

Foundation of Cryptography, Lecture 10

Pseudorandom Generator from One-Way Functions

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Section 1

Entropy

Different measures of entropy

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Equality iff X is **uniform**.

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Flattening Shannon entropy

Lemma 1

Let X be a rv over \mathcal{U} , let $t \in \mathbb{N}$ and let $\varepsilon > 0$. Then \exists rv Z that is $(\varepsilon + 2^{-t})$ -close to X^t , and $H_\infty(Z) \geq t \cdot H(X) - O(\sqrt{t \cdot \log(1/\varepsilon)} \cdot \log(|\mathcal{U}| \cdot t))$.

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Proof: ?

Pairwise independent hashing

Definition 2 (pairwise independent function family)

A function family $\mathcal{H} = \{h: \{0, 1\}^n \mapsto \{0, 1\}^m\}$ is **pairwise independent**, if $\forall x \neq x' \in \{0, 1\}^n$ and $y, y' \in \{0, 1\}^m$, it holds that $\Pr_{h \leftarrow \mathcal{H}}[h(x) = y \wedge h(x') = y'] = 2^{-2m}$.

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Example $\mathcal{H} = \{(A, b) \in \{0, 1\}^{m \times n} \times \{0, 1\}^m\}$ with $(A, b)(x) = A \times x + b$.

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Lemma 3 (leftover hash lemma)

Let X be a rv over $\{0, 1\}^n$ with $H_2(X) \geq k$ and let $\mathcal{H} = \{h: \{0, 1\}^n \mapsto \{0, 1\}^m\}$ be pairwise independent, then

$$\text{SD}((H, H(X)), (H, U_m)) \leq 2^{(m-k-2)/2},$$

where H is uniformly distributed over \mathcal{H} and U_m is uniformly distributed over $\{0, 1\}^m$.

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Definition 4

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- Examples
- Repeated sampling

Section 2

PRG from Regular OWF

Definition 5

Given a function $f: \{0, 1\}^n \mapsto \{0, 1\}^n$ and function family $\mathcal{H}: \{0, 1\}^n \mapsto \{0, 1\}^m$, let $g = g(f, \mathcal{H}): \mathcal{H} \times \{0, 1\}^n \mapsto \mathcal{H} \times \{0, 1\}^n \times \{0, 1\}^m$ be defined by $g(h, x) = (g(x), h, h(x))$.

In case f and \mathcal{H} are function families, we let $g(f, \mathcal{H}) = \{g(f_n, \mathcal{H}_n)\}_{n \in \mathbb{N}}$.

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Claim 6

Let f be a $2^{k=k(n)}$ -regular OWF, $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^n \mapsto \{0, 1\}^{m(n)=k(n)+\log n}\}$ be efficient family of pairwise independent hash function family, and let $g = g(f, \mathcal{H})$. Then

- 1 $H(g(U_n, H_n)) \geq n + H(H_n) - \frac{1}{n}$, where H_n is uniform over \mathcal{H}_n .
- 2 g is one-way.

g has high entropy

$$\begin{aligned}\text{CP}(g(U_n, H_n)) &:= \Pr_{w, w' \leftarrow \{0,1\}^n \times \mathcal{H}_n} [g(w) = g(w')] \\ &= \Pr_{h, h' \leftarrow \mathcal{H}_n} [h = h'] \cdot \Pr_{x, x' \leftarrow \{0,1\}^n} [f(x) = f(x')] \\ &\quad \cdot \Pr_{h \leftarrow \mathcal{H}_n; x, x' \leftarrow \mathcal{Z}^n} [h(x) = h(x') \mid f(x) = f(x')] \\ &= \text{CP}(H_n) \cdot \text{CP}(f(U_n)) \cdot (2^{-k} + (1 - 2^{-k}) \cdot 2^{-m}) \\ &\leq \text{CP}(H_n) \cdot \text{CP}(f(U_n)) \cdot (2^{-k} + 2^{-m}) \\ &\leq \text{CP}(H_n)(2^{-n} + 2^{-n - \log n}) = \text{CP}(H_n) \cdot \text{CP}(U_n) \cdot (1 + \frac{1}{n}).\end{aligned}$$

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Hence, $H_2(g(U_n, H_n)) \geq H_2(\mathcal{H}_n) + H_2(U_n) + \log \frac{1}{1 + \frac{1}{n}} \geq H(H_n) + n - \frac{1}{n}$.

