Application of Information Theory, Lecture 9

Parallel Repetition of Interactive Arguments

Iftach Haitner

Tel Aviv University.

December 15, 2011

Part I

Interactive Proofs and Arguments

$\mathcal{NP}$ as a Non-interactive Proofs

Definition 1 ($\mathcal{NP}$)

$\mathcal{L} \in \mathcal{NP}$ iff $\exists$ and poly-time algorithm V such that:

- $\forall x \in \mathcal{L}$ there exists $w \in \{0,1\}^*$ s.t. $\mathsf{V}(x, w) = 1$
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- Soundness holds unconditionally

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A protocol (P,V) is an interactive proof for $\mathcal{L}$, if V is a PPT and:

Completeness $\forall x \in \mathcal{L}$: $\Pr[(\mathsf{P},\mathsf{V})(x) = 1] \geq 2/3$.

Soundness $\forall x \notin \mathcal{L}$, and any algorithm P^*: $\Pr[(\mathsf{P}^*,\mathsf{V})(x) = 1] \leq 1/3$.

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- Games — no-input protocols.

Section 1

Interactive Proof for Graph Non-Isomorphism

Graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $\mathrm{G}_0 = ([m], E_0)$ and $\mathrm{G}_1 = ([m], E_1)$ are isomorphic, denoted $\mathrm{G}_0 \equiv \mathrm{G}_1$, if $\exists \pi \in \Pi_m$ such that
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  Idea: Beer tasting...

Interactive proof for $\mathcal{GNI}$

Protocol 4 ($(\mathsf{P},\mathsf{V})(\mathsf{G}_0 = ([m], E_0), \mathsf{G}_1 = ([m], E_1))$)

1. V chooses $b \leftarrow \{0,1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b)$ to P.[a]
2. P send b' to V (tries to set $b' = b$).
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Claim 5

The above protocol is IP for $\mathcal{GNI}$, with perfect completeness and soundness error $\frac{1}{2}$.

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Hence,

$\mathsf{G}_0 \equiv \mathsf{G}_1$: $\Pr[b' = b] \leq \frac{1}{2}$.

$\mathsf{G}_0 \not\equiv \mathsf{G}_1$: $\Pr[b' = b] = 1$ (i.e., P can, possibly inefficiently, extracted from $\pi(E_i)$)

□

Part II

Hardness Amplification

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- Parallel repetition does achieve optimal amplification rate for interactive proofs and public-coin interactive arguments
- Public-coin interactive proof/argument — in each round the verifier flips coins and sends them to the prover. To compute its output, the verifier applies some (fixed) function to the protocol's transcript.

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- In the following we focus on games (no input protocols)

Section 2

Parallel repetition of public-coin interactive argument

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- Assume for simplicity that $\widetilde{\mathsf{P}^{(k)}}$ is deterministic

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Proof plan: Let $\widetilde{\mathsf{P}^{(k)}}$ be $s^{(k)}$-size algorithm with $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $s^{(k)} \cdot \frac{mk^3 s_{\mathsf{V}}}{\varepsilon^{(k)}}$-size $\widetilde{\mathsf{P}}$ with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq (\varepsilon^{(k)})^{4/k}$.

- The $k/4$ in the exponent can be pushed to be almost k.
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Parallel repetition of public-coin interactive argument

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- Let $\mathbf{R} \sim \{0,1\}^{m \times (k\ell)}$

Algorithm $\widetilde{\mathsf{P}}$

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Algorithm 7 ($\widetilde{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. Upon getting the j'th message r from V, do:

   2.1 Let $R \leftarrow \{0,1\}^{m\times(k\ell)}$, conditioned that $R_{1,\ldots,j-1} = \widetilde{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = r$.

   2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$:

   2.2.1 Set $\widetilde{R}_j = R_j$

   2.2.2 Send a_{j,i^*} back to V, for a_j being the j'th message $\widetilde{\mathsf{P}^{(k)}}$ send to $\mathsf{V}^{(k)}$ in $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R))$.

   Else, GOTO Line 2.1

   2.3 Abort if the overall number of sampling exceeds $\lceil qm/\varepsilon^{(k)} \rceil$.

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- $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq \Pr\left[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) := (\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widetilde{\mathbf{R}})) = 1^k \wedge \widetilde{\mathbf{N}} \leq qm/\varepsilon^{(k)}\right]$.

