Application of Information Theory, Lecture 6

Counting

Handout Mode

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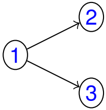
Section 1

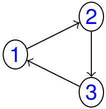
Graph Homomorphisms

Counting # of graph homomorphisms

- $T = (V_T, E_T)$ — directed graph (no self loops)
- $G = (V_G, E_G)$



- $H = (V_H, E_H)$



- (x_1, x_2, x_3) is an homomorphism of G in T, if $x_1, x_2, x_3 \in V_T$ and $(i,j) \in E_G \implies (x_i, x_j) \in E_T$ (might be $x_1 = x_2$)
- Example: see board
- $\mathrm{Hom}(X, T)$: all homomorphisms of X in T
- Claim $|\mathrm{Hom}(H, T)| \leq |\mathrm{Hom}(G, T)|$
- Trivial if G would be a subgraph of G
- Special case of a more general theorem

Proving the claim

- $(X_1, X_2, X_3) \leftarrow \mathrm{Hom}(H, T)$
- $$\begin{aligned}\log|\mathrm{Hom}(H,T)| &= H(X_1,X_2,X_3)\\ &= H(X_1) + H(X_2|X_1) + H(X_3|X_1,X_2)\\ &\leq H(X_1) + H(X_2|X_1) + H(X_3|X_2)\\ &= H(X_1) + 2\cdot H(X_2|X_1) && \text{(by symmetry of } H\text{)}\end{aligned}$$

- Let $D_2(x)$ be the distribution of $X_2|X_1 = x$, and let $X_2' \sim D_2(X_1)$
- $$\begin{aligned}H(X_1,X_2,X_2') &= H(X_1) + H(X_2|X_1) + H(X_2'|X_1,X_2)\\ &= H(X_1) + H(X_2|X_1) + H(X_2'|X_1)\\ &= H(X_1) + 2\cdot H(X_2|X_1)\end{aligned}$$

- $(X_1, X_2) \in E_T$ and $(X_1, X_2') \in E_T$

$\implies$ $(X_1, X_2, X_2') \in \mathrm{Hom}(G, T)$

$\implies$ $H(X_1, X_2, X_2') \leq \log|\mathrm{Hom}(G, T)|$

$\implies$ $\log|\mathrm{Hom}(H, T)| \leq \log|\mathrm{Hom}(G, T)|$. $\square$

Section 2

Perfect Matchings

Bregman's theorem

For bi-partite graph $G = (A, B, E)$, let $d(v) = |N(v) = \{u \in B \colon (v, u) \in E\}|$

Theorem 1

Let $G = (A, B, E)$ be bi-partite graph with $|A| = |B|$. Then $P(G)$ — the number of perfect matching in G — is at most $\prod_{v \in A} (d(v)!)^{1/d(v)}$.

- Let $A = B = [n] = \{1, \ldots, n\}$
- It is clear that $P(G) \leq \prod_{i \in [n]} d(i)$:
- Let $\mathcal{M}$ be the perfect matchings in G.
- For $m \in \mathcal{M}$ let $m(i)$ be the node in B matched with i by m.
- Let $M \leftarrow \mathcal{M}$. Hence,

$$\begin{aligned}
\log|\mathcal{M}| = H(M) &= H(M(1)) + H(M(2)|M(1)) + \ldots + H(M(n)|M(1), \ldots, M(n-1)) \\
&\leq H(M(1)) + H(M(2)) + \ldots + H(M(n)) \\
&\leq \log d(1) + \log d(2) + \ldots + \log d(n) \\
&= \prod_{i \in [n]} \log d(i)
\end{aligned}$$

Proving Bregman's theorem

- Key observations:
  $H(M(i|M(1),\ldots,M(i-1)) \leq \log|N(i) \setminus \{M(1),\ldots,M(i-1)\}|$
- Let $\mathcal{P}$ be the set of all permutation over $[n]$. For $p \in \mathcal{P}$:
  $H(M) = H(M(p(1))) + \ldots + H(M(p(n))|M(p(1)),\ldots,M(p(n-1)))$
- $\mathcal{S}_p(i) = \{1,\ldots,p^{-1}(i)-1\}$ — matchings appear above before i
- $H(M) = \sum_{i=1}^{n} H(M(i)|M(\mathcal{S}_p(i)))$
- For $m \in \mathcal{M}$ and $P \leftarrow \mathcal{P}$: $|N(i) \setminus m(\mathcal{S}_P(i))|$ is uniform over $\{1,\ldots,d(i)\}$

$\implies$ $\mathrm{E}_P\left[H(M(i) \mid M(\mathcal{S}_P(i)))\right] \leq \frac{1}{d(i)}\sum_{k=1}^{d(i)} \log k = \log\left((d(i)!)^{1/d(i)}\right)$

