Application of Information Theory, Lecture 3
Graph Covering, Differential Entropy

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Part I

Applications to Graph Covering

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Let G be a graph over $[n]$, let $Z \leftarrow \mathrm{nonIs}(G)$ and let $\hat{\chi}$ be a (valid) coloring of G such that $H(\hat{\chi}(Z))$ is minimal. Then $\mathrm{content}(G) := \frac{|\mathrm{nonIs}(G)|}{n} \cdot H(\hat{\chi}(Z))$.

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$$\begin{aligned} 0 = H(X|Y_1, \ldots, Y_t) &= H(X, Y_1, \ldots, Y_t) - H(Y_1, \ldots, Y_t) \\ &\geq H(X) + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i) \\ &\geq \log n + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i). \end{aligned}$$

- $Y_1, \ldots, Y_t$ are independent conditioned on X —
  $\Pr[Y_1 = y_1 \wedge Y_2 = y_2 \mid X = z] = \Pr[Y_1 = y_1 \mid X = z] \cdot \Pr[Y_2 = y_2 \mid X = z]$
- Hence, $H(Y_1, \ldots, Y_t|X) = \sum_i H(Y_i|X)$ (board)
- We conclude that $\sum_i H(Y_i) - \sum_i H(Y_i|X) \geq \log n$
- Since $H(Y_i) = H(\chi_i(Z_i))$ and $H(Y_i|X) = (1 - \frac{|\mathrm{nonIs}(G_i)|}{n}) \cdot H(\chi_i(Z_i))$,

  it follows that $\sum_i H(\chi_i(Z_i)) \frac{|\mathrm{nonIs}(G_i)|}{n} \geq \log n$. $\square$

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Proof: ?

Scrambling permutations

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- Hence, $|\mathcal{S}| \geq \frac{2}{\log e} \cdot \frac{n}{n-1} \cdot \log(n-1) \geq \frac{2}{\log e} \log n$

Part II

Differential Entropy

Entropy of continues random variable

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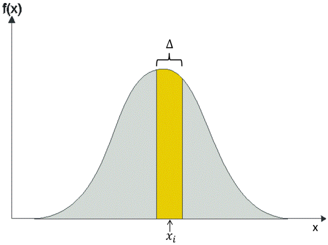

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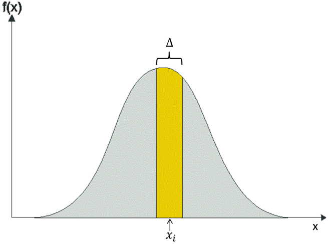

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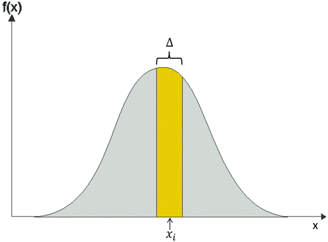

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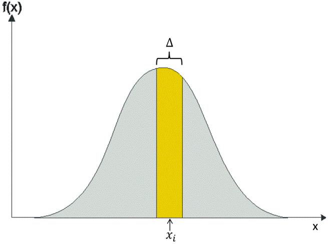

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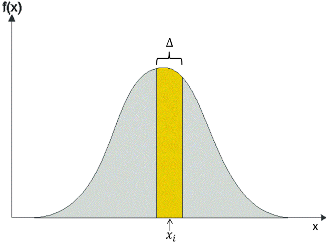

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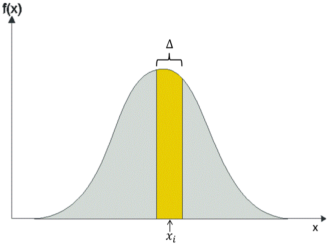

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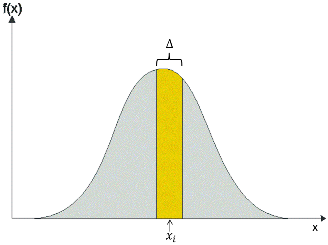

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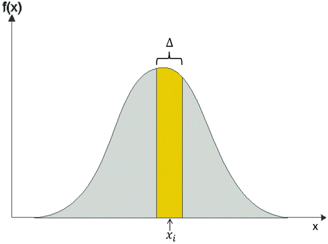

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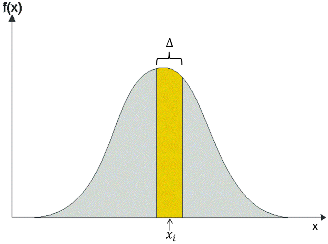

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- Used for comparing two distributions

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- In contradiction with “reversible laws"

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- CLT and the normal distribution where known and studied way before Shannon, yet this striking property was not known until his theory.

