

Foundation of Cryptography (0368-4162-01), Lecture 8

Encryption Schemes

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Section 1

Definitions

Correctness

Definition 1 (encryption scheme)

A trippet of PPT's (G, E, D) such that

- 1 $G(1^n)$ outputs a key $(e, d) \in \{0, 1\}^* \times \{0, 1\}^*$
- 2 $E(e, m)$ outputs a string in $c \in \{0, 1\}^*$
- 3 $D(d, c)$ outputs $m \in \{0, 1\}^*$

Correctness: $D(d, E(e, m)) = m$, for any $(e, d) \in \text{Supp}(G(1^n))$ and $m \in \{0, 1\}^*$

- e – encryption key, d – decryption key
- m – plaintext, $c = E(e, m)$ – ciphertext
- $E_e(m) \equiv E(e, m)$ and $D_d(c) \equiv D(d, c)$,
- public/private key

Security

- What would we like to achieve?
- Attempt: for any $m \in \{0, 1\}^*$:

$$(m, E_{G(1^n)_1}(m)) \equiv (m, U_{\ell(|m|)})$$

- Shannon – only for m with $|m| \leq |G(1^n)_1|$
- Other concerns, e.g., multiple encryptions, active adversary

Semantic Security

- 1 Ciphertext reveal “no information” about the plaintext
- 2 Formulate via the simulation paradigm
- 3 Cannot hide the message length

Semantic security – private-key model

Definition 2 (Semantic Security – private-key model)

An encryption scheme (G, E, D) is semantically secure in the private-key model, if for any PPT A , \exists PPT A' s.t. \forall poly-bounded dist. ensemble $\mathcal{M} = \{\mathcal{M}_n\}_{n \in \mathbb{N}}$ and poly-bounded functions $h, f: \{0, 1\}^* \mapsto \{0, 1\}^*$

$$\left| \Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1} [A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] - \Pr_{m \leftarrow \mathcal{M}_n} [A'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)] \right| = \text{neg}(n)$$

- poly-bounded? for simplicity we assume polynomial length
- 1^n and $1^{|m|}$ can be omitted
- Non-uniform definition
- Reflection to ZK
- public-key variant – A gets e

Indistinguishability of encryptions

- The encryption of two strings is indistinguishable
- Less intuitive than semantic security, but easier to work with

Indistinguishability of encryptions – private-key model

Definition 3 (Indistinguishability of encryptions – private-key model)

An encryption scheme (G, E, D) has indistinguishable encryptions in the private-key model, if for any $p, \ell \in \text{poly}$, $\{x_n, y_n \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$, $\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}$ and poly-time B ,

$$|\Pr_{e \leftarrow G(1^n)_1}[B(z_n, E_e(x_n)) = 1] - \Pr_{e \leftarrow G(1^n)_1}[B(z_n, E_e(y_n)) = 1]| = \text{neg}(n)$$

- Non-uniform definition
- Public-key variant

Equivalence of definitions

Theorem 4

An encryption scheme (G, E, D) is semantically secure iff it has indistinguishable encryptions.

We prove the private key case

Indistinguishability \implies Semantic Security

Fix \mathcal{M} , A , f and h , be as in Definition 2. We construct A' as

Algorithm 5 (A')

Input: 1^n , $1^{|m|}$ and $h(m)$

- 1 $e \leftarrow G(1^n)_1$
- 2 $c = E_e(1^{|m|})$
- 3 Output $A(1^n, 1^{|m|}, h(m), c)$

Claim 6

A' is a good simulator for A (according to Definition 2)

Proving Claim 6

Assume exists infinite $\mathcal{I} \subseteq \mathbb{N}$ and $p \in \text{poly}$ s.t. for any $n \in \mathcal{I}$:

$$\left| \Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1} [A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] - \Pr_{m \leftarrow \mathcal{M}_n} [A'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)] \right| > 1/p(n) \quad (1)$$

Fix $n \in \mathcal{I}$ and let $x_n \in \text{Supp}(\mathcal{M}_n)$ be a value that maximize Equation (3).

