

Foundation of Cryptography
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Interactive Proofs and Zero Knowledge

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Part I

Interactive Proofs

Interactive Vs. Interactive Proofs

Definition 1 (NP)

$\mathcal{L} \in \text{NP}$ iff $\exists \ell \in \text{poly}$ and poly-time algorithm V such that:

- $\forall x \in \mathcal{L} \cap \{0, 1\}^n$ there exists $w \in \{0, 1\}^{\ell(n)}$ s.t. $V(x, w) = 1$
- $V(x, \cdot) = 0$ for every $x \notin \mathcal{L}$

- *Non-interactive* proof
- Interactive proofs?

Interactive protocols

- Interactive algorithm
- Protocol $\pi = (A, B)$
- RV describing the parties joint output $\langle A(i_A), B(i_B) \rangle(i)$
- m -round algorithm, m -round protocol

Interactive Proofs

Definition 2 (Interactive Proof (IP))

A protocol (P, V) is an interactive proof for \mathcal{L} , if V is PPT and the following hold:

Completeness $\forall x \in \mathcal{L}, \Pr[\langle (P, V)(x) \rangle = \text{Accept}] \geq 2/3$

Soundness $\forall x \notin \mathcal{L}$, and *any* algorithm P^*
 $\Pr[\langle (P^*, V)(x) \rangle = \text{Accept}] \leq 1/3$

- $\text{IP} = \text{PSPACE}$
- We typically consider (and achieve) perfect completeness
- Negligible “soundness error” achieved via repetition.
- soundness only against PPT: *computationally sound proofs/interactive arguments*.
- efficient provers via “auxiliary input”

Section 1

IP for GNI

graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $G_0 = ([m], E_0)$ and $G_1 = ([m], E_1)$ are *isomorphic*, denoted $G_0 \equiv G_1$, if $\exists \pi \in \Pi_m$ such that

$(u, v) \in E_0$ iff $(\pi(u), \pi(v)) \in E_1$.

$GI = \{(G_0, G_1) : G_0 \equiv G_1\}$.

- Assume reasonable mapping from graphs to strings
- $GI \in NP$
- Does $GNI = \{(G_0, G_1) : G_0 \not\equiv G_1\} \in NP$?
- We will show a simple interactive proof for GNI Idea: Beer tasting...

IP for GNI

Protocol 4 ((P, V))

Common input $G_0 = ([m], E_0), G_1 = ([m], E_1)$

- 1 V chooses $b \leftarrow \{0, 1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b) = \{(\pi(u), \pi(v)) : (u, v) \in E_b\}$ to P
- 2 P send b' to V (tries to set $b' = b$)
- 3 V accepts iff $b' = b$

Claim 5

The above protocol is IP for GNI, with perfect completeness and soundness error $\frac{1}{2}$.

Proving Claim 5

- Graph isomorphism is an equivalence relation (separates the set of all graph pairs into separate subsets)
- $([m], \pi(E_i))$ is a random element in $[G_i]$ — the equivalence class of G_i

Hence,

$$G_0 \equiv G_1: \Pr[b' = b] \leq \frac{1}{2}.$$

$$G_0 \not\equiv G_1: \Pr[b' = b] = 1 \text{ (i.e., } i \text{ can, possibly inefficiently, extracted from } \pi(E_i))$$



Part II

Zero knowledge Proofs

The concept of zero knowledge

- Proving w/o revealing any additional information.
- What does it mean?
Simulation paradigm.

Zero knowledge Proof

Definition 6 (computational ZK)

An interactive proof (P, V) is computational zero-knowledge proof (CZKP) for \mathcal{L} , if \forall PPT V^* , \exists PPT S such that

$$\{ \langle (P, V^*)(x) \rangle \}_{x \in \mathcal{L}} \approx_c \{ S(x) \}_{x \in \mathcal{L}}.$$

Perfect ZK (PZKP)/statistical ZK (SZKP) – the above dist. are identically/statistically close, even for *unbounded* V^* .

