Application of Information Theory, Lecture 7

Relative Entropy

Handout Mode

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Section 1

Definition and Basic Facts

Definition

- For $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$, let

$$D(p\|q) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i}$$

  $0 \log \frac{0}{0} = 0$, $p \log \frac{p}{0} = \infty$

- The relative entropy of pair of rv's, is the relative entropy of their distributions.
- Names: Entropy of p relative to q, relative entropy, information divergence, Kullback-Leibler (KL) divergence/distance
- Many different interpretations
- Main interpretation: the information we gained about X, if we originally thought $X \sim q$ and now we learned $X \sim p$

Numerical Example

$D(p\|q) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i}$

- $p = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0)$, $q = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$
- $D(p\|q) = \frac{1}{4} \log \frac{\frac{1}{4}}{\frac{1}{2}} + \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{1}{4}} + \frac{1}{4} \log \frac{\frac{1}{4}}{\frac{1}{8}} + 0 \log 0 = \frac{1}{4} \cdot (-1) + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 1 = \frac{1}{2}$
- $D(q\|p) = \frac{1}{2} \log \frac{\frac{1}{2}}{\frac{1}{4}} + \frac{1}{4} \log \frac{\frac{1}{4}}{\frac{1}{2}} + \frac{1}{8} \log \frac{\frac{1}{8}}{\frac{1}{4}} + \frac{1}{8} \log \frac{\frac{1}{8}}{0} =$
  $\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot (-1) + \frac{1}{8} \cdot (-1) + \infty = \infty$

Supporting the interpretation

- X rv over $[m]$
- $H(X)$ — measure for amount of information we do not have about X
- $\log m - H(X)$ — measure for information we do have about X (just by knowing its distribution)
- Example $X = (X_1, X_2) \sim (\frac{1}{2}, 0, 0, \frac{1}{2})$ over $\{00, 01, 10, 11\}$
- $H(X) = 1$, $\log m - H(X) = 2 - 1 = 1$
- Indeed, we know $X_1 \oplus X_2$
- $$\begin{aligned} H(\sim [m]) - H(p_1, \ldots, p_m) &= \log m - H(p_1, \ldots, p_m) \\ &= \log m + \sum_i p_i \log p_i = \sum_i p_i(\log p_i - \log \frac{1}{m}) \\ &= \sum_i p_i \log \frac{p_i}{\frac{1}{m}} = D(p \| \sim [m]) \end{aligned}$$
- $D(X \| \sim [m])$ — measures the information we gained about X, if we originally thought it is $\sim [m]$ and now we learned it is $\sim p$

Supporting the interpretation, cont.

- (generally) $D(p\|q) \neq H(p) - H(q)$
- $H(q) - H(p)$ is not a good measure for information change
- Example: $q = (0.01, 0.99)$ and $p = (0.99, 0.01)$
- We were almost sure that $X = 1$ but learned that X is almost surely 0
- But $H(p) - H(q) = 0$
- Also, $H(q) - H(p)$ might be negative

- We understand $D(p\|q)$ as the information we gained about X, if we originally thought it is $\sim q$ and now we learned it is $\sim p$

Changing distribution

- What does it mean: originally thought $X \sim q$ and now we learned $X \sim p$?

  How can a distribution change?
- Typically, this happens by learning additional information
- $q_i = \Pr[X = i]$ and $p_i = \Pr[X = i|E]$
- Example $X \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$; someone saw X and tells us that $X \leq 2$
- The distribution changes to $X \sim (\frac{2}{3}, \frac{1}{3}, 0, 0)$
- Another example

$X \backslash Y$	1	2	3	4
0	$\frac{1}{4}$	$\frac{1}{4}$	0	0
1	$\frac{1}{4}$	0	$\frac{1}{4}$	0

- $Y \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$, but
- $Y \sim (\frac{1}{2}, \frac{1}{2}, 0, 0)$ conditioned on $X = 0$
- $Y \sim (\frac{1}{2}, 0, \frac{1}{2}, 0)$ conditioned on $X = 1$
- Generally, a distribution can change if we condition on event E

