Application of Information Theory, Lecture 3
Graph Covering, Differential Entropy

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Part I

Applications to Graph Covering

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Let G be a graph over $[n]$, let $Z \leftarrow \mathsf{nonIs}(G)$ and let $\hat{\chi}$ be a (valid) coloring of G such that $H(\hat{\chi}(Z))$ is minimal. Then $\mathsf{content}(G) := \frac{|\mathsf{nonIs}(G)|}{n} \cdot H(\hat{\chi}(Z))$.

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$$\begin{aligned} 0 = H(X|Y_1, \ldots, Y_t) &= H(X, Y_1, \ldots, Y_t) - H(Y_1, \ldots, Y_t) \\ &\geq H(X) + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i) \\ &= \log n + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i). \end{aligned}$$

- $Y_1, \ldots, Y_t$ are independent conditioned on X —
  $\Pr[Y_1 = y_1 \wedge Y_2 = y_2 \mid X = x] = \Pr[Y_1 = y_1 \mid X = x] \cdot \Pr[Y_2 = y_2 \mid X = x]$
- Hence, $H(Y_1, \ldots, Y_t|X) = \sum_i H(Y_i|X)$ (board)
- We conclude that $\sum_i H(Y_i) - \sum_i H(Y_i|X) \geq \log n$
- Since $H(Y_i) = H(\chi_i(Z_i))$ and $H(Y_i|X) = (1 - \frac{|\text{nonIs}(G_i)|}{n}) \cdot H(\chi_i(Z_i))$,
  it follows that $\sum_i H(\chi_i(Z_i)) \frac{|\text{nonIs}(G_i)|}{n} \geq \log n$. $\square$

Extension

Let $\alpha(G)$ be the size of the maximal independent set in G

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Theorem 5

Let $G, G_1, \ldots, G_t$ be graphs over $[n]$ with $\bigcup_{i=1}^{t} G_i = G$, then $\sum \mathrm{content}(G_i) \geq \log \frac{n}{\alpha(G)}$.

Proof: HW

Scrambling permutations

Scrambling permutations

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- Hence, $|\mathcal{S}| (n-1) \cdot \frac{\log e}{2} \geq n \cdot \log(n-1)$, and the proof follows.$\square$

Part II

Differential Entropy

Entropy of continues random variable

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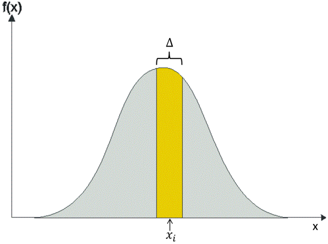

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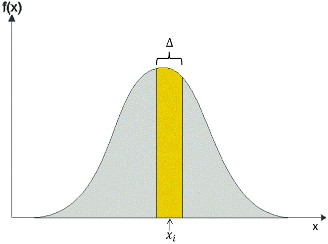

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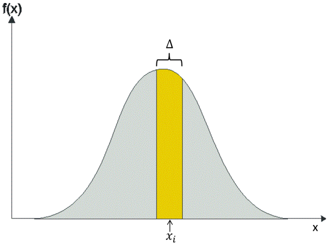

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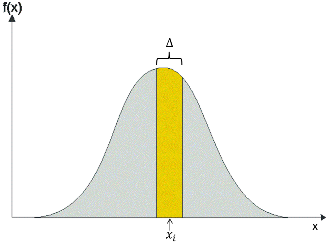

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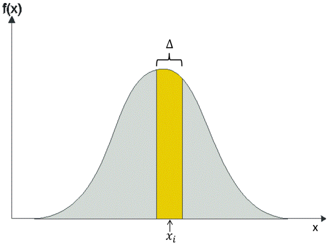

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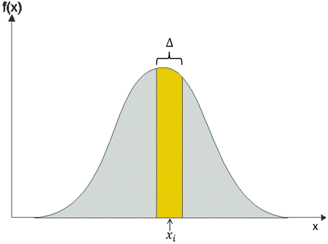

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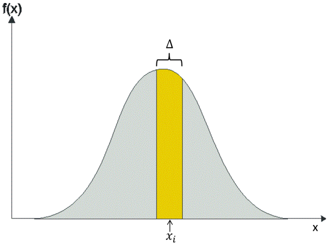

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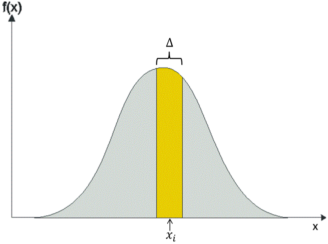

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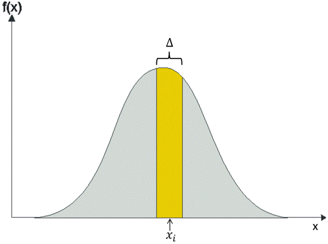

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- This suggests that distributions that are common in nature, are distributions of maximal entropy, under some constrains.

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- CLT holds also in many other variants: not id, not fully independent, ...

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- CLT and the normal distribution where known and studied way before Shannon, yet this striking property was not known until his theory.

The normal distribution, cont.

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Proof:

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- Hence, $\sum_i q_i \log p_i = \sum_i p_i \log p_i$. $\square$

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- $X \sim [a, b]$.
- $\mathrm{E}\, X = \frac{1}{2}(a+b)$ and $\mathrm{V}\, X = \frac{1}{12}(b-a)^2$
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Proof: HW

Using diff. entropy to bound discrete entropy

Using diff. entropy to bound discrete entropy

Proposition 12

Let $X \sim (p_1, p_2, \ldots)$, then $H(X) \leq \frac{\log 2\pi e}{2} \cdot \left(\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2 - \frac{1}{12}\right)$

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&= -\sum_{i=1}^{\infty} \left(\int_i^{i+1} f_{\tilde{X}}(x) dx \right) \cdot \log p_i
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Using diff. entropy to bound discrete entropy, cont.

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- Hence,

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\begin{aligned}
H(X) &= h(\tilde{X}) \\
&\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X})
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Using diff. entropy to bound discrete entropy, cont.

- Hence,

$$\begin{aligned} H(X) &= h(\tilde{X}) \\ &\leq \frac{1}{2}\log(2\pi e)\,\mathsf{V}(\tilde{X}) \\ &= \frac{1}{2}\log(2\pi e)\,(\mathsf{V}(X) + \mathsf{V}(U)) \end{aligned}$$

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H(X) &= h(\tilde{X}) \\
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&= \frac{\log 2\pi e}{2} \cdot \left(\left(\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2 \right) + \frac{1}{12} \right)
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$$
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- How good is this bound?

Using diff. entropy to bound discrete entropy, cont.

- Hence,

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- How good is this bound?
- Let $X \sim (\frac{1}{2}, \frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.

Using diff. entropy to bound discrete entropy, cont.

- Hence,

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- How good is this bound?
- Let $X \sim (\frac{1}{2}, \frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.
- Proposition 12 grantees that $H(X) \leq \frac{\log 2\pi e}{2}(\frac{1}{4} + \frac{1}{12}) \sim 1.255$