Application of Information Theory, Lecture 5

Channel Capacity and Isoperimetric Inequality

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Part I

Channel Capacity

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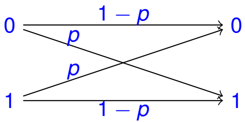

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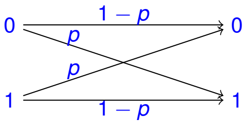

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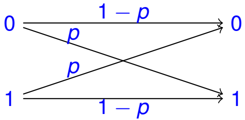

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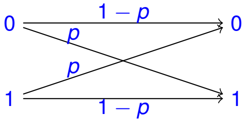

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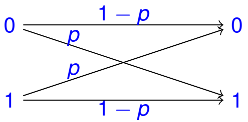

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- Can we send bits with smaller error?

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- Before Shannon it was believed that very small error rate requires very small transmission rate.

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- A revolution in EE and the whole world

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Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p,\ s.t.\ \forall \varepsilon > 0 \quad \exists m_\varepsilon,\ s.t.\ \forall m \geq m_\varepsilon$ *and* $n \geq m(\frac{1}{C_p} + \varepsilon)$,
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- We prove a weaker variant that holds w.h.p. over $\mathbf{x} \leftarrow \{0,1\}^m$

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- We sometimes just write $|\mathbf{y}|$.

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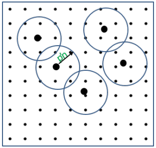

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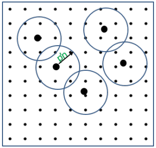

- We choose f uniformly at random (what does it mean?)

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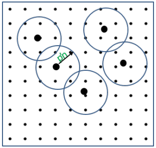

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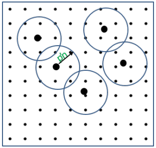

- We choose f uniformly at random (what does it mean?)
- Non-constructive proof
- Probabilistic method

Proving there exists good f

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- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0, 1\}^m$

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(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log\frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$

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Proving there exists good f

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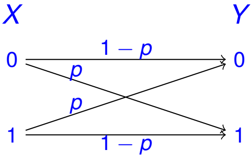

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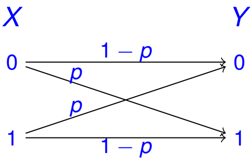

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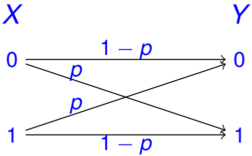

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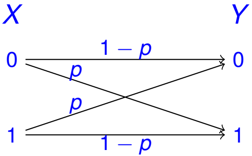

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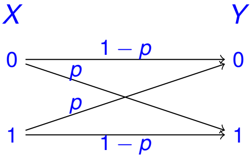

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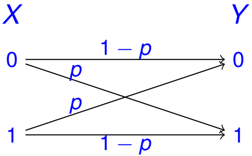

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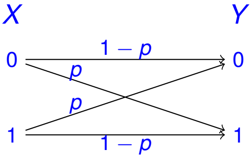

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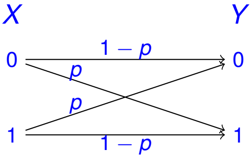

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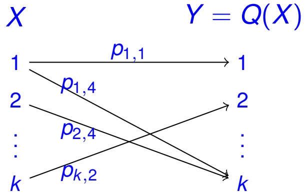

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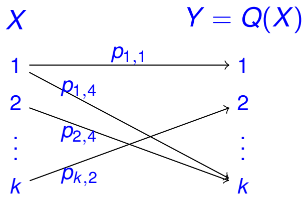

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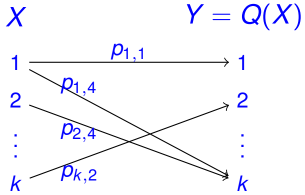

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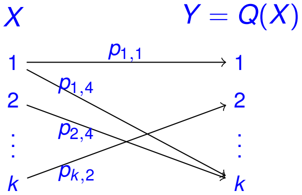

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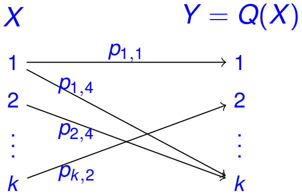

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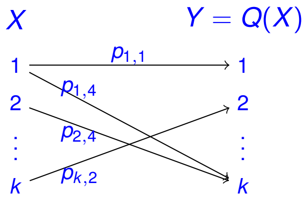

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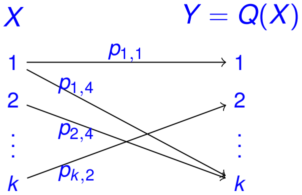

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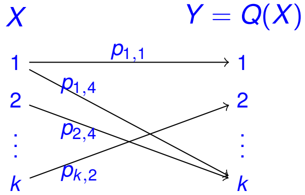

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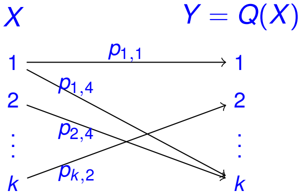

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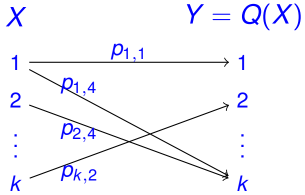

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- Proof: similar lines to the binary case, but more subtle distribution for f

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Part II

Hamming Ball

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Very useful estimation. Weaker variants follows by AEP or Stirling,

Hamming ball, cont.

The above bound yields the following concentration bound:

Corollary 4

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Very useful inequality. No Chernoff just IT

Part III

Combinatorial Applications

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Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
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- Hence, X is not determined by Y

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Lemma 7 (Lemma 6, recited)

$H(X_i|X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

We prove for $i = 1$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0, \mathbf{y}), (1, \mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$
- $|\mathcal{S}| = 2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|$
- $|E_1| = \left|\mathcal{S}^e_{-1}\right|$
- $H(X_1|X_{-1}) = \Pr\left[X_{-1} \in \mathcal{S}^e_{-1}\right] \cdot 1 = \frac{2\left|\mathcal{S}^e_{-1}\right|}{2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|}$

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