

Problem set 4*May 22, 2018*

Due: May 24

- Please submit the handout in class, or email the grader (quefumas at gmail.com).
- Write clearly and shortly using sub-claims if needed. The emphasize in most questions is on the proofs (no much point is writing a “solution” w/o proving its correctness)
- For Latex users, a solution example can be found in the course web site.
- It is allowed to work in (small) groups, but please write the id list of your partners in the solution file, and each student should write his solution by *himself* (joint effort is only allowed in the “thinking phase”)

1. Show that $\text{SD}(p, q) = \max_{S \subseteq [m]} (\sum_{i \in S} p_i - \sum_{i \in S} q_i)$ for any two distributions p, q over $[m]$.
2. Use the above to prove that $\text{SD}(p, q) = \max_D (\Pr_{X \sim p} [D(X) = 1] - \Pr_{X \sim q} [D(X) = 1])$, where the max is over all deterministic distinguishers. Try and extend the above to randomized distinguishers?
3. Relative entropy is not symmetric: give two distributions p, q such that $D(p||q) \neq D(q||p)$, and $D(p||q), D(q||p) < \infty$.
4. Relative entropy does not obey the triangle inequality: give three distributions p_1, p_2, p_3 such that $D(p_1||p_2) + D(p_2||p_3) < D(p_1||p_3)$
5. Relative entropy is non-negative: given two distributions p, q , show that $D(p||q) \geq 0$, with equality only if $p = q$.
6. Does Theorem 4 in Lecture 7 hold for any prefix code C with $\mathbb{E}_{i \leftarrow q} [|C(i)|] \leq H(q) + 1$?
(and not only for a code C with $C(x) = \left\lceil \log \frac{1}{q(x)} \right\rceil$, as stated in the theorem)
7. Prove the data processing inequality of relative entropy (Claim 7 in Lecture 7) for randomized functions.
8. Let U_1, \dots, U_k be uniform independent bits, let $\varepsilon > 0$ and let W be the event that at least $k(\frac{1}{2} + \varepsilon)$ of these bits are ones. Use Thm 10 in Lecture 7 to show that $\Pr [W] \leq 2^{-kc \cdot \varepsilon^2}$ for some universal (i.e., independent of n and ε) $c > 0$. BTW, does this inequality look familiar to you?