Application of Information Theory, Lecture 10

Hardcore Predicates

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Part I

Motivation and Definition

Hardcore predicates

- Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a "hard to invert" function, how unpredictable is x given $f(x)$

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- Parts of x might be (totally) predictable
- It turns out that there is an hardcore part in x.

Hardcore predicates, cont.

Definition 1 (hardcore predicates)

A predicate $b\colon \{0,1\}^n \mapsto \{0,1\}$ is (s,ε)-hardcore predicate of $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, if $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{P}(f(x)) = b(x)] \leq \frac{1}{2} + \varepsilon$, for any s-size P.

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- Is there a generic hardcore predicate for all hard to invert functions?

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- Is there a generic hardcore predicate for all hard to invert functions?

  Let f be a function and let b be a predicate, then b is typically not a hard-core predicate of $g(x) = (f(x), b(x))$.

Part II

The Information Theoretic Settings

Some definitions

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- In both examples $\mathrm{H}_\infty(Z) = k$

2-universal families

Definition 2 (2-universal families)

A function family $\mathcal{G} = \{g \colon \mathcal{D} \mapsto \mathcal{R}\}$ is 2-universal, if $\forall\ x \neq x' \in \mathcal{D}$ it holds that $\Pr_{g \leftarrow \mathcal{G}}[g(x) = g(x')] = \frac{1}{|\mathcal{R}|}$.

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Example: $\mathcal{D} = \{0,1\}^n$, $\mathcal{R} = \{0,1\}^m$ and $\mathcal{G} = \{A \in \{0,1\}^{m \times n}\}$ with $A(x) = A \times x \bmod 2$.

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Lemma 3 (leftover hash lemma)

Let X be a rv over $\{0,1\}^n$ with $\mathrm{H}_2(X) \geq k$ let $\mathcal{G} = \{g \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ be 2-universal and let $G \leftarrow \mathcal{G}$. Then $\mathrm{SD}((G, G(X)), (G, \sim \{0,1\}^m)) \leq \frac{1}{2} \cdot 2^{(m-k)/2}$.

Hardcore predicate for regular functions

Lemma 4

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be 2^k-regular function, let $\mathcal{G} = \{g\colon \{0,1\}^n \mapsto \{0,1\}\}$ be 2-universal and let $v\colon \{0,1\}^n \times \mathcal{G} \mapsto \{0,1\}^n \times \mathcal{G}$ be defined by $v(x,g) = (f(x), g)$.
Then $b(x,g) = g(x)$ is $(\infty, 2^{-(k-1)/2})$ hardcore-predicated of v.

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- b is an hardcore predicate of v (not of f)

Proving Lemma 4

Claim 5

$\mathrm{SD}\left((f(X), G, G(X)), (f(X), G, U)\right) \leq 2^{-(k-1)/2}$,
for $G \leftarrow \mathcal{G}$, $X \leftarrow \{0,1\}^n$ and $U \leftarrow \{0,1\}$.

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Lemma 6 (predicting to distinguishing)

Let (Y, Z) be rv over $\{0,1\}^* \times \{0,1\}$ and let P be an algorithm with $\Pr\left[\mathsf{P}(Y) = Z\right] \geq \frac{1}{2} + \varepsilon$. Then $\exists$ algorithm D, with essentially the same complexity as P, with $\Pr\left[\mathsf{D}(Y, Z) = 1\right] - \Pr\left[\mathsf{D}(Y, U) = 1\right] \geq \varepsilon$.

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Proof:

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Proof: $\mathsf{D}(y, z)$ outputs 1 if $\mathsf{P}(y) = z$ and 0 otherwise.$\square$

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Corollary 7

If $\mathrm{SD}((Y, Z), (Y, U)) < \varepsilon$, then $\Pr\left[\mathsf{P}(Y) = Z\right] < \frac{1}{2} + \varepsilon$ for any predictor P.

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Since $\mathrm{H}_\infty(X_y) = k$ for every $y \in Im(f)$, the leftover hash lemma yields that

$$
\begin{aligned}
\mathrm{SD}((G, G(X_y)), (G, U)) \leq& \frac{1}{2} \cdot 2^{(1 - \mathrm{H}_\infty(X_y)))} \\
&= 2^{(-k-1)/2}. \square
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Part III

The Computational Settings

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- f is hard $\implies$ predicting x from $f(x)$ is hard.
- But does any hard function has an hardcore predicate?
- f is injective and not hard $\implies$ f has no hardcore predicate.