Additional properties

- $0 \log \frac{0}{0} = 0$, $p \log \frac{p}{0} = \infty$ for $p > 0$
- $\exists i$ s.t. $p_i > 0$ and $q_i = 0$, then $D(p\|q) = \infty$
- If originally $\Pr[X = i] = 0$, then it cannot be more than 0 after we learned something.
- Hence, it make sense to think of it as infinite amount of information learnt
- Alteratively, we can define $D(p\|q)$ only for distribution with $q_i = 0 \implies p_i = 0$
  (recall that $\Pr[X = i] = 0 \implies \Pr[X = i|E] = 0$, for any event E
- If p_i is large and q_i is small, then $D(p\|q)$ is large
- $D(p\|q) \geq 0$, with equality iff $p = q$ (hw)

Example

- $q = (q_1, \ldots, q_m)$ with $\sum_{i=1}^{n} q_i = 2^{-k}$ (i.e., $n < m$)
- $p_i = \begin{cases} q_i/2^{-k}, & 1 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$
- $p = (p_1, \ldots, p_m)$ — the distribution of q conditioned on the event $i \in [n]$
- $D(p\|q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i \log 2^k = \sum_{i=1}^{n} p_i k = k$
- We gained k bits of information
- Example: $\sum_{i=1}^{n} q_i = \frac{1}{2}$, and we were told that $i \leq n$ or $i > n$, we got one bit of information

Section 2

Axiomatic Derivation

Axiomatic derivation

Let $\tilde{D}$ is a continuous and symmetric (wrt each distribution) function such that

1. $\tilde{D}(p\| \sim [m]) = \log m - H(p)$
2. $\tilde{D}((p_1,\ldots,p_m)\|(q_1,\ldots,q_m)) =$ $\tilde{D}((p_1,\ldots,p_{m-1},\alpha p_m,(1-\alpha)p_m)\|(q_1,\ldots,q_{m-1},\alpha q_m,(1-\alpha)q_m))$, for any $\alpha \in [0,1]$

then $\tilde{D} = D$.

Interpretation

Proof:

- $\tilde{D}(p\|q) = D((\alpha_{1,1}p_1,\ldots,\alpha_{1,k_1}p_1,\ldots,\alpha_{m,1}p_m,\ldots,\alpha_{m,k_m}p_m)\|$ $(\alpha_{1,1}q_1,\ldots,\alpha_{1,k_1}q_1,\ldots,\alpha_{m,1}q_m,\ldots,\alpha_{m,k_m}q_m))$, for $\sum_j \alpha_{i,j} = 1$ and $\alpha_{i,j\geq 0}$
- Taking α's s.t. $\alpha_{i,1} = \alpha_{i,2}\ldots,\alpha_{i,k_i} = \alpha_i$ and $\alpha_i q_i = \frac{1}{M}$, it follows that

$$\tilde{D}(p\|q) = \log M - H((\alpha_{1,1}p_1,\ldots,\alpha_{1,k_1}p_1,\ldots,\alpha_{m,1}p_m,\ldots,\alpha_{m,k_m}p_m))$$
$$= \sum p_i \log M + \sum_i p_i \log \alpha_i p_i = \sum_i p_i(\log M + \log \frac{p_i}{q_i M}) = \sum_i p_i \log \frac{p_i}{q_i}.$$

- Zeros and non-rational q_i's are dealt by continuity

Section 3

Relation to Mutual Information

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0,1\}$
- $(X|Y=0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X=i|Y=0]$
- $(X|Y=1) \sim p_1 = (p_{1,1}, \ldots, p_{1,m})$, $\qquad p_{1,i} = \Pr[X=i|Y=1]$
- If we learned $Y=j$, we gained $D(p_j \| q)$

$$
\begin{aligned}
\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y=0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
&+ \Pr[Y=1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m) \\
&= \Pr[Y=0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y=1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y=0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y=1] \cdot \sum p_{1,i} \log p_{1,i} \\
&- \Pr[Y=0] \cdot \sum p_{0,i} \log q_i - \Pr[Y=1] \cdot \sum p_{1,i} \log q_i \\
&= -H(X|Y) - \sum (\Pr[Y=0] \cdot p_{0,i} + \Pr[Y=1] \cdot p_{1,i}) \log q_i \\
&= -H(X|Y) + H(X) = I(X;Y)
\end{aligned}
$$

Equivalent definition for mutual information

- $(X, Y) \sim p$, then $I(X;Y) = D(p \| p_X p_Y)$
- Interpretation
- Proof:

$$\begin{aligned}
D(p \| p_X p_Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x) p_Y(y)} \\
&= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\
&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y) = I(X;Y)
\end{aligned}$$

- We will later see the relation between the above two facts.

