

Foundation of Cryptography, Lecture 7

Non-Interactive ZK and Proof of Knowledge

Handout Mode

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Part I

Non-Interactive Zero Knowledge

Interaction is crucial for \mathcal{ZK}

Claim 1

Assume that $\mathcal{L} \subseteq \{0, 1\}^*$ has a **one-message \mathcal{ZK}** proof (even computational), with standard completeness and soundness,^a then $\mathcal{L} \in \mathcal{BPP}$.

^aThat is, the completeness is $\frac{2}{3}$ and soundness error is $\frac{1}{3}$.

Proof: HW

- ➊ To reduce interaction we **relax** the zero-knowledge requirement
 - ➊ Witness Indistinguishability
$$\{\langle (P(w_x^1), V^*)(x) \rangle\}_{x \in \mathcal{L}} \approx_c \{\langle (P(w_x^2), V^*)(x) \rangle\}_{x \in \mathcal{L}},$$
for any $\{w_x^1 \in R_{\mathcal{L}}(x)\}_{x \in \mathcal{L}}$ and $\{w_x^2 \in R_{\mathcal{L}}(x)\}_{x \in \mathcal{L}}$
 - ➋ Witness Hiding
 - ➌ Non-interactive “zero knowledge”

Non-Interactive Zero Knowledge (\mathcal{NIZK})

Definition 2 (\mathcal{NIZK})

A pair of **non interactive** PPTM's (P, V) is a \mathcal{NIZK} for $\mathcal{L} \in \mathcal{NP}$, if $\exists \ell \in \text{poly}$ s.t.

- **Completeness:** $\Pr_{c \leftarrow \{0,1\}^{\ell(|x|)}} [V(x, c, P(x, w(x), c)) = 1] \geq 2/3$,
where $w(x) \in R_{\mathcal{L}}(x)$ for any $x \in \mathcal{L}$ (w is an arbitrary function)
- **Soundness:** $\Pr_{c \leftarrow \{0,1\}^{\ell(|x|)}} [V(x, c, P^*(x, c)) = 1] \leq 1/3$,
for any P^* and $x \notin \mathcal{L}$
- \mathcal{ZK} : \exists PPTM S s.t.
 $\{(x, c, P(x, w(x), c))\}_{x \in \mathcal{L}, c \leftarrow \{0,1\}^{\ell(|x|)}} \approx_c \{x, S(x)\}_{x \in \mathcal{L}}$

- c – common (random) reference string (CRS)
- CRS is chosen **by the simulator**
- What does the definition stand for?
- Amplification?

Section 1

NIZK in HBM

Hidden Bits Model (HBM)

A CRS is chosen at random, but **only the prover can see it**. The prover chooses which bits to reveal as part of the proof.

Let c^H be the “hidden” CRS:

- Prover sees c^H , and outputs a proof π and a set of indices \mathcal{I}
- Verifier only sees the bits in c^H that are indexed by \mathcal{I}
- Simulator outputs a proof π , a set of indices \mathcal{I} and a partially hidden CRS c^H
- Soundness, completeness and ZK are naturally defined.
- We give a \mathcal{NIZK} for \mathcal{HC} , Directed Graph Hamiltonicity, in the **HBM**, and then transfer it into a \mathcal{NIZK} for \mathcal{HC} in the **standard model**.
- The latter (standard model) \mathcal{NIZK} for \mathcal{HC} implies a \mathcal{NIZK} for all \mathcal{NP}

Useful Matrix

- **Permutation matrix**: an $n \times n$ Boolean matrix, where each row/column contains a single 1
- **Hamiltonian matrix**: an $n \times n$ adjacency matrix of a directed graph that consists of a single Hamiltonian cycle (note that this is also a permutation matrix)
- An $n^3 \times n^3$ Boolean matrix is **useful**: if it contains $n \times n$ Hamiltonian *generalized* sub-matrix, and all its other entries are zeros

Claim 3

Let T be a random $n^3 \times n^3$ Boolean matrix where each entry is 1 w.p n^{-5} . Then, $\Pr[T \text{ is useful}] \in \Omega(n^{-3/2})$.

