Application of Information Theory, Lecture 5
Channel Capacity and Isoperimetric Inequality

Iftach Haitner

Tel Aviv University.

April 12, 2018

Part I

Channel Capacity

The problem

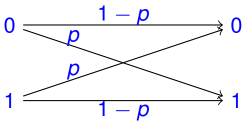
- We want to send a message $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$, but the communication channel is faulty

The problem

- We want to send a message $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$, but the communication channel is faulty
- Each bit is (independently) flipped w.p. p (e.g., 0.1)

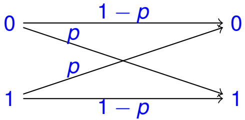
The problem

- We want to send a message $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$, but the communication channel is faulty
- Each bit is (independently) flipped w.p. p (e.g., 0.1)



The problem

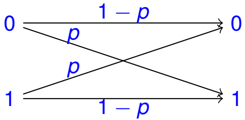
- We want to send a message $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$, but the communication channel is faulty
- Each bit is (independently) flipped w.p. p (e.g., 0.1)



- (expected) Error rate is p

The problem

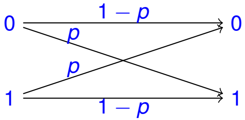
- We want to send a message $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$, but the communication channel is faulty
- Each bit is (independently) flipped w.p. p (e.g., 0.1)



- (expected) Error rate is p
- Such "channel" is called Binary Symmetric Channel (BSC)

The problem

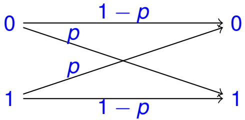
- We want to send a message $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$, but the communication channel is faulty
- Each bit is (independently) flipped w.p. p (e.g., 0.1)



- (expected) Error rate is p
- Such "channel" is called Binary Symmetric Channel (BSC)
- When sending m bits, we have $\approx pm$ errors

The problem

- We want to send a message $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$, but the communication channel is faulty
- Each bit is (independently) flipped w.p. p (e.g., 0.1)



- (expected) Error rate is p
- Such “channel” is called Binary Symmetric Channel (BSC)
- When sending m bits, we have $\approx pm$ errors
- Can we send bits with smaller error?

Solution

Solution

- Obvious solution is “error correction codes (ECC)”

Solution

- Obvious solution is “error correction codes (ECC)”
- Simplest example: send each bit three times, and take majority

Solution

- Obvious solution is “error correction codes (ECC)”
- Simplest example: send each bit three times, and take majority
- Error happens if the channel errs at least twice

Solution

- Obvious solution is “error correction codes (ECC)”
- Simplest example: send each bit three times, and take majority
- Error happens if the channel errs at least twice
- For $p = 0.1$: happens w.p.
  $3p^2(1-p) + p^3 = 3 \cdot 0.01 \cdot 0.9 + 0.001 = 0.028$

Solution

- Obvious solution is "error correction codes (ECC)"
- Simplest example: send each bit three times, and take majority
- Error happens if the channel errs at least twice
- For $p = 0.1$: happens w.p.
  $3p^2(1-p) + p^3 = 3 \cdot 0.01 \cdot 0.9 + 0.001 = 0.028$
- Error rate: $.028$

Solution

- Obvious solution is "error correction codes (ECC)"
- Simplest example: send each bit three times, and take majority
- Error happens if the channel errs at least twice
- For $p = 0.1$: happens w.p.
  $3p^2(1-p) + p^3 = 3 \cdot 0.01 \cdot 0.9 + 0.001 = 0.028$
- Error rate: $.028$
- Transmission rate: $1/3$ (i.e., # of bits recovered / #of bits transmitted)

Solution

- Obvious solution is “error correction codes (ECC)”
- Simplest example: send each bit three times, and take majority
- Error happens if the channel errs at least twice
- For $p = 0.1$: happens w.p.
  $3p^2(1-p) + p^3 = 3 \cdot 0.01 \cdot 0.9 + 0.001 = 0.028$
- Error rate: $.028$
- Transmission rate: $1/3$ (i.e., # of bits recovered / #of bits transmitted)
- We reduced the error rate by reducing the transmission rate.

Solution

- Obvious solution is “error correction codes (ECC)”
- Simplest example: send each bit three times, and take majority
- Error happens if the channel errs at least twice
- For $p = 0.1$: happens w.p.
  $3p^2(1-p) + p^3 = 3 \cdot 0.01 \cdot 0.9 + 0.001 = 0.028$
- Error rate: $.028$
- Transmission rate: $1/3$ (i.e., # of bits recovered / #of bits transmitted)
- We reduced the error rate by reducing the transmission rate.
- Can we reduce the error rate, without reducing the transmitting rate too much?

Solution

- Obvious solution is "error correction codes (ECC)"
- Simplest example: send each bit three times, and take majority
- Error happens if the channel errs at least twice
- For $p = 0.1$: happens w.p. $3p^2(1-p) + p^3 = 3 \cdot 0.01 \cdot 0.9 + 0.001 = 0.028$
- Error rate: $.028$
- Transmission rate: $1/3$ (i.e., # of bits recovered / #of bits transmitted)
- We reduced the error rate by reducing the transmission rate.
- Can we reduce the error rate, without reducing the transmitting rate too much?
- Before Shannon it was believed that very small error rate requires very small transmission rate.

Shannon's result

Shannon's result

- Shannon showed that you can reduce the error rate towards 0, without reducing the transmission rate towards 0

Shannon's result

- Shannon showed that you can reduce the error rate towards 0, without reducing the transmission rate towards 0
- For any $c < C_p$, exists a code with transmission rate c that is correct w.h.p.

Shannon's result

- Shannon showed that you can reduce the error rate towards 0, without reducing the transmission rate towards 0
- For any $c < C_p$, exists a code with transmission rate c that is correct w.h.p.
- Example: for $p = .1$, $C_p > \frac{1}{2}$.

Shannon's result

- Shannon showed that you can reduce the error rate towards 0, without reducing the transmission rate towards 0
- For any $c < C_p$, exists a code with transmission rate c that is correct w.h.p.
- Example: for $p = .1$, $C_p > \frac{1}{2}$.

  Hence, for sending $\mathbf{x} = (x_1, \ldots, x_m)$, one can send $2m$ bits, such that $\mathbf{x}$ is recovered w.p. close to 1

Shannon's result

- Shannon showed that you can reduce the error rate towards 0, without reducing the transmission rate towards 0
- For any $c < C_p$, exists a code with transmission rate c that is correct w.h.p.
- Example: for $p = .1$, $C_p > \frac{1}{2}$.

  Hence, for sending $\mathbf{x} = (x_1, \ldots, x_m)$, one can send $2m$ bits, such that $\mathbf{x}$ is recovered w.p. close to 1
- More generally, $\forall p \in [0,1] \quad \exists C_p$ such that for sending $\mathbf{x} \in \{0,1\}^m$, one can send $\approx \frac{m}{C_p}$ bits, and $\mathbf{x}$ is recovered w.p. close to 1

Shannon's result

- Shannon showed that you can reduce the error rate towards 0, without reducing the transmission rate towards 0
- For any $c < C_p$, exists a code with transmission rate c that is correct w.h.p.
- Example: for $p = .1$, $C_p > \frac{1}{2}$.

  Hence, for sending $\mathbf{x} = (x_1, \ldots, x_m)$, one can send $2m$ bits, such that $\mathbf{x}$ is recovered w.p. close to 1
- More generally, $\forall p \in [0, 1] \quad \exists C_p$ such that for sending $\mathbf{x} \in \{0, 1\}^m$, one can send $\approx \frac{m}{C_p}$ bits, and $\mathbf{x}$ is recovered w.p. close to 1
- C_p might be 0 (i.e., for $p = \frac{1}{2}$)

Shannon's result

- Shannon showed that you can reduce the error rate towards 0, without reducing the transmission rate towards 0
- For any $c < C_p$, exists a code with transmission rate c that is correct w.h.p.
- Example: for $p = .1$, $C_p > \frac{1}{2}$.

