Foundation of Cryptography, Lecture 7
Non-Interactive ZK and Proof of Knowledge

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Part I

Non-Interactive Zero Knowledge

Interaction is crucial for $\mathcal{ZK}$

Claim 1

Assume that $\mathcal{L} \subseteq \{0,1\}^*$ has a one-message $\mathcal{ZK}$ proof (even computational), with standard completeness and soundness,[a] then $\mathcal{L} \in \mathcal{BPP}$.

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Definition 2 ($\mathcal{NIZK}$)

A pair of non interactive PPTM's (P, V) is a $\mathcal{NIZK}$ for $\mathcal{L} \in \mathcal{NP}$, if $\exists \ell \in \text{poly}$ s.t.

- **Completeness:** $\Pr_{c \leftarrow \{0,1\}^{\ell(|x|)}}[\mathsf{V}(x, c, \mathsf{P}(x, w(x), c)) = 1] \geq 2/3$, where $w(x) \in R_{\mathcal{L}}(x)$ for any $x \in \mathcal{L}$ (w is an arbitrary function)
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Section 1

NIZK in HBM

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- Prover sees c^H, and outputs a proof π and a set of indices $\mathcal{I}$
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- The latter (standard model) $\mathcal{NIZK}$ for $\mathcal{HC}$ implies a $\mathcal{NIZK}$ for all $\mathcal{NP}$

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Claim 3

Let T be a random $n^3 \times n^3$ Boolean matrix where each entry is 1 w.p n^{-5}. Then, $\Pr[T \text{ is useful}] \in \Omega(n^{-3/2})$.

Proving Claim 3

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- Hence, wp $\theta(1/\sqrt{n})$ the matrix T contains a permutation matrix and all its other entries are zero.
- A random permutation matrix forms a cycle wp $1/n$ (there are $n!$ permutation matrices and $(n-1)!$ of them form a cycle)

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Algorithm 4 (P)

Input: G and a cycle C in G. A CRS $T \in \{0,1\}_{n^3 \times n^3}$

1. If T not useful, set $\mathcal{I} = n^3 \times n^3$ (i.e., reveal all T) and $\phi = \perp$

   Otherwise, let H be the (generalized) $n \times n$ sub-matrix containing the hamiltonian cycle in T.
2. Set $\mathcal{I} = T \setminus H$ (i.e., reveal the bits of T outside of H)
3. Choose $\phi \leftarrow \Pi_n$ s.t. C is mapped to the cycle in H
4. Add all the entries in H corresponding to non edges in G (with respect to ϕ) to $\mathcal{I}$
5. Output $\pi = (\mathcal{I}, \phi)$

$\mathcal{NIZK}$ for Hamiltonicity in HBM cont.

Algorithm 5 (V)

Input: a graph G, index set $\mathcal{I} \subseteq [n^3] \times [n^3]$, ordered set $\{T_i\}_{i \in \mathcal{I}}$ and a mapping ϕ

1. `Accept` if all the bits of T are revealed and T is not useful.

   Otherwise,
2. Verify that $\exists\ n \times n$ submatrix $H \subseteq T$ with all entries in $T \setminus H$ are zeros.
3. Verify that $\phi \in \Pi_n$, and that all entries of H not corresponding to edges of G (according to ϕ) are zeros

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Claim 6

The above protocol is a perfect $\mathcal{NIZK}$ for $\mathcal{HC}$ in the HBM, with perfect completeness and soundness error $1 - \Omega(n^{-3/2})$

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- For useful T, the location of H is uniform in the real and simulated case.
- ϕ is a random element in Π_n in both (real and simulated) cases
- Hence, the simulation is perfect!

Section 2

From HBM to Standard NIZK

Trapdoor Permutations

Definition 8 (trapdoor permutations)

A triplet $(\mathsf{G}, f, \mathsf{Inv})$, where G is a PPTM, and f and Inv are polynomial-time computable functions, is a family of trapdoor permutation (TDP), if:

1. On input 1^n, $\mathsf{G}(1^n)$ outputs a pair (sk, pk).
2. $f_{pk} = f(pk, \cdot)$ is a permutation over $\{0,1\}^n$, for every $n \in \mathbb{N}$ and $pk \in \mathrm{Supp}(\mathsf{G}(1^n)_2)$.
3. $\mathsf{Inv}(sk, \cdot) \equiv f_{pk}^{-1}$ for every $(sk, pk) \in \mathrm{Supp}(\mathsf{G}(1^n))$
4. For any PPTM A, $\Pr_{x \leftarrow \{0,1\}^n, pk \leftarrow \mathsf{G}(1^n)_2}[\mathsf{A}(pk, x) = f_{pk}^{-1}(x)] = \mathrm{neg}(n)$

