Application of Information Theory, Lecture 9

Parallel Repetition of Interactive Arguments

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Part I

Interactive Proofs and Arguments

$\mathcal{NP}$ as a Non-interactive Proofs

Definition 1 ($\mathcal{NP}$)

$\mathcal{L} \in \mathcal{NP}$ iff $\exists$ and poly-time algorithm V such that:

- $\forall x \in \mathcal{L}$ there exists $w \in \{0,1\}^*$ s.t. $V(x, w) = 1$
- $V(x, w) = 0$ for every $x \notin \mathcal{L}$ and $w \in \{0,1\}^*$

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- Soundness holds unconditionally

Interactive proofs/arguments

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A protocol (P, V) is an interactive proof for $\mathcal{L}$, if V is a PPT and:

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Soundness $\forall x \notin \mathcal{L}$, and any algorithm P^*: $\Pr[(P^*, V)(x) = 1] \leq 1/3$.

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- Games — no-input protocols.

Section 1

Interactive Proof for Graph Non-Isomorphism

Graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

> **Definition 3 (graph isomorphism)**
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> Graphs $G_0 = ([m], E_0)$ and $G_1 = ([m], E_1)$ are isomorphic, denoted $G_0 \equiv G_1$, if $\exists \pi \in \Pi_m$ such that
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  Idea: Beer tasting...

Interactive proof for $\mathcal{GNI}$

Protocol 4 $((P, V)(G_0 = ([m], E_0), G_1 = ([m], E_1)))$

1. V chooses $b \leftarrow \{0, 1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b)$ to P.[a]

2. P send b' to V   (tries to set $b' = b$).

3. V accepts iff $b' = b$.

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[a] $\pi(E) = \{(\pi(u), \pi(v): (u, v) \in E\}$.

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Claim 5

The above protocol is IP for $\mathcal{GNI}$, with perfect completeness and soundness error $\frac{1}{2}$.

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Hence,

$G_0 \equiv G_1$: $\Pr[b' = b] \leq \frac{1}{2}$.

$G_0 \not\equiv G_1$: $\Pr[b' = b] = 1$ (i.e., P can, possibly inefficiently, extracted from $\pi(E_i)$)

□

Part II

Hardness Amplification

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- Public-coin interactive proof/argument — in each round the verifier flips coins and sends them to the prover. To compute its output, the verifier applies some (fixed) function to the protocol's transcript.

Hardness amplification, cont.

- Give a protocol $\pi = (P, V)$ and $k \in \mathbb{N}$, let $\pi^{(k)} = (P^{(k)}, V^{(k)})$ be the k-fold parallel repetition of π: i.e., k parallel independent copies of π

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- Assume $\Pr\left[(\widetilde{P}, V) = 1\right] \leq \varepsilon$ for any s-size algorithm $\widetilde{P}$, we would like to prove that $\Pr\left[(\widetilde{P^{(k)}}, V^{(k)}) = 1^k\right] \leq f(\varepsilon)$ for any $s^{(k)}$-size algorithm $\widetilde{P^{(k)}}$.

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► Concrete security

► In the following we focus on games (no input protocols)

Section 2

Parallel repetition of public-coin interactive argument

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Theorem 6

Let $\pi = (\mathsf{P}, \mathsf{V})$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \leq \varepsilon$ for any s-size $\widetilde{\mathsf{P}}$, then $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] \leq \varepsilon^{k/4}$ for any $s \cdot \frac{\varepsilon^{k/4}}{mk^3 s_\mathsf{V}}$-size $\widetilde{\mathsf{P}^{(k)}}$, where s_V is V's size.

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Proof plan: Let $\widetilde{P^{(k)}}$ be $s^{(k)}$-size algorithm with $\Pr\left[(\widetilde{P^{(k)}}, V^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $s^{(k)} \cdot \frac{mk^3 s_V}{\varepsilon^{(k)}}$-size $\widetilde{P}$ with $\Pr\left[(\widetilde{P}, V) = 1\right] \ge (\varepsilon^{(k)})^{4/k}$.

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▶ The $k/4$ in the exponent can be pushed to be almost k.

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- The $k/4$ in the exponent can be pushed to be almost k.
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- Assume wlg. that V sends the first message in π and that in each round it samples and sends ℓ coins.

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> Let $\pi = (P, V)$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{P}, V) = 1\right] \leq \varepsilon$ for any s-size $\widetilde{P}$, then $\Pr\left[(\widetilde{P^{(k)}}, V^{(k)}) = 1^k\right] \leq \varepsilon^{k/4}$ for any $s \cdot \frac{\varepsilon^{k/4}}{mk^3 s_V}$-size $\widetilde{P^{(k)}}$, where s_V is V's size.