The Goldreich-Levin predicate

For $x, r \in \{0,1\}^n$, let $\langle x, r \rangle_2 := \left(\sum_{i=1}^{n} x_i \cdot r_i\right) \bmod 2 = \bigoplus_{i=1}^{n} x_i \cdot r_i$.

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Theorem 9 (Goldreich-Levin)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, define $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ by $g(x,r) = (f(x), r)$. Assume f is (s,ε)-hard, then $b(x,r) := \langle x, r\rangle_2$ is an $(\frac{\varepsilon}{n^2} \cdot s, \sqrt[3]{n\varepsilon})$-hardcore predicate of g.

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The Goldreich-Levin predicate

For $x, r \in \{0,1\}^n$, let $\langle x, r\rangle_2 := (\sum_{i=1}^n x_i \cdot r_i) \bmod 2 = \bigoplus_{i=1}^n x_i \cdot r_i$.

Theorem 9 (Goldreich-Levin)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, define $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ by $g(x,r) = (f(x), r)$. Assume f is (s,ε)-hard, then $b(x,r) := \langle x, r\rangle_2$ is an $(\frac{\varepsilon}{n^2} \cdot s, \sqrt[3]{n\varepsilon})$-hardcore predicate of g.

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- We prove $\exists$ $(\frac{n^2}{\delta^2} \cdot s')$-size Inv with $\Pr[\mathsf{Inv}(f(X)) = X] \in \Omega(\delta^3/n)$.

Focusing on a good set

Claim 10

There exists set $\mathcal{S} \subseteq \{0,1\}^n$ with

1. $\frac{|\mathcal{S}|}{2^n} \geq \frac{\delta}{2}$, and
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□

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We conclude the theorem's proof showing that there exists a $\frac{n^2}{\delta^2}$-size Inv with

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for every $x \in \mathcal{S}$.

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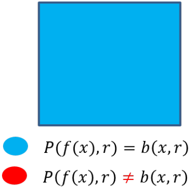
for every $x \in \mathcal{S}$. In the following we fix $x \in \mathcal{S}$.

The perfect case

$$\Pr\left[\mathsf{P}(f(x), R) = b(x, R)\right] = 1$$



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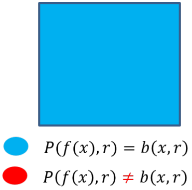

In particular, $\mathrm{P}(f(x), e^i) = b(x, e^i)$ for every $i \in [n]$, for $e^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})$.

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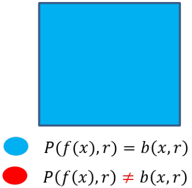

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Hence, $x_i = \langle x, e^i \rangle_2$

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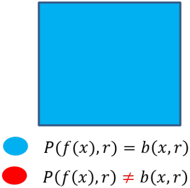

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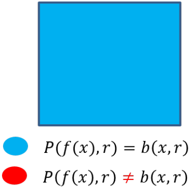
The perfect case

$\Pr[\mathsf{P}(f(x), R) = b(x, R)] = 1$



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Algorithm 11 (Inverter Inv on input $y \in \mathrm{Im}(f)$)

Return $(\mathsf{P}(y, e^1), \ldots, \mathsf{P}(y, e^n))$.

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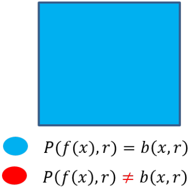

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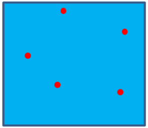
$\mathsf{Inv}(f(x)) = x$.