Ideal “attacker”

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For $j = 1$ to m:

1. Let $R \leftarrow \{0,1\}^{m\times(k\ell)}$, conditioned that $R_{1,\ldots,j-1} = \hat{R}_{1,\ldots,j-1}$.
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Lemma 9

$\Pr\left[\hat{\boldsymbol{N}} \leq qm/\varepsilon^{(k)}\right] \geq 1 - \frac{1}{q}$

Proving Lemma 9

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- For $(z_1, \ldots, z_m)$, let $z^j = (z_1, \ldots, z_m)$.

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Claim 10

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Hence, $\mathrm{E}\left[\frac{1}{v(Y^j)}\right] = \sum_{\mathbf{y} \in \operatorname{Supp}(Y^j)} \Pr[Y^j = \mathbf{y}] \cdot \frac{1}{v(\mathbf{y})}$

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$= \sum_{\mathbf{y}} \Pr[X^j = \mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}} \cdot \frac{1}{v(\mathbf{y})} = \frac{1}{\varepsilon^{(k)}} \cdot \sum_{\mathbf{y} \in \operatorname{Supp}(Y^j)} \Pr[X^j = \mathbf{y}] \leq \frac{1}{\varepsilon^{(k)}}$. $\square$

Proving Claim 10

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Note that

$$\Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\dots,j-1}}[y_j] = \sum_{\ell=1}^{\infty}(1 - v(\mathbf{y}_{1\dots,j-1}))^{\ell-1} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\dots,j-1}}[y_j] \cdot v(\mathbf{y}) \qquad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1\dots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\dots,j-1}}[y_j] \cdot v(\mathbf{y})$$

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The proof proceeds by induction on j.

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$\Pr_{Y^j}[\mathbf{y}] = \Pr_{Y^{j-1}}\left[\mathbf{y}_{1\dots,j-1}\right] \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j]$

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$$= \Pr_{X^{j-1}}\left[\mathbf{y}_{1,\ldots,j-1}\right] \cdot \frac{v(\mathbf{y}_{1,\ldots,j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1,\ldots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\ldots,j-1}}[y_j] \qquad \text{(Eq. (1))}$$

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The proof proceeds by induction on j.

$$\Pr_{Y^j}[\mathbf{y}] = \Pr_{Y^{j-1}}\left[\mathbf{y}_{1\dots,j-1}\right] \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j]$$

$$= \Pr_{X^{j-1}}\left[\mathbf{y}_{1\dots,j-1}\right] \cdot \frac{v(\mathbf{y}_{1\dots,j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] \qquad \text{(i.h.)}$$

$$= \Pr_{X^{j-1}}\left[\mathbf{y}_{1\dots,j-1}\right] \cdot \frac{v(\mathbf{y}_{1\dots,j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1\dots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] \qquad \text{(Eq. (1))}$$

$$= \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}.$$

Ideal “attacker”, variant

Ideal “attacker”, variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.

   2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.

   2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.

   2.4 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathrm{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m\times(k\ell)}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.

   2.2 If $(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.

   2.3 Let $R \leftarrow \{0,1\}^{m\times\ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.

   2.4 If $(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathrm{P}}$.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathrm{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.

   2.2 If $(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.

   2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.

   2.4 If $(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathrm{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(\mathbf{R}))=1^k}$

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m\times(k\ell)}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.

   2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.

   2.3 Let $R \leftarrow \{0,1\}^{m\times\ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.

   2.4 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:

   2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.

   2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.

   2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.

   2.4 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:
   - **2.1** Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.
   - **2.2** If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
   - **2.3** Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.
   - **2.4** If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Lemma 12 (Essentially the same proof as of Lemma 9)

$\Pr\left[\mathrm{win}(\widehat{\boldsymbol{R}}, \widehat{\boldsymbol{N}})\right] \geq 1 - \frac{1}{q}$

From ideal to real

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
3. Lemma 12 $\implies \alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
3. Lemma 12 $\implies \alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.
4. By (2), $\alpha \cdot \log \frac{\alpha}{\beta} + (1 - \alpha) \log \frac{1-\alpha}{1-\beta} \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\operatorname{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \operatorname{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
3. Lemma 12 $\implies \alpha := \Pr[\operatorname{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\operatorname{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.
4. By (2), $\alpha \cdot \log \frac{\alpha}{\beta} + (1 - \alpha) \log \frac{1-\alpha}{1-\beta} \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

