Application of Information Theory, Lecture 3
Graph Covering, Differential Entropy

Handout Mode

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Part I

Applications to Graph Covering

Graph Covering

- How may graphs of certain type it takes to cover the full graph?
- K_n — the complete graph over $[n]$
- Let $G_1, \ldots, G_t$ be bipartite graphs over $[n]$ with $\bigcup_i G_i = K_n$. What can we say about t?
- Clearly, $t \geq \frac{\binom{n}{2}}{(n/2)^2} \approx 2$, but can we give a better bound?

Theorem 1

Let $G_1, \ldots, G_t$ be bipartite graphs over $[n]$ with $\bigcup_{i=1}^t G_i = K_n$, then $t \geq \log n$.

Proof: Let $\chi(G)$ be the chromatic number of G.

- $\chi(G_i) \leq 2$ and $\chi(K_n) = n$.
- $\chi(G \cup G') \leq \chi(G) \cdot \chi(G')$.(?)

$\implies \chi(\bigcup_{i=1}^t G_i) \leq 2^t$

$\implies t \geq \log n$

Proving Thm 1 using entropy

- $G_i = (A_i, B_i, E_i)$
- $X \leftarrow [n]$
- $Y_i = \begin{cases} 0, & X \in A_i \\ 1, & X \in B_i \end{cases}$
- X is determined by $Y_1, \ldots, Y_t$ (?)

$$\begin{aligned} 0 = H(X|Y_1, \ldots, Y_t) &= H(X, Y_1, \ldots, Y_t) - H(Y_1, \ldots, Y_t) \\ &\geq H(X) - \sum_i H(Y_i) \\ &\geq \log n - t. \end{aligned}$$

Extensions

- $\text{nonIs}(G)$ — non-isolated vertices in G.

Theorem 2

Let $G_1, \ldots, G_t$ be bipartite graphs over $[n]$ with $\bigcup_{i=1}^{t} G_i = K_n$, then $\frac{1}{n}\sum_{i=1}^{t} |\text{nonIs}(G_i)| \geq \log n$.

Definition 3 (graph content)

Let G be a graph over $[n]$, let $Z \leftarrow \text{nonIs}(G)$ and let $\hat{\chi}$ be a (valid) coloring of G such that $H(\hat{\chi}(Z))$ is minimal. Then $\text{content}(G) := \frac{|\text{nonIs}(G)|}{n} \cdot H(\hat{\chi}(Z))$.

Theorem 4

Let $G_1, \ldots, G_t$ be graphs over $[n]$ with $\bigcup_{i=1}^{t} G_i = K_n$. Then $\sum \text{content}(G_i) \geq \log n$.

- Since $\text{content}(G) \leq \frac{|\text{nonIs}(G)|}{n}$ for bipartite G, Thm 4 yields Thm 2.

Proving Thm 4

- Let χ_i be a (valid) coloring of G_i.
- Let $X \leftarrow [n]$, and let
  $Y_i = \begin{cases} \chi_i(X) & X \in \text{nonIs}(G_i) \\ \chi_i(Z_i) & \text{otherwise, for } Z_i \leftarrow \text{nonIs}(G_i) \text{ (ind. of the other } Z\text{'s).} \end{cases}$
- X is determined by $Y_1, \ldots, Y_t$ (?)

$$\begin{aligned} 0 = H(X|Y_1, \ldots, Y_t) &= H(X, Y_1, \ldots, Y_t) - H(Y_1, \ldots, Y_t) \\ &\geq H(X) + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i) \\ &= \log n + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i). \end{aligned}$$

- $Y_1, \ldots, Y_t$ are independent conditioned on X —
  $\Pr[Y_1 = y_1 \wedge Y_2 = y_2 \mid X = x] = \Pr[Y_1 = y_1 \mid X = x] \cdot \Pr[Y_2 = y_2 \mid X = x]$
- Hence, $H(Y_1, \ldots, Y_t|X) = \sum_i H(Y_i|X)$ (board)
- We conclude that $\sum_i H(Y_i) - \sum_i H(Y_i|X) \geq \log n$
- Since $H(Y_i) = H(\chi_i(Z_i))$ and $H(Y_i|X) = (1 - \frac{|\text{nonIs}(G_i)|}{n}) \cdot H(\chi_i(Z_i))$,

  it follows that $\sum_i H(\chi_i(Z_i)) \frac{|\text{nonIs}(G_i)|}{n} \geq \log n$. $\square$

Extension

Let $\alpha(G)$ be the size of the maximal independent set in G.