$\implies$

$$\begin{aligned} H(M) &= \mathop{\mathrm{E}}_{P}\left[\sum_{i=1}^{n} H(M(i)|M(\mathcal{S}_P(i)))\right] \\ &= \sum_{i=1}^{n} \mathop{\mathrm{E}}_{P}\left[H(M(i)|J_P(i))\right] \\ &\leq \prod_{i\in[n]} \log\left((d(i)!)^{1/d(i)}\right). \end{aligned}$$

$\square$

Section 3

Shearer's Lemma

$H(X_1, X_2, X_3)$ Vs. $H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)$

- How does $H(X_1, X_2, X_3)$ compares to $H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)$?
- If X_1, X_2, X_3 are independence, then $H(X_1, X_2, X_3) = \frac{1}{2}\left(H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)\right)$
- In general: $H(X_1, X_2, X_3) \leq \frac{1}{2}\left(H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)\right)$
- A tighter bounds than $H(X_1) + H(X_2) + H(X_3)$
- Proof:

$$
\begin{aligned}
2H(X_1, X_2, X_3) &= 2H(X_1) && +2H(X_2|X_1) && +2H(X_3|X_1, X_2) \\
H(X_2, X_3) &= H(X_1) && +H(X_2|X_1) && \\
H(X_2, X_3) &= && +H(X_2) && +H(X_3|X_2) \\
H(X_1, X_3) &= H(X_1) && && +H(X_3|X_1)
\end{aligned}
$$

- but

$$
\begin{aligned}
H(X_2|X_1) &\leq H(X_2) \\
H(X_3|X_1, X_2) &\leq H(X_3|X_1) \\
H(X_3|X_1, X_2) &\leq H(X_3|X_2)
\end{aligned}
$$

Shearer's lemma

- Let $X = (X_1, \ldots, X_n)$
- For $\mathcal{S} = \{i_1, \ldots, i_k\} \subseteq [n]$, let $X_{\mathcal{S}} = (X_{i_1}, \ldots, X_{i_k})$
- Example: $X_{1,3} = (X_1, X_3)$

Lemma 2 (Shearer's lemma)

Let $X = (X_1, \ldots, X_n)$ be a rv and let $\mathcal{F}$ be a family of subset of $[n]$ s.t. each $i \in [n]$ appears in at least m subset of $\mathcal{F}$. Then $H(X) \leq \frac{1}{m} \sum_{F \in \mathcal{F}} H(X_F)$.

Proof:

- $H(X) = \sum_{i=1}^{n} H(X_i | \{X_\ell \colon \ell < i\})$
- $H(X_F) = \sum_{i \in F} H(X_i | \{X_\ell \colon \ell < i \wedge \ell \in F\})$
- Hence,

$$\sum_{F \in \mathcal{F}} H(X_F) \geq \sum_{i=1}^{n} \sum_{j=1}^{m} H(X_i | \{X_\ell \colon \ell < i \wedge \ell \in F_{i,m}\})$$

$$\geq m \cdot \sum_{i=1}^{n} H(X_i | \{X_\ell \colon \ell < i\}) = m \cdot H(X)$$

Corollary

Corollary 3

Let $\mathcal{F} = \{F \subseteq [n] \colon |F| = k\}$. *Then* $H(X) \leq \frac{n}{k} \cdot \frac{1}{\binom{n}{k}} \cdot \sum_{F \in \mathcal{F}} H(X_F) = \frac{n}{k} \cdot \mathrm{E}_{F \leftarrow \mathcal{F}}\left[H(X_F)\right]$.

Proof: $\frac{k}{n} \cdot \binom{n}{k}$ is the # of times i appears in $\mathcal{F}$.

Implications:

- Let $Q \subseteq \{0,1\}^n$ and $X = (X_1, \ldots, X_n) \leftarrow Q$
- $|Q| \leq 2^{\frac{n}{k} \cdot \mathrm{E}_{F \leftarrow \mathcal{F}}[H(X_F)]}$
- $\mathrm{E}_F\left[H(X_F)\right]$ is small $\implies$ Q is small
- Q is large $\implies$ $\mathrm{E}_F\left[H(X_F)\right]$ is large

Example

- $Q \subseteq \{0,1\}^n$ with $|Q| = 2^n/2 = 2^{n-1}$; $X \leftarrow Q$.
- $\mathcal{F} = \{F \subseteq [n] \colon |F| = k\}$
- By Corollary 3, $\log|Q| = n - 1 \leq \frac{n}{k} \cdot \mathrm{E}_{F \leftarrow \mathcal{F}}[H(X_F)]$

$\implies \mathrm{E}_F[H(X_F)] \geq k(1 - \frac{1}{n}) = n - \frac{k}{n}$

$\implies \exists F \in \mathcal{F}$ s.t. $H(X_F) \geq n - \frac{k}{n}$

- Assume $n = 1000$ and $k = 5$, hence $H(X_F) \geq 5 - \frac{1}{200}$
- X_F can take at least $2^{5-\frac{1}{200}} = 2^{-\frac{1}{200}} \cdot 2^5 > 31$ (and hence 32) values
- Stronger conclusion: X_F is close to the uniform distribution.

