Application of Information Theory, Lecture 1
Basic Definitions and Facts

Handout Mode

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The entropy function

X — Discrete random variable (finite number of values) over $\mathcal{X}$ with probability mass $p = p_X$. The entropy of X is defined by:

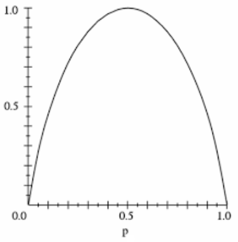
$$H(X) := -\sum_{x \in \mathcal{X}} \Pr[X = x] \cdot \log_2 \Pr[X = x]$$

taking $0 \cdot \log 0 = 0$.

- $H(X) = -\sum_x p(x) \log p(x) = \mathrm{E}_X \log \frac{1}{p(X)} = \mathrm{E}_{Y=p(X)} \log \frac{1}{Y}$
- $H(X)$ was introduced by Shannon as mesure for the uncertainty in X — number of bits requited to describe X, information we don't have about X.
- When using the natural logarithm, the quantity is called nats ("natural")
- Entropy is a function of p (sometimes refers to as $H(p)$).

Examples

1. $X \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$:
   (i.e., for some $x_1 \neq x_2 \neq x_3$, $\mathrm{P}_X(x_1) = \frac{1}{2}, \mathrm{P}_X(x_2) = \frac{1}{4}, \mathrm{P}_X(x_3) = \frac{1}{4}$)
   $H(X) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{4}\log\frac{1}{4} = \frac{1}{2} + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 = 1\frac{1}{2}$.
2. $H(X) = H(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.
3. X is uniformly distributed over $\{0,1\}^n$:
   $H(X) = -\sum_{i=1}^{2^n} \frac{1}{2^n}\log\frac{1}{2^n} = -\log\frac{1}{2^n} = n$.
   - n bits are needed to describe X
   - n bits are needed to sample X
4. $X = X_1, \ldots, X_n$ where X_i are iid over $\{0,1\}$, with $\mathrm{P}_{X_i}(1) := \Pr[X_i = 1] = \frac{1}{3}$. $H(X) = ?$
5. $X \sim (p, q)$, $p + q = 1$
   - $H(X) = H(p,q) = -p\log p - q\log q$
   - $H(1,0) = (0,1) = 0$
   - $H(\frac{1}{2}, \frac{1}{2}) = 1$
   - $h(p) := H(p, 1-p)$ is continuous



Applications

- Data compression
- Error correction codes
- Algorithm Analysis
- Protocols Analysis
- Cryptography
- Counting. Example # of gold coins in a cube
  - Projection of Q on xy — 6
  - Projection of Q on xz — 8
  - Projection of Q on yz — 12

  Can we bound $|Q|$?
- and more and more...

And all are rather simple to prove

Axiomatic derivation of the entropy function

Any other choices for defining entropy?
Shannon function is the only symmetric function (over probability distributions) satisfying the following three axioms:

- **A1** Continuity: $H(p, 1-p)$ is continuous function of p.
- **A2** Normalization: $H(\frac{1}{2}, \frac{1}{2}) = 1$
- **A3** Grouping axiom:
  $H(p_1, p_2, \ldots, p_m) = H(p_1 + p_2, p_3, \ldots, p_m) + (p_1 + p_2)H(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2})$

Why $A3$?

Not hard to prove that Shannon's entropy function satisfies above axioms, proving this is the only such function is more challenging.

Let H^* be a function that satisfying the above axioms.

We prove (assuming additional axiom) that H^* is the Shannon function H.

Generalization of the grouping axiom

Fix $p = (p_1, \ldots, p_m)$ and let $S_k = \sum_{i=1}^{k} p_i$.
Grouping axiom: $H^*(p_1, p_2, \ldots, p_m) = H^*(S_2, p_3, \ldots, p_m) + S_2 H^*(\frac{p_1}{S_2}, \frac{p_2}{S_2})$.

