Application of Information Theory, Lecture 7

Relative Entropy

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Part I

Statistical Distance

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- Interpretation

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Theorem 1 (this lecture)

Let X rv over $[m]$. Assume $H(X) \geq \log m - \varepsilon$, then
$\mathrm{SD}(X, \sim [m]) \leq \sqrt{\varepsilon \cdot \frac{\ln 2}{2}} = O(\sqrt{\varepsilon})$

Part II

Relative entropy Distance

Section 1

Definition and Basic Facts

Definition

- For $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$, let

$$D(p\|q) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i}$$

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- Many different interpretations
- Main interpretation: the information we gained about X, if we originally thought $X \sim q$ and now we learned $X \sim p$

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- We understand $D(p\|q)$ as the information we gained about X, if we originally thought it is $\sim q$ and now we learned it is $\sim p$

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1	$\frac{1}{4}$	0	$\frac{1}{4}$	0

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- $Y \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$, but

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- $D(p\|q) \geq 0$, with equality iff $p = q$ (hw)

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- We gained k bits of information
- Example: $\sum_{i=1}^{n} q_i = \frac{1}{2}$, and we were told that $i \leq n$ or $i > n$, we got one bit of information

Section 2

Axiomatic Derivation

Axiomatic derivation

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Let $\tilde{D}$ is a continuous and symmetric (wrt each distribution) function such that

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- Zeros and non-rational q_i's are dealt by continuity

Section 3

Relation to Mutual Information

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$$\begin{aligned}\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y = 0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\ &\quad + \Pr[Y = 1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m)\end{aligned}$$

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0,1\}$
- $(X|Y=0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X = i | Y = 0]$
- $(X|Y=1) \sim p_1 = (p_{1,1}, \ldots, p_{1,m})$, $\qquad p_{1,i} = \Pr[X = i | Y = 1]$
- If we learned $Y = j$, we gained $D(p_j \| q)$

$$
\begin{aligned}
\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y=0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
&\quad + \Pr[Y=1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m) \\
&= \Pr[Y=0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y=1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i}
\end{aligned}
$$

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0, 1\}$
- $(X|Y = 0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X = i|Y = 0]$
- $(X|Y = 1) \sim p_1 = (p_{1,1}, \ldots, p_{1,m})$, $\qquad p_{1,i} = \Pr[X = i|Y = 1]$
- If we learned $Y = j$, we gained $D(p_j \| q)$

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\begin{aligned}
\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y = 0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
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&= \Pr[Y = 0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y = 1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y = 0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y = 1] \cdot \sum p_{1,i} \log p_{1,i} \\
&\quad - \Pr[Y = 0] \cdot \sum p_{0,i} \log q_i - \Pr[Y = 1] \cdot \sum p_{1,i} \log q_i
\end{aligned}
$$

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0, 1\}$
- $(X|Y=0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X=i|Y=0]$
- $(X|Y=1) \sim p_1 = (p_{1,1}, \ldots, p_{1,m})$, $\qquad p_{1,i} = \Pr[X=i|Y=1]$
- If we learned $Y = j$, we gained $D(p_j \| q)$

$$
\begin{aligned}
\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y=0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
&+ \Pr[Y=1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m) \\
&= \Pr[Y=0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y=1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y=0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y=1] \cdot \sum p_{1,i} \log p_{1,i} \\
&- \Pr[Y=0] \cdot \sum p_{0,i} \log q_i - \Pr[Y=1] \cdot \sum p_{1,i} \log q_i \\
&= -H(X|Y) - \sum (\Pr[Y=0] \cdot p_{0,i} + \Pr[Y=1] \cdot p_{1,i}) \log q_i
\end{aligned}
$$

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0,1\}$
- $(X|Y=0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X=i|Y=0]$
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\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y=0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
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&= \Pr[Y=0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y=1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y=0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y=1] \cdot \sum p_{1,i} \log p_{1,i} \\
&\quad - \Pr[Y=0] \cdot \sum p_{0,i} \log q_i - \Pr[Y=1] \cdot \sum p_{1,i} \log q_i \\
&= -H(X|Y) - \sum (\Pr[Y=0] \cdot p_{0,i} + \Pr[Y=1] \cdot p_{1,i}) \log q_i \\
&= -H(X|Y) + H(X)
\end{aligned}
$$