Hardcore Predicates for Trapdoor Permutations

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A polynomial-time computable $b\colon \{0,1\}^n \mapsto \{0,1\}$ is a hardcore predicate of a TDP $(\mathsf{G}, f, \mathsf{Inv})$, if

$$\Pr_{e \leftarrow \mathsf{G}(1^n)_2, x \leftarrow \{0,1\}^n}[\mathsf{P}(e, f_e(x)) = b(x)] \leq \frac{1}{2} + \mathrm{neg}(n),$$

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Goldreich-Levin: any TDP has an hardcore predicate (ignoring padding issues)

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  In particular, $(x^e)^d \equiv x \bmod n$, for every $x \in \mathbb{Z}_n^*$, where $d \equiv e^{-1} \bmod \phi(n)$

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- $\mathbb{Z}_n = [n]$ and $\mathbb{Z}_n^* = \{x \in [n] \colon \gcd(x,n) = 1\}$
- $\phi(n) = |\mathbb{Z}_n^*|$ (equals $(p-1)(q-1)$ for $n = pq$ with $p, q \in \mathcal{P}$)
- For every $e \in \mathbb{Z}_{\phi(n)}^*$, the function $f(x) \equiv x^e$ is a permutation over $\mathbb{Z}_n^*$.
  In particular, $(x^e)^d \equiv x \bmod n$, for every $x \in \mathbb{Z}_n^*$, where $d \equiv e^{-1} \bmod \phi(n)$

Definition 10 (RSA)

- $\mathrm{G}(p,q)$ sets $pk = (n = pq, e)$ for some $e \in \mathbb{Z}_{\phi(n)}^*$, and $sk = (n, d \equiv e^{-1} \bmod \phi(n))$
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Factoring is easy $\implies$ RSA is easy.

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Factoring is easy $\implies$ RSA is easy. Other direction?

The transformation

- Let $(\mathsf{P}_H, \mathsf{V}_H)$ be a HBM $\mathcal{NIZK}$ for $\mathcal{L}$, and let $\ell(n)$ be the length of the CRS used for $x \in \{0,1\}^n$.

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We construct a $\mathcal{NIZK}$ (P, V) for $\mathcal{L}$, with the same completeness and “not too large" soundness error.

The protocol

Algorithm 11 (P)

Input: $x \in \mathcal{L}$, $w \in R_{\mathcal{L}}(x)$ and CRS $c = (c_1, \ldots, c_\ell) \in \{0,1\}^{n\ell}$, where $n = |x|$ and $\ell = \ell(n)$.

1. Choose $(sk, pk) \leftarrow \mathrm{G}(sk)$ and compute $c^H = (b(z_1 = f_{pk}^{-1}(c_1)), \ldots, b(z_{\ell(n)} = f_{pk}^{-1}(c_\ell)))$
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Algorithm 12 (V)

Input: $x \in \mathcal{L}$, CRS $c = (c_1, \ldots, c_\ell) \in \{0,1\}^{np}$, and $(\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}})$, where $n = |x|$ and $\ell = \ell(n)$.

1. Verify that $pk \in \{0,1\}^n$ and that $f_{pk}(z_i) = c_i$ for every $i \in \mathcal{I}$
2. Return $\mathrm{V}_H(x, \pi_H, \mathcal{I}, c^H)$, where $c_i^H = b(z_i)$ for every $i \in \mathcal{I}$.

Claim 13

Assuming that $(\mathsf{P}_H, \mathsf{V}_H)$ is a $\mathcal{NIZK}$ for $\mathcal{L}$ in the HBM with soundness error $2^{-n} \cdot \alpha$, then (P, V) is a $\mathcal{NIZK}$ for $\mathcal{L}$ with the same completeness, and soundness error α.

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For every $pk \in \{0,1\}^n$: $\left(b(f_{pk}^{-1}(c_1)), \ldots, b(f_{pk}^{-1}(c_\ell))\right)_{c \leftarrow \{0,1\}^{np}}$ is uniformly distributed in $\{0,1\}^\ell$.

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- Zero knowledge:?

Proving zero knowledge

Algorithm 14 (S)

Input: $x \in \{0,1\}^n$ of length n.

- Let $(\pi_H, \mathcal{I}, c^H) = \mathsf{S}_H(x)$, where S_H is the simulator of $(\mathsf{P}_H, \mathsf{V}_H)$
- Output $(c, (\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}}))$, where
  - $pk \leftarrow \mathsf{G}(U_n)$
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- Distinguishing $\mathsf{P}(x, w_x)$ from $\mathsf{S}(x)$ is hard

Section 3

Adaptive NIZK

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- **Completeness:** $\forall f \colon \{0,1\}^{\ell(n)} \mapsto \mathcal{L} \cap \{0,1\}^n$:
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$$\{(f(c), c, \mathsf{P}(f(c), w(f(c)), c \leftarrow \{0,1\}^{\ell(n)})\}_{n \in \mathbb{N}} \approx_c \{\mathsf{S}^f(n)\}_{n \in \mathbb{N}}.$$

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Assume TDP exist, then every $\mathcal{NP}$ language has an adaptive $\mathcal{NIZK}$ with perfect completeness and negligible soundness error.

