Foundation of Cryptography, Lecture 1
One-Way Functions

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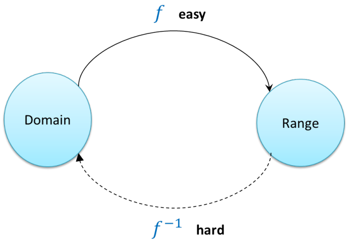
Section 1

One-Way Functions

Informal discussion



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- Easy to compute, everywhere
- Hard to invert, on the average

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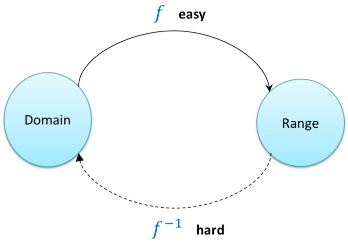

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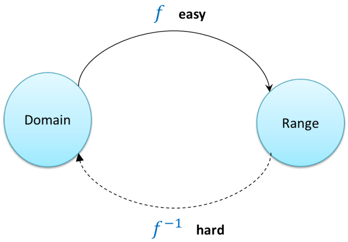

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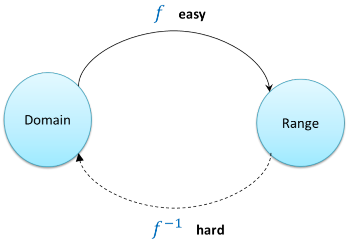

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- Easy to compute, everywhere
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- Why should we care about OWFs?
- Hidden in (almost) any cryptographic primitive: necessary for "cryptography"
- Sufficient for many cryptographic primitives

Formal definition

Definition 1 (one-way functions (OWFs))

A polynomial-time computable function $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ is one-way, if

$$\Pr_{x \xleftarrow{R} \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] = \operatorname{neg}(n)$$

for any PPT A.

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We typically omit 1^n from the input list of A

Formal definition cont.

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7. Non uniform OWFs

Definition 2 (Non-uniform OWF))

A polynomial-time computable function $f : \{0,1\}^* \mapsto \{0,1\}^*$ is non-uniformly one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[C_n(f(x)) \in f^{-1}(f(x))\right] = \operatorname{neg}(n)$$

for any polynomial-size family of circuits $\{C_n\}_{n\in\mathbb{N}}$.

Length-preserving functions

Definition 3 (length preserving functions)

A function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is length preserving, if $|f(x)| = |x|$ for every $x \in \{0,1\}^*$

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Proof idea: use the assumed OWF to create a length preserving one

Partial domain functions

Definition 5 (Partial domain functions)

For $m, \ell \colon \mathbb{N} \mapsto \mathbb{N}$, let $f \colon \{0,1\}^{m(n)} \mapsto \{0,1\}^{\ell(n)}$ denote a function defined over input lengths in $\{m(n)\}_{n \in \mathbb{N}}$, and maps strings of length $m(n)$ to strings of length $\ell(n)$.

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The definition of one-wayness naturally extends to such functions.

OWFs imply length-preserving OWFs cont.

Let $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ be a OWF, let $p \in \text{poly}$ be a bound on its computing-time and assume wlg. that p is monotony increasing (can we?).

OWFs imply length-preserving OWFs cont.

Let $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ be a OWF, let $p \in$ poly be a bound on its computing-time and assume wlg. that p is monotony increasing (can we?).

Construction 6 (the length preserving function)

Define $g\colon \{0,1\}^{p(n)} \mapsto \{0,1\}^{p(n)}$ as

$$g(x) = f(x_{1,\ldots,n}), 0^{p(n)-|f(x_{1,\ldots,n})|}$$

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Note that g is well defined, length preserving and efficient (why?).

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Answer: using reduction.