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0, 1\}$
- $(X|Y = 0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X = i|Y = 0]$
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&= \Pr[Y = 0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y = 1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y = 0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y = 1] \cdot \sum p_{1,i} \log p_{1,i} \\
&\quad - \Pr[Y = 0] \cdot \sum p_{0,i} \log q_i - \Pr[Y = 1] \cdot \sum p_{1,i} \log q_i \\
&= -H(X|Y) - \sum (\Pr[Y = 0] \cdot p_{0,i} + \Pr[Y = 1] \cdot p_{1,i}) \log q_i \\
&= -H(X|Y) + H(X) = I(X; Y)
\end{aligned}
$$

Equivalent definition for mutual information

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- $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$

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- Proof:

$$D(p \| p_X p_Y) = \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x) p_Y(y)}$$

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$$= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)}$$

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&= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\
&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y)
\end{aligned}
$$

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- $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$
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$$= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)}$$
$$= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y)$$
$$= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y)$$

Equivalent definition for mutual information

- $(X,Y) \sim p$, then $I(X;Y) = D(p\|p_X p_Y)$
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- Proof:

$$
\begin{aligned}
D(p\|p_X p_Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x)p_Y(y)} \\
&= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\
&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y)
\end{aligned}
$$

Equivalent definition for mutual information

- $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$
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&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y) = I(X; Y)
\end{aligned}
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- $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$
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- Proof:

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- We will later see the relation between the above two facts.

Section 4

Relation to Data Compression

Wrong code

Wrong code

Theorem 2

Let p *and* q *be distributions over* $[m]$*, and let* C *be code with* $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$*. Then*

$$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$$

Wrong code

Theorem 2

Let p *and* q *be distributions over* $[m]$*, and let* C *be code with* $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$*. Then*
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.

Wrong code

Theorem 2

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

Wrong code

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Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
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Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
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- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

$$\mathop{\mathrm{E}}_{i \leftarrow p}\left[\ell(i)\right] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil$$

Wrong code

Theorem 2

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

$$\mathop{\mathrm{E}}_{i \leftarrow p}\left[\ell(i)\right] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1)$$

Wrong code

Theorem 2

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

$$\mathop{\mathrm{E}}_{i \leftarrow p}\left[\ell(i)\right] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1)$$
$$= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i})$$

Wrong code

Theorem 2

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
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$$= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i}) = 1 + \sum_i p_i (\log \frac{p_i}{q_i}) + \sum_i p_i (\log \frac{1}{p_i})$$

Wrong code

Theorem 2

Let p *and* q *be distributions over* $[m]$, *and let* C *be code with* $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. *Then*
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

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$$\begin{aligned}
\mathop{\mathrm{E}}_{i \leftarrow p}\left[\ell(i)\right] &= \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1) \\
&= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i}) = 1 + \sum_i p_i (\log \frac{p_i}{q_i}) + \sum_i p_i (\log \frac{1}{p_i}) \\
&= 1 + D(p\|q) + H(p)
\end{aligned}$$

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Theorem 2

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
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&= 1 + D(p\|q) + H(p)
\end{aligned}
$$

- Can there be a (close) to optimal code for q that is better for p?

Wrong code

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Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
$H(p) + D(p\|q) \le \mathrm{E}_{i \leftarrow p}[\ell(i)] \le H(p) + D(p\|q) + 1$

- Recall that $H(q) \le \mathrm{E}_{i \leftarrow q}[\ell(i)] \le H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

$$\begin{aligned}
\mathop{\mathrm{E}}_{i \leftarrow p}[\ell(i)] &= \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1) \\
&= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i}) = 1 + \sum_i p_i (\log \frac{p_i}{q_i}) + \sum_i p_i (\log \frac{1}{p_i}) \\
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\end{aligned}$$

