

Application of Information Theory, Lecture 4

Asymptotic Equipartition Property, Data Compression & Gambling

Handout Mode

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Part I

Asymptotic Equipartition Theorem

Entropy as # of bits to describe random variable

- ▶ In what sense is it true?
- ▶ Let $k \leq n \in \mathbb{N}$ and $p = \frac{k}{n}$

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

$$\approx \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{k}{e}\right)^k \cdot \left(\frac{n-k}{e}\right)^{n-k}}$$

(Stirling approx: $m! \approx \left(\frac{m}{e}\right)^m$)

$$= \frac{n^n}{k^k (n-k)^{n-k}}$$

$$= \left(\frac{k}{n}\right)^{-k} \cdot \left(\frac{n-k}{n}\right)^{-(n-k)}$$

$$= p^{-pn} \cdot (1-p)^{-(1-p)n}$$

$$= 2^{-p \log(p)n} \cdot 2^{-(1-p) \log(1-p)n}$$

$$= 2^{n(-p \log p - (1-p) \log(1-p))}$$

$$= 2^{n \cdot h(p)}$$

- ▶ It takes about $n \cdot h(k/n)$ bits to describe a string of k zeros in $\{0, 1\}^n$.

Entropy as # of bits to describe random variable, cont.

- ▶ Let X_1, \dots, X_n be iid $\sim (p, 1 - p)$
- ▶ w.h.p. about pn of X_i 's are zeros (law of large numbers)
- ▶ Assume that exactly $k = pn$ of x_i 's are zeros
- ▶ There are $\binom{n}{k} \approx 2^{nh(p)}$ possibilities.
- ▶ We need $nh(p)$ bits to tell in which possibility we are.
- ▶ In other words: it takes about $nh(p)$ bits to describe $X = X_1, \dots, X_n$, which is $H(X)$!
- ▶ Describing X :
 - ▶ Send k — the number of zeros in X . ($\log n$ bits)
 - ▶ Send the index of X in the strings of k zeroes. (about $H(X)$ bits)
- ▶ Over all it takes about $H(X)$ bits

Entropy as # of bits to describe random variable, cont..

- ▶ Let k_1, \dots, k_ℓ with $\sum k_i = n$, and let $p_i = \frac{k_i}{n}$
- ▶ $\binom{n}{k_1, \dots, k_\ell} \approx 2^{n \cdot H(p_1, \dots, p_\ell)}$
- ▶ Let X_1, \dots, X_n be iid $\sim (p_1, \dots, p_\ell)$, and $n \gg \ell$
- ▶ w.h.p. we can describe $X = X_1, \dots, X_n$ using $H(X) = n \cdot H(p_1, \dots, p_\ell)$ bits.
 - ▶ $\forall j \in [\ell]$: Send the number of X_i 's that get the value j . ($\ell \cdot \log n$ bits)
 - ▶ Send the index of X among all strings of this characterization.
(about $n \cdot H(p_1, \dots, p_\ell) = H(X)$ bits)
- ▶ Over all it takes about $H(X)$ bits

Asymptotic equipartition theorem (AEP)

- ▶ A sequence $\{Z_i\}_{i=1}^{\infty}$ of rv's converges in **probability** to μ (denoted $Z_n \xrightarrow{P} \mu$), if $\lim_{n \rightarrow \infty} \Pr[|Z_n - \mu| > \varepsilon] = 0$ for all $\varepsilon > 0$
- ▶ Let X_1, \dots, X_n be iid $\sim p$ and let $\mu = E X_1$.
- ▶ *Weak law of large numbers:* $\frac{1}{n} \cdot \sum_{i=1}^n X_i \xrightarrow{P} \mu$
- ▶ Let $\mathbf{p}(x_1, \dots, x_n) = \prod_i p(x_i)$ and consider the rv $\mathbf{p}(X_1, \dots, X_n)$.
- ▶ Example $p = (.1, .9)$.

