# Foundation of Cryptography (0368-4162-01), Lecture 3

## Hardcore Predicates for Any One-way Function

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**Definition 1 (hardcore predicates)**

An efficiently computable function $b : \{0,1\}^n \mapsto \{0,1\}$ is an hardcore predicate of $f : \{0,1\}^n \mapsto \{0,1\}^n$, if

$$\Pr[P(f(U_n)) = b(U_n)] \leq \frac{1}{2} + \operatorname{neg}(n),$$

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**Theorem 2 (Goldreich-Levin)**

*Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a OWF, and define $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ as $g(x,r) = f(x), r$. Then $b(x,r) = \langle x, r\rangle_2$, is an hardcore predicate of $g$.*

Note that if $f$ is one-to-one, then so is $g$.

Section 1

# The Information Theoretic Case

**Definition 3 (min-entropy)**

The min entropy of a random variable $X$, is defined

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Examples

- $X$ is uniform over a set of size $2^k$
- $(X \mid f(X) = y)$, where $f \colon \{0,1\}^n \mapsto \{0,1\}^n$ is $2^k$ to 1 and $X$ is uniform over $\{0,1\}^n$

# Pairwise independent hashing

**Definition 4 (pairwise independent hash functions)**

A function family $\mathcal{H}$ from $\{0,1\}^n$ to $\{0,1\}^m$ is pairwise independent, if for every $x \neq x' \in \{0,1\}^n$ and $y, y' \in \{0,1\}^m$, it holds that $\Pr_{h \leftarrow \mathcal{H}}[h(x) = y \wedge h(x') = y')] = 2^{-2m}$.

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## Lemma 5 (leftover hash lemma)

*Let $X$ be a random variable over $\{0,1\}^n$ with $\mathrm{H}_\infty(X) \geq k$ and let $\mathcal{H}$ be a family of pairwise independent hash functions from $\{0,1\}^n$ to $\{0,1\}^m$, then*

$$\mathrm{SD}((h, h(x))_{h \leftarrow \mathcal{H}, x \leftarrow X}, (h, y)_{h \leftarrow \mathcal{H}, y \leftarrow \{0,1\}^m}) \leq 2^{(m-k-2))/2}.$$

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* We typically simply write $\mathrm{SD}((H, H(X)), (H, U_m))$, where $H$ is uniformly distributed over $\mathcal{H}$.

# efficient function families

**Definition 6 (efficient function family)**

An ensemble of function families $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is efficient, if the following hold:

**Samplable.** $\mathcal{F}$ is samplable in polynomial-time: there exists a PPT that given $1^n$, outputs (the description of) a uniform element in $\mathcal{F}_n$.

**Efficient.** There exists a polynomial-time algorithm that given $x \in \{0,1\}^n$ and (a description of) $f \in \mathcal{F}_n$, outputs $f(x)$.

# hardcore predicate for regular OWF

## Lemma 7

*Let $f : \{0,1\}^n \mapsto \{0,1\}^n$ be a $d(n) \in 2^{\omega(\log n)}$ regular function and let $\mathcal{H} = \{\mathcal{H}_n\}$ be an efficient family of Boolean pairwise independent hash functions over $\{0,1\}^n$. Define $g\colon \{0,1\}^n \times \mathcal{H}_n \mapsto \{0,1\}^n \times \mathcal{H}_n$ as*

$$g(x,h) = (f(x),h),$$

*then $b(x,h) = h(x)$ is an hardcore predicate of $g$.*

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Proof: We prove the claim by showing that

**Claim 8**

$\mathrm{SD}\left((f(U_n), H, H(U_n)), (f(U_n), H, U_1)\right) = \mathrm{neg}(n)$, where the rv $H = H(n)$ is uniformly distributed over $\mathcal{H}_n$.

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Does this conclude the proof?

# Proving Claim 8

Proof: For $y \in f(\{0,1\}^n) := \{f(x) \colon x \in \{0,1\}^n\}$, let the rv $X_y$ be uniformly distributed over $f^{-1}(y) := \{x \in \{0,1\}^n \colon f(x) = y\}$.

