

Application of Information Theory, Lecture 3

Graph Covering, Differential Entropy

Handout Mode

Iftach Haitner

Tel Aviv University.

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Part I

Applications to Graph Covering

Graph Covering

- ▶ How many graphs of certain type it takes to cover the full graph?
- ▶ K_n — the complete graph over $[n]$
- ▶ Let G_1, \dots, G_t be bipartite graphs over $[n]$ with $\bigcup_i G_i = K_n$.
What can we say about t ?
- ▶ Clearly, $t \geq \frac{\binom{n}{2}}{(n/2)^2} \approx 2$, but can we give a better bound?

Theorem 1

Let G_1, \dots, G_t be bipartite graphs over $[n]$ with $\bigcup_{i=1}^t G_i = K_n$, then $t \geq \log n$.

Proof: Let $\chi(G)$ be the chromatic number of G .

- ▶ $\chi(G_i) \leq 2$ and $\chi(K_n) = n$.
- ▶ $\chi(G \cup G') \leq \chi(G) \cdot \chi(G')$. (?)

$$\Rightarrow \chi\left(\bigcup_{i=1}^t G_i\right) \leq 2^t$$

$$\Rightarrow t \geq \log n$$

Proving Thm 1 using entropy

- ▶ $G_i = (A_i, B_i, E_i)$
- ▶ $X \leftarrow [n]$
- ▶ $Y_i = \begin{cases} 0, & X \in A_i \\ 1, & X \in B_i \end{cases}$
- ▶ X is determined by Y_1, \dots, Y_t (?)

$$\begin{aligned} 0 &= H(X|Y_1, \dots, Y_t) = H(X, Y_1, \dots, Y_t) - H(Y_1, \dots, Y_t) \\ &\geq H(X) - \sum_i H(Y_i) \\ &\geq \log n - t. \end{aligned}$$

Extensions

- $\text{nonls}(G)$ — non-isolated vertices in G .

Theorem 2

Let G_1, \dots, G_t be bipartite graphs over $[n]$ with $\bigcup_{i=1}^t G_i = K_n$, then $\frac{1}{n} \sum_{i=1}^t |\text{nonls}(G_i)| \geq \log n$.

Definition 3 (graph content)

Let G be a graph over $[n]$, let $Z \leftarrow \text{nonls}(G)$ and let $\hat{\chi}$ be a (valid) coloring of G such that $H(\hat{\chi}(Z))$ is minimal. Then $\text{content}(G) := \frac{|\text{nonls}(G)|}{n} \cdot H(\hat{\chi}(Z))$.

Theorem 4

Let G_1, \dots, G_t be graphs over $[n]$ with $\bigcup_{i=1}^t G_i = K_n$. Then $\sum \text{content}(G_i) \geq \log n$.

- Since $\text{content}(G) \leq \frac{|\text{nonls}(G)|}{n}$ for bipartite G , Thm 4 yields Thm 2.

Proving Thm 4

- ▶ Let χ_i be a (valid) coloring of G_i .
- ▶ Let $X \leftarrow [n]$, and let
$$Y_i = \begin{cases} \chi_i(X) & X \in \text{nonls}(G_i) \\ \chi_i(Z_i) & \text{otherwise, for } Z_i \leftarrow \text{nonls}(G_i) \text{ (ind. of the other } Z\text{'s).} \end{cases}$$
- ▶ X is **determined** by Y_1, \dots, Y_t (?)

$$\begin{aligned} 0 &= H(X|Y_1, \dots, Y_t) = H(X, Y_1, \dots, Y_t) - H(Y_1, \dots, Y_t) \\ &\geq H(X) + H(Y_1, \dots, Y_t|X) - \sum_i H(Y_i) \\ &= \log n + H(Y_1, \dots, Y_t|X) - \sum_i H(Y_i). \end{aligned}$$

- ▶ Y_1, \dots, Y_t are **independent** conditioned on X —

$$\Pr[Y_1 = y_1 \wedge Y_2 = y_2 | X = x] = \Pr[Y_1 = y_1 | X = x] \cdot \Pr[Y_2 = y_2 | X = x]$$

- ▶ Hence, $H(Y_1, \dots, Y_t|X) = \sum_i H(Y_i|X)$ (board)
- ▶ We conclude that $\sum_i H(Y_i) - \sum_i H(Y_i|X) \geq \log n$
- ▶ Since $H(Y_i) = H(\chi_i(Z_i))$ and $H(Y_i|X) = (1 - \frac{|\text{nonls}(G_i)|}{n}) \cdot H(\chi_i(Z_i))$,
it follows that $\sum_i H(\chi_i(Z_i)) \frac{|\text{nonls}(G_i)|}{n} \geq \log n$. \square

Extension

Let $\alpha(G)$ be the size of the maximal independent set in G .

