Information Theory, Fall 2014 Iftach Haitner

Problem set 3

December 9, 2014 Due: December 23

- Please submit the handout in class, or email the grader.
- Write clearly and shortly using sub-claims if needed. The emphasize in most questions is on the proofs (no much point is writing a “solution" w/o proving its correctness)
- For Latex users, a solution example can be found in the course web site.
- It is allowed to work in (small) groups, but please write the id list of your partners in the solution file, and each student should write his solution by *himself* (joint effort is only allowed in the “thinking phase")

1. Let $Q\colon \{0,1\} \mapsto \{0,1\} \cup \{\perp\}$ be the random function with $\Pr\left[Q(x) = \perp\right] = p$ and $\Pr\left[Q(x) = x\right] = 1-p$ for any $x \in \{0,1\}$. Find the capacity of the channel described by Q?

   That is, find the right value of C_p for which the natural adjustment of Shannon's theorem (Theorem 1 in lecture 5) for the noise model described by Q (i.e., Q is applied independently to each transmitted bit) can be proven.

2. Let G be the graph with set of nodes $\{0,1,2\}^n$, where two nodes $(x,y) \in \{0,1,2\}^n$ are connected by an edge iff they differ in exactly one coordinate. State and prove an isoperimetric inequality for the graph G. (Similar to the isoperimetric inequality for the hyper-cube we did in class).

3. Prove or give a counter example: For every rv's X_1, X_2, X_3, X_4:

$$\begin{aligned} &H(X_1,X_2,X_3) + H(X_2,X_3,X_4) + H(X_3,X_4,X_1) + H(X_4,X_1,X_2) \\ &\leq \frac{3}{2}\left[H(X_1,X_2) + H(X_2,X_3) + H(X_3,X_4) + H(X_4,X_1)\right] \end{aligned}$$

4. Show that $\mathrm{SD}(p,q) = \max_{\mathcal{S} \subseteq [m]} \left(\sum_{i \in \mathcal{S}} p_i - \sum_{i \in \mathcal{S}} q_i\right)$ for any two distributions p,q over $[m]$.

5. Relative entropy is not symmetric: given two distributions p,q such that $D(p\|q) \neq D(q\|p)$, and $D(p\|q), D(q\|p) < \infty$.

6. Relative entropy does not obey the triangle inequality: give three distributions p_1, p_2, p_3 such that $D(p_1\|p_2) + D(p_2\|p_3) < D(p_1\|p_3)$

7. Relative entropy is non-negative: given two distributions p,q, show that $D(p\|q) \geq 0$, with equality only if $p = q$.

8. Does Theorem 1 in Lecture 7 hold for any prefix code C with $\mathrm{E}_{i \leftarrow q}\left[|C(i)|\right] \leq H(q) + 1$?

   (and not only for a code C with $C(x) = \left\lceil \log \frac{1}{q(x)} \right\rceil$, as stated in the theorem)