Foundation of Cryptography, Lecture 7
Non-Interactive ZK and Proof of Knowledge

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Part I

Non-Interactive Zero Knowledge

Interaction is crucial for $\mathcal{ZK}$

Claim 1

Assume that $\mathcal{L} \subseteq \{0,1\}^*$ has a one-message $\mathcal{ZK}$ proof (even computational), with standard completeness and soundness,[a] then $\mathcal{L} \in \mathcal{BPP}$.

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Definition 2 ($\mathcal{NIZK}$)

A pair of non interactive PPTM's (P, V) is a $\mathcal{NIZK}$ for $\mathcal{L} \in \mathcal{NP}$, if $\exists \ell \in \text{poly}$ s.t.

- **Completeness:** $\Pr_{c \leftarrow \{0,1\}^{\ell(|x|)}}[\mathsf{V}(x, c, \mathsf{P}(x, w(x), c)) = 1] \geq 2/3$, for any $x \in \mathcal{L}$ and $w(x) \in R_{\mathcal{L}}(x)$.
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Section 1

NIZK in HBM

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- The latter implies a $\mathcal{NIZK}$ for all $\mathcal{NP}$.

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Claim 3

Let T be a random $n^3 \times n^3$ Boolean matrix where each entry is 1 w.p n^{-5}. Then, $\Pr[T \text{ is useful}] \in \Omega(n^{-3/2})$.

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- Hence, wp $\theta(1/\sqrt{n})$ the matrix T contains a permutation matrix and all its other entries are zero.
- A random permutation matrix forms a cycle wp $1/n$ (there are $n!$ permutation matrices and $(n-1)!$ of them form a cycle)

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Algorithm 4 (P)

Input: n-node graph G and a cycle C in G.

CRS: $T \in \{0,1\}_{n^3 \times n^3}$.

1. If T not useful, set $\mathcal{I} = n^3 \times n^3$ (i.e., reveal all T) and $\phi = \perp$.
2. Otherwise, let H be the (generalized) $n \times n$ sub-matrix containing the hamiltonian cycle in T.
   1. Set $\mathcal{I} = T \setminus H$ (i.e., reveal the bits of T outside of H).
   2. Choose $\phi \leftarrow \Pi_n$ s.t. C is mapped to the cycle in H.
   3. Add the entries in H corresponding to non edges in G (wrt. ϕ) to $\mathcal{I}$.
3. Output $\pi = (\mathcal{I}, \phi)$.

$\mathcal{NIZK}$ for Hamiltonicity in HBM cont.

Algorithm 5 (V)

Input: a graph G, index set $\mathcal{I} \subseteq [n^3] \times [n^3]$, ordered set $\{T_i\}_{i \in \mathcal{I}}$, a mapping ϕ.

`Accept` if all the bits of T are revealed and T is not useful.

Otherwise,

1. Verify that $\exists$ $n \times n$ submatrix $H \subseteq T$ with all entries in $T \setminus H$ are zeros.
2. Verify that $\phi \in \Pi_n$, and that all entries of H not corresponding to edges of G (according to ϕ) are zeros.

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Claim 6

The above protocol is a perfect $\mathcal{NIZK}$ for $\mathcal{HC}$ in the HBM, with perfect completeness and soundness error $1 - \Omega(n^{-3/2})$

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- Zero knowledge?

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- For useful T, the location of H is uniform in the real and simulated case.

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- For useful T, the location of H is uniform in the real and simulated case.
- ϕ is a random element in Π_n in both (real and simulated) cases

Algorithm 7 (S)

Input: G

1. Choose T at random (i.e., each entry is one wp n^{-5}).
2. If T is not useful, set $\mathcal{I} = n^3 \times n^3$ and $\phi = \perp$.
3. Otherwise,
   1. Set $\mathcal{I} = T \setminus H$ (where H is the hamiltonian sub-matrix in T).
   2. Let $\phi \leftarrow \Pi_n$. Replace all entries of H with zeros.
   3. Add the entries in H corresponding to non edges in G to $\mathcal{I}$.
4. Output $\pi = (T, \mathcal{I}, \phi)$.

