Application of Information Theory, Lecture 2
Joint & Conditional Entropy, Mutual Information

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Part I

Joint and Conditional Entropy

Joint entropy

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Joint entropy, cont.

- The joint entropy of $(X_1, \ldots, X_n) \sim p$, is

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Conditional entropy, cont.

- Example

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$$\begin{aligned} H(X|Y) &= \mathop{\mathrm{E}}_{y \leftarrow Y} H(X|_{Y=y}) \\ &= \frac{3}{4} H(X|_{Y=0}) + \frac{1}{4} H(X|_{Y=1}) \\ &= \frac{3}{4} H(\frac{1}{3}, \frac{2}{3}) + \frac{1}{4} H(1,0) = 0.6887 \neq H(Y|X). \end{aligned}$$

Conditional entropy, cont..

- $$H(X|Y,Z) = \mathop{\mathrm{E}}_{(y,z) \leftarrow (Y,Z)} H(X|_{Y=y,Z=z})$$

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- Note that $H(Y|X = x)$ might be larger than $H(Y)$ for some $x \in \mathrm{Supp}(X)$.

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- $H(Y|X) = H(X,Y) - H(X)$ is as an alternative definition for $H(Y|X)$.

Chain rule (for the entropy function)

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For rvs X, Y, it holds that $H(X, Y) = H(X) + H(Y|X)$.

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- Proof immediately follow by the grouping axiom:

Chain rule (for the entropy function)

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$H(Y|X) \leq H(Y)$

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Jensen inequality: for any concave function f, values $t_1, \ldots, t_k$ and $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_i \lambda_i = 1$, it holds that $\sum_i \lambda_i f(t_i) \leq f(\sum_i \lambda_i t_i)$.

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  Follows from $H(X,Y) = H(X) + H(Y|X)$.
  - Left inequality since $H(Y|X)$ is non negative.
  - Right inequality since $H(Y|X) \leq H(Y)$.
- $H(X,|Z) = H(X|Z) + H(Y|X,Z)$ (by chain rule)
- $H(X|Y,Z) \leq H(X|Y)$

  Proof:
$$\begin{aligned}
H(X|Y,Z) &= \mathop{\mathrm{E}}_{(z,y)\leftarrow(Z,Y)} H(X|_{(Y,Z)=(z,y)}) \\
&= \mathop{\mathrm{E}}_{y\leftarrow Y} \mathop{\mathrm{E}}_{z\leftarrow Z|_{Y=y}} H(X|_{(Y,Z)=(z,y)}) \\
&= \mathop{\mathrm{E}}_{y\leftarrow Y} \mathop{\mathrm{E}}_{z\leftarrow Z|_{Y=y}} H((X|_{Y=y})|_{Z=z}) \\
&\leq \mathop{\mathrm{E}}_{y\leftarrow Y} H(X|_{Y=y}) \\
&= H(X|Y).
\end{aligned}$$

Chain rule (for the entropy function), general case

Claim 2

For rvs $X_1, \ldots, X_k$, it holds that
$H(X_1, \ldots, X_k) = H(X_i) + H(X_2|X_1) + \ldots + H(X_k|X_1, \ldots, X_{k-1})$.

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$\implies \quad t \geq \log n! = \Theta(n \log n)$

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We are now certain that we drew the graph of the (two-dimensional) entropy function right...

Part II

Mutual Information

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Claim 4 (Chain rule for mutual information)

For rvs $X_1, \ldots, X_k, Y$, it holds that
$I(X_1, \ldots, X_k; Y) = I(X_1; Y) + I(X_2; Y|X_1) + \ldots + I(X_k; Y|X_1, \ldots, X_{k-1})$.

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Part III

Data processing

Data processing Inequality

Definition 5 (Markov Chain)

Rvs $(X, Y, Z) \sim p$ form a Markov chain, denoted $X \to Y \to Z$, if $p(x, y, z) = p_X(x) \cdot p_{Y|X}(y|x) \cdot p_{Z|Y}(z|y)$, for all x, y, z.

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- Since $I(X; Y|Z) \geq 0$, we conclude $I(X; Y) \geq I(X; Z)$.$\square$

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Theorem 7 (Fano's inequality)

For any rvs X and Y, and any (even random) g, it holds that

$$h(P_e) + P_e \log |\mathcal{X}| \geq H(X|\hat{X}) \geq H(X|Y)$$

for $\hat{X} = g(Y)$ and $P_e = \Pr\left[\hat{X} \neq X\right]$.

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- The inequality can be weekend to $1 + P_e \log|\mathcal{X}| \geq H(X|Y)$,

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- How well can we guess X from Y?
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