Application of Information Theory, Lecture 10

Hardcore Predicates

Iftach Haitner

Tel Aviv University.

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Part I

Motivation and Definition

Hardcore predicates

- Let $f \colon \{0,1\}^n \mapsto \{0,1\}^n$ be a “hard to invert" function, how unpredictable is x given $f(x)$

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- Parts of x might be (totally) predictable
- It turns out that there is an hardcore part in x.

Hardcore predicates, cont.

Definition 1 (hardcore predicates)

A predicate $b \colon \{0,1\}^n \mapsto \{0,1\}$ is (s,ε)-hardcore predicate of $f \colon \{0,1\}^n \mapsto \{0,1\}^n$, if $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{P}(f(x)) = b(x)] \leq \frac{1}{2} + \varepsilon$, for any s-size P.

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- Is there a generic hardcore predicate for all hard to invert functions?

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- Is there a generic hardcore predicate for all hard to invert functions?

  Let f be a function and let b be a predicate, then b is typically not a hard-core predicate of $g(x) = (f(x), b(x))$.

Part II

The Information Theoretic Settings

Some definitions

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- In both examples $\mathrm{H}_\infty(Z) = k$

2-universal families

Definition 2 (2-universal families)

A function family $\mathcal{G} = \{g \colon \mathcal{D} \mapsto \mathcal{R}\}$ is 2-universal, if $\forall\ x \neq x' \in \mathcal{D}$ it holds that $\Pr_{g \leftarrow \mathcal{G}}\left[g(x) = g(x')\right] = \frac{1}{|\mathcal{R}|}$.

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Example: $\mathcal{D} = \{0,1\}^n$, $\mathcal{R} = \{0,1\}^m$ and $\mathcal{G} = \{A \in \{0,1\}^{m \times n}\}$ with $A(x) = A \times x \bmod 2$.

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Lemma 3 (leftover hash lemma)

Let X be a rv over $\{0,1\}^n$ with $\mathrm{H}_2(X) \geq k$ let $\mathcal{G} = \{g \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ be 2-universal and let $G \leftarrow \mathcal{G}$. Then $\mathrm{SD}((G, G(X)), (G, \sim \{0,1\}^m)) \leq \frac{1}{2} \cdot 2^{(m-k)/2}$.

Hardcore predicate for regular functions

Lemma 4

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be 2^k-regular function, let $\mathcal{G} = \{g\colon \{0,1\}^n \mapsto \{0,1\}\}$ be 2-universal and let $v\colon \{0,1\}^n \times \mathcal{G} \mapsto \{0,1\}^n \times \mathcal{G}$ be defined by $v(x,g) = (f(x),g)$.
Then $b(x,g) = g(x)$ is $(\infty, 2^{-(k-1)/2})$ hardcore-predicated of v.

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Then $b(x,g) = g(x)$ *is* $(\infty, 2^{-(k-1)/2})$ *hardcore-predicated of* v.

- b is an hardcore predicate of v (not of f)

Proving Lemma 4

Claim 5

$\mathrm{SD}\left((f(X), G, G(X)), (f(X), G, U)\right) \leq 2^{-(k-1)/2}$,
for $G \leftarrow \mathcal{G}$, $X \leftarrow \{0,1\}^n$ and $U \leftarrow \{0,1\}$.

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Lemma 6 (predicting to distinguishing)

Let Y, Z be rvs over $\{0,1\}^* \times \{0,1\}$ and let P be an algorithm with $\Pr\left[\mathsf{P}(Y) = Z\right] \geq \frac{1}{2} + \varepsilon$. Then $\exists$ algorithm D, with essentially the same complexity as P, with $\Pr\left[\mathsf{D}(Y,Z) = 1\right] - \Pr\left[\mathsf{D}(Y,U) = 1\right] \geq \varepsilon$.

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Corollary 7

If $\mathrm{SD}((Y, Z), (Y, U)) < \varepsilon$, then $\Pr[\mathsf{P}(Y) = Z] < \frac{1}{2} + \varepsilon$ for any predictor P.