$$\text{▶ } (X_1, X_2) = \begin{cases} 00, & .01 \\ 01, & .09 \\ 10, & .09 \\ 11, & .81 \end{cases} \text{ and } \mathbf{p}(X_1, X_2) = \begin{cases} .01, & .01 \\ .09, & .18 \\ .81, & .81 \end{cases}$$

- ▶ $\log \mathbf{p}(x_1, \dots, x_n) = \log \prod_i p(x_i) = \sum_i \log p(x_i)$
- ▶ Hence, $E_{X_1, \dots, X_n} [-\log \mathbf{p}(X_1, \dots, X_n)] = -\sum_i E [\log p(X_i)] = H(X_1, \dots, X_n)$
- ▶ We will show that w.h.p. $-\log \mathbf{p}(X_1, \dots, X_n)$ is **close** to its expectation

Asymptotic equipartition theorem (AEP), cont.

- ▶ By weak law of large numbers:

$$\frac{1}{n} \log \mathbf{p}(X_1, \dots, X_n) = \frac{1}{n} \sum_i \log p(X_i) \xrightarrow{P} \mathbb{E} \log p(X_1) = -H(X_1)$$

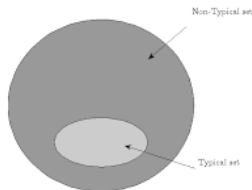
- ▶ That is, $\lim_{n \rightarrow \infty} \Pr \left[\left| -\frac{1}{n} \log(\mathbf{p}(X_1, \dots, X_n)) - H(X_1) \right| > \varepsilon \right] = 0$, for any $\varepsilon > 0$

Hence, $\forall \varepsilon > 0$:

- ▶ $\lim_{n \rightarrow \infty} \Pr \left[H(X_1) - \varepsilon \leq -\frac{1}{n} \log(\mathbf{p}(X_1, \dots, X_n)) \leq H(X_1) + \varepsilon \right] = 1$
- ▶ $\lim_{n \rightarrow \infty} \Pr \left[2^{-H(X_1, \dots, X_n) - \varepsilon n} \leq \mathbf{p}(X_1, \dots, X_n) \leq 2^{-H(X_1, \dots, X_n) + \varepsilon n} \right] = 1$
- ▶ What does it mean?

Typical values

- ▶ Let X_1, \dots, X_n be iid $\sim p$
- ▶ For $n \in \mathbb{N}$ and $\varepsilon > 0$, the **typical sequence** $A_{n,\varepsilon} := \{(a_1, \dots, a_n) : 2^{-n(H(X_1)+\varepsilon)} \leq \Pr[X_1 = a_1 \wedge \dots \wedge X_n = a_n] \leq 2^{-n(H(X_1)-\varepsilon)}\}$
- ▶ $\frac{1}{2} \cdot 2^{n(H(X_1)-\varepsilon)} \leq |A_{n,\varepsilon}| \leq 2^{n(H(X_1)+\varepsilon)}$ (on board)
(for the lower bound we assume $\Pr[(X_1, \dots, X_n) \in A_{n,\varepsilon}] \geq \frac{1}{2}$)
- ▶ Hence, $n(H(X_1) - \varepsilon) - 1 \leq \log |A_{n,\varepsilon}| \leq n(H(X_1) + \varepsilon)$
- ▶ $\lim_{n \rightarrow \infty} \Pr[(X_1, \dots, X_n) \notin A_{n,\varepsilon}] = 0$
- ▶ So roughly, (X_1, \dots, X_n) is close to **uniform** over $A_{n,\varepsilon}$ and $|A_{n,\varepsilon}| \approx 2^{n(H(X_1))}$
- ▶ $A_{n,\varepsilon}$ might be tiny, but still happens, with respect to X , with high probability.