Easy case

$$\Pr\left[\mathrm{P}(f(x), R) = b(x, R)\right] \geq 1 - \frac{1}{4n}$$



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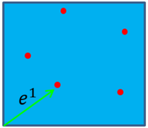

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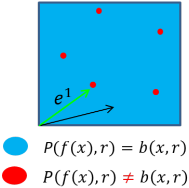

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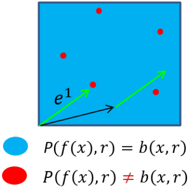

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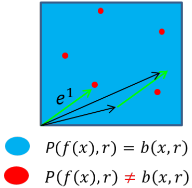

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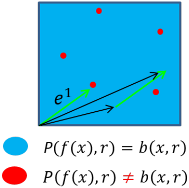

Fact 12

1. $b(x, w) \oplus b(x, y) = b(x, w \oplus y)$, *for every* $w, y \in \{0, 1\}^n$.

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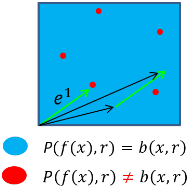

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1. $b(x,w) \oplus b(x,y) = b(x, w \oplus y)$, *for every* $w, y \in \{0,1\}^n$.
2. $\forall r \in \{0,1\}^n$, *the rv* $(R \oplus r)$ *is uniformly distributed over* $\{0,1\}^n$.

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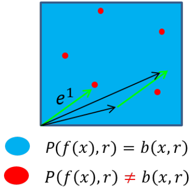

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Hence, $\forall i \in [n]$:

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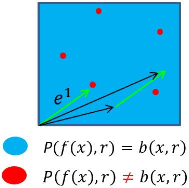

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Easy case

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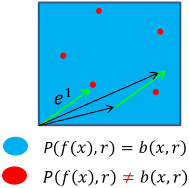

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Algorithm 13 (Inverter Inv on input y)

Return $(\mathsf{P}(y,R) \oplus \mathsf{P}(y,R \oplus e^1)),\ldots,\mathsf{P}(y,R) \oplus \mathsf{P}(y,R \oplus e^n))$.

Easy case

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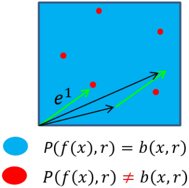

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$$\Pr\left[\mathsf{Inv}(f(x)) = x\right] \geq 1 - 2n \cdot \frac{1}{4n} = \frac{1}{2}$$

Proving Fact 12

1. For $w, y \in \{0,1\}^n$:

$$b(x,y) \oplus b(x,w) = \left(\bigoplus_{i=1}^{n} x_i \cdot y_i\right) \oplus \left(\bigoplus_{i=1}^{n} x_i \cdot w_i\right)$$
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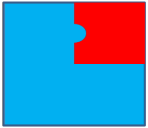
$$= b(x, y \oplus w)$$

2. For $r, y \in \{0,1\}^n$:

$$\Pr\left[R \oplus r = y\right] = \Pr\left[R = y \oplus r\right] = 2^{-n}$$

Intermediate case

$\Pr\left[\mathsf{P}(f(x), R) = b(x, R)\right] \geq \frac{3}{4} + \frac{\delta}{2}$



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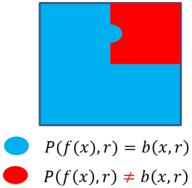

For any $i \in [n]$

$$\begin{aligned}
&\Pr[\mathsf{P}(f(x), R) \oplus \mathsf{P}(f(x), R \oplus e^i) = x_i] \\
&\geq \Pr[\mathsf{P}(f(x), R) = b(x, R) \wedge \mathsf{P}(f(x), R \oplus e^i) = b(x, R \oplus e^i)] \\
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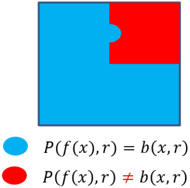

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\end{aligned}$$

Algorithm 14 ($\mathsf{Inv}(y)$)

For every $i \in [n]$:

1. Sample $r^1, \ldots, r^v \in \{0,1\}^n$ uniformly at random
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- Hence, the proof follows for $v = \left\lceil \log(n) \cdot \frac{1}{2\delta^2} \right\rceil + 1$.