Proof plan: Let $\widetilde{P^{(k)}}$ be $s^{(k)}$-size algorithm with $\Pr\left[(\widetilde{P^{(k)}}, V^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $s^{(k)} \cdot \frac{mk^3 s_V}{\varepsilon^{(k)}}$-size $\widetilde{P}$ with $\Pr\left[(\widetilde{P}, V) = 1\right] \geq (\varepsilon^{(k)})^{4/k}$.

- The $k/4$ in the exponent can be pushed to be almost k.
- Assume for simplicity that $\widetilde{P^{(k)}}$ is deterministic
- Assume wlg. that V sends the first message in π and that in each round it samples and sends ℓ coins.
- We view the coins of $V^{(k)}$ as a matrix $R \in \{0, 1\}^{m \times (k\ell)}$, letting R_j denote the coins of the j'th round, and $R_{1,\dots,j}$ the coins of the first j rounds.

Parallel repetition of public-coin interactive argument

> **Theorem 6**
>
> Let $\pi = (P, V)$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{P}, V) = 1\right] \le \varepsilon$ for any s-size $\widetilde{P}$, then $\Pr\left[(\widetilde{P^{(k)}}, V^{(k)}) = 1^k\right] \le \varepsilon^{k/4}$ for any $s \cdot \frac{\varepsilon^{k/4}}{mk^3 s_V}$-size $\widetilde{P^{(k)}}$, where s_V is V's size.

Proof plan: Let $\widetilde{P^{(k)}}$ be $s^{(k)}$-size algorithm with $\Pr\left[(\widetilde{P^{(k)}}, V^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $s^{(k)} \cdot \frac{mk^3 s_V}{\varepsilon^{(k)}}$-size $\widetilde{P}$ with $\Pr\left[(\widetilde{P}, V) = 1\right] \ge (\varepsilon^{(k)})^{4/k}$.

- The $k/4$ in the exponent can be pushed to be almost k.
- Assume for simplicity that $\widetilde{P^{(k)}}$ is deterministic
- Assume wlg. that V sends the first message in π and that in each round it samples and sends ℓ coins.
- We view the coins of $V^{(k)}$ as a matrix $R \in \{0, 1\}^{m \times (k\ell)}$, letting R_j denote the coins of the j'th round, and $R_{1,...,j}$ the coins of the first j rounds.
- Let $\mathbf{R} \sim \{0, 1\}^{m \times (k\ell)}$

Algorithm $\widetilde{\mathsf{P}}$

Algorithm $\widetilde{\mathsf{P}}$

Let $q = k^2$.

Algorithm $\widetilde{\mathsf{P}}$

Let $q = k^2$.

Algorithm 7 ($\widetilde{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.

2. Upon getting the j'th message r from V, do:

    2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned that $R_{1,\ldots,j-1} = \widetilde{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = r$.

    2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$:

        2.2.1 Set $\widetilde{R}_j = R_j$

        2.2.2 Send a_{j,i^*} back to V, for a_j being the j'th message $\widetilde{\mathsf{P}^{(k)}}$ send to $\mathsf{V}^{(k)}$ in $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R))$.

        Else, GOTO Line 2.1

    2.3 Abort if the overall number of sampling exceeds $\lceil qm/\varepsilon^{(k)} \rceil$.

Algorithm $\widetilde{\mathsf{P}}$

Let $q = k^2$.

> **Algorithm 7 ($\widetilde{\mathsf{P}}$)**
>
> **1.** Let $i^* \leftarrow [k]$.
> **2.** Upon getting the j'th message r from V, do:
>
>   **2.1** Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned that $R_{1,\dots,j-1} = \widetilde{R}_{1,\dots,j-1}$ and $R_{j,i^*} = r$.
>
>   **2.2** If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$:
>
>   **2.2.1** Set $\widetilde{R}_j = R_j$
>   **2.2.2** Send a_{j,i^*} back to V, for a_j being the j'th message $\widetilde{\mathsf{P}^{(k)}}$ send to $\mathsf{V}^{(k)}$ in $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R))$.
>
>   Else, GOTO Line 2.1
>
>   **2.3** Abort if the overall number of sampling exceeds $\lceil qm/\varepsilon^{(k)} \rceil$.

