Application of Information Theory, Lecture 3
Graph Covering, Differential Entropy

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November 11, 2014

Part I

Applications to Graph Covering

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- Hence, $H(Y_1, \ldots, Y_t|X) = \sum_i H(Y_i|X)$ (board)
- We conclude that $\sum_i H(Y_i|X) - \sum_i H(Y_i) \geq \log n$
- Since $H(Y_i|X) = (1 - \frac{|\text{nonIs}(G_i)|}{n}) \cdot H(Z_i)$, and since $H(Z_i) = H(Y_i)$

Proving Thm 4

- Let χ_i be a (valid) coloring of G_i.
- Let $X \leftarrow [n]$, and let $Y_i = \begin{cases} \chi_i(X) & X \text{ is non-isolated vertix of } G_i \\ \chi_i(Z_i) & \text{otherwise,} \end{cases}$
  where $Z_i \leftarrow \text{nonIs}(G_i)$ (independent of the other Z's)
- X is determined by $Y_1, \ldots, Y_t$ (?)

$$\begin{aligned} 0 = H(X|Y_1, \ldots, Y_t) &= H(X, Y_1, \ldots, Y_t) - H(Y_1, \ldots, Y_t) \\ &= H(X) + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i) \\ &\geq \log n + H(Y_1, \ldots, Y_t|X) - \sum_i H(Y_i). \end{aligned}$$

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Proof: ?

Scrambling permutations

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- Hence, $|\mathcal{S}| \geq \frac{2}{\log e} \cdot \frac{n}{n-1} \cdot \log(n-1) \geq \frac{2}{\log e} \log n$

Part II

Differential Entropy

Entropy of continues random variable

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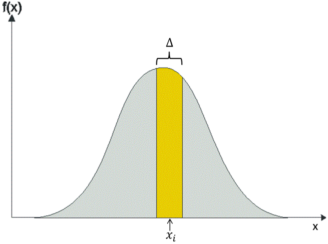

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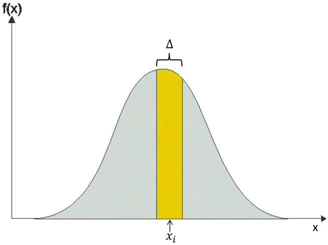

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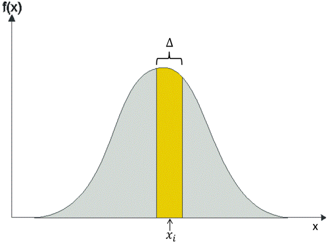

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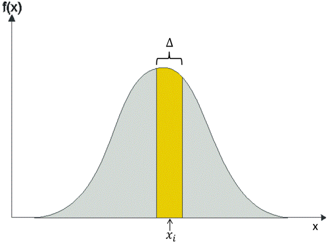

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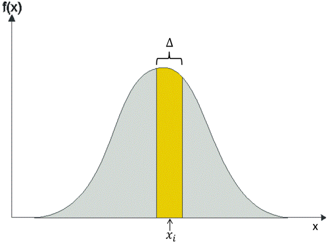

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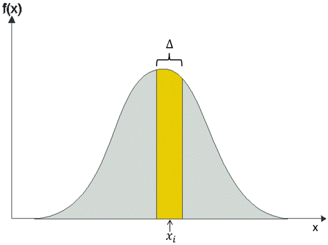

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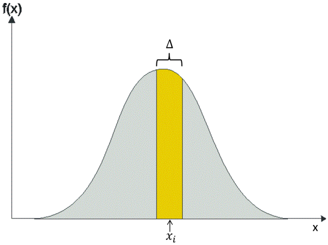

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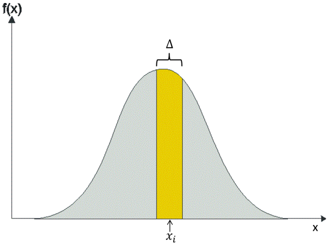

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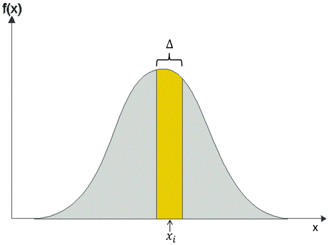

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- Used for comparing two distributions

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- In contradiction with “reversible laws"

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- CLT and the normal distribution where known and studied way before Shannon, yet this striking property was not known until his theory.

The normal distribution, cont.

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- Hence, $-\int g(x)\log g(x) \leq -\int g(x)\log q(x)$

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- Hence, $\sum_i q_i \log p_i = \sum_i p_i \log p_i$. $\square$

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Proof: ?

Differential entropy bound on discrete entropy

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Proposition 12

Let $X \sim (p_1, p_2, \ldots)$, then $H(X) \leq \frac{\log 2\pi e}{2} \cdot \left(\sum_{i=1}^{\infty} p_i i^2 - (\sum_{i=1}^{\infty} p_i i)^2 - \frac{1}{12}\right)$

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Differential entropy bound on discrete entropy, cont.

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Differential entropy bound on discrete entropy, cont.

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Differential entropy bound on discrete entropy, cont.

- Hence,

$$\begin{aligned} H(X) &= h(\tilde{X}) \\ &\leq \frac{1}{2}\log(2\pi e)\,\mathrm{V}(\tilde{X}) \end{aligned}$$

Differential entropy bound on discrete entropy, cont.

- Hence,

$$\begin{aligned} H(X) &= h(\tilde{X}) \\ &\leq \frac{1}{2}\log(2\pi e)\,\mathsf{V}(\tilde{X}) \\ &= \frac{1}{2}\log(2\pi e)\,(\mathsf{V}(X)+\mathsf{V}(U)) \end{aligned}$$

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&= \frac{\log 2\pi e}{2}\cdot\left(\left(\sum_{i=1}^{\infty} p_i i^2 - (\sum_{i=1}^{\infty} p_i i)^2\right)+\frac{1}{12}\right)
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- How good is this bound?

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- How good is this bound?
- Let $X \sim (\frac{1}{2}, \frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.

Differential entropy bound on discrete entropy, cont.

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- How good is this bound?
- Let $X \sim (\frac{1}{2},\frac{1}{2})$. Hence, $\mathrm{V}[X] = \frac{1}{4}$ and $H(X) = 1$.
- Proposition 12 grantees that $H(X) \le \frac{\log 2\pi e}{2}(\frac{1}{4}+\frac{1}{12}) \sim 1.255$