- 1 ZK is a property of the prover.
- 2 ZK only required to hold with respect to true statements.
- 3 wlg. V^* 's outputs is its "view".
- 4 Trivial to achieve for $\mathcal{L} \in \text{BPP}$
- 5 Extension: auxiliary input
- 6 The "standard" NP proof is typically not zero knowledge
- 7 Next class — ZK for all NP

Section 2

ZK Proof for GI

ZK Proof for Graph Isomorphism

Idea: route finding

Protocol 7 ((P, V))

Common input $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

P's input a permutation π such that $\pi(E_1) = E_0$

- ➊ P chooses $\pi' \leftarrow \Pi_m$ and sends $E = \pi'(E_0)$ to V
- ➋ V sends $b \leftarrow \{0, 1\}$ to P
- ➌ if $b = 0$, P sets $\pi'' = \pi'$, otherwise, it sends $\pi'' = \pi' \circ \pi$ to V
- ➍ V accepts iff $\pi''(E_b) = E$

Claim 8

The above protocol is SZKP for GI, with perfect completeness and soundness $\frac{1}{2}$.

Proving Claim 8

Completeness Clear

Soundness If exist $j \in \{0, 1\}$ for which $\nexists \pi' \in \Pi_m$ with $\pi'(E_j) = E$, then V rejects w.p. at least $\frac{1}{2}$.
Assuming V rejects w.p. less than $\frac{1}{2}$ and lett π_0 and π_1 be the values guaranteed by the above observation (i.e., mapping E_0 and E_1 to E respectively).

Then $\pi_0^{-1}(\pi_1(E_1)) = \pi_0 \implies (G_0, G_1) \in \text{GI}$.

ZK Idea: for $(G_0, G_1) \in \text{GI}$, it is easy to generate a random transcript for Steps 1-2, and to be able to open it with prob $\frac{1}{2}$.

The simulator

For a start we consider a deterministic cheating verifier V^* that never aborts.

Algorithm 9 (S)

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

Do $|x|$ times:

- 1 Choose $b' \leftarrow \{0, 1\}$ and $\pi \leftarrow \Pi_m$, and “send” $\pi(E_{b'})$ to $V^*(x)$.
- 2 Let b be V^* ’s answer. If $b = b'$, send π to V^* , output V^* ’s output and halt.
Otherwise, rewind the simulation to its first step.

Abort

Claim 10

$$\{\langle (P, V^*)(x) \rangle\}_{x \in \text{GI}} \approx \{S(x)\}_{x \in \text{GI}}$$

Proving Claim 10

Algorithm 11 (S')

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

Do $|x|$ times:

- 1 Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $V^*(x)$.
- 2 Let b be V^* 's answer.
W.p. $\frac{1}{2}$, find π' such that $E = \pi'(E_b)$ and send it to V^* ,
output V^* 's output and halt.
Otherwise, rewind the simulation to its first step.

Abort

Claim 12

$S(x) \equiv S'(x)$ for any $x \in \text{GI}$.

Proof: ?

Proving Claim 10 cont.

Algorithm 13 (S'')

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

- 1 Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $V^*(x)$.
- 2 Find π' such that $E = \pi'(E_b)$, send it to V^* , output V^* 's output and halt.

Claim 14

$\forall x \in \text{GI}$ it holds that

- 1 $\langle (P, V^*(x)) \rangle \equiv S''(x)$.
- 2 $\text{SD}(S''(x), S'(x)) \leq 2^{-|x|}$.

Proof: ? (1) is clear.

Proving Claim 14(2)

Fix (E, π') and let $\alpha = \Pr_{S''}[(E, \pi')]$.

It holds that

$$\begin{aligned}\Pr_{S'}[(E, \pi')] &= \alpha \cdot \sum_{i=1}^{|x|} \left(1 - \frac{1}{2}\right)^{i-1} \cdot \frac{1}{2} \\ &= (1 - 2^{-|x|}) \cdot \alpha\end{aligned}$$

Hence, $\text{SD}(S''(x), S'(x)) \leq 2^{-|x|} \square$

Remarks

- 1 Randomized verifiers
- 2 Aborting verifiers – Normalize aborting probability
- 3 Auxiliary input
- 4 Negligible soundness error? Sequential/Parallel composition
- 5 Perfect ZK for “expected time simulators”
- 6 “Black box” simulation

Section 3

Black-box ZK

Black-box simulators

Definition 15 (Black-box simulator)

(P, V) is CZKP with black-box simulation for \mathcal{L} , if \exists oracle-aided PPT S s.t. for every deterministic polynomial-time^a V^* :

$$\{(P(w_x), V^*(z))(x)\}_{x \in \mathcal{L}} \approx_c \{S^{V^*(x, z_x)}(x)\}_{x \in \mathcal{L}}$$

for any $\{(w_x, z_x) \in R_{\mathcal{L}}(x) \times \{0, 1\}^*\}_{x \in \mathcal{L}}$.