   $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i\in[k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
3. Lemma 12 $\implies \alpha := \Pr[\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.
4. By (2), $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log \frac{1-\alpha}{1-\beta} \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

   $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$
5. Since $q = k^2$: $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1-\alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i := \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
3. Lemma 12 $\implies \alpha := \Pr[\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.
4. By (2), $\alpha \cdot \log \frac{\alpha}{\beta} + (1 - \alpha) \log \frac{1-\alpha}{1-\beta} \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

   $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$
5. Since $q = k^2$: $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1 - \alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$
6. We conclude that $\beta \geq 2^{\frac{4}{k} \log \varepsilon^{(k)}} = \sqrt[k/4]{\varepsilon^{(k)}}$.$\square$

Proving Claim 13

Proving Claim 13

Let $\widehat{\mathsf{I}}$ and $\widetilde{\mathsf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}, \widehat{\mathbf{I}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{I}})$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathrm{P}}$ and $\widetilde{\mathrm{P}}$ respectively.

$$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}, \widehat{\mathbf{I}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{I}}) \qquad \text{(data-processing)}$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathrm{P}}$ and $\widetilde{\mathrm{P}}$ respectively.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i)
\end{aligned}
$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$
\begin{aligned}
D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}, \widehat{\mathbf{I}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{I}}) && \text{(data-processing)} \\
&= D(\widehat{\mathbf{I}} || \widetilde{\mathbf{I}}) + \frac{1}{k} \sum_{\in [k]} D(\widehat{\mathbf{R}}_i, \widehat{\mathbf{N}}_i || \widetilde{\mathbf{R}}_i, \widetilde{\mathbf{N}}_i) && \text{(chain rule)}
\end{aligned}
$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i)
\end{aligned}
$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i)
\end{aligned}$$

For $i \in [k]$, it holds that

$$D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) = D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_i}\left[D(\widehat{\mathbf{N}}_i|\widehat{\mathbf{R}}_i = r||\widetilde{\mathbf{N}}_i|\widetilde{\mathbf{R}}_i = r)\right]$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i)
\end{aligned}$$

For $i \in [k]$, it holds that

$$D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) = D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_i}\left[D(\widehat{\mathbf{N}}_i|\widehat{\mathbf{R}}_i = r||\widetilde{\mathbf{N}}_i|\widetilde{\mathbf{R}}_i = r)\right] \qquad \text{(chain rule)}$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i)
\end{aligned}$$

For $i \in [k]$, it holds that

$$\begin{aligned}
D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) &= D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_i}\left[D(\widehat{\mathbf{N}}_i|\widehat{\mathbf{R}}_i = r||\widetilde{\mathbf{N}}_i|\widetilde{\mathbf{R}}_i = r)\right] && \text{(chain rule)}\\
&= D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i)
\end{aligned}$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i)
\end{aligned}$$

For $i \in [k]$, it holds that

$$\begin{aligned}
D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) &= D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_i}\left[D(\widehat{\mathbf{N}}_i|\widehat{\mathbf{R}}_i = r||\widetilde{\mathbf{N}}_i|\widetilde{\mathbf{R}}_i = r)\right] \quad \text{(chain rule)}\\
&= D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i) \quad \text{(since } (\widehat{\mathbf{N}}_i|\widehat{\mathbf{R}}_i = r) \equiv (\widetilde{\mathbf{N}}_i|\widetilde{\mathbf{R}}_i = r) \text{ for every } r\text{)}
\end{aligned}$$

Proving Claim 13

Let $\widehat{\mathbf{I}}$ and $\widetilde{\mathbf{I}}$ be the values of i^* in $\widehat{\mathsf{P}}$ and $\widetilde{\mathsf{P}}$ respectively.

$$\begin{aligned} D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)} \\ &= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) && \text{(chain rule)} \\ &= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) \end{aligned}$$