▶ Let $\widetilde{\mathsf{P}}'$ be the non aborting variant of $\widetilde{\mathsf{P}}'$, let $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{N}}$ be the value of $\widetilde{R}$ and $\#$ of samples done in a random execution of $(\widetilde{\mathsf{P}}', \mathsf{V}^{(k)})$.

Algorithm $\widetilde{\mathsf{P}}$

Let $q = k^2$.

Algorithm 7 ($\widetilde{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.

2. Upon getting the j'th message r from V, do:

    2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned that $R_{1,\ldots,j-1} = \widetilde{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = r$.

    2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$:

        2.2.1 Set $\widetilde{R}_j = R_j$

        2.2.2 Send a_{j,i^*} back to V, for a_j being the j'th message $\widetilde{\mathsf{P}^{(k)}}$ send to $\mathsf{V}^{(k)}$ in $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R))$.

    Else, GOTO Line 2.1

    2.3 Abort if the overall number of sampling exceeds $\lceil qm/\varepsilon^{(k)} \rceil$.

▶ Let $\widetilde{\mathsf{P}}'$ be the non aborting variant of $\widetilde{\mathsf{P}}'$, let $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{N}}$ be the value of $\widetilde{R}$ and $\#$ of samples done in a random execution of $(\widetilde{\mathsf{P}}', \mathsf{V}^{(k)})$.

▶ $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq \Pr\left[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) := (\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widetilde{\mathbf{R}})) = 1^k \wedge \widetilde{\mathbf{N}} \leq qm/\varepsilon^{(k)}\right]$.

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

1. Let $R \leftarrow \{0, 1\}^{m \times \ell}$, conditioned that $R_{1,\ldots,j-1} = \hat{R}_{1,\ldots,j-1}$.

2. If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

1. Let $R \leftarrow \{0, 1\}^{m \times \ell}$, conditioned that $R_{1,\dots,j-1} = \hat{R}_{1,\dots,j-1}$.

2. If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- Let $\hat{\mathbf{R}}$ be the value of $\hat{R}$ in the end of a random execution of $\hat{\mathsf{P}}$.

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

1. Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned that $R_{1,\dots,j-1} = \hat{R}_{1,\dots,j-1}$.

2. If $\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(R))} = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- Let $\hat{\mathbf{R}}$ be the value of $\hat{R}$ in the end of a random execution of $\hat{\mathsf{P}}$.

- $\hat{\mathbf{R}} \sim \mathbf{R}|_{\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R}))} = 1^k}$

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

1. Let $R \leftarrow \{0, 1\}^{m \times \ell}$, conditioned that $R_{1,\dots,j-1} = \hat{R}_{1,\dots,j-1}$.

2. If $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- Let $\hat{\mathbf{R}}$ be the value of $\hat{R}$ in the end of a random execution of $\hat{\mathsf{P}}$.
- $\hat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R}))=1^k}$
- In particular, $\Pr\left[(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\hat{\mathbf{R}}) = 1^k\right] = 1$

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

   1. Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned that $R_{1,\ldots,j-1} = \hat{R}_{1,\ldots,j-1}$.

   2. If $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- ▶ Let $\hat{\mathbf{R}}$ be the value of $\hat{R}$ in the end of a random execution of $\hat{\mathsf{P}}$.

- ▶ $\hat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R}))=1^k}$

- ▶ In particular, $\Pr\left[(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\hat{\mathbf{R}}) = 1^k\right] = 1$

- ▶ Let $\hat{\mathbf{N}}$ be # of samples done in $\hat{\mathbf{R}}$.

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

1. Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned that $R_{1,\dots,j-1} = \hat{R}_{1,\dots,j-1}$.

2. If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- Let $\hat{\mathbf{R}}$ be the value of $\hat{R}$ in the end of a random execution of $\hat{\mathsf{P}}$.

- $\hat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$

- In particular, $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\hat{\mathbf{R}}) = 1^k\right] = 1$

- Let $\hat{\mathbf{N}}$ be # of samples done in $\hat{\mathbf{R}}$.

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

1. Let $R \leftarrow \{0, 1\}^{m \times \ell}$, conditioned that $R_{1,...,j-1} = \hat{R}_{1,...,j-1}$.

2. If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- Let $\hat{\mathbf{R}}$ be the value of $\hat{R}$ in the end of a random execution of $\hat{\mathsf{P}}$.
- $\hat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- In particular, $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\hat{\mathbf{R}}) = 1^k\right] = 1$
- Let $\hat{\mathbf{N}}$ be # of samples done in $\hat{\mathbf{R}}$.