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- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

$$\underset{i \leftarrow p}{\mathrm{E}}\left[\ell(i)\right] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1)$$
$$= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i}) = 1 + \sum_i p_i (\log \frac{p_i}{q_i}) + \sum_i p_i (\log \frac{1}{p_i})$$
$$= 1 + D(p\|q) + H(p)$$

- Can there be a (close) to optimal code for q that is better for p? HW

Section 5

Conditional Relative Entropy

Conditional relative entropy

Conditional relative entropy

Definition 3

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) := \sum_{x \in \mathcal{X}} p_{\mathcal{X}}(x) \cdot \sum_{y \in \mathcal{Y}} p_{\mathcal{Y}|\mathcal{X}}(y|x) \log \frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)}$$

Conditional relative entropy

Definition 3

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

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$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

Conditional relative entropy

Definition 3

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

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$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then
  $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} \left[D(Y_q|_{X_p = x} \| Y_q|_{X_q = x}) \right]$

Conditional relative entropy

Definition 3

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) := \sum_{x \in \mathcal{X}} p_{\mathcal{X}}(x) \cdot \sum_{y \in \mathcal{Y}} p_{\mathcal{Y}|\mathcal{X}}(y|x) \log \frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)}$$

$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} \left[D(Y_q|_{X_p=x} \| Y_q|_{X_q=x}) \right]$
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Conditional relative entropy

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Section 6

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- Hence, $D(f(X)\|f(Y)) \leq D(X\|Y)$.

Section 7

Relation to Statistical Distance

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- Corollary: For rv X over $[m]$ with $H(X) \geq m - \varepsilon$, it holds that $\mathrm{SD}(X, \sim [m]) \leq \sqrt{\frac{\ln 2}{2} \cdot (m - H(X))} = \sqrt{\frac{\ln 2}{2} \cdot \varepsilon}$

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  HW

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  $D(p\|q) = \alpha \log \frac{\alpha}{\beta} + (1-\alpha)\log\frac{1-\alpha}{1-\beta} \geq \frac{4}{2\ln 2}(\alpha-\beta)^2 = \frac{2}{\ln 2}\mathrm{SD}(p,q)^2$
- Let $g(\alpha,\beta) = \alpha \log \frac{\alpha}{\beta} + (1-\alpha)\log\frac{1-\alpha}{1-\beta} - \frac{4}{2\ln 2}(\alpha-\beta)^2$

$$\begin{aligned}\frac{\partial g(x,y)}{\partial y} &= -\frac{x}{y\ln 2} + \frac{1-x}{(1-y)\ln 2} - \frac{4}{2\ln 2}2(y-x)\\ &= \frac{y-x}{y(1-y)\ln 2} - \frac{4}{\ln 2}(y-x)\end{aligned}$$

- Since $y(1-y) \leq \frac{1}{4}$, $\frac{\partial g(x,y)}{\partial y} \leq 0$ for $y < x$.

Proving Thm 6, boolean case

- Let $p = (\alpha, 1-\alpha)$ and $q = (\beta, 1-\beta)$ and assume $\alpha \geq \beta$
- $\mathrm{SD}(p,q) = \alpha - \beta$
- We will show that
  $D(p\|q) = \alpha \log \frac{\alpha}{\beta} + (1-\alpha)\log \frac{1-\alpha}{1-\beta} \geq \frac{4}{2\ln 2}(\alpha-\beta)^2 = \frac{2}{\ln 2}\mathrm{SD}(p,q)^2$
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- Since $y(1-y) \leq \frac{1}{4}$, $\frac{\partial g(x,y)}{\partial y} \leq 0$ for $y < x$.
- Since $g(x,x) = 0$, $g(x,y) \geq 0$ for $y < x$. $\square$

Proving Thm 6, general case

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- Let $\mathcal{U} = \text{Supp}(p) \cup \text{Supp}(q)$

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$$D(p\|q) \geq D(\hat{P}\|\hat{Q})$$

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$D(p\|q) \geq D(\hat{P}\|\hat{Q})$ (data-proccessing inequality)