Theorem 5

Let $G, G_1, \ldots, G_t$ be graphs over $[n]$ with $\bigcup_{i=1}^{t} G_i = G$, then $\sum \text{content}(G_i) \geq \log \frac{n}{\alpha(G)}$.

Proof: HW

Scrambling permutations

Theorem 6

Let $\mathcal{S}$ be a set of permutations over $[n]$ s.t. for any triplet (i,j,k) of distinct elements of $[n]$, exists $\pi \in \mathcal{S}$ with $\pi(i) < \pi(j) < \pi(k)$ or $\pi(i) > \pi(j) > \pi(k)$. Then $|\mathcal{S}| \geq \frac{2}{\log e} \log n$.

- For $\pi \in \mathcal{S}$, the graph $G_\pi = (V, E_\pi)$ is defined by:
  - $V = \{(i,j) \in [n]^2 \colon i \neq j\}$
  - $E_\pi = \{((i,j),(k,j)) \in V^2 \colon \pi(i) < \pi(j) < \pi(k) \vee \pi(i) > \pi(j) > \pi(k)\}$
- $G = \bigcup_{\pi \in \mathcal{S}} G_\pi$ has n connected components, each consists of $(n-1)$-vertex cliques: $\{(i,j) \colon i \in [n] \setminus \{j\}\}$ for each $j \in [n]$.
- G_π consists of n complete bipartite graphs (two are empty): $\{(i,j) \colon \pi(i) \leq \pi(j)\}$ and $\{(i,j) \colon \pi(i) > \pi(j)\}$ for each $j \in [n]$.

  The sum of content of these bipartite graphs is

$$\sum_{i=0}^{n-1} h(\tfrac{i}{n-1}) = (n-1)\sum_{i=0}^{n-1} h(\tfrac{i}{n-1}) \cdot \tfrac{1}{n-1} \leq (n-1)\int_0^1 h(p)dp = (n-1) \cdot \tfrac{\log e}{2}.$$

- By Thm 5 (applied for each component) $|\mathcal{S}| \cdot \frac{\log e}{2} \cdot (n-1) \geq n \log(n-1)$.
- Hence, $|\mathcal{S}| \geq \frac{2}{\log e} \cdot \frac{n}{n-1} \cdot \log(n-1) \geq \frac{2}{\log e} \log n$. $\square$

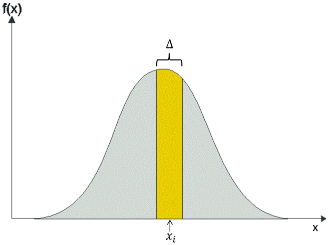
Part II

Differential Entropy

Entropy of continues random variable

- Entropy of discrete random variable: $H(X) = -\sum_i p_i \log p_i$
- Also used when X has infinite $\#$ of states (entropy might be infinite!)
- Continues random variable is defined by its density function: $f\colon \mathbb{R} \mapsto \mathbb{R}^+$,for which $\int_{\mathbb{R}} f(x)dx = 1$.
- $F_X(x) := \Pr[X \leq x] = \int_{-\infty}^{x} f(x)dx$
- $\mathrm{E}\,X = \int x \cdot f(x)dx$ and $\mathrm{V}\,X = \int x^2 \cdot f(x)dx - (\mathrm{E}\,X)^2$
- Examples: $X \sim [0,1]$, $X \sim N(0,1)$
- $H(X)$ must be infinite! it takes infinite number of bits to describe X
- The differential entropy of X is defined by $h(X) = -\int f(x)\log f(x)dx$.
- We focus on cases where $h(X)$ is well defined.
- Since h is a function of the density function, we sometimes write $h(f)$
- If not stated otherwise, we integrate over $\mathbb{R}$

