Application of Information Theory, Lecture 6

Counting

Handout Mode

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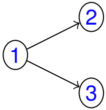
Section 1

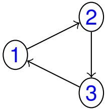
Graph Homomorphisms

Counting # of graph homomorphisms

- $T = (V_T, E_T)$ — directed graph (no self loops)
- $G = (V_G, E_G)$



- $H = (V_H, E_H)$



- (x_1, x_2, x_3) is an homomorphism of G in T, if $x_1, x_2, x_3 \in V_T$ and $(i, j) \in E_G \implies (x_i, x_j) \in E_T$ (might be $x_1 = x_2$)
- Example: see board
- $\mathrm{Hom}(X, T)$: all homomorphisms of X in T
- Claim: $|\mathrm{Hom}(H, T)| \leq |\mathrm{Hom}(G, T)|$
- Trivial if G would be a subgraph of H
- Special case of a more general theorem

Proving the claim

- $(X_1, X_2, X_3) \leftarrow \mathrm{Hom}(H, T)$
- $$\begin{aligned}\log|\mathrm{Hom}(H,T)| &= H(X_1,X_2,X_3)\\ &= H(X_1) + H(X_2|X_1) + H(X_3|X_1,X_2)\\ &\leq H(X_1) + H(X_2|X_1) + H(X_3|X_2)\\ &= H(X_1) + 2\cdot H(X_2|X_1) \qquad \text{(by symmetry of } H\text{)}\end{aligned}$$

- Let $D_2(x)$ be the distribution of $X_2|X_1 = x$, and let $X_2' \sim D_2(X_1)$
- $$\begin{aligned}H(X_1,X_2,X_2') &= H(X_1) + H(X_2|X_1) + H(X_2'|X_1,X_2)\\ &= H(X_1) + H(X_2|X_1) + H(X_2'|X_1)\\ &= H(X_1) + 2\cdot H(X_2|X_1)\end{aligned}$$

- $(X_1, X_2) \in E_T$ and $(X_1, X_2') \in E_T$

$\Longrightarrow$ $(X_1, X_2, X_2') \in \mathrm{Hom}(G, T)$

$\Longrightarrow$ $H(X_1, X_2, X_2') \leq \log|\mathrm{Hom}(G, T)|$

$\Longrightarrow$ $\log|\mathrm{Hom}(H, T)| \leq \log|\mathrm{Hom}(G, T)|$. $\square$

Section 2

Perfect Matchings. Skipped

Bregman's theorem

For bipartite graph $G = (A, B, E)$, let $d(v) = |N(v) = \{u \in B\colon (v, u) \in E\}|$

Theorem 1

Let $G = (A, B, E)$ be bi-partite graph with $|A| = |B|$, and let $\mathcal{M}$ be the perfect matchings in G. Then $|\mathcal{M}| \leq \prod_{v \in A} (d(v)!)^{1/d(v)}$.

- Let $A = B = [n] = \{1, \ldots, n\}$, and for $m \in \mathcal{M}$ let $m(i)$ be the node in B matched with i by m.
- It is clear that $|\mathcal{M}| \leq \prod_{i \in [n]} d(i)$:
- Let $M \leftarrow \mathcal{M}$. Hence,

$$\begin{aligned}
\log |\mathcal{M}| = H(M) &= H(M(1)) + H(M(2)|M(1)) + \ldots + H(M(n)|M(1), \ldots, M(n-1)) \\
&\leq H(M(1)) + H(M(2)) + \ldots + H(M(n)) \\
&\leq \log d(1) + \log d(2) + \ldots + \log d(n) \\
&= \log \prod_{i \in [n]} d(i)
\end{aligned}$$

Proving Bregman's theorem

- Key observations:
  $H(M(i)|M(1),\ldots,M(i-1)) \leq \log|N(i)\setminus\{M(1),\ldots,M(i-1)\}|$
- Let $\mathcal{P}$ be the set of all permutation over $[n]$. For $p\in\mathcal{P}$:
  $H(M) = H(M(p(1))) + \ldots + H(M(p(n))|M(p(1)),\ldots,M(p(n-1)))$
- $\mathcal{S}_p(i) = \{j\in[n]\colon p^{-1}(j) < p^{-1}(i)\}$ — matchings proceeding i w.r.t. p
- $H(M) = \sum_{i=1}^{n} H(M(i)|M(\mathcal{S}_p(i)))$
- For $m\in\mathcal{M}$ and $P\leftarrow\mathcal{P}$: $|N(i)\setminus m(\mathcal{S}_P(i))|$ is uniform over $\{1,\ldots,d(i)\}$