The normal distribution, cont.

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- By Jensen: $\forall t_1, \ldots, t_n$ and $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_i \lambda_i = 1$: $\sum_i \lambda_i \log t_i \leq \log \sum_i \lambda_i t_i$

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- Hence, $-\int g(x)\log g(x) \le -\int g(x)\log q(x)$

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- Let $X \sim (p_1, \ldots, p_m)$ and $Y \sim (q_1, \ldots, q_m)$.
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- Let $C = 1/\sum_i e^{-K \cdot E_i}$.

  Then
$$\begin{aligned}
\sum_i q_i \log p_i &= \sum_i q_i \log(C \cdot e^{-KE_i}) \\
&= \sum_i q_i \log C - \sum_i q_i \cdot KE_i \cdot \log e \\
&= \log C - K \cdot \log e \sum_i q_i E_i \\
&= \log C - K \cdot \log e \cdot \mathrm{E}\, X
\end{aligned}$$

- Hence, $\sum_i q_i \log p_i = \sum_i p_i \log p_i$. $\square$

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Proof: ?

Differential entropy bound on discrete entropy

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Proposition 12

Let $X \sim (p_1, p_2, \ldots)$, then $H(X) \leq \frac{\log 2\pi e}{2} \cdot \left(\sum_{i=1}^{\infty} p_i i^2 - (\sum_{i=1}^{\infty} p_i i)^2 - \frac{1}{12}\right)$

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Differential entropy bound on discrete entropy, cont.

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Differential entropy bound on discrete entropy, cont.

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Differential entropy bound on discrete entropy, cont.

- Hence,

$$
\begin{aligned}
H(X) &= h(\tilde{X}) \\
&\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X})
\end{aligned}
$$

Differential entropy bound on discrete entropy, cont.

- Hence,

$$\begin{aligned} H(X) &= h(\tilde{X}) \\ &\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X}) \\ &= \frac{1}{2}\log(2\pi e)\,(\mathrm{V}(X)+\mathrm{V}(U)) \end{aligned}$$

Differential entropy bound on discrete entropy, cont.

- Hence,

$$\begin{aligned} H(X) &= h(\tilde{X}) \\ &\leq \frac{1}{2}\log(2\pi e)\,\mathsf{V}(\tilde{X}) \\ &= \frac{1}{2}\log(2\pi e)\,(\mathsf{V}(X)+\mathsf{V}(U)) \\ &= \frac{\log 2\pi e}{2}\cdot\left(\left(\sum_{i=1}^{\infty} p_i i^2 - (\sum_{i=1}^{\infty} p_i i)^2\right)+\frac{1}{12}\right) \end{aligned}$$

Differential entropy bound on discrete entropy, cont.

- Hence,

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- How good is this bound?

Differential entropy bound on discrete entropy, cont.

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- How good is this bound?
- Let $X \sim (\frac{1}{2},\frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.

Differential entropy bound on discrete entropy, cont.

- Hence,

$$
\begin{aligned}
H(X) &= h(\tilde{X}) \\
&\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X}) \\
&= \frac{1}{2}\log(2\pi e)\,(\mathrm{V}(X) + \mathrm{V}(U)) \\
&= \frac{\log 2\pi e}{2} \cdot \left(\left(\sum_{i=1}^{\infty} p_i i^2 - (\sum_{i=1}^{\infty} p_i i)^2 \right) + \frac{1}{12} \right)
\end{aligned}
$$

- How good is this bound?
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- Proposition 12 grantees that $H(X) \leq \frac{\log 2\pi e}{2}(\frac{1}{4} + \frac{1}{12}) \sim 1.255$