Application of Information Theory, Lecture 12

Accessible Entropy and Statistically Hiding Commitments

Handout Mode

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January 14, 2018

Section 1

Commitment Schemes

Motivation

- Digital analogue of a safe
- Numerous applications (e.g., zero-knowledge, coin-flipping, secure computations)

Definition

μ is negligible, denoted $\mu(n) = \mathsf{neg}(n)$, if $\forall p \in \mathsf{poly}\ \exists n' \in \mathbb{N}$ s.t. $\mu(n) < \frac{1}{p(n)}$ forall $n > n'$.

Definition 1 (Commitment scheme)

An efficient two-stage protocol (S, R).

- Commit stage: The sender S has private input bit $b \in \{0, 1\}$ and a common input is 1^n. Let trans be the transcript of this stage.
- Reveal stage: S sends the pair (r, b) to R, and R accepts if trans is consistent with $\mathsf{S}(\sigma, r)$.

Hiding: Let $V_n^{\mathsf{R}^*}(b)$ be R^*'s *view* in (the commit stage of) $(\mathsf{S}(b), \mathsf{R}^*)(1^n)$.
Then for any R^*: $\Delta^{\mathsf{R}^*}(V_n^{\mathsf{R}^*}(0), V_n^{\mathsf{R}^*}(1)) = \mathsf{neg}(n)$.

Binding: The following happens with negligible probability for any S^*:

> $\mathsf{S}^*(1^n)$ *interacts with* $\mathsf{R}(1^n)$ *in the commit stage resulting in transcript* trans. *Then* S^* *outputs two strings* r_0 *and* r_1 *such that* $\mathsf{R}(\mathsf{trans}, r_0, 0) = \mathsf{R}(\mathsf{trans}, r_1, 1) = \texttt{Accept}$.

Alternative Binding definition: Assume that following the interaction S^* outputs a pair (r, b) with $\mathsf{R}(\mathsf{trans}, r, b) = \texttt{Accept}$. Let V^{S^*} be S^*'s view in (the commit stage of) $(\mathsf{S}^*, \mathsf{R}^*)(1^n)$. Then $\mathsf{H}(b | V^{\mathsf{S}^*}) = \mathsf{neg}(n)$.

Definition cont.

- Naturally extends to strings
- Hiding: Perfect, statistical, computational.
- Binding: Perfect, statistical, computational.
- Impossible to have simultaneously both properties to be statistical.
- OWF is necessary assumption
- OWFs imply both statistically binding and computationally hiding commitments, and (more difficult) computationally binding and statistically hiding commitments.
- We focus on computationally binding, and statistically hiding commitments (SHC)

Section 2

Inaccessible Entropy

Motivation

Definition 2 (collision resistant hash family (CRH))

Function family $\mathcal{H} = \{\mathcal{H}_n \colon \{0,1\}^n \mapsto \{0,1\}^{n/2}\}$ is collision resistant, if $\forall$ PPT A

$$\Pr_{\substack{h \leftarrow \mathcal{H}_n \\ (x,x') \leftarrow \mathsf{A}(1^n,h)}}[x \neq x' \in \{0,1\}^* \wedge h(x) = h(x')] = \operatorname{neg}(n)$$

- Implies SHC. (?) Believed not to be implied by OWFs.
- Assume for simplicity that $h \in \mathcal{H}_n$ is $2^{n/2}$ to 1 and that a PPT cannot find a collision in any $h \in \mathcal{H}_n$
- Given $h(U_n)$, the (min) entropy of U_n is $n/2$.
- Consider PPT A that on input h first outputs h, y, and then outputs $x \in h^{-1}(y)$ (possibly using additional random coins)
- What is the entropy of x given (h, y) and the coins A's used to sample y? (essentially) 0!
- The generator $G(h,x) = (h, h(x), x)$ has inaccessible entropy $n/2$
- Does inaccessible entropy generator implies SHC?
- Does OWF implies inaccessible entropy generator?

Real entropy of block generator

- Let $G\colon \{0,1\}^n \mapsto (\{0,1\}^{\ell(n)})^{m(n)}$ be an m-block generator
- Let $(G_1,\ldots,G_m) = G(U_n)$
- For $\mathbf{g} = (g_1,\ldots,g_m) \in \mathrm{Supp}(G_1,\ldots,G_m)$, let

$$\mathrm{RealH}_G(\mathbf{g}) := \sum_{i\in[m]} H_{G_i|G_{\le i-1}}(g_i|g_{\le i-1})$$

- The real Shannon entropy of G, wrt security parameter n, is

$$\mathop{\mathrm{E}}_{\mathbf{g}\leftarrow G(U_n)}\left[\mathrm{RealH}_{G,n}(\mathbf{g})\right]$$

