Foundation of Cryptography, Lecture 11
Black-Box Impossibility Results

Handout Mode

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June 10, 2014

Motivating example: Basing Key-Agreement on OWFs

- Key-Agreement protocols (KA) can be based on the existence of TDP, RSA or discrete log assumptions, and ...
- We don't know how to base KA on the existence of OWFs/OWPs.
- Can we base KA on OWFs/OWPs?
- Proving unconditional negative result seems beyond reach.

  Assume RSA assumption holds.

  $\Longrightarrow$ key-agreement protocols exist.

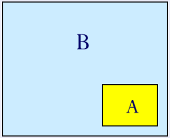
  $\Longrightarrow$ OWFs imply the existence of key-agreement protocols in a trivial sense.

(Fully) Black-box constructions

Definition 1 (A fully Black-box construction of *B* from *A*)

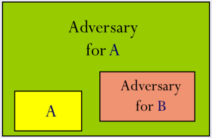
Black-box **construction**:

A oracle-aided PPT I such that I^{O} implements *B* for any algorithm O implementing *A*



Black-box **proof of security**:

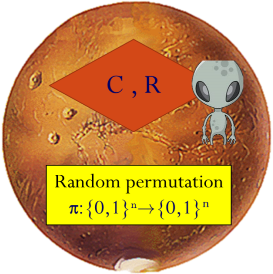
A oracle-aided PPT R such that $\mathsf{R}^{\mathsf{O},\mathsf{D}}$ breaks *B*, for any algorithms O implementing *A*, and D breaking *B*.



- Fully-black-box constructions relativize: hold relative to any oracle.
- Most constructions in cryptography are (fully) black-box, e.g., pseudorandom generator from OWF.
- Few “non black-box" techniques that apply in restricted settings (typically using ZK proofs)

Approach for proving BB impossibility result

Assume $\exists$ fully-BB construction (I, R) of a KA from OWP.



I^{π} is an efficient KA protocol and $\mathsf{R}^{\pi,\mathsf{D}}$ should invert π for any D breaking I^{π}.

Assume $\exists$ (even inefficient) algorithm D that breaks any efficient KA with oracle access to π, but is not useful for inverting π.

This yields a contradiction, implying that (I, R) does not exist.

Section 1

Random Permutations

Before we begin

- Let Π_n — set of all permutations over $\{0,1\}^n$, and let $N = 2^n$.
- $|\Pi_n| = N! \implies$ it takes $\log(N!)$ bits to describe $\pi \in \Pi_n$.
- How many bits it takes to describe $\mathcal{S} \subseteq \{0,1\}^n$ of (known) size a?
  There are $\binom{N}{a}$ such sets, so it takes $\log\binom{N}{a}$ bits to describe $\mathcal{S}$.
- For integer $b \geq a$: $a! \geq (\frac{a}{e})^a$ and $\binom{b}{a} := \frac{b!}{(b-a)!\cdot a!} \leq (\frac{eb}{a})^a$
- For $a = 2^{\alpha n}$-size set $\mathcal{S} \subseteq \{0,1\}^n$:
  - It takes at most $a \cdot ((1-\alpha)n + O(1))$ bits to describe .
  - It takes at least $a(\alpha n - O(1))$ bits to describe a permutation over $\mathcal{S}$.
  - The latter is larger in for $\alpha > \frac{1}{2}$.
- We are in the number of M-size oracle-circuits .

Claim 2

The number of M-size oracle-circuits mapping n-bit strings to n-bit strings, with oracle access to a function n-bit strings to n-bit strings, is at most $2^{2M+(M+1)n(\log(Mn+n)+1)}$.

Proof: ?

