Application of Information Theory, Lecture 3
Graph Covering, Differential Entropy

Iftach Haitner

Tel Aviv University.

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Part I

Applications to Graph Covering

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$$\begin{aligned} 0 = H(X|Y_1, \ldots, Y_t) &= H(X, Y_1, \ldots, Y_t) - H(Y_1, \ldots, Y_t) \\ &\geq H(X) + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i) \\ &= \log n + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i). \end{aligned}$$
- $Y_1, \ldots, Y_t$ are independent conditioned on X —
$\Pr[Y_1 = y_1 \wedge Y_2 = y_2 \mid X = x] = \Pr[Y_1 = y_1 \mid X = x] \cdot \Pr[Y_2 = y_2 \mid X = x]$
- Hence, $H(Y_1, \ldots, Y_t|X) = \sum_i H(Y_i|X)$ (board)
- We conclude that $\sum_i H(Y_i) - \sum_i H(Y_i|X) \geq \log n$
- Since $H(Y_i) = H(\chi_i(Z_i))$ and $H(Y_i|X) = (1 - \frac{|\text{nonIs}(G_i)|}{n}) \cdot H(\chi_i(Z_i))$,

  it follows that $\sum_i H(\chi_i(Z_i)) \frac{|\text{nonIs}(G_i)|}{n} \geq \log n$. $\square$

Extension

Let $\alpha(G)$ be the size of the maximal independent set in G

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Theorem 5

Let $G, G_1, \ldots, G_t$ be graphs over $[n]$ with $\bigcup_{i=1}^{t} G_i = G$, then $\sum \text{content}(G_i) \geq \log \frac{n}{\alpha(G)}$.

Proof: HW

Scrambling permutations

Scrambling permutations

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- Hence, $|\mathcal{S}| \geq \frac{2}{\log e} \cdot \frac{n}{n-1} \cdot \log(n-1) \geq \frac{2}{\log e}\log n$. $\square$

Part II

Differential Entropy

Entropy of continues random variable

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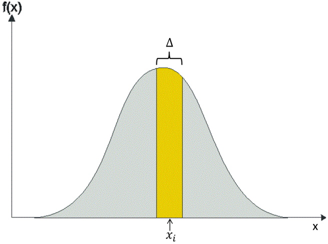

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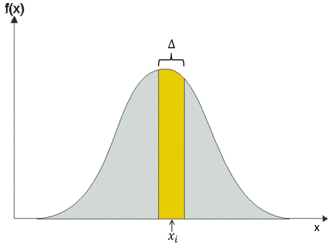

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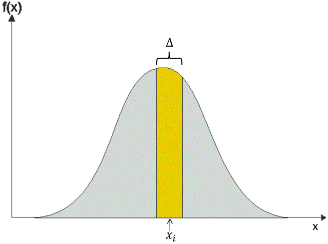

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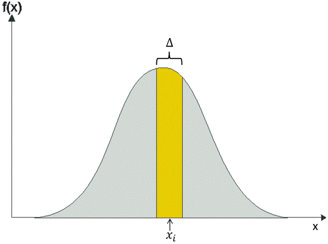

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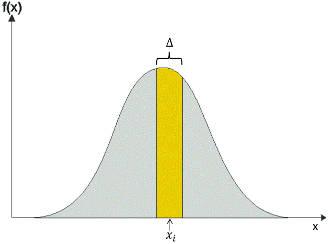

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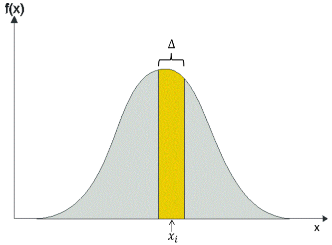

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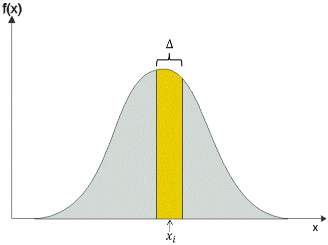