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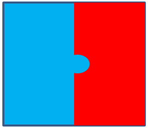

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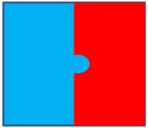

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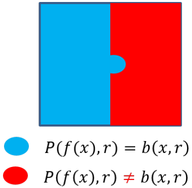

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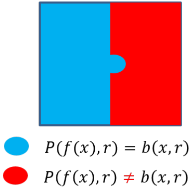

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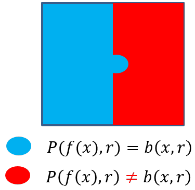

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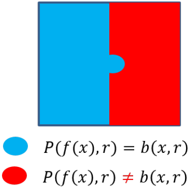

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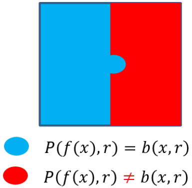

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- Solution: choose the samples in a correlated manner

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&\Pr[R^{\mathcal{L}} = w \wedge R^{\mathcal{L}'} = w'] \\
&= \sum_{(t^2,\ldots,t^\ell) \in \{0,1\}^{(\ell-1)n}} \Pr[(T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)] \cdot \Pr[R^{\mathcal{L}} = w \wedge R^{\mathcal{L}'} = w' \mid (T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)] \\
&= \sum_{(t^2,\ldots,t^\ell)\colon (\bigoplus_{i \in \mathcal{L}} t^i) = w} \Pr[(T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)] \cdot \Pr[R^{\mathcal{L}'} = w' \mid (T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)] \\
&= \sum_{(t^2,\ldots,t^\ell)\colon (\bigoplus_{i \in \mathcal{L}} t^i) = w} \Pr[(T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)] \cdot 2^{-n} \\
&= 2^{-n} \cdot 2^{-n}
\end{aligned}$$

Analyzing Inv's success probability

1. Let $T^1, \ldots, T^\ell$ be iid and uniform over $\{0,1\}^n$.
2. For $\mathcal{L} \subseteq [\ell]$, let $R^{\mathcal{L}} = \bigoplus_{i \in \mathcal{L}} T^i$.

Claim 18

1. $\forall \mathcal{L} \subseteq [\ell]$, $R^{\mathcal{L}}$ is uniformly distributed over $\{0,1\}^n$.
2. $\forall\, w, w' \in \{0,1\}^n$ and $\mathcal{L} \neq \mathcal{L}' \subseteq [\ell]$, it holds that $\Pr[R^{\mathcal{L}} = w \wedge R^{\mathcal{L}'} = w'] = \Pr[R^{\mathcal{L}} = w] \cdot \Pr[R^{\mathcal{L}'} = w'] = 2^{-2n}$.

Proof: (1) is clear. For (2), assume wlg. that $1 \in (\mathcal{L}' \setminus \mathcal{L})$.

$$\Pr[R^{\mathcal{L}} = w \wedge R^{\mathcal{L}'} = w']$$

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$$= 2^{-n} \cdot 2^{-n} = \Pr[R^{\mathcal{L}} = w] \cdot \Pr[R^{\mathcal{L}'} = w']. \square$$

Pairwise independence variables

Definition 19 (pairwise independent random variables)

A sequence of rv's $X^1, \ldots, X^v$ is pairwise independent, if $\forall i \neq j \in [v]$ and $\forall a, b$, it holds that $\Pr[X^i = a \wedge X^j = b] = \Pr[X^i = a] \cdot \Pr[X^j = b]$.

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- Hence, also $W^{\mathcal{L}}$ and $W^{\mathcal{L}'}$ are.

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Lemma 20 (Chebyshev's inequality)

Let $X^1, \ldots, X^v$ be pairwise-independent random variables with expectation μ and variance σ^2. Then, for every $\alpha > 0$: $\Pr\left[\left|\frac{\sum_{j=1}^{v} X^j}{v} - \mu\right| \geq \alpha\right] \leq \frac{\sigma^2}{\alpha^2 v}$.

Inv's success provability, cont.

- Assuming that Inv always guesses $\{b(x, t^i)\}$ correctly, then $\forall \mathcal{L} \subseteq [\ell]$:

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  - $\mathrm{E}[W^{\mathcal{L}}] \geq \frac{1}{2} + \frac{\delta}{2}$

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- Recalling that we guaranteed to work well on $\frac{\delta}{2}$ of the x's. We conclude that $\Pr[\mathsf{Inv}(f(x)) = x] \in \Theta(\delta^3/n)$.

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$\implies$ (by GL) $\exists$ Inv that guesses X from nothing, with prob $\alpha^{O(1)} > 2^{-k}$

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- The difference comparing to Goldreich-Levin — no control over the R's.