Random permutations are hard to invert

Theorem 3 (Gennarro-Tevisan, '01)

For any large enough $n \in \mathbb{N}$ and $2^{n/5}$-query circuit D,

$$\Pr_{\pi \leftarrow \Pi_n}\left[\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{D}(\pi(x)) = x] > 2^{-n/5}\right] \leq 2^{-2^{\frac{3}{5}n}/2}$$

- In words: Random permutations are (extremely) hard even for exponential-size circuits.
- Constants are somewhat arbitrary and non tight.
- By Claim 2 the number of $2^{n/5}$-size circuits (of the right form) is bounded by $2^{\tilde{O}(2^{n/5})}$. Thus Thm 3 yields that

$$\Pr_{\pi \leftarrow \Pi_n}\left[\exists\ 2^{n/5}\text{-size circuit } \mathsf{C} \text{ with } \Pr_{x \leftarrow \{0,1\}^n}[\mathsf{D}(\pi(x)) = x] > 2^{-n/5}\right] \leq 2^{-2^{n/2}}$$

- In words: Random permutations are (extremely) hard simultaneously, for all exponential-size circuits.

Proving GT theorem (Thm 3)

Lemma 4 (compression lemma)

For any q-query circuit D and $\varepsilon > 0$, exist algorithms Enc and Dec such that: Let $\pi \in \Pi_n$ be such that $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{D}^{\pi}(\pi(x)) = x] > \varepsilon$, then

- $\mathsf{Dec}(\mathsf{Enc}(\pi)) = \pi$
- $|\mathsf{Enc}(\pi)| \leq \log((2^n - a)!) + 2 \cdot \log\binom{N}{a}$, *for some* $a \geq \frac{\varepsilon N}{q+1}$.

- The description of π using $\mathsf{Enc}(\pi)$ “saves" $\log(a!) - \log\binom{N}{a}$ bits.
- Let D be a $2^{n/5}$-query circuit. Lemma 4 yields that the fraction of $\pi \in \Pi_n$ with $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{D}(\pi(x)) = x] > 2^{-n/5}$, is (for large enough n) at most

$$\frac{(N - 2^{\frac{3}{5}n})! \cdot \binom{N}{2^{\frac{3}{5}n}}^2}{N!} = \frac{\binom{N}{2^{\frac{3}{5}n}}}{2^{\frac{3}{5}n}!} \leq 2^{-2^{\frac{3}{5}n}/2},$$

proving Thm 3.

Proving Compression Lemma (Lemma 4)

Let D be q-query circuit, and let π be such that $\Pr_{x \leftarrow \{0,1\}^n}[D(\pi(x)) = x] > \varepsilon$.

Construction 5 (Useful set $\mathcal{Y} \subseteq \{0,1\}^n$)

1. Set $\mathcal{Y} = \emptyset$ and $\mathcal{I} = \{y \in \{0,1\}^n \colon D^\pi(\pi(x)) = \pi\}$.
2. While $\mathcal{I} \neq \emptyset$, let y be the smallest lexicographic element in $\mathcal{I}$.
   - (a) Add y to $\mathcal{Y}$.
   - (b) Remove y and all π-queries $D^\pi(y)$ makes, from $\mathcal{I}$.

Algorithm 6 ($\mathsf{Enc}(\pi)$)

Output (description of) $\mathcal{Y}$ and $\mathcal{V} = \{(x, \pi(x)) \colon \pi(x) \notin \mathcal{Y}\}$.

(Under proper encoding) $|\mathsf{Enc}(\pi)| \leq \log((N-a)!) + 2 \cdot \log\binom{N}{a}$ for $a = |Y| \geq \frac{\varepsilon N}{q+1}$.

Algorithm 7 ($\mathsf{Dec}(\mathcal{Y}, \mathcal{V})$)

For all $y \in \mathcal{Y}$ in lex. order:

1. Emulate $D^\pi(y)$.
2. If D makes a π-query x that is undefined in $\mathcal{V}$, add (x, y) to $\mathcal{V}$.
   Otherwise, add $(D^{\pi, \mathsf{Sam}_r^\pi}(y), y)$ to $\mathcal{V}$.

Use $\mathcal{V}$ to reconstruct π.

Remarks

- The short description argument is an incredibly useful paradigm (see next).
- Immediately yields same result for algorithms (replacing size with running time).
- Alternative compression argument.
- Similar results can be proven for random variants of OWF, TDP, CRH.

