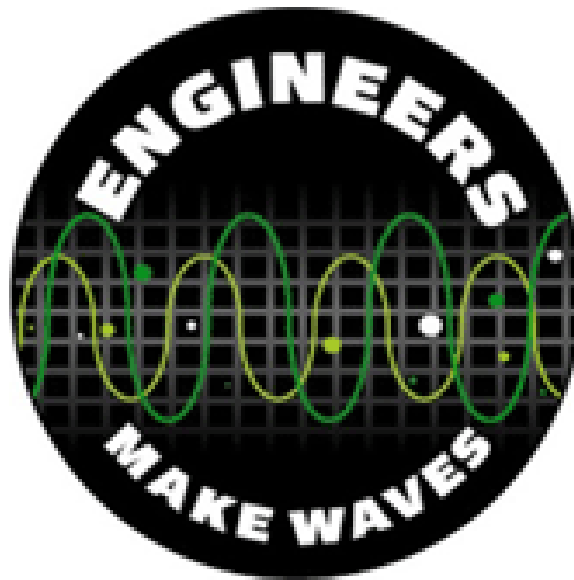


# Chapter 2

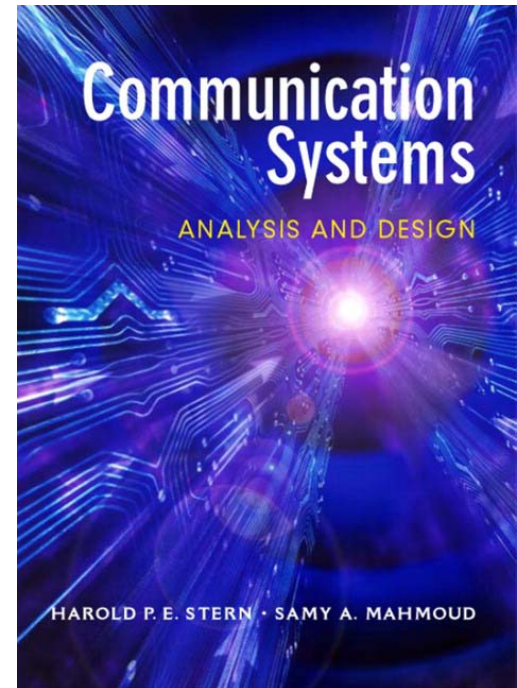
## Frequency Domain Analysis



## Chapter 2

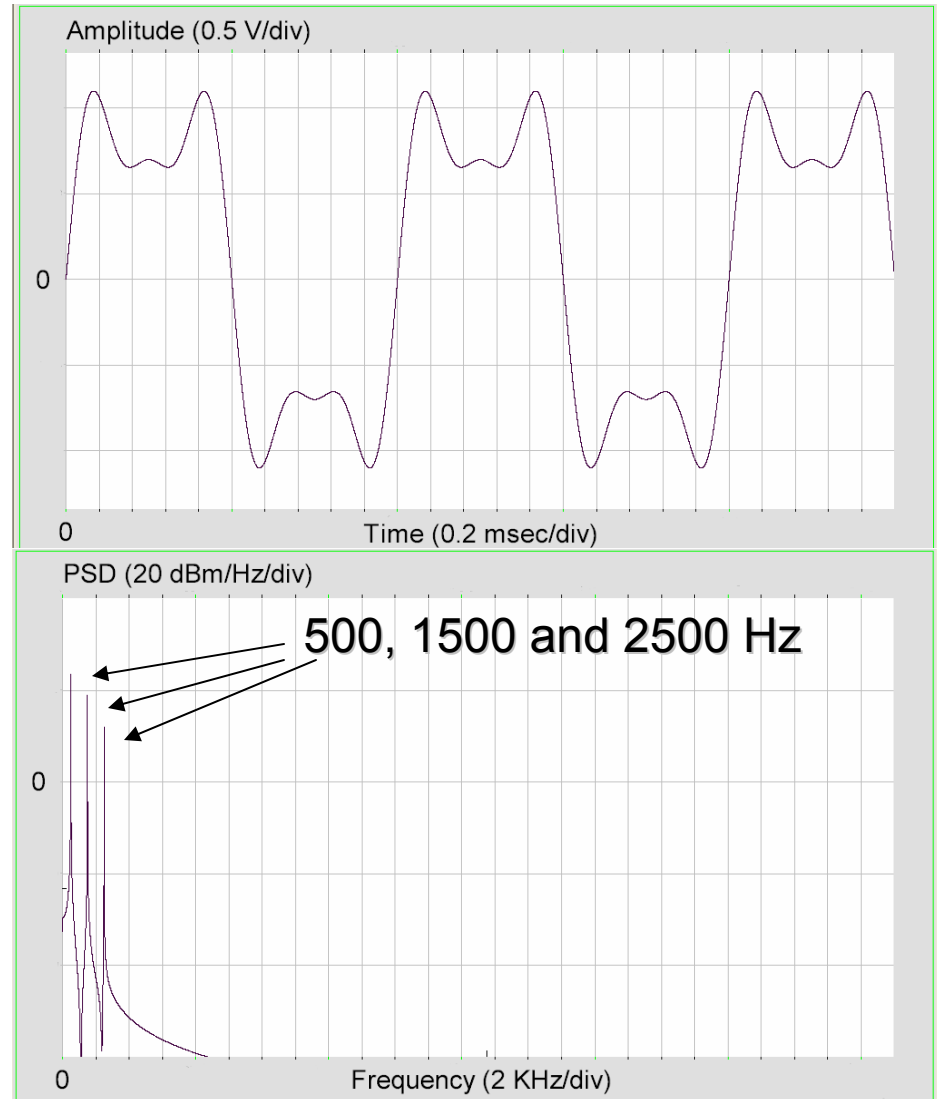
# Frequency Domain Analysis

- *Why Study Frequency Domain Analysis?*
- Pages 6-13



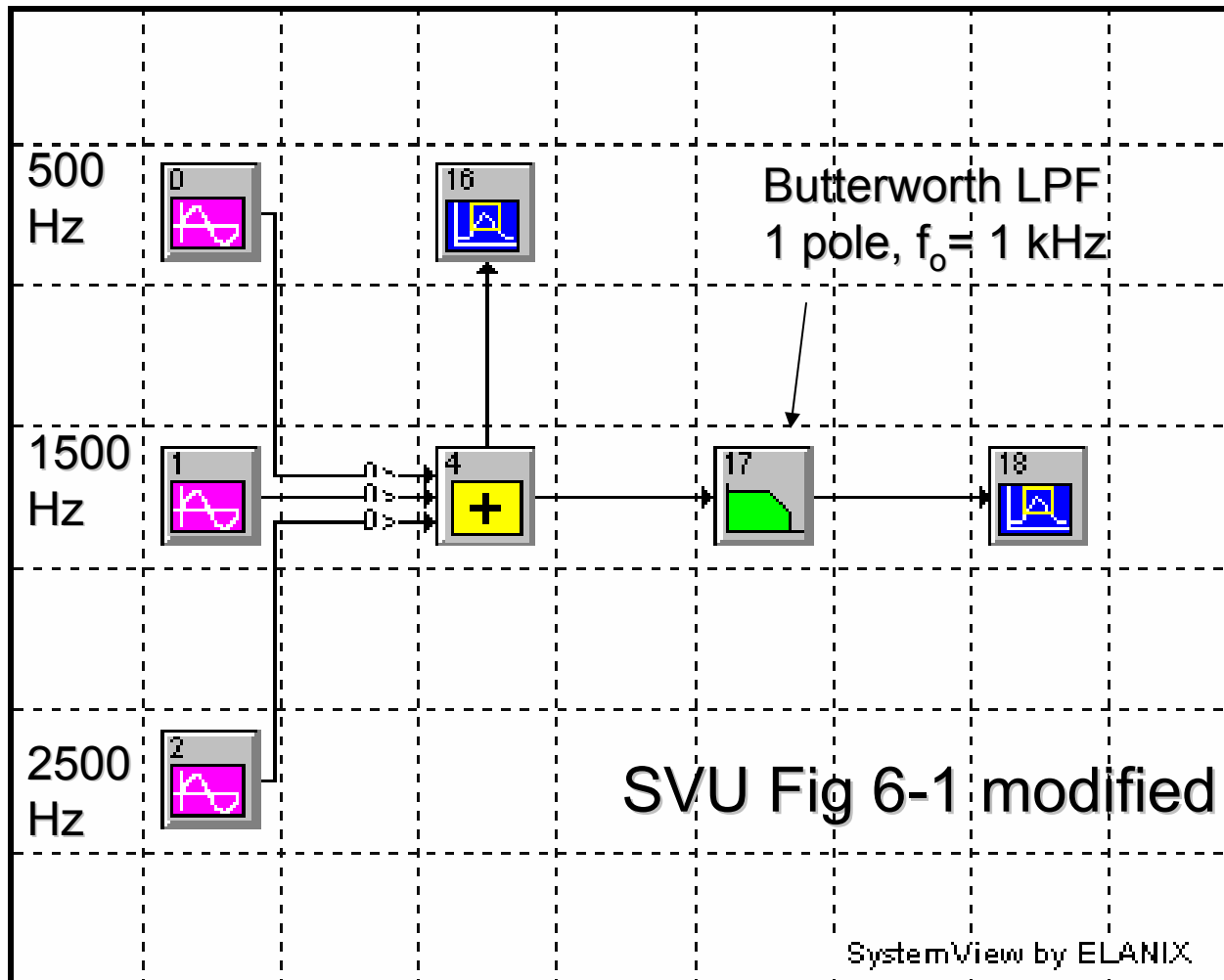
## *Why frequency domain analysis?*

- Allows simple algebra rather than time-domain differential equations to be used
- Transfer functions can be applied to transmitter, communication channel and receiver
- Channel bandwidth, noise and power are easier to evaluate

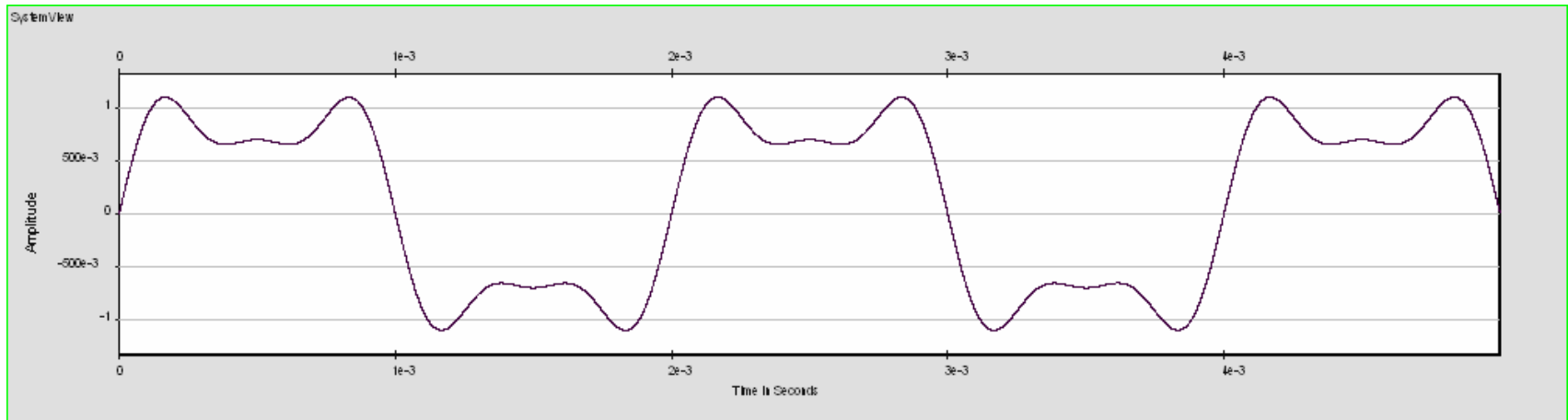


SVU Figure 6.2 and Figure 6.3

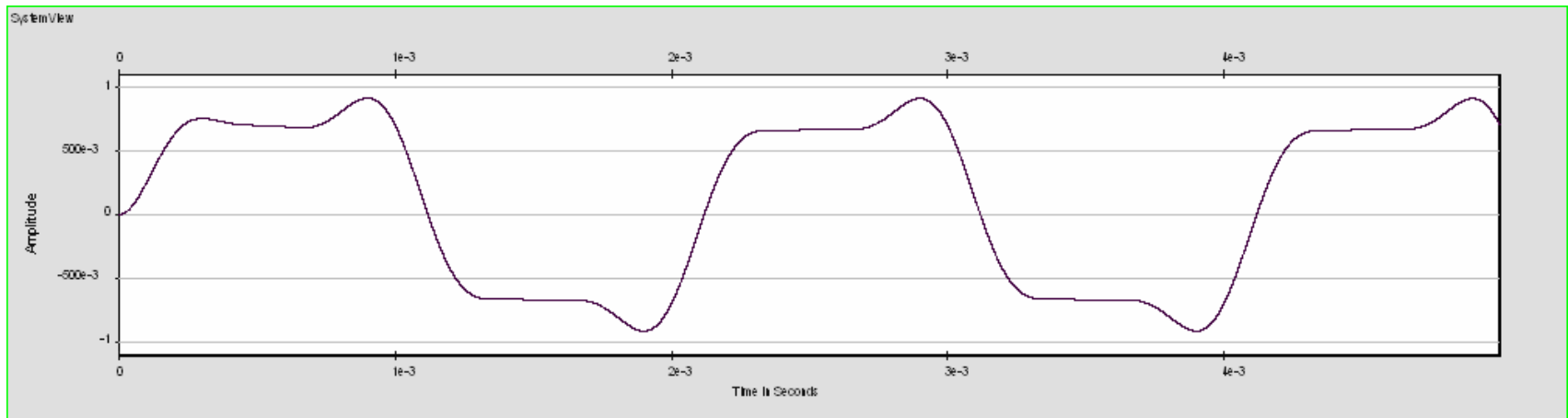
- Example 2.1 Input sum of three sinusoids



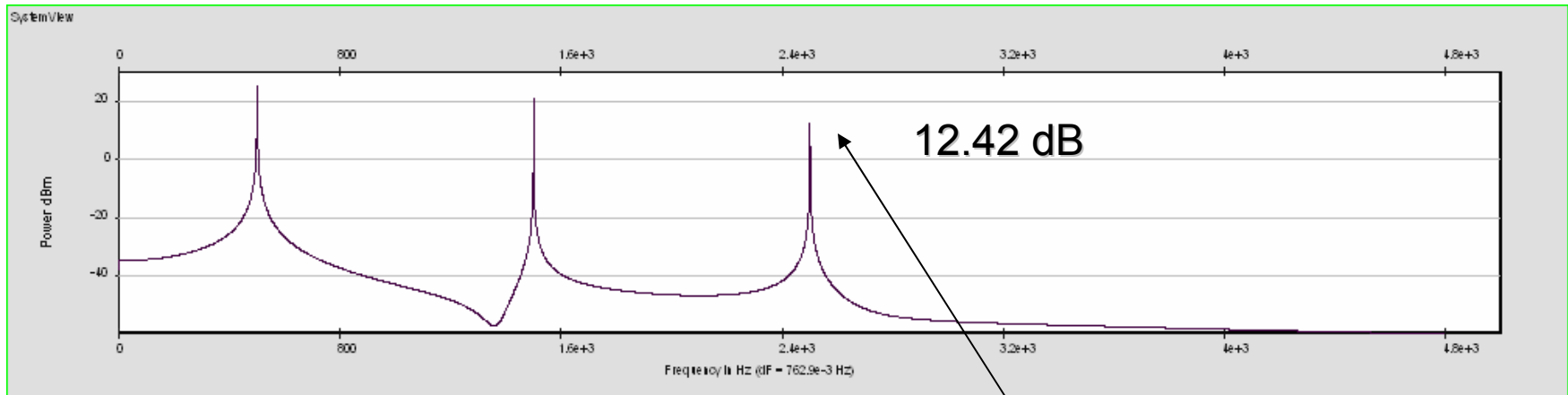
- Example 2.1 Input sum of three sinusoids