  Hence, for sending $\mathbf{x} = (x_1, \ldots, x_m)$, one can send $2m$ bits, such that $\mathbf{x}$ is recovered w.p. close to 1
- More generally, $\forall p \in [0,1] \quad \exists C_p$ such that for sending $\mathbf{x} \in \{0,1\}^m$, one can send $\approx \frac{m}{C_p}$ bits, and $\mathbf{x}$ is recovered w.p. close to 1
- C_p might be 0 (i.e., for $p = \frac{1}{2}$)
- A revolution in EE and the whole world

Error correction code

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f\colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad (n > m)$

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f\colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad (n > m)$
- Decoding scheme: $g\colon \{0,1\}^n \mapsto \{0,1\}^m$

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f \colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad$ ($n > m$)
- Decoding scheme: $g \colon \{0,1\}^n \mapsto \{0,1\}^m$
- $\frac{m}{n}$ — transmission rate

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f\colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad (n > m)$
- Decoding scheme: $g\colon \{0,1\}^n \mapsto \{0,1\}^m$
- $\frac{m}{n}$ — transmission rate
- Sender sends $f(x)$ rather than x

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f\colon \{0,1\}^m \mapsto \{0,1\}^n$ $(n > m)$
- Decoding scheme: $g\colon \{0,1\}^n \mapsto \{0,1\}^m$
- $\frac{m}{n}$ — transmission rate
- Sender sends $f(x)$ rather than x
- Receiver decodes the message by applying g

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f \colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad (n > m)$
- Decoding scheme: $g \colon \{0,1\}^n \mapsto \{0,1\}^m$
- $\frac{m}{n}$ — transmission rate
- Sender sends $f(x)$ rather than x
- Receiver decodes the message by applying g
- $\underbrace{\mathbf{x}}_{m \text{ bits}} \xrightarrow{\text{encoding}} \underbrace{f(\mathbf{x})}_{n \text{ bits}} \xrightarrow{\text{channel}} \underbrace{f(\mathbf{x}) \oplus Z}_{\text{bitwise XOR}} \xrightarrow{\text{decoding}} g(f(\mathbf{x}) \oplus Z)$

  $Z = (Z_1, \ldots, Z_n)$ where $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$ (i.e., over $\{0,1\}$ with $\Pr[Z_i = 1] = p$)

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f\colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad (n > m)$
- Decoding scheme: $g\colon \{0,1\}^n \mapsto \{0,1\}^m$
- $\frac{m}{n}$ — transmission rate
- Sender sends $f(x)$ rather than x
- Receiver decodes the message by applying g
- $\underbrace{\mathbf{x}}_{m \text{ bits}} \stackrel{\text{encoding}}{\longrightarrow} \underbrace{f(\mathbf{x})}_{n \text{ bits}} \stackrel{\text{channel}}{\longrightarrow} \underbrace{f(\mathbf{x}) \oplus Z}_{\text{bitwise XOR}} \stackrel{\text{decoding}}{\longrightarrow} g(f(\mathbf{x}) \oplus Z)$

  $Z = (Z_1, \ldots, Z_n)$ where $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$ (i.e., over $\{0,1\}$ with $\Pr[Z_i = 1] = p$)
- We hope $g(f(\mathbf{x}) \oplus Z) = \mathbf{x}$

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f\colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad (n > m)$
- Decoding scheme: $g\colon \{0,1\}^n \mapsto \{0,1\}^m$
- $\frac{m}{n}$ — transmission rate
- Sender sends $f(x)$ rather than x
- Receiver decodes the message by applying g
- $\underbrace{\mathbf{x}}_{m \text{ bits}} \xrightarrow{\text{encoding}} \underbrace{f(\mathbf{x})}_{n \text{ bits}} \xrightarrow{\text{channel}} \underbrace{f(\mathbf{x}) \oplus Z}_{\text{bitwise XOR}} \xrightarrow{\text{decoding}} g(f(\mathbf{x}) \oplus Z)$

  $Z = (Z_1, \ldots, Z_n)$ where $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$ (i.e., over $\{0,1\}$ with $\Pr[Z_i = 1] = p$)
- We hope $g(f(\mathbf{x}) \oplus Z) = \mathbf{x}$
- ECCs are everywhere

Error correction code

- Message to send $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding scheme: $f \colon \{0,1\}^m \mapsto \{0,1\}^n$ $\qquad (n > m)$
- Decoding scheme: $g \colon \{0,1\}^n \mapsto \{0,1\}^m$
- $\frac{m}{n}$ — transmission rate
- Sender sends $f(x)$ rather than x
- Receiver decodes the message by applying g
- $\underbrace{\mathbf{x}}_{m \text{ bits}} \xrightarrow{\text{encoding}} \underbrace{f(\mathbf{x})}_{n \text{ bits}} \xrightarrow{\text{channel}} \underbrace{f(\mathbf{x}) \oplus Z}_{\text{bitwise XOR}} \xrightarrow{\text{decoding}} g(f(\mathbf{x}) \oplus Z)$

  $Z = (Z_1, \ldots, Z_n)$ where $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$ (i.e., over $\{0,1\}$ with $\Pr[Z_i = 1] = p$)
- We hope $g(f(\mathbf{x}) \oplus Z) = \mathbf{x}$
- ECCs are everywhere
- ECC Vs compression

Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p,$ s.t. $\forall \varepsilon > 0 \quad \exists m_\varepsilon,$ s.t. $\forall m \geq m_\varepsilon$ *and* $n \geq m(\frac{1}{C_p} + \varepsilon)$,
$\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ *and* $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\forall \boldsymbol{x} \in \{0,1\}^m$:

$$\Pr_{z \leftarrow Z = (Z_1,\ldots,Z_n)} [g(f(\boldsymbol{x}) \oplus z) \neq \boldsymbol{x}] \leq \varepsilon$$

for $Z_1, \ldots, Z_n$ *iid* $\sim (1-p, p)$.

Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p$, s.t. $\forall \varepsilon > 0 \quad \exists m_\varepsilon$, s.t. $\forall m \geq m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$,
$\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\forall \boldsymbol{x} \in \{0,1\}^m$:

$$\Pr_{z \leftarrow Z = (Z_1,\ldots,Z_n)} [g(f(\boldsymbol{x}) \oplus z) \neq \boldsymbol{x}] \leq \varepsilon$$

for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$.

- $C_p = 1 - h(p)$ — the channel capacity

Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p$, s.t. $\forall \varepsilon > 0 \quad \exists m_\varepsilon$, s.t. $\forall m \geq m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$,
$\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\forall \boldsymbol{x} \in \{0,1\}^m$:

$$\Pr_{z \leftarrow Z = (Z_1,\ldots,Z_n)} [g(f(\boldsymbol{x}) \oplus z) \neq \boldsymbol{x}] \leq \varepsilon$$

for $Z_1,\ldots,Z_n$ iid $\sim (1-p,p)$.

- $C_p = 1 - h(p)$ — the channel capacity

  $p = .1 \implies C_p = 0.5310 > \frac{1}{2}$

Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p,\ s.t.\ \ \forall \varepsilon > 0 \quad \exists m_\varepsilon,\ s.t.\ \ \forall m \geq m_\varepsilon$ *and* $n \geq m(\frac{1}{C_p} + \varepsilon)$,
$\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ *and* $g \colon \{0,1\}^n \mapsto \{0,1\}^m,\ s.t.\ \ \forall \boldsymbol{x} \in \{0,1\}^m$:

$$\Pr_{z \leftarrow Z = (Z_1,\ldots,Z_n)} [g(f(\boldsymbol{x}) \oplus z) \neq \boldsymbol{x}] \leq \varepsilon$$

for $Z_1, \ldots, Z_n$ *iid* $\sim (1-p, p)$.

- $C_p = 1 - h(p)$ — the channel capacity

  $p = .1 \implies C_p = 0.5310 > \frac{1}{2}$

  $p = .25 \implies C_p \approx \frac{1}{5}$

Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p$, s.t. $\forall \varepsilon > 0 \quad \exists m_\varepsilon$, s.t. $\forall m \geq m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$,
$\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\forall \boldsymbol{x} \in \{0,1\}^m$:

$$\Pr_{z \leftarrow Z = (Z_1,\ldots,Z_n)} [g(f(\boldsymbol{x}) \oplus z) \neq \boldsymbol{x}] \leq \varepsilon$$

for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$.