Section 2

BB Impossibility for Efficient OWF based PRG

BB Impossibility for OWF based PRG

Definition 8 (pseudorandom generators (PRGs))

Poly-time $G\colon \{0,1\}^n \mapsto \{0,1\}^{\ell(n)}$ is a pseudorandom generator, if

- G is length extending (i.e., $\ell(n) > n$ for any n)
- $G(U_n)$ is pseudorandom (i.e., $\{G(U_n)\}_{n\in\mathbb{N}} \approx_c \{U_{\ell(n)}\}_{n\in\mathbb{N}}$)

We focus on BB constructions of efficient length-doubling PRGs.

Theorem 9

In any fully-BB construction of length-doubling PRG over n-bits string from OWP over $\{0,1\}^n$, the construction makes $\Omega(n/\log n)$ oracle calls.

- Matches known upper bounds.
- Without the restriction on the OWP input length, yields an optimal $n^{\Omega(1)}/\log n$ bound.

Proving Thm 9

- Let (I, R) be a fully-BB reduction of a $q(n) \in o(n/\log n)$-query, length-doubling PRG over $\{0,1\}^n$, to OWP over $\{0,1\}^n$.

  We assume wlg. that I makes distinct queries.(?)
- For $t = t(n) = \lceil n/2q(n) \rceil$, consider the following generator $G\colon \{0,1\}^{\frac{3}{2}n} \mapsto \{0,1\}^{2n}$:

Algorithm 10 ($G(x)$)

1. Emulate $\mathrm{C}^{\pi}(x_{1,\dots,n})$, while answering the i'th query z of I to π, with $x_{n+i\cdot t+1,\dots,n+(i+1)\cdot t} \circ z_{t+1,\dots,n}$.
2. Output the same output that I does.

- Let $\Pi_{n,t}$ be the set of all permutations over $\{0,1\}^n$ that are identity over the last $n-t$ bits (i.e., $\pi(x)_{n-t+1,\dots,n} = x_{n-t+1,\dots,n}$).

Claim 11

$G(U_{3n/2}) \equiv (\mathrm{I}^{\pi}(U_n))_{\pi \leftarrow \Pi_{n,t}}$.

Proving Thm 9, cont.

- $\exists$ algorithm D that distinguishes $G(U_{3n/2})$ from U_{2n} with advantage $1 - 2^{-n/4} > \frac{1}{2}$. (?)

  $\implies$ wlg. $\Pr_{\pi \leftarrow \Pi_{t,n}}[\mathsf{D}(\mathrm{I}^{\pi}(U_n)) = 1] - \Pr[\mathsf{D}(U_{2n}) = 1] > \frac{1}{2}$

  $\implies \Pr_{\pi \leftarrow \Pi_{t,n}}\left[\Pr[\mathsf{D}(\mathrm{I}^{\pi}(U_n)) = 1] - \Pr[\mathsf{D}(U_{2n}) = 1] > \frac{1}{4}\right] \geq \frac{1}{4}$

  $\implies \Pr_{\pi \leftarrow \Pi_{t,n}}\left[\mathsf{R}^{\pi,\mathsf{D}} \text{ inverts } \pi \text{ with non-negligible prob.}\right] \geq \frac{1}{4}$
- Let $n' = t(n) \in \omega(\log n)$.
- By the above, exists $2^{o(n')}$-query circuit R', such that

$$\Pr_{\pi \leftarrow \Pi_{n'}}[\mathsf{R}'^{\pi} \text{ inverts } \pi \text{ with non-negligible prob.}] \geq \frac{1}{4},$$

  in contradiction to Thm 3.