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- $\mathsf{SD}(\hat{P}, \hat{Q}) = \Pr[P \in \mathcal{S}] - \Pr[Q \in \mathcal{S}] = \mathsf{SD}(p, q)$

$$\begin{aligned} D(p\|q) &\geq D(\hat{P}\|\hat{Q}) && \text{(data-proccessing inequality)} \\ &\geq \frac{2}{\ln 2} \cdot \mathsf{SD}(\hat{P}, \hat{Q})^2 \end{aligned}$$

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D(p\|q) &\geq D(\hat{P}\|\hat{Q}) && \text{(data-proccessing inequality)}\\
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\end{aligned}
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$$\begin{aligned}
D(p\|q) &\geq D(\hat{P}\|\hat{Q}) && \text{(data-proccessing inequality)} \\
&\geq \frac{2}{\ln 2} \cdot \mathrm{SD}(\hat{P}, \hat{Q})^2 && \text{(the Boolean case)} \\
&= \frac{2}{\ln 2} \cdot \mathrm{SD}(p, q)^2. \ \square
\end{aligned}$$

Section 8

Conditioned Distributions

Main theorem

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Theorem 7

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{U}$*, and let* $Y = (Y_1, \ldots, Y_k)$ *be rv over* $\mathcal{U}^k$*. Then* $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

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For rv Z, let $Z(z) = \Pr[Z = z]$.

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We prove for $k = 2$, general case follows similar lines.

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$$D(Y \| X) = \sum_{\mathbf{y} \in \mathcal{U}^2} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{X(\mathbf{y})}$$

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D(Y \| X) &= \sum_{\mathbf{y} \in \mathcal{U}^2} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{X(\mathbf{y})} = \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(y_1)} \frac{Y_2(y_2)}{X_2(y_2)} \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(x_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(x_1)} \\
&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)}
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&= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(x_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(x_1)} \\
&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)} \\
&= D(Y_1\|X_1) + D(Y_2\|X_2) + I(Y_1; Y_2)
\end{aligned}
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Let $X_1, \ldots, X_k$ be iid over $\mathcal{U}$, and let $Y = (Y_1, \ldots, Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^k D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

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We prove for $k = 2$, general case follows similar lines. Let $X = (X_1, X_2)$

$$D(Y\|X) = \sum_{\mathbf{y} \in \mathcal{U}^2} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{X(\mathbf{y})} = \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(y_1)} \frac{Y_2(y_2)}{X_2(y_2)} \frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)}$$

$$= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(x_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(x_1)}$$

$$+ \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)}$$

$$= D(Y_1\|X_1) + D(Y_2\|X_2) + I(Y_1; Y_2) \geq D(Y_1\|X_1) + D(Y_2\|X_2)$$

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&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= D(Y_1 \| X_1) + D(Y_2 \| X_2) + I(Y_1; Y_2) \geq D(Y_1 \| X_1) + D(Y_2 \| X_2)
\end{aligned}
$$

Conditioning distributions, relative entropy case

Conditioning distributions, relative entropy case

Theorem 8

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{X}$, *let* $X = (X_1, \ldots, X_k)$ *and let* W *be an event (i.e., Boolean rv). Then* $\sum_{j=1}^{k} D((X_j|_W)\|X_j) \leq D((X|_W)\|X) \leq \log \frac{1}{\Pr[W]}$.

Conditioning distributions, relative entropy case

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$$\sum_{j=1}^{k} D((X_j|_W)\|X_j) \leq D((X|_W)\|X) \qquad \text{(Thm 7)}$$

$$= \sum_{\mathbf{x} \in \mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{(X|_W)(\mathbf{x})}{X(\mathbf{x})}$$

Conditioning distributions, relative entropy case

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$$\sum_{j=1}^{k} D((X_j|_W)\|X_j) \leq D((X|_W)\|X) \qquad \text{(Thm 7)}$$

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$$= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{\Pr[W|X=\mathbf{x}])}{\Pr[W]}$$