- $C_p = 1 - h(p)$ — the channel capacity

  $p = .1 \implies C_p = 0.5310 > \frac{1}{2}$

  $p = .25 \implies C_p \approx \frac{1}{5}$

- Tight theorem

Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p,\ s.t.\ \forall \varepsilon > 0 \quad \exists m_\varepsilon,\ s.t.\ \forall m \geq m_\varepsilon$ *and* $n \geq m(\frac{1}{C_p} + \varepsilon)$,
$\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ *and* $g \colon \{0,1\}^n \mapsto \{0,1\}^m,\ s.t.\ \forall \boldsymbol{x} \in \{0,1\}^m$:

$$\Pr_{z \leftarrow Z = (Z_1,\ldots,Z_n)} [g(f(\boldsymbol{x}) \oplus z) \neq \boldsymbol{x}] \leq \varepsilon$$

for $Z_1,\ldots,Z_n$ *iid* $\sim (1-p, p)$.

- $C_p = 1 - h(p)$ — the channel capacity

  $p = .1 \implies C_p = 0.5310 > \frac{1}{2}$

  $p = .25 \implies C_p \approx \frac{1}{5}$
- Tight theorem
- We prove a weaker variant that holds w.h.p. over $\mathbf{x} \leftarrow \{0,1\}^m$

Hamming distance

Hamming distance

- For $\mathbf{y} = (y_1, \ldots, y_n) \in \{0,1\}^n$, let $\|\mathbf{y}\|_1 = \sum_i y_i$ — Hamming weight of $\mathbf{y}$

Hamming distance

- For $\mathbf{y} = (y_1, \ldots, y_n) \in \{0,1\}^n$, let $\|\mathbf{y}\|_1 = \sum_i y_i$ — Hamming weight of $\mathbf{y}$
- $\|\mathbf{y} - \mathbf{y}'\|_1 = \|\mathbf{y} \oplus \mathbf{y}'\|_1$ — Hamming distance of $\mathbf{y}$ from $\mathbf{y}'$; # of places differ.

Hamming distance

- For $\mathbf{y} = (y_1, \ldots, y_n) \in \{0,1\}^n$, let $\|\mathbf{y}\|_1 = \sum_i y_i$ — Hamming weight of $\mathbf{y}$
- $\|\mathbf{y} - \mathbf{y}'\|_1 = \|\mathbf{y} \oplus \mathbf{y}'\|_1$ — Hamming distance of $\mathbf{y}$ from $\mathbf{y}'$; # of places differ.
- We sometimes just write $|\mathbf{y}|$.

Proving the theorem

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis.

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis.

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f\colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\operatorname{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f\colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\operatorname{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\}\colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f\colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\operatorname{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\}\colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$
-

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\operatorname{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\} \colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$
-

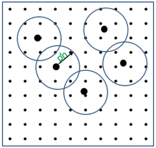
Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\operatorname{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t.
  $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\} \colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$
- Idea: for $p' > p$ to be determined later, find f s.t. w.h.p. over $\mathbf{x}$ and Z:
  (1) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x})\|_1 \leq p'n$
  (2) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x}')\|_1 > p'n$ for all $\mathbf{x}' \neq \mathbf{x}$

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\mathrm{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\} \colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$
- Idea: for $p' > p$ to be determined later, find f s.t. w.h.p. over $\mathbf{x}$ and Z:

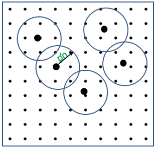
  (1) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x})\|_1 \leq p'n$
  (2) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x}')\|_1 > p'n$ for all $\mathbf{x}' \neq \mathbf{x}$



Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\operatorname{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\} \colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$
- Idea: for $p' > p$ to be determined later, find f s.t. w.h.p. over $\mathbf{x}$ and Z:

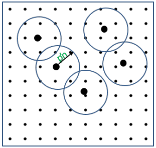
  (1) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x})\|_1 \leq p'n$
  (2) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x}')\|_1 > p'n$ for all $\mathbf{x}' \neq \mathbf{x}$



- We choose f uniformly at random (what does it mean?)

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\mathrm{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\} \colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$
- Idea: for $p' > p$ to be determined later, find f s.t. w.h.p. over $\mathbf{x}$ and Z:
  (1) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x})\|_1 \leq p'n$
  (2) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x}')\|_1 > p'n$ for all $\mathbf{x}' \neq \mathbf{x}$

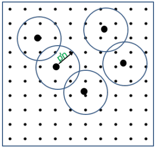


- We choose f uniformly at random (what does it mean?)
- Non-constructive proof

Proving the theorem

- Fix $p \in [0, \frac{1}{2})$ and $\varepsilon > 0$, and let $m > m_\varepsilon$ and $n \geq m(\frac{1}{C_p} + \varepsilon)$, for m_ε to be determined by the analysis. (Recall $C_p = 1 - h(p)$).
- We show $\exists\, f \colon \{0,1\}^m \mapsto \{0,1\}^n$ and $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \varepsilon$
- $g(y)$ returns $\operatorname{argmin}_{\mathbf{x}' \in \{0,1\}^m} \|y - f(\mathbf{x}')\|_1$
- So it all boils down to finding f s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\} \colon \|f(\mathbf{x}) - y\|_1 < \|f(\mathbf{x}') - y\|_1] \geq 1 - \varepsilon$
- Idea: for $p' > p$ to be determined later, find f s.t. w.h.p. over $\mathbf{x}$ and Z:

  (1) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x})\|_1 \leq p'n$
  (2) $\|f(\mathbf{x}) \oplus Z, f(\mathbf{x}')\|_1 > p'n$ for all $\mathbf{x}' \neq \mathbf{x}$



- We choose f uniformly at random (what does it mean?)
- Non-constructive proof
- Probabilistic method

Proving there exists good f

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \|y' - y\|_1 \leq p'n\}$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N} \text{ s.t. } \forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \ \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N} \text{ s.t. } \forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\Longrightarrow \quad \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \ \Pr_f\left[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] = \frac{b(p')}{2^n}$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \; \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N} \text{ s.t. } \forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \; \Pr_f[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] = \frac{b(p')}{2^n}$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \; \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \; \Pr_f[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \Pr_f\left[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$$

$$\implies \beta_{m,n} := \Pr_f\left[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] \leq 2^{m - nC_{p'}}$$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \ \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \ \Pr_f\left[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\implies \beta_{m,n} := \Pr_f\left[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] \leq 2^{m - nC_{p'}}$

$\implies \beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log \frac{\varepsilon}{2})$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \ \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \ \Pr_f[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\implies \beta_{m,n} := \Pr_f[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] \leq 2^{m - nC_{p'}}$

$\implies \beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log \frac{\varepsilon}{2})$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \ \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \ \Pr_f[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\implies \beta_{m,n} := \Pr_f[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] \leq 2^{m - nC_{p'}}$

$\implies \beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log \frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m\colon\ \Pr_f[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$$

$$\implies \beta_{m,n} := \Pr_f[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] \leq 2^{m - nC_{p'}}$$

$$\implies \beta_{m,n} \leq \frac{\varepsilon}{2}, \text{ for } n \geq \frac{1}{C_{p'}}(m - \log \frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$$

(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log \frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and

$$n \geq m(\frac{1}{C_p} + \varepsilon)$$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \; \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$$\Longrightarrow \quad \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \; \Pr_f\left[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$$

$$\Longrightarrow \quad \beta_{m,n} := \Pr_f\left[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] \leq 2^{m - nC_{p'}}$$

$$\Longrightarrow \quad \beta_{m,n} \leq \frac{\varepsilon}{2}, \text{ for } n \geq \frac{1}{C_{p'}}(m - \log\frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log\frac{\varepsilon}{2}}{mC_{p'}})$$

(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log\frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and

$$n \geq m(\frac{1}{C_p} + \varepsilon)$$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \ \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \ \Pr_f\left[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\implies \beta_{m,n} := \Pr_f\left[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] \leq 2^{m - nC_{p'}}$

$\implies \beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log\frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log\frac{\varepsilon}{2}}{mC_{p'}})$

(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log\frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and

$$n \geq m(\frac{1}{C_p} + \varepsilon) \geq m(\frac{1}{C_p} + \frac{\varepsilon}{2} + \frac{-\log\frac{\varepsilon}{2}}{mC_{p'}})$$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \Pr_f[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\implies \beta_{m,n} := \Pr_f[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] \leq 2^{m - nC_{p'}}$

$\implies \beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log\frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log\frac{\varepsilon}{2}}{mC_{p'}})$

(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log\frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and

$$n \geq m(\frac{1}{C_p} + \varepsilon) \geq m(\frac{1}{C_p} + \frac{\varepsilon}{2} + \frac{-\log\frac{\varepsilon}{2}}{mC_{p'}}) \geq m(\frac{1}{C_{p'}} - \frac{\log\frac{\varepsilon}{2}}{mC_{p'}})$$

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \; \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\Longrightarrow$ $\forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m$: $\Pr_f[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\Longrightarrow$ $\beta_{m,n} := \Pr_f[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))] \leq 2^{m - nC_{p'}}$

$\Longrightarrow$ $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log \frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$

(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log \frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and

$$n \geq m(\frac{1}{C_p} + \varepsilon) \geq m(\frac{1}{C_p} + \frac{\varepsilon}{2} + \frac{-\log \frac{\varepsilon}{2}}{mC_{p'}}) \geq m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$$

- Fix $m > m_\varepsilon := \max\{m', n'\}$ and $n > m(\frac{1}{C_p} + \varepsilon)$.