## Proving Claim 8

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$$\begin{aligned}
&\mathrm{SD}((f(U_n), H, H(U_n)), (f(U_n), H, U_1)) \\
&\quad = \sum_{y \in f(\{0,1\}^n)} \Pr[f(U_n) = y] \cdot \mathrm{SD}\big((f(U_n), H, H(U_n) \mid f(U_n) = y) \\
&\qquad\qquad , (f(U_n), H, U_1 \mid f(U_n) = y)\big)
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&= \sum_{y \in f(\{0,1\}^n)} \Pr[f(U_n) = y] \cdot \mathrm{SD}\left((y, H, H(X_y)), (y, H, U_1)\right) \\
&\leq \max_{y \in f(\{0,1\}^n)} \mathrm{SD}((y, H, H(X_y)), (y, H, U_1))
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Since $\mathrm{H}_\infty(X_y) = \log(d(n))$ for any $y \in f(\{0,1\}^n)$,

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Since $\mathrm{H}_\infty(X_y) = \log(d(n))$ for any $y \in f(\{0,1\}^n)$,
The leftover hash lemma yields that

$$\begin{aligned} \mathrm{SD}((H, H(X_y)), (H, U_1)) &\leq 2^{(1-\mathrm{H}_\infty(X_y)-2))/2} \\ &= 2^{(1-\log(d(n)))/2} = \mathrm{neg}(n). \qquad \square \end{aligned}$$

## Further remarks

**Remark 9**

- We can output $\Theta(\log d(n))$ bits,
- $g$ and $b$ are not defined over all input length.

Section 2

# The Computational Case

## Proving Goldreich-Levin Theorem

**Theorem 10 (Goldreich-Levin)**

*Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a OWF, and define $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ as $g(x,r) = f(x), r$. Then $b(x,r) = \langle x, r\rangle_2$, is an hardcore predicate of $g$.*

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Proof: Assume $\exists$ PPT A, $p \in$ poly and infinite set $\mathcal{I} \subseteq \mathbb{N}$ with

$$\Pr[\mathrm{A}(g(U_n, R_n)) = b(U_n, R_n)] \geq \frac{1}{2} + \frac{1}{p(n)}, \tag{1}$$

for any $n \in \mathcal{I}$, where $U_n$ and $R_n$ are uniformly (and independently) distributed over $\{0,1\}^n$.

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We show $\exists$ PPT B and $p' \in$ poly with

$$\Pr_{y \leftarrow f(U_n)}[\mathsf{B}(y) \in f^{-1}(y) \geq \frac{1}{p'(n)}, \tag{2}$$

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for every $n \in \mathcal{I}$. In the following fix $n \in \mathcal{I}$.

## Focusing on a good set

### Claim 11

There exists a set $\mathcal{S} \subseteq \{0,1\}^n$ with

1. $\frac{|\mathcal{S}|}{2^n} \geq \frac{1}{2p(n)}$, and
2. $\alpha(x) := \Pr[\mathsf{A}(f(x), R_n) = b(x, R_n)] \geq \frac{1}{2} + \frac{1}{2p(n)}, \forall x \in S$.

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$$\Pr[\mathsf{A}(g(U_n, R_n)) = b(U_n, R_n)] \quad \leq \Pr[U_n \notin \mathcal{S}] \cdot \left(\frac{1}{2} + \frac{1}{2p(n)}\right) + \Pr[U_n \in \mathcal{S}]$$

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We will present $q \in$ poly and a PPT $B$ such that

$$\Pr[\mathsf{B}(y = f(x)) \in f^{-1}(y) \geq \frac{1}{q(n)}, \tag{3}$$

for every $x \in \mathcal{S}$.

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$$\Pr[\mathsf{B}(y = f(x)) \in f^{-1}(y) \geq \frac{1}{q(n)}, \qquad (3)$$

for every $x \in \mathcal{S}$. Fix $x \in \mathcal{S}$.