Intuition for definition of h



- Let X^{Δ} be rounding of X for precision Δ:
  $X^{\Delta} \sim (\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$,
  where $p_i = \int_{i\cdot\Delta}^{(i+1)\cdot\Delta} f(x)dx = f(x_i)\cdot\Delta$
  for some $x_i \in [i\cdot\Delta, (i+1)\cdot\Delta]$ (?)

- $H(X^{\Delta}) = -\sum_{i=-\infty}^{\infty} p_i \log p_i$

$$H(X^{\Delta}) = -\sum_{i=-\infty}^{\infty} f(x_i)\cdot\Delta\cdot\log(f(x_i)\cdot\Delta) = -\sum_{i=-\infty}^{\infty} f(x_i)\cdot\Delta\cdot(\log f(x_i) + \log\Delta)$$

$$= -\sum_{i=-\infty}^{\infty} f(x_i)\cdot\log f(x_i)\cdot\Delta - \left(\sum_{i=-\infty}^{\infty} f(x_i)\cdot\Delta\right)\log\Delta$$

- $\lim_{\Delta\to 0} H(X^{\Delta}) = h(X) - \log\Delta$
- Hence, $\lim_{\Delta\to 0} H(X^{\Delta}) + \log\Delta = h(x)$
- Intuitively, $h(X)$ is the entropy of X plus const ($-\log\Delta$).
- Note that $\lim_{\Delta\to 0} -\log\Delta = \infty$

Properties of the entropy function

- Shift invariant: $h(f) = h(g)$ for $g(x) = f(x + a)$
- $h(X) = -\int f(x) \log f(x) dx$ might be infinite (?)
- Example $f(x) = 2^{-2^i}$ with probability $1/2^i$ (i.e., over segment of length $2^{-i}/2^{-2^i}$)
- $h(X)$ might be negative
- Example: $X \sim [0, a]$ – $f(x) = \frac{1}{a}$ on $[1, a]$

  $-\int f(x) \log f(x) dx = -\log \frac{1}{a} = \log a$. Negative for $a < 1$.
- $h(X)$ should be interpreted as the uncertainty up to a certain constant
- Used for comparing two distributions

Common distribution (in nature)

- The uniform distribution: $X \sim [a, b]$
- Normal (Gaussian) distribution: (we focus on $\mathsf{E} = 0$ and $\mathsf{V} = 1$)
  $X \sim N(0, 1)$: $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$
- Boltzmann (Gibbs) distribution:
  $X \in \{E_1, E_2, \ldots, E_m\}$, $\Pr[X = E_i] = C \cdot e^{-\beta E_i}$ for $\beta > 0$ (the *distribution constant*) and $C = 1/\sum_i e^{-\beta E_i}$.
  - Describes a (discrete) physical system that can take states $\{1, \ldots, m\}$ with energies $E_1, \ldots, E_m$.
  - Probability is inverse to the energy
- Why are these distributions so common?
- What is common to these distributions?

Historical background

- Shannon (1948) $H = -\sum_i p_i \log p_i$
- But the notion of entropy already existed in statistical physics
- There, entropy — energy that cannot used, statistical disorder
- Clausius (1865), who coined the name *entropy*, based on Carnot (1824), $H = \int_t \frac{\delta Q}{T} dt$ (Q is *heat* and T is *temperature*)
- Boltzmann (1877) $H = \log S$, for S being the number of states a system can be in (after measuring the macro parameters: pressure, temperature)
- $\log \#$ of states is Shannon entropy of the uniform distribution
- Shannon looked for a name for his measure, von Neumann pointed out the relation to physics and suggested the name entropy.
- Today it is accepted that Shannon's entropy is the right notion also in statistical mechanic. Measures the uncertainty of a system — energy that cannot be used.