Lemma 9

$\Pr\left[\hat{\mathbf{N}} \le qm/\varepsilon^{(k)}\right] \ge 1 - \frac{1}{q}$

Proving Lemma 9

Proving Lemma 9

▶ Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

Proving Lemma 9

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

- $v(\mathbf{y} = (y_1, \ldots, y_j)) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m)) = 1^k \mid X^j = \mathbf{y}\right]$

  (letting $X^j = (X_1, \ldots, X_j)$)

Proving **Lemma 9**

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

- $v(\mathbf{y} = (y_1, \ldots, y_j)) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m)) = 1^k \mid X^j = \mathbf{y} \right]$
  (letting $X^j = (X_1, \ldots, X_j)$)

- Conditioned on $Y^j = \mathbf{y} = (y_1, \ldots, y_j)$, the expected $\#$ of samples done in $(j+1)$'th round of $\widehat{\mathsf{P}}$ is $\frac{1}{v(\mathbf{y})}$.

Proving Lemma 9

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

- $v(\mathbf{y} = (y_1, \ldots, y_j)) := \Pr\left[\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(X^m))} = 1^k \mid X^j = \mathbf{y}\right]$
  (letting $X^j = (X_1, \ldots, X_j)$)

- Conditioned on $Y^j = \mathbf{y} = (y_1, \ldots, y_j)$, the expected $\#$ of samples done in $(j+1)$'th round of $\widehat{\mathsf{P}}$ is $\frac{1}{v(\mathbf{y})}$.

- We prove Lemma 9 showing that $\mathsf{E}\left[\frac{1}{v(Y^j)}\right] \leq \frac{1}{\varepsilon^{(k)}}$ for every $j \in \{0, \ldots, m-1\}$

Proving Lemma 9

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

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Proving Lemma 9

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

- $v(\mathbf{y} = (y_1, \ldots, y_j)) := \Pr\left[\left(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) \right) = 1^k \mid X^j = \mathbf{y} \right]$
  (letting $X^j = (X_1, \ldots, X_j)$)

- Conditioned on $Y^j = \mathbf{y} = (y_1, \ldots, y_j)$, the expected $\#$ of samples done in $(j+1)$'th round of $\widehat{\mathsf{P}}$ is $\frac{1}{v(\mathbf{y})}$.

- We prove Lemma 9 showing that $\mathsf{E}\left[\frac{1}{v(Y^j)} \right] \leq \frac{1}{\varepsilon^{(k)}}$ for every $j \in \{0, \ldots, m-1\}$

Claim 10

For $j \in \{0, \ldots, m-1\}$ and $\mathbf{y} \in \mathrm{Supp}(Y^j)$ it holds that $\Pr_{Y^j}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Proving Lemma 9

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

- $v(\mathbf{y} = (y_1, \ldots, y_j)) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m)) = 1^k \mid X^j = \mathbf{y}\right]$
  (letting $X^j = (X_1, \ldots, X_j)$)

- Conditioned on $Y^j = \mathbf{y} = (y_1, \ldots, y_j)$, the expected $\#$ of samples done in $(j+1)$'th round of $\widehat{\mathsf{P}}$ is $\frac{1}{v(\mathbf{y})}$.

- We prove Lemma 9 showing that $\mathsf{E}\left[\frac{1}{v(Y^j)}\right] \leq \frac{1}{\varepsilon^{(k)}}$ for every $j \in \{0, \ldots, m-1\}$

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For $j \in \{0, \ldots, m-1\}$ and $\mathbf{y} \in \mathrm{Supp}(Y^j)$ it holds that $\Pr_{Y^j}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Hence, $\mathsf{E}_{Y^j}\left[\frac{1}{v(Y^j)}\right] = \sum_{\mathbf{y} \in \mathrm{Supp}(Y^j)} \Pr[Y^j = \mathbf{y}] \cdot \frac{1}{v(\mathbf{y})}$

Proving **Lemma 9**

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

- $v(\mathbf{y} = (y_1, \ldots, y_j)) := \Pr\left[\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(X^m))} = 1^k \mid X^j = \mathbf{y}\right]$
  (letting $X^j = (X_1, \ldots, X_j)$)

- Conditioned on $Y^j = \mathbf{y} = (y_1, \ldots, y_j)$, the expected $\#$ of samples done in $(j+1)$'th round of $\widehat{\mathsf{P}}$ is $\frac{1}{v(\mathbf{y})}$.