Section 4

Relation to Data Compression

Wrong code

Theorem 1

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

$$\begin{aligned}
\mathop{\mathrm{E}}_{i \leftarrow p}\left[\ell(i)\right] &= \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1) \\
&= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i}) = 1 + \sum_i p_i (\log \frac{p_i}{q_i}) + \sum_i p_i (\log \frac{1}{p_i}) \\
&= 1 + D(p\|q) + H(p)
\end{aligned}$$

- Can there be a (close) to optimal code for q that is better for p? HW

Section 5

Conditional Relative Entropy

Conditional relative entropy

Definition 2

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) := \sum_{x \in \mathcal{X}} p_{\mathcal{X}}(x) \cdot \sum_{y \in \mathcal{Y}} p_{\mathcal{Y}|\mathcal{X}}(y|x) \log \frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)}$$

$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} [D(X_q | X_p = x \| Y_q | X_q = x)]$

- Example: $p =$

X \ Y	0	1
0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{4}$	$\frac{1}{2}$

$q =$

X \ Y	0	1
0	$\frac{1}{8}$	$\frac{1}{4}$
1	$\frac{1}{2}$	$\frac{1}{8}$

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \frac{1}{4} \cdot D((\frac{1}{2}, \frac{1}{2}) \| (\frac{1}{3}, \frac{2}{3})) + \frac{3}{4} \cdot D((\frac{1}{3}, \frac{2}{3}) \| (\frac{4}{5}, \frac{1}{5}))$$

$$= \ldots$$

Chain rule

Claim 3

For any two distributions p and q over $\mathcal{X} \times \mathcal{Y}$, it holds that $D(p\|q) = D(p_{\mathcal{X}}\|q_{\mathcal{X}}) + D(p_{\mathcal{Y}|\mathcal{X}}\|q_{\mathcal{Y}|\mathcal{X}})$

- Proof:

$$\begin{aligned} D(p\|q) &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\log\frac{p(x,y)}{q(x,y)} \\ &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\log\frac{p_{\mathcal{X}}(x)p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{X}}(x)q_{\mathcal{Y}|\mathcal{X}}(y|x)} \\ &= \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\log\frac{p_{\mathcal{X}}(x)}{q_{\mathcal{X}}(x)} + \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p(x,y)\log\frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)} \\ &= D(p_{\mathcal{X}}\|q_{\mathcal{X}}) + D(p_{\mathcal{Y}|\mathcal{X}}\|q_{\mathcal{Y}|\mathcal{X}}) \end{aligned}$$

- It follows that for $(X;Y) \sim p$: $I(X,Y) = D(p\|p_X p_Y) = D(p_X\|p_X) + \mathrm{E}_{x\leftarrow X}\left[D(p_{Y|X=x}, p_Y)\right] = \mathrm{E}_{x\leftarrow X}\left[D(p_{Y|X=x}, p_Y)\right]$

Section 6

Data-processing inequality

Data-processing inequality

Claim 4

For any rv's X and Y and function f: $D(X\|Y) \geq D(f(X)\|f(Y))$.

- Analogues to $H(X) \geq H(f(X))$
- Proof:
- $D(X, f(X)\|Yf(Y)) = D(X\|Y)$
- $D(X, f(X)\|Yf(Y)) =$
  $D(f(X)\|f(Y)) + \mathrm{E}_{z \leftarrow f(X)}\left[D(X|f(X) = z\|Y|f(X) = z))\right] \leq D(f(X)\|f(Y))$
- Hence, $D(f(X)\|f(Y)) \geq D(X\|Y)$.

