Foundation of Cryptography, Lecture 11
Black-Box Impossibility Results

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June 10, 2014

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  $\Longrightarrow$ OWFs imply the existence of key-agreement protocols in a trivial sense.

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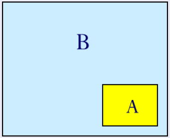
A oracle-aided PPT I such that I^{O} implements B for any algorithm O implementing A



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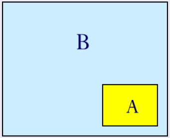

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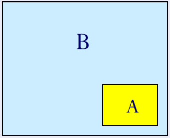

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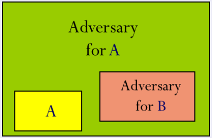
A oracle-aided PPT R such that $\mathrm{R}^{\mathrm{O},\mathrm{D}}$ breaks *B*, for any algorithms O implementing *A*, and D breaking *B*.



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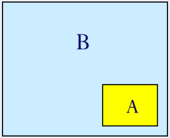

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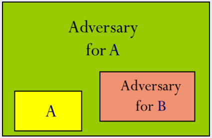

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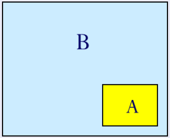

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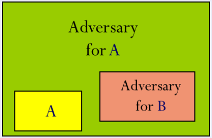

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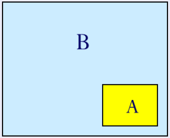

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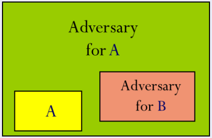

- Fully-black-box constructions relativize: hold relative to any oracle.
- Most constructions in cryptography are (fully) black-box, e.g., pseudorandom generator from OWF.
- Few "non black-box" techniques that apply in restricted settings (typically using ZK proofs)

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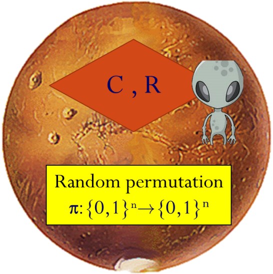

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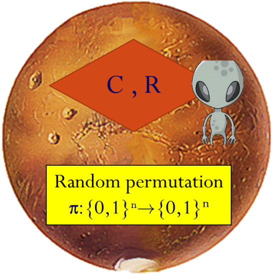

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This yields a contradiction, implying that (I, R) does not exist.

Section 1

Random Permutations

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Proof: ?

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For any large enough $n \in \mathbb{N}$ and $2^{n/5}$-query circuit D,

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For any q-query circuit D and $\varepsilon > 0$, exist algorithms Enc and Dec such that: Let $\pi \in \Pi_n$ be such that $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{D}^{\pi}(\pi(x)) = x] > \varepsilon$, then

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- Let D be a $2^{n/5}$-query circuit.

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For any q-query circuit D and $\varepsilon > 0$, exist algorithms Enc and Dec such that: Let $\pi \in \Pi_n$ be such that $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{D}^{\pi}(\pi(x)) = x] > \varepsilon$, then

- $\mathsf{Dec}(\mathsf{Enc}(\pi)) = \pi$
- $|\mathsf{Enc}(\pi)| \leq \log((2^n - a)!) + 2 \cdot \log \binom{N}{a}$, *for some* $a \geq \frac{\varepsilon N}{q+1}$.

- The description of π using $\mathsf{Enc}(\pi)$ “saves" $\log(a!) - \log \binom{N}{a}$ bits.
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$$\frac{(N - 2^{\frac{3}{5}n})! \cdot \binom{N}{2^{\frac{3}{5}n}}^2}{N!} = \frac{\binom{N}{2^{\frac{3}{5}n}}}{2^{\frac{3}{5}n}!} \leq 2^{-2^{\frac{3}{5}n}/2},$$

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Construction 5 (Useful set $\mathcal{Y} \subseteq \{0,1\}^n$)

1. Set $\mathcal{Y} = \emptyset$ and $\mathcal{I} = \{y \in \{0,1\}^n \colon \mathsf{D}^{\pi}(\pi(x)) = \pi\}$.
2. While $\mathcal{I} \neq \emptyset$, let y be the smallest lexicographic element in $\mathcal{I}$.
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Output (description of) $\mathcal{Y}$ and $\mathcal{V} = \{(x, \pi(x)) : \pi(x) \notin \mathcal{Y}\}$.