Claim 1 (Generalized grouping axiom)

$H^*(p_1, p_2, \ldots, p_m) = H^*(S_k, p_{k+1}, \ldots, p_m) + S_k \cdot H^*(\frac{p_1}{S_k}, \ldots, \frac{p_k}{S_k})$

Proof: Let $h(q) = H^*(q, 1-q)$.

$$\begin{aligned} H^*(p_1, p_2, \ldots, p_m) &= H^*(S_2, p_3, \ldots, p_m) + S_2 h(\frac{p_2}{S_2}) \qquad (1) \\ &= H^*(S_3, p_4, \ldots, p_m) + S_3 h(\frac{p_3}{S_3}) + S_2 h(\frac{p_2}{S_2}) \\ &\vdots \\ &= H^*(S_k, p_{k+1}, \ldots, p_m) + \sum_{i=2}^{k} S_i h(\frac{p_i}{S_i}) \end{aligned}$$

Hence,

$$H^*(\frac{p_1}{S_k}, \ldots, \frac{p_k}{S_k}) = H^*(\frac{S_{k-1}}{S_k}, \frac{p_k}{S_k}) + \sum_{i=2}^{k-1} \frac{S_i}{S_k} h(\frac{p_i/S_k}{S_i/S_k}) = \frac{1}{S_k} \sum_{i=2}^{k} S_i h(\frac{p_i}{S_i}) \qquad (2)$$

Claim follows by combining the above equations. □

Further generalization of the grouping axiom

Let $1 = k_1 < k_2 < \ldots < k_q < m$ and let $C_t = \sum_{i=k_t}^{k_{t+1}-1} p_i$ (letting $k_{q+1} = m + 1$).

Claim 2 (Generalized^{++} grouping axiom)

$H^*(p_1, p_2, \ldots, p_m) =$
$H^*(C_1, \ldots, C_q) + C_1 \cdot H^*(\frac{p_1}{C_1}, \ldots, \frac{p_{k_2-1}}{C_1}) + \ldots + C_q \cdot H^*(\frac{p_{k_q+1}}{C_q}, \ldots, \frac{p_m}{C_q})$

Proof: Follow by the extended group axiom and the symmetry of H $\square$

Implication: Let $f(m) = H^*(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m})$

- $f(3^2) = 2f(3) = 2H^*(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

  $\implies f(3^n) = nf(3)$.
- $f(mn) = f(m) + f(n)$

  $\implies f(m^k) = kf(m)$

$f(m) = \log m$

We give a proof under the additional axiom

A4 $f(m) < f(m+1)$

(you can Google for a proof using only $A1$–$A3$)

- For $n \in \mathbb{N}$, let $k = \lfloor \log 3^n = n \log 3 \rfloor$.
- Since, $2^k < 3^n < 2^{k+1}$, by $A4$: $f(2^k) < f(3^n) < f(2^{k+1})$.
- By grouping axiom, $k < nf(3) < k+1$.

$\implies \frac{\lfloor n \log 3 \rfloor}{n} < f(3) < \frac{\lfloor n \log 3 \rfloor + 1}{n}$ for any $n \in \mathbb{N}$

$\implies f(3) = \log 3$.

- Proof extends to any integer (not only 3)

$H^*(p,q) = -p\log p - q\log q$

- For rational p, q, let $p = \frac{k}{m}$ and $q = \frac{m-k}{m}$, where m is the smallest common multiplier.
- By grouping axiom, $f(m) = H^*(p,q) + p \cdot f(k) + q \cdot f(m-k)$.
- Hence,

$$\begin{aligned}
H^*(p,q) &= \log m - p\log k - q\log(m-k) \\
&= p(\log m - \log k) + q(\log m - \log(m-k)) \\
&= -p\log\frac{m}{k} - q\log\frac{m-k}{m} = -p\log p - q\log q
\end{aligned}$$

- By continuity axiom, holds for every p, q.

$H^*(p_1, p_2, \ldots, p_m) = -\sum_i^m p_i \log p_i$

We prove for $m = 3$. Proof for arbitrary m follows the same lines.