- We prove **Lemma 9** showing that $\mathsf{E}\left[\frac{1}{v(Y^j)}\right] \leq \frac{1}{\varepsilon^{(k)}}$ for every $j \in \{0, \ldots, m-1\}$

> **Claim 10**
>
> For $j \in \{0, \ldots, m-1\}$ and $\mathbf{y} \in \mathrm{Supp}(Y^j)$ it holds that $\Pr_{Y^j}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Hence, $\mathsf{E}_{Y^j}\left[\frac{1}{v(Y^j)}\right] = \sum_{\mathbf{y} \in \mathrm{Supp}(Y^j)} \Pr[Y^j = \mathbf{y}] \cdot \frac{1}{v(\mathbf{y})}$

$= \sum_{\mathbf{y}} \Pr[X^j = \mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}} \cdot \frac{1}{v(\mathbf{y})} = \frac{1}{\varepsilon^{(k)}} \cdot \sum_{\mathbf{y} \in \mathrm{Supp}(Y^j)} \Pr[X^j = \mathbf{y}] \leq \frac{1}{\varepsilon^{(k)}}$. $\square$

Proving Claim 10

Proving Claim 10

Note that

$$\Pr_{Y_j \mid Y^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{1\ldots,j-1}))^{\ell-1} \cdot \Pr_{X_j \mid X^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \cdot v(\mathbf{y}) \quad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j \mid X^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \cdot v(\mathbf{y})$$

Proving Claim 10

Note that

$$\Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] = \sum_{\ell=1}^{\infty}(1 - v(\mathbf{y}_{1\ldots,j-1}))^{\ell-1} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \cdot v(\mathbf{y}) \quad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

Proving Claim 10

Note that

$$\Pr_{Y_j \mid Y^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{1\ldots,j-1}))^{\ell-1} \cdot \Pr_{X_j \mid X^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \cdot v(\mathbf{y}) \quad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j \mid X^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\Pr_{Y^j} [\mathbf{y}] = \Pr_{Y^{j-1}} [\mathbf{y}_{1\ldots,j-1}] \cdot \Pr_{Y_j \mid Y^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j]$$

Proving Claim 10

Note that

$$\Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] = \sum_{\ell=1}^{\infty}(1-v(\mathbf{y}_{1\ldots,j-1}))^{\ell-1} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \cdot v(\mathbf{y}) \quad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\Pr_{Y^j}[\mathbf{y}] = \Pr_{Y^{j-1}}[\mathbf{y}_{1\ldots,j-1}] \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j]$$

$$= \Pr_{X^{j-1}}[\mathbf{y}_{1\ldots,j-1}] \cdot \frac{v(\mathbf{y}_{1\ldots,j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \quad \text{(i.h.)}$$

Proving Claim 10

Note that

$$\Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] = \sum_{\ell=1}^{\infty}(1-v(\mathbf{y}_{1\ldots,j-1}))^{\ell-1} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \cdot v(\mathbf{y}) \quad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\Pr_{Y^j}[\mathbf{y}] = \Pr_{Y^{j-1}}[\mathbf{y}_{1\ldots,j-1}] \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j]$$

$$= \Pr_{X^{j-1}}[\mathbf{y}_{1\ldots,j-1}] \cdot \frac{v(\mathbf{y}_{1\ldots,j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \quad \text{(i.h.)}$$

$$= \Pr_{X^{j-1}}[\mathbf{y}_{1\ldots,j-1}] \cdot \frac{v(\mathbf{y}_{1\ldots,j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \quad \text{(Eq. (1))}$$

Proving Claim 10

Note that

$$\Pr_{Y_j \mid Y^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{1\ldots,j-1}))^{\ell-1} \cdot \Pr_{X_j \mid X^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \cdot v(\mathbf{y}) \quad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j \mid X^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\Pr_{Y^j} [\mathbf{y}] = \Pr_{Y^{j-1}} [\mathbf{y}_{1\ldots,j-1}] \cdot \Pr_{Y_j \mid Y^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j]$$

$$= \Pr_{X^{j-1}} [\mathbf{y}_{1\ldots,j-1}] \cdot \frac{v(\mathbf{y}_{1\ldots,j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j \mid Y^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \qquad \text{(i.h.)}$$

$$= \Pr_{X^{j-1}} [\mathbf{y}_{1\ldots,j-1}] \cdot \frac{v(\mathbf{y}_{1\ldots,j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j \mid X^{j-1} = \mathbf{y}_{1\ldots,j-1}} [y_j] \quad \text{(Eq. (1))}$$

$$= \Pr_{X^j} [\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}.$$

Ideal "attacker", variant

Ideal "attacker", variant

Experiment 11 ($\widehat{P}$)

1. Let $i^* \leftarrow [k]$.
2. For for $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\dots,j-1} = \widehat{R}_{1,\dots,j-1}$.
   2.2 If $(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
   2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\dots,j-1} = \widehat{R}_{1,\dots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.
   2.4 If $(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

Ideal "attacker", variant

1. Let $i^* \leftarrow [k]$.
2. For for $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.
   2.2 If $(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
   2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and
       $R_{j,i^*} = \widehat{R}_{j,i^*}$.
   2.4 If $(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{P}$.