Proving Claim 3

- The expected # of ones (entries) in T is $n^6 \cdot n^{-5} = n$.
- By extended Chernoff bound, T contains **exactly** n ones w.p. $\theta(1/\sqrt{n})$.
- Each row/column of T contain more than a single one entry with probability at most $\binom{n^3}{2} \cdot n^{-10} < n^{-4}$.
Hence, wp at least $1 - 2 \cdot n^3 \cdot n^{-4} = 1 - O(n^{-1})$, no row or column of T contains more than a single one entry.
- Hence, wp $\theta(1/\sqrt{n})$ the matrix T contains a permutation matrix and all its other entries are zero.
- A random permutation matrix forms a cycle wp $1/n$ (there are $n!$ permutation matrices and $(n-1)!$ of them form a cycle)

\mathcal{NIZK} for Hamiltonicity in HBM

- Common input: a directed graph $G = ([n], E)$
- we assume w.l.g. that n is a power of 2
- Common reference string T viewed as a $n^3 \times n^3$ Boolean matrix, where each entry is 1 w.p n^{-5} ??

Algorithm 4 (P)

Input: G and a cycle C in G . A CRS $T \in \{0, 1\}_{n^3 \times n^3}$

- 1 If T not useful, set $\mathcal{I} = n^3 \times n^3$ (i.e., reveal all T) and $\phi = \perp$
Otherwise, let H be the (generalized) $n \times n$ sub-matrix containing the hamiltonian cycle in T .
- 2 Set $\mathcal{I} = T \setminus H$ (i.e., reveal the bits of T outside of H)
- 3 Choose $\phi \leftarrow \Pi_n$ s.t. C is mapped to the cycle in H
- 4 Add all the entries in H corresponding to non edges in G (with respect to ϕ) to \mathcal{I}
- 5 Output $\pi = (\mathcal{I}, \phi)$

\mathcal{NIZK} for Hamiltonicity in HBM cont.

Algorithm 5 (V)

Input: a graph G , index set $\mathcal{I} \subseteq [n^3] \times [n^3]$, ordered set $\{T_i\}_{i \in \mathcal{I}}$ and a mapping ϕ

- 1 **Accept** if all the bits of T are revealed and T is **not useful**.
Otherwise,
- 2 Verify that $\exists n \times n$ submatrix $H \subseteq T$ with all entries in $T \setminus H$ are zeros.
- 3 Verify that $\phi \in \Pi_n$, and that all entries of H **not corresponding to edges of G** (according to ϕ) are zeros

Claim 6

The above protocol is a **perfect** \mathcal{NIZK} for \mathcal{HC} in the HBM, with **perfect** completeness and soundness error $1 - \Omega(n^{-3/2})$

Proving Claim 6

- Completeness: Clear
- Soundness: Assume T is useful and V accepts. Then ϕ^{-1} maps the unrevealed “edges” of H to the edges of G .
Hence, ϕ^{-1} maps the cycle in H to an Hamiltonian cycle in G
- Zero knowledge?

Algorithm 7 (S)

Input: G

- 1 Choose T at random (i.e., each entry is one wp n^{-5}).
- 2 If T is not useful, set $\mathcal{I} = n^3 \times n^3$ and $\phi = \perp$.
Otherwise,
- 3 Set $\mathcal{I} = T \setminus H$ (where H is the hamiltonian sub-matrix in T)
- 4 Let $\phi \leftarrow \Pi_n$. Replace all the entries of H not corresponding to edges of G (according to ϕ) with zeros
- 5 Add the entries in H corresponding to non edges in G to \mathcal{I}
- 6 Output $\pi = (T, \mathcal{I}, \phi)$

- Perfect simulation for non useful T 's.
- For useful T , the location of H is uniform in the real and simulated case.
- ϕ is a random element in Π_n in both (real and simulated) cases
- Hence, the simulation is perfect!

