

# Foundation of Cryptography (0368-4162-01), Lecture 8

## Encryption Schemes

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January 3 – 17, 2012

# Section 1

## Definitions

# Correctness

## Definition 1 (encryption scheme)

A triplet of PPT's  $(G, E, D)$  such that

- 1  $G(1^n)$  outputs a key  $(e, d) \in \{0, 1\}^* \times \{0, 1\}^*$
- 2  $E(e, m)$  outputs a string in  $c \in \{0, 1\}^*$
- 3  $D(d, c)$  outputs  $m \in \{0, 1\}^*$

**Correctness:**  $D(d, E(e, m)) = m$ , for any  $(e, d) \in \text{Supp}(G(1^n))$  and  $m \in \{0, 1\}^*$

- $e$  – encryption key,  $d$  – decryption key
- $m$  – plaintext,  $c = E(e, m)$  – ciphertext
- $E_e(m) \equiv E(e, m)$  and  $D_d(c) \equiv D(d, c)$ ,
- public/private key

# Security

- What would we like to achieve?
- Attempt: for any  $m \in \{0, 1\}^*$ :

$$(m, E_{G(1^n)_1}(m)) \equiv (m, U_{\ell(|m|)})$$

- Shannon – only for  $m$  with  $|m| \leq |G(1^n)_1|$
- Other concerns, e.g., multiple encryptions, active adversary

# Semantic Security

- 1 Ciphertext reveal “no information” about the plaintext
- 2 Formulate via the simulation paradigm
- 3 Cannot hide the message length

## Semantic security – private-key model

### Definition 2 (Semantic Security – private-key model)

An encryption scheme  $(G, E, D)$  is semantically secure in the private-key model, if for any PPT  $A$ ,  $\exists$  PPT  $A'$  s.t.  $\forall$  poly-bounded dist. ensemble  $\mathcal{M} = \{\mathcal{M}_n\}_{n \in \mathbb{N}}$  and poly-bounded functions  $h, f: \{0, 1\}^* \mapsto \{0, 1\}^*$

$$\left| \Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1} [A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \right. \\ \left. - \Pr_{m \leftarrow \mathcal{M}_n} [A'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)] \right| = \text{neg}(n)$$

- poly-bounded? for simplicity we assume polynomial length
- $1^n$  and  $1^{|m|}$  can be omitted
- Non-uniform definition
- Reflection to ZK
- public-key variant –  $A$  gets  $e$

# Indistinguishability of encryptions

- The encryption of two strings is indistinguishable
- Less intuitive than semantic security, but easier to work with

## Indistinguishability of encryptions – private-key model

### Definition 3 (Indistinguishability of encryptions – private-key model)

An encryption scheme  $(G, E, D)$  has indistinguishable encryptions in the private-key model, if for any  $p, \ell \in \text{poly}$ ,  $\{x_n, y_n \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$ ,  $\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}$  and poly-time  $B$ ,

$$|\Pr_{e \leftarrow G(1^n)_1}[B(z_n, E_e(x_n)) = 1] - \Pr_{e \leftarrow G(1^n)_1}[B(z_n, E_e(y_n)) = 1]| \\ = \text{neg}(n)$$

- Non-uniform definition
- Public-key variant



## Equivalence of definitions

### Theorem 4

*An encryption scheme  $(G, E, D)$  is semantically secure iff it has indistinguishable encryptions.*

We prove the private key case

**Indistinguishability  $\implies$  Semantic Security**

Fix  $\mathcal{M}$ ,  $A$ ,  $f$  and  $h$ , be as in Definition 2. We construct  $A'$  as

**Algorithm 5 ( $A'$ )**

**Input:**  $1^n$ ,  $1^{|m|}$  and  $h(m)$

- 1  $e \leftarrow G(1^n)_1$
- 2  $c = E_e(1^{|m|})$
- 3 Output  $A(1^n, 1^{|m|}, h(m), c)$

**Claim 6**

$A'$  is a good simulator for  $A$  (according to Definition 2)

## Proving Claim 6

For  $n \in \mathbb{N}$ , let

$$\delta(n) := \left| \Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1} [A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \right. \\ \left. - \Pr_{m \leftarrow \mathcal{M}_n} [A'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)] \right|$$