Part II

Data Compression

Data compression

- ▶ Let X_1, \dots, X_n be iid $\sim p$
- ▶ To describe (X_1, \dots, X_n) with negligible error, we need $H(X_1, \dots, X_n) + \varepsilon n$ bits, for any $\varepsilon > 0$ and $n \rightarrow \infty$
- ▶ So $H(X_1, \dots, X_n)$ is approximately the number of bits it takes to describe X_1, \dots, X_n

Lower bound

- ▶ Encoding function $f: \{0, 1\}^n \mapsto \{0, 1\}^m$ and decoding function $g: \{0, 1\}^m \mapsto \{0, 1\}^n$ (typically $m < n$)
- ▶ X rv over $\{0, 1\}^n$, $Y = f(X)$
- ▶ $X \rightarrow Y \rightarrow g(Y)$
- ▶ Assume $\Pr[g(Y) = X] \geq 1 - \varepsilon$ — g restores X w.h.p.
- ▶ By Fano, $H(X | Y)$ is small: $H(X|Y) \leq h(\varepsilon) + \varepsilon \log(2^n - 1) \leq \varepsilon n + 1$
- ▶ Hence,
$$H(X) - \varepsilon n - 1 \leq H(X) - H(X|Y) = I(X; Y) = H(Y) - H(Y|X) \leq H(Y) \leq m$$
- ▶ Thus, $m \geq H(X) - \varepsilon n - 1$
- ▶ In case $H(X) = nH(X_1)$, then $m \geq n(H(X_1) - \varepsilon) - 1$

Codes

Definition 1 (Codes)

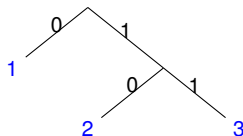
A code for random variable X over \mathcal{X} is a mapping $C: \mathcal{X} \mapsto \Sigma^*$.

- ▶ We call $\{C(x): x \in \mathcal{X}\}$ the **codewords** of C (with respect to X)
- ▶ C is **nonsingular**, if it is **injective** over \mathcal{X} .
- ▶ For $\mathbf{x} = (x_1, x_2, \dots, x_k) \in \mathcal{X}^k$, let $C(\mathbf{x}) = C(x_1)C(x_2) \dots C(x_k)$
- ▶ C is **uniquely decodable**, if it is nonsingular over \mathcal{X}^*
- ▶ Uniquely decodable \implies nonsingular (other direction is not true)
- ▶ A code is **prefix code** (or instantaneous code), if no codeword is a prefix of another codeword
- ▶ Prefix code \implies uniquely decodable
- ▶ We focus on binary prefix codes ($\Sigma = \{0, 1\}$)

Examples

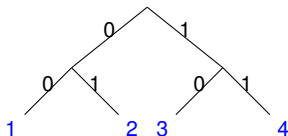
- ▶ $X \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ (i.e., $\Pr[X = i] = p_i$).
- ▶ We can use one bit to tell whether $X = 1$ or $X \in \{2, 3\}$, and another bit to tell whether $X = 2$ or $X = 3$
- ▶ The code

x	$C(x)$
1	0
2	10
3	11

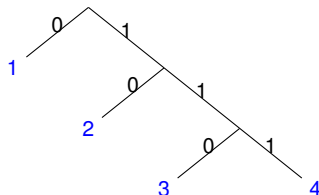


- ▶ Expected encoding length: $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = 1\frac{1}{2}$

- ▶ $X \sim (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$



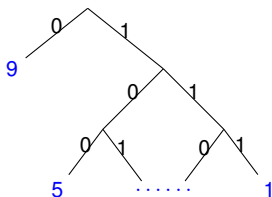
Or



- ▶ All are **prefix** codes: no codeword is a prefix of another codeword

Prefix codes

- ▶ Let $X \sim (p_1, \dots, p_m)$ (i.e., $\Pr[X = i] = p_i$)
- ▶ We want to place $\{1, \dots, m\}$ on the **leaves** of a binary tree T (not necessarily in order):



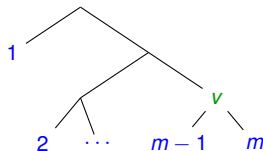
- ▶ Every symbol is encoded by the bits on the path leading to it.
- ▶ This yields a binary prefix code.
- ▶ Every prefix code can be uniquely represented as such a tree
- ▶ We identify prefix codes with their trees.
- ▶ Encoding/decoding is clear (and highly efficient)