More generally

- $|Q| \geq \frac{1}{2^d} \cdot 2^n$; $X \leftarrow Q$
- $\mathcal{F} = \{F \subseteq [n] \colon |F| = k\}$
- $n - d \leq H(X) \leq \frac{n}{k} \cdot \frac{1}{|\mathcal{F}|} \cdot \sum_{F \in \mathcal{F}} H(X_F)$

$\implies \frac{1}{|\mathcal{F}|} \cdot \sum_{F \in \mathcal{F}} H(X_F) \geq k - \frac{dk}{n}$

$\implies \mathrm{E}_{F \leftarrow \mathcal{F}} [H(X_F)] \geq k - \frac{dk}{n}$

- If $dk << n$, then a typical X_F is close to the uniform distribution

Section 4

Statistical Distance

Statistical distance

- Let $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$ be distributions over $[m]$
- Their statistical distance (also known as, variation distance) is defined by
$$\mathrm{SD}(p, q) := \frac{1}{2} \sum_{i \in [m]} |p_i - q_i|$$
- This is simply the L_1 norm between the distribution vectors
- We will see other “distance" measures for distributions next lecture
- For $Z \sim p$ and $Y \sim q$, let $\mathrm{SD}(X, Y) = \mathrm{SD}(p, q)$
- Claim (HW): $\mathrm{SD}(p, q) = \max_{\mathcal{S} \subseteq [m]} \left(\sum_{i \in \mathcal{S}} p_i - \sum_{i \in \mathcal{S}} q_i \right)$
- Hence, $\mathrm{SD}(p, q) = \max_{\mathrm{D}} \left(\Pr_{X \sim p} [\mathrm{D}(X) = 1] - \Pr_{X \sim q} [\mathrm{D}(X) = 1] \right)$
- Interpretation

Distance from the uniform distribution

- Let X be rv over $[m]$
- $H(X) \leq \log m$
- $H(X) = m \longleftrightarrow X$ is uniform over $[m]$

Theorem 4 (Next lecture)

Let X rv over $[m]$. Assume $H(X) \geq \log m - \varepsilon$, then
$\mathrm{SD}(X, \sim [m]) \leq \sqrt{\varepsilon \cdot 2 \cdot \ln 2} = O(\varepsilon)$

Section 5

Gold Coins

of gold coins in a cube

- Q — (finite) set of points in $\mathbb{R}^3$
  - Projection of Q on xy — 6
  - Projection of Q on xz — 8
  - Projection of Q on yz — 12
- Can we bound $|Q|$?
- The real story
- $X = (X_1, X_2, X_3) \leftarrow Q$
- $$\begin{aligned}\log|Q| = H(X) &\leq \frac{1}{2}(H(X_1,X_2) + H(X_1,X_3) + H(X_2,X_3))\\ &\leq \frac{1}{2}(\log 6 + \log 8 + \log 12)\\ &\leq \frac{1}{2}(\log 6 \cdot 8 \cdot 12)\end{aligned}$$
- Hence, $|Q| \leq \sqrt{6 \cdot 8 \cdot 12} = 24$
- Tight! $2 \times 3 \times 4$

of gold coins, the hyperspace case

- Q — (finite) set of points in $\mathbb{R}^n$
- m_i— # of coins in projection on $(1,\ldots,i-1,i+1,\ldots,n)$
- Claim: $|Q| \leq (\prod_{i\in[n]} m_i)^{1/(n-1)}$
- Proof: $X = (X_1,\ldots,X_n) \leftarrow Q$, $X_{-i} = (X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_n)$
- $\log|Q| = H(X) \leq \frac{1}{n-1}\sum_i H(X_{-i}) \leq \frac{1}{n-1}\sum_i \log m_i$

Section 6

Intersecting Graphs

Another corollary of Shearer's lemma

Corollary 5

Let $\mathcal{A}$ and $\mathcal{F}$ be collections of subsets of $[n]$, and for $F \in \mathcal{F}$ let $\mathcal{A}_F$ be the collection $\{A \cap F \colon A \in \mathcal{A}\}$. Assume that each element of $[n]$ appears in at least m subsets of $\mathcal{F}$, then $|\mathcal{A}|^m \leq \prod_{F \in \mathcal{F}} |\mathcal{A}_F|$.