- Carnot was also an engineer...

Second law of thermodynamics

- The entropy of a closed physical system never decreases.
- If we wait enough time, the system tends to be in maximal entropy.
- If there are constrains, the it tends to be in maximal entropy under this constrains.
- This suggests that distributions that are common in nature, are distributions of maximal entropy, under some constrains.
- In contradiction with “reversible laws"

The normal distribution

- $X \sim N(0,1)$: $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$
- Why is it so common?
- Answer: the central limit theorem (CLT):

  Let $X_1, \ldots, X_n$ be iid with $\mathrm{E}\, X_i = 0$ and $\mathrm{V}\, X_i = 1$. Then $\lim_{n\to\infty} \frac{\sum_i X_i}{\sqrt{n}} = N(0,1)$.
- But why does it converge to $N(0,1)$??
- CLT holds also in many other variants: not id, not fully independent, ...
- We know that $\mathrm{E}\, \frac{\sum_i X_i}{\sqrt{n}} = 0$ and $\mathrm{V}\, \frac{\sum_i X_i}{\sqrt{n}} = 1$, but it could have converge to any other distribution with these constraints.
- The reason is that $N(0,1)$ has the highest entropy among all distribution with $\mathrm{E} = 0$ and $\mathrm{V} = 1$.
- CLT and the normal distribution where known and studied way before Shannon, yet this striking property was not known until his theory.

The normal distribution, cont.

Theorem 7

$h(X) \leq h(N(0,1))$, *for any rv* X *with* $\mathrm{V}\,X = 1$.

- Among the distributions of $\mathrm{V} = 1$, the distribution $N(0,1)$ has maximal entropy.
- Generalizes to any variance:

  $h(X) \leq h(N(0,\mathrm{V}(X))) = \frac{1}{2} \cdot \log(2\pi e) \cdot \mathrm{V}(X)$

Let g be a density function with $\int g(x)x^2dx = 1$, and let $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$. We will show that

1. $-\int g(x)\log g(x)dx \leq -\int g(x)\log f(x)dx$
2. $-\int g(x)\log f(x)dx = -\int f(x)\log f(x)$

$-\int g(x) \log g(x) dx \leq -\int g(x) \log f(x) dx$

Claim 8

$-\int g(x) \log g(x) dx \leq -\int g(x) \log q(x) dx$ for any two density functions q, g.

Proof: (the continuous version of $Q3$ in handout 1)

- Jensen: For any function t and density function λ:
  $\int \lambda(x) \log t(x) \leq \log \int \lambda(x) t(x) dx$
- Assume for simplicity that $g(x) > 0$ for all x.
- By Jensen, $\int g(x) \log \frac{q(x)}{g(x)} \leq \log \int g(x) \frac{q(x)}{g(x)} dx = \log 1 = 0$
- Hence, $-\int g(x) \log g(x) \leq -\int g(x) \log q(x)$

$-\int g(x)\log f(x)dx = -\int f(x)\log f(x)dx$

Claim 9

Exists $c \in \mathbb{R}$ such that $-\int g(x)\log f(x)dx = c$ for any density function g with $\int g(x)x^2 dx = 1$.

Hence, $-\int g(x)\log f(x)dx = -\int f(x)\log f(x)dx$.

Proof:
$$
\begin{aligned}
-\int g(x)\log f(x)dx &= -\int g(x)\log \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}dx \\
&= -\int g(x)\left(\log \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2} \cdot \log e\right) \\
&= -\log \frac{1}{\sqrt{2\pi}} \int g(x)dx + \frac{\log e}{2} \int g(x)x^2 dx \\
&= -\log \frac{1}{\sqrt{2\pi}} + \frac{\log e}{2}.
\end{aligned}
$$