Ideal "attacker", variant

1. Let $i^* \leftarrow [k]$.
2. For for $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.

   2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.

   2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.

   2.4 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k}$

Ideal "attacker", variant

Experiment 11 ($\widehat{P}$)

1. Let $i^* \leftarrow [k]$.

2. For for $j = 1$ to m:

    2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.

    2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.

    2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.

    2.4 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- Let $\widehat{\mathbf{N}}$ be the # of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For for $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\dots,j-1} = \widehat{R}_{1,\dots,j-1}$.
   2.2 If $\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(R))} = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
   2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\dots,j-1} = \widehat{R}_{1,\dots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.
   2.4 If $\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(R))} = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R}))}=1^k}$
- Let $\widehat{\mathbf{N}}$ be the # of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Ideal "attacker", variant

- Let $\widehat{R}$ be the final value of $\widehat{R}$ in $\widehat{P}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{\widetilde{(P^{(k)}, V^{(k)}(\mathbf{R}))} = 1^k}$
- Let $\widehat{\mathbf{N}}$ be the # of Step-2.3-samples done in $\widehat{P}$.

Lemma 12

$\Pr\left[\mathsf{win}(\widehat{R}, \widehat{N})\right] \geq 1 - \frac{1}{q}$

From ideal to real

From ideal to real
Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} \| \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i \| \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} \| \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} \| \mathbf{R})$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

▶ Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

- Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathbf{P}^{(k)}}, \mathbf{V}^{(k)}(\mathbf{R}))=1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

- Hence, $D(\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

- Thm. 7 in Lecture 7 $\implies$ $D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

- Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

- Claim 12 $\implies$ $\alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

> **Claim 13**
>
> $D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

> **Claim 14**
>
> $\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

- Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

- Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

- Claim 12 $\implies \alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.

- Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log(1-\alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

- ▶ Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathbf{P}^{(k)}}, \mathbf{V}^{(k)}(\mathbf{R}))=1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

- ▶ Hence, $D(\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

- ▶ Claim 12 $\implies \alpha := \Pr[\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.

- ▶ Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log(1-\alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

  $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

► Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathbf{P}^{(k)}}, \mathbf{V}^{(k)}(\mathbf{R}))=1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

► Hence, $D(\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

► Claim 12 $\implies \alpha := \Pr[\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.

► Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log(1-\alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

$\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$

► Recalling $q = k^2$, $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1 - \alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i}$ $(= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

- Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

- Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

- Claim 12 $\implies \alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.

- Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log(1-\alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

  $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$

- Recalling $q = k^2$, $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1-\alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$

- We conclude that $\beta \geq 2^{\frac{4}{k} \log \varepsilon^{(k)}} = \sqrt[k/4]{\varepsilon^{(k)}}.\square$

Proving Claim 13

HW...

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids, let W be an event, and let

$D_i(z) := \prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.

Then $\sum_{i=1}^{k} D(Z_W || D_i) \leq D(Z_W || Z)$.

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids, let W be an event, and let

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Then $\sum_{i=1}^{k} D(Z_W \| D_i) \leq D(Z_W \| Z)$.

Letting $Z = \mathbf{R}$ and W be the event $\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R}))} = 1^k$, Lemma 15 yields that $\sum_{i \in [k]} D(\widehat{\mathbf{R}} \| \widetilde{\mathbf{R}}_i) = \sum_{i \in [k]} D(\mathbf{R}|_W \| \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W \| \mathbf{R}) = D(\widehat{\mathbf{R}} \| \mathbf{R})$. $\square$

Proving Claim 14

> **Lemma 15**
>
> Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids, let W be an event, and let
>
> $D_i(z) := \prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\dots,j-1} = z_{1,\dots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.
>
> Then $\sum_{i=1}^{k} D(Z_W \| D_i) \leq D(Z_W \| Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} \| \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W \| \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W \| \mathbf{R}) = D(\widehat{\mathbf{R}} \| \mathbf{R})$. $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

Proving Claim 14

> **Lemma 15**
>
> Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids, let W be an event, and let
>
> $D_i(z) := \prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{j,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.
>
> Then $\sum_{i=1}^{k} D(Z_W || D_i) \le D(Z_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R}))} = 1^k$, Lemma 15 yields that $\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i \in [k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \le D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$

Proving Claim 14

> **Lemma 15**
>
> Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids, let W be an event, and let
>
> $D_i(z) := \prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.
>
> Then $\sum_{i=1}^{k} D(Z_W \| D_i) \leq D(Z_W \| Z)$.