Section 2

From HBM to Standard NIZK

Trapdoor Permutations

Definition 8 (trapdoor permutations)

A triplet (G, f, Inv) , where G is a PPTM, and f and Inv are polynomial-time computable functions, is a family of trapdoor permutation (TDP), if:

- 1 On input 1^n , $G(1^n)$ outputs a pair (sk, pk) .
- 2 $f_{pk} = f(pk, \cdot)$ is a permutation over $\{0, 1\}^n$, for every $n \in \mathbb{N}$ and $pk \in \text{Supp}(G(1^n)_2)$.
- 3 $\text{Inv}(sk, \cdot) \equiv f_{pk}^{-1}$ for every $(sk, pk) \in \text{Supp}(G(1^n))$
- 4 For any PPTM A , $\Pr_{x \leftarrow \{0, 1\}^n, pk \leftarrow G(1^n)_2} [A(pk, x) = f_{pk}^{-1}(x)] = \text{neg}(n)$

Hardcore Predicates for Trapdoor Permutations

Definition 9 (hardcore predicates for TDP)

A polynomial-time computable $b: \{0, 1\}^n \mapsto \{0, 1\}$ is a hardcore predicate of a TDP (G, f, Inv) , if

$$\Pr_{e \leftarrow G(1^n)_2, x \leftarrow \{0, 1\}^n} [P(e, f_e(x)) = b(x)] \leq \frac{1}{2} + \text{neg}(n),$$

for any PPTM P .

Goldreich-Levin: any TDP has an hardcore predicate (ignoring padding issues)

example, RSA

In the following $n \in \mathbb{N}$ and all operations are modulo n .

- $\mathbb{Z}_n = [n]$ and $\mathbb{Z}_n^* = \{x \in [n]: \gcd(x, n) = 1\}$
- $\phi(n) = |\mathbb{Z}_n^*|$ (equals $(p-1)(q-1)$ for $n = pq$ with $p, q \in \mathcal{P}$)
- For every $e \in \mathbb{Z}_{\phi(n)}^*$, the function $f(x) \equiv x^e$ is a permutation over \mathbb{Z}_n^* .

In particular, $(x^e)^d \equiv x \pmod n$, for every $x \in \mathbb{Z}_n^*$, where $d \equiv e^{-1} \pmod{\phi(n)}$

Definition 10 (RSA)

- $G(p, q)$ sets $pk = (n = pq, e)$ for some $e \in \mathbb{Z}_{\phi(n)}^*$, and $sk = (n, d \equiv e^{-1} \pmod{\phi(n)})$
- $f(pk, x) = x^e \pmod n$
- $\text{Inv}(sk, x) = x^d \pmod n$

Factoring is easy \implies RSA is easy. Other direction?

The transformation

- Let (P_H, V_H) be a HBM \mathcal{NIZK} for \mathcal{L} , and let $\ell(n)$ be the length of the CRS used for $x \in \{0, 1\}^n$.
- Let (G, f, Inv) be a TDP and let b be an hardcore bit for it. For simplicity we assume $G(1^n)$ chooses (sk, pk) as follows
 - 1 $sk \leftarrow \{0, 1\}^n$
 - 2 $pk = PK(sk)$

where $PK: \{0, 1\}^n \mapsto \{0, 1\}^n$ is a polynomial-time computable function.

We construct a $\mathcal{NIZK}(P, V)$ for \mathcal{L} , with the same completeness and “not too large” soundness error.

The protocol

Algorithm 11 (P)

Input: $x \in \mathcal{L}$, $w \in R_{\mathcal{L}}(x)$ and CRS $c = (c_1, \dots, c_\ell) \in \{0, 1\}^{n\ell}$, where $n = |x|$ and $\ell = \ell(n)$.