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \ \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N} \text{ s.t. } \forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies \forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \ \Pr_f\left[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\implies \beta_{m,n} := \Pr_f\left[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] \leq 2^{m - nC_{p'}}$

$\implies \beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log \frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$

(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log \frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and

$$n \geq m(\frac{1}{C_p} + \varepsilon) \geq m(\frac{1}{C_p} + \frac{\varepsilon}{2} + \frac{-\log \frac{\varepsilon}{2}}{mC_{p'}}) \geq m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$$

- Fix $m > m_\varepsilon := \max\{m', n'\}$ and $n > m(\frac{1}{C_p} + \varepsilon)$.
- $\Pr_f\left[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}\right] \leq \alpha_n + \beta_{m,n} \leq \varepsilon$.

Proving there exists good f

- Fix $p' > p$ such that $0 < \frac{1}{C_{p'}} - \frac{1}{C_p} \leq \frac{\varepsilon}{2}$ and $x \in \{0,1\}^m$
- For $y \in \{0,1\}^n$, let $B_{p'}(y) = \{y \in \{0,1\}^n \colon \ \|y' - y\|_1 \leq p'n\}$

(1) By weak low of large numbers, $\exists n' \in \mathbb{N}$ s.t. $\forall n \geq n'$
$\alpha_n := \Pr_{z \leftarrow Z}\left[(f(\mathbf{x}) \oplus z) \notin B_{p'}(f(\mathbf{x}))\right] \leq \frac{\varepsilon}{2}$ (for any fixed f)

- Fact (proved later): $b(p') := |B_{p'}(y)| \leq 2^{n \cdot h(p')}$

$\implies$ $\forall \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon \ \Pr_f\left[f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] = \frac{b(p')}{2^n} \leq \frac{2^{n \cdot h(p')}}{2^n} = 2^{-nC_{p'}}$

$\implies$ $\beta_{m,n} := \Pr_f\left[\exists \mathbf{x}' \neq \mathbf{x} \in \{0,1\}^m \colon f(\mathbf{x}) \oplus Z \in B_{p'}(f(\mathbf{x}'))\right] \leq 2^{m - nC_{p'}}$

$\implies$ $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $n \geq \frac{1}{C_{p'}}(m - \log \frac{\varepsilon}{2}) = m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$

(2) $\beta_{m,n} \leq \frac{\varepsilon}{2}$, for $m \geq m' := \left\lceil \frac{-\log \frac{\varepsilon}{2}}{\frac{\varepsilon}{2} \cdot C_{p'}} \right\rceil$ and

$$n \geq m(\frac{1}{C_p} + \varepsilon) \geq m(\frac{1}{C_p} + \frac{\varepsilon}{2} + \frac{-\log \frac{\varepsilon}{2}}{mC_{p'}}) \geq m(\frac{1}{C_{p'}} - \frac{\log \frac{\varepsilon}{2}}{mC_{p'}})$$

- Fix $m > m_\varepsilon := \max\{m', n'\}$ and $n > m(\frac{1}{C_p} + \varepsilon)$.
- $\Pr_f\left[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}\right] \leq \alpha_n + \beta_{m,n} \leq \varepsilon$.

$\implies$ $\exists f$ s.t. $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}\left[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}\right] \leq \varepsilon$. $\square$

Tightness

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f\colon \{0,1\}^m \mapsto \{0,1\}^n$, $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f \colon \{0,1\}^m \mapsto \{0,1\}^n$, $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$*, then* $nC_p \geq m(1-\varepsilon) - 1$*.*

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1,\ldots,Z_n)$, for $Z_1,\ldots,Z_n$ iid $\sim (1-p,p)$, let $f\colon \{0,1\}^m \mapsto \{0,1\}^n$, $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr\left[g(Y) = X\right] \geq 1-\varepsilon$, *then* $nC_p \geq m(1-\varepsilon)-1$.

- Recall that Thm 1 allows $nC_p = m(1+\varepsilon C_p)$.

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f \colon \{0,1\}^m \mapsto \{0,1\}^n$, $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$, *then* $nC_p \geq m(1-\varepsilon) - 1$.

- Recall that Thm 1 allows $nC_p = m(1 + \varepsilon C_p)$.
- Hence, $\lim_{\varepsilon \to 0} \frac{m}{n} = C_p$

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f \colon \{0,1\}^m \mapsto \{0,1\}^n$, $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$, *then* $nC_p \geq m(1-\varepsilon) - 1$.

- Recall that Thm 1 allows $nC_p = m(1 + \varepsilon C_p)$.
- Hence, $\lim_{\varepsilon \to 0} \frac{m}{n} = C_p$

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f \colon \{0,1\}^m \mapsto \{0,1\}^n$, $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$, *then* $nC_p \geq m(1-\varepsilon) - 1$.

- Recall that Thm 1 allows $nC_p = m(1 + \varepsilon C_p)$.
- Hence, $\lim_{\varepsilon \to 0} \frac{m}{n} = C_p$

Proof:

- $\underbrace{X}_{m \text{ bits}} \longrightarrow \underbrace{f(X)}_{n \text{ bits}} \longrightarrow \underbrace{f(X) \oplus Z}_{Y} \longrightarrow \underbrace{g(f(X) \oplus Z)}_{g(Y)}$

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f \colon \{0,1\}^m \mapsto \{0,1\}^n$, $g \colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$*, then* $nC_p \geq m(1-\varepsilon) - 1$*.*

- Recall that Thm 1 allows $nC_p = m(1 + \varepsilon C_p)$.
- Hence, $\lim_{\varepsilon \to 0} \frac{m}{n} = C_p$

Proof:

- $\underbrace{X}_{m \text{ bits}} \longrightarrow \underbrace{f(X)}_{n \text{ bits}} \longrightarrow \underbrace{f(X) \oplus Z}_{Y} \longrightarrow \underbrace{g(f(X) \oplus Z)}_{g(Y)}$
- By Fano, $H(X|Y) \leq h(\varepsilon) + \varepsilon \cdot \log 2^m \leq 1 + \varepsilon m$

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1,\ldots,Z_n)$, for $Z_1,\ldots,Z_n$ iid $\sim (1-p,p)$, let $f\colon \{0,1\}^m \mapsto \{0,1\}^n$, $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr\left[g(Y) = X\right] \geq 1-\varepsilon$*, then* $nC_p \geq m(1-\varepsilon) - 1$*.*

- Recall that Thm 1 allows $nC_p = m(1+\varepsilon C_p)$.
- Hence, $\lim_{\varepsilon\to 0} \frac{m}{n} = C_p$

Proof:

- $\underbrace{X}_{m\text{ bits}} \longrightarrow \underbrace{f(X)}_{n\text{ bits}} \longrightarrow \underbrace{f(X)\oplus Z}_{Y} \longrightarrow \underbrace{g(f(X)\oplus Z)}_{g(Y)}$
- By Fano, $H(X|Y) \leq h(\varepsilon) + \varepsilon\cdot\log 2^m \leq 1+\varepsilon m$
- $I(X;Y) = H(X) - H(X|Y) \geq m - \varepsilon m - 1 = m(1-\varepsilon) - 1$

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1,\ldots,Z_n)$, for $Z_1,\ldots,Z_n$ iid $\sim (1-p,p)$, let $f\colon \{0,1\}^m \mapsto \{0,1\}^n$, $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1-\varepsilon$, *then* $nC_p \geq m(1-\varepsilon) - 1$.

