Application of Information Theory, Lecture 8

Kolmogorov Complexity and Other Entropy Measures

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Part I

Other Entropy Measures

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- $\mathrm{H}_\infty(X|Y=1) = 0$ and $\mathrm{H}_\infty(X|Y=0) = n$. But $\mathrm{H}_\infty(X) = 1$.

Section 1

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Let $Z^1, \ldots, Z^n$ be iids over $[0,1]$ with expectation μ. Then, $\Pr\left[\left|\frac{\sum_{j=i}^n Z^j}{n} - \mu\right| \ge \varepsilon\right] \le 2 \cdot e^{-2\varepsilon^2 n}$ for every $\varepsilon > 0$.

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Let $Z^1, \ldots, Z^n$ be iids over $[0,1]$ with expectation μ. Then, $\Pr\left[\left|\frac{\sum_{j=i}^n Z^j}{n} - \mu\right| \geq \varepsilon\right] \leq 2 \cdot e^{-2\varepsilon^2 n}$ for every $\varepsilon > 0$.

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$\exists$ rv W that is $(2 \cdot e^{-2\varepsilon^2 n})$-close to X^n, and $\mathrm{H}_\infty(W) \geq n(H(X) - \varepsilon)$.

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Given rv $X \sim p$, let X^n denote n independent copies of X, and let $p^n(x_1 \ldots, x_n) = \prod_{i=1}^n p(x_i)$.

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Let $X \sim p$ and let $\varepsilon > 0$. Then $\Pr\left[-\log p^n(X^n) \leq n \cdot (\mathrm{H}(X) - \varepsilon)\right] < 2 \cdot e^{-2\varepsilon^2 n}$.

Proof: (quantitative) AEP.

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Shannon to Min entropy, conditional version

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Let $(X, Y) \sim p$ *let* $\varepsilon > 0$*. Then*
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Proof: ?

Section 2

Renyi-entropy to Uniform Distribution

Pairwise independent hashing

Definition 6 (pairwise independent function family)

A function family $\mathcal{G} = \{g \colon \mathcal{D} \mapsto \mathcal{R}\}$ is pairwise independent, if $\forall\ x \neq x' \in \mathcal{D}$ and $y, y' \in \mathcal{R}$, it holds that $\Pr_{g \leftarrow \mathcal{G}}\left[g(x) = y \wedge g(x') = y')\right] = (\frac{1}{|\mathcal{R}|})^2$.

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- Example: for $\mathcal{D} = \{0,1\}^n$ and $\mathcal{R} = \{0,1\}^m$ let $\mathcal{G} = \{(A, b) \in \{0,1\}^{m \times n} \times \{0,1\}^m\}$ with $(A, b)(x) = A \times x + b$.

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Leftover hash lemma

Lemma 7 (leftover hash lemma)

Let X be a rv over $\{0,1\}^n$ with $\mathrm{H}_2(X) \geq k$ let $\mathcal{G} = \{g \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ be 2-universal and let $G \leftarrow \mathcal{G}$. Then
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Let p be a distribution over $\mathcal{U}$ with $CP(p) \leq \frac{1+\delta}{|\mathcal{U}|}$, then $\mathrm{SD}(p, \sim \mathcal{U}) \leq \frac{\sqrt{\delta}}{2}$.

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- $\|p-q\|_2^2 = \sum_{u \in \mathcal{U}} (p(u) - q(u))^2 = \|p\|_2^2 + \|q\|_2^2 - 2\langle p, q\rangle = \mathrm{CP}(p) - \frac{1}{|\mathcal{U}|} \leq \frac{\delta}{|\mathcal{U}|}$
- Chebyshev Sum Inequality: $(\sum_{i=1}^n a_i)^2 \leq n \sum_{i=1}^n a_i^2$
- Hence, $\|p-q\|_1^2 \leq |\mathcal{U}| \cdot \|p-q\|_2^2$
- Thus, $\mathrm{SD}(p,q) = \frac{1}{2}\|p-q\|_1 \leq \frac{\sqrt{\delta}}{2}$.□

To deuce the proof of Lemma 7, we notice that $\mathrm{CP}(G, G(X)) \leq \frac{1}{|\mathcal{G}|} \cdot (2^{-k} + 2^{-m}) = \frac{1+2^{m-k}}{|\mathcal{G} \times \{0,1\}^m|}$

Part II

Kolmogorov Complexity

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- Solution: the word "described" above in the definition of s is not well defined

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- In particular, a random sequence has Kolmogorov complexity $\approx n$

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- But for most numbers $k(x) \geq n - 1$

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- Example: coin flip $(0.7, 0.3)$ then whp we get a string with $K(x) \approx n \cdot h(0.3)$

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- This is not a paradox, since the description of s is not short.

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- Take C such that $C > \log C + D$
- If T_C stops and outputs x, then $k(x) < \log C + D < C$, a contradiction to the fact that $\exists$ proof that $k(x) > C$.

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