Proof:

- Let $Y \leftarrow \mathcal{A}$, let $X_i = 1$ iff $i \in Y$, and $X = (X_1, \ldots, X_n)$.
- $\log |\mathcal{A}| = H(Y) = H(X)$ $\quad$ ($Y \longleftrightarrow X$)
- $\log |\mathcal{A}_F| \geq H(X_F)$ $\quad$ ($\mathrm{Supp}(X_F) \subseteq \mathcal{A}_F$)
- By Shearer's lemma, $H(X) \leq \frac{1}{m} \sum_{F \in \mathcal{F}} H(X_F)$. $\square$

of intersecting graphs

Theorem 6

Let $\mathcal{G}$ be a family of graphs over $[n]$, s.t. $G \cap G'$ contains a triangle for each $G, G' \in \mathcal{G}$. Then $|\mathcal{G}| \leq 2^{\binom{n}{2}-2}$.

This improves over $|\mathcal{G}| \leq 2^{\binom{n}{2}-1}$, which follows from $G \cap G' \neq \emptyset$.
(wlg. all graph shares the same edge)

Proof:

- For $\lfloor n/2 \rfloor$-size set $\mathcal{S} \subset [n]$, let $F = F(\mathcal{S})$ be the union of the cliques $\mathcal{S}$ and $[n] \setminus \mathcal{S}$
- $F \cap G \cap G' \neq \emptyset$ for any $G, G' \in \mathcal{G}$ and $\mathcal{S}$ as above
- Hence $|\mathcal{G}_F := G \cap F \colon G \in \mathcal{G}| \leq 2^{|F|-1}$
- Let $m = \binom{n}{2}$ and $m' = |F|$
- Each edge over $[n] \times [n]$, appears in $\frac{m'}{m}$ of graphs $\{F(\mathcal{S})\}_{\mathcal{S} \subset [n] \colon |\mathcal{S}| = \lfloor n/2 \rfloor}$.
- By Corollary 5, $|\mathcal{G}|^{\frac{m'}{m} \cdot \binom{n}{\lfloor n/2 \rfloor}} \leq (2^{m'-1})^{\binom{n}{\lfloor n/2 \rfloor}}$
- Hence, $|\mathcal{G}| \leq 2^{m - \frac{m}{m'}} \leq 2^{\binom{n}{2}-2}$

Section 7

Independent Sets

of independent sets in bi-partite graphs

Theorem 7

Let $G = (A, B, E)$ be an n-regular graph with $|A| = |B| = m$. Then the number of independent sets in G is at most $(2^{n+1} - 1)m$.

Proof: $\mathcal{I}$ — set of independent sets in G.

- Let $I \leftarrow \mathcal{I}$, let $X_v = 1$ iff $v \in I$, and $X_{\mathcal{S}} = \{X_v \colon v \in \mathcal{S}\}$.
- $$\begin{aligned} H(I) &= H(X_A|X_B) + H(X_B) \\ &\leq \sum_{v \in A} H(X_v|X_B) + \frac{1}{n} \sum_{v \in A} H(X_{N(v)}) \qquad \text{(rhs by Sherer's Lemma)} \\ &\leq \sum_{v \in A} \left(H(X_v|N(v)) + \frac{1}{n} H(X_{N(v)}) \right) \end{aligned}$$
- Fix $v \in A$. Let $\chi_v = \begin{cases} 0, & X_{N(v)} = 0^{|N(v)|} \\ 1, & \text{otherwise.} \end{cases}$, and $p = p(v) = \Pr[\chi_v = 0]$
- $H(X_v|X_{N(v)}) \leq H(X_v|\chi_v) \leq p$
- $H(X_{N(v)}) = H(X_{N(v)}\chi_v) = H(\chi_v) + H(X_{N(v)}|\chi_v) \leq h(p) + (1-p)\log(2^n - 1)$
- Hence $H(I) \leq \sum_{v \in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1 - p(v))\log(2^n - 1)\right)$

of independent sets in bi-partite graphs, cont.

- $\log|\mathcal{I}| = H(I) \leq \sum_{v\in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1-p(v))\log(2^n-1)\right)$
- Let $f(t) := t + \frac{1}{n}\left(h(t) + (1-p(t))\log(2^n-1)\right)$
- By calculus, $\max_{t\in[0,1]} f(t) = \frac{1}{n}\log(2^{n+1}-1)$
- Hence, $\log|\mathcal{I}| \leq \frac{m}{n}\log(2^{n+1}-1)$. $\square$