Assume exists algorithm B that contradicts the indistinguishability of the scheme with respect to

$$\{(x_n, y_n = 1^{|x_n|})\}_{n \in \mathbb{N}} \text{ and } \{z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n))\}_{n \in \mathbb{N}}.$$

Algorithm 7 (B)

Input: $z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n)), c$

Output 1 iff $A(1^n, 1^{|x_n|}, h(x + n), c) = f(1^n, x_n)$

Semantic Security \implies Indistinguishability

Assume $\exists B, \{x_n, y_n \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$ and a $\{z_n\}_{n \in \mathbb{N}}$, such that (wlg.) for infinitely many n 's:

$$\Pr_{e \leftarrow G(1^n)}[B(z_n, E_e(x_n)) = 1] - \Pr_{e \leftarrow G(1^n)}[B(z_n, E_e(y_n)) = 1] \geq \frac{1}{p(n)}$$

- Let \mathcal{M}_n be x_n wp $\frac{1}{2}$ and y_n otherwise.
- Let $f(1^n, x_n) = 1$, $f(1^n, y_n) = 0$ and $h(1^n, \cdot) = z_n$.
- Define $A(1^n, 1^{\ell(n)}, z_n, c)$ to return $B(z_n, c)$.

$$\Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)}[A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \geq \frac{1}{2} + \frac{1}{p(n)}$$

For any A'

$$\Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)}[A(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \leq \frac{1}{2}$$

Security Under Multiple Encryptions

Definition 8 (Indistinguishability for multiple encryptions – private-key model)

An encryption scheme (G, E, D) has indistinguishable encryptions for multiple messages in the private-key model, if for any $p, \ell, t \in \text{poly}$,

$\{x_{n,1}, \dots, x_{n,t(n)}, y_{n,1}, \dots, y_{n,t(n)} \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$,
 $\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}$ and polynomial-time B ,

$$\left| \Pr_{e \leftarrow G(1^n)} [B(z_n, E_e(x_{n,1}), \dots, E_e(x_{n,t(n)})) = 1] \right. \\ \left. - \Pr_{e \leftarrow G(1^n)} [B(z_n, E_e(y_{n,1}), \dots, E_e(y_{n,t(n)})) = 1] \right| = \text{neg}(n)$$

Extensions:

- Different length messages
- Semantic security version
- Public-key definition

Multiple Encryption in the Public-Key Model

Theorem 9

A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.

Proof: Assume (G, E, D) is public-key secure for a single message and not for multiple messages with respect to B ,

$$\{x_{1,t(n)}, \dots, x_{n,t(n)}, y_{n,1}, \dots, y_{n,t(n)} \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}},$$

$$\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}.$$

It follows that for some function $i(n) \in [t(n)]$

$$\begin{aligned} & \left| \Pr[B(1^n, e, E_e(x_{n,1}), \dots, E_e(x_{n,i-1}), E_e(y_{n,i}), \dots, E_e(y_{n,t(n)})) = 1] \right. \\ & \left. - \Pr[B(1^n, e, E_e(x_{n,1}), \dots, E_e(x_{n,i}), E_e(y_{n,i+1}), \dots, E_e(y_{n,t(n)})) = 1] \right| \\ & > \text{neg}(n) \end{aligned}$$

where in both cases $e \leftarrow G(1^n)_1$

Algorithm 10 (B')

Input: $1^n, z_n = (i(n), x_{1,t(n)}, \dots, x_{n,t(n)}, y_{n,1}, \dots, y_{n,t(n)}, e, c$

Return $B(c, E_e(x_{n,1}), \dots, E_e(x_{n,i-1}), c, E_e(y_{n,i+1}), \dots, E_e(y_{n,t(n)}))$

B' is critically using the public key

Multiple Encryption in the Private-Key Model

Fact 11

Assuming (non uniform) OWFs exists, there exists an encryption scheme that has private-key indistinguishable encryptions for a single messages, but not for multiple messages

Proof: Let $g: \{0, 1\}^n \mapsto \{0, 1\}^{n+1}$ be a (non-uniform) PRG, and for $i \in \mathbb{N}$ let g^i be its "iterated extension" to output of length i (see Lecture 2, Construction 15).