Section 7

Relation to Statistical Distance

Relation to statistical distance

- $D(p\|q)$ is used many time to measure the distance from p to q
- It is not a distance in the mathematical sense: $D(p\|q) \neq D(q\|p)$ and no triangle inequality
- However,

Theorem 5

$\mathrm{SD}(p,q) \leq \sqrt{\frac{\ln 2}{2} \cdot D(p\|q)}$

- Corollary: For rv X over $[m]$ with $H(X) \geq m - \varepsilon$, it holds that $\mathrm{SD}(X, \sim [m]) \leq \sqrt{\frac{\ln 2}{2} \cdot (m - H(X))} = \sqrt{\frac{\ln 2}{2} \cdot \varepsilon}$
- Other direction is incorrect: $\mathrm{SD}(p,q)$ might be small but $D(p\|q) = \infty$
- Does $\mathrm{SD}(p, \ [m])$ being small imply $D(p\| \ [m]) = \log m - H(p)$ is small?

Proving Thm 5, boolean case

- Let $p = (\alpha, 1-\alpha)$ and $q = (\beta, 1-\beta)$ and assume $\alpha \geq \beta$
- We will show that
  $D(p\|q) = \alpha \log \frac{\alpha}{\beta} + (1-\alpha)\log\frac{1-\alpha}{1-\beta} \geq \frac{4}{2\ln 2}(\alpha-\beta)^2 = \frac{2}{\ln 2}\mathsf{SD}(p,q)^2$
- Let $g(\alpha,\beta) = \alpha \log \frac{\alpha}{\beta} + (1-\alpha)\log\frac{1-\alpha}{1-\beta} - \frac{4}{2\ln 2}(\alpha-\beta)^2$

$$\begin{aligned}
\frac{\partial g(\alpha,\beta)}{\partial\beta} &= -\frac{\alpha}{\beta\ln 2} + \frac{1-\alpha}{(1-\beta)\ln 2} - \frac{4}{2\ln 2}2(\beta-\alpha)\\
&= \frac{\beta-\alpha}{\beta(1-\beta)\ln 2} - \frac{4}{\ln 2}(\beta-\alpha)\\
&\leq 0 \qquad\qquad (\text{since } \beta(1-\beta)\leq \tfrac{1}{4} \text{ and } \beta<\alpha)
\end{aligned}$$

- $g(\alpha,\alpha) = 0$, and hence $g(\alpha,\beta)\geq 0$ for $\beta<\alpha$. $\square$

Proving Thm 5, general case

- Let $\mathcal{U} = \text{Supp}(p) \cup \text{Supp}(q)$
- Let $\mathcal{S} = \{u \in \mathcal{U} \colon p(u) > q(u)\}$
- $\text{SD}(p, q) = \Pr_p[\mathcal{S}] - \Pr_q[\mathcal{S}]$ (by homework)
- Let $P \sim p$, and let the indicator $\hat{P}$ be 1 iff $P \in \mathcal{S}$.
- Let $Q \sim q$, and let the indicator $\hat{Q}$ be 1 iff $Q \in \mathcal{S}$.
- $\text{SD}(\hat{P}, \hat{Q}) = \Pr[P \in \mathcal{S}] - \Pr[Q \in \mathcal{S}] = \text{SD}(p, q)$

$$\begin{aligned} D(p\|q) &\geq D(\hat{P}\|\hat{Q}) && \text{(data-proccessing inequality)} \\ &\geq \frac{2}{\ln 2} \cdot \text{SD}(\hat{P}, \hat{Q})^2 && \text{(the Boolean case)} \\ &= \frac{2}{\ln 2} \cdot \text{SD}(p, q)^2. \ \square \end{aligned}$$

Section 8

Conditioned Distributions

Main theorem

Theorem 6

Let $X_1, \ldots, X_k$ be iid over $\mathcal{U}$, and let $Y = (Y_1, \ldots, Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

We prove for $k = 2$, general case follows similar lines.