Theorem 5

Let G, G_1, \dots, G_t be graphs over $[n]$ with $\bigcup_{i=1}^t G_i = G$, then
$$\sum \text{content}(G_i) \geq \log \frac{n}{\alpha(G)}.$$

Proof: HW

Scrambling permutations

Theorem 6

Let \mathcal{S} be a set of permutations over $[n]$ s.t. for any triplet (i, j, k) of distinct elements of $[n]$, exists $\pi \in \mathcal{S}$ with $\pi(i) < \pi(j) < \pi(k)$ or $\pi(i) > \pi(j) > \pi(k)$. Then $|\mathcal{S}| \geq \frac{2}{\log e} \log n$.

- ▶ For $\pi \in \mathcal{S}$, the graph $G_\pi = (V, E_\pi)$ is defined by:
 - ▶ $V = \{(i, j) \in [n]^2 : i \neq j\}$
 - ▶ $E_\pi = \{((i, j), (k, j)) \in V^2 : \pi(i) < \pi(j) < \pi(k) \vee \pi(i) > \pi(j) > \pi(k)\}$
- ▶ $G = \bigcup_{\pi \in \mathcal{S}} G_\pi$ has n connected components, each consists of $(n-1)$ -vertex cliques: $\{(i, j) : i \in [n] \setminus \{j\}\}$ for each $j \in [n]$.
- ▶ G_π consists of n complete bipartite graphs (two are empty): $\{(i, j) : \pi(i) \leq \pi(j)\}$ and $\{(i, j) : \pi(i) > \pi(j)\}$ for each $j \in [n]$.

The sum of content of these bipartite graphs is

$$\sum_{i=0}^{n-1} h\left(\frac{i}{n-1}\right) = (n-1) \sum_{i=0}^{n-1} h\left(\frac{i}{n-1}\right) \cdot \frac{1}{n-1} \leq (n-1) \int_0^1 h(p) dp = (n-1) \cdot \frac{\log e}{2}.$$

- ▶ By Thm 5 (applied for each component) $|\mathcal{S}| \cdot \frac{\log e}{2} \cdot (n-1) \geq n \log(n-1)$.
- ▶ Hence, $|\mathcal{S}| \geq \frac{2}{\log e} \cdot \frac{n}{n-1} \cdot \log(n-1) \geq \frac{2}{\log e} \log n$. \square

Part II

Differential Entropy

Entropy of continuous random variable

- ▶ Entropy of **discrete** random variable: $H(X) = -\sum_i p_i \log p_i$
- ▶ Also used when X has **infinite** # of states (entropy might be infinite!)
- ▶ Continuous random variable is defined by its **density function**:
 $f: \mathbb{R} \mapsto \mathbb{R}^+$, for which $\int_{\mathbb{R}} f(x) dx = 1$.
- ▶ $F_X(x) := \Pr[X \leq x] = \int_{-\infty}^x f(x) dx$
- ▶ $E X = \int x \cdot f(x) dx$ and $V X = \int x^2 \cdot f(x) dx - (E X)^2$
- ▶ Examples: $X \sim [0, 1]$, $X \sim N(0, 1)$
- ▶ $H(X)$ must be infinite! it takes infinite number of bits to describe X
- ▶ The **differential entropy** of X is defined by $h(X) = -\int f(x) \log f(x) dx$.
- ▶ We focus on cases where $h(X)$ is **well defined**.
- ▶ Since h is a function of the density function, we sometimes write $h(f)$
- ▶ If not stated otherwise, we integrate over \mathbb{R}

Intuition for definition of h

- ▶ Let X^Δ be **rounding** of X for precision Δ :

$$X^\Delta \sim (\dots, p_{-2}, p_{-1}, p_0, p_1, p_2, \dots),$$

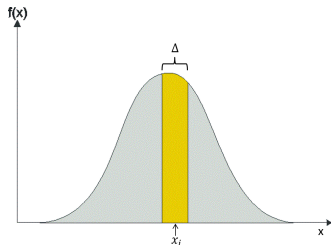
$$\text{where } p_i = \int_{i \cdot \Delta}^{(i+1) \cdot \Delta} f(x) dx = f(x_i) \cdot \Delta$$

for some $x_i \in [i \cdot \Delta, (i+1) \cdot \Delta]$ (?)