## The perfect case $\alpha(x) = 1$

For every $i \in [n]$, it holds that

$$A(f(x), e^i) = b(x, e^i),$$

where $e^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})$.

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We let $B(f(x)) = (A(f(x), e^1), \ldots, A(f(x), e^n))$

# Easy case: $\alpha(x) \geq 1 - \mathrm{neg}(n)$

## Fact 12

1. $\forall r \in \{0,1\}^n$, *the rv* $(r \oplus R_n)$ *is uniformly dist. over* $\{0,1\}^n$
2. $\forall w, y \in \{0,1\}^n$, *it holds that* $b(x,w) \oplus b(x,y) = b(x, w \oplus y)$

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Hence, $\forall i \in [n]$:

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Hence, $\forall i \in [n]$:

1. $\forall r \in \{0,1\}^n$ it holds that $x_i = b(x,r) \oplus b(x, r \oplus e^i)$
2. $\Pr[\mathsf{A}(f(x), R_n) = b(x, R_n) \wedge \mathsf{A}(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)]$
$\geq 1 - \text{neg}(n)$

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Hence, $\forall i \in [n]$:

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2. $\Pr[\mathrm{A}(f(x), R_n) = b(x, R_n) \wedge \mathrm{A}(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \geq 1 - \mathrm{neg}(n)$

We let $\mathrm{B}(f(x)) = (\mathrm{A}(f(x), R_n) \oplus \mathrm{A}(f(x), R_n \oplus e^1)), \ldots, \mathrm{A}(f(x), R_n) \oplus \mathrm{A}(f(x), R_n \oplus e^n))$.

# Intermediate case: $\alpha(x) \geq \frac{3}{4} + \frac{1}{q(n)}$

For any $i \in [n]$, it holds that

$$\begin{aligned}
&\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \quad (4)\\
&\geq \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)]\\
&\geq \frac{1}{2} + \frac{2}{q(n)}
\end{aligned}$$

**Algorithm 13 (B)**

Input: $f(x) \in \{0,1\}^n$

1. For every $i \in [n]$
   - Sample $r^1, \ldots, r^v \in \{0,1\}^n$ uniformly at random
   - Let $m_i = \mathrm{maj}_{j \in [v]}\{(A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i))\}$
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## Intermediate case: $\alpha(x) \geq \frac{3}{4} + \frac{1}{q(n)}$

For any $i \in [n]$, it holds that

$$\begin{aligned}
&\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \qquad (4)\\
&\quad \geq \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)]\\
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**Algorithm 13 (B)**

Input: $f(x) \in \{0,1\}^n$

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# B's success provability

The following holds for "large enough" $v = v(n)$.

**Claim 14**

For every $i \in [n]$, it holds that $\Pr[m_i = x_i] \geq 1 - \text{neg}(n)$.

Proof: For $j \in [v]$, let the indicator rv $W^j$ be 1, iif $A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i) = x_i$.

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**Lemma 15 (Hoeffding's inequality)**

*Let $X^1, \ldots, X^v$ be iid over $[0,1]$ with expectation $\mu$. Then,*
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We complete the proof taking $X^j = W^j$, $\varepsilon = 1/4q(n)$ and $v \in \omega(\log(n) \cdot q(n)^2)$.

## The actual case: $\alpha(x) \geq \frac{1}{2} + \frac{1}{q(n)}$

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- Problem: negligible success probability
- Solution: choose the samples in a *correlated* manner

## Algorithm B

Fix $\ell = \ell(n)$ (will be $O(\log n)$) and set $v = 2^{\ell} - 1$.

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## Algorithm 16 (B)

Input: $f(x) \in \{0,1\}^n$

1. Sample uniformly (and independently) $t_1, \ldots, t_\ell \in \{0,1\}^n$
2. For all $\mathcal{L} \subseteq [\ell]$, set $r^{\mathcal{L}} = \bigoplus_{i \in \mathcal{L}} t^i$
3. Guess $\{b(x, t^i)\}$, and compute $\{b(x, r^{\mathcal{L}})\}$ (how?)
4. For all $i \in [n]$, let
   $m_i = \mathrm{maj}_{\mathcal{L} \subseteq \{0,1\}^n} \{\mathsf{A}(f(x), r^{\mathcal{L}} \oplus e^i) \oplus b(x, r^{\mathcal{L}})\}$
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We want to lower-bound $\Pr[\sum_{\mathcal{L} \subseteq [\ell]} W^{\mathcal{L}} \leq \frac{v}{2}]$
Problem: the $W^{\mathcal{L}}$'s are *dependent*!