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Algorithm 7 (Dec($\mathcal{Y}, \mathcal{V}$))

For all $y \in \mathcal{Y}$ in lex. order:

1. Emulate $D^{\pi}(y)$.
2. If D makes a π-query x that is undefined in $\mathcal{V}$, add (x, y) to $\mathcal{V}$.
   Otherwise, add $(D^{\pi, \text{Sam}_r^{\pi}}(y), y)$ to $\mathcal{V}$.

Use $\mathcal{V}$ to reconstruct π.

Remarks

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- Alternative compression argument.
- Similar results can be proven for random variants of OWF, TDP, CRH.

Section 2

BB Impossibility for Efficient OWF based PRG

BB Impossibility for OWF based PRG

Definition 8 (pseudorandom generators (PRGs))

Poly-time $G\colon \{0,1\}^n \mapsto \{0,1\}^{\ell(n)}$ is a pseudorandom generator, if

- G is length extending (i.e., $\ell(n) > n$ for any n)
- $G(U_n)$ is pseudorandom (i.e., $\{G(U_n)\}_{n\in\mathbb{N}} \approx_c \{U_{\ell(n)}\}_{n\in\mathbb{N}}$)

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- Without the restriction on the OWP input length, yields an optimal $n^{\Omega(1)}/\log n$ bound.

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- Let (I, R) be a fully-BB reduction of a $q(n) \in o(n/\log n)$-query, length-doubling PRG over $\{0,1\}^n$, to OWP over $\{0,1\}^n$.

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1. Emulate $\mathrm{C}^{\pi}(x_{1,\ldots,n})$, while answering the i'th query z of I to π, with $x_{n+i\cdot t+1,\ldots,n+(i+1)\cdot t} \circ z_{t+1,\ldots,n}$.

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Claim 11

$G(U_{3n/2}) \equiv (\mathrm{I}^{\pi}(U_n))_{\pi \leftarrow \Pi_{n,t}}$.

Proving Thm 9, cont.

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- $\exists$ algorithm D that distinguishes $G(U_{3n/2})$ from U_{2n} with advantage $1 - 2^{-n/4} > \frac{1}{2}$. (?)

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- Results can be easily extended to OWFs/TDPs.
- Using similar means, one can prove lower bound on fully-BB constructions of encryption schemes, signature schemes and universal-one-way-hash-functions (UOWHFs), from OWFs/OWPs/TDPs

Section 3

BB Impossibility for Basing CRH on OWF

Basing CRH on OWF

Definition 12 (collision resistant hash family (CRH))

A function family $\mathcal{H} = \{\mathcal{H}_n \colon \{0,1\}^* \mapsto \{0,1\}^n\}$ is collision resistant, if

$$\Pr_{\substack{h \leftarrow \mathcal{H}_n \\ (x,x') \leftarrow \mathsf{A}(1^n,h)}} \left[x \neq x' \in \{0,1\}^* \wedge h(x) = h(x')\right] = \operatorname{neg}(n)$$

for any PPT A.

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- In a BB construction of CRH family, both the sampling and evaluation algorithms makes use of the oracle.
- wlg. the sampling algorithm outputs a string h, independent of the oracle, and $h \in \mathcal{H}_n$ is an oracle circuit.

Basing CRH on OWF

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A function family $\mathcal{H} = \{\mathcal{H}_n \colon \{0,1\}^* \mapsto \{0,1\}^n\}$ is collision resistant, if

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for any PPT A.