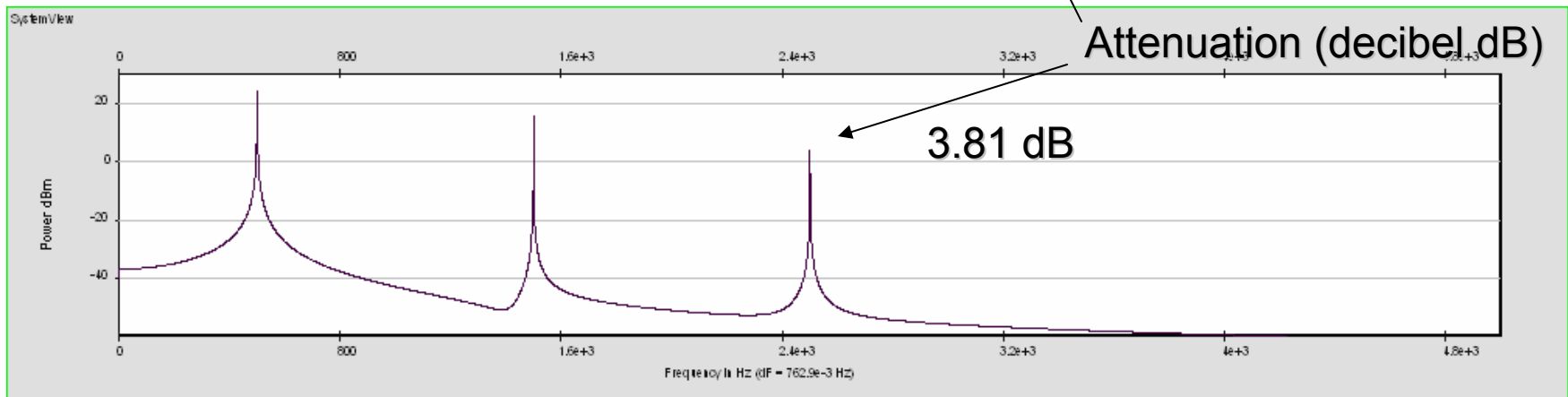
- Output after Butterworth LPF

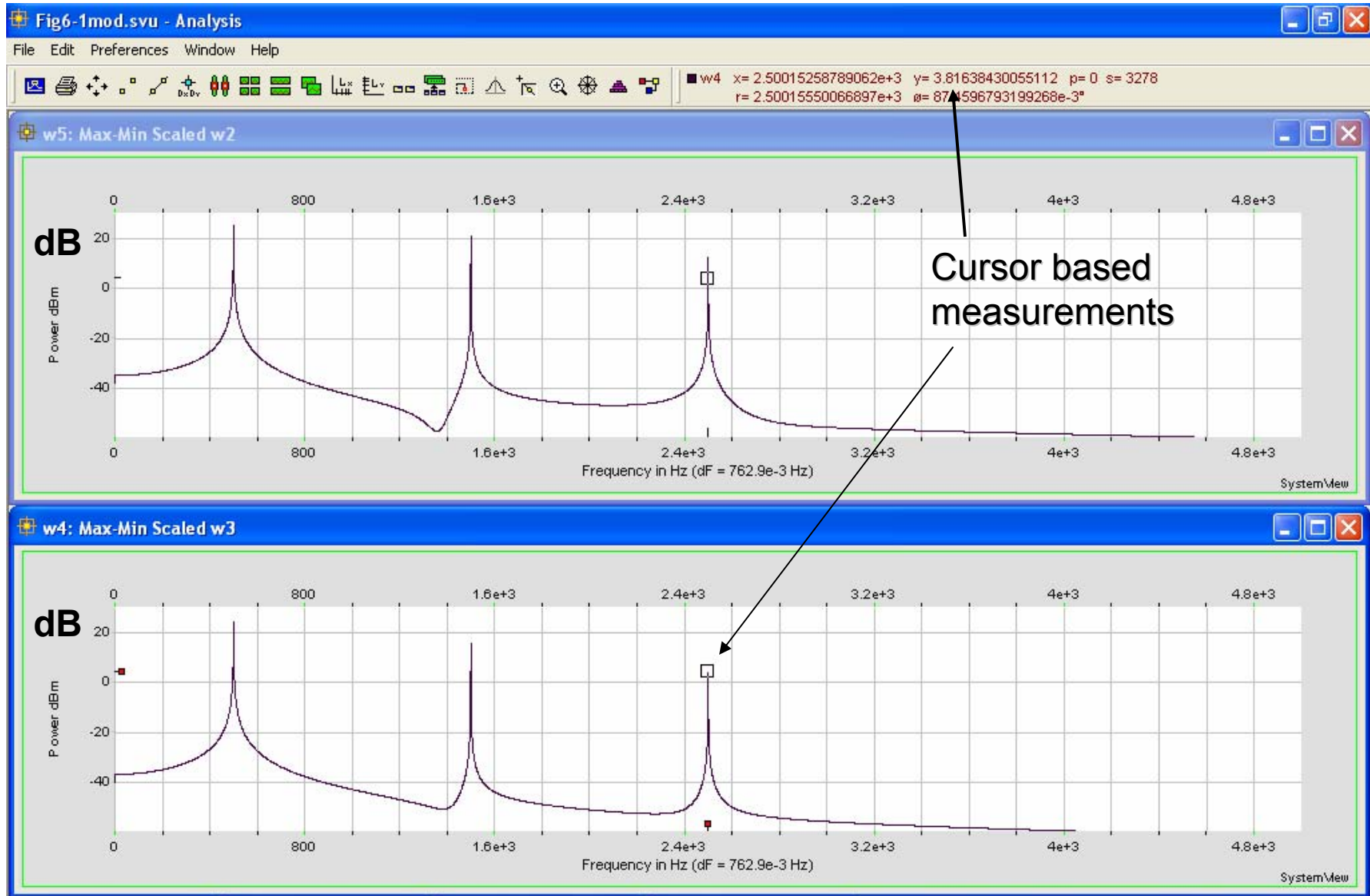


- Input power spectral density of the sum of three sinusoids



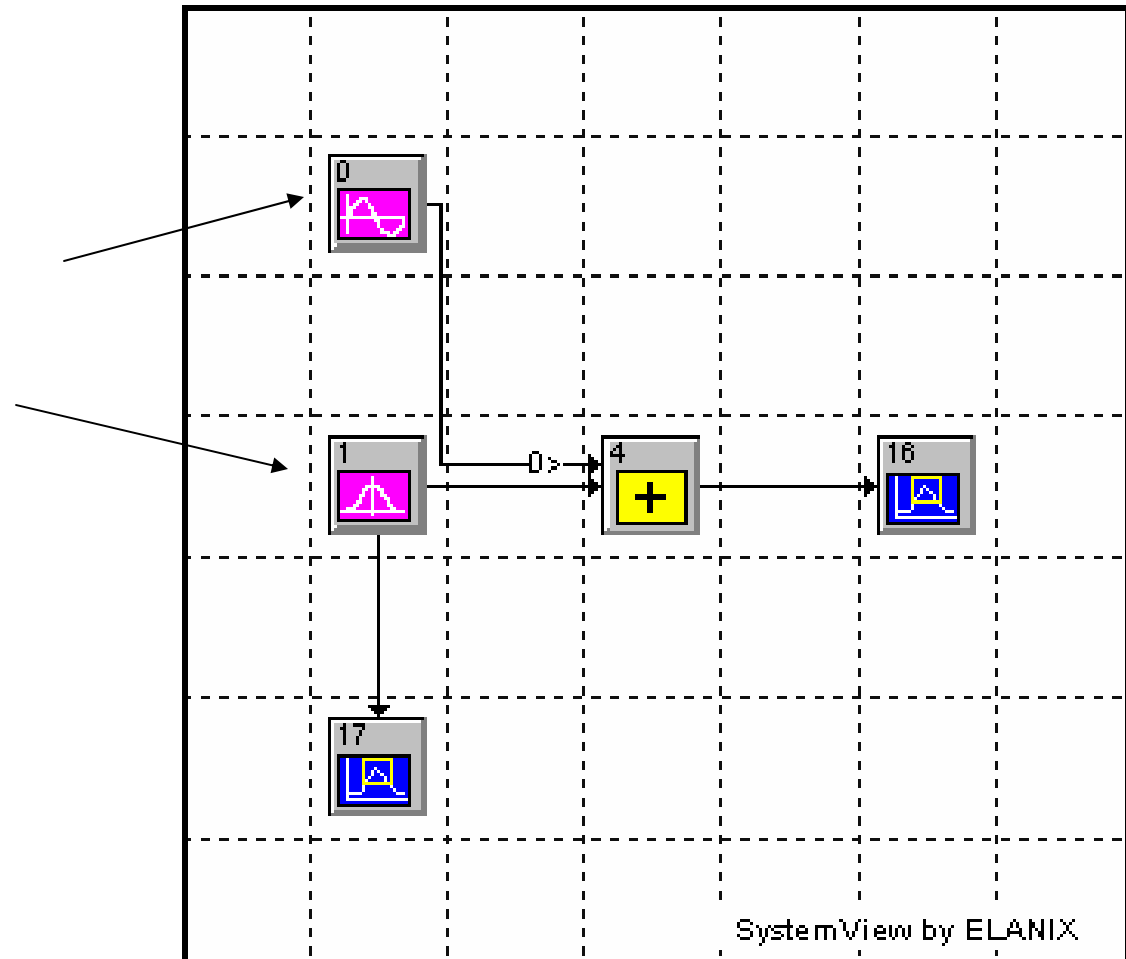
- Output power spectral density after Butterworth LPF





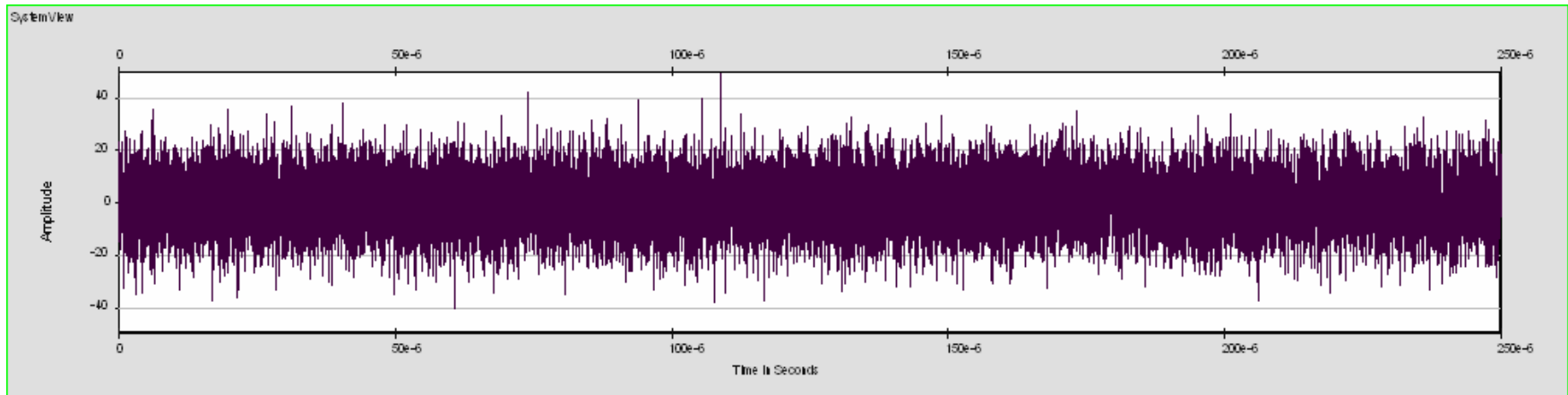
- Example 2.2

10 MHz sinusoid  
with additive  
white Gaussian  
noise (AWGN)

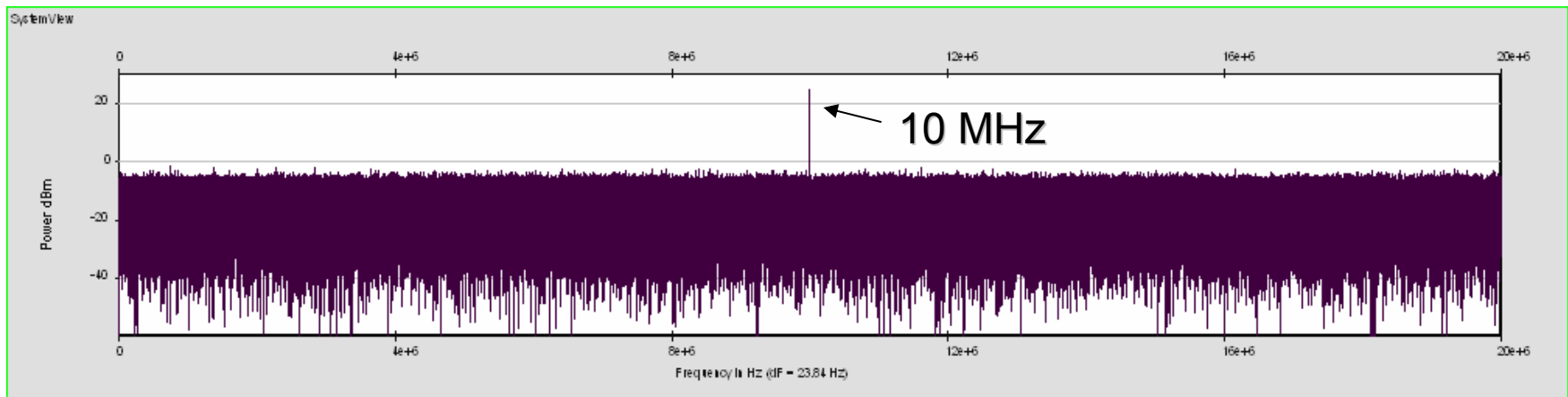




- Example 2.2 10 MHz sinusoid with AWGN



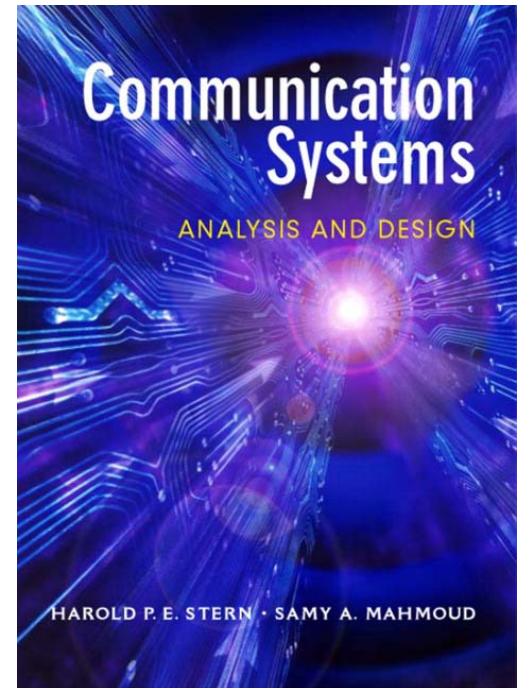
- Power spectral density of 10 MHz sinusoid with AWGN



# Chapter 2

## Frequency Domain Analysis

- *The Fourier Series*
- Pages 13-38



- Fourier Series

Jean Baptiste Joseph Fourier was a French mathematician and physicist who is best known for initiating the investigation of Fourier Series and its application to problems of heat flow. The Fourier transform is also named in his honor.



1768-1830

$$s(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi n f_o t + \varphi_n)$$

- Fourier series coefficients:

trigonometric  $a_n$   $b_n$

polar  $X_n$

complex  $c_n$

$$X_0 = a_0 \quad X_n = \sqrt{a_n^2 + b_n^2}$$

$$|c_n| = X_n / 2 \quad X_n = |2 c_n|$$

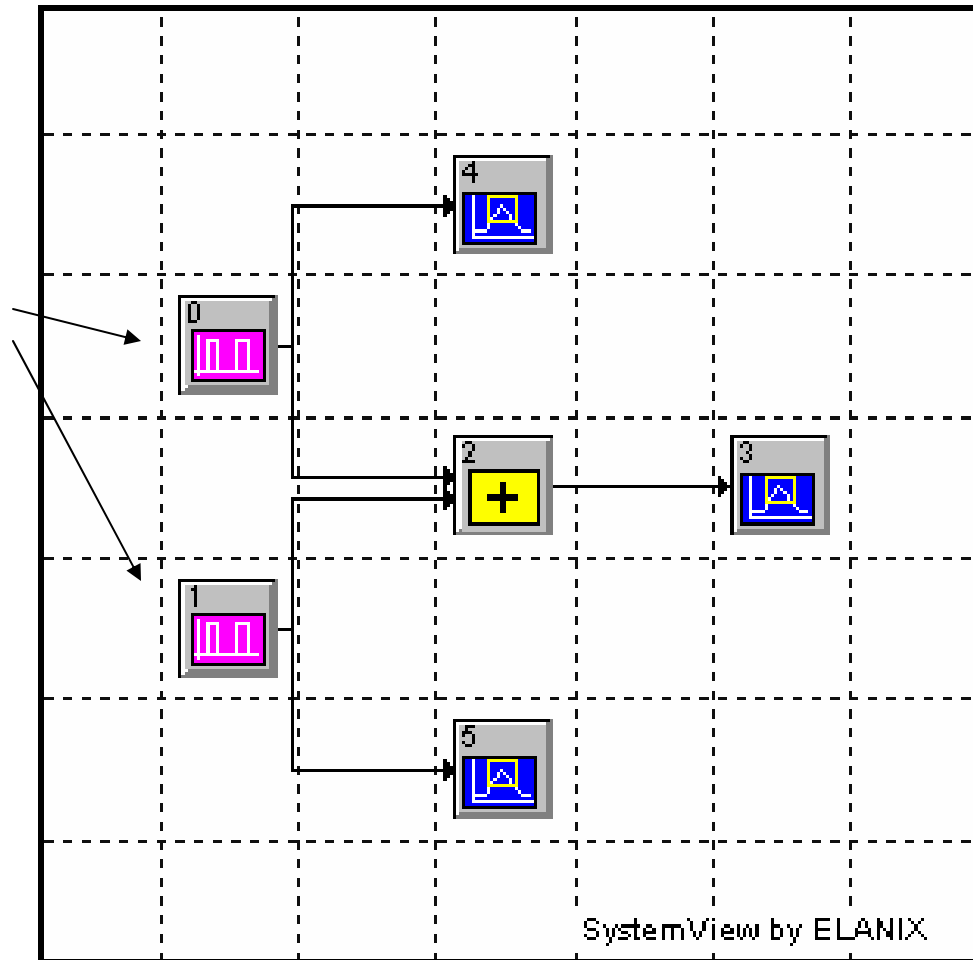
*SystemVue* simulation can provide the magnitude of the complex Fourier series coefficients for any periodic waveform.

