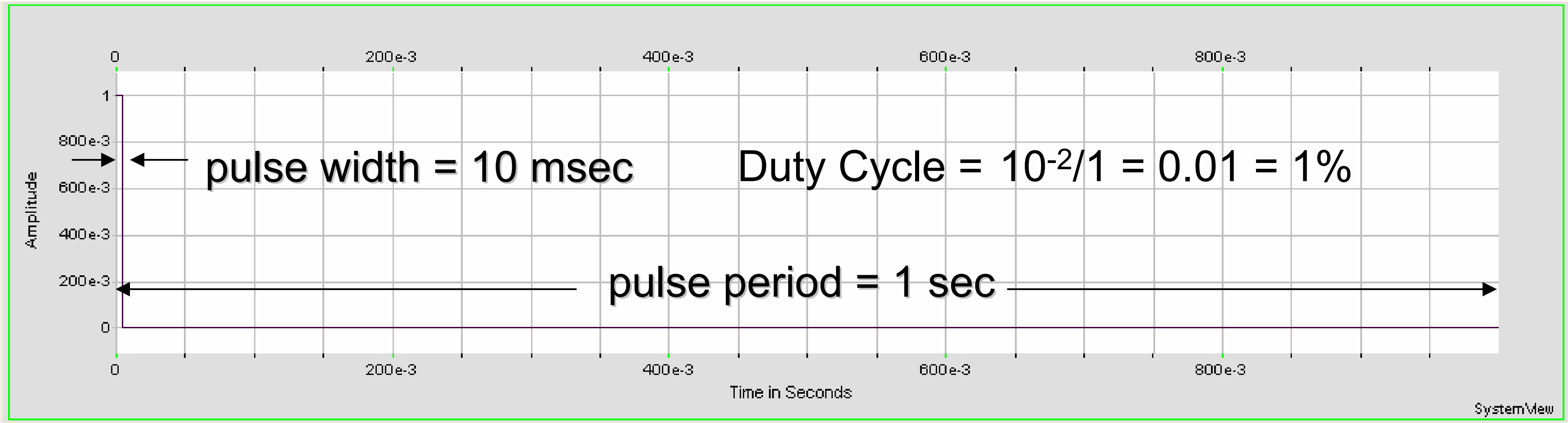
The magnitude of the Fourier Transform of a single pulse is *continuous* and *not discrete* since there is no Fourier series representation. In the *SystemVue* simulation the data points are very dense and virtually display a continuous plot.



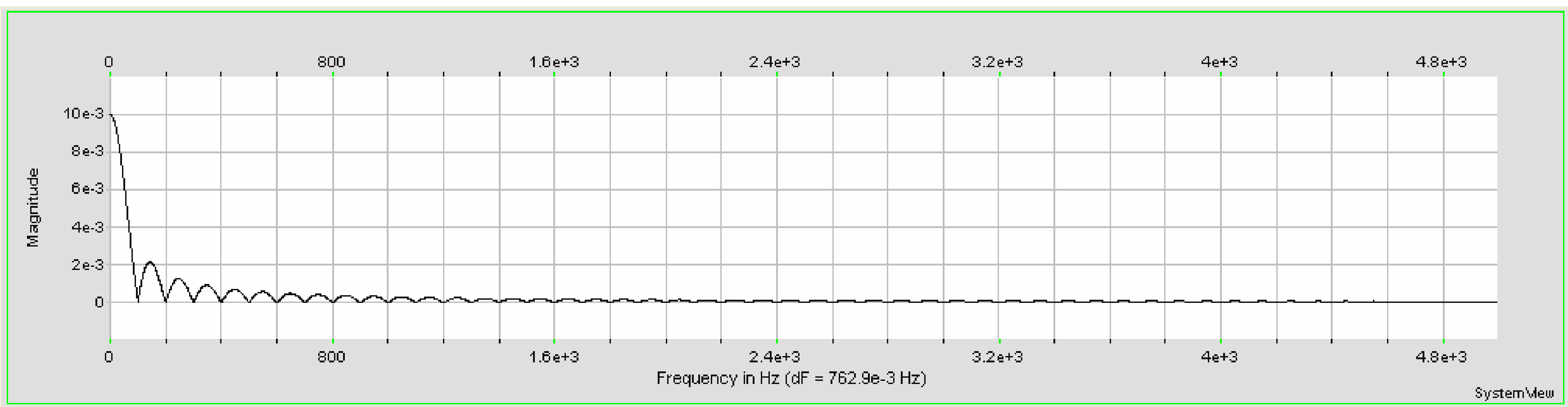
- Example 2.12 Simulated single pulse | FFT |