Remarks

- We showed a lower bound on the efficiency of fully-BB constructions of length-doubling PRG from OWPs.
- Actually we ruled out a **less restricted** type of BB-construction, called weakly-BB construction:

  If O is a secure implementation of *A*, then I^{O} is a secure implementation of *B* against adversaries with no access direct to O.
- Results can be easily extended to OWFs/TDPs.
- Using similar means, one can prove lower bound on fully-BB constructions of encryption schemes, signature schemes and universal-one-way-hash-functions (UOWHFs), from OWFs/OWPs/TDPs

Section 3

BB Impossibility for Basing CRH on OWF

Basing CRH on OWF

Definition 12 (collision resistant hash family (CRH))

A function family $\mathcal{H} = \{\mathcal{H}_n \colon \{0,1\}^* \mapsto \{0,1\}^n\}$ is collision resistant, if

$$\Pr_{\substack{h \leftarrow \mathcal{H}_n \\ (x,x') \leftarrow \mathsf{A}(1^n,h)}} [x \neq x' \in \{0,1\}^* \wedge h(x) = h(x')] = \operatorname{neg}(n)$$

for any PPT A.

- In a BB construction of CRH family, both the sampling and evaluation algorithms makes use of the oracle.
- wlg. the sampling algorithm outputs a string h, independent of the oracle, and $h \in \mathcal{H}_n$ is an oracle circuit.
- For simplicity, assume that $h \in \mathcal{H}_n$ only queries the oracle on inputs of length n.

Theorem 13

There exists no fully BB-construction of CRH from OWP.

- Seems harder to prove: CRH exists relative to random permutations!

Proving Thm 13

Fix $n \in \mathbb{N}$.

We construct an (inefficient) algorithm Sam, that finds collision in any CRH, but non-useful for inverting random permutation.

Algorithm 14 (Sam^π)

Input: An n-bit input circuit C.
Oracle: $\pi \in \Pi_n$.

1. Sample $x \leftarrow \{0,1\}^n$.
2. Find the first (in a random order) random $x' \in \{0,1\}^n$ with $\mathsf{C}^\pi(x) = \mathsf{C}^\pi(x')$.
3. Return (x, x').

- Let Sam_r be the variant of Sam whose coins are fixed to r.
- In the actual implementation Sam uses independent randomness per input query C.

No CRH relative to Sam

Let $\mathcal{H}_n$ be a length-decreasing oracle-aided circuit family.

Claim 15

For any $h \in \mathcal{H}_n$ and $\pi \in \Pi_n$: $\Pr_{(x,x') \leftarrow \mathsf{Sam}^{\pi}(h)}[x \neq x' \wedge h^{\pi}(x) = h^{\pi}(x')] \geq \frac{1}{4}$.

Proof: It suffices to prove that for any length decreasing function g over $\{0,1\}^n$, $\Pr_{x \leftarrow \{0,1\}^n}\left[\left|g^{-1}(g(x))\right| = 1\right] \leq \frac{1}{2}$.$\square$

The following algorithm breaks the collision resistance of any Black-box construction of a CRH from OWP.

Algorithm 16 ($\mathsf{D}^{\mathsf{Sam}^{\pi}}$)

On input $h \in \mathcal{H}_n$, return $\mathsf{Sam}^{\pi}(h)$.

Random permutations are hard relative to Sam

Proof via the GT paradigm, but $D^{\pi,\mathsf{Sam}_r^\pi}$, via Sam_r^π, might make more than 2^n queries...

Idea: focus only on the “collision" queries made by Sam_r^π.

Definition 17

The augmented number of queries an oracle-aided circuit/algorithm with Sam-gate does, is the number of queries it makes **directly**, plus twice the number of queries the circuits it **passes** to Sam do.

Theorem 18

For any large enough $n \in \mathbb{N}$ and $2^{n/5}/2$-augmented-query circuit D:

$$\Pr_{\pi \leftarrow \Pi_n; r \leftarrow \{0,1\}^*}\left[\Pr_{x \leftarrow \{0,1\}^n}\left[D^{\pi,\mathsf{Sam}_r^\pi}(\pi(x)) = x\right] > 2 \cdot 2^{-n/5}\right] \leq 2 \cdot 2^{-2^{\frac{3}{5}n}}$$

- Almost the same result as in the non-Sam case.
- Hence, random permutations are (extremely) hard for exponential-size circuits with oracle access to Sam.