Construction 12

- $G(1^n)$ outputs $e \leftarrow \{0, 1\}^n$,
- $E_e(m)$ outputs $g^{|m|}(e) \oplus m$
- $D_e(c)$ outputs $g^{|c|}(e) \oplus c$

Claim 13

(G, E, D) has private-key indistinguishable encryptions for a single message

Proof: Assume not, and let $B, \{x_n, y_n \in \{0, 1\}^{\ell(n)}\}_{n \in \mathbb{N}}$ and $\{z_n \in \{0, 1\}^{p(n)}\}_{n \in \mathbb{N}}$ be the triplet that realizes it. Wlog,

$$|\Pr[B(z_n, g^{|x_n|}(U_n) \oplus x_n) = 1] - \Pr[B(z_n, U_{|x_n|} \oplus x_n) = 1]| > \text{neg}(n) \quad (2)$$

Hence, B implies a (non-uniform) distinguisher for g

Claim 14

(G, E, D) does not have a private-key indistinguishable encryptions for multiple messages

Proof: Take $x_{n,1} = x_{n,2}, y_{n,1} \neq y_{n,2}$ and $D(c_1, c_2)$ outputs 1 iff $c_1 = c_2$

Section 2

Constructions

Private key indistinguishable encryptions for multiple messages

Suffice to encrypt messages of a single length (here the length is n).

Let \mathcal{F} be a (non-uniform) length preserving PRF

Construction 15

- $G(1^n)$: output $e \leftarrow \mathcal{F}_n$,
- $E_e(m)$: choose $r \leftarrow \{0, 1\}^n$ and output $(r, e(r) \oplus m)$
- $D_e(r, c)$: output $e(r) \oplus c$

Claim 16

(G, E, D) has private-key indistinguishable encryptions for a multiple messages

Proof:

Public key indistinguishable encryptions for multiple messages

Let (G, f, Inv) be a (non-uniform) family of trapdoor permutations (see Lecture 6, Def 8) and let b be an hardcore predicate for f .

Construction 17 (bit encryption)

- $G(1^n)$: output $(e, d) \leftarrow G(1^n)$
- $E_e(m)$: choose $r \leftarrow \{0, 1\}^n$ and output $(y = f_e(r), c = b(r) \oplus m)$
- $D_d(y, c)$: output $b(\text{Inv}_d(y)) \oplus c$

Claim 18

(G, E, D) has public-key indistinguishable encryptions for a multiple messages

- We believe that public-key encryptions are of different complexity than private-key ones

Section 3

Active Adversaries

Active Adversaries

- Chosen plaintext attack (CPA):
The adversary can ask for encryption and choose the messages to distinguish accordingly
- Chosen ciphertext attack (CCA):
The adversary can also ask for *decryptions* of certain messages
- In the public-key settings, the adversary is also given the public key
- We focus on indistinguishability, but each of the above definitions has an equivalent semantic security variant.