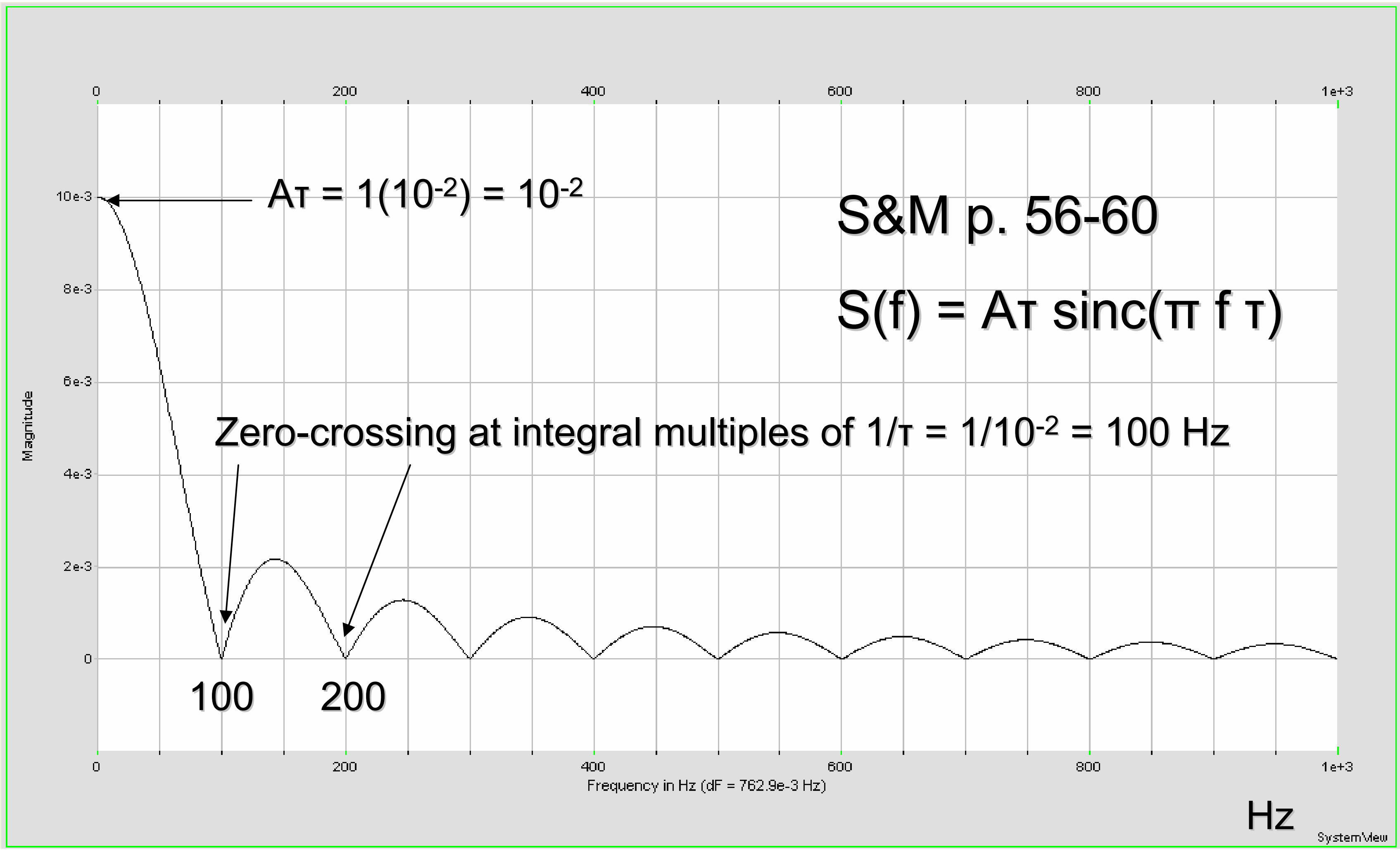
- Example 2.12a Simulated single pulse



- Magnitude of the Fourier Transform | FFT |



- Example 2.12a Simulated single pulse | FFT |



- Properties of the Fourier Transform

Linearity

$$a f_1(t) + b f_2(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad a F_1(\omega) + b F_2(\omega)$$

Convolution

$$f_1(t) * f_2(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad F_1(\omega) F_2(\omega)$$