- Perfect simulation for non-useful T's.
- For useful T, the location of H is uniform in the real and simulated case.
- ϕ is a random element in Π_n in both (real and simulated) cases
- Hence, the simulation is perfect!

Section 2

From HBM to Standard NIZK

Trapdoor permutations

Definition 8 (trapdoor permutations)

A triplet $(\mathsf{G}, f, \mathsf{Inv})$, where G is a PPTM, and f and Inv are poly-time computable, is a family of trapdoor permutation (TDP), if:

1. On input 1^n, $\mathsf{G}(1^n)$ outputs a pair (sk, pk).
2. $f_{pk} = f(pk, \cdot)$ is a permutation over $\{0,1\}^n$, for every $n \in \mathbb{N}$ and $pk \in \mathrm{Supp}(\mathsf{G}(1^n)_2)$.
3. $\mathsf{Inv}_{sk} = \mathsf{Inv}(sk, \cdot) \equiv f_{pk}^{-1}$ for every $(sk, pk) \in \mathrm{Supp}(\mathsf{G}(1^n))$
4. For any PPTM A,
$\Pr_{x \leftarrow \{0,1\}^n, pk \leftarrow \mathsf{G}(1^n)_2}\left[\mathsf{A}(pk, x) = f_{pk}^{-1}(x)\right] = \mathrm{neg}(n)$

Hardcore Predicates for Trapdoor Permutations

Definition 9 (hardcore predicates for TDP)

A polynomial-time computable $b\colon \{0,1\}^n \mapsto \{0,1\}$ is a hardcore predicate of a TDP $(\mathsf{G}, f, \mathsf{Inv})$, if

$$\Pr_{pk \leftarrow \mathsf{G}(1^n)_2, x \leftarrow \{0,1\}^n}[\mathsf{P}(pk, f_{pk}(x)) = b(x)] \leq \frac{1}{2} + \mathrm{neg}(n),$$

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Goldreich-Levin: any TDP has an hardcore predicate (ignoring padding issues)

Example, RSA

In the following $n \in \mathbb{N}$ and all operations are modulo n.

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- For every $e \in \mathbb{Z}_{\phi(n)}^*$, the function $f(x) \equiv x^e \bmod n$ is a permutation over $\mathbb{Z}_n^*$.

  In particular, $(x^e)^d \equiv x \bmod n$, for every $x \in \mathbb{Z}_n^*$, where $d \equiv e^{-1} \bmod \phi(n)$

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Definition 10 (RSA)

- $\mathsf{G}(p, q)$ sets $pk = (n = pq, e)$ for some $e \in \mathbb{Z}^*_{\phi(n)}$, and $sk = (n, d \equiv e^{-1} \bmod \phi(n))$
- $f(pk, x) = x^e \bmod n$
- $\mathsf{Inv}(sk, x) = x^d \bmod n$

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Factoring is easy $\implies$ RSA is easy.

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- $f(pk, x) = x^e \bmod n$
- $\mathsf{Inv}(sk, x) = x^d \bmod n$

Factoring is easy $\implies$ RSA is easy. The other direction?

The transformation

- Let $(\mathsf{P}_H, \mathsf{V}_H)$ be a HBM $\mathcal{NIZK}$ for $\mathcal{L}$, and let $\ell(n)$ be the length of the CRS used for $x \in \{0,1\}^n$.

The transformation

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- Let $(\mathsf{G}, f, \mathsf{Inv})$ be a TDP and let b be an hardcore bit for it.

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We construct a $\mathcal{NIZK}$ (P, V) for $\mathcal{L}$, with the same completeness and “not too large" soundness error.