- 1 Choose $(sk, pk) \leftarrow G(sk)$ and compute $c^H = (b(z_1 = f_{pk}^{-1}(c_1)), \dots, b(z_{\ell(n)} = f_{pk}^{-1}(c_\ell)))$
- 2 Let $(\pi_H, \mathcal{I}) \leftarrow P_H(x, w, c^H)$ and output $(\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}})$

Algorithm 12 (V)

Input: $x \in \mathcal{L}$, CRS $c = (c_1, \dots, c_\ell) \in \{0, 1\}^{np}$, and $(\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}})$, where $n = |x|$ and $\ell = \ell(n)$.

- 1 Verify that $pk \in \{0, 1\}^n$ and that $f_{pk}(z_i) = c_i$ for every $i \in \mathcal{I}$
- 2 Return $V_H(x, \pi_H, \mathcal{I}, c^H)$, where $c_i^H = b(z_i)$ for every $i \in \mathcal{I}$.

Claim 13

Assuming that (P_H, V_H) is a \mathcal{NIZK} for \mathcal{L} in the HBM with soundness error $2^{-n} \cdot \alpha$, then (P, V) is a \mathcal{NIZK} for \mathcal{L} with the same completeness, and soundness error α .

Proof: Assume for simplicity that b is unbiased (i.e., $\Pr[b(U_n) = 1] = \frac{1}{2}$). For every $pk \in \{0, 1\}^n$: $\left(b(f_{pk}^{-1}(c_1)), \dots, b(f_{pk}^{-1}(c_\ell))\right)_{c \leftarrow \{0, 1\}^{np}}$ is uniformly distributed in $\{0, 1\}^\ell$.

- Completeness: clear
- Soundness: follows by a union bound over all possible choice of $pk \in \{0, 1\}^n$.
- Zero knowledge:?

Proving zero knowledge

Algorithm 14 (S)

Input: $x \in \{0, 1\}^n$ of length n .

- Let $(\pi_H, \mathcal{I}, c^H) = S_H(x)$, where S_H is the simulator of (P_H, V_H)
- Output $(c, (\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}}))$, where
 - ▶ $pk \leftarrow G(U_n)$
 - ▶ Each z_i is chosen at random in $\{0, 1\}^n$ such that $b(z_i) = c_i^H$
 - ▶ $c_i = f_{pk}(z_i)$ for $i \in \mathcal{I}$, and a random value in $\{0, 1\}^n$ otherwise.

- Exists efficient M s.t. $M(S_H(x)) \equiv S(x)$ and $M(P_H(x, w_x)) \approx_c P(x, w_x)$
- Distinguishing $P(x, w_x)$ from $S(x)$ is hard

Section 3

Adaptive NIZK

Adaptive \mathcal{NIZK}

x is chosen *after* the CRS.

- **Completeness:** $\forall f: \{0, 1\}^{\ell(n)} \mapsto \mathcal{L} \cap \{0, 1\}^n$:
 $\Pr_{c \leftarrow \{0, 1\}^{\ell(n)}} [V(f(c), c, P(f(c), w(f(c)), c)) = 1] \geq 2/3$
- **Soundness:** $\forall f: \{0, 1\}^{\ell(n)} \mapsto \{0, 1\}^n$ and P^*
 $\Pr_{c \leftarrow \{0, 1\}^{\ell(n)}} [V(f(c), c, P^*(c)) = 1 \wedge f(c) \notin \mathcal{L}] \leq 1/3$
- **\mathcal{ZK} :** \exists pair of PPTM's (S_1, S_2) s.t. $\forall f: \{0, 1\}^{\ell(n)} \mapsto \mathcal{C} \cap \{0, 1\}^n$

$$\{(f(c), c, P(f(c), w(f(c)), c) \mid c \leftarrow \{0, 1\}^{\ell(n)})\}_{n \in \mathbb{N}} \approx_c \{S^f(n)\}_{n \in \mathbb{N}}.$$

where $S^f(n)$ is the output of the following process

- 1 $(c, s) \leftarrow S_1(1^n)$
- 2 $x = f(c)$
- 3 Output $(x, c, S_2(x, c, s))$

- Adaptive completeness and soundness are easy to achieve from any non-adaptive \mathcal{NIZK} .
- Not every \mathcal{NIZK} is adaptive (but the above protocol is).