For rv Z, let $Z(z) = \Pr[Z = z]$. Let $X = (X_1, X_2)$

$$
\begin{aligned}
D(Y \| X) &= \sum_{\mathbf{y} \in \mathcal{U}^2} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{X(\mathbf{y})} = \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(y_1)} \frac{Y_2(y_2)}{X_2(y_2)} \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(x_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(x_1)} \\
&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= D(Y_1 \| X_1) + D(Y_2 \| X_2) + I(Y_1; Y_2) \geq D(Y_1 \| X_1) + D(Y_2 \| X_2)
\end{aligned}
$$

Conditioning distributions, relative entropy case

Theorem 7

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event (i.e., Boolean rv). Then $\sum_{j=1}^{k} D((X_j|W)\|X_j) \leq \log \frac{1}{\Pr[W]}$.

Let $X = (X_1, \ldots, X_k)$.

$$\begin{aligned}
\sum_{j=1}^{k} D((X_j|W)\|X_j) &\leq D((X|W)\|X) && \text{(Thm 6)} \\
&= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|W)(\mathbf{x}) \log \frac{(X|W)(\mathbf{x})}{X(\mathbf{x})} \\
&= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|W)(\mathbf{x}) \log \frac{\Pr[W|X = \mathbf{x}])}{\Pr[W]} && \text{(Bayes)} \\
&= \log \frac{1}{\Pr[W]} + \sum_{\mathbf{x}\in\mathcal{X}^k} (X|W)(\mathbf{x}) \log \Pr[W|X = \mathbf{x}]) \\
&\leq \log \frac{1}{\Pr[W]}
\end{aligned}$$

Conditioning distributions, statistical distance case

Theorem 8

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathrm{SD}((X_j|W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Proof: follows by Thm 5, and Thm 6.□

Using $(\sum_{j=1}^{k} a_i)^2 \leq k \cdot \sum_{j=1}^{k} a_i^2$, it follows that

Corollary 9

$\sum_{j=1}^{k} \mathrm{SD}((X_j|W), X_j) \leq \sqrt{k \log(\frac{1}{\Pr[W]})}$, *and*

$\mathrm{E}_{j \leftarrow k]} \mathrm{SD}((X_j|W), X_j) \leq \sqrt{\frac{1}{k} \log(\frac{1}{\Pr[W]})}$

Extraction

Numerical example

- Let $X = (X_1, \ldots, X_k) \leftarrow \{0,1\}^{40}$ and let $f\colon \{0,1\}^{40} \mapsto 0$ be such that $\Pr[f(X) = 0] = 2^{-10}$.
- $\mathrm{E}_{j \leftarrow [40]} \mathrm{SD}((X_j | f(X) = 0), \sim \{0,1\}) \leq \sqrt{\frac{1}{40} \cdot 10} = \frac{1}{2}$
- Typical bits are not too biassed, even when conditioning on a very unlikely event.

Extension

Theorem 10

Let $X = (X_1, \ldots, X_k)$, T *and* V *be rv's over* $\mathcal{X}^k$, $\mathcal{T}$ *and* $\mathcal{V}$ *respectively. Let* W *be an event and assume that the* X_i*'s are iid conditioned on* T*. Then* $\sum_{j=1}^{k} D((TVX_j|W) \| (TV|W)X'_j(T)) \leq \log \frac{1}{\Pr[W]} + \log |\mathrm{Supp}(V|W)|$, *where* $X'_j(t)$ *is distributed according to* $X_j|T = t$.

Interpretation.

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|T = t$.

$$\begin{aligned}
&\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X'_j(T)) \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t\|(X_j|T = t)\big)\Big] && \text{(chain rule)} \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t\|(X_j|T = t)\big)\Big] \\
&\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\log \frac{1}{\Pr\left[W \wedge V = v|T = t\right]}\Big] && \text{(Thm 7)} \\
&\leq \log \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)} \frac{1}{\Pr\left[W \wedge V = v|T = t\right]} && \text{(Jensen's inequality)} \\
&= \log \sum_{(t,v)\in \mathrm{Supp}(TV|W)} \frac{\Pr\left[T = t\right]}{\Pr\left[W\right]} \leq \log \frac{\|\mathrm{Supp}(V|W)\|}{\Pr\left[W\right]}. && \square
\end{aligned}$$