Agilent EEsof EDA  
**SystemVue**  
System Design Software



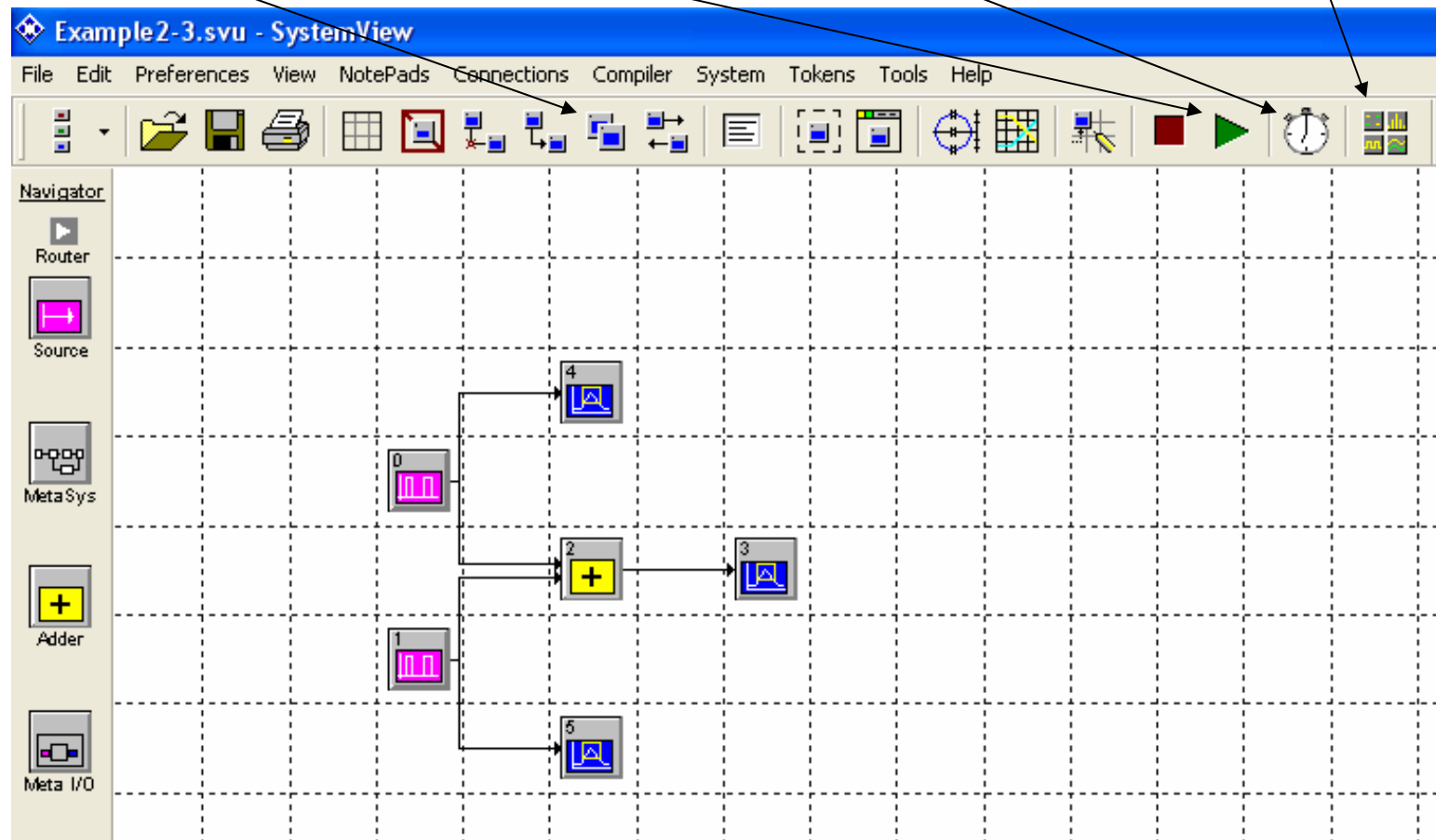
- Example 2.3

Complex pulse  
as the addition of  
two periodic pulses



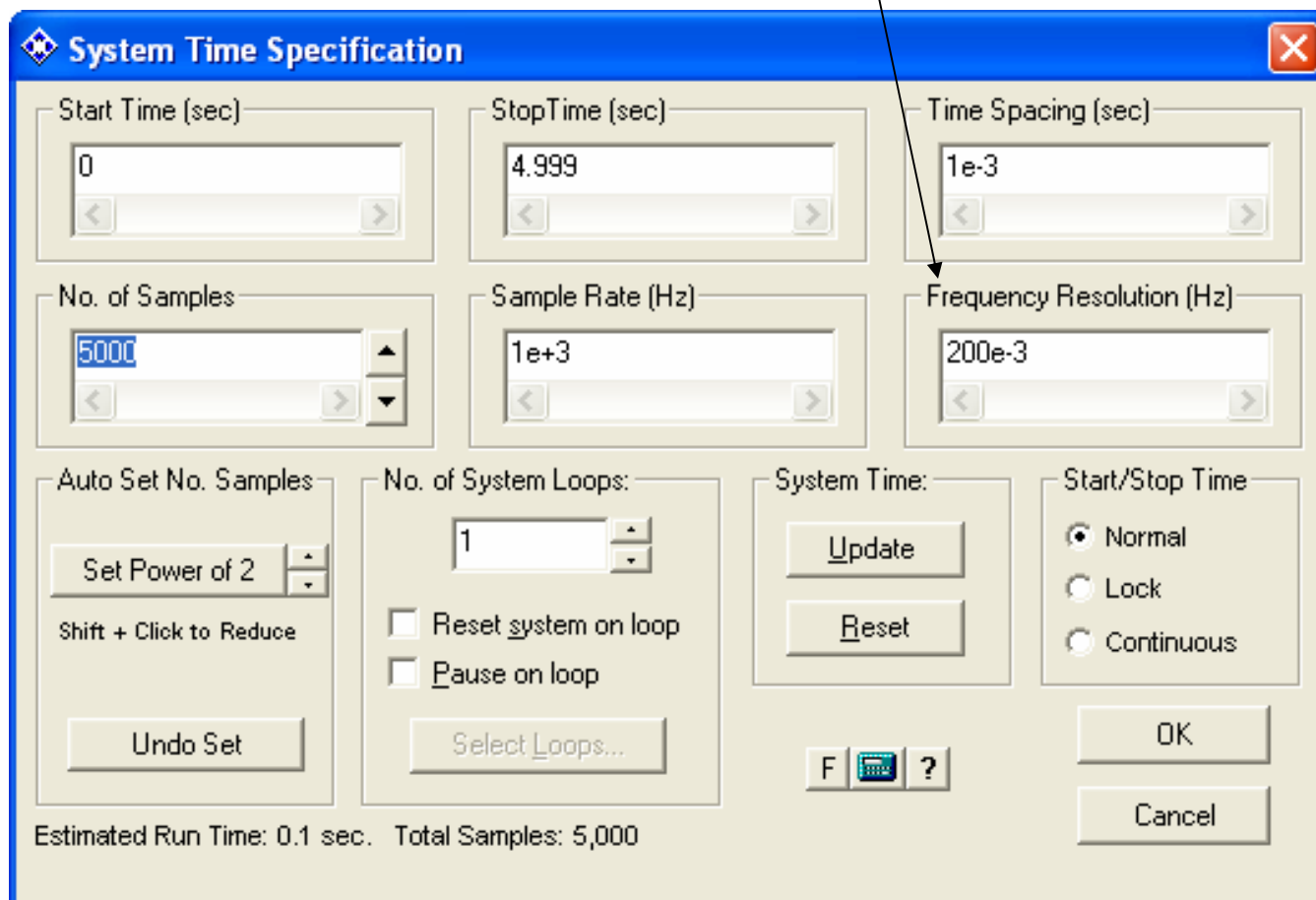
- Example 2.3 SystemVue Design Window

Editing   Simulate   System Time   Analysis Window



- Example 2.3 SystemVue System Time

Fundamental frequency  $f_0 = 0.2$  Hz,  $T_0 = 5$  sec

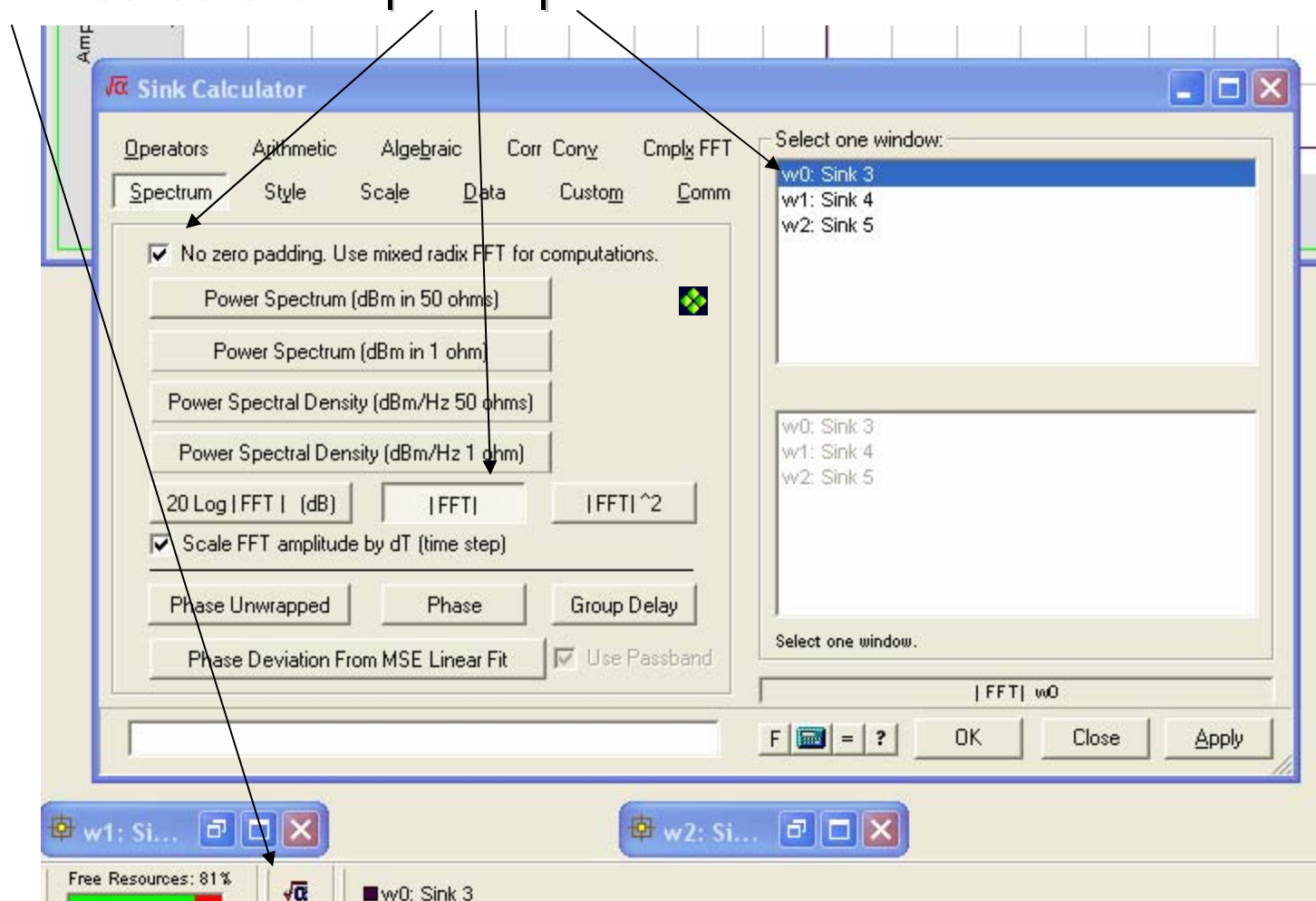


The image shows a 'System Time Specification' dialog box with the following fields and controls:

- Start Time (sec):** 0
- StopTime (sec):** 4.999
- Time Spacing (sec):** 1e-3 (indicated by an arrow from the text above)
- No. of Samples:** 5000
- Sample Rate (Hz):** 1e+3
- Frequency Resolution (Hz):** 200e-3
- Auto Set No. Samples:** Set Power of 2 (Shift + Click to Reduce), Undo Set
- No. of System Loops:** 1, Reset system on loop, Pause on loop, Select Loops...
- System Time:** Update, Reset
- Start/Stop Time:** Normal (selected), Lock, Continuous
- Buttons:** OK, Cancel
- Footer:** Estimated Run Time: 0.1 sec. Total Samples: 5,000