Prefect and statistical variants are defined analogously.

^aLength of auxiliary input does not count for the running time.

- 1 "Most simulators" are black box
- 2 Strictly weaker than general simulation!

Section 4

Zero Knowledge for all NP

CZKP for 3COL

- Assuming that OWFs exists, we give a CZKP for 3COL .
- We show how to transform it for any $\mathcal{L} \in \text{NP}$ (using that $3\text{COL} \in \text{NPC}$).

Definition 16 (3COL)

$G = (M, E) \in 3\text{COL}$, if $\exists \phi: M \mapsto [3]$ s.t. $\phi(u) \neq \phi(v)$ for every $(u, v) \in E$.

We use commitment schemes.

The protocol

Let π_3 be the set of all permutations over $[3]$. We use perfectly binding commitment Com (statistically binding?).

Protocol 17 ((P, V))

Common input: Graph $G = (M, E)$ with $n = |G|$

P's input: a (valid) coloring ϕ of G

- 1 P chooses $\pi \leftarrow \Pi_3$ and sets $\psi = \pi \circ \phi$
- 2 $\forall v \in M$: P commits to $\psi(v)$ using $\text{Com}(1^n)$.
Let c_v and d_v be the resulting commitment and decommitment.
- 3 V sends $e = (u, v) \leftarrow E$ to P
- 4 P sends $(d_u, \psi(u)), (d_v, \psi(v))$ to V
- 5 V verifies that (1) both decommitments are valid, (2) $\psi(u), \psi(v) \in [3]$ and (3) $\psi(u) \neq \psi(v)$.

Claim 18

The above protocol is a CZKP for 3COL, with perfect completeness and soundness $1/|E|$.

Completeness: Clear

Soundness: Let $\{c_v\}_{v \in M}$ be the commitments resulting from an interaction of V with an arbitrary P^* .

Define $\phi: M \mapsto [3]$ as follows:

$\forall v \in M$: let $\phi(v)$ be the (single) value that it is possible to decommit c_v into (if not in $[3]$, set $\phi(v) = 1$).

If $G \notin 3COL$, then $\exists (u, v) \in E$ s.t. $\phi(u) \neq \phi(v)$.

Hence V rejects such x w.p. at least $1/|E|$

Proving ZK

Fix a deterministic, non-aborting V^* that gets no auxiliary input.

Algorithm 19 (S)

Input: A graph $G = (M, E)$ with $n = |G|$

Do $n \cdot |E|$ times:

- 1 Choose $e' = (u, v) \leftarrow E$. Set $\psi(u) \leftarrow [3]$,
 $\psi(v) \leftarrow [3] \setminus \{\psi(u)\}$, and $\psi(w) = 1$ for $w \in M \setminus \{u, v\}$
- 2 $\forall v \in M$: commit to $\psi(v)$ to V^* (resulting in c_v and d_v)
- 3 Let e be the edge sent by V^* .
If $e = e'$, send $(d_u, \psi(u)), (d_v, \psi(v))$ to V^* , output V^* 's
output and halt.
Otherwise, rewind the simulation to its first step.

Abort

Proving ZK cont.

Claim 20

$\{(P(w_x), V^*)(x)\}_{x \in 3\text{COL}} \approx_c \{S^{V^*(x)}(x)\}_{x \in 3\text{COL}}$, for any $\{w_x \in R_{3\text{COL}}(x)\}_{x \in 3\text{COL}}$.