Proving Thm 18

Fix large enough n. The proof follows by the next two claims.

Definition 19

For circuit D, $\pi \in \Pi_n$, $r \in \{0,1\}^*$, an $y \in \{0,1\}^n$, let $\mathrm{hit}^{\pi}_{\mathsf{D};r}(y)$ be one, if $\mathsf{D}^{\pi,\mathrm{Sam}^{\pi}_r}(y)$ makes a query $(x, x') = \mathrm{Sam}^{\pi}_r(\mathsf{C})$, and either $\mathsf{C}^{\pi}(x)$ or $\mathsf{C}^{\pi}(x')$ query π on $\pi^{-1}(y)$.

The following probabilities are over $\pi \leftarrow \Pi_n$, $r \leftarrow \{0,1\}^*$ and $x \leftarrow \{0,1\}^n$.

Claim 20

For any t-augQuery circuit D, $\exists$ $2t$-augQuery circuit $\widetilde{\mathsf{D}}$ such that:

$$\Pr_{\pi;r}\left[\Pr_x\left[\mathrm{hit}^{\pi}_{\mathsf{D};r}(\pi(x))\right] > \varepsilon\right] \le$$
$$\Pr_{\pi;r}\left[\Pr_x\left[\widetilde{\mathsf{D}}^{\pi,\mathrm{Sam}^{\pi}_r}(\pi(x)) = x \wedge \neg\mathrm{hit}^{\pi}_{\widetilde{\mathsf{D}};r}(p(x))\right] > \varepsilon/2\right]$$ for any $\varepsilon \ge 0$.

Claim 21

For any $2^{n/5}/2$-augQuery circuit D:

$$\Pr_{\pi;r}\left[\Pr_x\left[\mathsf{D}^{\pi,\mathrm{Sam}^{\pi}_r}(\pi(x)) = x\right] > 2^{-n/5} \wedge \neg\mathrm{hit}^{\pi}_{\mathsf{D};r}\right] \le 2^{-2^{3n/5}}.$$

Proving Claim 20

For any t-augQuery circuit D, $\exists$ $2t$-augQuery circuit $\widetilde{\mathsf{D}}$ such that:
$\Pr_{\pi;r}\left[\Pr_x\left[\mathsf{hit}^{\pi}_{\mathsf{D};r}(\pi(x))\right] > \varepsilon\right] \leq$
$\Pr_{\pi;r}\left[\Pr_x\left[\widetilde{\mathsf{D}}^{\pi,\mathsf{Sam}^{\pi}_r}(\pi(x)) = x \wedge \neg\mathsf{hit}^{\pi}_{\widetilde{\mathsf{D}};r}(p(x))\right] > \varepsilon/2\right]$ **for any $\varepsilon \geq 0$.**

Proof: We describe a random circuit $\widetilde{\mathsf{D}}$, and its deterministic variant follows by fixing the best coins.

Algorithm 22 ($\widetilde{\mathsf{D}}^{\pi,\mathsf{Sam}^{\pi}_r}(y)$)

Emulate $\mathsf{D}^{\pi,\mathsf{Sam}^{\pi}_r}(y)$. Before any query of $\mathsf{Sam}^{\pi}_r(\mathsf{C})$:
Evaluate $\mathsf{C}^{\pi}(z)$ for $z \leftarrow \{0,1\}^n$. If $\mathsf{C}^{\pi}(z)$ makes a query $\pi(x) = y$, return x and halt.

