Application of Information Theory, Lecture 2
Joint & Conditional Entropy, Mutual Information

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Part I

Joint and Conditional Entropy

Joint entropy

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Joint entropy, cont.

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Conditional entropy, cont.

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$$\begin{aligned} H(X|Y) &= \mathop{\mathrm{E}}_{y \leftarrow Y} H(X|_{Y=y}) \\ &= \frac{3}{4}H(X|_{Y=0}) + \frac{1}{4}H(X|_{Y=1}) \\ &= \frac{3}{4}H(\frac{1}{3},\frac{2}{3}) + \frac{1}{4}H(1,0) = 0.6887 \neq H(Y|X). \end{aligned}$$

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- $H(Y|X) = H(X,Y) - H(X)$ is as an alternative definition for $H(Y|X)$.

Chain rule (for the entropy function)

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- Proof immediately follow by the grouping axiom:

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Jensen inequality: for any concave function f, values $t_1, \ldots, t_k$ and $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_i \lambda_i = 1$, it holds that $\sum_i \lambda_i f(t_i) \leq f(\sum_i \lambda_i t_i)$.

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  - Left inequality since $H(Y|X)$ is non negative.
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- $H(X, Y|Z) = H(X|Z) + H(Y|X, Z)$ (by chain rule)
- $H(X|Y, Z) \leq H(X|Y)$

Proof:

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H(X|Y,Z) &= \mathop{\mathrm{E}}_{(z,y)\leftarrow(Z,Y)} H(X|_{(Y,Z)=(z,y)}) \\
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&= \mathop{\mathrm{E}}_{y\leftarrow Y} \mathop{\mathrm{E}}_{z\leftarrow Z|_{Y=y}} H((X|_{Y=y})|_{Z=z}) \\
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Chain rule (for the entropy function), general case

Claim 2

For rvs $X_1, \ldots, X_k$, it holds that
$H(X_1, \ldots, X_k) = H(X_i) + H(X_2|X_1) + \ldots + H(X_k|X_1, \ldots, X_{k-1})$.

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- Interpretation
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Applications cont.

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$\implies \quad t \geq \log n! = \Theta(n \log n)$

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We are now certain that we drew the graph of the (two-dimensional) entropy function right...

Part II

Mutual Information

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Numerical example

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$X \backslash Y$	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
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Chain rule for mutual information

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Claim 4 (Chain rule for mutual information)

For rvs $X_1, \ldots, X_k, Y$, it holds that
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Part III

Data Processing

Data processing inequality

Definition 5 (Markov Chain)

Rvs $(X, Y, Z) \sim p$ form a Markov chain, denoted $X \to Y \to Z$, if $p(x, y, z) = p_X(x) \cdot p_{Y|X}(y|x) \cdot p_{Z|Y}(z|y)$, for all x, y, z.

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Example: random walk on graph.

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- Since $I(X; Y|Z) \geq 0$, we conclude $I(X; Y) \geq I(X; Z)$.$\square$

Fano's Inequality

- How well can we guess X from Y?

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for $\hat{X} = g(Y)$ and $P_e = \Pr\left[\hat{X} \neq X\right]$.

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- We call $\hat{X}$ an estimator for X (from Y).

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