### Claim 7

For every  $n \in \mathbb{N}$ , exists  $x_n \in \text{Supp}(\mathcal{M}_n)$  with

$$\delta(n) \leq \left| \Pr_{e \leftarrow G(1^n)_1} [A(1^n, 1^{|x_n|}, h(1^n, x_n), E_e(x_n)) = f(1^n, x_n)] \right. \\ \left. - \Pr [A'(1^n, 1^{|x_n|}, h(1^n, x_n)) = f(1^n, x_n)] \right|$$

Proof: Write the lhs and rhs terms in the definition of  $\delta(n)$  as sums over the different choices of  $m \in \text{Supp}(\mathcal{M}_n)$ , and use  $|a + b| \leq |a| + |b|$

Assume  $\exists$  an infinite  $\mathcal{I} \subseteq \mathbb{N}$  and  $p \in \text{poly}$  s.t.  $\delta(n) > 1/p(n)$  for every  $n \in \mathcal{I}$ .

The following algorithm contradicts the indistinguishability of  $(G, E, D)$  with respect to  $\{(x_n, y_n = 1^{|x_n|})\}_{n \in \mathbb{N}}$  and  $\{z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n))\}_{n \in \mathbb{N}}$ .

### Algorithm 8 (B)

**Input:**  $z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n)), c$

**Output** 1 iff  $A(1^n, 1^{|x_n|}, h(x_n), c) = f(1^n, x_n)$

# Semantic Security $\implies$ Indistinguishability

Assume  $\exists$  PPT  $B$ ,  $\{x_n, y_n \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$  and a  $\{z_n\}_{n \in \mathbb{N}}$ , such that (wlg) for infinitely many  $n$ 's:

(1)

$$\Pr_{e \leftarrow G(1^n)_1} [B(z_n, E_e(x_n)) = 1] - \Pr_{e \leftarrow G(1^n)_1} [B(z_n, E_e(y_n)) = 1] \geq \frac{1}{p(n)}$$

- Let  $\mathcal{M}_n$  be  $x_n$  wp  $\frac{1}{2}$  and  $y_n$  otherwise.
- Let  $f(1^n, x_n) = 1$ ,  $f(1^n, y_n) = 0$  and  $h(1^n, \cdot) = z_n$ .
- Define  $A(1^n, 1^{\ell(n)}, z_n, c)$  to return  $B(z_n, c)$ .

(2)

$$\Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1} [A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \geq \frac{1}{2} + \frac{1}{p(n)}$$

where for any  $A'$

(3)

$$\Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1} [A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \leq \frac{1}{2}$$

## Security Under Multiple Encryptions

### Definition 9 (Indistinguishability for multiple encryptions – private-key model)

An encryption scheme  $(G, E, D)$  has indistinguishable encryptions for multiple messages in the private-key model, if for any  $p, \ell, t \in \text{poly}$ ,

$\{x_{n,1}, \dots, x_{n,t(n)}, y_{n,1}, \dots, y_{n,t(n)} \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$ ,  
 $\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}$  and polynomial-time  $B$ ,

$$\left| \Pr_{e \leftarrow G(1^n)} [B(z_n, E_e(x_{n,1}), \dots, E_e(x_{n,t(n)})) = 1] \right. \\ \left. - \Pr_{e \leftarrow G(1^n)} [B(z_n, E_e(y_{n,1}), \dots, E_e(y_{n,t(n)})) = 1] \right| = \text{neg}(n)$$

### Extensions:

- Different length messages
- Semantic security version
- Public-key definition

## Multiple Encryption in the Public-Key Model

### Theorem 10

*A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.*

Proof: Assume  $(G, E, D)$  is public-key secure for a single message and not for multiple messages with respect to  $B$ ,

$$\{x_{1,t(n)}, \dots, x_{n,t(n)}, y_{n,1}, \dots, y_{n,t(n)} \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}},$$

$$\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}.$$

It follows that for some function  $i(n) \in [t(n)]$

$$\begin{aligned} & \left| \Pr[B(1^n, e, E_e(x_{n,1}), \dots, E_e(x_{n,i-1}), E_e(y_{n,i}), \dots, E_e(y_{n,t(n)})) = 1] \right. \\ & \left. - \Pr[B(1^n, e, E_e(x_{n,1}), \dots, E_e(x_{n,i}), E_e(y_{n,i+1}), \dots, E_e(y_{n,t(n)})) = 1] \right| \\ & > \text{neg}(n) \end{aligned}$$

where in both cases  $e \leftarrow G(1^n)_1$

**Algorithm 11 (B')**

**Input:**  $1^n, z_n = (i(n), x_{1,t(n)}, \dots, x_{n,t(n)}, y_{n,1}, \dots, y_{n,t(n)}, e, c$

Return  $B(c, E_e(x_{n,1}), \dots, E_e(x_{n,i-1}), c, E_e(y_{n,i+1}), \dots, E_e(y_{n,t(n)}))$