Letting $Z = \mathbf{R}$ and W be the event $\widetilde{(\mathrm{P}^{(k)}, \mathrm{V}^{(k)}(\mathbf{R})) = 1^k}$, Lemma 15 yields that
$\sum_{i \in [k]} D(\widehat{\mathbf{R}} \| \widetilde{\mathbf{R}}_i) = \sum_{i \in [k]} D(\mathbf{R}|_W \| \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W \| \mathbf{R}) = D(\widehat{\mathbf{R}} \| \mathbf{R})$. $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}[(x_1, x_2, y_1, y_2)]$

Proving Claim 14

> **Lemma 15**
>
> Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids, let W be an event, and let
>
> $D_i(z) := \prod_{j=1}^m \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.
>
> Then $\sum_{i=1}^k D(Z_W || D_i) \leq D(Z_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k}$, Lemma 15 yields that $\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i \in [k]} D(\mathbf{R}|_w || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_w || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}[(x_1, x_2, y_1, y_2)]$
- $C(x_1, x_2, y_1, y_1) := (X|_w)(x_1, x_2, y_1, y_1)$

Proving Claim 14

> **Lemma 15**
>
> Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids, let W be an event, and let
>
> $D_i(z) := \prod_{j=1}^m \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right].$
>
> Then $\sum_{i=1}^k D(Z_W \| D_i) \leq D(Z_W \| Z).$

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R})) = 1^k$, Lemma 15 yields that
$\sum_{i\in[k]} D(\widehat{\mathbf{R}} \| \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_w \| \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_w \| \mathbf{R}) = D(\widehat{\mathbf{R}} \| \mathbf{R}).$ $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}\left[(x_1, x_2, y_1, y_2)\right]$
- $C(x_1, x_2, y_1, y_1) := (X|_w)(x_1, x_2, y_1, y_1)$
- $Q(x_1, x_2, y_1, y_1) := \Pr\left[X_1 = x_1 | W\right] \cdot \Pr\left[X_2 = x_2 | W\right] \cdot$
  $\Pr\left[Y_1 = y_1 | W, X = (x_1, x_2)\right] \cdot \Pr\left[Y_2 = y_2 | W, X = (x_1, x_2)\right]$

Proving Claim 14

> **Lemma 15**
>
> Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids, let W be an event, and let
>
> $D_i(z) := \prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.
>
> Then $\sum_{i=1}^{k} D(Z_W \| D_i) \leq D(Z_W \| Z)$.

Letting $Z = \mathbf{R}$ and W be the event $\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R}))} = 1^k$, Lemma 15 yields that
$\sum_{i \in [k]} D(\widehat{\mathbf{R}} \| \widetilde{\mathbf{R}}_i) = \sum_{i \in [k]} D(\mathbf{R}|_W \| \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W \| \mathbf{R}) = D(\widehat{\mathbf{R}} \| \mathbf{R})$. $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}[(x_1, x_2, y_1, y_2)]$
- $C(x_1, x_2, y_1, y_1) := (X|_w)(x_1, x_2, y_1, y_1)$
- $Q(x_1, x_2, y_1, y_1) := \Pr[X_1 = x_1 | W] \cdot \Pr[X_2 = x_2 | W] \cdot$
  $\Pr[Y_1 = y_1 | W, X = (x_1, x_2)] \cdot \Pr[Y_2 = y_2 | W, X = (x_1, x_2)]$
- We write $\frac{C(x_1, x_2, y_1, y_1)}{U(x_1, x_2, y_1, y_1)} =$
  $\frac{\Pr[X_1=x_1|W] \cdot \Pr[Y_1=y_1|W,X=(x_1,x_2)]}{\Pr[X_1=x_1] \cdot \Pr[Y_1=y_1]} \cdot \frac{\Pr[X_2=x_2|W] \cdot \Pr[Y_2=y_2|W,X=(x_1,x_2)]}{\Pr[X_2=x_2] \cdot \Pr[Y_2=y_2]} \cdot \frac{C(x_1,x_2,y_1,y_1)}{Q(x_1,x_2,y_1,y_1)}$