Proving that g is one-way

Proof:

Assume that g is not one-way. Namely, there exists PPT A, $q \in \text{poly}$ and infinite set $\mathcal{I} \subseteq \{p(n) \colon n \in \mathbb{N}\}$, with

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, y) \in g^{-1}(g(x))\right] > 1/q(n) \qquad (1)$$

for every $n \in \mathcal{I}$.

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We show how to use A for inverting f.

Algorithm 8 (The inverter B)

Input: 1^n and $y \in \{0,1\}^*$

1. Let $x = \mathsf{A}(1^{p(n)}, y, 0^{p(n)-|y|})$
2. Return $x_{1,\ldots,n}$

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Let $\mathcal{I}' := \{n \in \mathbb{N} \colon p(n) \in \mathcal{I}\}$. Then

1. $\mathcal{I}'$ is infinite
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This contradict the assumed one-wayness of f. $\square$

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From partial-domain OWFs to OWFs

Construction 10

Given a function $f\colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^{\ell(n)}$, define $f_{\mathrm{all}}\colon \{0,1\}^* \mapsto \{0,1\}^*$ as

$$f_{\mathrm{all}}(x) = f(x_{1,\ldots,k}), 0^{n-k}$$

where $n = |x|$ and $k := \max\{\ell(n') \leq n\colon n' \in [n]\}$.

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Assume f and ℓ are efficiently computable, f is one-way, and ℓ satisfies $1 \leq \frac{\ell(n+1)}{\ell(n)} \leq p(n)$ for some $p \in \mathrm{poly}$, then f_{all} is one-way function.

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Proof: ?

Few Remarks

More “security-preserving" reductions exits.

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Convention for rest of the talk

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a one-way function.

Weak One Way Functions

Definition 12 (weak one-way functions)

A poly-time computable function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is α-one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] \leq \alpha(n)$$

for any PPT A and large enough $n \in \mathbb{N}$.

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2. Can we "amplify" weak OWF to strong ones?

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Proof:

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Proof: For a OWF f, let

$$g(x) = \begin{cases} (1, f(x)), & x_1 = 1; \\ 0, & \text{otherwise } (x_1 = 0). \end{cases}$$

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Assume there exist $(1-\delta)$-weak OWFs with $\delta(n) \geq 1/q(n)$ for some $q \in \mathrm{poly}$, then there exist (strong) one-way functions.

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- Fortunately, parallel repetition does amplify weak OWFs :-)

Amplification via Parallel Repetition

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Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$*, and for* $t(n) := \left\lceil \frac{\log^2 n}{\delta(n)} \right\rceil$ *define* $g\colon (\{0,1\}^n)^{t(n)} \mapsto (\{0,1\}^n)^{t(n)}$ *as*

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Proving that g is One-Way – the Naive Approach

Assume A attacks each of the t outputs of g independently: $\exists$ PPT A$'$ such that $\mathsf{A}(z_1, \ldots, z_t) = \mathsf{A}'(z_1) \ldots, \mathsf{A}'(z_t)$

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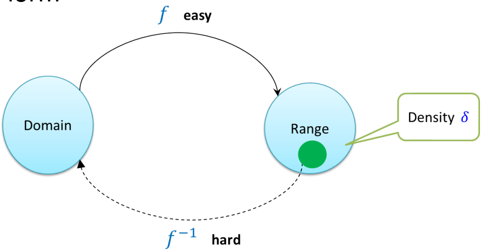
Any idea?

Hardcore Sets

Assume f is of the form



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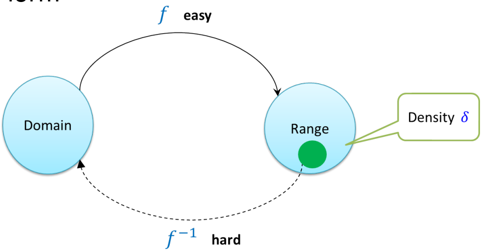