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In the following, when saying adaptive $\mathcal{NIZK}$, we mean negligible completeness and soundness error.

Section 4

Simulation Sound NIZK

Simulation Soundness

A $\mathcal{NIZK}$ system (P,V) for $\mathcal{L}$ has (one-time) simulation soundness, if $\exists$ a pair of PPTM's $\mathsf{S} = (\mathsf{S}_1,\mathsf{S}_2)$ satisfying the $\mathcal{ZK}$ property of P with respect to $\mathcal{L}$, such that the following holds $\forall$ pair of PPTM's $(\mathsf{P}_1^*,\mathsf{P}_2^*)$: let

Experiment 16 ($\mathsf{Exp}^n_{\mathsf{V},\mathsf{S},\mathsf{P}^*}$)

1. $(c,s) \leftarrow \mathsf{S}_1(1^n)$
2. $(x,p) \leftarrow \mathsf{P}_1^*(1^n,c)$
3. $\pi \leftarrow \mathsf{S}_2(x,c,s)$
4. $(x',\pi') \leftarrow \mathsf{P}_2^*(p,\pi)$
5. Output (c,x,π,x',π')

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4. $(x', \pi') \leftarrow \mathsf{P}_2^*(p, \pi)$
5. Output (c, x, π, x', π')

We require $\Pr[(c, x, \pi, x', \pi') \leftarrow \mathsf{Exp}^n_{\mathsf{V},\mathsf{S},\mathsf{P}^*} : x' \notin \mathcal{L} \wedge \mathsf{V}(x', \pi', c) = 1 \wedge (x', \pi') \neq (x, \pi)] = \mathrm{neg}(n)$.

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- Does the adaptive $\mathcal{NIZK}$ we seen in class have simulation soundness?

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We present a simulation sound $\mathcal{NIZK}$ (P, V) for $\mathcal{L} \in \mathcal{NP}$

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Algorithm 17 (P)

Input: $x \in \mathcal{L}$ and $w \in R_{\mathcal{L}}(x)$, and CRS $c = (c_1, c_2)$

1. $(sk, vk) \leftarrow \mathrm{Gen}(1^{|x|})$
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Input: $x \in \{0,1\}^*$, $\pi = (vk, \pi_A, \sigma)$ and a CRS $c = (c_1, c_2)$
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Claim 19

The proof system (P, V) is an adaptive $\mathcal{NIZK}$ for $\mathcal{L}$ with one-time simulation soundness.

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  - **Adaptive soundness:** Implicit in the proof of simulation soundness, given below

Proving simulation soundness

Let $\mathsf{P}^* = (\mathsf{P}^*_1, \mathsf{P}^*_2)$ be a pair of PPTM's attacking the simulation soundness of (V, S) with respect to $\mathcal{L}$, and let $c = (c_1, c_2)$, x , π, x' and $\pi' = (vk', \pi'_A, \sigma')$ be the values generated by a random execution of $\mathsf{Exp}^n_{\mathsf{V},\mathsf{S},\mathsf{P}^*}$.

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Since c_2 was chosen at random by S_1, the adaptive soundness of $(\mathsf{P}_A, \mathsf{V}_A)$ yields that $\Pr[\mathsf{V}_A(x'_A, c_2, \pi'_A) = 1] = \mathsf{neg}(n)$.

Adaptive soundness?

Part II

Proof of Knowledge

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Let (P, V) be an interactive proof $\mathcal{L} \in \mathcal{NP}$. A probabilistic machine E is a knowledge extractor for (P, V) and $R_{\mathcal{L}}$ with error $\eta \colon \mathbb{N} \mapsto \mathbb{R}$, if $\exists t \in \mathrm{poly}$ s.t. $\forall x \in \mathcal{L}$ and deterministic algorithm P^*, $\mathsf{E}^{\mathsf{P}^*}(x)$ runs in expected time bounded by $\frac{t(|x|)}{\delta(x)-\eta(|x|)}$ and outputs $w \in R_{\mathcal{L}}(x)$, where $\delta(x) = \Pr[(\mathsf{P}^*, \mathsf{V})(x) = 1]$.

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Examples

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The $\mathcal{ZK}$ proof we've seen in class for $\mathcal{GI}$, has a knowledge extractor with error $\frac{1}{2}$.

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