- In a BB construction of CRH family, both the sampling and evaluation algorithms makes use of the oracle.
- wlg. the sampling algorithm outputs a string h, independent of the oracle, and $h \in \mathcal{H}_n$ is an oracle circuit.
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- Seems harder to prove: CRH exists relative to random permutations!

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Input: An n-bit input circuit C.
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1. Sample $x \leftarrow \{0,1\}^n$.
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- In the actual implementation Sam uses independent randomness per input query C.

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For any $h \in \mathcal{H}_n$ and $\pi \in \Pi_n$: $\Pr_{(x,x') \leftarrow \mathsf{Sam}^{\pi}(h)}[x \neq x' \wedge h^{\pi}(x) = h^{\pi}(x')] \geq \frac{1}{4}$.

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The augmented number of queries an oracle-aided circuit/algorithm with Sam-gate does, is the number of queries it makes **directly**, plus twice the number of queries the circuits it **passes** to Sam do.

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For any large enough $n \in \mathbb{N}$ and $2^{n/5}/2$-augmented-query circuit D:

$$\Pr_{\pi\leftarrow\Pi_n; r\leftarrow\{0,1\}^*}\left[\Pr_{x\leftarrow\{0,1\}^n}\left[D^{\pi,\mathrm{Sam}_r^\pi}(\pi(x)) = x\right] > 2\cdot 2^{-n/5}\right] \leq 2\cdot 2^{-2^{\frac{3}{5}n}}$$

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- Almost the same result as in the non-Sam case.
- Hence, random permutations are (extremely) hard for exponential-size circuits with oracle access to Sam.

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Fix large enough n. The proof follows by the next two claims.

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For circuit D, $\pi \in \Pi_n$, $r \in \{0,1\}^*$, an $y \in \{0,1\}^n$, let $\mathsf{hit}^{\pi}_{\mathsf{D};r}(y)$ be one, if $\mathsf{D}^{\pi,\mathsf{Sam}^{\pi}_r}(y)$ makes a query $(x, x') = \mathsf{Sam}^{\pi}_r(\mathsf{C})$, and either $\mathsf{C}^{\pi}(x)$ or $\mathsf{C}^{\pi}(x')$ query π on $\pi^{-1}(y)$.

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For any t-augQuery circuit D, $\exists$ $2t$-augQuery circuit $\widetilde{\mathsf{D}}$ such that:
$\Pr_{\pi;r}\left[\Pr_x\left[\mathsf{hit}^{\pi}_{\mathsf{D};r}(\pi(x))\right] > \varepsilon\right] \le$
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Algorithm 22 ($\widetilde{\mathsf{D}}^{\pi,\mathsf{Sam}^{\pi}_{r}}(y)$)

Emulate $\mathsf{D}^{\pi,\mathsf{Sam}^{\pi}_{r}}(y)$. Before any query of $\mathsf{Sam}^{\pi}_{r}(\mathsf{C})$:
Evaluate $\mathsf{C}^{\pi}(z)$ for $z \leftarrow \{0,1\}^n$. If $\mathsf{C}^{\pi}(z)$ makes a query $\pi(x) = y$, return x and halt.

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- The augmented query complexity of $\widetilde{\mathsf{D}}$ is at most twice that of D.

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For any t-augQuery circuit D, $\exists$ $2t$-augQuery circuit $\widetilde{\mathrm{D}}$ such that:
$\Pr_{\pi;r}\left[\Pr_x\left[\mathrm{hit}^{\pi}_{\mathrm{D};r}(\pi(x))\right] > \varepsilon\right] \leq$
$\Pr_{\pi;r}\left[\Pr_x\left[\widetilde{\mathrm{D}}^{\pi,\mathrm{Sam}^{\pi}_r}(\pi(x)) = x \wedge \neg\mathrm{hit}^{\pi}_{\widetilde{\mathrm{D}};r}(p(x))\right] > \varepsilon/2\right]$ **for any $\varepsilon \geq 0$.**

Proof: We describe a random circuit $\widetilde{\mathrm{D}}$, and its deterministic variant follows by fixing the best coins.