Code length

- ▶ For a prefix code C over \mathcal{X} , let $\ell_C(x) = |C(x)|$ (i.e., # of bits in x)
- ▶ Since C a prefix code, $\ell_C(x)$ is the depth of x in the code tree of C
- ▶ $L_X(C) := E[\ell_C(X)]$ is the **average code length** (of C with respect to X)
- ▶ We sometimes speak about $L_X(T)$ where T is the tree representation of C .
- ▶ When clear from the context we omit the subscripts X and C
- ▶ $L(X)$ is the (average) code length of the **optimal** prefix code for X
- ▶ How small can $L(X)$ be?
- ▶ It turns out that $H(X) \leq L(X) \leq H(X) + 1$!

Huffman code

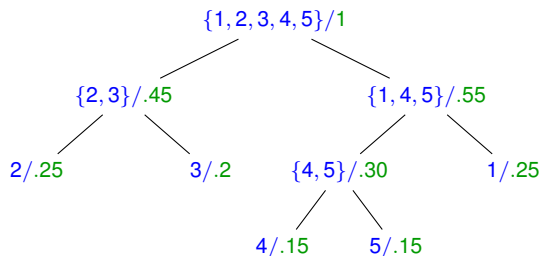
- ▶ Story...
- ▶ Suppose T is optimal tree for $X \sim (p_1, \dots, p_m)$ (wlg. $p_1 \geq p_2 \geq \dots \geq p_m$)
- ▶ Let v be (one of) the deepest internal vertex in T
- ▶ wlg. the descendants of v are $m-1$ and m
(o/w, we can change it to, w/o increasing $L_X(T)$)



- ▶ T' – generated from T by replacing the sub-tree rooted in v with the symbol $\{m-1, m\}$
- ▶ $L_X(T) = L_{X'}(T') + (p_{m-1} + p_m) \cdot 1$, for $X' \sim (p_1, \dots, p_{m-1} + p_m)$
- ▶ T' is optimal tree for X' .
(o/w, we can improve T' and hence improve T)
- ▶ Huffman algorithm:
 1. Sort p_1, \dots, p_m
 2. Find (via recursions) the best tree for $(p_1, \dots, p_{m-1} + p_m)$
 3. Replace leaf $\{m-1, m\}$ with the depth-one tree of leaves $m-1, m$
- ▶ Huffman is an optimal binary prefix code. Proof: ?

Huffman code, example

- ▶ $X \sim (.25, .25, .2, .15, .15)$



- ▶ On board...

Kraft inequality

Theorem 2 (Kraft inequality)

Let \mathcal{C} be (binary) prefix code. Then its codewords lengths ℓ_1, \dots, ℓ_m satisfy

$$\sum_{i \in [m]} 2^{-\ell_i} \leq 1.$$

Conversely, for any ℓ_1, \dots, ℓ_m satisfying the inequality, there exists a prefix code with these lengths.

Theorem extends to the infinite case.

First part:

- ▶ Denote the i 'th codeword by i
- ▶ Let Y the leaf reached by a uniform random walk on the code tree, taking the value \perp if reaches *empty leaf*.
- ▶ $\Pr[Y = i] = 2^{-\ell_i}$.
- ▶ Hence, $\sum_{i \in [m]} 2^{-\ell_i} = \sum_i \Pr[Y = i] \leq 1$

Kraft inequality. cont.