- $\mathrm{E}_{\mathbf{g}\leftarrow G(U_n)}\left[\mathrm{RealH}_{G,n}(\mathbf{g})\right] = \sum_{i\in[m]} H(G_i|G_{\le i-1}) = H(G(U_n))$

Accessible entropy of block generator

- Let G be an m-block generator.
- Let $\widetilde{G}$ be an m-block generator, that uses coins r_i before outputting its i'th block g_i.
- $\widetilde{G}$ is consistent with respect to G, if its output is always in the support of G

  Hereafter, we only consider consistent generators
- $\widetilde{T} = (\widetilde{R}_1, \widetilde{G}_1, \ldots, \widetilde{R}_m, \widetilde{G}_m)$— the rv's induced by random execution of $\widetilde{G}(1^n)$
- $$\begin{aligned}\mathrm{AccH}_{\widetilde{G},n}(\mathbf{t}) &:= \sum_{i\in[m]} H_{\widetilde{G}_i|\widetilde{R}_1,\widetilde{G}_1,\ldots,\widetilde{R}_{i-1},\widetilde{G}_{r-1}}(g_i|r_1,g_1,\ldots,r_{i-1},g_{i-1})\\ &= \sum_{i\in[m]} H_{\widetilde{G}_i|\widetilde{R}_{i-1}}(g_i|r_{\leq i-1})\end{aligned}$$
- The accessible entropy of $\widetilde{G}$ (wrt G), and n, is at most k, if

  $\Pr_{\mathbf{t}\leftarrow\widetilde{T}}\left[\mathrm{AccH}_{\widetilde{G},n}(\mathbf{t}) > k\right] \leq \mathrm{neg}(n)$. Why not $\mathrm{E}_{\mathbf{t}\leftarrow\widetilde{T}}\left[\mathrm{AccH}_{\widetilde{G},n}(\mathbf{t})\right]$?
- inaccessible entropy
- We will omit n when clear from the contex

Example

- Let $\mathcal{H} = \{\mathcal{H}_n \colon \{0,1\}^n \mapsto \{0,1\}^{n/2}\}$ be $2^{n/2}$-to-1 collision resistant, and assume for simplicity that a PPT cannot find a collision for any $h \in \mathcal{H}_n$.
- Let G be the 3-block generator $G(h,x) = (h, h(x), x)$
- Real entropy of G is $\log|\mathcal{H}_n| + n$
- Accessible entropy of G is $\log|\mathcal{H}_n| + \frac{n}{2}$

Section 3

Manipulating Inaccessible Entropy

Entropy equalization

Let G be m-bit generator.

For $\ell \in \text{poly}$ let $G^{\bigotimes \ell}$ be the following $(\ell - 1) \cdot m$-bit generator

$$G^{\bigotimes \ell}(x_1, \ldots, x_\ell, i) = G(x_1)_i, \ldots, G(x_1)_m, \ldots, G(x_\ell)_1, \ldots, G(x_\ell)_{i-1}$$

- Assume the accessible entropy of G is (at most) k_A, then $k_A^{\bigotimes \ell}$, the accessible entropy of $G^{\bigotimes \ell}$, is at most $k_A(\ell - 2) + m$.
- Assume the real entropy of G is k_R, then
  1. For any $i \in [(\ell - 1) \cdot m]$ and $(g_{\leq i-1}) \in \text{Supp}(G^{\bigotimes \ell}_{\leq i-1})$:

     $$H(G_i^{\bigotimes \ell} | G^{\bigotimes \ell}_{\leq i-1}) \geq k_R / m$$

  2. $k_R^{\bigotimes \ell}$, the real entropy of $G^{\bigotimes \ell}$, is at least $(\ell - 1)K_R$
- Assume $k_R \geq k_A + 1$, then for $\ell = m + 2$, it holds that $k_R^{\bigotimes \ell} \geq k_A^{\bigotimes \ell} + 1$

Parallel repetition

Let G be an m-block generator and for $\ell \in \mathsf{poly}$, let G^ℓ be the ℓ-fold parallel repetition of G.

- Assume accessible entropy of G is (at most) k_A, then the accessible entropy of G is at most $k_A^\ell = \ell k_A$.
- Assume $H(G_i|G_{\leq i-1}) = k_R$ for any $i \in [m]$, then for any $i \in [m]$ and $(g^\ell_{\leq i-1}) \in \mathsf{Supp}(G^\ell_{\leq i-1})$ it holds that

$$k^\ell_{min} = \mathrm{H}_\infty(G^\ell_i|G^\ell_{\leq i-1}) \approx \ell k_R$$

- If $k_A \leq k_R - 1$, then $\forall n \in \mathsf{poly}\ \exists \ell \in \mathsf{poly}$ such that $\ell k^\ell_{min} > k^\ell_A + n$

Section 4

Inaccessible Entropy from OWF

The generator

Definition 3

Given a function $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, let G be the $(n+1)$-block generator

$$G(x) = f(x)_1, \ldots, f(x)_n, x$$

Lemma 4

Assume that f is a OWF then G has accessible entropy at most $n - \log n$.