- Recall that Thm 1 allows $nC_p = m(1+\varepsilon C_p)$.
- Hence, $\lim_{\varepsilon\to 0} \frac{m}{n} = C_p$

Proof:

- $\underbrace{X}_{m \text{ bits}} \longrightarrow \underbrace{f(X)}_{n \text{ bits}} \longrightarrow \underbrace{f(X)\oplus Z}_{Y} \longrightarrow \underbrace{g(f(X)\oplus Z)}_{g(Y)}$
- By Fano, $H(X|Y) \leq h(\varepsilon) + \varepsilon \cdot \log 2^m \leq 1 + \varepsilon m$
- $I(X;Y) = H(X) - H(X|Y) \geq m - \varepsilon m - 1 = m(1-\varepsilon) - 1$
- $H(Y|X) = H(X,Y) - H(X) = H(X,Z) - H(X) = H(Z) = nh(p)$

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f\colon \{0,1\}^m \mapsto \{0,1\}^n$, $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$, *then* $nC_p \geq m(1-\varepsilon) - 1$.

- Recall that Thm 1 allows $nC_p = m(1 + \varepsilon C_p)$.
- Hence, $\lim_{\varepsilon \to 0} \frac{m}{n} = C_p$

Proof:

- $\underbrace{X}_{m \text{ bits}} \longrightarrow \underbrace{f(X)}_{n \text{ bits}} \longrightarrow \underbrace{f(X) \oplus Z}_{Y} \longrightarrow \underbrace{g(f(X) \oplus Z)}_{g(Y)}$
- By Fano, $H(X|Y) \leq h(\varepsilon) + \varepsilon \cdot \log 2^m \leq 1 + \varepsilon m$
- $I(X;Y) = H(X) - H(X|Y) \geq m - \varepsilon m - 1 = m(1-\varepsilon) - 1$
- $H(Y|X) = H(X,Y) - H(X) = H(X,Z) - H(X) = H(Z) = nh(p)$
- $I(X;Y) = H(Y) - H(Y|X) \leq n - nh(p)$

Tightness

Let $X \leftarrow \{0,1\}^m$, $Z = (Z_1, \ldots, Z_n)$, for $Z_1, \ldots, Z_n$ iid $\sim (1-p, p)$, let $f\colon \{0,1\}^m \mapsto \{0,1\}^n$, $g\colon \{0,1\}^n \mapsto \{0,1\}^m$, and let $Y = f(X)$.

Theorem 2

Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$, *then* $nC_p \geq m(1-\varepsilon) - 1$.

- Recall that Thm 1 allows $nC_p = m(1 + \varepsilon C_p)$.
- Hence, $\lim_{\varepsilon \to 0} \frac{m}{n} = C_p$

Proof:

- $\underbrace{X}_{m \text{ bits}} \longrightarrow \underbrace{f(X)}_{n \text{ bits}} \longrightarrow \underbrace{f(X) \oplus Z}_{Y} \longrightarrow \underbrace{g(f(X) \oplus Z)}_{g(Y)}$
- By Fano, $H(X|Y) \leq h(\varepsilon) + \varepsilon \cdot \log 2^m \leq 1 + \varepsilon m$
- $I(X;Y) = H(X) - H(X|Y) \geq m - \varepsilon m - 1 = m(1-\varepsilon) - 1$
- $H(Y|X) = H(X,Y) - H(X) = H(X,Z) - H(X) = H(Z) = nh(p)$
- $I(X;Y) = H(Y) - H(Y|X) \leq n - nh(p)$
- Hence, $m(1-\varepsilon) \leq I(X;Y) + 1 = n(1 - h(p)) + 1 = nC_p + 1$

Why $C_p = 1 - h(p)$?, intuitive explanation

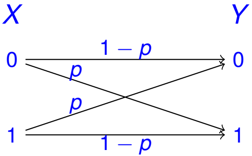
Why $C_p = 1 - h(p)$?, intuitive explanation

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p, p)$ and $Y = X \oplus Z$

A different proof:

Why $C_p = 1 - h(p)$?, intuitive explanation

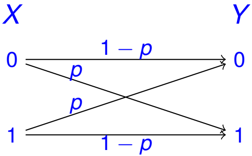
- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p, p)$ and $Y = X \oplus Z$



A different proof:

Why $C_p = 1 - h(p)$?, intuitive explanation

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p, p)$ and $Y = X \oplus Z$

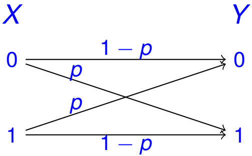


- $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = 1 - h(p) = C_p$

A different proof:

Why $C_p = 1 - h(p)$?, intuitive explanation

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p,p)$ and $Y = X \oplus Z$

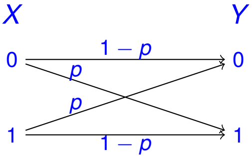


- $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = 1 - h(p) = C_p$
- Received bit “gives" C_p information about transmitted bit

A different proof:

Why $C_p = 1 - h(p)$?, intuitive explanation

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p, p)$ and $Y = X \oplus Z$

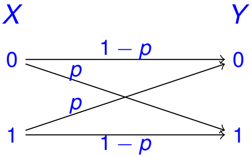


- $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = 1 - h(p) = C_p$
- Received bit “gives" C_p information about transmitted bit
- Hence, to recover m bits, we need to send at least $m \cdot \frac{1}{C_p}$ bits

A different proof:

Why $C_p = 1 - h(p)$?, intuitive explanation

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p, p)$ and $Y = X \oplus Z$



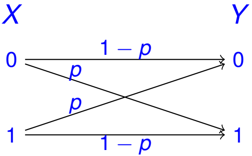
- $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = 1 - h(p) = C_p$
- Received bit “gives" C_p information about transmitted bit
- Hence, to recover m bits, we need to send at least $m \cdot \frac{1}{C_p}$ bits

A different proof:

- Let $X \leftarrow \{0,1\}^m$, $Y = f(X)$ and assume that $g(Y) = X$.

Why $C_p = 1 - h(p)$?, intuitive explanation

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p, p)$ and $Y = X \oplus Z$



- $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = 1 - h(p) = C_p$
- Received bit “gives" C_p information about transmitted bit
- Hence, to recover m bits, we need to send at least $m \cdot \frac{1}{C_p}$ bits

A different proof:

- Let $X \leftarrow \{0,1\}^m$, $Y = f(X)$ and assume that $g(Y) = X$.
- Hence, $n \geq H(Y) = H(X, Z) = m + nh(p)$

Why $C_p = 1 - h(p)$?, intuitive explanation

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p,p)$ and $Y = X \oplus Z$

$$\begin{array}{ccc} X & & Y \\ 0 & \xrightarrow{1-p} & 0 \\ 0 & \xrightarrow{p} & 1 \\ 1 & \xrightarrow{p} & 0 \\ 1 & \xrightarrow{1-p} & 1 \end{array}$$

- $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = 1 - h(p) = C_p$
- Received bit “gives" C_p information about transmitted bit
- Hence, to recover m bits, we need to send at least $m \cdot \frac{1}{C_p}$ bits

A different proof:

- Let $X \leftarrow \{0,1\}^m$, $Y = f(X)$ and assume that $g(Y) = X$.
- Hence, $n \geq H(Y) = H(X,Z) = m + nh(p)$

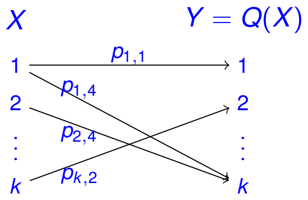
$\implies \quad n(1-h(p)) = nC_p \geq m$

General communication channel

General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr\left[Q(i) = j\right]$

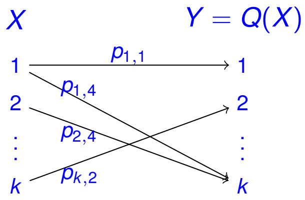


General communication channel

$Q \colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr\left[Q(i) = j\right]$

- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$

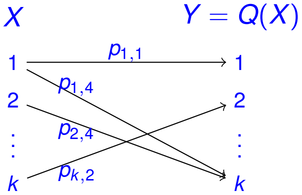


General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr[Q(i) = j]$

- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1, \ldots, k\}^n$

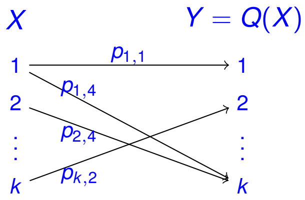


General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr[Q(i) = j]$

- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1,\ldots,k\}^n$
- Decoding function $g\{1,\ldots,k\}^n \mapsto \{0,1\}^m$