$B'$  is critically using the public key



## Multiple Encryption in the Private-Key Model

### Fact 12

*Assuming (non uniform) OWFs exists, there exists an encryption scheme that has private-key indistinguishable encryptions for a single messages, but not for multiple messages*

Proof: Let  $g: \{0, 1\}^n \mapsto \{0, 1\}^{n+1}$  be a (non-uniform) PRG, and for  $i \in \mathbb{N}$  let  $g^i$  be its "iterated extension" to output of length  $i$  (see Lecture 2, Construction 15).

### Construction 13

- $G(1^n)$  outputs  $e \leftarrow \{0, 1\}^n$ ,
- $E_e(m)$  outputs  $g^{|m|}(e) \oplus m$
- $D_e(c)$  outputs  $g^{|c|}(e) \oplus c$

**Claim 14**

$(G, E, D)$  has private-key indistinguishable encryptions for a single message

Proof: Assume not, and let  $B, \{x_n, y_n \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$  and  $\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}$  be the triplet that realizes it. Wlog,

$$|\Pr[B(z_n, g^{|x_n|}(U_n) \oplus x_n) = 1] - \Pr[B(z_n, U_{|x_n|} \oplus x_n) = 1]| > \text{neg}(n) \quad (4)$$

Hence,  $B$  implies a (non-uniform) distinguisher for  $g$

**Claim 15**

$(G, E, D)$  does not have a private-key indistinguishable encryptions for multiple messages

Proof: Take  $x_{n,1} = x_{n,2}, y_{n,1} \neq y_{n,2}$  and  $D(c_1, c_2)$  outputs 1 iff  $c_1 = c_2$

## Section 2

# Constructions

## Private key indistinguishable encryptions for multiple messages

Suffice to encrypt messages of some fixed length (here the length is  $n$ ).

Let  $\mathcal{F}$  be a (non-uniform) length preserving PRF

### Construction 16

- $G(1^n)$ : output  $e \leftarrow \mathcal{F}_n$ ,
- $E_e(m)$ : choose  $r \leftarrow \{0, 1\}^n$  and output  $(r, e(r) \oplus m)$
- $D_e(r, c)$ : output  $e(r) \oplus c$

### Claim 17

$(G, E, D)$  has private-key indistinguishable encryptions for a multiple messages

Proof:

## Public-key indistinguishable encryptions for multiple messages

Let  $(G, f, \text{Inv})$  be a (non-uniform) TDP, and let  $b$  be an hardcore predicate for  $f$ .

### Construction 18 (bit encryption)

- $G(1^n)$ : output  $(e, d) \leftarrow G(1^n)$
- $E_e(m)$ : choose  $r \leftarrow \{0, 1\}^n$  and output  $(y = f_e(r), c = b(r) \oplus m)$
- $D_d(y, c)$ : output  $b(\text{Inv}_d(y)) \oplus c$

### Claim 19

$(G, E, D)$  has public-key indistinguishable encryptions for a multiple messages

- We believe that public-key encryptions schemes are “more complex” than private-key ones

## Section 3

# Active Adversaries

## Active Adversaries

- Chosen plaintext attack (CPA):  
The adversary can ask for encryption and choose the messages to distinguish accordingly
- Chosen ciphertext attack (CCA):  
The adversary can also ask for *decryptions* of certain messages
- In the public-key settings, the adversary is also given the public key
- We focus on indistinguishability, but each of the above definitions has an equivalent semantic security variant.

