

Application of Information Theory, Lecture 1

Basic Definitions and Facts

Handout Mode

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March 08, 2018

The entropy function

X — Discrete random variable (finite number of values) over \mathcal{X} with probability mass $p = p_X$. The **entropy** of X is defined by:

$$H(X) := - \sum_{x \in \mathcal{X}} \Pr[X = x] \cdot \log \Pr[X = x] = \sum_{x \in \mathcal{X}} \Pr[X = x] \cdot H_X(x)$$

letting $\log = \log_2$, and taking $0 \cdot \log 0 = 0$.

- ▶ When using the natural logarithm, the quantity is called **nats** (“natural”)
- ▶ $H_X(x) := -\log \Pr[X = x]$ is the **sample entropy** of x wrt X
- ▶ $H(X) = -\sum_x p(x) \cdot H_X(x) = \mathbb{E}_{x \leftarrow X} H_X(x)$
- ▶ $H(X) = \mathbb{E}_X \log \frac{1}{p(X)} = \mathbb{E}_{Y=p(X)} \log \frac{1}{Y}$
- ▶ $H(X)$ was introduced by Shannon as measure for the uncertainty in X : number of **bits** it takes to describe X or the information we don't have about X .
- ▶ Entropy is a function of p (sometimes refers to as $H(p)$).

Examples

1. $X \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$:

(i.e., for some $x_1 \neq x_2 \neq x_3$, $P_X(x_1) = \frac{1}{2}$, $P_X(x_2) = \frac{1}{4}$, $P_X(x_3) = \frac{1}{4}$)

$$H(X) = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{4} \log \frac{1}{4} - \frac{1}{4} \log \frac{1}{4} = \frac{1}{2} + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 2 = 1\frac{1}{2}.$$

2. $H(X) = H(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

3. X is uniformly distributed over $\{0, 1\}^n$:

$$H(X) = -\sum_{i=1}^{2^n} \frac{1}{2^n} \log \frac{1}{2^n} = -\log \frac{1}{2^n} = n.$$

▶ n bits are needed to describe X

▶ n bits are needed to sample X

4. $X = X_1, \dots, X_n$ where X_i are iid over $\{0, 1\}$, with $P_{X_i}(1) := \Pr[X_i = 1] = \frac{1}{3}$. $H(X) = ?$

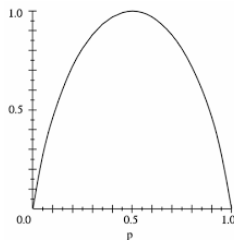
5. $X \sim (p, q)$, $p + q = 1$

▶ $H(X) = H(p, q) = -p \log p - q \log q$

▶ $H(1, 0) = (0, 1) = 0$

▶ $H(\frac{1}{2}, \frac{1}{2}) = 1$

▶ $h(p) := H(p, 1 - p)$ is continuous



Infinite random variables

- ▶ For infinite (discrete) random variable $X \sim (p_1, p_2, \dots)$,

$$H(X) = \sum_{i=1}^{\infty} p_i \log p_i$$

- ▶ Might be finite: $p_i = 2^{-i}$

$$\text{Hence, } H(X) = - \sum_{i=1}^{\infty} 2^{-i} \cdot i < \infty$$

- ▶ Or infinite, $X \sim \{p_{i,j}\}_{i \in \mathbb{N}, j \in \{1, \dots, 2^{2^i - i}\}}$, for $p_{i,j} = 2^{-2^i}$.

$$\text{Hence, } H(X) = \sum_{i=1}^{\infty} \sum_{j=1}^{2^{2^i - i}} 2^{-2^i} \cdot 2^i = \sum_{i=1}^{\infty} 2^{-2^i} \cdot 2^i = \infty$$

- ▶ We will typically restrict our attention to **finite** random variables.

Applications

- ▶ Data compression
- ▶ Error correction codes
- ▶ Algorithm Analysis
- ▶ Protocols Analysis
- ▶ Cryptography
- ▶ Counting. Example # of gold coins in a cube
 - ▶ Projection of Q on xy — 6
 - ▶ Projection of Q on xz — 8
 - ▶ Projection of Q on yz — 12

Can we bound $|Q|$?

- ▶ and more and more...

And all are rather simple to prove

Axiomatic derivation of the entropy function

Any other choices for defining entropy?

Shannon function is the **only** symmetric function (over probability distributions) satisfying the following three axioms:

A1 Continuity: $H(p, 1 - p)$ is continuous function of p .

A2 Normalization: $H(\frac{1}{2}, \frac{1}{2}) = 1$

A3 Grouping axiom:

$$H(p_1, p_2, \dots, p_m) = H(p_1 + p_2, p_3, \dots, p_m) + (p_1 + p_2)H\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right)$$

Why **A3**?

Not hard to prove that Shannon's entropy function satisfies above axioms, proving this is the only such function is more challenging.

Let H^* be a function that satisfying the above axioms.

We prove (assuming additional axiom) that H^* is the Shannon function H .

