Information Theory, Spring 2018 — Iftach Haitner

Problem set 1

March 27, 2018 — Due: March 29

- Please submit the handout in class, or email the grader (quefumas at gmail.com).
- Write clearly and shortly using sub-claims if needed. The emphasize in most questions is on the proofs (no much point is writing a “solution" w/o proving its correctness)
- For Latex users, a solution example can be found in the course web site.
- It is allowed to work in (small) groups, but please write the id list of your partners in the solution file, and each student should write his solution by *himself* (joint effort is only allowed in the “thinking phase")

1. Prove that the (Shanon) entropy function satisfies the (basic) grouping axioms $A3$ (also verify for yourself that it satisfies the axioms $A1, A2, A4$).

2. In Lecture 1, slide 10, we prove that $H^*(p_1, p_2, p_3) = H(p_1, p_2, p_3)$ for any *rational* p_1, p_2, p_3, and say that the non-rational case follows by the continuity of H^*. Prove it.

3. Let $(p_1, \ldots, p_m)$ and $(q_1, \ldots, q_m)$ be probability distributions (i.e., $p_i \geq 0$ for all i and $\sum_i p_i = 1$). Prove that
$$-\sum_i p_i \log p_i \leq -\sum_i p_i \log q_i$$

4. For random variables X and Y, and arbitrary deterministic functions f and g, what is larger? Prove your answers (you can use any of the inequalities stated in first two lectures),

   (a) $H(X|Y)$ or $H(f(X)|Y)$?

   (b) $H(X|Y)$ or $H(X|g(Y))$?

   (c) $H(X|Y)$ or $H(f(X,Y)|Y)$?

   (d) $H(X|Y)$ or $H(X|g(X,Y))$?

5. For a finite set $\mathcal{S}$ of random variables, let $H(\mathcal{S})$ denote the joint entropy of all random variables in $\mathcal{S}$. Prove that for any two finite sets of random variables $\mathcal{S}$ and $\mathcal{U}$, it holds that $H(\mathcal{S} \cup \mathcal{U}) + H(\mathcal{S} \cap \mathcal{U}) \leq H(\mathcal{S}) + H(\mathcal{U})$.