## CPA Security

Let  $(G, E, D)$  be an encryption scheme. For a pair of algorithms  $A = (A_1, A_2)$ ,  $n \in \mathbb{N}$ ,  $z \in \{0, 1\}^*$  and  $b \in \{0, 1\}$ , let:

### Experiment 20 ( $\text{Exp}_{A,n,z}^{\text{CPA}}(b)$ )

- 1  $(e, d) \leftarrow G(1^n)$
- 2  $(m_0, m_1, s) \leftarrow A_1^{E_e(\cdot)}(1^n, z)$
- 3  $c \leftarrow E_e(m_b)$
- 4 Output  $A_2^{E_e(\cdot)}(1^n, s, c)$

### Definition 21 (private key CPA)

$(G, E, D)$  has indistinguishable encryptions in the private-key model under CPA attack, if  $\forall$  PPT  $A_1, A_2$ , and poly-bounded  $\{z_n\}_{n \in \mathbb{N}}$ :

$$|\Pr[\text{Exp}_{A,n,z_n}^{\text{CPA}}(0) = 1] - \Pr[\text{Exp}_{A,n,z_n}^{\text{CPA}}(1) = 1]| = \text{neg}(n)$$



- public-key variant...
- The scheme from Construction 16 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)
- The scheme from Construction 18 has indistinguishable encryptions in the public-key model (for short, public-key CPA secure)
- In both cases, definitions are *not* equivalent

# CCA Security

## Experiment 22 ( $\text{Exp}_{A,n,Z}^{\text{CCA1}}(b)$ )

- 1  $(e, d) \leftarrow G(1^n)$
- 2  $(m_0, m_1, s) \leftarrow A_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
- 3  $c \leftarrow E_e(m_b)$
- 4 Output  $A_2^{E_e(\cdot)}(1^n, s, c)$

## Experiment 23 ( $\text{Exp}_{A,n,Z_n}^{\text{CCA2}}(b)$ )

- 1  $(e, d) \leftarrow G(1^n)$
- 2  $(x_0, x_1, s) \leftarrow A_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
- 3  $c \leftarrow E_e(x_b)$
- 4 Output  $A_2^{E_e(\cdot), D_d^{-c}(\cdot)}(1^n, s, c)$

**Definition 24 (private key CCA1/CCA2)**

$(G, E, D)$  has indistinguishable encryptions in the private-key model under  $x \in \{\text{CCA1}, \text{CCA2}\}$  attack, if  $\forall$  PPT  $A_1, A_2$ , and poly-bounded  $\{z_n\}_{n \in \mathbb{N}}$ :

$$|\Pr[\text{Exp}_{A,n,z_n}^x(0) = 1] - \Pr[\text{Exp}_{A,n,z_n}^x(1) = 1]| = \text{neg}(n)$$

- The public key definition is analogous

## Private-key CCA2

- Is the scheme from Construction 16 private-key CCA1 secure?
- CCA2 secure?

Let  $(G, E, D)$  be a private key CPA scheme, and let  $(\text{Gen}_M, \text{Mac}, \text{Vrfy})$  be an existential unforgeable strong MAC.

### Construction 25

- $G'(1^n)$ : Output  $(e \leftarrow G_E(1^n), k \leftarrow \text{Gen}_M(1^n))$ .<sup>a</sup>
- $E'_{d,k}(m)$ : let  $c = E_e(m)$  and output  $(c, t = \text{Mac}_k(c))$
- $D_{e,k}(c, t)$ : if  $\text{Vrfy}_k(c, t) = 1$ , output  $D_e(c)$ . Otherwise, output  $\perp$

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<sup>a</sup>We assume for simplicity that the encryption and decryption keys are the same.

**Theorem 26**

*Construction 25 is a private-key CCA2-secure encryption scheme.*

Proof: ?

# Public-key CCA1

Let  $(G, E, D)$  be a public-key CPA scheme and let  $(P, V)$  be a NIZK for  $\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) : \exists(m, z_0, z_1) \text{ s.t. } c_0 = E_{pk_0}(m, z_0) \wedge c_1 = E_{pk_1}(m, z_1)\}$

## Construction 27 (The Naor-Yung Paradigm)

- $G'(1^n)$ :
  - ① For  $i \in \{0, 1\}$ : set  $(sk_i, pk_i) \leftarrow G(1^n)$ .
  - ② Let  $r \leftarrow \{0, 1\}^{\ell(n)}$ , and output  $pk' = (pk_0, pk_1, r)$  and  $sk' = (pk', sk_0, sk_1)$
- $E'_{pk'}(m)$ :
  - ① For  $i \in \{0, 1\}$ :  $c_i = E_{pk_i}(m, z_i)$ , where  $z_i$  is a uniformly chosen string of the right length
  - ②  $\pi \leftarrow P((c_0, c_1, pk_0, pk_1), (m, z_0, z_1), r)$
  - ③ Output  $(c_0, c_1, \pi)$ .
- $D'_{sk'}(c_0, c_1, \pi)$ : If  $V((c_0, c_1, pk_0, pk_1), \pi, r) = 1$ , return  $D_{sk_0}(c_0)$ . Otherwise, return  $\perp$

## Omitted details:

- We assume for simplicity that the encryption key output by  $G(1^n)$  is of length at least  $n$ .
- $\ell$  is an arbitrary polynomial, and determines the maximum message length to encrypt using "security parameter"  $n$ .