- Example 2.3 SystemVue Analysis Window

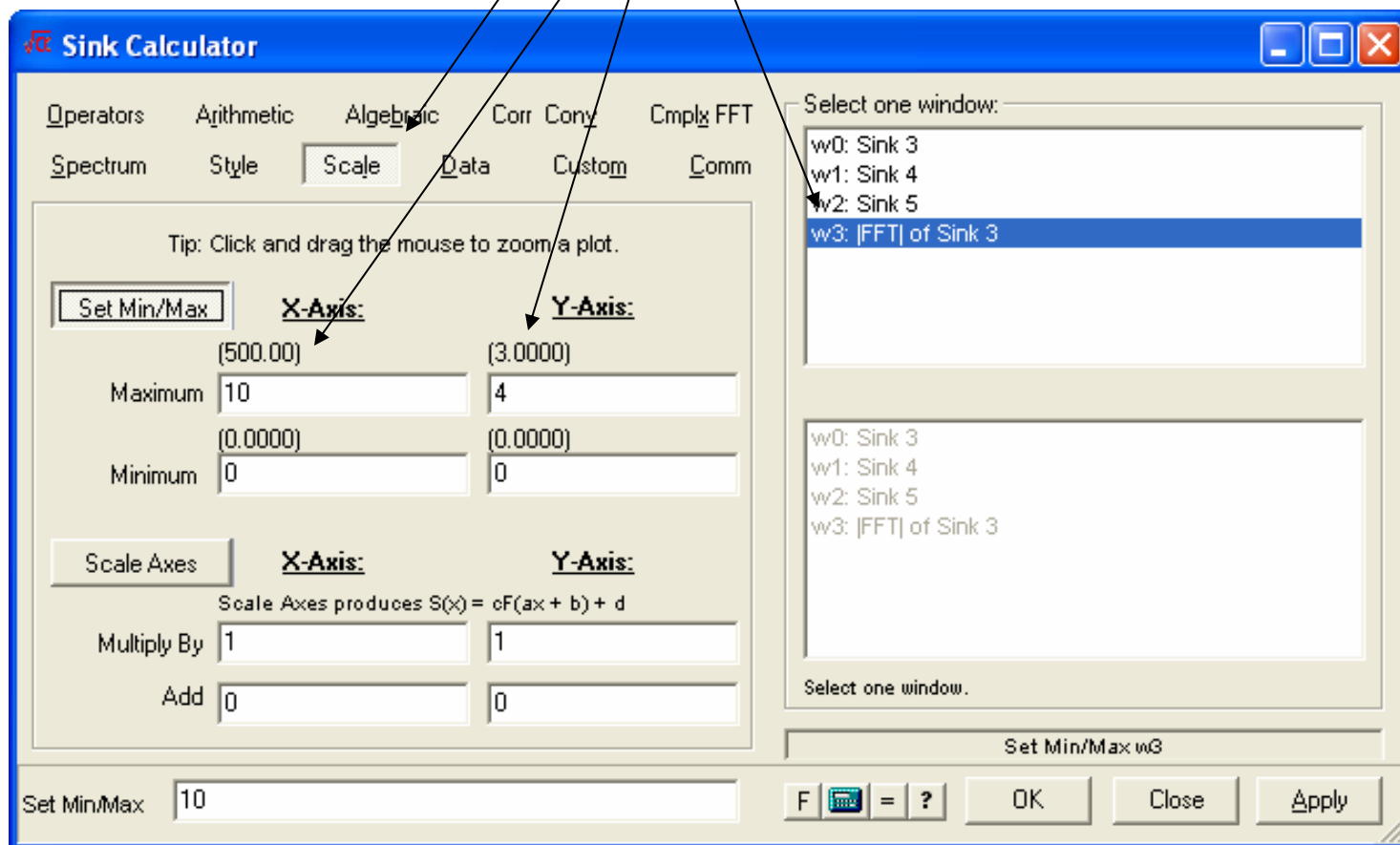
Sink calculator | FFT |



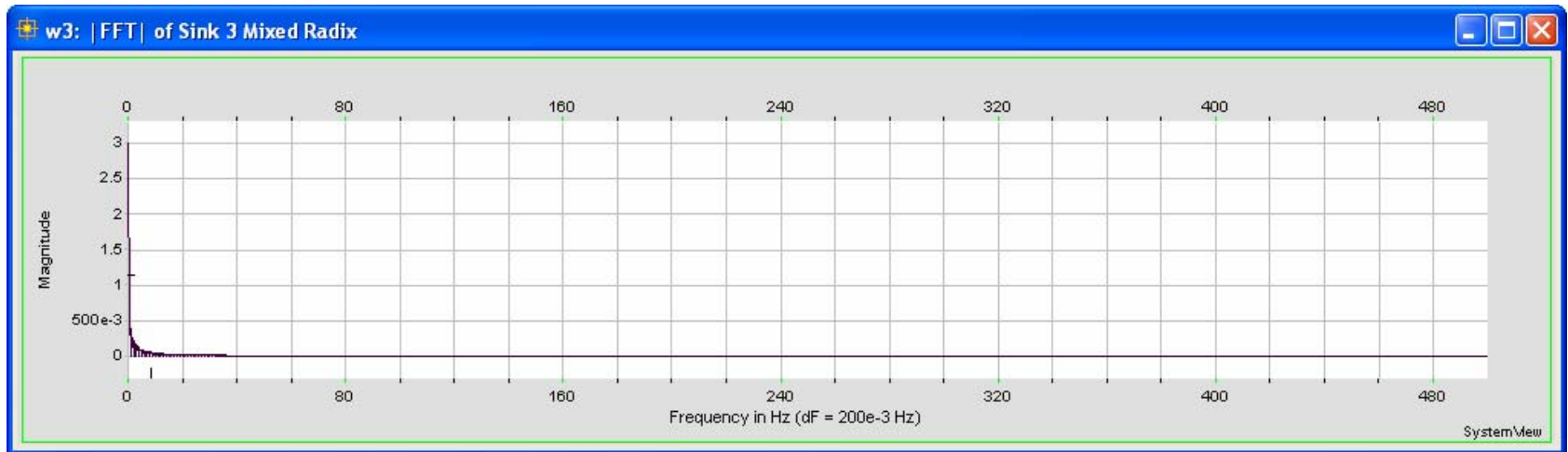


- Example 2.3 SystemVue Analysis Window

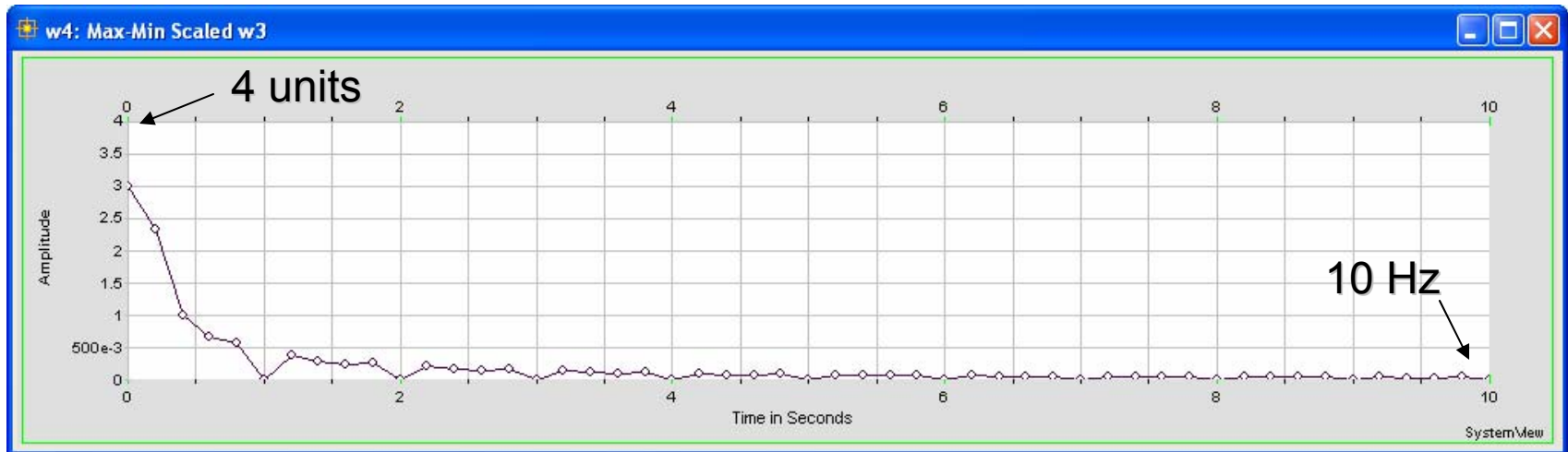
Sink calculator    Scale    Display



- Example 2.3 Unscaled | FFT |

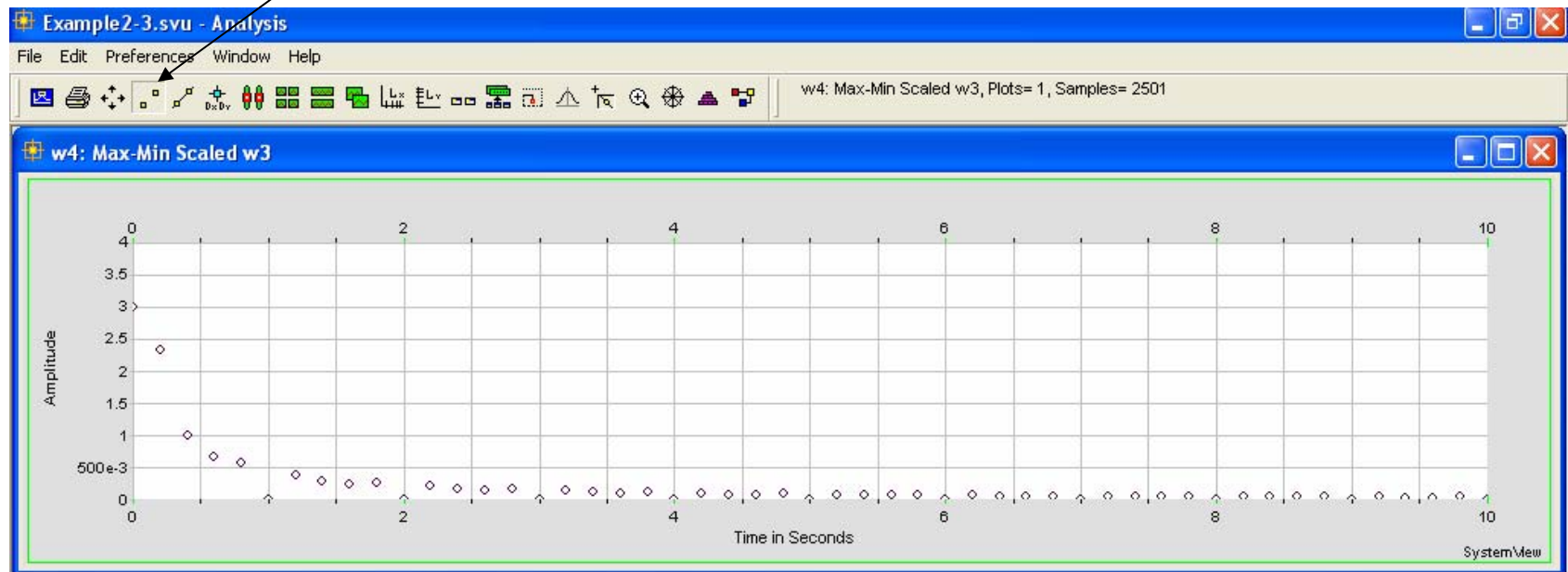


- Scaled | FFT |

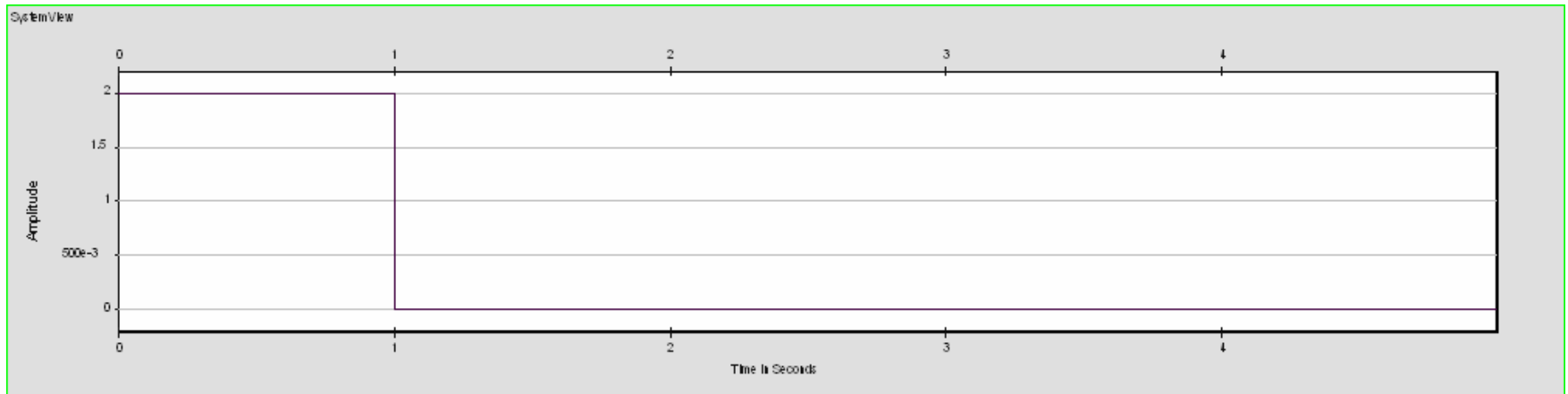


- Example 2.3 Scaled | FFT |

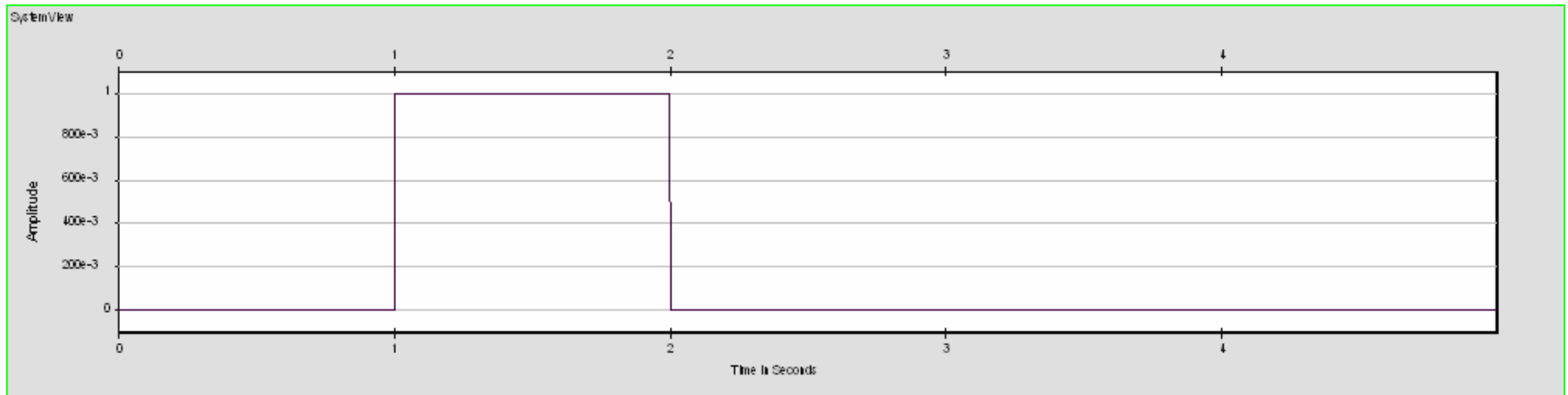
The Fourier series components are *discrete*. In the *SystemVue* Analysis Window the connection between data points can be eliminated if warranted.



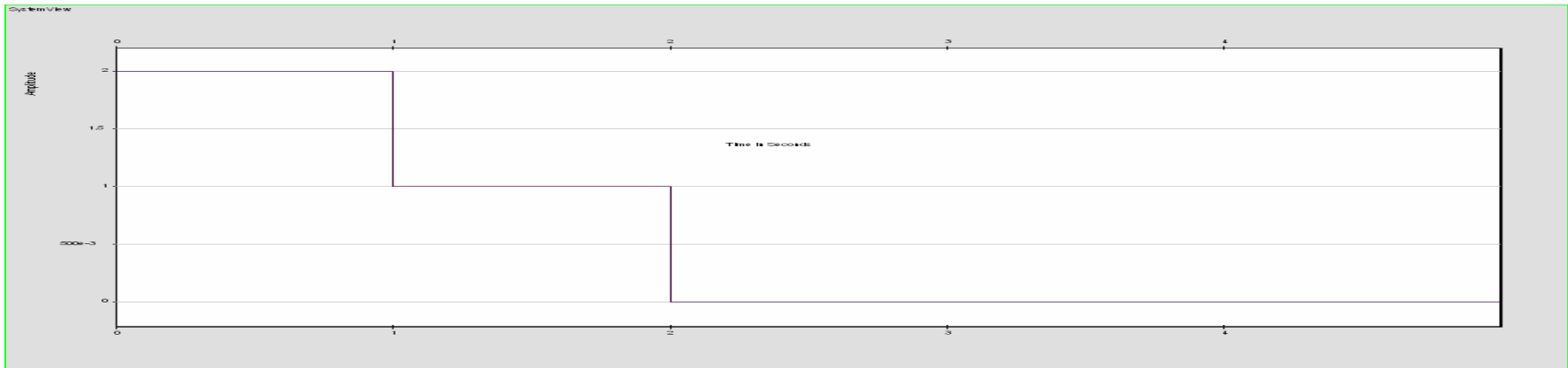
- Example 2.3 First periodic pulse



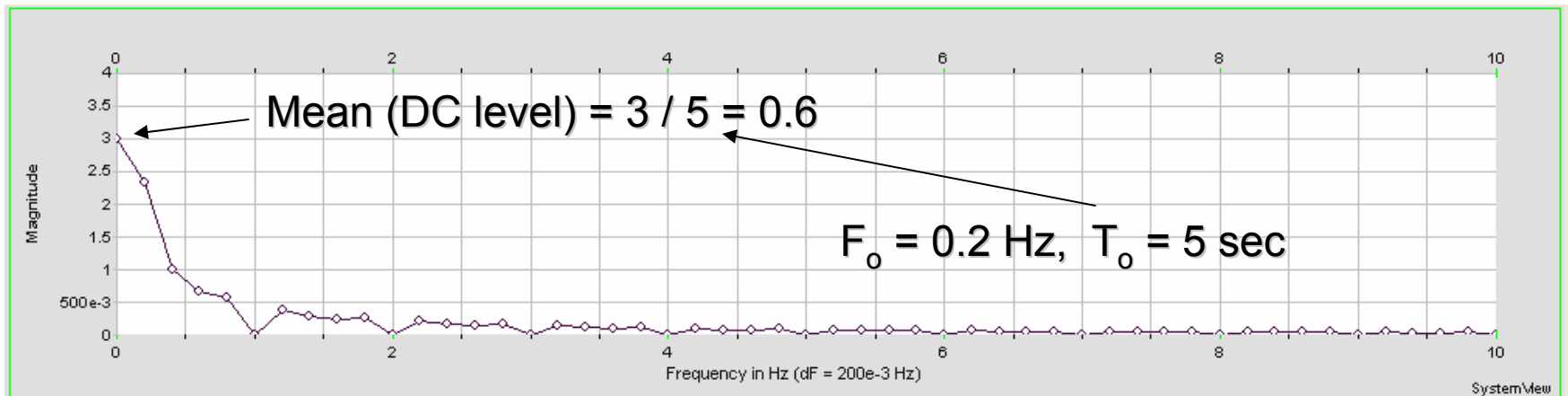
- Second periodic pulse



- Example 2.3 Sum of first and second periodic pulses

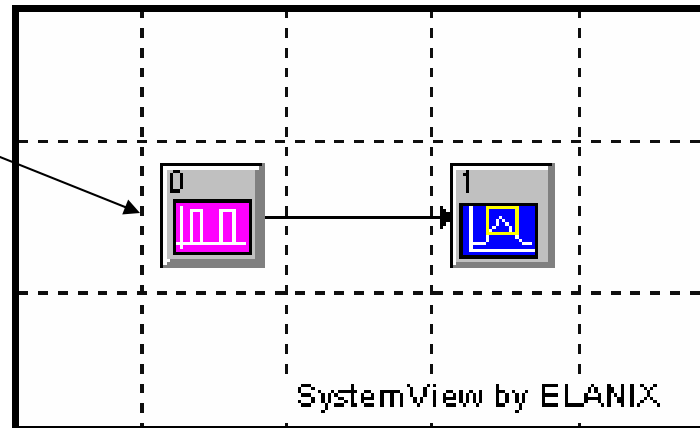


- Magnitude of the Fourier Transform | FFT |



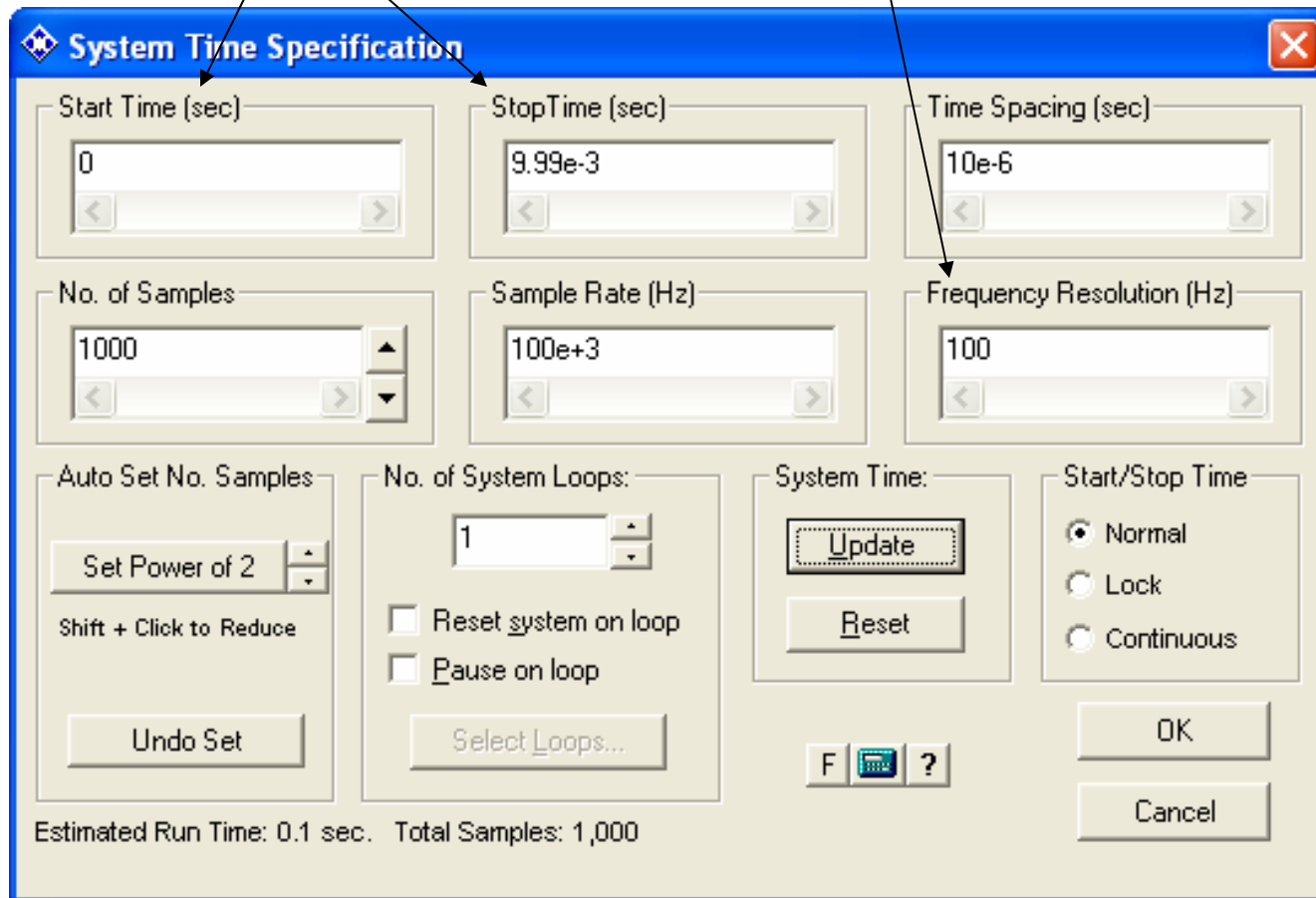
- Example 2.7

Rectangular pulse  
train



- Example 2.7 SystemVue System Time

Period  $T_0 \approx 10$  msec, fundamental frequency  $f_0 = 100$  Hz



The image shows a 'System Time Specification' dialog box with various settings. Arrows from the text above point to specific fields: 'Start Time (sec)' is 0, 'StopTime (sec)' is 9.99e-3, and 'Time Spacing (sec)' is 10e-6. Other settings include 'No. of Samples' (1000), 'Sample Rate (Hz)' (100e+3), and 'Frequency Resolution (Hz)' (100). The 'Auto Set No. Samples' section has 'Set Power of 2' and 'Undo Set' buttons. The 'No. of System Loops' is 1, with checkboxes for 'Reset system on loop' and 'Pause on loop', and a 'Select Loops...' button. The 'System Time' section has 'Update' and 'Reset' buttons. The 'Start/Stop Time' section has radio buttons for 'Normal' (selected), 'Lock', and 'Continuous'. At the bottom, there are 'F', a computer icon, and a '?' button, along with 'OK' and 'Cancel' buttons. The status bar at the bottom indicates 'Estimated Run Time: 0.1 sec. Total Samples: 1,000'.

**System Time Specification**

Start Time (sec): 0

StopTime (sec): 9.99e-3

Time Spacing (sec): 10e-6

No. of Samples: 1000

Sample Rate (Hz): 100e+3

Frequency Resolution (Hz): 100

Auto Set No. Samples: Set Power of 2 (Shift + Click to Reduce), Undo Set

No. of System Loops: 1, Reset system on loop, Pause on loop, Select Loops...

