Application of Information Theory, Lecture 7

Relative Entropy

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Part I

Statistical Distance

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Theorem 1 (this lecture)

Let X rv over $[m]$. Assume $H(X) \geq \log m - \varepsilon$, then $\mathrm{SD}(X, \sim [m]) \leq \sqrt{\varepsilon \cdot \frac{\ln 2}{2}} = O(\sqrt{\varepsilon})$

Part II

Relative Entropy

Section 1

Definition and Basic Facts

Definition

- For $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$, let

$$D(p\|q) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i}$$

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- Many different interpretations
- Main interpretation: the information we gained about X, if we originally thought $X \sim q$ and now we learned $X \sim p$

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- We understand $D(p\|q)$ as the information we gained about X, if we originally thought it is $\sim q$ and now we learned it is $\sim p$

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- $Y \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$, but

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- Generally, a distribution can change if we condition on event E

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- $D(p\|q) \geq 0$, with equality iff $p = q$ (hw)

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- Example: $\sum_{i=1}^{n} q_i = \frac{1}{2}$, and we were told that $i \le n$ or $i > n$, we got one bit of information

Section 2

Axiomatic Derivation

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Let $\tilde{D}$ is a continuous and symmetric (wrt each distribution) function such that

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- Zeros and non-rational q_i's are dealt by continuity

Section 3

Relation to Mutual Information

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$$\mathop{\mathrm{E}}_{Y}\left[D(p_Y \| q)\right] = \Pr\left[Y=0\right] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) + \Pr\left[Y=1\right] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m)$$

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$$\begin{aligned}\mathop{\mathrm{E}}_{Y}\left[D(p_Y \| q)\right] &= \Pr\left[Y = 0\right] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\ &\quad + \Pr\left[Y = 1\right] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m)\end{aligned}$$

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- $(X|_{Y=j}) \sim p_j = (p_{j,1}, \ldots, p_{j,m})$, $\qquad p_{j,i} = \Pr\left[X = i | Y = j\right]$

$$\begin{aligned}
\mathop{\mathrm{E}}_{Y}\left[D(p_Y \| q)\right] &= \Pr\left[Y=0\right] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
&\quad + \Pr\left[Y=1\right] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m) \\
&= \Pr\left[Y=0\right] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr\left[Y=1\right] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i}
\end{aligned}$$

Mutual information as expected relative entropy

Claim 2

$\mathrm{E}_{y \leftarrow Y}\left[D(X|_{Y=y} \| X)\right] = I(X;Y)$.

Proof:

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&= \Pr\left[Y=0\right] \cdot \sum p_{0,i} \log p_{0,i} + \Pr\left[Y=1\right] \cdot \sum p_{1,i} \log p_{1,i} \\
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&= \Pr\left[Y=0\right]\cdot \sum p_{0,i}\log p_{0,i} + \Pr\left[Y=1\right]\cdot\sum p_{1,i}\log p_{1,i} \\
&\quad - \Pr\left[Y=0\right]\cdot \sum p_{0,i}\log q_i - \Pr\left[Y=1\right]\cdot\sum p_{1,i}\log q_i \\
&= -H(X|Y) - \sum(\Pr\left[Y=0\right]\cdot p_{0,i} + \Pr\left[Y=1\right]\cdot p_{1,i})\log q_i
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&\quad - \Pr\left[Y=0\right] \cdot \sum p_{0,i} \log q_i - \Pr\left[Y=1\right] \cdot \sum p_{1,i} \log q_i \\
&= -H(X|Y) - \sum (\Pr\left[Y=0\right] \cdot p_{0,i} + \Pr\left[Y=1\right] \cdot p_{1,i}) \log q_i \\
&= -H(X|Y) + H(X)
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Equivalent definition for mutual information

Claim 3

Let $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$.

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$$D(p \| p_X p_Y) = \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p_X(x) p_Y(y)}$$

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$$= \sum_{x,y} p(x, y) \log \frac{p_{X|Y}(x|y)}{p_X(x)}$$

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&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y)
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&= -\sum_{x,y} p(x, y) \log p_X(x) + \sum_{x,y} p(x, y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y)
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&= -\sum_{x,y} p(x, y) \log p_X(x) + \sum_{x,y} p(x, y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y)
\end{aligned}$$

Equivalent definition for mutual information

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Let $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$.