Conditioning distributions, relative entropy case

Theorem 8

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$, let $X = (X_1, \ldots, X_k)$ and let W be an event (i.e., Boolean rv). Then $\sum_{j=1}^{k} D((X_j|_W) \| X_j) \leq D((X|_W) \| X) \leq \log \frac{1}{\Pr[W]}$.

$$\sum_{j=1}^{k} D((X_j|_W) \| X_j) \leq D((X|_W) \| X) \qquad \text{(Thm 7)}$$

$$= \sum_{\mathbf{x} \in \mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{(X|_W)(\mathbf{x})}{X(\mathbf{x})}$$

$$= \sum_{\mathbf{x} \in \mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{\Pr[W|X = \mathbf{x}])}{\Pr[W]} \qquad \text{(Bayes)}$$

Conditioning distributions, relative entropy case

Theorem 8

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$, let $X = (X_1, \ldots, X_k)$ and let W be an event (i.e., Boolean rv). Then $\sum_{j=1}^{k} D((X_j|_W)\|X_j) \leq D((X|_W)\|X) \leq \log \frac{1}{\Pr[W]}$.

$$
\begin{aligned}
\sum_{j=1}^{k} D((X_j|_W)\|X_j) &\leq D((X|_W)\|X) && \text{(Thm 7)} \\
&= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{(X|_W)(\mathbf{x})}{X(\mathbf{x})} \\
&= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{\Pr[W|X=\mathbf{x}])}{\Pr[W]} && \text{(Bayes)} \\
&= \log \frac{1}{\Pr[W]} + \sum_{\mathbf{x}\in\mathcal{X}^k} (X|_W)(\mathbf{x}) \log \Pr[W|X=\mathbf{x}])
\end{aligned}
$$

Conditioning distributions, relative entropy case

Theorem 8

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$, let $X = (X_1, \ldots, X_k)$ and let W be an event (i.e., Boolean rv). Then $\sum_{j=1}^{k} D((X_j|_W)\|X_j) \leq D((X|_W)\|X) \leq \log \frac{1}{\Pr[W]}$.

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\begin{aligned}
\sum_{j=1}^{k} D((X_j|_W)\|X_j) &\leq D((X|_W)\|X) && \text{(Thm 7)}\\
&= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{(X|_W)(\mathbf{x})}{X(\mathbf{x})}\\
&= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|_W)(\mathbf{x}) \log \frac{\Pr\left[W|X=\mathbf{x}\right])}{\Pr\left[W\right]} && \text{(Bayes)}\\
&= \log \frac{1}{\Pr\left[W\right]} + \sum_{\mathbf{x}\in\mathcal{X}^k} (X|_W)(\mathbf{x}) \log \Pr\left[W|X=\mathbf{x}\right])\\
&\leq \log \frac{1}{\Pr\left[W\right]}
\end{aligned}
$$

Conditioning distributions, statistical distance case

Conditioning distributions, statistical distance case

Theorem 9

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{X}$ *and let* W *be an event. Then* $\sum_{j=1}^{k} \mathsf{SD}((X_j|_W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Conditioning distributions, statistical distance case

Theorem 9

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathsf{SD}((X_j|_W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Proof: follows by Thm 6, and Thm 7.$\square$

Conditioning distributions, statistical distance case

Theorem 9

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathrm{SD}((X_j|_W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Proof: follows by Thm 6, and Thm 7.□

Using $(\sum_{j=1}^{k} a_i)^2 \leq k \cdot \sum_{j=1}^{k} a_i^2$, it follows that

Corollary 10

$\sum_{j=1}^{k} \mathrm{SD}((X_j|_W), X_j) \leq \sqrt{k \log(\frac{1}{\Pr[W]})}$, *and*

$\mathrm{E}_{j \leftarrow k} \mathrm{SD}((X_j|_W), X_j) \leq \sqrt{\frac{1}{k} \log(\frac{1}{\Pr[W]})}$

Conditioning distributions, statistical distance case

Theorem 9

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathrm{SD}((X_j|_W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Proof: follows by Thm 6, and Thm 7.$\square$

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Extraction

Numerical example

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- Let $X = (X_1, \ldots, X_k) \leftarrow \{0,1\}^{40}$ and let $f \colon \{0,1\}^{40} \mapsto 0$ be such that $\Pr[f(X) = 0] = 2^{-10}$.