# Analyzing B's success probability

1. Let $T^1, \ldots, T^\ell$ be iid over $\{0,1\}^n$.
2. For every $\mathcal{L} \subseteq [\ell]$, let $R^{\mathcal{L}} = \bigoplus_{i \in \mathcal{L}} T^i$ .

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## Fact 17

1. $\forall \mathcal{L} \subseteq [\ell]$, $R^\mathcal{L}$ *is uniformly distributed over* $\{0,1\}^n$
2. $\forall w, y \in \{0,1\}^n$ *and* $\forall \mathcal{L} \neq \mathcal{L}' \subseteq [\ell]$, *it holds that* $\Pr[R^\mathcal{L} = w \wedge R^{\mathcal{L}'} = y] = \Pr[R^\mathcal{L} = w] \cdot \Pr[R^{\mathcal{L}'} = y]$

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That is, the $R^{\mathcal{L}}$'s are *pairwise independent*.

# Proving Fact 17(2)

Assume wlg. that $1 \in (\mathcal{L}' \setminus \mathcal{L})$.

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\end{aligned}
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## Pairwise independence variables

**Definition 18 (pairwise independent random variables)**

A sequence of random variables $X^1, \ldots, X^v$ is pairwise independent, if $\forall i \neq j \in [v]$ and $\forall a, b$, it holds that

$$\Pr[X^i = a \wedge X^j = b] = \Pr[X^i = a] \cdot \Pr[X^j = b]$$

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**Lemma 19 (Chebyshev's inequality)**

*Let $X^1, \ldots, X^v$ be pairwise-independent random variables with expectation $\mu$ and variance $\sigma^2$. Then, for every $\varepsilon > 0$,*

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## B's success provability cont

Assuming that B always guesses $\{b(x, t^i)\}$ correctly, then for every $\mathcal{L} \subseteq [\ell]$

- $\mathrm{E}[W^{\mathcal{L}}] \geq \frac{1}{2} + \frac{1}{q(n)}$
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Taking the guessing into account, yields that B outputs $x$ with probability at least $2^{-\ell-1} \in \Omega(n/q(n)^2)$.

## Reflections

**Hardcore functions.** Similar ideas allows to output $\log n$ “pseudorandom bits"

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2. $\exists$A that predicts $\langle R_n, X\rangle_2$ given $R_n$ with prob $\frac{1}{2} + \alpha$
3. (by GL) $\exists$B that guesses $X$ "from nothing", with prob $\alpha^{O(1)} > 2^{-t}$

# Reflections cont.

**List decoding.** An efficient encoding $C\colon \{0,1\}^n \mapsto \{0,1\}^m$, and a decoder $D$. Such that the following holds for any $x \in \{0,1\}^n$ and $c$ of hamming distance $\frac{1}{2} - \delta$ from $C(x)$:
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The code we used here is known as the *Hadamard* code

# Reflections cont.

**List decoding.** An efficient encoding $C\colon \{0,1\}^n \mapsto \{0,1\}^m$, and a decoder $D$. Such that the following holds for any $x \in \{0,1\}^n$ and $c$ of hamming distance $\frac{1}{2} - \delta$ from $C(x)$:
$D(c, \delta)$ outputs a list of size at most poly$(1/\delta)$ that whp. contains $x$
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**LPN - learning parity with noise.** Find $x$ given polynomially many samples of $\langle x, R_n \rangle_2 + N$, where $\Pr[N = 1] \leq \frac{1}{2} - \delta$.

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The difference comparing to Goldreich-Levin – no control over the $R_n$'s.