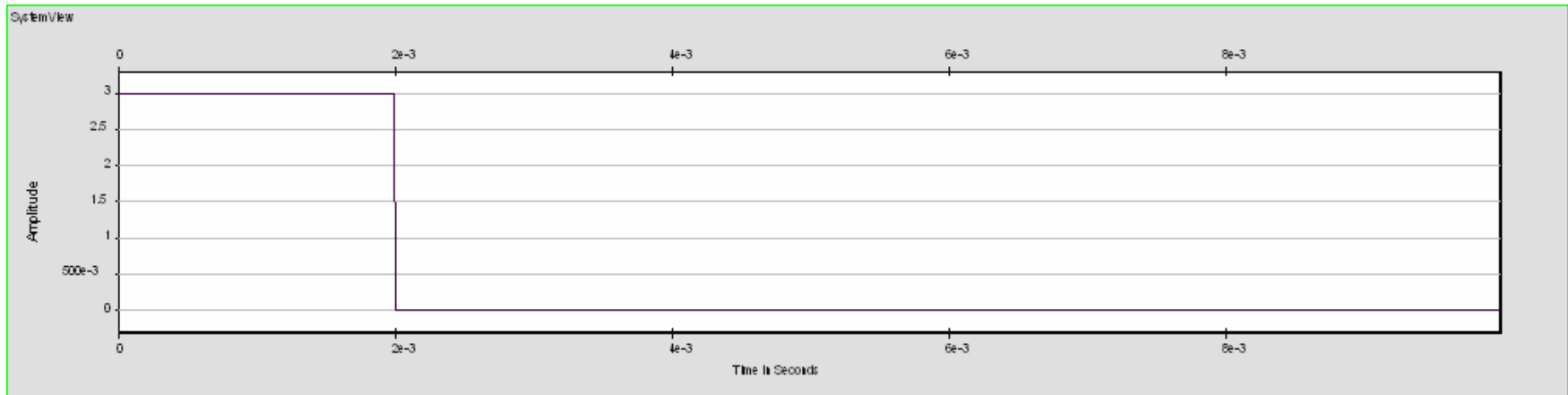
System Time: Update, Reset

Start/Stop Time: Normal (selected), Lock, Continuous

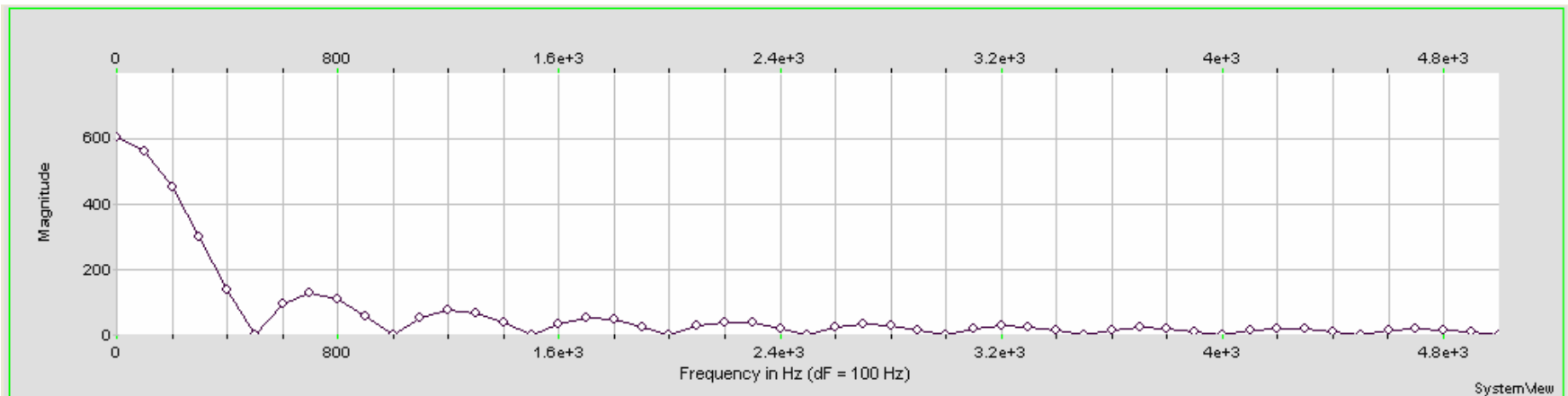
Estimated Run Time: 0.1 sec. Total Samples: 1,000

OK, Cancel

- Example 2.7 One cycle of periodic pulse

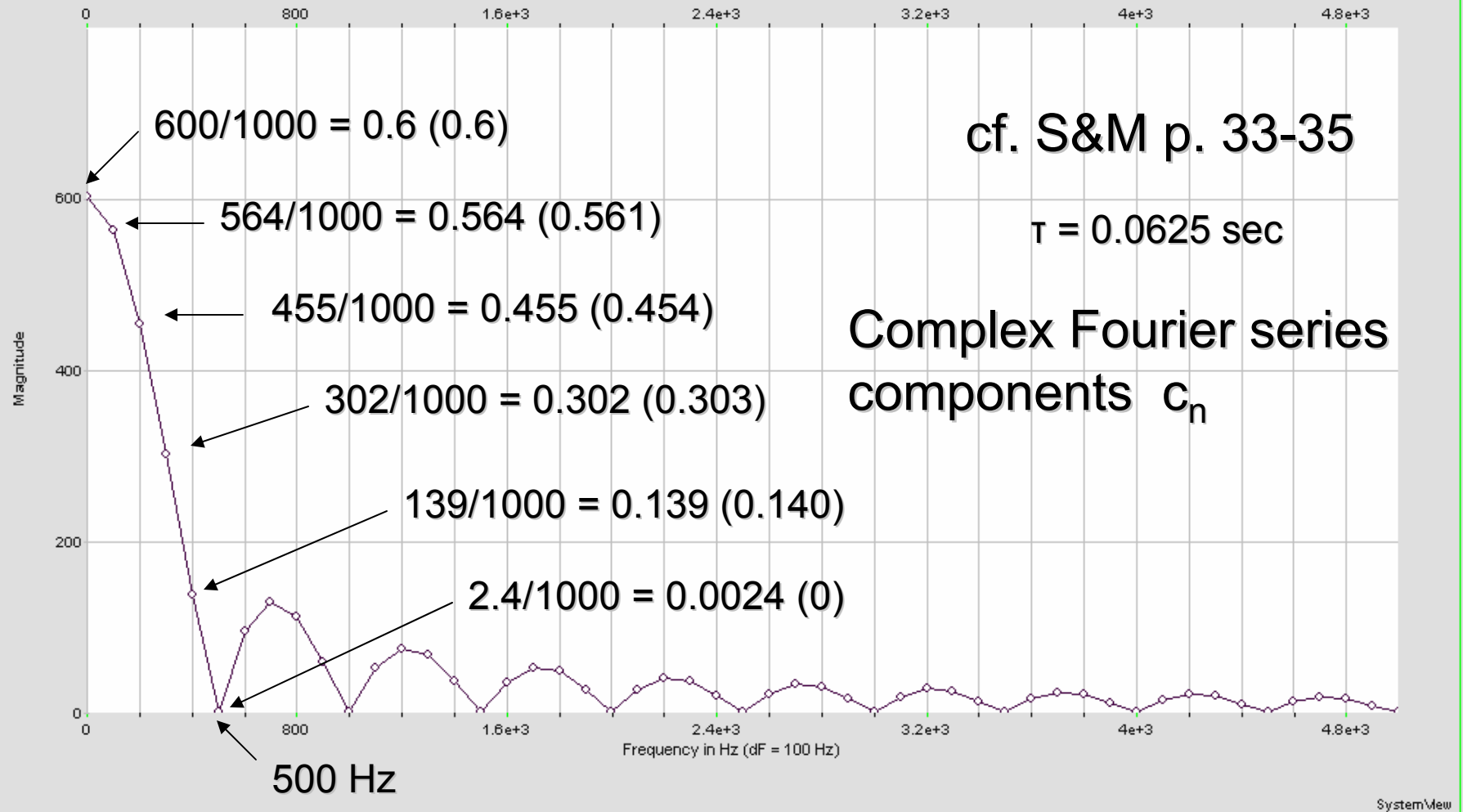


- Magnitude of the Fast Fourier Transform | FFT |





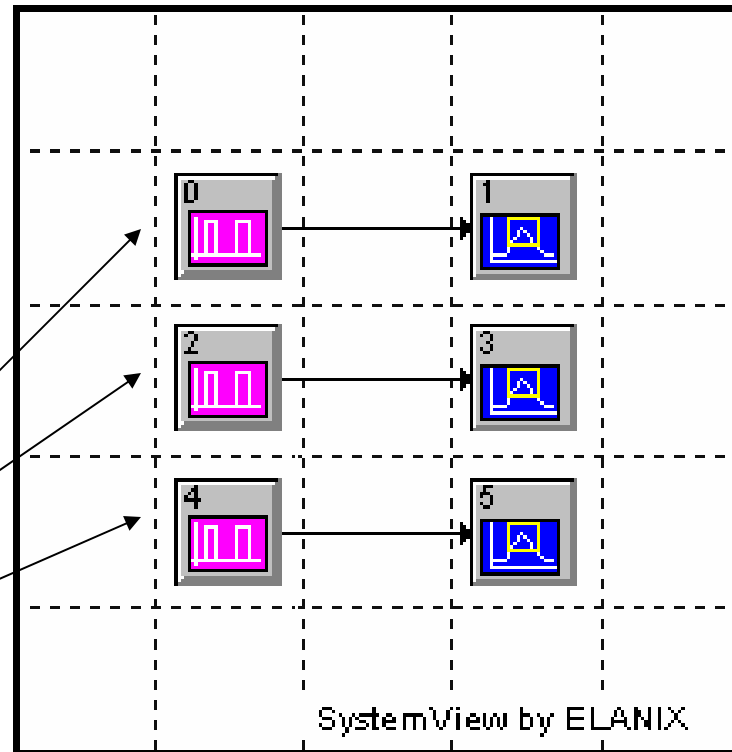
- Example 2.7 | FFT |



- Example 2.8

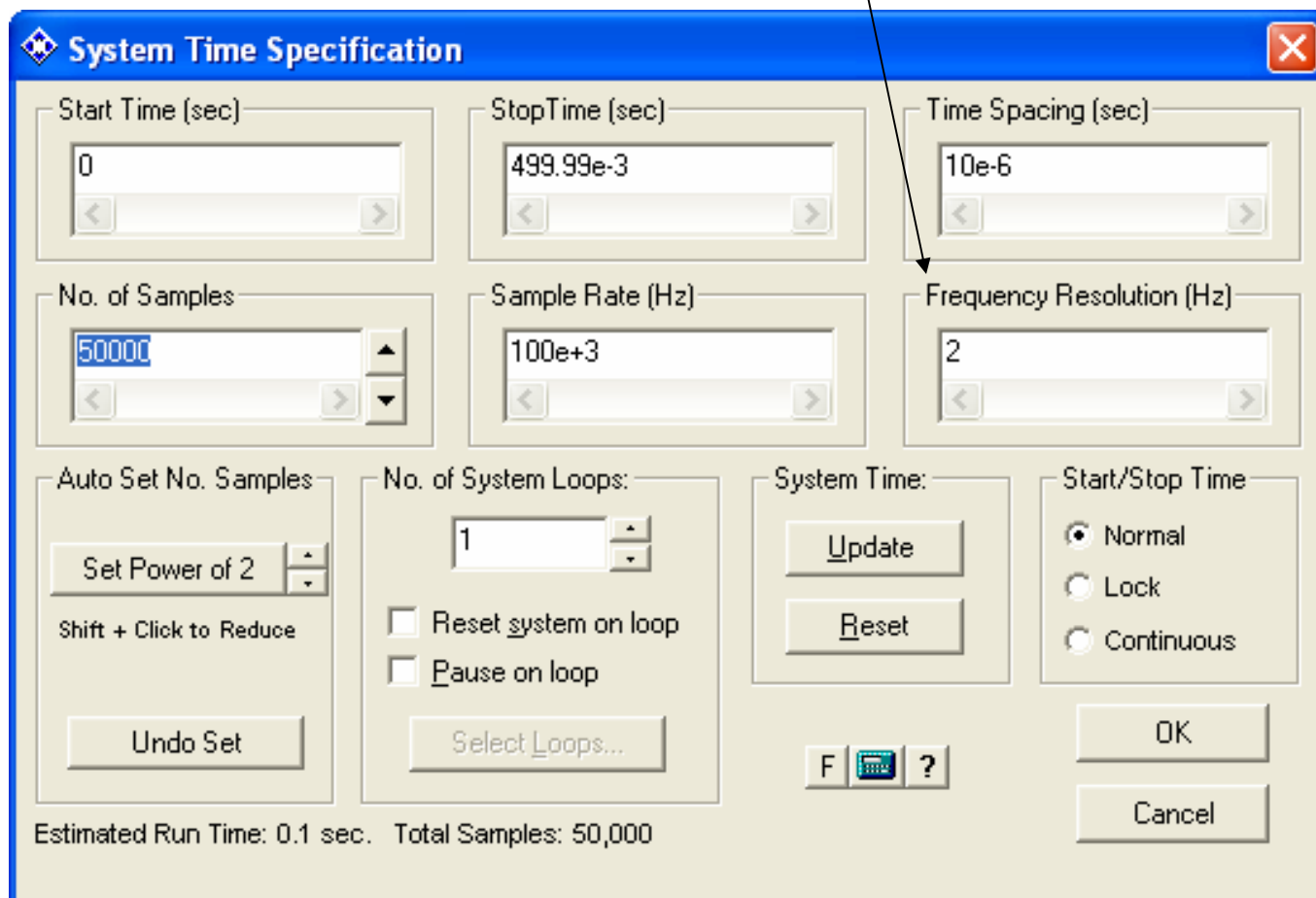
Rectangular pulse  
trains with pulse  
period of 0.5 sec  
and pulse widths of

- 0.0625 sec
- 0.125 sec
- 0.250 sec



- Example 2.8 SystemVue System Time

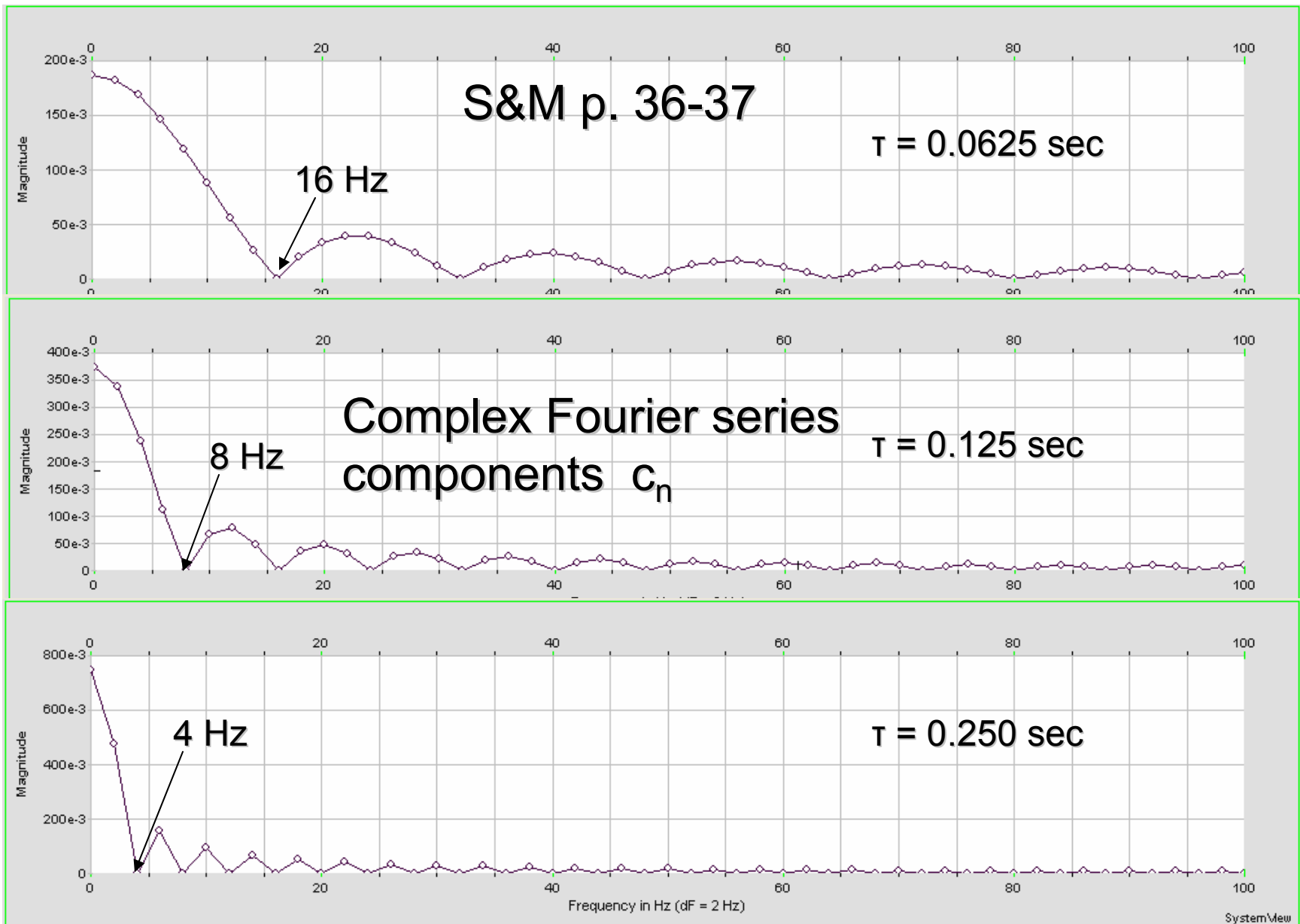
Fundamental frequency  $f_0 = 2$  Hz,  $T_0 = 0.5$  sec



The image shows a 'System Time Specification' dialog box with the following fields and controls:

- Start Time (sec):** 0
- StopTime (sec):** 499.99e-3
- Time Spacing (sec):** 10e-6
- No. of Samples:** 50000
- Sample Rate (Hz):** 100e+3
- Frequency Resolution (Hz):** 2
- Auto Set No. Samples:** Set Power of 2 (Shift + Click to Reduce), Undo Set
- No. of System Loops:** 1, Reset system on loop, Pause on loop, Select Loops...
- System Time:** Update, Reset
- Start/Stop Time:** Normal (selected), Lock, Continuous
- Buttons:** OK, Cancel
- Footer:** Estimated Run Time: 0.1 sec. Total Samples: 50,000

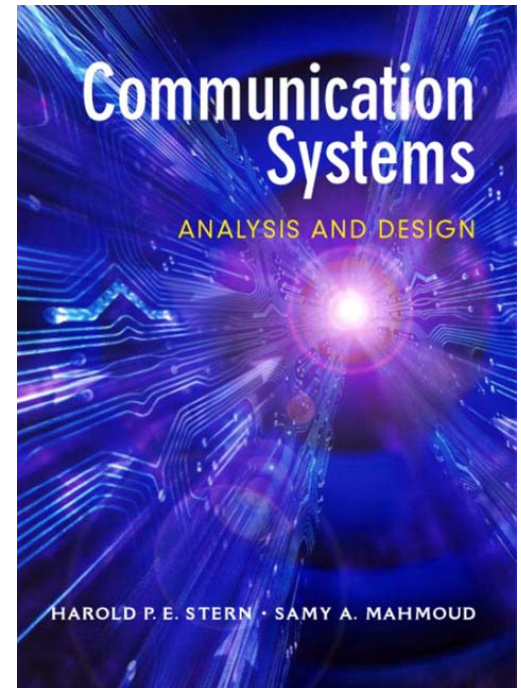
An arrow points from the text 'Fundamental frequency  $f_0 = 2$  Hz,  $T_0 = 0.5$  sec' to the 'Frequency Resolution (Hz)' field, which is set to 2.



## Chapter 2

# Frequency Domain Analysis

- *Power in the Frequency Domain*
- Pages 38-52



- Average Normalized ( $R = 1\Omega$ ) Power

Periodic signal as a frequency domain representation

$$s(t) = X_0 + \sum_{n=1}^{\infty} X_n \cos(2\pi n f_0 t + \varphi_n)$$

Average normalized power in the signal as a time domain or frequency domain representation

$$P_s = \frac{1}{T} \int_{t_0}^{t_0+T} s^2(t) dt = X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2}$$

*Parseval's Theorem*

- *Parseval's Theorem*

Marc-Antoine Parseval des Chênes was a French mathematician, most famous for what is now known as Parseval's Theorem, which presaged the equivalence of the Fourier Transform. A monarchist opposed to the French Revolution, Parseval fled the country after being imprisoned in 1792 by Napoleon for publishing tracts critical of the government.