Proving Lemma 15, cont.

$$D(C\|U) = \operatorname*{E}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr\left[X_1=x_1|W\right]\cdot\Pr\left[Y_1=y_1|W,X=(x_1,x_2)\right]}{\Pr\left[X_1=x_1\right]\cdot\Pr\left[Y_1=y_1\right]}\right]$$

$$+\operatorname*{E}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr\left[X_2=x_2|W\right]\cdot\Pr\left[Y_2=y_2|W,X=(x_1,x_2)\right]}{\Pr\left[X_2=x_2\right]\cdot\Pr\left[Y_2=y_2\right]}\right]$$

$$+\operatorname*{E}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{C(x_1,x_2,y_1,y_2)}{Q(x_1,x_2,y_1,y_2)}\right].$$

Proving Lemma 15, cont.

$$D(C\|U) = \mathop{E}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr[X_1 = x_1|W]\cdot\Pr[Y_1 = y_1|W, X = (x_1,x_2)]}{\Pr[X_1 = x_1]\cdot\Pr[Y_1 = y_1]}\right]$$

$$+ \mathop{E}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr[X_2 = x_2|W]\cdot\Pr[Y_2 = y_2|W, X = (x_1,x_2)]}{\Pr[X_2 = x_2]\cdot\Pr[Y_2 = y_2]}\right]$$

$$+ \mathop{E}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{C(x_1,x_2,y_1,y_2)}{Q(x_1,x_2,y_1,y_2)}\right].$$

It follows that

$$D(C\|U) = D(X_1|_W, X_2|_{W,X_1}, Y_1|_{W,X}, Y_2|_{W,X,Y_1}\|X_1, X_2|_{W,X_1}, Y_1, Y_2|_{W,X,Y_1})$$
$$+ D(X_2|_W, X_1|_{W,X_2}, Y_2|_{W,X}, Y_1|_{W,X,Y_2}\|X_2, X_1|_{W,X_2}, Y_2, Y_1|_{W,X,Y_2})$$
$$+ D(C\|Q),$$

Proving Lemma 15, cont.

$$D(C||U) = \mathop{E}_{(x_1,x_2,y_1,y_2) \leftarrow C} \left[\log \frac{\Pr[X_1 = x_1 | W] \cdot \Pr[Y_1 = y_1 | W, X = (x_1, x_2)]}{\Pr[X_1 = x_1] \cdot \Pr[Y_1 = y_1]} \right]$$

$$+ \mathop{E}_{(x_1,x_2,y_1,y_2) \leftarrow C} \left[\log \frac{\Pr[X_2 = x_2 | W] \cdot \Pr[Y_2 = y_2 | W, X = (x_1, x_2)]}{\Pr[X_2 = x_2] \cdot \Pr[Y_2 = y_2]} \right]$$

$$+ \mathop{E}_{(x_1,x_2,y_1,y_2) \leftarrow C} \left[\log \frac{C(x_1, x_2, y_1, y_2)}{Q(x_1, x_2, y_1, y_2)} \right].$$

It follows that

$$D(C||U) = D(X_1|_W, X_2|_{W,X_1}, Y_1|_{W,X}, Y_2|_{W,X,Y_1} || X_1, X_2|_{W,X_1}, Y_1, Y_2|_{W,X,Y_1})$$
$$+ D(X_2|_W, X_1|_{W,X_2}, Y_2|_{W,X}, Y_1|_{W,X,Y_2} || X_2, X_1|_{W,X_2}, Y_2, Y_1|_{W,X,Y_2})$$
$$+ D(C||Q),$$

and the proof follows since $D(\cdot || \cdot) \geq 0$. $\square$

Parallel repetition of interactive proofs

Parallel repetition of interactive proofs

▶ Similar proof to the public-coin proof we gave above.

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- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}})$, till he gets an accepting execution.

Parallel repetition of interactive proofs

- Similar proof to the public-coin proof we gave above.

- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)})$, till he gets an accepting execution.

- Why fails us to extend this approach for non-public-coin interactive arguments?

Section 3

Parallel amplification for any interactive argument

Parallel amplification theorem for any protocol

Parallel amplification theorem for any protocol

► Can we amplify the security of any interactive argument "in parallel"?

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument "in parallel"?
- Yes we can!