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- $\mathrm{E}_{j \leftarrow [40]} \mathrm{SD}((X_j|_{f(X)=0}), \sim \{0,1\}) \leq \sqrt{\frac{1}{40} \cdot 10} = \frac{1}{2}$

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- Typical bits are not too biassed, even when conditioning on a very unlikely event.

Extension

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Theorem 11

Let $X = (X_1, \ldots, X_k)$, T *and* V *be rv's over* $\mathcal{X}^k$, $\mathcal{T}$ *and* $\mathcal{V}$ *respectively. Let* W *be an event and assume that the* X_i*'s are iid conditioned on* T*. Then* $\sum_{j=1}^{k} D((TVX_j)|_W \| (TV)|_W X'_j(T)) \leq \log \frac{1}{\Pr[W]} + \log |\mathrm{Supp}(V|_W)|$,
where $X'_j(t)$ *is distributed according to* $X_j|_{T=t}$.

Extension

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Interpretation.

Proving Thm 11

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Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|_{T=t}$.

Proving Thm 11

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|_{T=t}$.

$$\sum_{j=1}^{k} D((TVX_j)|_W \| (TV)|_W X'_j(T))$$
$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum_{j=1}^{k} D\big(X_j|_{W,V=v,T=t} \| (X_j|_{T=t})\big)\Big]$$

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$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\left[\sum_{j=1}^{k} D\left(X_j|_{W,V=v,T=t} \| (X_j|_{T=t}\right)\right] \qquad \text{(chain rule)}$$

Proving Thm 11

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X_j'(t)$ be distributed according to the distribution of $X_j|_{T=t}$.

$$\sum_{j=1}^{k} D((TVX_j)|_W \| (TV)|_W X_j'(T))$$

$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum_{j=1}^{k} D\big(X_j|_{W,V=v,T=t} \| (X_j|_{T=t})\big)\Big] \qquad \text{(chain rule)}$$

$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum D\big((X_j|\underbrace{W, V=v}_{W'})|_{T=t} \| (X_j|T=t)\big)\Big]$$

Proving Thm 11

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$$\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\log \frac{1}{\Pr[W \wedge V = v | T = t]}\Big]$$

Proving Thm 11

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|_{T=t}$.

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$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum D\big((X_j|_{\underbrace{W,\, V=v}_{W'}})|_{T=t} \| (X_j|T=t\big)\Big]$$

$$\le \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\log\frac{1}{\Pr[W \wedge V=v | T=t]}\Big] \qquad \text{(Thm 8)}$$

Proving Thm 11

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X_j'(t)$ be distributed according to the distribution of $X_j|_{T=t}$.

$$
\begin{aligned}
&\sum_{j=1}^{k} D((TVX_j)|_W \| (TV)|_W X_j'(T)) \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum_{j=1}^{k} D\big(X_j|_{W,V=v,T=t} \| (X_j|_{T=t})\big)\Big] && \text{(chain rule)} \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\sum D\big((X_j|_{\underbrace{W,\, V=v}_{W'}})|_{T=t} \| (X_j|T=t)\big)\Big] \\
&\le \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W}\Big[\log \frac{1}{\Pr[W \wedge V=v | T=t]}\Big] && \text{(Thm 8)} \\
&\le \log \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W} \frac{1}{\Pr[W \wedge V=v | T=t]}
\end{aligned}
$$

Proving Thm 11

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|_{T=t}$.

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Proving Thm 11

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&\le \log \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV)|_W} \frac{1}{\Pr[W \wedge V = v | T = t]} && \text{(Jensen's inequality)} \\
&= \log \sum_{(t,v)\in \mathrm{Supp}((TV)|_W)} \frac{\Pr[T = t]}{\Pr[W]} \le \log \frac{||\mathrm{Supp}(V|_W)||}{\Pr[W]}. && \square
\end{aligned}
$$