Algorithm 22 ($\widetilde{\mathrm{D}}^{\pi,\mathrm{Sam}^{\pi}_r}(y)$)

Emulate $\mathrm{D}^{\pi,\mathrm{Sam}^{\pi}_r}(y)$. Before any query of $\mathrm{Sam}^{\pi}_r(\mathrm{C})$:
Evaluate $\mathrm{C}^{\pi}(z)$ for $z \leftarrow \{0,1\}^n$. If $\mathrm{C}^{\pi}(z)$ makes a query $\pi(x) = y$, return x and halt.

- The augmented query complexity of $\widetilde{\mathrm{D}}$ is at most twice that of D.
- Fix π and y, and let $\delta_i = \Pr_r[\mathrm{D}(y)$ makes first hit on i'th Sam query$]$.
  $\Longrightarrow$ $\Pr_r[\mathrm{D}(y)$ makes first hit on the "x-part" of i'th Sam query$] \geq \delta_i/2$
  $\Longrightarrow$
  $\Pr_r\left[\widetilde{\mathrm{D}}(y) \text{ finds } \pi^{-1}(y) \text{ just before } i\text{'th Sam query, w/o hitting}\right] \geq \delta_i/2.\square$

Proving Claim 21

For any $2^{n/5}/2$-augQuery circuit D:
$\Pr_{\pi;r}\left[\Pr_x\left[\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x)) = x\right] > 2^{-n/5} \land \neg\mathsf{hit}_{\mathsf{D};r}^\pi\right] \leq 2^{-2^{3n/5}}$.

Proving Claim 21

For any $2^{n/5}/2$-augQuery circuit D:
$\Pr_{\pi;r}\left[\Pr_x\left[\mathsf{D}^{\pi,\mathsf{Sam}_r^{\pi}}(\pi(x)) = x\right] > 2^{-n/5} \wedge \neg\mathsf{hit}^{\pi}_{\mathsf{D};r}\right] \leq 2^{-2^{3n/5}}$.

The proof is similar to the non-Sam case.

Proving Claim 21

For any $2^{n/5}/2$-augQuery circuit D:
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The proof is similar to the non-Sam case.

Lemma 23 (compression lemma, Sam variant)

For q-augQuery circuit D, $r \in \{0,1\}^*$ and $\varepsilon > 0$, exist algorithms Enc and Dec such that: Let $\pi \in \Pi_n$ be with
$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x)) = x \wedge \neg\mathsf{hit}^\pi_{\mathsf{D};r}(p(x))\right] > \varepsilon$, *then*

- $\mathsf{Dec}(\mathsf{Enc}(\pi)) = \pi$
- $|\mathsf{Enc}(\pi)| \le \log((N-a)!) + 2\cdot\log\binom{N}{a}$, *for* $a \ge \frac{\varepsilon N}{q+1}$

Proving Lemma 23

Proving Lemma 23

Definition 24

Assume $\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x))(y)$ makes a query $\mathsf{Sam}_r^\pi(C)$ and get answer (x,x'), we call the π-queries done by $C^\pi(x)$ and $C^\pi(x')$, indirect queries of D.

Construction 25 (Useful set $\mathcal{Y} \subseteq \{0,1\}^n$)

1. Set $\mathcal{Y} = \emptyset$ and $\mathcal{I} = \{y \in \{0,1\}^n \colon \mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x)) = \pi \wedge \neg \mathsf{hit}^\pi_{\mathsf{D};r}(y)\}$.
2. While $\mathcal{I} \neq \emptyset$, let y be the smallest lex. element in $\mathcal{I}$.
    1. Add y to $\mathcal{Y}$.
    2. Remove y and all direct & indirect π-queries $\mathsf{D}(y)$ makes from $\mathcal{I}$.