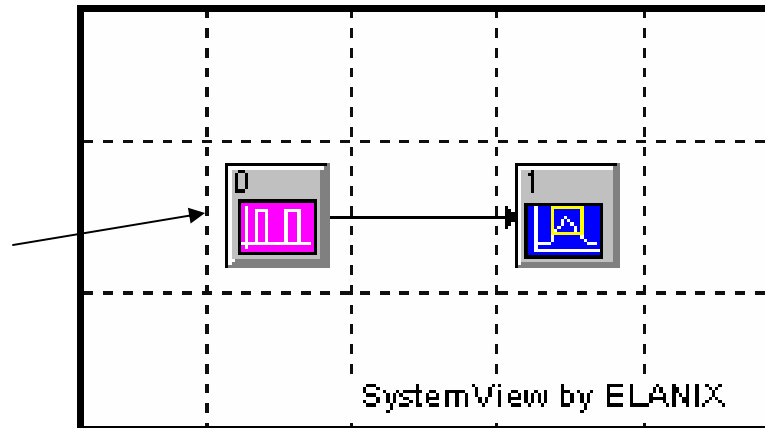


1755-1836

$$P_s = \frac{1}{T} \int_{t_0}^{t_0+T} s^2(t) dt = X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2}$$

- Example 2.9

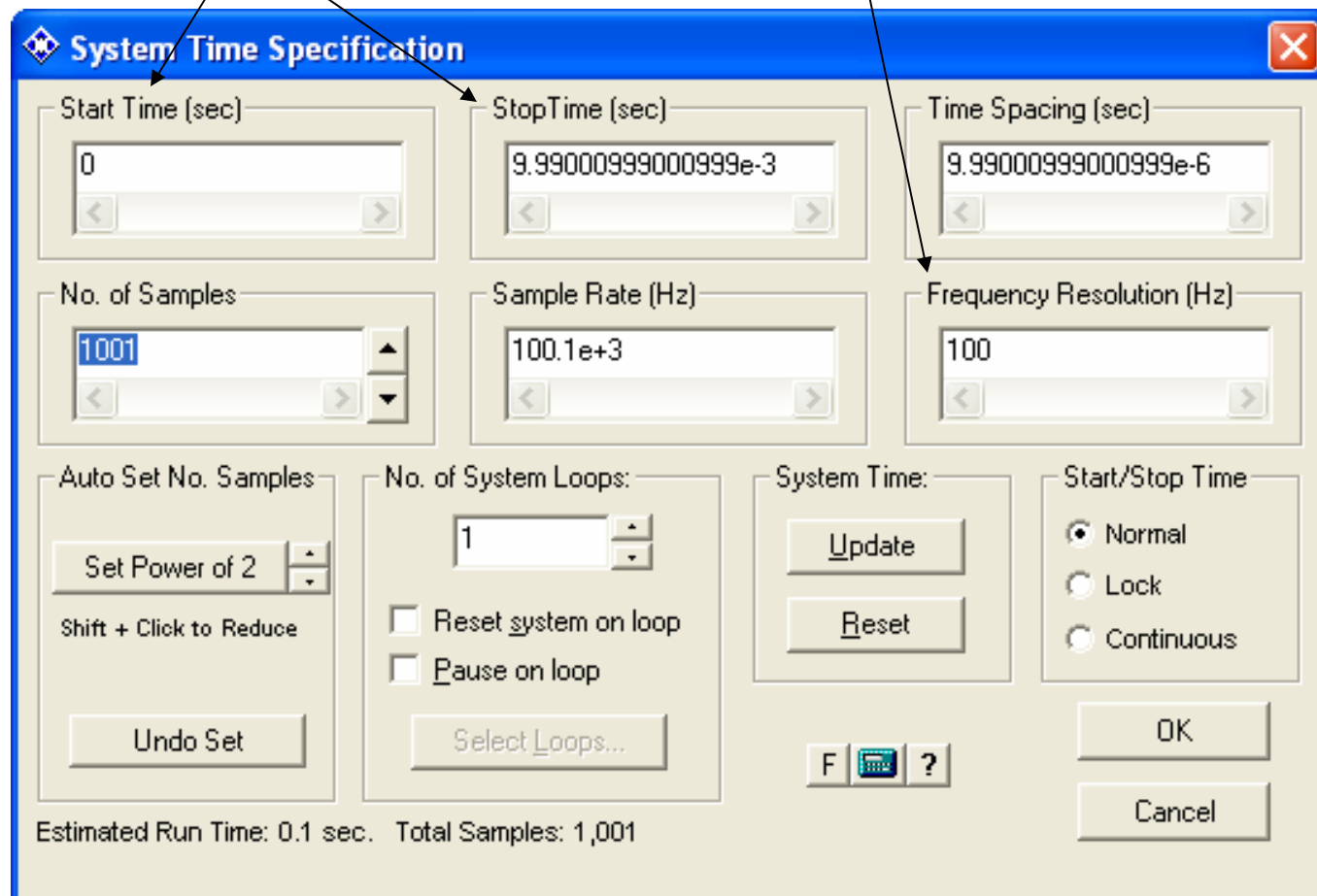
Normalized power spectrum of a periodic rectangular pulse train





- Example 2.8 SystemVue System Time

Period  $T_0 \approx 10$  msec, frequency resolution = 100 Hz



The image shows a 'System Time Specification' dialog box with various input fields and controls. Arrows from the text above point to specific fields: 'Start Time (sec)' is set to 0, 'StopTime (sec)' is set to 9.99000999000999e-3, and 'Time Spacing (sec)' is set to 9.99000999000999e-6. Other fields include 'No. of Samples' (1001), 'Sample Rate (Hz)' (100.1e+3), and 'Frequency Resolution (Hz)' (100). The dialog also features buttons for 'Update', 'Reset', 'OK', 'Cancel', and 'Undo Set', along with checkboxes for 'Reset system on loop' and 'Pause on loop'. A status bar at the bottom indicates 'Estimated Run Time: 0.1 sec. Total Samples: 1,001'.

**System Time Specification**

Start Time (sec): 0

StopTime (sec): 9.99000999000999e-3

Time Spacing (sec): 9.99000999000999e-6

No. of Samples: 1001

Sample Rate (Hz): 100.1e+3

Frequency Resolution (Hz): 100

Auto Set No. Samples: Set Power of 2 (Shift + Click to Reduce)

No. of System Loops: 1

System Time: Update, Reset

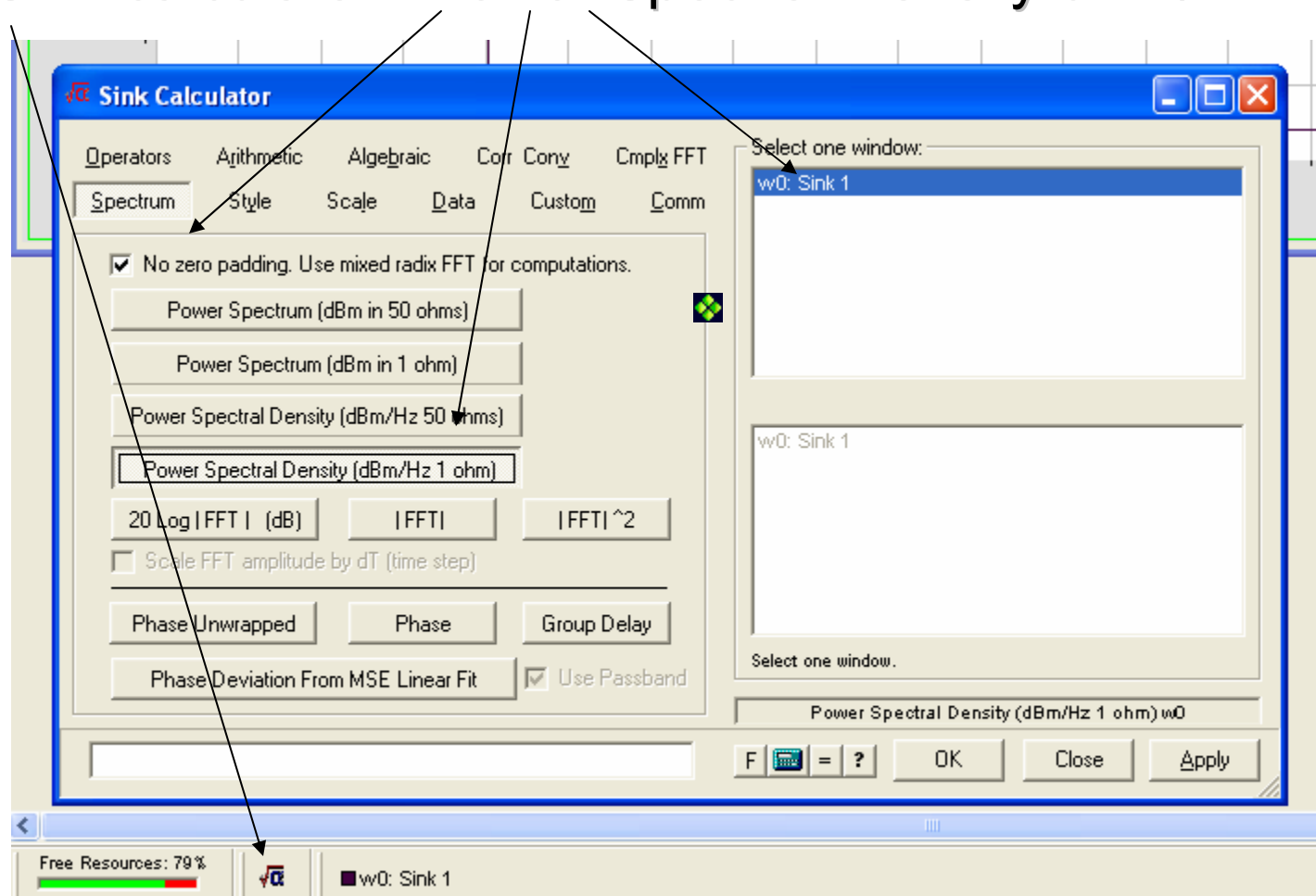
Start/Stop Time: ☒ Normal, ☐ Lock, ☐ Continuous

OK, Cancel

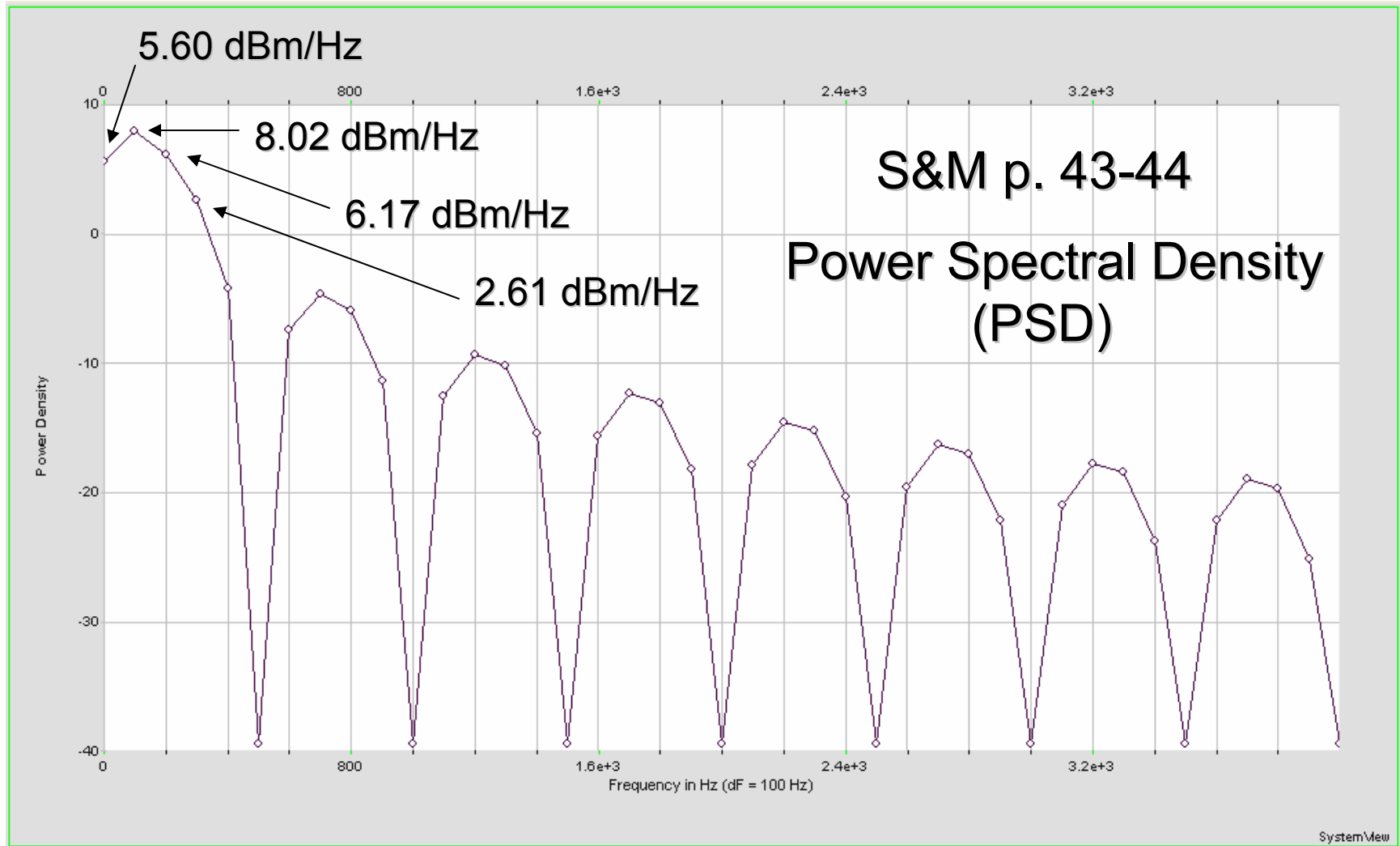
Estimated Run Time: 0.1 sec. Total Samples: 1,001

- Example 2.9 SystemVue Analysis Window

Sink calculator Power Spectral Density dBm/Hz 1Ω

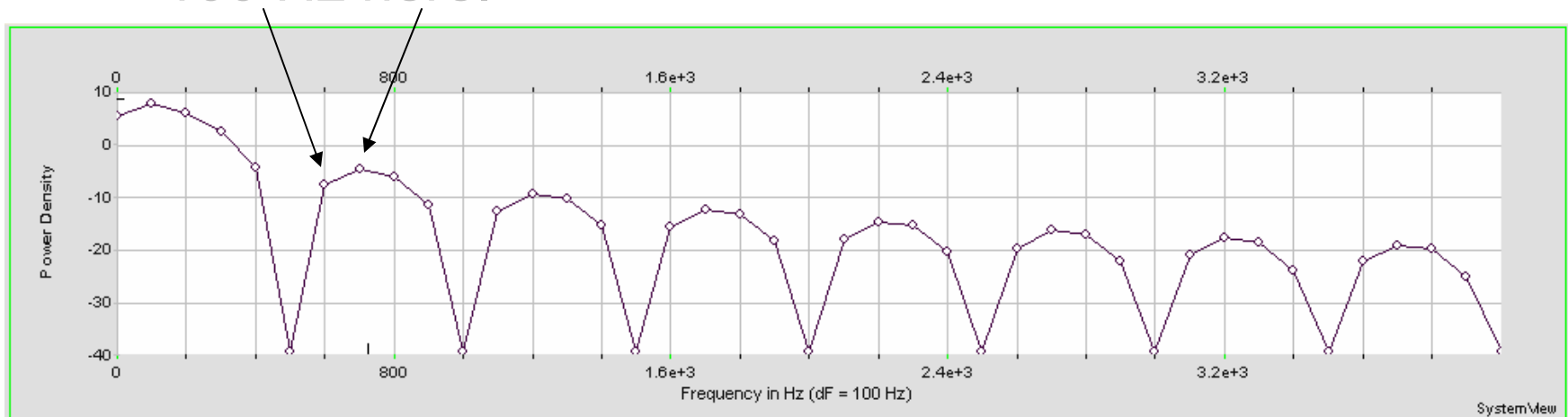


- Example 2.9 PSD dBm/Hz  $1\Omega$



- Power Spectral Density dBm/Hz 1Ω

The power spectral density (PSD) for periodic signals is *discrete* because of the fundamental frequency  $f_0 = 1/T_0 = 100$  Hz here.



However, for aperiodic signals the PSD is conceptually *continuous*. Periodic signals contain *no information* and only aperiodic signals are, in fact, communicated.

- Power Spectral Density dBm/Hz  $1\Omega$

dBm is decibel (dB) referenced to 1 normalized milliwatt ( $\text{mW} = 10^{-3} \text{ W}$ , normalized  $V^2/R$ ,  $R = 1\Omega$ )

$$\text{dBm} = 10 \log (\text{Power} / 10^{-3} \text{ V}^2) \quad \text{normalized } R = 1\Omega$$

$$5.6 \text{ dBm/Hz} \quad f_o = 100 \text{ Hz}$$

$$5.6 = 10 \log (\text{Power/Hz} / 10^{-3})$$

$$\text{Power/Hz} = 10^{5.6/10} (10^{-3}) = 3.63 \times 10^{-3} \text{ V}^2/\text{Hz}$$

$$\text{Power} = (\text{Power/Hz})(f_o \text{ Hz})$$

$$\text{Power} = (3.63 \times 10^{-3} \text{ V}^2/\text{Hz})(100 \text{ Hz})$$

$$\text{Power} = 0.36 \text{ V}^2 \quad (0.36 \text{ V}^2, \text{ S \& M p. 43})$$

- de·ci·bel (dēs'ə-bəl, -bĕl')  
*n.* (*Abbr.* dB)

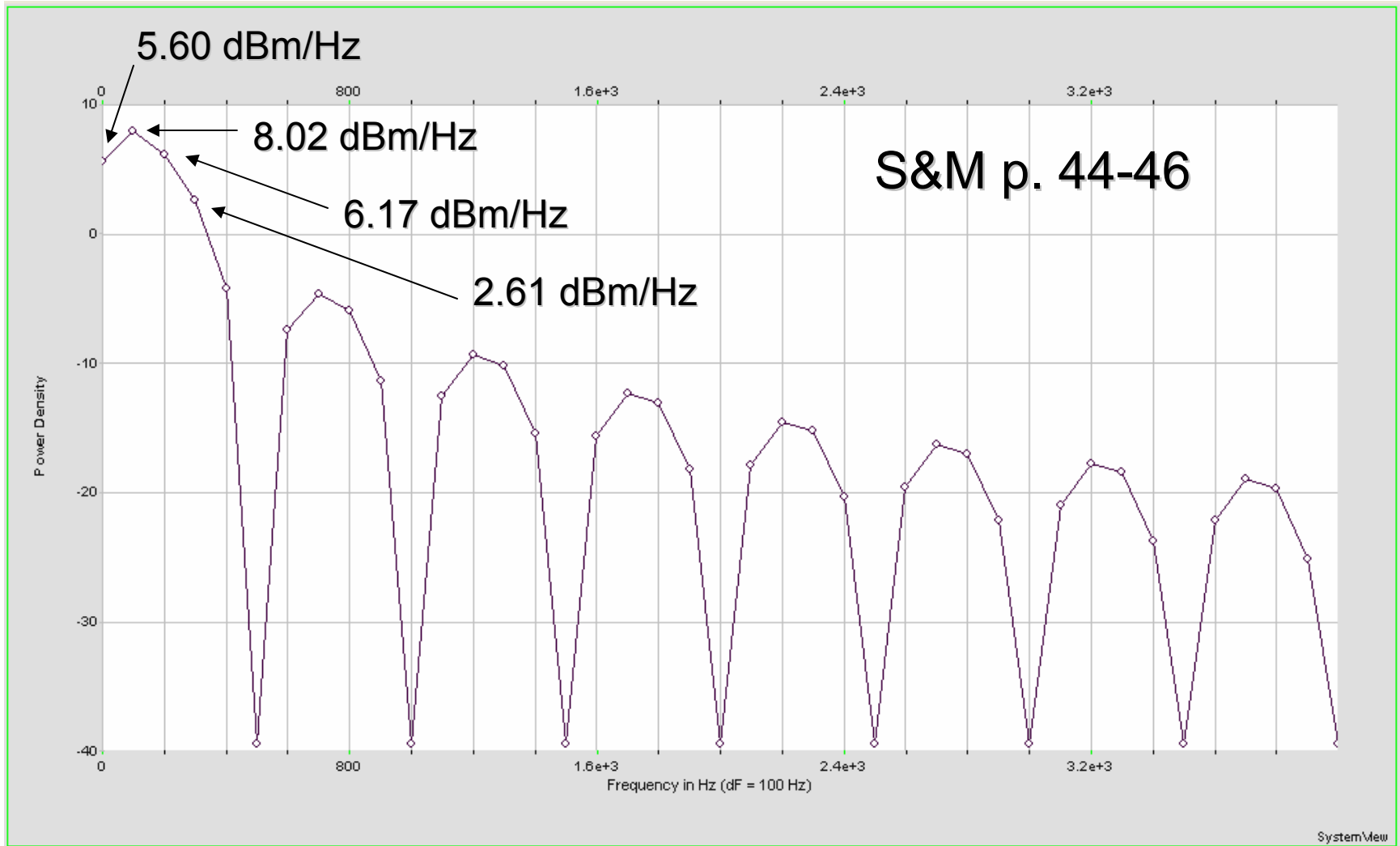
A unit used to express relative difference in power or intensity, usually between two acoustic or electric signals, equal to ten times the common logarithm of the ratio of the two levels.