- Proof:

$$\begin{aligned}
D(p\|p_X p_Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x)p_Y(y)} \\
&= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\
&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y) = I(X;Y). \square
\end{aligned}$$

Equivalent definition for mutual information

Claim 3

Let $(X, Y) \sim p$, then $I(X;Y) = D(p||p_X p_Y)$.

- Proof:

$$\begin{aligned}
D(p||p_X p_Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x)p_Y(y)} \\
&= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\
&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y) = I(X;Y). \square
\end{aligned}$$

- We will later relate the above two claims.

Section 4

Relation to Data Compression

Wrong code

Wrong code

Theorem 4

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

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- Proof of upperbound (lowerbound is proved similarly)

$$\mathop{\mathrm{E}}_{i\leftarrow p}\left[\ell(i)\right] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil$$

Wrong code

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Let p *and* q *be distributions over* $[m]$*, and let* C *be code with* $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$*. Then*
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.
- Proof of upperbound (lowerbound is proved similarly)

$$\mathop{\mathrm{E}}_{i \leftarrow p}\left[\ell(i)\right] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1)$$

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&= 1 + \sum_i p_i(\log \frac{p_i}{q_i} \frac{1}{p_i})
\end{aligned}$$

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- Can there be a (close) to optimal code for q that is better for p?

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- Can there be a (close) to optimal code for q that is better for p? HW

Section 5

Conditional Relative Entropy

Conditional relative entropy

Conditional relative entropy

For dist. p over $\mathcal{X} \times \mathcal{Y}$, let $p_{\mathcal{X}}$ and $p_{\mathcal{Y}|\mathcal{X}}$ be its marginal and conditional dist.

Conditional relative entropy

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Definition 5

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) := \sum_{x \in \mathcal{X}} p_{\mathcal{X}}(x) \cdot \sum_{y \in \mathcal{Y}} p_{\mathcal{Y}|\mathcal{X}}(y|x) \log \frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)}$$

Conditional relative entropy

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- $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$

Conditional relative entropy

For dist. p over $\mathcal{X} \times \mathcal{Y}$, let $p_{\mathcal{X}}$ and $p_{\mathcal{Y}|\mathcal{X}}$ be its marginal and conditional dist.

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- $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{(X,Y)\sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$
- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} \left[D(Y_q|_{X_p=x} \| Y_q|_{X_q=x}) \right]$$

Conditional relative entropy

For dist. p over $\mathcal{X} \times \mathcal{Y}$, let $p_{\mathcal{X}}$ and $p_{\mathcal{Y}|\mathcal{X}}$ be its marginal and conditional dist.

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For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

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- Numerical example:

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X \ Y	0	1
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For dist. p over $\mathcal{X} \times \mathcal{Y}$, let $p_\mathcal{X}$ and $p_{\mathcal{Y}|\mathcal{X}}$ be its marginal and conditional dist.

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For any two distributions p and q over $\mathcal{X} \times \mathcal{Y}$, it holds that
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Section 6

Data-processing inequality

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For any rv's X and Y and function f, it holds that $D(f(X) \| f(Y)) \leq D(X \| Y)$.

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Relation to Statistical Distance

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- Corollary: For rv X over $[m]$ with $H(X) \geq \log m - \varepsilon$, it holds that $\mathrm{SD}(X, \sim [m]) \leq \sqrt{\frac{\ln 2}{2} \cdot (\log m - H(X))} = \sqrt{\frac{\ln 2}{2} \cdot \varepsilon}$

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  HW

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- Since $y(1-y) \leq \frac{1}{4}$, $\frac{\partial g(x,y)}{\partial y} \leq 0$ for $y < x$.