The protocol

Algorithm 11 (P)

Input: $x \in \mathcal{L}$, $w \in R_{\mathcal{L}}(x)$ and CRS $c = (c_1, \ldots, c_\ell) \in \{0,1\}^{n\ell}$, where $n = |x|$ and $\ell = \ell(n)$.

1. Choose $(sk, pk) \leftarrow \mathsf{G}(sk)$ and compute $c^H = (b(z_1 = f_{pk}^{-1}(c_1)), \ldots, b(z_{\ell(n)} = f_{pk}^{-1}(c_\ell)))$
2. Let $(\pi_H, \mathcal{I}) \leftarrow \mathsf{P}_H(x, w, c^H)$ and output $(\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}})$

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Algorithm 12 (V)

Input: $x \in \mathcal{L}$, CRS $c = (c_1, \ldots, c_\ell) \in \{0,1\}^{np}$, and $(\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}})$, where $n = |x|$ and $\ell = \ell(n)$.

1. Verify that $pk \in \{0,1\}^n$ and that $f_{pk}(z_i) = c_i$ for every $i \in \mathcal{I}$
2. Return $\mathsf{V}_H(x, \pi_H, \mathcal{I}, c^H)$, where $c_i^H = b(z_i)$ for every $i \in \mathcal{I}$.

Claim 13

Assuming that $(\mathsf{P}_H, \mathsf{V}_H)$ is a $\mathcal{NIZK}$ for $\mathcal{L}$ in the HBM with soundness error $2^{-n} \cdot \alpha$, then (P, V) is a $\mathcal{NIZK}$ for $\mathcal{L}$ with the same completeness, and soundness error α.

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- Zero knowledge:?

Proving zero knowledge

Algorithm 14 (S)

Input: $x \in \{0,1\}^n$ of length n.

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- The above implicitly describes an efficient M s.t. $M(\mathsf{S}_H(x)) \equiv \mathsf{S}(x)$ and $M(\mathsf{P}_H(x, w(x))) \approx_c \mathsf{P}(x, w(x))$

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- Direct solution for our $\mathcal{NIZK}$

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- Direct solution for our $\mathcal{NIZK}$
- An “adaptive" $\mathcal{NIZK}$

Section 3

Adaptive NIZK

Adaptive $\mathcal{NIZK}$

x is chosen after the CRS.

Adaptive $\mathcal{NIZK}$

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- **Completeness:** $\forall f \colon \{0,1\}^{\ell(n)} \mapsto \mathcal{L} \cap \{0,1\}^n$ and $w(x) \in R_{\mathcal{L}}(x)$: $\Pr_{c \leftarrow \{0,1\}^{\ell(n)}; x=f(c)}[\mathsf{V}(x, c, \mathsf{P}(x, w(x), c)) = 1] \geq 2/3$

Adaptive $\mathcal{NIZK}$

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- **Soundness:** $\forall f\colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^n$ and P^*
  $\Pr_{c \leftarrow \{0,1\}^{\ell(n)}; x = f(c)}[\mathsf{V}(x, c, \mathsf{P}^*(c)) = 1 \wedge x \notin \mathcal{L}] \leq 1/3$

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- $\mathcal{ZK}$: $\exists$ pair of PPTM's $(\mathsf{S}_1, \mathsf{S}_2)$ s.t. $\forall f \colon \{0,1\}^{\ell(n)} \mapsto \mathcal{L} \cap \{0,1\}^n$

$$\{(c \leftarrow \{0,1\}^{\ell(n)}, x = f(c), \mathsf{P}(x, w(x)))\}_{n \in \mathbb{N}} \approx_c \{\mathsf{S}^f(n)\}_{n \in \mathbb{N}}.$$

where $\mathsf{S}^f(n)$ is the output of the following process

Adaptive $\mathcal{NIZK}$

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- **Soundness:** $\forall f\colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^n$ and P^*
  $\Pr_{c \leftarrow \{0,1\}^{\ell(n)}; x=f(c)}[\mathsf{V}(x, c, \mathsf{P}^*(c)) = 1 \wedge x \notin \mathcal{L}] \leq 1/3$
- $\mathcal{ZK}$**:** $\exists$ pair of PPTM's $(\mathsf{S}_1, \mathsf{S}_2)$ s.t. $\forall f\colon \{0,1\}^{\ell(n)} \mapsto \mathcal{L} \cap \{0,1\}^n$

