# Application of Information Theory, Lecture 5

# Channel Capacity and Isoperimetric Inequality

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# Part I

# Channel Capacity

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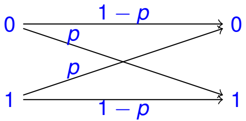

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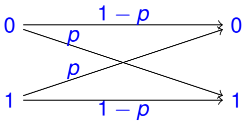


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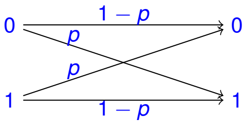


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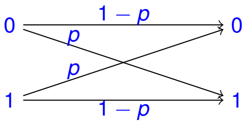


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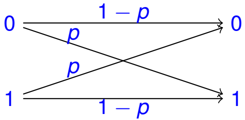


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- Before Shannon it was believed that very small error rate requires very small transmission rate.

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- $C_p$ might be $0$ (i.e., for $p = \frac{1}{2}$)
- A revolution in EE and the whole world

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- ECC Vs compression

# Shannon's theorem

**Theorem 1**

$\forall p \quad \exists C_p,\ s.t.\ \forall \varepsilon > 0 \quad \exists m_\varepsilon,\ s.t.\ \forall m \geq m_\varepsilon$ *and* $n \geq m(\frac{1}{C_p} + \varepsilon)$,
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- We prove a weaker variant that holds w.h.p. over $\mathbf{x} \leftarrow \{0,1\}^m$

# Hamming distance

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- $|\mathbf{y} - \mathbf{y}'| = |\mathbf{y} \oplus \mathbf{y}'|$ — Hamming distance of $\mathbf{y}$ from $\mathbf{y}'$; # of places differ.

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- So it all boils down to finding $f$ s.t.
  $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m; y = f(\mathbf{x}) \oplus Z}[\forall \mathbf{x}' \in \{0,1\}^m \setminus \{\mathbf{x}\} \colon |f(\mathbf{x}) - y| < |f(\mathbf{x}') - y|] \geq 1 - \varepsilon$

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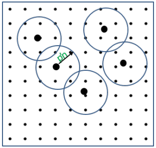

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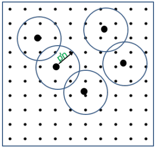


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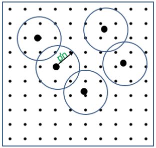

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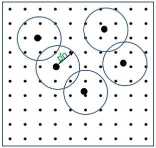


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- Hence, for $m > m_\varepsilon := \max\{m', n'\}$ and $n > m(\frac{1}{C_p} + \varepsilon)$, it holds that $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}] \leq \alpha_n + \beta_{m,n} \leq \varepsilon$. $\square$

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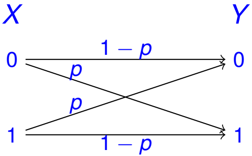

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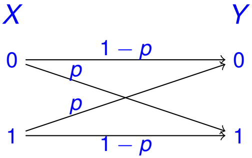


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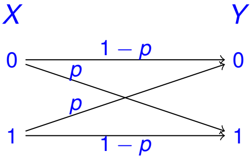


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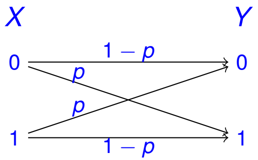


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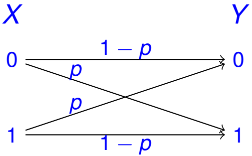


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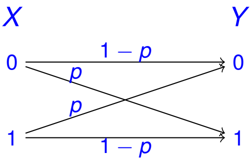


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For $p \in [0, \frac{1}{2}]$ and $n \in \mathbb{N}$: it holds that $\sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{n \cdot h(p)}$

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Very useful estimation. Weaker variants follows by AEP or Stirling,

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- Alternative proof

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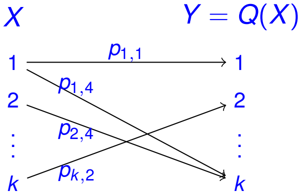

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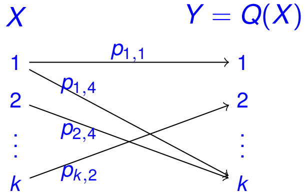

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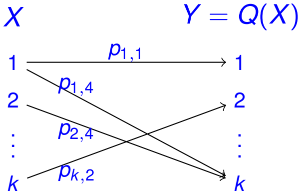


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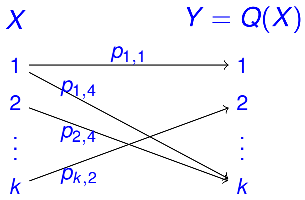


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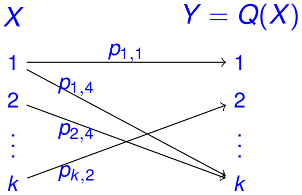


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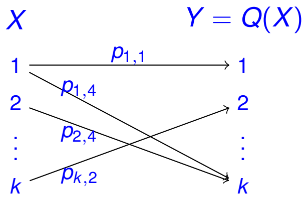


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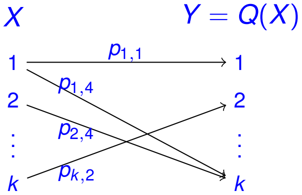


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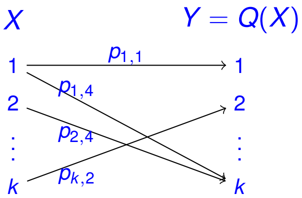


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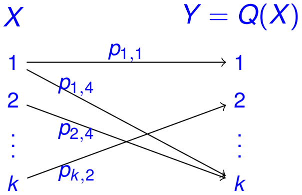


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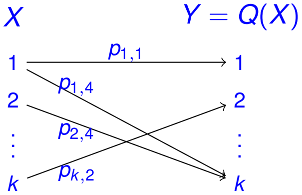


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- Proof: similar lines to the binary case, but more subtle distribution for $f$

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Part II

# Combinatorial Applications

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- Hence, $X$ is not determined by $Y$

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- Very useful inequality. No Chernoff, just IT

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