$\implies$ $\mathrm{E}_P\left[H(M(i)\mid M(\mathcal{S}_P(i)))\right] \leq \frac{1}{d(i)}\sum_{k=1}^{d(i)}\log k = \log\left((d(i)!)^{1/d(i)}\right)$

$\implies$

$$H(M) = \mathop{\mathrm{E}}_{P}\left[\sum_{i=1}^{n} H(M(i)|M(\mathcal{S}_P(i)))\right] = \sum_{i=1}^{n}\mathop{\mathrm{E}}_{P}\left[H(M(i)|\mathcal{S}_P(i))\right]$$
$$\leq \log\prod_{i\in[n]}\left((d(i)!)^{1/d(i)}\right). \qquad \square$$

Section 3

Shearer's Lemma

$H(X_1, X_2, X_3)$ Vs. $H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)$

- How does $H(X_1, X_2, X_3)$ compares to $H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)$?
- If X_1, X_2, X_3 are independent, then $H(X_1, X_2, X_3) = \frac{1}{2}\left(H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)\right)$
- In general: $H(X_1, X_2, X_3) \le \frac{1}{2}\left(H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)\right)$
- A tighter bound than $H(X_1) + H(X_2) + H(X_3)$
- Proof:

$$\begin{aligned}
2H(X_1, X_2, X_3) &= 2H(X_1) && +2H(X_2|X_1) && +2H(X_3|X_1, X_2)\\
H(X_1, X_2) &= H(X_1) && +H(X_2|X_1) && \\
H(X_2, X_3) &= && +H(X_2) && +H(X_3|X_2)\\
H(X_1, X_3) &= H(X_1) && && +H(X_3|X_1)
\end{aligned}$$

- but

$$\begin{aligned}
H(X_2|X_1) &\le H(X_2)\\
H(X_3|X_1, X_2) &\le H(X_3|X_1)\\
H(X_3|X_1, X_2) &\le H(X_3|X_2)
\end{aligned}$$

Shearer's lemma

- Let $X = (X_1, \ldots, X_n)$
- For $\mathcal{S} = \{i_1, \ldots, i_k\} \subseteq [n]$, let $X_{\mathcal{S}} = (X_{i_1}, \ldots, X_{i_k})$
- Example: $X_{1,3} = (X_1, X_3)$

Lemma 2 (Shearer's lemma)

Let $X = (X_1, \ldots, X_n)$ be a rv and let $\mathcal{F}$ be a family of subset of $[n]$ s.t. each $i \in [n]$ appears in at least m subset of $\mathcal{F}$. Then $H(X) \leq \frac{1}{m} \sum_{F \in \mathcal{F}} H(X_F)$.

Proof:

- $H(X) = \sum_{i=1}^{n} H(X_i | \{X_\ell \colon \ell < i\})$
- $H(X_F) = \sum_{i \in F} H(X_i | \{X_\ell \colon \ell < i \wedge \ell \in F\})$
- Let $F_{i,j}$ be the j'th family that contains X_i. Hence,

$$\begin{aligned}\sum_{F \in \mathcal{F}} H(X_F) &\geq \sum_{i=1}^{n} \sum_{j=1}^{m} H(X_i | \{X_\ell \colon \ell < i \wedge \ell \in F_{i,j}\}) \\ &\geq m \cdot \sum_{i=1}^{n} H(X_i | \{X_\ell \colon \ell < i\}) = m \cdot H(X)\end{aligned}$$

Corollary

Corollary 3

Let $\mathcal{F} = \{F \subseteq [n] \colon |F| = k\}$. *Then*
$H(X) \leq \frac{n}{k} \cdot \frac{1}{\binom{n}{k}} \cdot \sum_{F \in \mathcal{F}} H(X_F) = \frac{n}{k} \cdot \mathrm{E}_{F \leftarrow \mathcal{F}}\left[H(X_F)\right]$.

Proof: $\frac{k}{n} \cdot \binom{n}{k}$ is the # of times i appears in $\mathcal{F}$.