- Recall f is OWF if
  $\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{Inv}(f(x)) \in f^{-1}(f(x))\right] = \operatorname{neg}(n)$ for any PPT Inv.
- The real entropy of G is n
- Hence, inaccessible entropy gap is $\log n$
- Proof idea

Proving Lemma 4

Let $\widetilde{G}$ be a PPT, and assume $\Pr\left[\mathrm{AccH}_{G,\widetilde{G}}(\widetilde{T}) \geq n - \log n\right] \geq \varepsilon = \frac{1}{\mathrm{poly}(n)}$.
(recall $\widetilde{T} = (\widetilde{R}_1, \widetilde{G}_1, \ldots, \widetilde{R}_m, \widetilde{G}_m)$ is the coins and output blocks of $\widetilde{G}$)

Algorithm 5 ($\mathsf{Inv}(z)$)

1. For $i = 1$ to n, do the following for n^2/ε times:
   1.1 Sample r_i uniformly at random and let g_i be the i'th output block of $\widetilde{G}(r_1, \ldots, r_i)$.
   1.2 If $g_i = z_i$, move to next value of i.
2. Finish the execution of $\widetilde{G}(r_1, \ldots, r_{n+1})$, and output its $(n+1)$ output block.

- We start by assuming that Inv is unbounded (replace n^2/ε with ∞)
- $\widehat{T} = (\widehat{R}_1, \widehat{G}_1, \ldots, \widehat{R}_{n+1}, \widehat{G}_{n+1})$ is the (final) values of $(r_1, g_1, \ldots, r_{n+1}, g_{n+1})$ in a random execution of $\mathsf{Inv}(f(U_n))$.

$\widetilde{T}$ vs. $\widehat{T}$

- Fix $\mathbf{t} = (r_1, g_1, \ldots, r_{n+1}, g_{n+1}) \in \operatorname{Supp}(\widetilde{T})$
- Let $P(\mathbf{t}) = \prod_{i=1}^{n+1} \Pr\left[\widetilde{R}_i = r_i \mid (\widetilde{R}_{\leq i-1}, \widetilde{G}_i) = (r_{\leq i-1}, g_i)\right]$

$$\begin{aligned}
\Pr_{\widetilde{T}}[t] &= \Pr[\widetilde{G}_1 = g_1] \cdot \Pr[\widetilde{R}_1 = r_1 | \widetilde{G}_1 = g_1] \cdot \Pr[\widetilde{G}_2 = g_2 | \widetilde{R}_1 = r_1] \\
&\cdot \quad \Pr[\widetilde{R}_2 = r_2 | \widetilde{G}_2 = g_2] \cdots \\
&= P(\mathbf{t}) \cdot \Pr[\widetilde{G}_1 = g_1] \cdot \Pr[\widetilde{G}_2 = g_2 | \widetilde{R}_1 = r_1] \cdots \\
&= P(\mathbf{t}) \cdot 2^{-\sum_{i=1}^{m} H_{\widetilde{G}_i | \widetilde{R}_{\leq i-1}}(g_i | r_{\leq i-1})} \\
&= P(\mathbf{t}) \cdot 2^{-\operatorname{AccH}_{\widetilde{G}}(\mathbf{t})}
\end{aligned}$$

- $\Pr_{\widehat{T}}[\mathbf{t}] = \Pr[f(U_n) = g_{\leq n}] \cdot \Pr\left[\widetilde{G}_{n+1} = g_{n+1} | \widetilde{R}_{\leq n} = r_{\leq n}\right] \cdot P(\mathbf{t})$
- $\Pr_{\widehat{T}}[\mathbf{t}] = \frac{\Pr\left[f(U_n)=g_{\leq n}\right] \cdot \Pr\left[\widetilde{G}_{n+1}=g_{n+1} | \widetilde{R}_{\leq n}=r_{\leq n}\right]}{2^{-\operatorname{AccH}_{G,\widetilde{G}}(\mathbf{t})}} \cdot \Pr_{\widetilde{T}}[\mathbf{t}]$

$\widetilde{T}$ vs. $\widehat{T}$ cont.