Proving Thm 8, Boolean case

- Let $p = (\alpha, 1-\alpha)$ and $q = (\beta, 1-\beta)$ and assume $\alpha \geq \beta$
- $\mathrm{SD}(p,q) = \alpha - \beta$
- We will show that
  $D(p\|q) = \alpha \log \frac{\alpha}{\beta} + (1-\alpha)\log\frac{1-\alpha}{1-\beta} \geq \frac{4}{2\ln 2}(\alpha-\beta)^2 = \frac{2}{\ln 2}\mathrm{SD}(p,q)^2$
- Let $g(x,y) = x\log\frac{x}{y} + (1-x)\log\frac{1-x}{1-y} - \frac{4}{2\ln 2}(x-y)^2$

$$\begin{aligned}\frac{\partial g(x,y)}{\partial y} &= -\frac{x}{y\ln 2} + \frac{1-x}{(1-y)\ln 2} - \frac{4}{2\ln 2}2(y-x)\\ &= \frac{y-x}{y(1-y)\ln 2} - \frac{4}{\ln 2}(y-x)\end{aligned}$$

- Since $y(1-y) \leq \frac{1}{4}$, $\frac{\partial g(x,y)}{\partial y} \leq 0$ for $y < x$.
- Since $g(x,x) = 0$, $g(x,y) \geq 0$ for $y < x$. $\square$

Proving Thm 8, general case

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$D(p\|q) \geq D(\hat{P}\|\hat{Q})$ (data-proccessing inequality)

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$$\begin{aligned} D(p\|q) &\geq D(\hat{P}\|\hat{Q}) && \text{(data-proccessing inequality)} \\ &\geq \frac{2}{\ln 2} \cdot \mathsf{SD}(\hat{P},\hat{Q})^2 \end{aligned}$$

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&\geq \frac{2}{\ln 2} \cdot \mathsf{SD}(\hat{P}, \hat{Q})^2 && \text{(the Boolean case)}\\
&= \frac{2}{\ln 2} \cdot \mathsf{SD}(p, q)^2. \ \square
\end{aligned}$$

Section 8

Conditioned Distributions

Main theorem

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Theorem 9

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{U}$*, and let* $Y = (Y_1, \ldots, Y_k)$ *be rv over* $\mathcal{U}^k$*. Then* $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$*.*

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For rv Z, let $Z(z) = \Pr[Z = z]$.

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&= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(y_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(y_1)} \\
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Let $X_1,\ldots,X_k$ be iid over $\mathcal{U}$, and let $Y=(Y_1,\ldots,Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^{k} D(Y_j\|X_j)\leq D(Y\|(X_1,\ldots,X_k))$.

For rv Z, let $Z(z)=\Pr\left[Z=z\right]$.

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D(Y\|X) &= \sum_{\mathbf{y}\in\mathcal{U}^2} Y(\mathbf{y})\log\frac{Y(\mathbf{y})}{X(\mathbf{y})} = \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y})\log\frac{Y_1(y_1)}{X_1(y_1)}\frac{Y_2(y_2)}{X_2(y_2)}\frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)}\\
&= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y})\log\frac{Y_1(y_1)}{X_1(y_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y})\log\frac{Y_2(y_2)}{X_2(y_1)}\\
&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y})\log\frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)}\\
&= D(Y_1\|X_1)+D(Y_2\|X_2)+I(Y_1;Y_2)\geq D(Y_1\|X_1)+D(Y_2\|X_2)
\end{aligned}$$

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&= D(Y_1 \| X_1) + D(Y_2 \| X_2) + I(Y_1; Y_2) \geq D(Y_1 \| X_1) + D(Y_2 \| X_2)
\end{aligned}
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Conditioning distributions, relative entropy case

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Theorem 10

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{X}$, *let* $X = (X_1, \ldots, X_k)$ *and let* W *be an event (i.e., Boolean rv). Then* $\sum_{j=1}^{k} D((X_j|_W) \| X_j) \leq D((X|_W) \| X) \leq \log \frac{1}{\Pr[W]}$.