CPA Security

Let (G, E, D) be an encryption scheme. For a pair of algorithms $A = (A_1, A_2)$, $n \in \mathbb{N}$, $z \in \{0, 1\}^*$ and $b \in \{0, 1\}$, let:

Experiment 19 ($\text{Exp}_{A,n,z}^{\text{CPA}}(b)$)

- 1 $(e, d) \leftarrow G(1^n)$
- 2 $(m_0, m_1, s) \leftarrow A_1^{E_e(\cdot)}(1^n, z)$
- 3 $c \leftarrow E_e(m_b)$
- 4 Output $A_2^{E_e(\cdot)}(1^n, s, c)$

Definition 20 (private key CPA)

(G, E, D) has indistinguishable encryptions in the private-key model under CPA attack, if \forall PPT A_1, A_2 , and poly-bounded $\{z_n\}_{n \in \mathbb{N}}$:

$$|\Pr[\text{Exp}_{A,n,z_n}^{\text{CPA}}(0) = 1] - \Pr[\text{Exp}_{A,n,z_n}^{\text{CPA}}(1) = 1]| = \text{neg}(n)$$

- public-key variant...
- The scheme from Construction 15 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)
- The scheme from Construction 17 has indistinguishable encryptions in the public-key model (for short, public-key CPA secure)
- In both cases, definitions are *not* equivalent

CCA Security

Experiment 21 ($\text{Exp}_{A,n,Z}^{\text{CCA1}}(b)$)

- 1 $(e, d) \leftarrow G(1^n)$
- 2 $(m_0, m_1, s) \leftarrow A_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
- 3 $c \leftarrow E_e(m_b)$
- 4 Output $A_2^{E_e(\cdot)}(1^n, s, c)$

Experiment 22 ($\text{Exp}_{A,n,Z_n}^{\text{CCA2}}(b)$)

- 1 $(e, d) \leftarrow G(1^n)$
- 2 $(x_0, x_1, s) \leftarrow A_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
- 3 $c \leftarrow E_e(x_b)$
- 4 Output $A_2^{E_e(\cdot), D_d^{-c}(\cdot)}(1^n, s, c)$

Definition 23 (private key CCA1/CCA2)

(G, E, D) has indistinguishable encryptions in the private-key model under $x \in \{\text{CCA1}, \text{CCA2}\}$ attack, if \forall PPT A_1, A_2 , and poly-bounded $\{z_n\}_{n \in \mathbb{N}}$:

$$|\Pr[\text{Exp}_{A,n,z_n}^x(0) = 1] - \Pr[\text{Exp}_{A,n,z_n}^x(1) = 1]| = \text{neg}(n)$$

- The public key definition is analogous

Private-key CCA2

- Is the scheme from Construction 15 private-key CCA1 secure?
- CCA2 secure?

Let (G, E, D) be a private key CPA scheme, and let $(\text{Gen}_M, \text{Mac}, \text{Vrfy})$ be an existential unforgeable strong MAC.

Construction 24

- $G'(1^n)$: Output $(e \leftarrow G_E(1^n), k \leftarrow \text{Gen}_M(1^n))$.^a
- $E'_{d,k}(m)$: let $c = E_e(m)$ and output $(c, t = \text{Mac}_k(c))$
- $D_{e,k}(c, t)$: if $\text{Vrfy}_k(c, t) = 1$, output $D_e(c)$. Otherwise, output \perp

^aWe assume for simplicity that the encryption and decryption keys are the same.

Theorem 25

Construction 24 is a private-key CCA2-secure encryption scheme.

Proof: ?

Public-key CCA1

Let (G, E, D) be a public-key CPA scheme and let (P, V) be a NIZK for $\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) : \exists(m, z_0, z_1) \text{ s.t. } c_0 = E_{pk_0}(m, z_0) \wedge c_1 = E_{pk_1}(m, z_1)\}$

Construction 26 (The Naor-Yung Paradigm)

- $G'(1^n)$:
 - ① For $i \in \{0, 1\}$: set $(sk_i, pk_i) \leftarrow G(1^n)$.
 - ② Let $r \leftarrow \{0, 1\}^{\ell(n)}$, and output $pk' = (pk_0, pk_1, r)$ and $sk' = (pk', sk_0, sk_1)$
- $E'_{pk'}(m)$:
 - ① For $i \in \{0, 1\}$: $c_i = E_{pk_i}(m, z_i)$, where z_i is a uniformly chosen string of the right length
 - ② $\pi \leftarrow P((c_0, c_1, pk_0, pk_1), (m, z_0, z_1), r)$
 - ③ Output (c_0, c_1, π) .
- $D'_{sk'}(c_0, c_1, \pi)$: If $V((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, return $D_{sk_0}(c_0)$. Otherwise, return \perp

Omitted details:

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n .
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using "security parameter" n .