Theorem 15

Assume TDP exist, then every \mathcal{NP} language has an adaptive \mathcal{NIZK} with perfect completeness and negligible soundness error.

In the following, when saying adaptive \mathcal{NIZK} , we mean negligible completeness and soundness error.

Section 4

Simulation Sound NIZK

Simulation Soundness

A \mathcal{NIZK} system (P, V) for \mathcal{L} has *(one-time) simulation soundness*, if \exists a pair of PPTM's $S = (S_1, S_2)$ satisfying the \mathcal{ZK} property of P with respect to \mathcal{L} , such that the following holds \forall pair of PPTM's (P_1^*, P_2^*) : let

Experiment 16 (Exp_{V,S,P^*}^n)

- 1 $(c, s) \leftarrow S_1(1^n)$
- 2 $(x, p) \leftarrow P_1^*(1^n, c)$
- 3 $\pi \leftarrow S_2(x, c, s)$
- 4 $(x', \pi') \leftarrow P_2^*(p, \pi)$
- 5 Output (c, x, π, x', π')

We require $\Pr[(r, x, \pi, x', \pi') \leftarrow \text{Exp}_{V,S,P^*}^n : x' \notin \mathcal{L} \wedge V(x', \pi', c) = 1 \wedge (x', \pi') \neq (x, \pi)] = \text{neg}(n)$.

- Even for $x \notin \mathcal{L}$, hard to generate additional false proofs
- Definition only considers efficient provers
- (P, V) might be adaptive or non-adaptive
- Adaptive \mathcal{NIZK} guarantees weak type of simulation soundness
- Does the adaptive \mathcal{NIZK} we seen in class have simulation soundness?

Construction

We present a simulation sound $\mathcal{NIZK}(\mathcal{P}, \mathcal{V})$ for $\mathcal{L} \in \mathcal{NP}$ **Ingredients:**

- 1 Strong signature scheme $(\text{Gen}, \text{Sign}, \text{Vrfy})$ (one time suffice)
- 2 Non-interactive, perfectly-binding commitment Com
 - ▶ Pseudorandom range: for some $\ell \in \text{poly}$
 $\{\text{Com}(s, r \leftarrow \{0, 1\}^{\ell(|s|)})\}_{s \in \{0, 1\}^*} \approx_c \{u \leftarrow \{0, 1\}^{\ell(|s|)}\}_{s \in \{0, 1\}^*}$
* implied by OWP (or TDP)
 - ▶ Negligible support: a random string is a valid commitment only with negligible probability.
* achieved from any commitment scheme by committing to the same value many times
- 3 Adaptive $\mathcal{NIZK}(\mathcal{P}_A, \mathcal{V}_A)$ for
 $\mathcal{L}_A := \{(x, c, s) : x \in \mathcal{L} \vee \exists z \in \{0, 1\}^* : c = \text{Com}(s, z)\}$ *adaptive
WI suffices

Algorithm 17 (P)

Input: $x \in \mathcal{L}$ and $w \in R_{\mathcal{L}}(x)$, and CRS $r = (r_1, r_2)$

- 1 $(sk, vk) \leftarrow \text{Gen}(1^{|x|})$
- 2 $\pi_A \leftarrow P_A((x, r_1, vk), w, r_2)$
- 3 $\sigma \leftarrow \text{Sign}_{sk}(x, \pi_A)$
- 4 Output $\pi = (vk, \pi_A, \sigma)$

Algorithm 18 (V)

Input: $x \in \{0, 1\}^*$, $\pi = (vk, \pi_A, \sigma)$ and a CRS $r = (r_1, r_2)$

Verify that $\text{Vrfy}_{vk}((x, \pi), \sigma) = 1$ and $V_A((x, r_1, vk), r_2, \pi_A) = 1$

Claim 19

The proof system (P, V) is an adaptive \mathcal{NIZK} for \mathcal{L} with one-time simulation soundness.