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$$= \sum_{\mathbf{x} \in \mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{(X|_W)(\mathbf{x})}{X(\mathbf{x})}$$

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$$= \sum_{\mathbf{x} \in \mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{\Pr\left[W | X = \mathbf{x}\right])}{\Pr\left[W\right]} \qquad \text{(Bayes)}$$

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$$= \log \frac{1}{\Pr[W]} + \sum_{\mathbf{x} \in \mathcal{X}^k} (X|_W)(\mathbf{x}) \log \Pr[W|X = \mathbf{x}])$$

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$$\leq \log \frac{1}{\Pr[W]}$$

Conditioning distributions, statistical distance case

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Theorem 11

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathrm{SD}((X_j|_W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

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Proof: follows by Thm 8, and Thm 9.□

Conditioning distributions, statistical distance case

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Proof: follows by Thm 8, and Thm 9.□

Using $(\sum_{j=1}^{k} a_i)^2 \leq k \cdot \sum_{j=1}^{k} a_i^2$, it follows that

Corollary 12

$\sum_{j=1}^{k} \mathrm{SD}((X_j|_W), X_j) \leq \sqrt{k \log(\frac{1}{\Pr[W]})}$, *and*

$\mathrm{E}_{j \leftarrow k} \mathrm{SD}((X_j|_W), X_j) \leq \sqrt{\frac{1}{k} \log(\frac{1}{\Pr[W]})}$

Conditioning distributions, statistical distance case

Theorem 11

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathrm{SD}((X_j|_W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Proof: follows by Thm 8, and Thm 9.$\square$

Using $(\sum_{j=1}^{k} a_i)^2 \leq k \cdot \sum_{j=1}^{k} a_i^2$, it follows that

Corollary 12

$\sum_{j=1}^{k} \mathrm{SD}((X_j|_W), X_j) \leq \sqrt{k \log(\frac{1}{\Pr[W]})}$, *and*

$\mathrm{E}_{j \leftarrow k} \mathrm{SD}((X_j|_W), X_j) \leq \sqrt{\frac{1}{k} \log(\frac{1}{\Pr[W]})}$

Extraction

Numerical example

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- Let $X = (X_1, \ldots, X_{40}) \leftarrow \{0,1\}^{40}$ and let $f \colon \{0,1\}^{40} \mapsto \{0,1\}$ be such that $\Pr[f(X) = 0] = 2^{-10}$.

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- Typical bits are not too biassed, even when conditioning on a very unlikely event.

Extension

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Theorem 13

Let $X = (X_1, \ldots, X_k)$, T *and* V *be rv's over* $\mathcal{X}^k$, $\mathcal{T}$ *and* $\mathcal{V}$ *respectively. Let* W *be an event and assume that the* X_i*'s are iid conditioned on* T*. Then*
$\sum_{j=1}^{k} D((TVX_j)|_W \| (TV)|_W X_j'(T)) \leq \log \frac{1}{\Pr[W]} + \log |\mathrm{Supp}(V|_W)|$,
where $X_j'(t)$ *is distributed according to* $X_j|_{T=t}$.

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Interpretation.

Proving Thm 13

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Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|_{T=t}$.

Proving Thm 13

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$$\sum_{j=1}^{k} D((TVX_j)|_W \| (TV)|_W X_j'(T))$$
$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum_{j=1}^{k} D\big(X_j|_{W,V=v,T=t} \| (X_j|_{T=t}\big)\Big]$$

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$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\left[\sum D((X_j|\underbrace{W, V = v}_{W'})|_{T=t} \| (X_j|T = t)\right]$$

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&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum_{j=1}^{k} D\big(X_j|_{W,V=v,T=t} \| (X_j|_{T=t}\big)\Big] && \text{(chain rule)} \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum D\big((X_j|\underbrace{W, V=v}_{W'})|_{T=t} \| (X_j|T=t\big)\Big] \\
&\le \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\log \frac{1}{\Pr\left[W \wedge V=v | T=t\right]}\Big]
\end{aligned}
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$$\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow (TV)|_W} \Big[\log \frac{1}{\Pr[W \wedge V = v | T = t]}\Big] \qquad \text{(Thm 10)}$$

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$$= \log \sum_{(t,v)\in \mathrm{Supp}((TV)|_W)} \frac{\Pr[T = t]}{\Pr[W]}$$

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&= \log \sum_{(t,v)\in \mathrm{Supp}((TV)|_W)} \frac{\Pr[T = t]}{\Pr[W]} \leq \log \frac{||\mathrm{Supp}(V|_W)||}{\Pr[W]}. && \square
\end{aligned}
$$