Application of Information Theory, Lecture 6
Counting

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Section 1

Graph Homomorphisms

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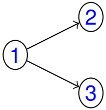

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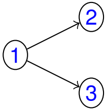

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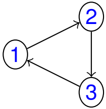

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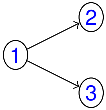

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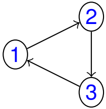

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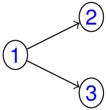

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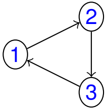

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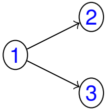

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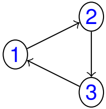

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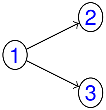

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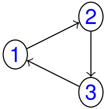

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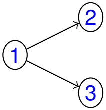

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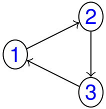

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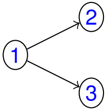

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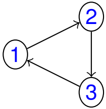

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- Special case of a more general theorem

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Section 2

Perfect Matchings

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&\leq H(M(1)) + H(M(2)) + \ldots + H(M(n)) \\
&\leq \log d(1) + \log d(2) + \ldots + \log d(n)
\end{aligned}
$$

Bregman's theorem

For bi-partite graph $G = (A, B, E)$, let $d(v) = |N(v) = \{u \in B\colon (v, u) \in E\}|$

Theorem 1

Let $G = (A, B, E)$ be bi-partite graph with $|A| = |B|$, and let $\mathcal{M}$ be the perfect matchings in G. Then $|\mathcal{M}| \leq \prod_{v \in A} (d(v)!)^{1/d(v)}$.

- Let $A = B = [n] = \{1, \ldots, n\}$, and for $m \in \mathcal{M}$ let $m(i)$ be the node in B matched with i by m.
- It is clear that $|\mathcal{M}| \leq \prod_{i \in [n]} d(i)$:
- Let $M \leftarrow \mathcal{M}$. Hence,

$$\begin{aligned}
\log|\mathcal{M}| = H(M) &= H(M(1)) + H(M(2)|M(1)) + \ldots + H(M(n)|M(1), \ldots, M(n-1)) \\
&\leq H(M(1)) + H(M(2)) + \ldots + H(M(n)) \\
&\leq \log d(1) + \log d(2) + \ldots + \log d(n) \\
&= \log \prod_{i \in [n]} d(i)
\end{aligned}$$

Proving Bregman's theorem

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- Key observations:
  $H(M(i|M(1),\ldots,M(i-1)) \leq \log|N(i) \setminus \{M(1),\ldots,M(i-1)\}|$

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$$\leq \log\prod_{i\in[n]}\left((d(i)!)^{1/d(i)}\right). \qquad \square$$

Section 3

Shearer's Lemma

$H(X_1, X_2, X_3)$ **Vs.** $H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)$

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- Proof:

$$2H(X_1, X_2, X_3) = 2H(X_1) \qquad +2H(X_2|X_1) \qquad +2H(X_3|X_1, X_2)$$

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\end{aligned}$$

- but

$$\begin{aligned}
H(X_2|X_1) &\leq H(X_2) \\
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Lemma 2 (Shearer's lemma)

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- Stronger conclusion: X_F is close to the uniform distribution.

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- If $dk << n$, then $\exists F \in \mathcal{F}$ s.t. X_F is close to the uniform distribution (over k bits)

Section 4

Gold Coins

of gold coins in a cube

of gold coins in a cube

- Q — (finite) set of points in $\mathbb{R}^3$

of gold coins in a cube

- Q — (finite) set of points in $\mathbb{R}^3$
  - Projection of Q on xy — 6

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of gold coins in a cube

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- Can we bound $|Q|$?