- ▶ $H(X^\Delta) = - \sum_{i=-\infty}^{\infty} p_i \log p_i$

$$\begin{aligned} H(X^\Delta) &= - \sum_{i=-\infty}^{\infty} f(x_i) \cdot \Delta \cdot \log(f(x_i) \cdot \Delta) = - \sum_{i=-\infty}^{\infty} f(x_i) \cdot \Delta \cdot (\log f(x_i) + \log \Delta) \\ &= - \sum_{i=-\infty}^{\infty} f(x_i) \cdot \log f(x_i) \cdot \Delta - \left(\sum_{i=-\infty}^{\infty} f(x_i) \cdot \Delta \right) \log \Delta \end{aligned}$$

- ▶ $\lim_{\Delta \rightarrow 0} H(X^\Delta) = h(X) - \log \Delta$
- ▶ Hence, $\lim_{\Delta \rightarrow 0} H(X^\Delta) + \log \Delta = h(x)$
- ▶ Intuitively, $h(X)$ is the entropy of X plus const ($-\log \Delta$).
- ▶ Note that $\lim_{\Delta \rightarrow 0} -\log \Delta = \infty$



Properties of the entropy function

- ▶ Shift invariant: $h(f) = h(g)$ for $g(x) = f(x + a)$
- ▶ $h(X) = - \int f(x) \log f(x) dx$ might be infinite (?)
- ▶ Example $f(x) = 2^{-2^i}$ with probability $1/2^i$ (i.e., over segment of length $2^{-i}/2^{-2^i}$)
- ▶ $h(X)$ might be negative
- ▶ Example: $X \sim [0, a] - f(x) = \frac{1}{a}$ on $[1, a]$
 $-\int f(x) \log f(x) dx = -\log \frac{1}{a} = \log a$. Negative for $a < 1$.
- ▶ $h(X)$ should be interpreted as the uncertainty up to a certain constant
- ▶ Used for comparing two distributions

Common distribution (in nature)

- ▶ The uniform distribution: $X \sim [a, b]$
- ▶ Normal (Gaussian) distribution: (we focus on $E = 0$ and $V = 1$)
 $X \sim N(0, 1): f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$
- ▶ Boltzmann (Gibbs) distribution:
 $X \in \{E_1, E_2, \dots, E_m\}$, $\Pr[X = E_i] = C \cdot e^{-\beta E_i}$ for $\beta > 0$ (the *distribution constant*) and $C = 1 / \sum_i e^{-\beta E_i}$.
 - ▶ Describes a (discrete) physical system that can take states $\{1, \dots, m\}$ with energies E_1, \dots, E_m .
 - ▶ Probability is inverse to the energy
- ▶ Why are these distributions so common?
- ▶ What is common to these distributions?

Historical background

- ▶ Shannon (1948) $H = - \sum_i p_i \log p_i$
- ▶ But the notion of entropy already existed in statistical physics
- ▶ There, entropy — energy that cannot be used, statistical disorder
- ▶ Clausius (1865), who coined the name *entropy*, based on Carnot (1824),
 $H = \int_t \frac{\delta Q}{T} dt$ (Q is *heat* and T is *temperature*)
- ▶ Boltzmann (1877) $H = \log S$, for S being the number of states a system can be in (after measuring the macro parameters: pressure, temperature)
- ▶ $\log \#$ of states is Shannon entropy of the uniform distribution
- ▶ Shannon looked for a name for his measure, von Neumann pointed out the relation to physics and suggested the name entropy.
- ▶ Today it is accepted that Shannon's entropy is the right notion also in statistical mechanics. Measures the uncertainty of a system — energy that cannot be used.
- ▶ Carnot was also an engineer...