Is the scheme CCA1 secure? We need the NIZK to be "adaptive secure".

Theorem 27

Assuming that (P, V) is adaptive secure, then Construction 26 is a public-key CCA1 secure encryption scheme.

Proof: Given an attacker A' for the CCA1 security of (G', E', D') , we use it to construct an attacker A on the CPA security of (G, E, D) .

Let $S = (S_1, S_2)$ be the (adaptive) simulator for (P, V, \mathcal{L})

Algorithm 28 (A)

Input: $(1^n, pk)$

- ❶ let $j \leftarrow \{0, 1\}$, $pk_{1-j} = pk$, $(pk_j, sk_j) \leftarrow G(1^n)$ and $(r, s) \leftarrow S_1(1^n)$
- ❷ Emulate $A'(1^n, pk' = (pk_0, pk_1, r))$ as follows:
- ❸ On query (c_0, c_1, π) of A' to D' :
If $V((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, answer $D_{sk_j}(c_j)$.
Otherwise, answer \perp .
- ❹ Output the same pair (m_0, m_1) as A' does
- ❺ On challenge $c (= E_{pk}(m_b))$:
 - Set $c_{1-j} = c$, $a \leftarrow \{0, 1\}$, $c_j = E_{pk_j}(m_a)$, and $\pi \leftarrow S_2((c_0, c_1, pk_0, pk_1), r, s)$
 - Send $c' = (c_0, c_1, \pi)$ to A'
- ❻ Output the same value that A' does

Claim 29

Assume that A' breaks the CCA1 security of (G', E', D') with probability $\delta(n)$, then A breaks the CPA security of (G, E, D) with probability $(\delta(n) - \text{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V) , yields that

$$\Pr[A' \text{ "makes" } A(1^n) \text{ decrypt an invalid cipher}] = \text{neg}(n) \quad (3)$$

Hence, only negligible information leaks about j .

Let $A'(1^n, a^*, b^*)$ be the output of $A'(1^n)$ in the emulation induced by A , where $a = a^*$ and $b = b^*$. It holds that

- ① $A'(1^n, 0, 1) \equiv A'(1^n, 1, 0)$
- ② The adaptive zero-knowledge of (P, V) yields that $|\Pr[A'(1^n, 1, 1) = 1] - \Pr[A'(1^n, 0, 0) = 1]| \geq \delta(n) - \text{neg}(n)$

Let $A(b)$ be the outputs of A when the challenge is b .

$$\begin{aligned} & |\Pr[A(1) = 1] - \Pr[A(0) = 1]| \\ &= \left| \frac{1}{2}(\Pr[A'(0, 1) = 1] + \Pr[A'(1, 1) = 1]) \right. \\ &\quad \left. - \frac{1}{2}(\Pr[A'(0, 0) = 1] + \Pr[A'(1, 0) = 1]) \right| \\ &\geq \frac{1}{2} |\Pr[A'(1, 1) = 1] - \Pr[A'(0, 0) = 1]| \\ &\quad - \frac{1}{2} |\Pr[A'(1, 0) = 1] - \Pr[A'(0, 1) = 1]| \\ &\geq (\delta(n) - \text{neg}(n))/2 \end{aligned}$$

Public-key CCA2

- Is Construction 26 CCA2 secure?
- **Problem:** Soundness might not hold with respect to the simulated CRS, after seeing a proof for an *invalid* statement
- **Solution:** use *simulation sound* NIZK