Is the scheme CCA1 secure? We need the NIZK to be "adaptive secure".

### Theorem 28

*Assuming that  $(P, V)$  is adaptive secure, then Construction 27 is a public-key CCA1 secure encryption scheme.*

Proof: Given an attacker  $A'$  for the CCA1 security of  $(G', E', D')$ , we use it to construct an attacker  $A$  on the CPA security of  $(G, E, D)$ .

Let  $S = (S_1, S_2)$  be the (adaptive) simulator for  $(P, V, \mathcal{L})$

## Algorithm 29 (A)

**Input:**  $(1^n, pk)$

- ① let  $j \leftarrow \{0, 1\}$ ,  $pk_{1-j} = pk$ ,  $(pk_j, sk_j) \leftarrow G(1^n)$  and  $(r, s) \leftarrow S_1(1^n)$
- ② Emulate  $A'(1^n, pk' = (pk_0, pk_1, r))$  as follows:
- ③ On query  $(c_0, c_1, \pi)$  of  $A'$  to  $D'$ :  
If  $V((c_0, c_1, pk_0, pk_1), \pi, r) = 1$ , answer  $D_{sk_j}(c_j)$ .  
Otherwise, answer  $\perp$ .
- ④ Output the same pair  $(m_0, m_1)$  as  $A'$  does
- ⑤ On challenge  $c (= E_{pk}(m_b))$ :
  - Set  $c_{1-j} = c$ ,  $a \leftarrow \{0, 1\}$ ,  $c_j = E_{pk_j}(m_a)$ , and  $\pi \leftarrow S_2((c_0, c_1, pk_0, pk_1), r, s)$
  - Send  $c' = (c_0, c_1, \pi)$  to  $A'$
- ⑥ Output the same value that  $A'$  does



### Claim 30

Assume that  $A'$  breaks the CCA1 security of  $(G', E', D')$  with probability  $\delta(n)$ , then  $A$  breaks the CPA security of  $(G, E, D)$  with probability  $(\delta(n) - \text{neg}(n))/2$ .

The adaptive soundness and adaptive zero-knowledge of  $(P, V)$ , yields that

$$\Pr[A' \text{ "makes" } A(1^n) \text{ decrypt an invalid cipher}] = \text{neg}(n) \quad (5)$$

Hence, only negligible information leaks about  $j$ .

Let  $A'(1^n, a^*, b^*)$  be the output of  $A'(1^n)$  in the emulation induced by  $A$ , where  $a = a^*$  and  $b = b^*$ . It holds that

- ①  $A'(1^n, 0, 1) \equiv A'(1^n, 1, 0)$
- ② The adaptive zero-knowledge of  $(P, V)$  yields that  $|\Pr[A'(1^n, 1, 1) = 1] - \Pr[A'(1^n, 0, 0) = 1]| \geq \delta(n) - \text{neg}(n)$

Let  $A(b)$  be the outputs of  $A$  when the challenge is  $b$ .

$$\begin{aligned}
 & |\Pr[A(1) = 1] - \Pr[A(0) = 1]| \\
 &= \left| \frac{1}{2}(\Pr[A'(0, 1) = 1] + \Pr[A'(1, 1) = 1]) \right. \\
 &\quad \left. - \frac{1}{2}(\Pr[A'(0, 0) = 1] + \Pr[A'(1, 0) = 1]) \right| \\
 &\geq \frac{1}{2} |\Pr[A'(1, 1) = 1] - \Pr[A'(0, 0) = 1]| \\
 &\quad - \frac{1}{2} |\Pr[A'(1, 0) = 1] - \Pr[A'(0, 1) = 1]| \\
 &\geq (\delta(n) - \text{neg}(n))/2
 \end{aligned}$$

## Public-key CCA2

- Is Construction 27 CCA2 secure?
- **Problem:** Soundness might not hold with respect to the simulated CRS, after seeing a proof for an *invalid* statement
- **Solution:** use *simulation sound* NIZK