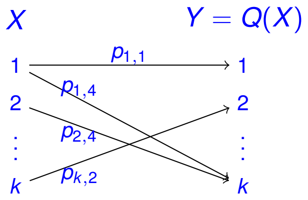


General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr[Q(i) = j]$

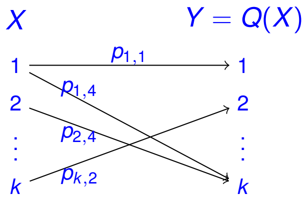
- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1,\ldots,k\}^n$
- Decoding function $g\{1,\ldots,k\}^n \mapsto \{0,1\}^m$
- $\mathbf{x} \xrightarrow{\text{encoding}} f(\mathbf{x}) \xrightarrow{\text{channel}} Q(f(\mathbf{x})) \xrightarrow{\text{decoding}} g(Q(f(\mathbf{x})))$



General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr\left[Q(i) = j\right]$

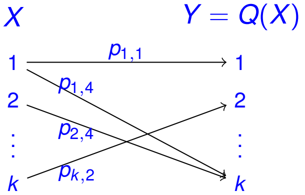


- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1, \ldots, k\}^n$
- Decoding function $g\{1, \ldots, k\}^n \mapsto \{0,1\}^m$
- $\mathbf{x} \xrightarrow{\text{encoding}} f(\mathbf{x}) \xrightarrow{\text{channel}} Q(f(\mathbf{x})) \xrightarrow{\text{decoding}} g(Q(f(\mathbf{x})))$
- We hope for $g(Q(f(\mathbf{x}))) = \mathbf{x}$

General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr[Q(i) = j]$

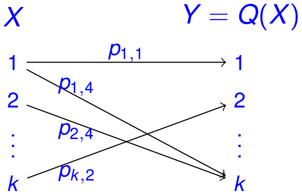


- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1,\ldots,k\}^n$
- Decoding function $g\{1,\ldots,k\}^n \mapsto \{0,1\}^m$
- $\mathbf{x} \overset{\text{encoding}}{\longrightarrow} f(\mathbf{x}) \overset{\text{channel}}{\longrightarrow} Q(f(\mathbf{x})) \overset{\text{decoding}}{\longrightarrow} g(Q(f(\mathbf{x})))$
- We hope for $g(Q(f(\mathbf{x}))) = \mathbf{x}$
- Channel capacity $C_Q = \max_p C(p)$, for $c(p) := I(X_p; Q(X_p))$ and $X_p \sim p$.

General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr[Q(i) = j]$

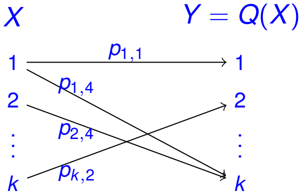


- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1,\ldots,k\}^n$
- Decoding function $g\{1,\ldots,k\}^n \mapsto \{0,1\}^m$
- $\mathbf{x} \overset{\text{encoding}}{\longrightarrow} f(\mathbf{x}) \overset{\text{channel}}{\longrightarrow} Q(f(\mathbf{x})) \overset{\text{decoding}}{\longrightarrow} g(Q(f(\mathbf{x})))$
- We hope for $g(Q(f(\mathbf{x}))) = \mathbf{x}$

- Channel capacity $C_Q = \max_p C(p)$, for $c(p) := I(X_p; Q(X_p))$ and $X_p \sim p$.
- The maximal information Y gives on X

General communication channel

$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

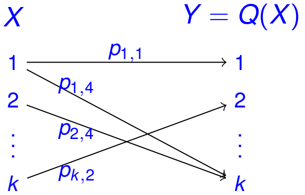


$p_{i,j} = \Pr[Q(i) = j]$

- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1,\ldots,k\}^n$
- Decoding function $g\{1,\ldots,k\}^n \mapsto \{0,1\}^m$
- $\mathbf{x} \overset{\text{encoding}}{\longrightarrow} f(\mathbf{x}) \overset{\text{channel}}{\longrightarrow} Q(f(\mathbf{x})) \overset{\text{decoding}}{\longrightarrow} g(Q(f(\mathbf{x})))$
- We hope for $g(Q(f(\mathbf{x}))) = \mathbf{x}$

- Channel capacity $C_Q = \max_p C(p)$, for $c(p) := I(X_p; Q(X_p))$ and $X_p \sim p$.
- The maximal information Y gives on X
- Shannon theorem: $\forall Q$ and $\forall \varepsilon > 0$, $\exists m_\varepsilon \colon \forall m > m_\varepsilon$ and $\forall n > m(\frac{1}{C_Q} + \varepsilon)$: $\exists f, g$ as above $\;\text{s.t.}\;$ $\Pr_Q[g(Q(f(\mathbf{x}))) \neq \mathbf{x}] \leq \varepsilon$, for all $\mathbf{x} \in \{0,1\}^m$.

General communication channel



$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr[Q(i) = j]$

- $\mathbf{x} = (x_1, \ldots, x_m) \in \{0,1\}^m$
- Encoding function $f\colon \{0,1\}^m \mapsto \{1,\ldots,k\}^n$
- Decoding function $g\{1,\ldots,k\}^n \mapsto \{0,1\}^m$
- $\mathbf{x} \overset{\text{encoding}}{\longrightarrow} f(\mathbf{x}) \overset{\text{channel}}{\longrightarrow} Q(f(\mathbf{x})) \overset{\text{decoding}}{\longrightarrow} g(Q(f(\mathbf{x})))$
- We hope for $g(Q(f(\mathbf{x}))) = \mathbf{x}$
- Channel capacity $C_Q = \max_p C(p)$, for $c(p) := I(X_p; Q(X_p))$ and $X_p \sim p$.
- The maximal information Y gives on X
- Shannon theorem: $\forall Q$ and $\forall \varepsilon > 0$, $\exists m_\varepsilon \colon \forall m > m_\varepsilon$ and $\forall n > m(\frac{1}{C_Q} + \varepsilon)$: $\exists f, g$ as above s.t. $\Pr_Q[g(Q(f(\mathbf{x}))) \neq \mathbf{x}] \leq \varepsilon$, for all $\mathbf{x} \in \{0,1\}^m$.
- Proof: similar lines to the binary case, but more subtle distribution for f

Discussion

Discussion

- Tight result

Discussion

- Tight result
- Non-constructive

Discussion

- Tight result
- Non-constructive
- Coding theory: design explicit (and efficient) code achieving the same bounds

Discussion

- Tight result
- Non-constructive
- Coding theory: design explicit (and efficient) code achieving the same bounds
- Application: faulty communication, storage

Discussion

- Tight result
- Non-constructive
- Coding theory: design explicit (and efficient) code achieving the same bounds
- Application: faulty communication, storage
- Combination of data compression and ECC

Part II

Hamming Ball

Why $H(X_1, \ldots, X_n) \le \sum_i H(X_i)$ so useful?

Why $H(X_1, \ldots, X_n) \leq \sum_i H(X_i)$ so useful?

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$

Why $H(X_1, \ldots, X_n) \le \sum_i H(X_i)$ so useful?

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\log |\mathcal{S}| = H(X) \le \sum_i H(X_i)$ implies $|\mathcal{S}| \le 2^{\sum_i H(X_i)}$

Why $H(X_1, \ldots, X_n) \leq \sum_i H(X_i)$ so useful?

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\log |\mathcal{S}| = H(X) \leq \sum_i H(X_i)$ implies $|\mathcal{S}| \leq 2^{\sum_i H(X_i)}$
- If $\sum_i H(X_i)$ is small, then $\mathcal{S}$ is small

Why $H(X_1,\ldots,X_n) \le \sum_i H(X_i)$ so useful?

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1,\ldots,X_n) \leftarrow \mathcal{S}$
- $\log|\mathcal{S}| = H(X) \le \sum_i H(X_i)$ implies $|\mathcal{S}| \le 2^{\sum_i H(X_i)}$
- If $\sum_i H(X_i)$ is small, then $\mathcal{S}$ is small

  X_i are unbalanced, e.g., $\sim (0.1, 0.9)$, implies $|\mathcal{S}| \le 2^{n\cdot h(0.1)} \le 2^{n/2}$

Why $H(X_1,\ldots,X_n) \leq \sum_i H(X_i)$ so useful?