□

The Boltzmann distribution

- States $\{1, \ldots, m\}$, energies $E_1, \ldots, E_m$.
- $\Pr[X = E_i] = C \cdot e^{-\beta E_i}$ for $\beta > 0$ and $C = 1/\sum_i e^{-\beta \cdot E_i}$
- We will denote it by $\sim B(\beta, E_1, \ldots, E_m)$
- Like the exponential distribution (i.e., $f(x) = \lambda e^{-\lambda x}$), but discrete.
  - Describes a (discrete) physical system that can take states $\{1, \ldots, m\}$ with energies $E_1, \ldots, E_m$.
  - Probability is inverse to energy

Theorem 10

Let $X \sim B(\beta, E_1, \ldots, E_m)$. Then $H(Y) \leq H(X)$ for any rv Y over $\{E_1, \ldots, E_m\}$, with $\mathrm{E}\, Y = \mathrm{E}\, X$.

- The Boltzmann distribution is maximal among all distributions of the same energy.

Proving Theorem 10

- $\sim B(\beta, E_1, \ldots, E_m)$ and $\mathrm{E}\,Y = \mathrm{E}\,X$
- Let $X \sim (p_1, \ldots, p_m)$ and $Y \sim (q_1, \ldots, q_m)$.
- $H(Y) \leq \sum_i q_i \log p_i$ (*Q*3 in Handout 1)
- Let $C = 1/\sum_i e^{-\beta \cdot E_i}$.

  Then
$$\begin{aligned}\sum_i q_i \log p_i &= \sum_i q_i \log(C \cdot e^{-\beta E_i}) \\ &= \sum_i q_i \log C - \sum_i q_i \cdot \beta E_i \cdot \log e \\ &= \log C - \beta \cdot \log e \cdot \sum_i q_i E_i \\ &= \log C - \beta \cdot \log e \cdot \mathrm{E}\,X\end{aligned}$$

- Hence, $\sum_i q_i \log p_i = \sum_i p_i \log p_i$. $\square$

The uniform distribution

- $X \sim [a, b]$.
- $\mathrm{E}\, X = \frac{1}{2}(a + b)$ and $\mathrm{V}\, X = \frac{1}{12}(b - a)^2$
- What come to mind when saying "X takes values in $[0, 1]$".

Theorem 11

$h(X) \leq -h(\sim [a, b])$, for any RV with $\mathrm{Supp}(X) \subseteq [a, b]$.

Proof: HW

Using diff. entropy to bound discrete entropy

Proposition 12

Let $X \sim (p_1, p_2, \ldots)$, then $H(X) \leq \frac{\log 2\pi e}{2} \cdot \left(\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2 - \frac{1}{12}\right)$

We assume wlg. that $p_i = \Pr[X = i]$.

- Let $U \sim [0, 1]$, let $\tilde{X} = X + U$ and let $f_{\tilde{X}}$ be the density function of $\tilde{X}$.

$$\begin{aligned}
H(X) &= -\sum_{i=1}^{\infty} p_i \log p_i \\
&= -\sum_{i=1}^{\infty} \left(\int_i^{i+1} f_{\tilde{X}}(x)dx\right) \cdot \log p_i = -\sum_{i=1}^{\infty} \int_i^{i+1} f_{\tilde{X}}(x) \log p_i dx \\
&= -\sum_{i=1}^{\infty} \int_i^{i+1} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x)dx \qquad (f_{\tilde{X}}(x) = p_i \text{ for all } x \in [i, i+1]) \\
&= -\int_1^{\infty} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x)dx \\
&= h(\tilde{X})
\end{aligned}$$

Using diff. entropy to bound discrete entropy, cont.

- Hence,

$$\begin{aligned}
H(X) &= h(\tilde{X}) \\
&\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X}) \\
&= \frac{1}{2}\log(2\pi e)\,(\mathrm{V}(X) + \mathrm{V}(U)) \\
&= \frac{\log 2\pi e}{2} \cdot \left(\left(\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2\right) + \frac{1}{12}\right)
\end{aligned}$$

- How good is this bound?
- Let $X \sim (\frac{1}{2}, \frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.
- Proposition 12 grantees that $H(X) \leq \frac{\log 2\pi e}{2}(\frac{1}{4} + \frac{1}{12}) \sim 1.255$