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Since $\text{H}_\infty(X_y) = k$ for every $y \in Im(f)$, the leftover hash lemma yields that

$$
\begin{aligned}
\text{SD}((G, G(X_y)), (G, U)) &\leq \frac{1}{2} \cdot 2^{(1 - \text{H}_\infty(X_y)))} \\
&= 2^{(-k-1)/2}. \square
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Part III

The Computational Settings

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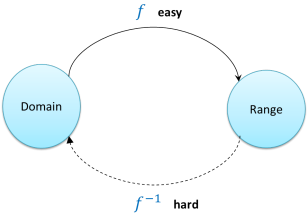

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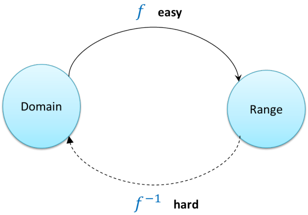

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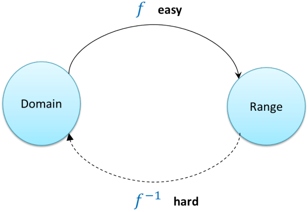

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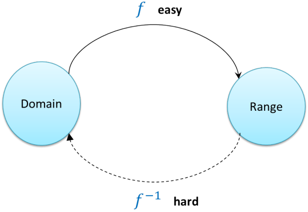

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- Hidden in (almost) any cryptographic primitive: necessary for "cryptography"
- Sufficient for many cryptographic primitives

One-way functions, cont.

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A poly-time $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ is (s,ε)-one-way, if $\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{Inv}(f(x)) \in f^{-1}(f(x))\right] = \varepsilon$ for any s-size Inv.

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- But does any one-way function has an hardcore predicate?
- Such hardcore predicates have many cryptographic applications
- f is injective and not one-way $\implies$ f has no hardcore predicate.

Direct product predicate

Direct product predicate

Theorem 9

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$*, define* $g(x,i) = (f(x), i)$ *and* $b(x,i) = x_i$*. Assuming* f *is* $(s, \frac{1}{2})$*-one way, then* b *is* $(\frac{s}{n}, \frac{1}{2} - \frac{1}{2n})$*-hardcore predicate of* g*.*

Direct product predicate

Theorem 9

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$*, define* $g(x,i) = (f(x),i)$ *and* $b(x,i) = x_i$*. Assuming* f *is* $(s,\frac{1}{2})$*-one way, then* b *is* $(\frac{s}{n},\frac{1}{2}-\frac{1}{2n})$*-hardcore predicate of* g*.*

Namely, $\Pr_{x\leftarrow\{0,1\}^n, i\leftarrow[n]}\left[\mathsf{P}(f(x),i) = x_i\right] \leq 1 - \frac{1}{2n}$ for any $\frac{s}{n}$-size P.

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3. Construction is "inefficient"

The Goldreich-Levin predicate

For $x, r \in \{0,1\}^n$, let $\langle x, r\rangle_2 := (\sum_{i=1}^{n} x_i \cdot r_i) \bmod 2 = \bigoplus_{i=1}^{n} x_i \cdot r_i$.

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- The proof does not rely on the fact that f is efficiently computable.

Focusing on a good set

Claim 11

There exists set $\mathcal{S} \subseteq \{0,1\}^n$ with

1. $\frac{|\mathcal{S}|}{2^n} \geq \frac{\delta}{2}$, and
2. $\Pr\left[\mathsf{P}(f(x), R) = b(x, R)\right]\right] \geq \frac{1}{2} + \frac{\delta}{2}, \qquad \forall x \in S$.

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&\leq \left(\frac{1}{2} + \frac{\delta}{2}\right) + \Pr[X \in \mathcal{S}]. \qquad \square
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We conclude the theorem's proof showing that there exists a $\frac{n^2}{\delta^2}$-size Inv with

$$\Pr\left[\mathsf{Inv}(f(x)) = x\right] \in \Omega(\delta^2/n)$$

for every $x \in \mathcal{S}$.

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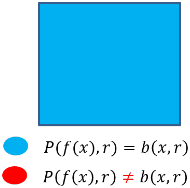
for every $x \in \mathcal{S}$. In the following we fix $x \in \mathcal{S}$.