- ▶ Let $\ell_1 \leq \dots \leq \ell_m$ be such that $\sum_{i \in [m]} 2^{-\ell_i} \leq 1$
 - ▶ We construct a tree of m codewords with the above lengths.
 1. Start with a full binary tree of depth ℓ_m
 2. At step i , assign an **unassigned** node of depth ℓ_i to the i 'th codeword, and remove node's descendants from the tree.
 - ▶ If completed, the algorithm yields the desired code.
 - ▶ Claim: the algorithm always completes.
 - ▶ $\mathcal{S}(\ell, j)$ — nodes of depth $\ell \geq \ell_j$ that the assignment of node to the j 'th codeword made **unavailable**.
 - ▶ $|\mathcal{S}(\ell, j)| = 2^{\ell - \ell_j}$
 - ▶ $\mathcal{Z}(i) := \bigcup_{j=1}^{i-1} \mathcal{S}(\ell_i, j)$ — nodes of depth ℓ_i unavailable at the **beginning** of step i
 - ▶ $|\mathcal{Z}(i)| \cdot 2^{-\ell_i} = (\sum_{j \in [i-1]} 2^{\ell_i - \ell_j}) \cdot 2^{-\ell_i} = \sum_{j \in [i-1]} 2^{-\ell_j} < 1$
- $\implies |\mathcal{Z}(i)| < 2^{\ell_i}$
- \implies At beginning of step i exists an available depth- ℓ_i node.

Optimal code

Theorem 3

$H(X) \leq L(X) < H(X) + 1$ for any rv X .

Proving lower bound:

- ▶ Let C be a binary prefix code for $X \sim p = (p_1, \dots, p_m)$, and let $\ell_i = |C(i)|$. (As usual, we assume wlg. that $p_i = \Pr[X = i]$).
- ▶ Let $q_1 = 2^{-\ell_1}, \dots, q_m = 2^{-\ell_m}$. By Kraft. $\sum q_i \leq 1$
- ▶ By Jensen (HW 1) $-\sum_{i \in [m]} p_i \log p_i \leq -\sum p_i \log q_i = \sum_i p_i \ell_i = L_X(C)$
- ▶ Hence $H(X) \leq L_X(C)$.

Proving upper bound:

- ▶ $\ell_i = \lceil -\log p_i \rceil$.
- ▶ $\sum_{i \in [m]} 2^{-\ell_i} \leq \sum_{i \in [m]} p_i \leq 1$
- ▶ By Kraft, \exists boolean prefix code C over \mathcal{X} with $|C(i)| = \ell_i$
- ▶ $L_X(C) = \sum_i p_i \ell_i < \sum_i p_i (-\log p_i + 1) = -\sum_i p_i \log p_i + \sum_i p_i = H(X) + 1$

Discrete distribution generation

Definition 4

Algorithm G generates the rv $X \sim \{p_1, \dots, p_m\}$ if the following holds: in each step, G either stops or flips a coin $\sim (q_i, 1 - q_i)$.^a After it stop, G outputs a value in \mathbb{N} . The probability that G outputs i is p_i .

^a q_i can be a function of previous coins outcome.

Proposition 5

Let X be rv, and let $g(X)$ be the expected number of coins used by its best generating algorithm. Then $H(X) \leq g(X) < H(X) + 1$. If each p_i is a power of 2 (i.e., 2^{-k} for some $k \in \mathbb{Z}$), then $g(X) = H(X)$.

Proof: ? HW

Proposition 6 (proof omitted)

Let X be a rv, and let $g_b(X)$ be the expected number of coins used by its best generating algorithm that only flips uniform coins. Then $H(X) \leq g_b(X) \leq H(X) + 2$.

Part III

Gambling

Horse racing

- ▶ Horses $\{1, \dots, m\}$
- ▶ If horse i wins, gambler get payoff o_i per 1 \$
- ▶ Gambler strategy $\mathbf{b} = (b_1, \dots, b_m)$ — b_i is the fraction of gambler wealth invested in horse i ($b_i \geq 0$ and $\sum_i b_i = 1$)
- ▶ If horse i wins, gamblers' wealth is multiplied by $b_i o_i$
- ▶ Let $X \sim \mathbf{p} = (p_1, \dots, p_m)$ be the outcome of a random race.
- ▶ $S(X) := \mathbf{b}(X)\mathbf{o}(X)$ is the factor in which gamblers' wealth is multiplied in a single race (letting $\mathbf{z}(i) = \mathbf{z}_i$)
- ▶ We are interested in $S_n := \prod_{i=1}^n S(X_i)$, where X_i 's are iid $\sim p$