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  where $\mathsf{S}^f(n)$ is the output of the following process

  1. $(c, s) \leftarrow \mathsf{S}_1(1^n)$

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Why do we need s?

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Assume TDP exist, then every $\mathcal{NP}$ language has an adaptive $\mathcal{NIZK}$ with perfect completeness and negligible soundness error.

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In the following, when saying adaptive $\mathcal{NIZK}$, we mean negligible completeness and soundness error.

Section 4

Simulation-Sound NIZK

Simulation soundness

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for any pair of PPTM's $\mathsf{P}^* = (\mathsf{P}^*_1, \mathsf{P}^*_2)$.

Experiment 16 ($\mathrm{Exp}^n_{\mathsf{V},\mathsf{S},\mathsf{P}^*}$)

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- Does the adaptive $\mathcal{NIZK}$ we seen have simulation soundness?

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We present a simulation sound $\mathcal{NIZK}$ (P, V) for $\mathcal{L} \in \mathcal{NP}$

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Input: $x \in \mathcal{L}$ and $w \in R_{\mathcal{L}}(x)$, and CRS $c = (c_1, c_2)$

1. $(sk, vk) \leftarrow \text{Gen}(1^{|x|})$
2. $\pi_A \leftarrow \text{P}_A((x, c_1, vk), w, c_2)$
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Input: $x \in \{0,1\}^*$, $\pi = (vk, \pi_A, \sigma)$ and a CRS $c = (c_1, c_2)$
Verify that $\mathrm{Vrfy}_{vk}((x, \pi), \sigma) = 1$ and $\mathrm{V}_A((x, c_1, vk), c_2, \pi_A) = 1$

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Claim 19

The proof system (P, V) is an adaptive $\mathcal{NIZK}$ for $\mathcal{L}$, with one-time simulation soundness.

Proving Claim 19

Recall $\mathcal{L}_A := \{(x, \mathrm{com}, w) \colon x \in \mathcal{L} \vee \exists r \in \{0,1\}^* \colon \mathrm{com} = \mathrm{Com}(w, r)\}$.

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    1. Let $(sk, vk) \leftarrow \mathsf{Gen}(1^n)$, $z \leftarrow \{0,1\}^{\ell(n)}$ and $c_1 = \mathrm{Com}(vk, z)$.
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- **Adaptive soundness:** Implicit in the proof of simulation soundness, given next slide.

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Adaptive soundness?

Part II

Proof of Knowledge

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Let (P,V) be an interactive proof for $\mathcal{L} \in \mathcal{NP}$. A probabilistic algorithm E is a knowledge extractor for (P,V) and $R_{\mathcal{L}}$ with error $\eta \colon \mathbb{N} \mapsto \mathbb{R}$, if $\exists t \in \text{poly}$ s.t. $\forall x \in \mathcal{L}$ and deterministic algorithm P^*, $\mathsf{E}^{\mathsf{P}^*}(x)$ runs in expected time bounded by $\frac{t(|x|)}{\delta(x)-\eta(|x|)}$ and outputs $w \in R_{\mathcal{L}}(x)$, where $\delta(x) = \Pr[(\mathsf{P}^*,\mathsf{V})(x) = 1]$.

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Examples

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The $\mathcal{ZK}$ proof we've seen in class for $\mathcal{GI}$, has a knowledge extractor with error $\frac{1}{2}$.

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