For $i \in [k]$, it holds that

$$\begin{aligned} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_i||\widetilde{\mathbf{R}}_i,\widetilde{\mathbf{N}}_i) &= D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_i}\left[D(\widehat{\mathbf{N}}_i|\widehat{\mathbf{R}}_i = r||\widetilde{\mathbf{N}}_i|\widetilde{\mathbf{R}}_i = r)\right] && \text{(chain rule)} \\ &= D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i) \quad \text{(since } (\widehat{\mathbf{N}}_i|\widehat{\mathbf{R}}_i = r) \equiv (\widetilde{\mathbf{N}}_i|\widetilde{\mathbf{R}}_i = r) \text{ for every } r\text{)} \end{aligned}$$

Hence, $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_i||\widetilde{\mathbf{R}}_i)$ $\square$

Proving Claim 14

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids and let* W *be an event. For* $z \in \operatorname{Supp}(Z)$*, let* $D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$*.*
Then $\sum_{i=1}^{k} D(Z|_W || D_i) \le D(Z|_W || Z)$*.*

Proving Claim 14

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Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids and let W be an event. For $z \in \operatorname{Supp}(Z)$, let $D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.
Then $\sum_{i=1}^{k} D(Z|_W || D_i) \leq D(Z|_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. $\square$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids and let W be an event. For $z \in \mathrm{Supp}(Z)$, let
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.
Then $\sum_{i=1}^{k} D(Z|_W||D_i) \leq D(Z|_W||Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathrm{P}^{(k)}, \mathrm{V}^{(k)}}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids and let W be an event. For $z \in \mathrm{Supp}(Z)$, let $D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.

Then $\sum_{i=1}^{k} D(Z|_W || D_i) \leq D(Z|_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathrm{P}^{(k)}, \mathrm{V}^{(k)}}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i \in [k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. □

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$

Proving Claim 14

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Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids and let W be an event. For $z \in \mathrm{Supp}(Z)$, let $D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.

Then $\sum_{i=1}^{k} D(Z|_W||D_i) \leq D(Z|_W||Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathrm{P}^{(k)}, \mathrm{V}^{(k)}}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. $\square$

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}\left[(x_1, x_2, y_1, y_2)\right]$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids and let W be an event. For $z \in \operatorname{Supp}(Z)$, let $D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.

Then $\sum_{i=1}^{k} D(Z|_W || D_i) \leq D(Z|_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. $\square$

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- Let $X = Z_1$ and $Y = Z_2$
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- $C(x_1, x_2, y_1, y_1) := \Pr_{(X,Y)|_W}\left[(x_1, x_2, y_1, y_2)\right]$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids and let* W *be an event. For* $z \in \mathrm{Supp}(Z)$*, let*
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Then $\sum_{i=1}^{k} D(Z|_W||D_i) \leq D(Z|_W||Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. □

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}\left[(x_1, x_2, y_1, y_2)\right]$
- $C(x_1, x_2, y_1, y_1) := \Pr_{(X,Y)|_W}\left[(x_1, x_2, y_1, y_2)\right]$
- $Q(x_1, x_2, y_1, y_1) := \Pr\left[X_1 = x_1 | W\right] \cdot \Pr\left[X_2 = x_2 | W\right] \cdot \Pr\left[Y_1 = y_1 | W, X = (x_1, x_2)\right] \cdot \Pr\left[Y_2 = y_2 | W, X = (x_1, x_2)\right]$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids and let* W *be an event. For* $z \in \mathrm{Supp}(Z)$, *let*
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Then $\sum_{i=1}^{k} D(Z|_W || D_i) \leq D(Z|_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. $\square$

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- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}\left[(x_1, x_2, y_1, y_2)\right]$
- $C(x_1, x_2, y_1, y_1) := \Pr_{(X,Y)|_W}\left[(x_1, x_2, y_1, y_2)\right]$
- $Q(x_1, x_2, y_1, y_1) := \Pr\left[X_1 = x_1 | W\right] \cdot \Pr\left[X_2 = x_2 | W\right] \cdot \Pr\left[Y_1 = y_1 | W, X = (x_1, x_2)\right] \cdot \Pr\left[Y_2 = y_2 | W, X = (x_1, x_2)\right]$
- We write $\frac{C(x_1,x_2,y_1,y_1)}{U(x_1,x_2,y_1,y_1)} = \frac{\Pr[X_1=x_1|W]\cdot\Pr[Y_1=y_1|W,X=(x_1,x_2)]}{\Pr[X_1=x_1]\cdot\Pr[Y_1=y_1]} \cdot \frac{\Pr[X_2=x_2|W]\cdot\Pr[Y_2=y_2|W,X=(x_1,x_2)]}{\Pr[X_2=x_2]\cdot\Pr[Y_2=y_2]} \cdot \frac{C(x_1,x_2,y_1,y_1)}{Q(x_1,x_2,y_1,y_1)}$

Proving Lemma 15, cont.