Doubling rate

For gambling strategy $\mathbf{b} = (b_1, \dots, b_m)$, and race outcome distribution $\mathbf{p} = (p_1, \dots, p_m)$, $S_n := \prod_{i=1}^n S(X_i) = \prod_{i=1}^n \mathbf{b}(X_i) \mathbf{o}(X_i)$, where X_i 's are iid $\sim \mathbf{p}$

Definition 7 (doubling rate)

The **doubling rate** is $W(\mathbf{b}, \mathbf{p}) = \sum_{i=1}^m p_i \log(b_i o_i)$

Theorem 8

For race outcome $\sim \mathbf{p}$ and gambling strategy \mathbf{b} , it holds that $S_n \xrightarrow{n} 2^{nW(\mathbf{b}, \mathbf{p})}$

Proof:

- ▶ fix \mathbf{p} and \mathbf{b} and let X_1, \dots, X_m be iid $\sim \mathbf{p}$
- ▶ $\log S(X_1), \dots, \log S(X_n)$ are iid
- ▶ By weak law of large numbers,

$$\frac{1}{n} \log S_n = \frac{1}{n} \sum_i \log(S(X_i)) \xrightarrow{n} \mathbb{E}(\log S(X_1)) = W(\mathbf{b}, \mathbf{p})$$

Maximal doubling rate

Theorem 9

Let $W^*(\mathbf{p}) = \max_{\mathbf{b}} W(\mathbf{b}, \mathbf{p})$, then $W^*(\mathbf{p}) = W(\mathbf{p}, \mathbf{p}) = \sum_i p_i \log o_i - H(\mathbf{p})$

Roughly, best strategy is to follow the distribution (ignoring the payoffs)!

$$\begin{aligned} W(\mathbf{b}, \mathbf{p}) &= \sum_{i=1}^m p_i \log(b_i o_i) \\ &= \sum_i p_i \log\left(\frac{b_i}{p_i} p_i o_i\right) \\ &= \sum_i p_i \log o_i - H(\mathbf{p}) - \sum_i p_i \cdot \log \frac{b_i}{p_i} \\ &= \sum_i p_i \log o_i - H(\mathbf{p}) - D(\mathbf{p} \parallel \mathbf{b}) \\ &\leq \sum_i p_i \log o_i - H(\mathbf{b}) = W(\mathbf{p}, \mathbf{p}) \end{aligned}$$

where $D(\mathbf{p} \parallel \mathbf{b})$, the **relative entropy** from \mathbf{p} to \mathbf{b} , is known to be non-negative.

Gambling with side information

- ▶ Let $(X, Y) \sim p$ be the outcome of a race and a side information, and let \mathbf{o} be the race payoffs.
- ▶ $W^*(X) := \max_{\mathbf{b}} \sum_x p_X(x) (\mathbf{b}(x) \mathbf{o}(x))$
The best strategy for (X, \mathbf{o})
- ▶ $W^*(X|Y) := \max_{\mathbf{b}} \sum_{x,y} p(x, y) \log(\mathbf{b}_y(x) \mathbf{o}(x))$
The best strategy for (X, \mathbf{o}) , when Y is known
- ▶ $\Delta W := W^*(X|Y) - W^*(X)$

Theorem 10

$$\Delta W = I(X; Y).$$

- ▶ $W^*(X) = \sum_x p_X(x) \log \mathbf{o}(x) - H(X)$
- ▶ $W^*(X|Y) = \mathbb{E}_{y \leftarrow Y} [\sum_x p_{X|Y}(x|y) \log \mathbf{o}(x) - H(X|_{Y=y})] = \sum p_X(x) \log \mathbf{o}(x) - H(X|Y)$
- ▶ Hence, $\Delta W = H(X) - H(X|Y) = I(X; Y)$. □