Generalization of the grouping axiom

Fix $p = (p_1, \dots, p_m)$ and let $S_k = \sum_{i=1}^k p_i$.

Grouping axiom: $H^*(p_1, p_2, \dots, p_m) = H^*(S_2, p_3, \dots, p_m) + S_2 H^*(\frac{p_1}{S_2}, \frac{p_2}{S_2})$.

Claim 1 (Generalized grouping axiom)

$$H^*(p_1, p_2, \dots, p_m) = H^*(S_k, p_{k+1}, \dots, p_m) + S_k \cdot H^*(\frac{p_1}{S_k}, \dots, \frac{p_k}{S_k})$$

Proof: Let $h(q) = H^*(q, 1 - q)$.

$$\begin{aligned} H^*(p_1, p_2, \dots, p_m) &= H^*(S_2, p_3, \dots, p_m) + S_2 h(\frac{p_2}{S_2}) \\ &= H^*(S_3, p_4, \dots, p_m) + S_3 h(\frac{p_3}{S_3}) + S_2 h(\frac{p_2}{S_2}) \\ &\vdots \\ &= H^*(S_k, p_{k+1}, \dots, p_m) + \sum_{i=2}^k S_i h(\frac{p_i}{S_i}) \end{aligned} \tag{1}$$

Hence,

$$H^*(\frac{p_1}{S_k}, \dots, \frac{p_k}{S_k}) = H^*(\frac{S_{k-1}}{S_k}, \frac{p_k}{S_k}) + \sum_{i=2}^{k-1} \frac{S_i}{S_k} h(\frac{p_i/S_k}{S_i/S_k}) = \frac{1}{S_k} \sum_{i=2}^k S_i h(\frac{p_i}{S_i}) \tag{2}$$

Claim follows by combining the above equations. \square

Further generalization of the grouping axiom

Let $1 = k_1 < k_2 < \dots < k_q < m$ and let $C_t = \sum_{i=k_t}^{k_{t+1}-1} p_i$ (letting $k_{q+1} = m + 1$).

Claim 2 (Generalized⁺⁺ grouping axiom)

$$H^*(p_1, p_2, \dots, p_m) = \\ H^*(C_1, \dots, C_q) + C_1 \cdot H^*\left(\frac{p_1}{C_1}, \dots, \frac{p_{k_2-1}}{C_1}\right) + \dots + C_q \cdot H^*\left(\frac{p_{k_q+1}}{C_q}, \dots, \frac{p_m}{C_q}\right)$$

Proof: Follow by the extended group axiom and the symmetry of H \square

Implication: Let $f(m) := H^*\left(\underbrace{\frac{1}{m}, \dots, \frac{1}{m}}_m\right)$

$$\triangleright f(3^2) = 2f(3) = 2H^*\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$\implies f(3^n) = nf(3).$$

$$\triangleright f(mn) = f(m) + f(n)$$

$$\implies f(m^k) = kf(m)$$

$$f(m) = \log m$$

We give a proof under the additional axiom

$$\mathbf{A4} \quad f(m) \leq f(m+1)$$

(you can Google for a proof using only **A1–A3**)

- ▶ For $n \in \mathbb{N}$, let $k = \lfloor \log 3^n = n \log 3 \rfloor$.
- ▶ Since, $2^k \leq 3^n \leq 2^{k+1}$, by **A4**: $f(2^k) \leq f(3^n) \leq f(2^{k+1})$.
- ▶ By grouping + normalization axiom, $k \leq nf(3) \leq k+1$.

$$\implies \frac{\lfloor n \log 3 \rfloor}{n} \leq f(3) \leq \frac{\lfloor n \log 3 \rfloor + 1}{n} \text{ for any } n \in \mathbb{N}$$

$$\implies f(3) = \log 3.$$

- ▶ Proof extends to any integer (not only 3)

$$H^*(p, q) = -p \log p - q \log q$$

- ▶ For **rational** p, q , let $p = \frac{k}{m}$ and $q = \frac{m-k}{m}$, where m is the smallest common multiplier.
- ▶ By grouping axiom, $f(m) = H^*(p, q) + p \cdot f(k) + q \cdot f(m - k)$.
- ▶ Hence,

$$\begin{aligned} H^*(p, q) &= \log m - p \log k - q \log(m - k) \\ &= p(\log m - \log k) + q(\log m - \log(m - k)) \\ &= -p \log \frac{m}{k} - q \log \frac{m - k}{m} = -p \log p - q \log q \end{aligned}$$

- ▶ By continuity axiom, holds for **every** p, q .

$$H^*(p_1, p_2, \dots, p_m) = - \sum_i^m p_i \log p_i$$

We prove for $m = 3$. Proof for arbitrary m follows the same lines.