Proving Claim 19

- **Adaptive Completeness:** Clear

- **Adaptive \mathcal{ZK} :**

- ▶ $S_1(1^n)$:

- 1 Let $(sk, vk) \leftarrow \text{Gen}(1^n)$, $z \leftarrow \{0, 1\}^{\ell(n)}$ and $r_1 = \text{Com}(vk, z)$.
- 2 Output $(r = (r_1, r_2), s = (z, sk, vk))$, where r_2 is chosen uniformly at random

- ▶ $S_2(x, r, s = (z, sk, vk))$:

- 1 let $\pi_A \leftarrow P_A((x, r_1, vk), z, r_2)$
- 2 $\sigma \leftarrow \text{Sign}_{sk}(x, \pi_A)$
- 3 Output $\pi = (vk, \pi_A, \sigma)$

Proof follows by the adaptive WI of (P_A, V_A) and the pseudorandomness of Com

- ▶ **Adaptive soundness:** Implicit in the proof of simulation soundness, given below

Proving simulation soundness

Let $P^* = (P_1^*, P_2^*)$ be a pair of PPTM's attacking the simulation soundness of (V, S) with respect to \mathcal{L} , and let $r = (r_1, r_2)$, x , π , x' and $\pi' = (vk', \pi'_A, \sigma')$ be the values generated by a random execution of Exp_{V,S,P^*}^n .

Assuming $\text{Vrfy}_{vk'}((x', \pi'_A), \sigma') = 1$, $x' \notin \mathcal{L}$ and $(x', \pi') \neq (x, \pi)$, then with save but negligible probability:

- vk' is not the signing key in π
- $\nexists z \in \{0, 1\}^*$ s.t. $r_1 = \text{Com}(vk', z)$
- $x'_A = (x', r_1, vk') \notin \mathcal{L}_A$

Since r_2 was chosen at random by S_1 , the adaptive soundness of (P_A, V_A) yields that $\Pr[V_A(x'_A, r_2, \pi'_A) = 1] = \text{neg}(n)$.

Part II

Proof of Knowledge

Proof of Knowledge

The protocol (P, V) is a *proof of knowledge* for $\mathcal{L} \in \mathcal{NP}$, if P convinces V to accept x , only if it “knows” $w \in R_{\mathcal{L}}(x)$.

Definition 20 (knowledge extractor)

Let (P, V) be an interactive proof $\mathcal{L} \in \mathcal{NP}$. A probabilistic machine E is a knowledge extractor for (P, V) and $R_{\mathcal{L}}$ with error $\eta: \mathbb{N} \mapsto \mathbb{R}$, if $\exists t \in \text{poly}$ s.t. $\forall x \in \mathcal{L}$ and deterministic algorithm P^* , $E^{P^*}(x)$ runs in expected time bounded by $\frac{t(|x|)}{\delta(x) - \eta(|x|)}$ and outputs $w \in R_{\mathcal{L}}(x)$, where $\delta(x) = \Pr[(P^*, V)(x) = 1]$.

If (P, V) is a proof of knowledge (with error η), is it has a knowledge extractor with such error.

- A property of V
- Why do we need it? Proving that you know the password
- Relation to \mathcal{ZK}

Examples

Claim 21

The \mathcal{ZK} proof we've seen in class for \mathcal{GI} , has a knowledge extractor with error $\frac{1}{2}$.

Proof: ?

Claim 22

The \mathcal{ZK} proof we've seen in class for 3COL , has a knowledge extractor with error $\frac{1}{|E|}$.

Proof: ?