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1,\ldots,X_n) \leftarrow \mathcal{S}$
- $\log|\mathcal{S}| = H(X) \leq \sum_i H(X_i)$ implies $|\mathcal{S}| \leq 2^{\sum_i H(X_i)}$
- If $\sum_i H(X_i)$ is small, then $\mathcal{S}$ is small

  X_i are unbalanced, e.g., $\sim (0.1, 0.9)$, implies $|\mathcal{S}| \leq 2^{n \cdot h(0.1)} \leq 2^{n/2}$
- $\mathcal{S}$ is large implies $\sum_i H(X_i)$ is large, hence most X_i are almost balanced

Why $H(X_1,\ldots,X_n) \leq \sum_i H(X_i)$ so useful?

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1,\ldots,X_n) \leftarrow \mathcal{S}$
- $\log|\mathcal{S}| = H(X) \leq \sum_i H(X_i)$ implies $|\mathcal{S}| \leq 2^{\sum_i H(X_i)}$
- If $\sum_i H(X_i)$ is small, then $\mathcal{S}$ is small

  X_i are unbalanced, e.g., $\sim (0.1, 0.9)$, implies $|\mathcal{S}| \leq 2^{n \cdot h(0.1)} \leq 2^{n/2}$
- $\mathcal{S}$ is large implies $\sum_i H(X_i)$ is large, hence most X_i are almost balanced
- $|\mathcal{S}| \geq 2^n/2$ implies $\mathrm{E}_{i \leftarrow [n]}[H(X_i)] \geq 1 - \frac{1}{n}$

Why $H(X_1,\ldots,X_n) \leq \sum_i H(X_i)$ so useful?

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1,\ldots,X_n) \leftarrow \mathcal{S}$
- $\log|\mathcal{S}| = H(X) \leq \sum_i H(X_i)$ implies $|\mathcal{S}| \leq 2^{\sum_i H(X_i)}$
- If $\sum_i H(X_i)$ is small, then $\mathcal{S}$ is small

  X_i are unbalanced, e.g., $\sim (0.1, 0.9)$, implies $|\mathcal{S}| \leq 2^{n \cdot h(0.1)} \leq 2^{n/2}$
- $\mathcal{S}$ is large implies $\sum_i H(X_i)$ is large, hence most X_i are almost balanced
- $|\mathcal{S}| \geq 2^n/2$ implies $\mathrm{E}_{i \leftarrow [n]}[H(X_i)] \geq 1 - \frac{1}{n}$
- Most X_i are close to uniform

Hamming ball

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i
- Hence, $\Pr[X_i = 1] \leq p$ or every i.

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i
- Hence, $\Pr[X_i = 1] \leq p$ or every i.

$\implies$ $H(X_i) \leq h(p)$ for every i

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i
- Hence, $\Pr[X_i = 1] \leq p$ or every i.

$\implies$ $H(X_i) \leq h(p)$ for every i

$\implies$ $|\mathcal{S}| \leq 2^{\sum_i H(X_i)} \leq 2^{nh(p)}$

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i
- Hence, $\Pr[X_i = 1] \leq p$ or every i.

$\implies$ $H(X_i) \leq h(p)$ for every i

$\implies$ $|\mathcal{S}| \leq 2^{\sum_i H(X_i)} \leq 2^{nh(p)}$

$\implies$ $\sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{nh(p)}$

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i
- Hence, $\Pr[X_i = 1] \leq p$ or every i.

$\implies$ $H(X_i) \leq h(p)$ for every i

$\implies$ $|\mathcal{S}| \leq 2^{\sum_i H(X_i)} \leq 2^{nh(p)}$

$\implies$ $\sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{nh(p)}$

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i
- Hence, $\Pr[X_i = 1] \leq p$ or every i.

$\implies \quad H(X_i) \leq h(p)$ for every i

$\implies \quad |\mathcal{S}| \leq 2^{\sum_i H(X_i)} \leq 2^{nh(p)}$

$\implies \quad \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{nh(p)}$

Corollary 3

For $y \in \{0,1\}^n$ and $p \in [0, \frac{1}{2}]$, let $B_p(y) = \{y \in \{0,1\}^n \colon \left\|y' - y\right\|_1 \leq pn\}$. Then $|B_p(y)| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{n \cdot h(p)}$

Hamming ball

- $p \leq \frac{1}{2}$; $\mathcal{S} = \{(a_1, \ldots, a_n) \in \{0,1\}^n \colon \sum_i a_i \leq pn\}$
- $|\mathcal{S}| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k}$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $\sum_i X_i \leq pn \implies \mathrm{E}\left[\sum X_i\right] \leq pn$, and by symmetry $\mathrm{E}\left[X_i\right] \leq p$ for every i
- Hence, $\Pr[X_i = 1] \leq p$ or every i.

$\implies$ $H(X_i) \leq h(p)$ for every i

$\implies$ $|\mathcal{S}| \leq 2^{\sum_i H(X_i)} \leq 2^{nh(p)}$

$\implies$ $\sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{nh(p)}$

Corollary 3

For $y \in \{0,1\}^n$ *and* $p \in [0, \frac{1}{2}]$, *let* $B_p(y) = \{y \in \{0,1\}^n \colon \|y' - y\|_1 \leq pn\}$. *Then* $|B_p(y)| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{n \cdot h(p)}$

Very useful estimation. Weaker variants follows by AEP or Stirling,

Hamming ball, cont.

The above bound yields the following concentration bound:

Corollary 4

Let $X_1, \ldots, X_n$ be iid uniform bits and let $p \in [0, \frac{1}{2}]$, then
$\Pr\left[\sum_i X_i \leq pn\right] = \Pr\left[(X_1, \ldots, X_n) \in \mathcal{S}\right] \leq 2^{nh(p)} \cdot 2^{-n} = 2^{-n(1-h(p))}$.

Hamming ball, cont.

The above bound yields the following concentration bound:

Corollary 4

Let $X_1, \ldots, X_n$ be iid uniform bits and let $p \in [0, \frac{1}{2}]$, then
$\Pr\left[\sum_i X_i \leq pn\right] = \Pr\left[(X_1, \ldots, X_n) \in \mathcal{S}\right] \leq 2^{nh(p)} \cdot 2^{-n} = 2^{-n(1-h(p))}$.

Very useful inequality. No Chernoff just IT

Part III

Combinatorial Applications

Movies

Movies

- 2^n people, $m = 3n$ movies.

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(0.9, \frac{0.1}{3}, \frac{0.1}{3}, \frac{0.1}{3}) \leq \frac{2}{3}$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(0.9, \frac{0.1}{3}, \frac{0.1}{3}, \frac{0.1}{3}) \leq \frac{2}{3}$
- $H(Y = (Y_1, \ldots, Y_m)) \leq H(Y_1, Y_2) + H(Y_3, Y_4) + \ldots + H(Y_{m-1}, Y_m)$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(0.9, \frac{0.1}{3}, \frac{0.1}{3}, \frac{0.1}{3}) \leq \frac{2}{3}$
- $H(Y = (Y_1, \ldots, Y_m)) \leq H(Y_1, Y_2) + H(Y_3, Y_4) + \ldots + H(Y_{m-1}, Y_m)$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(0.9, \frac{0.1}{3}, \frac{0.1}{3}, \frac{0.1}{3}) \leq \frac{2}{3}$
- $H(Y = (Y_1, \ldots, Y_m)) \leq H(Y_1, Y_2) + H(Y_3, Y_4) + \ldots + H(Y_{m-1}, Y_m) < \frac{3n}{2} \cdot \frac{2}{3}$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(0.9, \frac{0.1}{3}, \frac{0.1}{3}, \frac{0.1}{3}) \leq \frac{2}{3}$
- $H(Y = (Y_1, \ldots, Y_m)) \leq H(Y_1, Y_2) + H(Y_3, Y_4) + \ldots + H(Y_{m-1}, Y_m) < \frac{3n}{2} \cdot \frac{2}{3} = n$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(0.9, \frac{0.1}{3}, \frac{0.1}{3}, \frac{0.1}{3}) \leq \frac{2}{3}$
- $H(Y = (Y_1, \ldots, Y_m)) \leq H(Y_1, Y_2) + H(Y_3, Y_4) + \ldots + H(Y_{m-1}, Y_m) < \frac{3n}{2} \cdot \frac{2}{3} = n = H(X)$