The *bel* (B) as a unit of measurement was originally proposed in 1929 by W. H. Martin of Bell Labs. The bel was too large for everyday use, so the decibel (dB), equal to 0.1 B, became more commonly used.

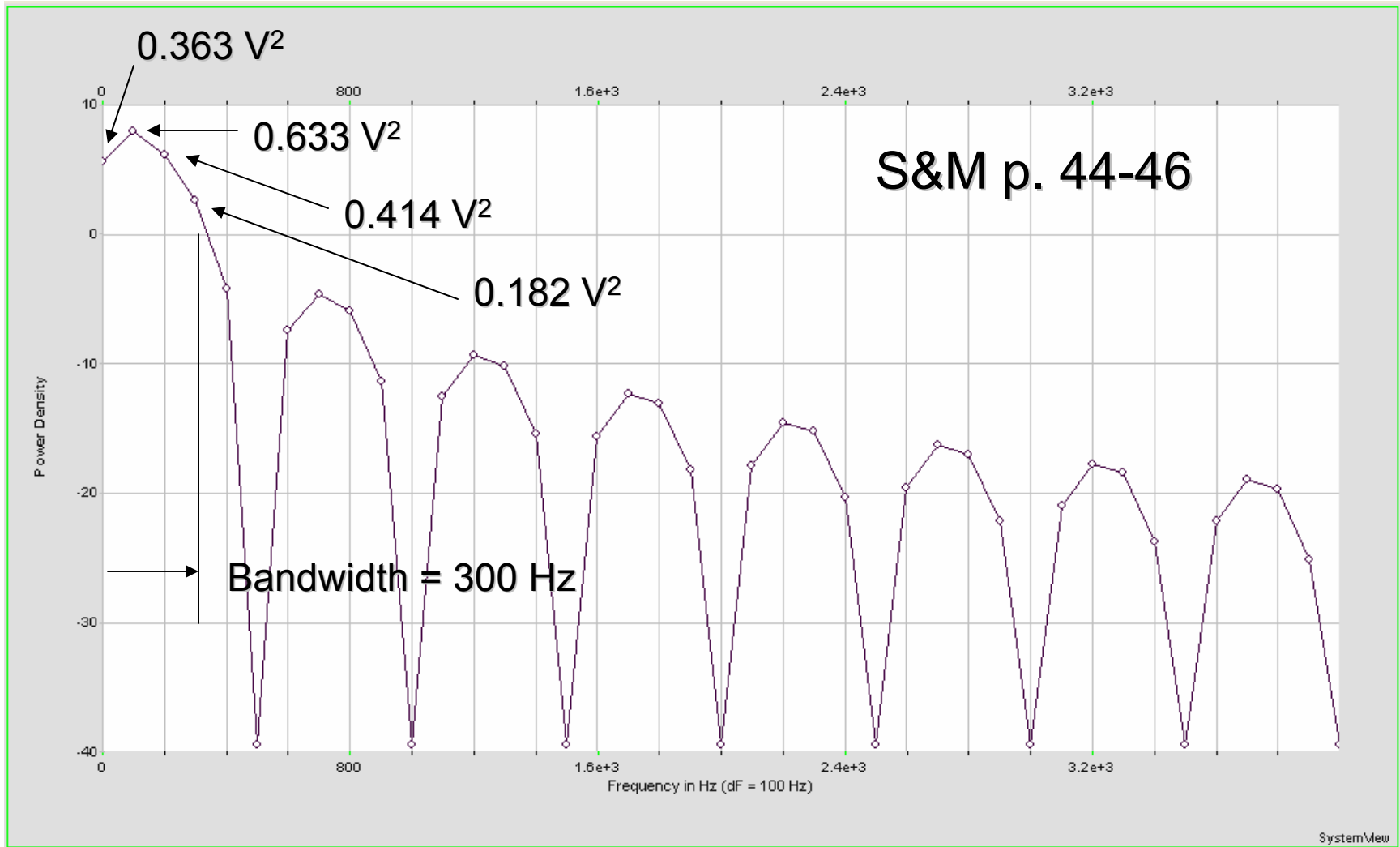


Alexander Graham Bell  
1847-1922

- Example 2.10 PSD dBm/Hz  $1\Omega$



- Example 2.10 PSD dBm/Hz  $1\Omega$  Converted to  $V^2$





- Bandwidth

The *bandwidth* of a signal is the width of the frequency band in Hertz that contains a sufficient number of the signal's frequency components to reproduce the signal with an acceptable amount of distortion.

Bandwidth is a nebulous term and communication engineers must always define what is meant by “bandwidth” in the context of use.



- Total Power in the Signal

Parseval's Theorem allows us to determine the *total normalized power* in the signal without the infinite sum of Fourier series components by integrating in the temporal domain:

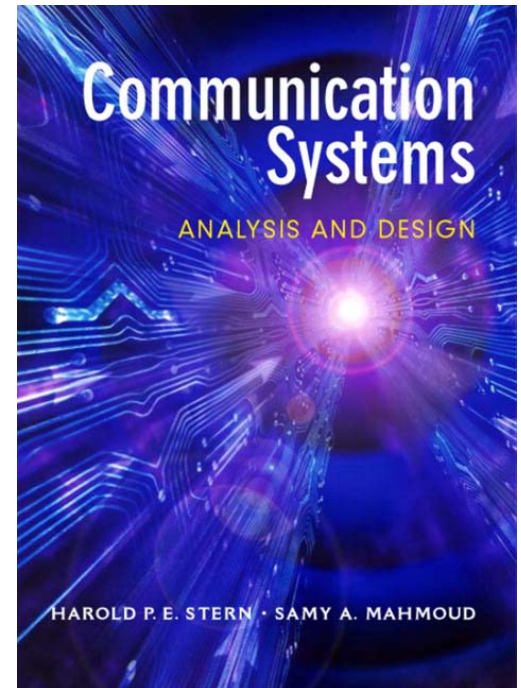
$$P_S = \frac{1}{T} \int_{t_0}^{t_0+T} s^2(t) dt = X_0^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{2}$$

The total power in the signal then is 1.8 V<sup>2</sup> and the percentage of the total power in the signal in a bandwidth of 300 Hz then is approximately 88% (S&M p. 45) .

## Chapter 2

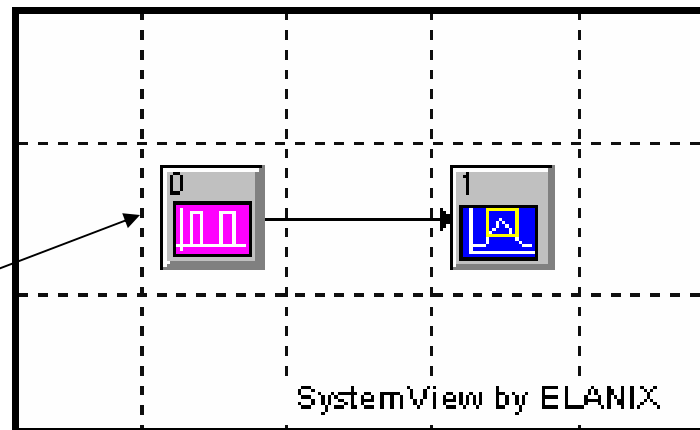
# Frequency Domain Analysis

- *The Fourier Transform*
- Pages 52-69



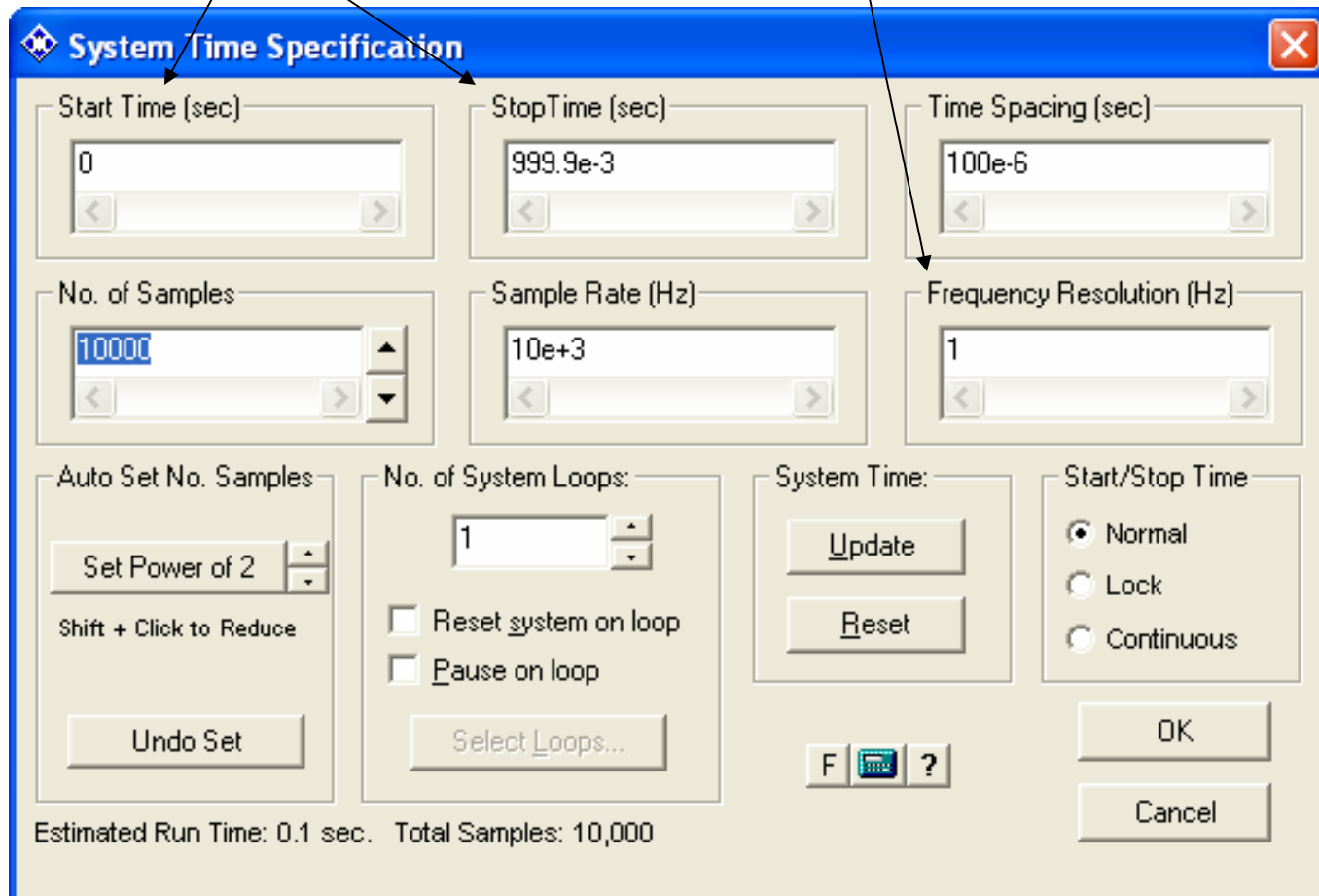
- Example 2.12

Spectrum of a  
simulated single  
pulse from a very  
low duty cycle  
rectangular pulse  
train



- Example 2.7 SystemVue System Time

Period  $T_0 \approx 1$  sec, fundamental frequency  $f_0 = 1$  Hz



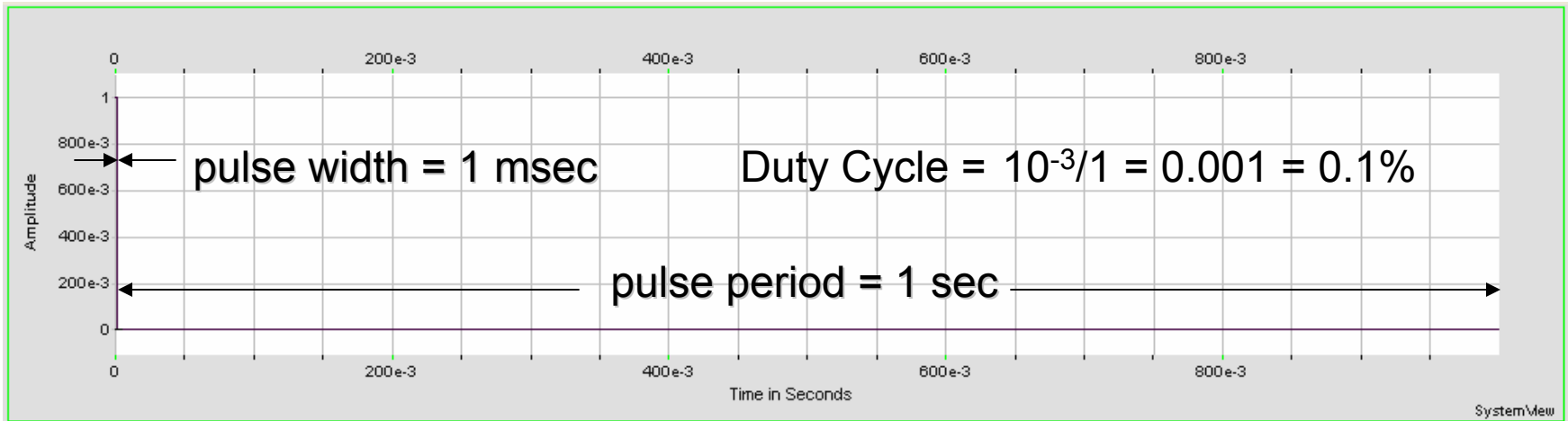
The image shows a screenshot of the 'System Time Specification' dialog box in SystemVue. The dialog box has a blue title bar with a gear icon and a close button. It contains several input fields and buttons for configuring system time parameters. Arrows from the text above point to the 'Start Time (sec)' and 'Time Spacing (sec)' fields.

Parameter	Value
Start Time (sec)	0
StopTime (sec)	999.9e-3
Time Spacing (sec)	100e-6
No. of Samples	10000
Sample Rate (Hz)	10e+3
Frequency Resolution (Hz)	1

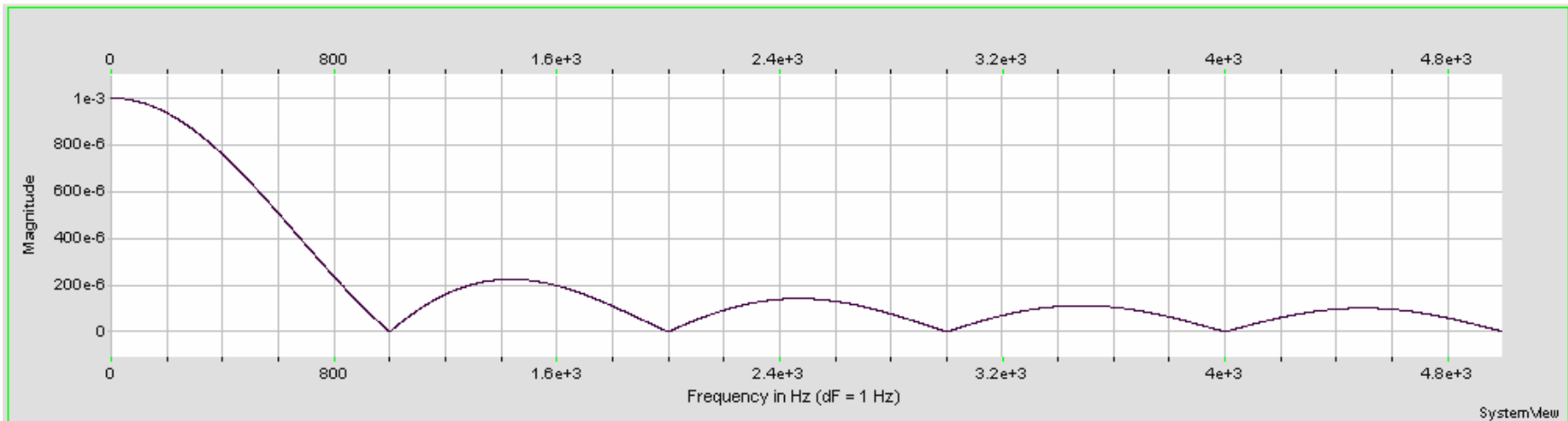
Additional controls include:

- Auto Set No. Samples:** Set Power of 2 (Shift + Click to Reduce), Undo Set button.
- No. of System Loops:** 1, Reset system on loop (checkbox), Pause on loop (checkbox), Select Loops... button.
- System Time:** Update, Reset buttons.
- Start/Stop Time:** Normal (selected), Lock, Continuous radio buttons.
- Buttons:** OK, Cancel.
- Footer:** Estimated Run Time: 0.1 sec. Total Samples: 10,000.