- ▶ For rational p_1, p_2, p_3 , let $p_1 = \frac{k_1}{m}$, $p_2 = \frac{k_2}{m}$ and $p_3 = \frac{k_3}{m}$, where $m = k_1 + k_2 + k_3$ is the smallest common multiplier.
- ▶ $f(m) = H^*(p_1, p_2, p_3) + p_1 f(k_1) + p_2 f(k_2) + p_3 f(k_3)$
- ▶ Hence,

$$\begin{aligned} H^*(p_1, p_2, p_3) &= \log m - p_1 \log k_1 - p_2 \log k_2 - p_3 \log k_3 \\ &= -p_1 \log \frac{k_1}{m} - p_2 \log \frac{k_2}{m} - p_3 \log \frac{k_3}{m} \\ &= -p_1 \log p_1 - p_2 \log p_2 - p_3 \log p_3 \end{aligned}$$

- ▶ By continuity axiom, holds for every p_1, p_2, p_3 .



Section 1

Basic Properties

$$0 \leq H(p_1, \dots, p_m) \leq \log m$$

► Tight bounds

- $H(p_1, \dots, p_m) = 0$ for $(p_1, \dots, p_m) = (1, 0, \dots, 0)$.
- $H(p_1, \dots, p_m) = \log m$ for $(p_1, \dots, p_m) = (\frac{1}{m}, \dots, \frac{1}{m})$.

► Non negativity is clear.

- A function f is **concave** (“keura”) if $\forall t_1, t_2, \lambda \in [0, 1] \leq 1$
 $\lambda f(t_1) + (1 - \lambda)f(t_2) \leq f(\lambda t_1 + (1 - \lambda)t_2)$

$$\Rightarrow \text{(by induction)} \quad \forall t_1, \dots, t_k, \lambda_1, \dots, \lambda_k \in [0, 1] \text{ with } \sum_i \lambda_i = 1$$

$$\sum_i \lambda_i f(t_i) \leq f(\sum_i \lambda_i t_i)$$

$$\Rightarrow \text{(Jensen inequality): } E f(X) \leq f(E X) \text{ for any random variable } X.$$

- $\log(x)$ is (strictly) concave for $x > 0$, since its second derivative $(-\frac{1}{x^2})$ is always negative.
- Hence, $H(p_1, \dots, p_m) = \sum_i p_i \log \frac{1}{p_i} \leq \log \sum_i p_i \frac{1}{p_i} = \log m$
- Alternatively, for X over $\{1, \dots, m\}$,
 $H(X) = E_X \log \frac{1}{P_X(X)} \leq \log E_X \frac{1}{P_X(X)} = \log m$
- What if $\text{Supp}(X) := \{x: P_X(x) > 0\} \subsetneq [m]$?

$$H(g(X)) \leq H(X)$$

Let X be a random variable, and let g be over $\text{Supp}(X)$.

► $H(Y = g(X)) \leq H(X)$.

Proof:

$$\begin{aligned} H(X) &= - \sum_x P_X(x) \log P_X(x) = - \sum_y \sum_{x: g(x)=y} P_X(x) \log P_X(x) \\ &\geq - \sum_y P_Y(y) \cdot \max_{x: g(x)=y} \log P_X(x) \\ &\geq - \sum_y P_Y(y) \cdot \log P_Y(y) = H(Y) \end{aligned}$$

- Or use the group axiom...
- If g is injective, then $H(Y) = H(X)$.

Proof: $p_X(X) = P_Y(Y)$.

- If g is non-injective (over $\text{Supp}(X)$), then $H(Y) < H(X)$. Proof: ?
- $H(X) = H(2^X)$.
- $H(\sin(X)) < H(X)$, if $0, \pi \in \text{Supp}(X)$.

Historical background

- ▶ Shannon (1948) $H = - \sum_i p_i \log p_i$
- ▶ But the notion of entropy already existed in statistical physics
- ▶ There, entropy — energy that cannot be used, statistical disorder
- ▶ Clausius (1865), who coined the name *entropy*, based on Carnot (1824),
 $H = \int_t \frac{\delta Q}{T} dt$ (Q is *heat* and T is *temperature*)
- ▶ Boltzmann (1877) $H = \log S$, for S being the number of states a system can be in (after measuring the macro parameters: pressure, temperature)
- ▶ $\log \#$ of states is Shannon entropy of the uniform distribution
- ▶ Shannon looked for a name for his measure, von Neumann pointed out the relation to physics and suggested the name entropy.
- ▶ Today it is accepted that Shannon's entropy is the right notion also in statistical mechanics. Measures the uncertainty of a system — energy that cannot be used.
- ▶ Carnot was also an engineer...

Notation

- ▶ $[n] = \{1, \dots, n\}$
- ▶ $P_X(x) = \Pr[X = x]$
- ▶ $\text{Supp}(X) := \{x: P_X(x) > 0\}$
- ▶ For random variable X over \mathcal{X} , let $p(x)$ be its density function:
 $p(x) = P_X(x)$.
In other words, $X \sim p(x)$.
- ▶ For random variable Y over \mathcal{Y} , let $p(y)$ be its density function:
 $p(y) = P_Y(y) \dots$