- For rational p_1, p_2, p_3, let $p_1 = \frac{k_1}{m}$, $q = \frac{k_2}{m}$ and $p_3 = \frac{k_3}{m}$, where $m = k_1 + k_2 + k_3$ is the smallest common multiplier.
- $f(m) = H^*(p_1, p_2, p_3) + p_1 f(k_1) + p_2 f(k_2) + p_3 f(k_3)$
- Hence,

$$
\begin{aligned}
H^*(p_1, p_2, p_3) &= \log m - p_1 \log k_1 - p_2 \log k_2 - p_3 \log k_3 \\
&= -p_1 \log \frac{k_1}{m} - p_2 \log \frac{k_2}{m} - p_3 \frac{k_3}{m} \\
&= -p_1 \log p_1 - p_2 \log p_2 - p_3 \log p_3
\end{aligned}
$$

- By continuity axiom, holds for every p_1, p_2, p_3.

$\square$

Section 1

Basic Properties

$0 \leq H(p_1, \ldots, p_m) \leq \log m$

- Tight bounds
  - $H(p_1, \ldots, p_m) = 0$ for $(p_1, \ldots, p_m) = (1, 0, \ldots, 0)$.
  - $H(p_1, \ldots, p_m) = \log m$ for $(p_1, \ldots, p_m) = (\frac{1}{m}, \ldots, \frac{1}{m})$.
- Non negativity is clear.
- A function f is concave ("keura") if $\forall\ t_1, t_2$, $\lambda \in [0, 1] \leq 1$
  $\lambda f(t_1) + (1 - \lambda) f(t_2) \leq f(\lambda t_1 + (1 - \lambda) t_2)$

$\Longrightarrow$ (by induction) $\forall\ t_1, \ldots, t_k$, $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_i \lambda_i = 1$
$\sum_i \lambda_i f(\lambda_i t_i) \leq f(\sum_i \lambda_i t_i)$

$\Longrightarrow$ (Jensen inequality): $\mathrm{E}\, f(X) \leq f(\mathrm{E}\, X)$ for any random variable X.

- $\log(x)$ is (strictly) concave for $x > 0$, since its second derivative $(-\frac{1}{x^2})$ is always negative.
- Hence, $H(p_1, \ldots, p_m) = \sum_i p_i \log \frac{1}{p_i} \leq \log \sum_i p_i \frac{1}{p_i} = \log m$
- Alternatively, for X over $\{1, \ldots, m\}$,
  $H(X) = \mathrm{E}_X \log \frac{1}{\mathrm{P}_X(X)} \leq \log \mathrm{E}_X \frac{1}{\mathrm{P}_X(X)} = \log m$

$H(g(X)) \le H(X)$

Let X be a random variable, and let g be over $\mathrm{Supp}(X) := \{x \colon \mathrm{P}_X(x) < 0\}$.

- $H(Y = g(X)) \le H(X)$.

  Proof:

$$\begin{aligned} H(X) &= -\sum_x \mathrm{P}_X(x) \log \mathrm{P}_X(x) = -\sum_y \sum_{x \colon g(x)=y} \mathrm{P}_X(x) \log \mathrm{P}_X(x) \\ &\ge -\sum_y \mathrm{P}_Y(y) \cdot \max_{x \colon g(x)=y} \log \mathrm{P}_X(x) \\ &\ge -\sum_y \mathrm{P}_Y(y) \cdot \log \mathrm{P}_Y(y) = H(Y) \end{aligned}$$

- Or use the group axiom...
- If g is injective, then $H(Y) = H(X)$.

  Proof: $p_X(X) = P_Y(Y)$.
- If g is non-injective (over $\mathrm{Supp}(X)$), then $H(Y) < H(X)$.

  Proof: ?
- $H(X) = H(2^X)$.
- $H(X) < H(\cos(X))$, if $0, \pi \in \mathrm{Supp}(X)$.

Notation

- $[n] = \{1, \ldots, n\}$
- $\mathrm{P}_X(x) = \Pr[X = x]$
- $\mathrm{Supp}(X) := \{x \colon \mathrm{P}_X(x) < 0\}$
- For random variable X over $\mathcal{X}$, let $p(x)$ be its density function: $p(x) = \mathrm{P}_X(x)$.

  In other words, $X \sim p(x)$.
- For random variable Y over $\mathcal{Y}$, let $p(y)$ be its density function: $p(y) = \mathrm{P}_Y(y)\ldots$