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Thus, $H(g(U_n, H_n)) \geq H(H_n) + n - \frac{1}{n}$.

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Let $t = t(n) = k(n) - 2 \lceil \log(p(n)) \rceil$.

Algorithm 7 (B)

Input: $y \in \{0, 1\}^n$.

Sample $h \leftarrow \mathcal{H}_n$ and $z \leftarrow \{0, 1\}^t$, and return $D(y, h, z)$

Algorithm 8 (D)

Input: $y \in \{0, 1\}^n$, $h \in \mathcal{H}_n$ and $z_1 \in \{0, 1\}^t$.

For all $z_2 \in \{0, 1\}^{m-t}$:

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$$\Pr_{x \leftarrow \{0,1\}^n; h \leftarrow \mathcal{H}_n} [D(f(x), h, h(x)_{1,\dots,t}) \in f^{-1}(f(x))] \geq \frac{1}{p(n)} \quad (1)$$

g is one-way, cont.

By the leftover hash lemma(?)

$$\text{SD}((f(x), h, h(x)_1, \dots, t)_{x \leftarrow \{0,1\}, h \leftarrow \mathcal{H}_n}, (f(x), h, U_t)_{x \leftarrow \{0,1\}, h \leftarrow \mathcal{H}_n}) \leq \frac{1}{2p(n)} \quad (2)$$

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Hence,

$$\Pr_{x \leftarrow \{0,1\}^n} [\mathbf{B}(f(x)) \in f^{-1}(f(x))] \geq \frac{1}{p(n)} - \frac{1}{2p(n)} = \frac{1}{2p(n)}.$$

The generator

Claim 9

Let $f: \{0, 1\}^n \mapsto \{0, 1\}^m$ be a OWF with $H(f(U_n)) \geq n - \frac{1}{2}$, and let b be an hardcore predicate for f . Then $g(x) = f(x) \circ b(x)$ has pseudoentropy $n + \frac{1}{2}$.

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Proof: by the flattening lemma, taking $\varepsilon = 2^{-\log^2 n}$ and $t = n$.

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Let $f: \{0, 1\}^n \mapsto \{0, 1\}^m$ be a OWF with $H(f(U_n)) \geq n - \frac{1}{2}$, and let b be an hardcore predicate for f . Then $g(x) = f(x) \circ b(x)$ has pseudoentropy $n + \frac{1}{2}$.

Proof: ?

We call such g a **pseudo-entropy** generator.

Claim 10

The function $g^{n^2}(x_1, \dots, x_{n^2}) = g(x_1), \dots, g(x_{n^2})$ has **pseudo min-entropy** $n(n + \frac{1}{2}) - O(\sqrt{n \log^2 n \cdot \log(n^2)}) \geq n^2 + n/2 - O(n^{2/3})$.

Proof: by the flattening lemma, taking $\varepsilon = 2^{-\log^2 n}$ and $t = n$.

Claim 11

Let $\mathcal{H}: \{0, 1\}^{n^2+n} \mapsto \{0, 1\}^{n^2+n/4}$ be an efficient pairwise hash function, then $G: \{0, 1\}^{n^2} \times \mathcal{H}_n$ defined by $G(x_1, \dots, x_{n^2}, h) = (h, h(g^{n^2}(x_1, \dots, x_{n^2})))$, is a PRG.