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = & \mathop{\mathsf{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr[X_1 = x_1 | W] \cdot \Pr[Y_1 = y_1 | W, X = (x_1,x_2)]}{\Pr[X_1 = x_1] \cdot \Pr[Y_1 = y_1]}\right] \\
& + \mathop{\mathsf{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr[X_2 = x_2 | W] \cdot \Pr[Y_2 = y_2 | W, X = (x_1,x_2)]}{\Pr[X_2 = x_2] \cdot \Pr[Y_2 = y_2]}\right] \\
& + \mathop{\mathsf{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{C(x_1,x_2,y_1,y_2)}{Q(x_1,x_2,y_1,y_2)}\right].
\end{aligned}
$$

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = & \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr[X_1 = x_1 | W] \cdot \Pr[Y_1 = y_1 | W, X = (x_1,x_2)]}{\Pr[X_1 = x_1] \cdot \Pr[Y_1 = y_1]}\right] \\
& + \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr[X_2 = x_2 | W] \cdot \Pr[Y_2 = y_2 | W, X = (x_1,x_2)]}{\Pr[X_2 = x_2] \cdot \Pr[Y_2 = y_2]}\right] \\
& + \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{C(x_1,x_2,y_1,y_2)}{Q(x_1,x_2,y_1,y_2)}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D(C||U) = & D(X_1|_W, X_2|_{W,X_1}, Y_1|_{W,X}, Y_2|_{W,X,Y_1} || X_1, X_2|_{W,X_1}, Y_1, Y_2|_{W,X,Y_1}) \\
& + D(X_2|_W, X_1|_{W,X_2}, Y_2|_{W,X}, Y_1|_{W,X,Y_2} || X_2, X_1|_{W,X_2}, Y_2, Y_1|_{W,X,Y_2}) \\
& + D(C||Q)
\end{aligned}
$$

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = & \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr[X_1 = x_1|W]\cdot \Pr[Y_1 = y_1|W, X = (x_1,x_2)]}{\Pr[X_1 = x_1]\cdot \Pr[Y_1 = y_1]}\right] \\
& + \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr[X_2 = x_2|W]\cdot \Pr[Y_2 = y_2|W, X = (x_1,x_2)]}{\Pr[X_2 = x_2]\cdot \Pr[Y_2 = y_2]}\right] \\
& + \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{C(x_1,x_2,y_1,y_2)}{Q(x_1,x_2,y_1,y_2)}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D(C||U) = & D(X_1|_W, X_2|_{W,X_1}, Y_1|_{W,X}, Y_2|_{W,X,Y_1}||X_1, X_2|_{W,X_1}, Y_1, Y_2|_{W,X,Y_1}) \\
& + D(X_2|_W, X_1|_{W,X_2}, Y_2|_{W,X}, Y_1|_{W,X,Y_2}||X_2, X_1|_{W,X_2}, Y_2, Y_1|_{W,X,Y_2}) \\
& + D(C||Q)
\end{aligned}
$$

and the proof follows since $D(C||Q) \geq 0$. $\square$

Parallel repetition of interactive proofs

Parallel repetition of interactive proofs

- Similar proof to the public-coin proof we gave above.

Parallel repetition of interactive proofs

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- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $\widetilde{(\mathsf{P}^{(k)}}, \mathsf{V}^{(k)})$, till he gets an accepting execution.

Parallel repetition of interactive proofs

- Similar proof to the public-coin proof we gave above.
- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}})$, till he gets an accepting execution.
- Why fails us to extend this approach for non-public-coin interactive arguments?

Section 3

Parallel amplification for any interactive argument

Parallel amplification theorem for any protocol

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument "in parallel"?

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument “in parallel”?
- Yes we can!