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- $$\begin{aligned}\log|Q| = H(X) &\leq \frac{1}{2}(H(X_1, X_2) + H(X_1, X_3) + H(X_2, X_3))\\ &\leq \frac{1}{2}(\log 6 + \log 8 + \log 12)\\ &\leq \frac{1}{2}(\log 6 \cdot 8 \cdot 12)\end{aligned}$$
- Hence, $|Q| \leq \sqrt{6 \cdot 8 \cdot 12} = 24$
- Can be 24 or less

of gold coins in a cube

- Q — (finite) set of points in $\mathbb{R}^3$
  - Projection of Q on xy — 6
  - Projection of Q on xz — 8
  - Projection of Q on yz — 12
- Can we bound $|Q|$?
- The real story
- $X = (X_1, X_2, X_3) \leftarrow Q$
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Section 5

Independent Sets

of independent sets in bi-partite graphs

of independent sets in bi-partite graphs

Theorem 4

Let $G = (A, B, E)$ be an n-regular graph with $|A| = |B| = m$. Then the number of independent sets in G is at most $(2^{n+1} - 1)m$.

of independent sets in bi-partite graphs

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- Hence $H(I) \leq \sum_{v \in A} p(v) + \frac{1}{n}(h(p(v)) + (1 - p(v))\log(2^n - 1))$

of independent sets in bi-partite graphs, cont.

- $\log|\mathcal{I}| = H(I) \leq \sum_{v \in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1 - p(v))\log(2^n - 1)\right)$

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- By calculus, $\max_{t\in[0,1]} f(t) = \frac{1}{n}\log(2^{n+1}-1)$
- Hence, $\log|\mathcal{I}| \leq \frac{m}{n}\log(2^{n+1}-1)$. $\square$

Section 6

Intersecting Graphs

Another corollary of Shearer's lemma

Corollary 5

Let $\mathcal{A}$ and $\mathcal{F}$ be collections of subsets of $[n]$, and for $F \in \mathcal{F}$ let $\mathcal{A}_F$ be the collection $\{A \cap F \colon A \in \mathcal{A}\}$. Assume that each element of $[n]$ appears in at least m subsets of $\mathcal{F}$, then $|\mathcal{A}|^m \leq \prod_{F \in \mathcal{F}} |\mathcal{A}_F|$.

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Proof:

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Proof:

- Let $X = (X_1, \ldots, X_n) \leftarrow \mathcal{A}$.
- $\log |\mathcal{A}_F| \geq H(X_F)$ ($\mathrm{Supp}(X_F) \subseteq \mathcal{A}_F$)

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- $\log |\mathcal{A}_F| \geq H(X_F)$ $\quad$ ($\mathrm{Supp}(X_F) \subseteq \mathcal{A}_F$)
- By Shearer's lemma, $\log |\mathcal{A}| = H(X) \leq \frac{1}{m} \sum_{F \in \mathcal{F}} H(X_F)$. $\square$

of intersecting graphs

of intersecting graphs

Theorem 6

Let $\mathcal{G}$ be a family of graphs over $[n]$, s.t. $G \cap G'$ contains a triangle for each $G, G' \in \mathcal{G}$. Then $|\mathcal{G}| \leq 2^{\binom{n}{2}-2}$.

of intersecting graphs

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Proof:

- We focus on even n, and view graphs over $[n]$ as subsets $[\binom{n}{2}]$

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Proof:

- We focus on even n, and view graphs over $[n]$ as subsets $[\binom{n}{2}]$
- For $\frac{n}{2}$-size set $\mathcal{S} \subset [n]$, let $F = F(\mathcal{S})$ be union of the cliques $\mathcal{S}$ and $[n] \setminus \mathcal{S}$

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This improves over $|\mathcal{G}| \leq 2^{\binom{n}{2}-1}$, which follows from $G \cap G' \neq \emptyset$.
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- We focus on even n, and view graphs over $[n]$ as subsets $[\binom{n}{2}]$
- For $\frac{n}{2}$-size set $\mathcal{S} \subset [n]$, let $F = F(\mathcal{S})$ be union of the cliques $\mathcal{S}$ and $[n] \setminus \mathcal{S}$
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- Hence, $|\mathcal{G}| \leq 2^{m - \frac{m}{m'}} \leq 2^{\binom{n}{2}-2}$

Section 7

Statistical Distance

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- Interpretation

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Theorem 7 (Next lecture)

Let X rv over $[m]$. Assume $H(X) \geq \log m - \varepsilon$, then
$\mathrm{SD}(X, \sim [m]) \leq \sqrt{\varepsilon \cdot \frac{\ln 2}{2}} = O(\sqrt{\varepsilon})$