Movies

- 2^n people, $m = 3n$ movies.
- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(0.9, \frac{0.1}{3}, \frac{0.1}{3}, \frac{0.1}{3}) \leq \frac{2}{3}$
- $H(Y = (Y_1, \ldots, Y_m)) \leq H(Y_1, Y_2) + H(Y_3, Y_4) + \ldots + H(Y_{m-1}, Y_m) < \frac{3n}{2} \cdot \frac{2}{3} = n = H(X)$
- Hence, X is not determined by Y

Isoperimetric inequality

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0, 1\}^n$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon \ |u - v| = 1\}$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon\ |u-v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is “face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon\ |u-v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is “face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon \ |u-v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is “face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is “face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon\ |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is “face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon \ |u-v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad$ $(E = \dot\bigcup_{i \in [n]} E_i)$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x}, \mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad$ ($E = \dot{\bigcup}_{i \in [n]} E_i$)
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $(E = \dot\bigcup_{i \in [n]} E_i)$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x}, \mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad$ ($E = \bigcup_{i \in [n]} E_i$)
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Lemma 6

$H(X_i | X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon \; |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad$ ($E = \dot\bigcup_{i \in [n]} E_i$)
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Lemma 6

$H(X_i | X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

Proving Thm **??**:

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon \; |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x}, \mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad$ ($E = \dot\bigcup_{i \in [n]} E_i$)
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Lemma 6

$H(X_i | X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

Proving Thm **??**:

$$\log |\mathcal{S}| = H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_1, X_2, \ldots, X_{n-1})$$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u-v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log|\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad$ $(E = \dot{\bigcup}_{i \in [n]} E_i)$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Lemma 6

$H(X_i | X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

Proving Thm **??**:

$$\log|\mathcal{S}| = H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_1, X_2, \ldots, X_{n-1})$$

$$\geq H(X_1|X_{-1}) + H(X_2|X_{-2}) + \ldots H(X_n|X_{-n})$$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is “face" : $\mathcal{S} = \{(\mathbf{x}, \mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad$ $(E = \dot\bigcup_{i \in [n]} E_i)$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Lemma 6

$H(X_i | X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

Proving Thm **??**:

$$\log |\mathcal{S}| = H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_1, X_2, \ldots, X_{n-1})$$

$$\geq H(X_1|X_{-1}) + H(X_2|X_{-2}) + \ldots H(X_n|X_{-n}) = \sum_i \frac{2\,|E_i|}{|\mathcal{S}|}$$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon \ |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad (E = \dot\bigcup_{i \in [n]} E_i)$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Lemma 6

$H(X_i | X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

Proving Thm **??**:

$$\log |\mathcal{S}| = H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_1, X_2, \ldots, X_{n-1})$$
$$\geq H(X_1|X_{-1}) + H(X_2|X_{-2}) + \ldots H(X_n|X_{-n}) = \sum_i \frac{2\,|E_i|}{|\mathcal{S}|} = \frac{2\,|E|}{|\mathcal{S}|}$$

Isoperimetric inequality

- $\mathcal{S} \subseteq \{0,1\}^n$
- Edges of $\mathcal{S}$ — $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$

Theorem 5

$|E| \leq \frac{1}{2} \cdot |\mathcal{S}| \cdot \log |\mathcal{S}|$

- Equality if $\mathcal{S}$ is "face" : $\mathcal{S} = \{(\mathbf{x},\mathbf{y}) \colon \mathbf{y} \in \{0,1\}^d\}$ for some $\mathbf{x} \in \{0,1\}^{n-d}$
- Example: $\mathcal{S}$ is a **face** of the 3-dimensional cube

  $n = 3$, $|\mathcal{S}| = 4$, implies $|E| \leq \frac{1}{2} \cdot 4 \cdot \log 4 = 4$
- E_i — edges of E in direction i $\qquad (E = \dot\bigcup_{i \in [n]} E_i)$
- $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$ and $X_{-i} = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Lemma 6

$H(X_i | X_{-i}) = \frac{2|E_i|}{|\mathcal{S}|}$

Proving Thm **??**:

$$\begin{aligned}\log |\mathcal{S}| &= H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + \ldots + H(X_n|X_1, X_2, \ldots, X_{n-1}) \\ &\geq H(X_1|X_{-1}) + H(X_2|X_{-2}) + \ldots H(X_n|X_{-n}) = \sum_i \frac{2\,|E_i|}{|\mathcal{S}|} = \frac{2\,|E|}{|\mathcal{S}|}. \quad \square\end{aligned}$$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon\ |u - v| = 1\}$ and E_1 contains edges of E in direction 1

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon\ |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0, \mathbf{y}), (1, \mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0,\mathbf{y}), (1,\mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$
- $|\mathcal{S}| = 2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0,\mathbf{y}), (1,\mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$
- $|\mathcal{S}| = 2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|$
- $|E_1| = \left|\mathcal{S}^e_{-1}\right|$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1,\ldots,X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x,\mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2,\ldots,n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0,\mathbf{y}),(1,\mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$
- $|\mathcal{S}| = 2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|$
- $|E_1| = \left|\mathcal{S}^e_{-1}\right|$
- $H(X_1|X_{-1}) = \Pr\left[X_{-1} \in \mathcal{S}^e_{-1}\right] \cdot 1$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}_{-1}^{e} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0,\mathbf{y}), (1,\mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}_{-1}^{\neg e} = \mathcal{S}_{-1} \setminus \mathcal{S}_{-1}^{e}$
- $|\mathcal{S}| = 2\left|\mathcal{S}_{-1}^{e}\right| + \left|\mathcal{S}_{-1}^{\neg e}\right|$
- $|E_1| = \left|\mathcal{S}_{-1}^{e}\right|$
- $H(X_1|X_{-1}) = \Pr\left[X_{-1} \in \mathcal{S}_{-1}^{e}\right] \cdot 1$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0, \mathbf{y}), (1, \mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$
- $|\mathcal{S}| = 2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|$
- $|E_1| = \left|\mathcal{S}^e_{-1}\right|$
- $H(X_1|X_{-1}) = \Pr\left[X_{-1} \in \mathcal{S}^e_{-1}\right] \cdot 1 = \frac{2\left|\mathcal{S}^e_{-1}\right|}{2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|}$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon\ |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x, \mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0,\mathbf{y}), (1,\mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$
- $|\mathcal{S}| = 2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|$
- $|E_1| = \left|\mathcal{S}^e_{-1}\right|$
- $H(X_1|X_{-1}) = \Pr\left[X_{-1} \in \mathcal{S}^e_{-1}\right] \cdot 1 = \frac{2\left|\mathcal{S}^e_{-1}\right|}{2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|} = \frac{2|E_1|}{|\mathcal{S}|}$ $\square$

Proving Lemma ??

We prove $H(X_1|X_{-1}) = \frac{2|E_1|}{|\mathcal{S}|}$

- $\mathcal{S} \subseteq \{0,1\}^n$; $X = (X_1, \ldots, X_n) \leftarrow \mathcal{S}$
- $E = \{(u,v) \in \mathcal{S} \colon |u - v| = 1\}$ and E_1 contains edges of E in direction 1
- $\mathcal{S}_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon \exists x \in \{0,1\} \text{ s.t. } (x,\mathbf{y}) \in \mathcal{S}\}$.
  ($\mathcal{S}$ projected on $(2, \ldots, n)$)
- $\mathcal{S}^e_{-1} := \{\mathbf{y} \in \{0,1\}^{n-1} \colon (0,\mathbf{y}),(1,\mathbf{y}) \in \mathcal{S}\}$ and $\mathcal{S}^{\neg e}_{-1} = \mathcal{S}_{-1} \setminus \mathcal{S}^e_{-1}$
- $|\mathcal{S}| = 2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|$
- $|E_1| = \left|\mathcal{S}^e_{-1}\right|$
- $H(X_1|X_{-1}) = \Pr\left[X_{-1} \in \mathcal{S}^e_{-1}\right] \cdot 1 = \frac{2\left|\mathcal{S}^e_{-1}\right|}{2\left|\mathcal{S}^e_{-1}\right| + \left|\mathcal{S}^{\neg e}_{-1}\right|} = \frac{2|E_1|}{|\mathcal{S}|}$ $\square$