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Proof: by the leftover hash lemma

Section 3

PRG from any OWF

Inefficient construction

Definition 12

Given a function $f: \{0, 1\}^n \mapsto \{0, 1\}^m$ and $x \in \{0, 1\}^n$, let

$$d_f(x) = \lceil \log(|f^{-1}(f(x))|) + \log n \rceil.$$

Given $\mathcal{H}: \{0, 1\}^n \mapsto \{0, 1\}^{n+\log n}$, let

$g = g(f, \mathcal{H}): \mathcal{H} \times \{0, 1\}^n \mapsto \mathcal{H} \times \{0, 1\}^n \times \{0, 1\}^{n+\log n}$ be defined by
 $g(h, x) = g(x), h, h(x)_{1, \dots, d_f(x)}, 1^{n+\log n - d_f(x)}.$

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Let f be a OWF, $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^n \mapsto \{0, 1\}^{n+\log n}\}$ be efficient family of pairwise independent hash function family, and let $g = g(f, \mathcal{H})$. Then

- ① $H(g(U_n, H_n)) \geq n + H(H_n) - \frac{1}{n}$, where H_n is uniform over \mathcal{H}_n .
- ② Assume d_f is poly-time computable, then g is a one-way function.

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Hence, if d_f is poly-time computable, then building a PRG from f follows the same lines we used for regular OWF.

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Hence, if d_f is poly-time computable, then building a PRG from f follows the same lines we used for regular OWF.

Should we expect d_f to be poly-time computable?

Efficient construction, first approach

Definition 14

For $f: \{0, 1\}^n \mapsto \{0, 1\}^n$ and $\mathcal{H} = \{h: \{0, 1\}^n \mapsto \{0, 1\}^{n+\log n}\}$, let $g = g(f, \mathcal{H}): \mathcal{H} \times [n] \times \{0, 1\}^n \mapsto \mathcal{H} \times [n] \times \{0, 1\}^n \times \{0, 1\}^{n+\log n}$ be defined by $g(h, i, x) = f(x), h, i, h(x)_{1, \dots, i+\log n}, 1^{n+\log n-i}$.

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Claim 15

Assume f is OWF and that \mathcal{H} is the Matrix-based pairwise-independent hash functions. Then the **pseudo** Shannon-entropy of $g(H_n, I_n, U_n)$, where I_n is uniform over $[n]$, is larger by at least $1/n$ than its (real) Shannon entropy.

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Proof: Define

$$g'(h, i, x) = \begin{cases} f(x), h, i, h(x)_{1, \dots, i+\log n-1}, U, 1^{n+\log n-i}, & i = d_f(x) \\ g(h, i, x), & \text{otherwise.} \end{cases}$$

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- ① $g(H_n, I_n, U_n) \approx_c g'(H_n, I_n, U_n)$
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False-pseudoentropy generator to PRG

- 1 Using repetition convert the Shannon pseudoentropy of the output of g into min pseudoentropy.

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Very complicated an inefficient construction. Seed length of PRG is $\Theta(n^8)$.

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For $f: \{0, 1\}^n \mapsto \{0, 1\}^n$, and the Matrix-based pairwise-independent hash functions $\mathcal{H} = \{h: \{0, 1\}^n \mapsto \{0, 1\}^{n+\log n}\}$, let $g: \mathcal{H} \times \{0, 1\}^n \mapsto \mathcal{H} \times \{0, 1\}^n \times \{0, 1\}^{n+\log n}$ be defined by $g(h, x) = (f(x), h, h(x))$.

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But g is **invertible** and thus its output pseudoentropy is as large as its real entropy.(?)

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Right, but not in the eyes of an **online observer**.

Next-block pseudoentropy generator

Definition 18 (next-block pseudoentropy)

$X = (X_1, \dots, X_m)$ has next-block pseudoentropy at least k , \exists rv $Y = (Y_1, \dots, Y_m)$, (jointly distributed with X), such that:

- 1 $\forall i, (X_1, X_2, \dots, X_{i-1}, X_i) \approx_c (X_1, X_2, \dots, X_{i-1}, Y_i).$
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Quantitative generalization of unpredictability: measures how hard it is to predict X_i from X_1, X_2, \dots, X_{i-1} (for $i \leftarrow [k]$).

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Assume f is OWF, then $g(U_n, H_n)$ has next-block pseudoentropy $n + |h| + 1$.

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$g(U_n, H_n)$ realizes the next-block pseudoentropy of $g(U_n, H_n)$.