- The augmented query complexity of $\widetilde{\mathsf{D}}$ is at most twice that of D.
- Fix π and y, and let $\delta_i = \Pr_r[\mathsf{D}(y)$ makes first hit on i'th Sam query$]$.
  $\implies$ $\Pr_r[\mathsf{D}(y)$ makes first hit on the "x-part" of i'th Sam query$] \geq \delta_i/2$
  $\implies$
  $\Pr_r\left[\widetilde{\mathsf{D}}(y) \text{ finds } \pi^{-1}(y) \text{ just before } i\text{'th Sam query, w/o hitting}\right] \geq \delta_i/2.\square$

Proving Claim 21

For any $2^{n/5}/2$-augQuery circuit D:

$\Pr_{\pi;r}\left[\Pr_x\left[\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x)) = x\right] > 2^{-n/5} \wedge \neg\mathsf{hit}^\pi_{\mathsf{D};r}\right] \leq 2^{-2^{3n/5}}$.

The proof is similar to the non-Sam case.

Lemma 23 (compression lemma, Sam variant)

For q-augQuery circuit D, $r \in \{0,1\}^*$ and $\varepsilon > 0$, exist algorithms Enc and Dec such that: Let $\pi \in \Pi_n$ be with $\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x)) = x \wedge \neg\mathsf{hit}^\pi_{\mathsf{D};r}(p(x))\right] > \varepsilon$, *then*

- $\mathsf{Dec}(\mathsf{Enc}(\pi)) = \pi$
- $|\mathsf{Enc}(\pi)| \leq \log((N-a)!) + 2 \cdot \log\binom{N}{a}$, *for* $a \geq \frac{\varepsilon N}{q+1}$

Proving Lemma 23

Definition 24

Assume $\mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x))(y)$ makes a query $\mathsf{Sam}_r^\pi(C)$ and get answer (x, x'), we call the π-queries done by $C^\pi(x)$ and $C^\pi(x')$, indirect queries of D.

Construction 25 (Useful set $\mathcal{Y} \subseteq \{0,1\}^n$)

1. Set $\mathcal{Y} = \emptyset$ and $\mathcal{I} = \{y \in \{0,1\}^n \colon \mathsf{D}^{\pi,\mathsf{Sam}_r^\pi}(\pi(x)) = \pi \wedge \neg\mathsf{hit}^\pi_{\mathsf{D};r}(y)\}$.
2. While $\mathcal{I} \neq \emptyset$, let y be the smallest lex. element in $\mathcal{I}$.
   1. Add y to $\mathcal{Y}$.
   2. Remove y and all direct & indirect π-queries $\mathsf{D}(y)$ makes from $\mathcal{I}$.

Algorithm 26 ($\mathsf{Enc}(\pi)$)

Output (description of) $\mathcal{Y}$ and $\mathcal{V} = \{(x, \pi(x)) \colon \pi(x) \notin \mathcal{Y}\}$.

Under proper encoding, $|\mathsf{Enc}(\pi)| \leq \log((N-a)!) + 2 \cdot \log\binom{N}{a}$ for $a = |Y| \geq \frac{\varepsilon N}{q+1}$.

Proving Lemma 23 cont.

Algorithm 27 ($\mathrm{Dec}(\mathcal{Y}, \mathcal{V})$)

For all $y \in \mathcal{Y}$ in lex. order:

1. Emulate $\mathrm{D}^{\pi,\mathrm{Sam}_r^\pi}(y)$.
    1. Answer π-query using $\mathcal{V}$.
    2. On Sam-query $\mathrm{Sam}_r^\pi(\mathrm{C})$: choose x according to r, and let x' be the first element in $\{0,1\}^n$ for which the π-queries of $\mathrm{C}^\pi(x')$ are defined, and $\mathrm{C}^\pi(x') = \mathrm{C}^\pi(x)$.
2. If D makes a π-query x that is undefined in $\mathcal{V}$, add (x, y) to $\mathcal{V}$.

   Otherwise, add $(\mathrm{D}^{\pi,\mathrm{Sam}_r^\pi}(y), y)$ to $\mathcal{V}$.

Use $\mathcal{V}$ to reconstruct π

Correctness holds since $\mathrm{hit}^\pi_{\mathrm{D};r}(y) = 0$ for all $y \in \mathcal{Y}$, and thus answer to all Sam-queries are defined.

Remarks

- Results extends to OWFs and to TDPs.
- Making Sam use independent randomness per input query C?