Second law of thermodynamics

- ▶ The entropy of a closed physical system **never** decreases.
- ▶ If we wait enough time, the system tends to be in **maximal** entropy.
- ▶ If there are constraints, the it tends to be in maximal entropy under this constraints.
- ▶ This suggests that distributions that are common in nature, are distributions of maximal entropy, under some constraints.
- ▶ In contradiction with “reversible laws”

The normal distribution

- ▶ $X \sim N(0, 1): f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$

- ▶ Why is it so common?

- ▶ Answer: the central limit theorem (CLT):

Let X_1, \dots, X_n be iid with $E X_i = 0$ and $V X_i = 1$. Then

$$\lim_{n \rightarrow \infty} \frac{\sum_i X_i}{\sqrt{n}} = N(0, 1).$$

- ▶ But why does it converge to $N(0, 1)$??

- ▶ CLT holds also in many other variants: not id, not fully independent, ...

- ▶ We know that $E \frac{\sum_i X_i}{\sqrt{n}} = 0$ and $V \frac{\sum_i X_i}{\sqrt{n}} = 1$, but it could have converge to any other distribution with these constraints.

- ▶ The reason is that $N(0, 1)$ has the **highest** entropy among all distribution with $E = 0$ and $V = 1$.

- ▶ CLT and the normal distribution where known and studied way before Shannon, yet this striking property was not known until his theory.

The normal distribution, cont.

Theorem 7

$h(X) \leq h(N(0, 1))$, for any rv X with $V X = 1$.

- ▶ Among the distributions of $V = 1$, the distribution $N(0, 1)$ has maximal entropy.
- ▶ Generalizes to any variance:

$$h(X) \leq h(N(0, V(X))) = \frac{1}{2} \cdot \log(2\pi e) \cdot V(X)$$

Let g be a density function with $\int g(x)x^2 dx = 1$, and let $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$.

We will show that

1. $-\int g(x) \log g(x) dx \leq -\int g(x) \log f(x) dx$
2. $-\int g(x) \log f(x) dx = -\int f(x) \log f(x) dx$

$$-\int g(x) \log g(x) dx \leq -\int g(x) \log f(x) dx$$

Claim 8

$-\int g(x) \log g(x) dx \leq -\int g(x) \log q(x) dx$ for any two density functions q, g .

Proof: (the continuous version of Q3 in handout 1)

- ▶ Jensen: For any function t and density function λ :

$$\int \lambda(x) \log t(x) \leq \log \int \lambda(x) t(x) dx$$
- ▶ Assume for simplicity that $g(x) > 0$ for all x .
- ▶ By Jensen, $\int g(x) \log \frac{q(x)}{g(x)} \leq \log \int g(x) \frac{q(x)}{g(x)} dx = \log 1 = 0$
- ▶ Hence, $-\int g(x) \log g(x) \leq -\int g(x) \log q(x)$

$$-\int g(x) \log f(x) dx = -\int f(x) \log f(x) dx$$

Claim 9

Exists $c \in \mathbb{R}$ such that $-\int g(x) \log f(x) dx = c$ for any density function g with $\int g(x) x^2 dx = 1$.

Hence, $-\int g(x) \log f(x) dx = -\int f(x) \log f(x) dx$.

Proof:

$$\begin{aligned} -\int g(x) \log f(x) dx &= -\int g(x) \log \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} dx \\ &= -\int g(x) \left(\log \frac{1}{\sqrt{2\pi}} - \frac{x^2}{2} \cdot \log e \right) \\ &= -\log \frac{1}{\sqrt{2\pi}} \int g(x) dx + \frac{\log e}{2} \int g(x) x^2 dx \\ &= -\log \frac{1}{\sqrt{2\pi}} + \frac{\log e}{2}. \end{aligned}$$

□

The Boltzmann distribution

- ▶ States $\{1, \dots, m\}$, energies E_1, \dots, E_m .
- ▶ $\Pr[X = E_i] = C \cdot e^{-\beta E_i}$ for $\beta > 0$ and $C = 1 / \sum_i e^{-\beta \cdot E_i}$
- ▶ We will denote it by $\sim B(\beta, E_1, \dots, E_m)$
- ▶ Like the exponential distribution (i.e., $f(x) = \lambda e^{-\lambda x}$), but **discrete**.
 - ▶ Describes a (discrete) physical system that can take states $\{1, \dots, m\}$ with energies E_1, \dots, E_m .
 - ▶ Probability is inverse to energy

Theorem 10

Let $X \sim B(\beta, E_1, \dots, E_m)$. Then $H(Y) \leq H(X)$ for any rv Y over $\{E_1, \dots, E_m\}$, with $\mathbb{E} Y = \mathbb{E} X$.