The perfect case

$$\Pr[\mathsf{P}(f(x), R) = b(x, R)] = 1$$



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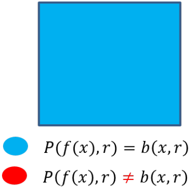

In particular, $\mathsf{P}(f(x), e^i) = b(x, e^i)$ for every $i \in [n]$, for $e^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})$.

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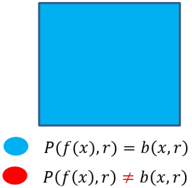

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Hence, $x_i = \langle x, e^i \rangle_2$

The perfect case

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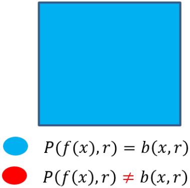

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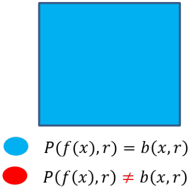

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Algorithm 12 (Inverter Inv on input $y \in \operatorname{Im}(f)$)

Return $(\mathsf{P}(y, e^1), \ldots, \mathsf{P}(y, e^n))$.

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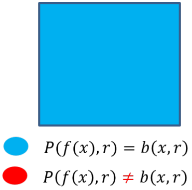

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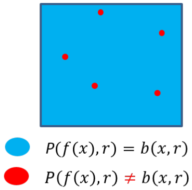
$\mathsf{Inv}(f(x)) = x$.

Easy case

$$\Pr\left[\mathrm{P}(f(x), R) = b(x, R)\right] \geq 1 - \frac{1}{4n}$$



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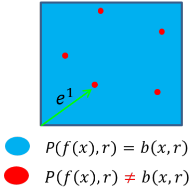

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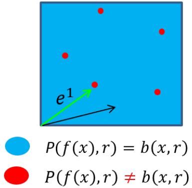

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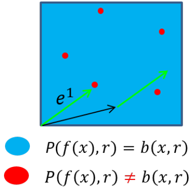

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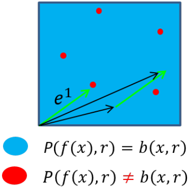

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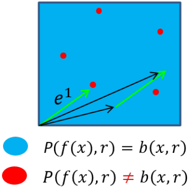

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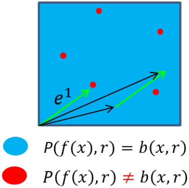

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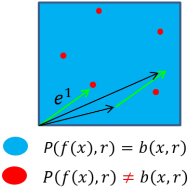

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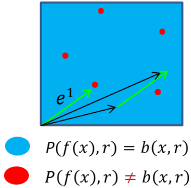

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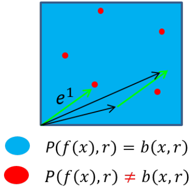

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Algorithm 14 (Inverter Inv on input y)

Return $(\mathrm{P}(y, R) \oplus \mathrm{P}(y, R \oplus e^1)), \ldots, \mathrm{P}(y, R) \oplus \mathrm{P}(y, R \oplus e^n))$.

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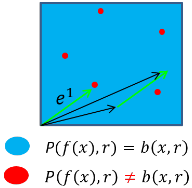

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Proving Fact 13

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$$b(x,y) \oplus b(x,w) = \left(\bigoplus_{i=1}^{n} x_i \cdot y_i\right) \oplus \left(\bigoplus_{i=1}^{n} x_i \cdot w_i\right)$$
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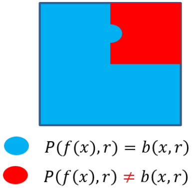
Intermediate Case

$$\Pr\left[\mathsf{P}(f(x), R) = b(x, R)\right] \geq \frac{3}{4} + \frac{\delta}{2}$$



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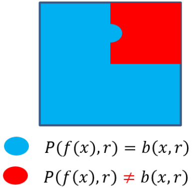

For any $i \in [n]$

$$\begin{aligned}
&\Pr[\mathsf{P}(f(x), R) \oplus \mathsf{P}(f(x), R \oplus e^i) = x_i] \\
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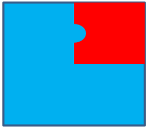

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Algorithm 15 ($\mathrm{Inv}(y)$)

For every $i \in [n]$:

1. Sample $r^1, \ldots, r^v \in \{0,1\}^n$ uniformly at random
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The following claim holds for “large enough" v.