Implications:

- Let $Q \subseteq \{0,1\}^n$ and $X = (X_1, \ldots, X_n) \leftarrow Q$
- $|Q| \leq 2^{\frac{n}{k} \cdot \mathrm{E}_{F \leftarrow \mathcal{F}}[H(X_F)]}$
- $\mathrm{E}_F\left[H(X_F)\right]$ is small $\implies$ Q is small
- Q is large $\implies$ $\mathrm{E}_F\left[H(X_F)\right]$ is large

Example

- $Q \subseteq \{0,1\}^n$ with $|Q| = 2^n/2 = 2^{n-1}$; $X \leftarrow Q$.
- $\mathcal{F} = \{F \subseteq [n] \colon |F| = k\}$
- By Corollary 3, $\log |Q| = n - 1 \leq \frac{n}{k} \cdot \mathrm{E}_{F \leftarrow \mathcal{F}}[H(X_F)]$

$$\implies \mathrm{E}_F[H(X_F)] \geq k(1 - \tfrac{1}{n}) = k - \tfrac{k}{n}$$

$$\implies \exists F \in \mathcal{F} \text{ s.t. } H(X_F) \geq k - \tfrac{k}{n}$$

- Assume $n = 1000$ and $k = 5$, hence $H(X_F) \geq 5 - \frac{1}{200}$
- X_F takes at least $2^{5-\frac{1}{200}} = 2^{-\frac{1}{200}} \cdot 2^5 > 31$ (and hence 32) values
- Stronger conclusion: X_F is close to the uniform distribution.

More generally

- $|Q| \geq \frac{1}{2^d} \cdot 2^n$; $X \leftarrow Q$
- $\mathcal{F} = \{F \subseteq [n] \colon |F| = k\}$
- $n - d \leq H(X) \leq \frac{n}{k} \cdot \frac{1}{|\mathcal{F}|} \cdot \sum_{F \in \mathcal{F}} H(X_F)$

$\implies \quad \frac{1}{|\mathcal{F}|} \cdot \sum_{F \in \mathcal{F}} H(X_F) \geq k - \frac{dk}{n}$

$\implies \quad \mathrm{E}_{F \leftarrow \mathcal{F}} [H(X_F)] \geq k - \frac{dk}{n}$

- If $dk << n$, then $\exists F \in \mathcal{F}$ s.t. X_F is close to the uniform distribution (over k bits)

Section 4

Gold Coins

of gold coins in a cube

- Q — (finite) set of points in $\mathbb{R}^3$
  - $|Q_{x,y}| = 6$ (i.e., # of projection points of Q on xy)
  - $|Q_{x,z}| = 8$
  - $|Q_{y,z}| = 12$
- Can we bound $|Q|$?
- The real story
- $X = (X_1, X_2, X_3) \leftarrow Q$
- $$\begin{aligned}\log|Q| = H(X) &\leq \frac{1}{2}(H(X_1, X_2) + H(X_1, X_3) + H(X_2, X_3))\\ &\leq \frac{1}{2}(\log 6 + \log 8 + \log 12)\\ &= \frac{1}{2}(\log 6 \cdot 8 \cdot 12).\end{aligned}$$
- Hence, $|Q| \leq \sqrt{6 \cdot 8 \cdot 12} = 24$
- Can it be 24 ? What is the minimal number?

of gold coins, the hyperspace case

- Q — (finite) set of points in $\mathbb{R}^n$
- m_i— # of coins in projection on $(1, \ldots, i-1, i+1, \ldots, n)$
- Claim: $|Q| \leq (\prod_{i \in [n]} m_i)^{1/(n-1)}$
- Proof: $X = (X_1, \ldots, X_n) \leftarrow Q$, $X_{-i} = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$
- $\log |Q| = H(X) \leq \frac{1}{n-1} \sum_i H(X_{-i}) \leq \frac{1}{n-1} \sum_i \log m_i$

Section 5

Independent Sets

of independent sets in bipartite graphs

Theorem 4

Let $G = (A, B, E)$ be an n-regular bipartite graph with $|A| = |B| = m$. Then the number of independent sets in G is at most $2^{m(1+1/n)}$.

Proof: $\mathcal{I}$ — set of independent sets in G.