Proving Lemma 23

Definition 24

Assume $D^{\pi,\mathrm{Sam}_r^\pi}(\pi(x))(y)$ makes a query $\mathrm{Sam}_r^\pi(C)$ and get answer (x, x'), we call the π-queries done by $C^\pi(x)$ and $C^\pi(x')$, indirect queries of D.

Construction 25 (Useful set $\mathcal{Y} \subseteq \{0,1\}^n$)

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Algorithm 26 ($\mathrm{Enc}(\pi)$)

Output (description of) $\mathcal{Y}$ and $\mathcal{V} = \{(x, \pi(x)) \colon \pi(x) \notin \mathcal{Y}\}$.

Proving Lemma 23

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Assume $\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x))(y)$ makes a query $\mathsf{Sam}_r^\pi(C)$ and get answer (x,x'), we call the π-queries done by $C^\pi(x)$ and $C^\pi(x')$, indirect queries of D.

Construction 25 (Useful set $\mathcal{Y} \subseteq \{0,1\}^n$)

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Algorithm 26 ($\mathsf{Enc}(\pi)$)

Output (description of) $\mathcal{Y}$ and $\mathcal{V} = \{(x,\pi(x)) \colon \pi(x) \notin \mathcal{Y}\}$.

Under proper encoding, $|\mathsf{Enc}(\pi)| \leq \log((N-a)!) + 2 \cdot \log \binom{N}{a}$ for $a = |Y| \geq \frac{\varepsilon N}{q+1}$.

Proving Lemma 23 cont.

Proving Lemma 23 cont.

Algorithm 27 ($\mathsf{Dec}(\mathcal{Y}, \mathcal{V})$)

For all $y \in \mathcal{Y}$ in lex. order:

1. Emulate $\mathsf{D}^{\pi, \mathsf{Sam}_r^{\pi}}(y)$.
   1. Answer π-query using $\mathcal{V}$.
   2. On Sam-query $\mathsf{Sam}_r^{\pi}(\mathsf{C})$: choose x according to r, and let x' be the first element in $\{0,1\}^n$ for which the π-queries of $\mathsf{C}^{\pi}(x')$ are defined, and $\mathsf{C}^{\pi}(x') = \mathsf{C}^{\pi}(x)$.
2. If D makes a π-query x that is undefined in $\mathcal{V}$, add (x, y) to $\mathcal{V}$.

   Otherwise, add $(\mathsf{D}^{\pi, \mathsf{Sam}_r^{\pi}}(y), y)$ to $\mathcal{V}$.

Use $\mathcal{V}$ to reconstruct π

Proving Lemma 23 cont.

Algorithm 27 ($\mathsf{Dec}(\mathcal{Y}, \mathcal{V})$)

For all $y \in \mathcal{Y}$ in lex. order:

1. Emulate $\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(y)$.
   1. Answer π-query using $\mathcal{V}$.
   2. On Sam-query $\mathsf{Sam}_r^\pi(\mathsf{C})$: choose x according to r, and let x' be the first element in $\{0,1\}^n$ for which the π-queries of $\mathsf{C}^\pi(x')$ are defined, and $\mathsf{C}^\pi(x') = \mathsf{C}^\pi(x)$.
2. If D makes a π-query x that is undefined in $\mathcal{V}$, add (x,y) to $\mathcal{V}$.

   Otherwise, add $(\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(y), y)$ to $\mathcal{V}$.

Use $\mathcal{V}$ to reconstruct π

Correctness holds since $\mathsf{hit}_{\mathsf{D};r}^\pi(y) = 0$ for all $y \in \mathcal{Y}$, and thus answer to all Sam-queries are defined.

Remarks

- Results extends to OWFs and to TDPs.

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- Making Sam use independent randomness per input query C?