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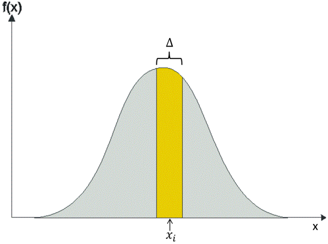

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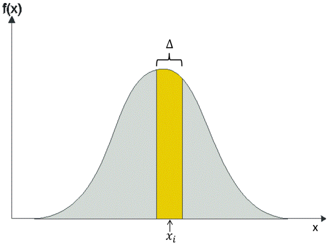

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- Shannon looked for a name for his measure, von Neumann pointed out the relation to physics and suggested the name entropy.

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- But the notion of entropy already existed in statistical physics
- There, entropy — energy that cannot used, statistical disorder
- Clausius (1865), who coined the name *entropy*, based on Carnot (1824), $H = \int_t \frac{\delta Q}{T} dt$ (Q is *heat* and T is *temperature*)
- Boltzmann (1877) $H = \log S$, for S being the number of states a system can be in (after measuring the macro parameters: pressure, temperature)
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- Carnot was also an engineer...

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- In contradiction with “reversible laws"

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- CLT and the normal distribution where known and studied way before Shannon, yet this striking property was not known until his theory.

The normal distribution, cont.

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Proof: (the continuous version of $Q3$ in handout 1)

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- Let $C = 1/\sum_i e^{-\beta \cdot E_i}$.

  Then
$$\sum_i q_i \log p_i = \sum_i q_i \log(C \cdot e^{-\beta E_i})$$

Proving Theorem 10

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- Hence, $\sum_i q_i \log p_i = \sum_i p_i \log p_i$. $\square$

The uniform distribution

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Proof: HW

Using diff. entropy to bound discrete entropy

Using diff. entropy to bound discrete entropy

Proposition 12

Let $X \sim (p_1, p_2, \ldots)$, then $H(X) \leq \frac{\log 2\pi e}{2} \cdot \left(\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2 - \frac{1}{12}\right)$

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Using diff. entropy to bound discrete entropy, cont.

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- Hence,

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Using diff. entropy to bound discrete entropy, cont.

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Using diff. entropy to bound discrete entropy, cont.

- Hence,

$$
\begin{aligned}
H(X) &= h(\tilde{X}) \\
&\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X})
\end{aligned}
$$

Using diff. entropy to bound discrete entropy, cont.

- Hence,

$$\begin{aligned} H(X) &= h(\tilde{X}) \\ &\leq \frac{1}{2}\log(2\pi e)\,\mathsf{V}(\tilde{X}) \\ &= \frac{1}{2}\log(2\pi e)\,(\mathsf{V}(X) + \mathsf{V}(U)) \end{aligned}$$

Using diff. entropy to bound discrete entropy, cont.

- Hence,

$$
\begin{aligned}
H(X) &= h(\tilde{X}) \\
&\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X}) \\
&= \frac{1}{2}\log(2\pi e)\,(\mathrm{V}(X) + \mathrm{V}(U)) \\
&= \frac{\log 2\pi e}{2} \cdot \left(\left(\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2 \right) + \frac{1}{12} \right)
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- How good is this bound?

Using diff. entropy to bound discrete entropy, cont.

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- How good is this bound?
- Let $X \sim (\frac{1}{2}, \frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.

Using diff. entropy to bound discrete entropy, cont.

- Hence,

$$
\begin{aligned}
H(X) &= h(\tilde{X}) \\
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&= \frac{\log 2\pi e}{2} \cdot \left(\left(\sum_{i=1}^{\infty} p_i \cdot i^2 - (\sum_{i=1}^{\infty} p_i \cdot i)^2 \right) + \frac{1}{12} \right)
\end{aligned}
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- How good is this bound?
- Let $X \sim (\frac{1}{2}, \frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.
- Proposition 12 grantees that $H(X) \leq \frac{\log 2\pi e}{2}(\frac{1}{4} + \frac{1}{12}) \sim 1.255$