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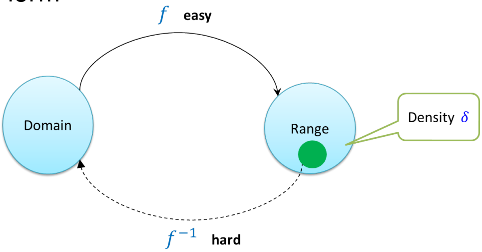

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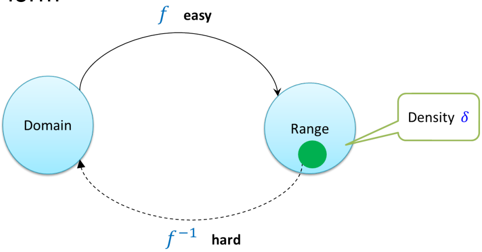

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Unfortunately, we do not know how to prove that f has hardcore set :-<

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A function $f \colon \{0,1\}^n \mapsto \{0,1\}^n$ has a δ-failing set for a pair (A, q) of algorithm and polynomial, if exists $\mathcal{S} = \{\mathcal{S}_n \subseteq \{0,1\}^n\}$, such that the following holds for large enough n:

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Let f be a $(1-\delta)$-OWF, then f has a $\delta/2$-failing set, for any pair of PPT A and $q \in \text{poly}$.

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We'll use A to contradict the hardness of f.

Using A to Invert f

For $n \in \mathbb{N}$, let $\mathcal{S}_n := \{y \in \{0,1\}^n \colon \Pr\left[\mathsf{A}(y) \in f^{-1}(y)\right] < 1/q(n)\}$.

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$\exists$ infinite $\mathcal{I} \subseteq \mathbb{N}$ with $\Pr_{x \xleftarrow{R} \{0,1\}^n}[f(x) \in \mathcal{S}_n] < \delta(n)/2$ for every $n \in \mathcal{I}$.

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Namely, f is not $(1-\delta)$-one-way $\square$

Proving g is One-Way cont.

We show that is g is not one way, then f has no $\delta/2$ flailing-set for some PPT B and $q \in$ poly.

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Assume $\exists$ PPT A, $p \in$ poly and an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that

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Namely, f has no $\delta/2$ failing set for $(\mathsf{B}, q = 2t(n)p(n))$

The No Failing-Set Algorithm

Algorithm 23 (Inverter B on input $y \in \{0,1\}^n$)

1. Choose $w \xleftarrow{R} (\{0,1\}^n)^{t(n)}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \xleftarrow{R} [t]$
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Proof: Assume for simplicity that A is deterministic.

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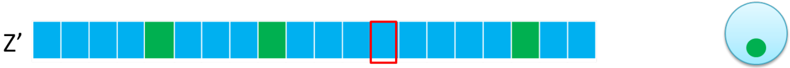

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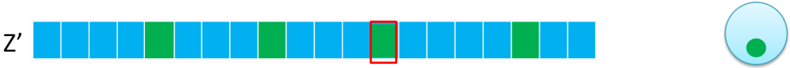

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The No Failing-Set Algorithm

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Claim 24

$\Pr_{x \xleftarrow{R} \{0,1\}^n | y = f(x) \in \mathcal{S}_n}\left[\mathsf{B}(y) \in f^{-1}(y)\right] \geq \frac{1}{t(n) \cdot p(n)} - n^{-\log n}$.

Proof: Assume for simplicity that A is deterministic.

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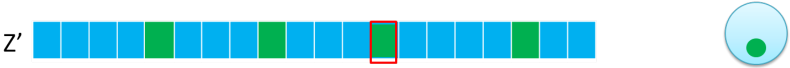

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To conclude the proof take $\mathcal{L} = \{v \in \{0,1\}^{t(n) \cdot n} \colon \mathsf{A}(v) \in g^{-1}(v)\}$

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- Similar hardness amplification theorems for other cryptographic primitives (e.g., Captchas, general protocols)?
- What properties of the weak OWFs have we used in the proof?