- Let $I \leftarrow \mathcal{I}$, let $X_v = 1$ iff $v \in I$, and $X_{\mathcal{S}} = \{X_v \colon v \in \mathcal{S}\}$.
- $$\begin{aligned} H(I) &= H(X_A|X_B) + H(X_B) \\ &\leq \sum_{v \in A} H(X_v|X_B) + \frac{1}{n}\sum_{v \in A} H(X_{N(v)}) \qquad \text{(rhs by Sherer's Lemma)} \\ &\leq \sum_{v \in A}\left(H(X_v|X_{N(v)}) + \frac{1}{n} H(X_{N(v)})\right) \end{aligned}$$
- Fix $v \in A$. Let $\chi_v = \begin{cases} 0, & X_{N(v)} = 0^{|N(v)|} \\ 1, & \text{otherwise.} \end{cases}$, and $p = p(v) = \Pr[\chi_v = 0]$
- $H(X_v|X_{N(v)}) \leq H(X_v|\chi_v) \leq p$
- $H(X_{N(v)}) = H(\chi_v, X_{N(v)}) = H(\chi_v) + H(X_{N(v)}|\chi_v) \leq h(p) + (1-p)\log(2^n - 1)$
- Hence $H(I) \leq \sum_{v \in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1 - p(v))\log(2^n - 1)\right)$

of independent sets in bipartite graphs, cont.

- $\log |\mathcal{I}| = H(I) \leq \sum_{v \in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1 - p(v)) \log(2^n - 1)\right)$
- Let $f(t) := t + \frac{1}{n}\left(h(t) + (1 - t) \log(2^n - 1)\right)$
- By calculus, $\max_{t \in [0,1]} f(t) = \frac{1}{n} \log(2^{n+1} - 1)$
- Hence, $\log |\mathcal{I}| \leq \frac{m}{n} \log(2^{n+1} - 1)$. $\square$

Section 6

Intersecting Graphs, Skipped

Another corollary of Shearer's lemma

Corollary 5

Let $\mathcal{A}$ and $\mathcal{F}$ be collections of subsets of $[n]$, and for $F \in \mathcal{F}$ let $\mathcal{A}_F$ be the collection $\{A \cap F \colon A \in \mathcal{A}\}$. Assume that each element of $[n]$ appears in at least m subsets of $\mathcal{F}$, then $|\mathcal{A}|^m \leq \prod_{F \in \mathcal{F}} |\mathcal{A}_F|$.

Proof:

- Let $X = (X_1, \ldots, X_n) \leftarrow \mathcal{A}$.
- $\log |\mathcal{A}_F| \geq H(X_F)$ ($\mathrm{Supp}(X_F) \subseteq \mathcal{A}_F$)
- By Shearer's lemma, $\log |\mathcal{A}| = H(X) \leq \frac{1}{m} \sum_{F \in \mathcal{F}} H(X_F)$. $\square$

of intersecting graphs

Theorem 6

Let $\mathcal{G}$ be a family of graphs over $[n]$, s.t. $G \cap G'$ contains a triangle for each $G, G' \in \mathcal{G}$. Then $|\mathcal{G}| \leq 2^{\binom{n}{2}-2}$.

This improves over $|\mathcal{G}| \leq 2^{\binom{n}{2}-1}$, which follows from $G \cap G' \neq \emptyset$.
(wlg. all graph shares the same edge)

Proof:

- We focus on even n, and view graphs over $[n]$ as subsets $[\binom{n}{2}]$
- For $\frac{n}{2}$-size set $\mathcal{S} \subset [n]$, let $F = F(\mathcal{S})$ be union of the cliques $\mathcal{S}$ and $[n] \setminus \mathcal{S}$
- $F \cap G \cap G' \neq \emptyset$, for any $G, G' \in \mathcal{G}$ and $\mathcal{S}$ as above
- Hence $|\mathcal{G}_F := G \cap F \colon G \in \mathcal{G}| \leq 2^{|F|-1}$
- Let $m = \binom{n}{2}$ and $m' = |F| = n(\frac{n}{2} - 1)$
- Each edge over $[n] \times [n]$, appears in $\frac{m'}{m}$ of graphs $\{F(\mathcal{S})\}_{\mathcal{S} \subset [n] \colon |\mathcal{S}| = \frac{n}{2}}$.
- By Corollary 5, $|\mathcal{G}|^{\frac{m'}{m} \cdot \binom{n}{n/2}} \leq (2^{m'-1})^{\binom{n}{n/2}}$
- Hence, $|\mathcal{G}| \leq 2^{m - \frac{m}{m'}} \leq 2^{\binom{n}{2}-2}$