Conjugation

$$\overline{f(t)} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \overline{F(-\omega)}$$

Scaling

$$f(at) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{|a|} F\left(\frac{\omega}{a}\right), \qquad a \in \mathbb{R}, a \neq 0$$

Time reversal

$$f(-t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad F(-\omega)$$

Time shift

$$f(t - t_0) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad e^{-i\omega t_0} F(\omega)$$

- Properties of the Fourier Transform

  Modulation principle

Modulation (multiplication by complex exponential)

$$f(t) \cdot e^{i\omega_0 t} \overset{\mathcal{F}}{\Longleftrightarrow} F(\omega - \omega_0) \qquad \omega_0 \in \mathbb{R},$$

Multiplication by sin ω_0t

$$f(t)\sin\omega_0 t \overset{\mathcal{F}}{\Longleftrightarrow} \frac{i}{2}[F(\omega+\omega_0) - F(\omega-\omega_0)]$$

Multiplication by cos ω_0t

$$f(t)\cos\omega_0 t \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{2}[F(\omega+\omega_0) + F(\omega-\omega_0)]$$

Integration

$$\int_{-\infty}^{t} f(u)\,du \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{i\omega}F(\omega) + \pi F(0)\delta(\omega)$$

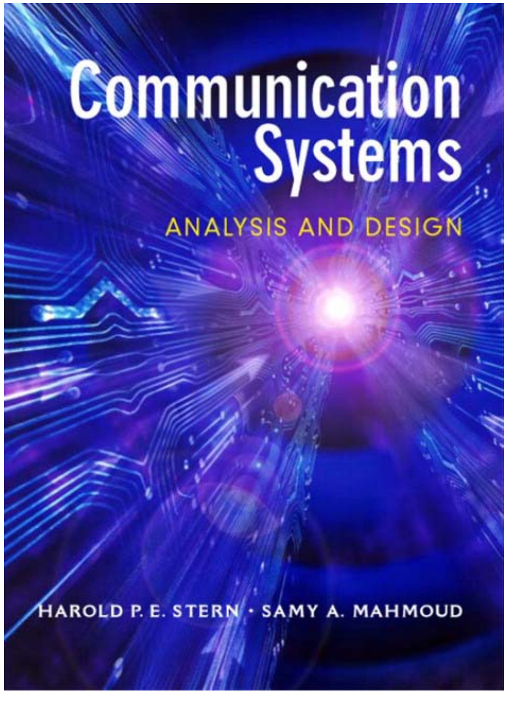
Parseval's theorem

$$\int_{\mathbb{R}} f(t) \cdot \overline{g(t)}\,dt = \int_{\mathbb{R}} F(\omega) \cdot \overline{G(\omega)}\,d\omega$$

Chapter 2

Frequency Domain Analysis

- *Normalized Energy Spectral Density*
- Pages 60-65



• Normalized Energy

If s(t) is a *non-periodic, finite energy signal* (a single pulse) then the average normalized power P_S is 0:

$$P_S = \lim_{T \to \infty} \frac{1}{T} \int_{t_o}^{t_o + T} s^2(t)\, dt = \lim_{T \to \infty} \frac{\text{finite value}}{T} = 0 \;\; V^2$$

$$E_S = \int_{-\infty}^{\infty} s^2(t)\, dt \quad V^2\text{-sec}$$

However, the normalized energy E_S for the same s(t) is *non-zero by definition* (S&M p. 60-61).

- Parseval's Energy Theorem

Parseval's energy theorem follows directly then from the discussion of power in a periodic signal:

$$E_S = \int_{-\infty}^{\infty} s^2(t)\,dt = \int_{-\infty}^{\infty} |S^2(f)|\,df \quad V^2\text{-sec}$$

- Energy Spectral Density

Analogous to the power spectral density is the energy spectral density (ESD) ψ(f). For a linear, time-invariant (LTI) system with a transfer function H(f), the output ESD which is the energy flow through the system is:

$$\psi_{OUT}(f) = \psi_{IN}(f)\,|\,H(f)\,|^2$$

- Energy Spectral Density

The energy spectral density (ESD) $\psi(f)$ is the magnitude squared of the Fourier transform $S(f)$ of a pulse signal $s(t)$:

$$\psi(f) = | S(f) |^2$$

The ESD can be *approximated* by the magnitude squared of the Fast Fourier Transform (FFT) in a *SystemVue* simulation as described in Chapter 3.

$$\psi(f) \approx | FFT |^2$$

End of Chapter 2

Frequency Domain Analysis