Consider the following (inefficient simulator)

Algorithm 21 (S')

Input: $G = (V, E)$ with $n = |G|$

Find (using brute force) a valid coloring ϕ of G

Do $n \cdot |E|$ times

① Act as the honest prover does given private input ϕ

② Let e be the edge sent by V^* .

W.p. $1/|E|$, S' sends $(\psi(u), d_u), (\psi(v), d_v)$ to V^* , output V^* 's output and halt.

Otherwise, rewind the simulation to its first step.

Abort

Claim 22

$$\{S^{V^*(x)}(x)\}_{x \in 3\text{COL}} \approx_c \{S'^{V^*(x)}(x)\}_{x \in 3\text{COL}}$$

Proof: ?

Proving Claim 22

Assume \exists PPT D , $p \in \text{poly}$ and an infinite set $\mathcal{I} \subseteq 3\text{COL}$ s.t.

$$\left| \Pr[D(|x|, S^{V^*}(x)) = 1] - \Pr[D(|x|, S'^{V^*}(x)) = 1] \right| \geq 1/p(|x|)$$

for all $x \in \mathcal{I}$.

Hence, \exists PPT R^* and $b \neq b' \in [3]$ such that

$$\{\text{View}_{R^*}(S(b), R^*(x))(1^{|x|})\}_{x \in \mathcal{I}} \not\approx_c \{\text{View}_{R^*}(S(b'), R^*(x))(1^{|x|})\}_{x \in \mathcal{I}}$$

where S is the sender in Com .

We critically used the non-uniform security of Com

S' is a good simulator

Claim 23

$\{(P(w_x), V^*)(x)\}_{x \in 3\text{COL}} \approx_c \{S'^{V^*(x)}(x)\}_{x \in 3\text{COL}}$, for any $\{w_x \in R_{\text{GI}}(x)\}_{x \in 3\text{COL}}$.

Proof: ?

Remarks

- Aborting verifiers
- Auxiliary inputs
- Soundness amplification
- Non-uniform hiding guarantee

Extending to all $\mathcal{L} \in \text{NP}$

Let (P, V) be a CZKP for 3COL, and let Map_X and Map_W be two poly-time functions s.t.

- $\forall x \in \{0, 1\}^*: x \in \mathcal{L} \iff \text{Map}_X(x) \in 3\text{COL},$
- $\forall x \in \mathcal{L} \text{ and } w \in R_L(x): \text{Map}_W(x, w) \in R_{3\text{COL}}(\text{Map}_X(x))$

Protocol 24 $((P_{\mathcal{L}}, V_{\mathcal{L}}))$

Common input: $x \in \{0, 1\}^*$

$P_{\mathcal{L}}$'s input: $w \in R_{\mathcal{L}}(x)$

- 1 The two parties interact in $\langle (P(\text{Map}_W(x, w)), V(\text{Map}_X(x))) \rangle$, where $P_{\mathcal{L}}$ and $V_{\mathcal{L}}$ taking the role of P and V respectively.
- 2 $V_{\mathcal{L}}$ accepts iff V accepts in the above execution.

Extending to all $\mathcal{L} \in \text{NP}$ cont.

Claim 25

$(P_{\mathcal{L}}, V_{\mathcal{L}})$ is a CZKP for \mathcal{L} with the same completeness and soundness as (P, V) as for 3COL.

- **Completeness and soundness:** Clear.
- **Zero knowledge:** Let S (an efficient) ZK simulator for (P, V) (for 3COL).
 Define $S_{\mathcal{L}}(x)$ to output $S(\text{Map}_X(x))$, while replacing the string $\text{Map}_X(x)$ in the output of S with x .
 $\{(P(w_x), V^*)(x)\}_{x \in \mathcal{L}} \not\approx_c \{S_{\mathcal{L}}^{V^*(x)}(x)\}_{x \in \mathcal{L}}$ for some $V_{\mathcal{L}}^*$,
 implies $\{(P(\text{Map}_W(x, w_x)), V^*)(x)\}_{x \in 3\text{COL}} \not\approx_c \{S^{V^*(x)}(x)\}_{x \in 3\text{COL}}$,
- $V^*(x)$: find $x^{-1} = \text{Map}_X^{-1}(x)$ and act like $V_{\mathcal{L}}^*(x^{-1})$