Application of Information Theory, Lecture 1
Basic Definitions and Facts

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- Entropy is a function of p (sometimes refers to as $H(p)$).

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Examples

1. $X \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$:
   (i.e., for some $x_1 \neq x_2 \neq x_3$, $\mathrm{P}_X(x_1) = \frac{1}{2}, \mathrm{P}_X(x_2) = \frac{1}{4}, \mathrm{P}_X(x_3) = \frac{1}{4}$)
   $H(X) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{4}\log\frac{1}{4} = \frac{1}{2} + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 = 1\frac{1}{2}$.
2. $H(X) = H(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.
3. X is uniformly distributed over $\{0,1\}^n$:
   $H(X) = -\sum_{i=1}^{2^n} \frac{1}{2^n}\log\frac{1}{2^n} = -\log\frac{1}{2^n} = n$.
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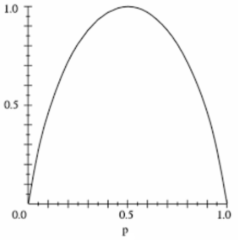

Axiomatic derivation of the entropy function

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Shannon function is the only *symmetric* function (over probability distributions) satisfying the following three axioms:

A1 Continuity: $H(p, 1-p)$ is continuous function of p.

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A3 Grouping axiom:
$H(p_1, p_2, \ldots, p_m) = H(p_1 + p_2, p_3, \ldots, p_m) + (p_1 + p_2)H(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2})$

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Let H be a symmetric function that satisfying the above axioms.

We prove (assuming additional axiom) that H is the Shannon function.

Generalization of the grouping axiom

Fix $p = (p_1, \ldots, p_m)$ and let $S_k = \sum_{i=1}^{k} p_i$.

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Claim follows by combining the above equations. □

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Let $1 = k_1 < k_2 < \ldots < k_q < m$ and let $C_t = \sum_{i=k_t}^{k_{t+1}-1} p_i$ (letting $k_{q+1} = m+1$).

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- Proof extends to any integer (not only 3)

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- Hence, $H(p_1, \ldots, p_m) = \sum_i p_i \log \frac{1}{p_i} \leq \log \sum_i p_i \frac{1}{p_i} = \log m$
- Alternatively, for X over $\{1, \ldots, m\}$,
  $H(X) = \mathrm{E}_X \log \frac{1}{\mathrm{P}_X(X)} \leq \log \mathrm{E}_X \frac{1}{\mathrm{P}_X(X)} = \log m$

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- If g is injective, then $H(Y) = H(X)$.

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- If g is injective, then $H(Y) = H(X)$.

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- $H(X) = H(2^X)$.
- $H(X) < H(\cos(X))$, if $0, 2\pi \in \mathrm{Supp}(X)$.

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