- $\mathbf{t} = (r_1, g_1, \ldots, r_{n+1}, g_{n+1}) \in \mathrm{Supp}(\widetilde{T})$
- $\Pr_{\widehat{T}}[\mathbf{t}] = \frac{\Pr\left[f(U_n)=g_{\le n}\right]\cdot\Pr\left[\widetilde{G}_{n+1}=g_{n+1}|\widetilde{R}_{\le n}=r_{\le n}\right]}{2^{-\mathrm{AccH}_{G,\widetilde{G}}(\mathbf{t})}} \cdot \Pr_{\widetilde{T}}[\mathbf{t}]$
- Note that $\Pr\left[f(U_n) = g_{\le n}\right] \cdot \frac{1}{\left|f^{-1}(g_{\le n})\right|} = 2^{-n}$
- Hence, for $\mathbf{t}$ with

  1. $\mathrm{AccH}_{G,\widetilde{G}}(\mathbf{t}) \ge n - \log n$, and
  2. $\Pr\left[\widetilde{G}_{n+1} = g_{n+1}|\widetilde{R}_{\le n} = r_{\le n}\right] \ge \frac{\alpha}{\left|f^{-1}(g_{\le n})\right|}$.

  It holds that

$$\Pr_{\widetilde{T}}[\mathbf{t}] \ge \frac{\alpha}{n} \cdot \Pr_{\widehat{T}}[\mathbf{t}] \tag{1}$$

Inv's success probability

Let $\mathcal{S} \subseteq \mathrm{Supp}(\widetilde{T})$ denote the set of transcripts $\mathbf{t} = (r_1, g_1, \ldots, r_{n+1}, g_{n+1})$ with

1. $\mathrm{AccH}_{\widetilde{G}}(\mathbf{t}) \geq n - \log n$,
2. $H_{\widetilde{G}_i \mid \widetilde{R}_{\leq i-1}}(g_i \mid r_{\leq i-1}) \leq \log(\frac{4n}{\varepsilon})$ for all $i \in [n]$,
3. $H_{\widetilde{G}_{n+1} \mid \widetilde{R}_{\leq n}}(g_{n+1} \mid r_{\leq n}) \leq \log(\frac{4}{\varepsilon} \cdot \left|f^{-1}(g_{\leq n})\right|$.

- $\Pr_{\widetilde{T}}\left[\exists i \in [n] \colon H_{\widetilde{G}_i \mid \widetilde{R}_{\leq i-1}}(g_i \mid r_{\leq i-1}) > \log(\frac{4n}{\varepsilon})\right] \leq n \cdot \frac{\varepsilon}{4n} = \varepsilon/4$
- $\Pr_{\widetilde{T}}\left[H_{\widetilde{G}_{n+1} \mid \widetilde{R}_{\leq n}}(g_{n+1} \mid r_{\leq n}) > \log(\frac{4}{\varepsilon} \cdot \left|f^{-1}(g_{\leq n})\right|\right] \leq \varepsilon/4$
- $\Pr_{\widetilde{T}}[\mathcal{S}] \geq \Pr\left[\mathrm{AccH}_{G,\widetilde{G}}(T) \geq n - \log n\right] - 2 \cdot \frac{\varepsilon}{4} \geq \frac{\varepsilon}{2}$
- By Eq. (1): $\Pr_{\widehat{T}}[\mathcal{S}] \geq \frac{\varepsilon/4}{n} \cdot \Pr_{\widetilde{T}}[\mathcal{S}] \geq \frac{\varepsilon^2}{8n} \ldots$

Back the bounded version of Inv.

- For $z \in \{0,1\}^n$ for which $\exists (r_1, z_1, \ldots, r_n, z_n, \ldots) \in \mathcal{S}$:
  $\Pr[\mathrm{Inv}(z) \text{ aborts }] \leq n \cdot (1 - \frac{\varepsilon}{4n})^{n^2/\varepsilon} = O(n \cdot 2^{-n}) \leq \frac{1}{2}$
- Hence, $\Pr_{\widehat{T}}[\mathcal{S}] \geq \frac{\varepsilon^2}{16n} \implies \Pr_{x \leftarrow \{0,1\}^n}\left[\mathrm{Inv}(f(x)) \in f^{-1}(f(x))\right] \geq \frac{\varepsilon^2}{16n}$

Section 5

Statistically Hiding Commitment from Inaccessible Entropy Generator

High-level description

- Entropy equalization + gap amplification to get generator that has the same min-entropy in each block and whose accessible entropy is n-bit smaller than the sum of the min entropies.
- Use "hashing protocol" to get a “generator” with zero accessible entropy block
- Use a a random block to mask the committed bit, to get a weakly binding SHC
- Amplify the above into full-fledged SHC