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$\Pr\left[\left|\frac{\sum_{j=i}^{v} X^j}{v} - \mu\right| \geq \alpha\right] \leq 2 \cdot \exp(-2\alpha^2 v)$ *for every $\alpha > 0$.*

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- Hence, the proof follows for $v = \left\lceil \log(n) \cdot \frac{1}{2\delta^2} \right\rceil + 1$.

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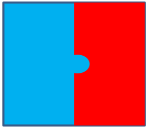

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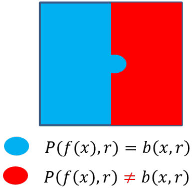

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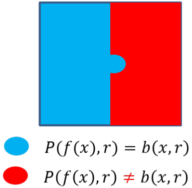

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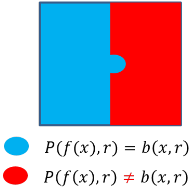

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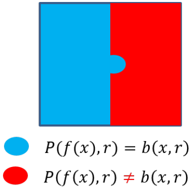

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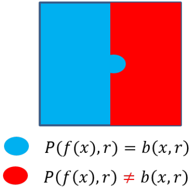

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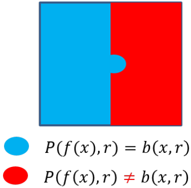

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- Solution: choose the samples in a correlated manner

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- Fix $i \in [n]$, and let $W^{\mathcal{L}}$ be 1 iff $\mathsf{P}(f(x), r^{\mathcal{L}} \oplus e^i) \oplus b(x, r^{\mathcal{L}}) = x_i$.

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4. For all $i \in [n]$, let $m_i = \mathrm{maj}_{\mathcal{L} \subseteq [\ell]}\{\mathsf{P}(f(x), r^{\mathcal{L}} \oplus e^i) \oplus b(x, r^{\mathcal{L}})\}$
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- Problem: the $W^{\mathcal{L}}$'s are dependent!

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Pairwise independence variables

Definition 20 (pairwise independent random variables)

A sequence of rv's $X^1, \ldots, X^v$ is pairwise independent, if $\forall i \neq j \in [v]$ and $\forall a, b$, it holds that $\Pr[X^i = a \wedge X^j = b] = \Pr[X^i = a] \cdot \Pr[X^j = b]$.

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Lemma 21 (Chebyshev's inequality)

Let $X^1, \ldots, X^v$ be pairwise-independent random variables with expectation μ and variance σ^2. Then, for every $\alpha > 0$: $\Pr\left[\left|\frac{\sum_{j=1}^{v} X^j}{v} - \mu\right| \geq \alpha\right] \leq \frac{\sigma^2}{\alpha^2 v}$.

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- Recalling that we guaranteed to work well on $\frac{\delta}{2}$ of the x's. We conclude that $\Pr[\mathrm{Inv}(f(x)) = x] \in \Theta(\delta^3/n)$.

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  Similar ideas allows to output $\log n$ “pseudorandom bits"

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  Let X be a rv with over $\{0,1\}^n$ with $H_\infty(X) \geq t$, and assume $SD((R, \langle R, X\rangle_2), (R, U)) > \alpha = 2^{-c \cdot t}$ for some universal $c > 0$.

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$\Longrightarrow$ (by GL) $\exists$ Inv that guesses X from nothing, with prob $\alpha^{O(1)} > 2^{-t}$

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  - Encoder $f\colon \{0,1\}^n \mapsto \{0,1\}^m$ and decoder g, such that for any $x \in \{0,1\}^n$ and c of hamming distance at most $(\frac{1}{2} - \delta)$ from $f(x)$: g examines $\mathrm{poly}(1/\delta)$ symbols of c and outputs a $\mathrm{poly}(1/\delta)$-size list that whp contains x

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- The difference comparing to Goldreich-Levin — no control over the R's.