- ▶ The Boltzmann distribution is **maximal** among all distributions of the same energy.

Proving Theorem 10

- ▶ $\sim B(\beta, E_1, \dots, E_m)$ and $E Y = E X$
- ▶ Let $X \sim (p_1, \dots, p_m)$ and $Y \sim (q_1, \dots, q_m)$ over $\{E_1, \dots, E_m\}$.
- ▶ $H(Y) \leq \sum_i q_i \log p_i$ (Q3 in Handout 1)
- ▶ Let $C = 1 / \sum_i e^{-\beta \cdot E_i}$.

Then

$$\begin{aligned}\sum_i q_i \log p_i &= \sum_i q_i \log(C \cdot e^{-\beta E_i}) \\&= \sum_i q_i \log C - \sum_i q_i \cdot \beta E_i \cdot \log e \\&= \log C - \beta \cdot \log e \cdot \sum_i q_i E_i \\&= \log C - \beta \cdot \log e \cdot E X\end{aligned}$$

- ▶ Hence, $\sum_i q_i \log p_i = \sum_i p_i \log p_i$. \square

The uniform distribution

- ▶ $X \sim [a, b]$.
- ▶ $E X = \frac{1}{2}(a + b)$ and $V X = \frac{1}{12}(b - a)^2$
- ▶ What come to mind when saying “ X takes values in $[0, 1]$ ”.

Theorem 11

$h(X) \leq -h(\sim [a, b])$, for any RV with $\text{Supp}(X) \subseteq [a, b]$.

Proof: HW

Using diff. entropy to bound discrete entropy

Proposition 12

Let $X \sim (p_1, p_2, \dots)$, then $H(X) \leq \frac{\log 2\pi e}{2} \cdot (\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2 - \frac{1}{12})$

We assume wlg. that $p_i = \Pr[X = i]$.

- ▶ Let $U \sim [0, 1]$, let $\tilde{X} = X + U$ and let $f_{\tilde{X}}$ be the density function of \tilde{X} .

$$\begin{aligned} H(X) &= - \sum_{i=1}^{\infty} p_i \log p_i \\ &= - \sum_{i=1}^{\infty} \left(\int_i^{i+1} f_{\tilde{X}}(x) dx \right) \cdot \log p_i = - \sum_{i=1}^{\infty} \int_i^{i+1} f_{\tilde{X}}(x) \log p_i dx \\ &= - \sum_{i=1}^{\infty} \int_i^{i+1} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x) dx \quad (f_{\tilde{X}}(x) = p_i \text{ for all } x \in [i, i+1]) \\ &= - \int_1^{\infty} f_{\tilde{X}}(x) \log f_{\tilde{X}}(x) dx \\ &= h(\tilde{X}) \end{aligned}$$

Using diff. entropy to bound discrete entropy, cont.

- Hence,

$$\begin{aligned} H(X) &= h(\tilde{X}) \\ &\leq \frac{1}{2} \log(2\pi e) V(\tilde{X}) \\ &= \frac{1}{2} \log(2\pi e) (V(X) + V(U)) \\ &= \frac{\log 2\pi e}{2} \cdot \left(\left(\sum_{i=1}^{\infty} p_i \cdot i^2 - \left(\sum_{i=1}^{\infty} p_i \cdot i \right)^2 \right) + \frac{1}{12} \right) \end{aligned}$$

- How good is this bound?
- Let $X \sim (\frac{1}{2}, \frac{1}{2})$. Hence, $V[X] = \frac{1}{4}$ and $H(X) = 1$.
- **Proposition 12** grants that $H(X) \leq \frac{\log 2\pi e}{2} (\frac{1}{4} + \frac{1}{12}) \sim 1.255$