- Example 2.12 Simulated single pulse

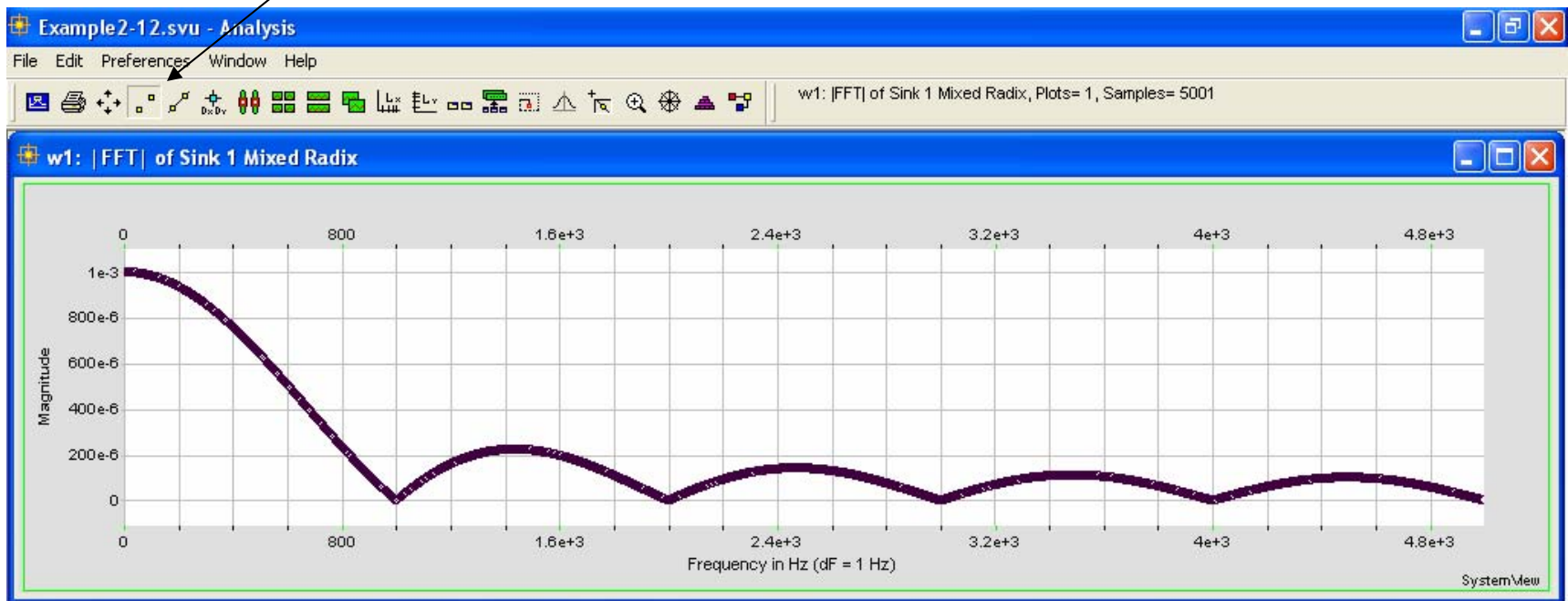


- Magnitude of the Fourier Transform | FFT |

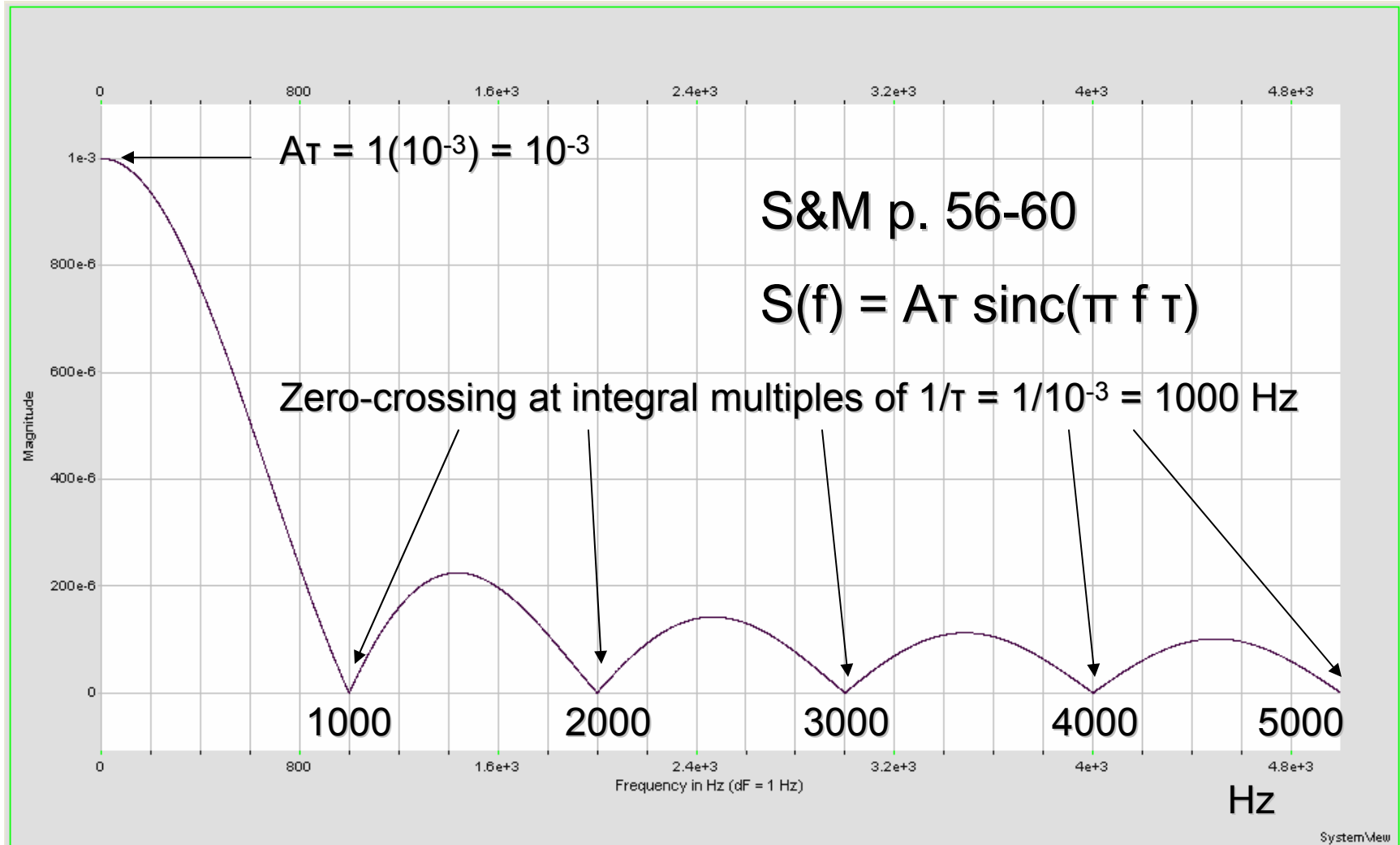


- Example 2.12 Simulated single pulse

The magnitude of the Fourier Transform of a single pulse is *continuous* and *not discrete* since there is no Fourier series representation. In the *SystemVue* simulation the data points are very dense and virtually display a continuous plot.

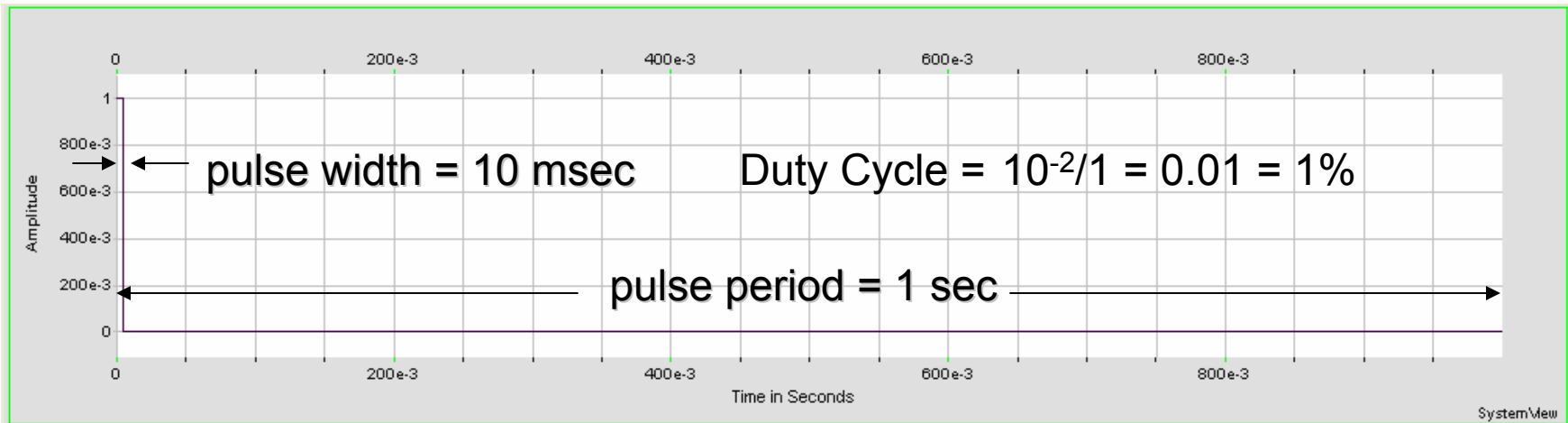


- Example 2.12 Simulated single pulse | FFT |

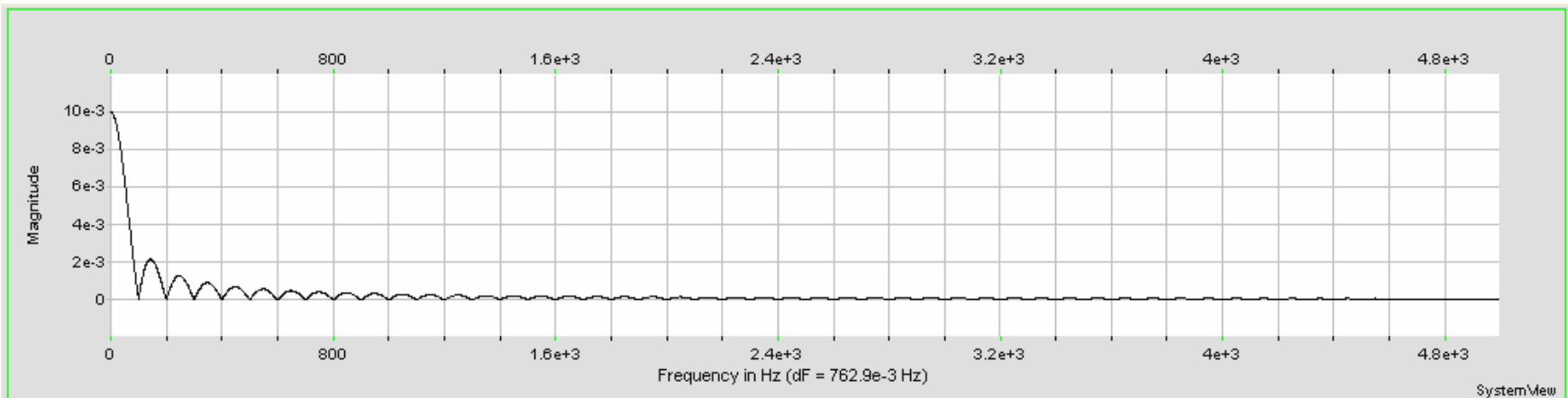




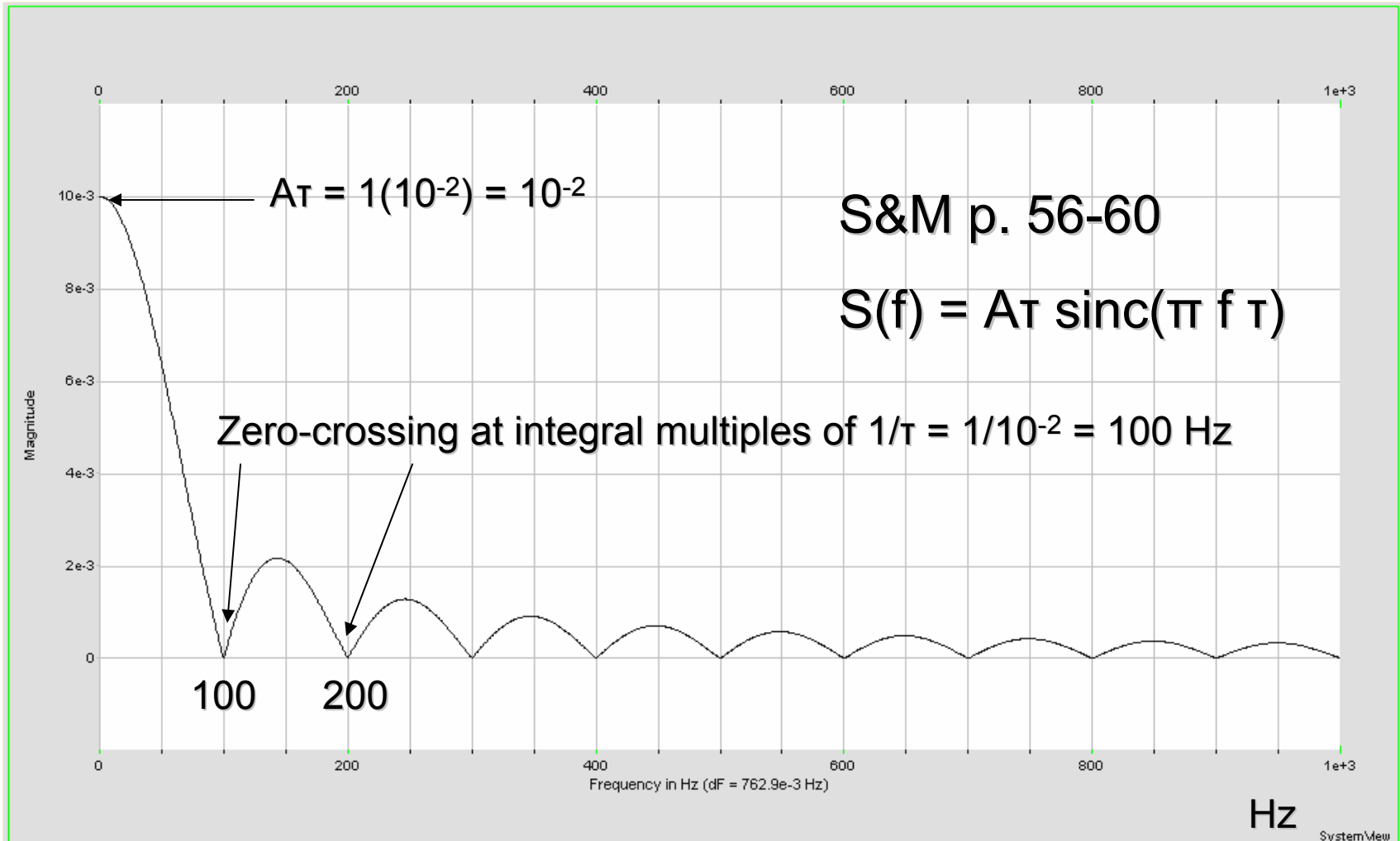
- Example 2.12a Simulated single pulse



- Magnitude of the Fourier Transform | FFT |



- Example 2.12a Simulated single pulse | FFT |



- Properties of the Fourier Transform

Linearity

$$af_1(t) + bf_2(t) \xLeftrightarrow{\mathcal{F}} aF_1(\omega) + bF_2(\omega)$$

Convolution

$$f_1(t) * f_2(t) \xLeftrightarrow{\mathcal{F}} F_1(\omega)F_2(\omega)$$

Conjugation

$$\overline{f(t)} \xLeftrightarrow{\mathcal{F}} \overline{F(-\omega)}$$

Scaling

$$f(at) \xLeftrightarrow{\mathcal{F}} \frac{1}{|a|} F\left(\frac{\omega}{a}\right), \quad a \in \mathbb{R}, a \neq 0$$

Time reversal

$$f(-t) \xLeftrightarrow{\mathcal{F}} F(-\omega)$$

Time shift

$$f(t - t_0) \xLeftrightarrow{\mathcal{F}} e^{-i\omega t_0} F(\omega)$$

- Properties of the Fourier Transform

Modulation principle

Modulation (multiplication by complex exponential)

$$f(t) \cdot e^{i\omega_0 t} \xLeftrightarrow{\mathcal{F}} F(\omega - \omega_0) \quad \omega_0 \in \mathbb{R},$$

Multiplication by  $\sin \omega_0 t$

$$f(t) \sin \omega_0 t \xLeftrightarrow{\mathcal{F}} \frac{i}{2} [F(\omega + \omega_0) - F(\omega - \omega_0)]$$

Multiplication by  $\cos \omega_0 t$

$$f(t) \cos \omega_0 t \xLeftrightarrow{\mathcal{F}} \frac{1}{2} [F(\omega + \omega_0) + F(\omega - \omega_0)]$$

Integration

$$\int_{-\infty}^t f(u) du \xLeftrightarrow{\mathcal{F}} \frac{1}{i\omega} F(\omega) + \pi F(0) \delta(\omega)$$

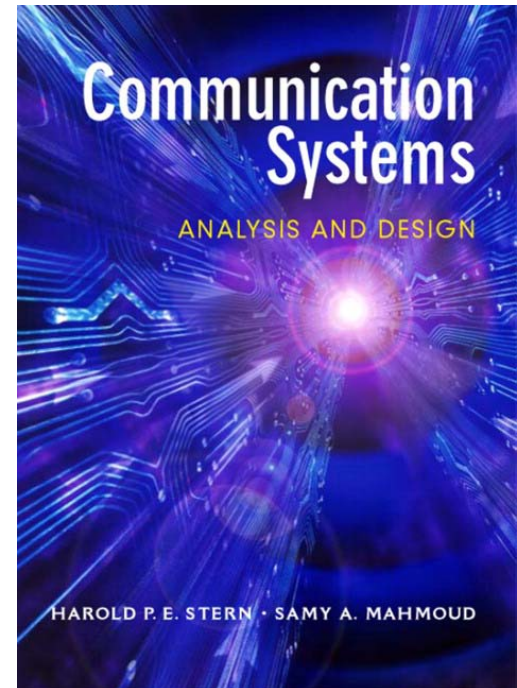
Parseval's theorem

$$\int_{\mathbb{R}} f(t) \cdot \overline{g(t)} dt = \int_{\mathbb{R}} F(\omega) \cdot \overline{G(\omega)} d\omega$$

## Chapter 2

# Frequency Domain Analysis

- *Normalized Energy Spectral Density*
- Pages 60-65



- Normalized Energy

If  $s(t)$  is a *non-periodic, finite energy signal* (a single pulse) then the average normalized power  $P_S$  is 0:

$$P_S = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} s^2(t) dt = \lim_{T \rightarrow \infty} \frac{\text{finite value}}{T} = 0 \quad V^2$$

$$E_S = \int_{-\infty}^{\infty} s^2(t) dt \quad V^2 \cdot \text{sec}$$

However, the normalized energy  $E_S$  for the same  $s(t)$  is *non-zero by definition* (S&M p. 60-61).

- Parseval's Energy Theorem

Parseval's energy theorem follows directly then from the discussion of power in a periodic signal:

$$E_s = \int_{-\infty}^{\infty} s^2(t) dt = \int_{-\infty}^{\infty} |S^2(f)| df \quad V^2 \cdot \text{sec}$$

- Energy Spectral Density

Analogous to the power spectral density is the energy spectral density (ESD)  $\psi(f)$ . For a linear, time-invariant (LTI) system with a transfer function  $H(f)$ , the output ESD which is the energy flow through the system is:

$$\psi_{\text{OUT}}(f) = \psi_{\text{IN}}(f) |H(f)|^2$$

- Energy Spectral Density

The energy spectral density (ESD)  $\psi(f)$  is the magnitude squared of the Fourier transform  $S(f)$  of a pulse signal  $s(t)$ :

$$\psi(f) = | S(f) |^2$$

The ESD can be *approximated* by the magnitude squared of the Fast Fourier Transform (FFT) in a *SystemVue* simulation as described in Chapter 3.

$$\psi(f) \approx | \text{FFT} |^2$$



# End of Chapter 2

# Frequency Domain Analysis

