Application of Information Theory, Lecture 9

Parallel Repetition of Interactive Arguments

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Part I

Interactive Proofs and Arguments

$\mathcal{NP}$ as a Non-interactive Proofs

Definition 1 ($\mathcal{NP}$)

$\mathcal{L} \in \mathcal{NP}$ iff $\exists$ and poly-time algorithm V such that:

- $\forall x \in \mathcal{L}$ there exists $w \in \{0,1\}^*$ s.t. $\mathsf{V}(x,w) = 1$
- $\mathsf{V}(x,w) = 0$ for every $x \notin \mathcal{L}$ and $w \in \{0,1\}^*$

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- Soundness holds unconditionally

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Protocols between efficient verifier and unbounded/efficent prover.

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A protocol (P, V) is an interactive proof for $\mathcal{L}$, if V is a `PPT` and:

Completeness $\forall x \in \mathcal{L}$: $\Pr[(\mathsf{P}, \mathsf{V})(x) = 1] \geq 2/3$.

Soundness $\forall x \notin \mathcal{L}$, and any algorithm P^*: $\Pr[(\mathsf{P}^*, \mathsf{V})(x) = 1] \leq 1/3$.

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- Games — no-input protocols.

Section 1

Interactive Proof for Graph Non-Isomorphism

Graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $\mathsf{G}_0 = ([m], E_0)$ and $\mathsf{G}_1 = ([m], E_1)$ are isomorphic, denoted $\mathsf{G}_0 \equiv \mathsf{G}_1$, if $\exists \pi \in \Pi_m$ such that
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  Idea: Beer tasting...

Interactive proof for $\mathcal{GNI}$

Protocol 4 $((\mathsf{P}, \mathsf{V})(\mathsf{G}_0 = ([m], E_0), \mathsf{G}_1 = ([m], E_1)))$

1. V chooses $b \leftarrow \{0,1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b)$ to P.[a]
2. P send b' to V (tries to set $b' = b$).
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Claim 5

The above protocol is IP for $\mathcal{GNI}$, with perfect completeness and soundness error $\frac{1}{2}$.

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Hence,

$\mathrm{G}_0 \equiv \mathrm{G}_1$: $\Pr[b' = b] \leq \frac{1}{2}$.

$\mathrm{G}_0 \not\equiv \mathrm{G}_1$: $\Pr[b' = b] = 1$ (i.e., b can be extracted from $\pi(E_i)$)

□

Part II

Hardness Amplification

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- Parallel repetition does achieve optimal amplification rate for interactive proofs and public-coin interactive arguments
- Public-coin interactive proof/argument — in each round the verifier flips coins and sends them to the prover. To compute its output, the verifier applies some (fixed) function to the protocol's transcript.

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- In the following we focus on games (no input protocols)

Section 2

Parallel repetition of public-coin interactive argument

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Theorem 6

Let $\pi = (\mathsf{P}, \mathsf{V})$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \leq \varepsilon$ for any s-size $\widetilde{\mathsf{P}}$, then $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] \leq \varepsilon^{k/4}$ for any $s \cdot \frac{\varepsilon^{k/4}}{mk^3 s_{\mathsf{V}}}$-size $\widetilde{\mathsf{P}^{(k)}}$, where s_{V} is V's size.

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Proof plan: Let $\widetilde{\mathsf{P}^{(k)}}$ be $s^{(k)}$-size algorithm with $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $s^{(k)} \cdot \frac{mk^3 s_{\mathsf{V}}}{\varepsilon^{(k)}}$-size $\widetilde{\mathsf{P}}$ with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq (\varepsilon^{(k)})^{4/k}$.

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- Assume for simplicity that $\widetilde{\mathsf{P}^{(k)}}$ is deterministic

Parallel repetition of public-coin interactive argument

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Let $\pi = (\mathsf{P}, \mathsf{V})$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \leq \varepsilon$ for any s-size $\widetilde{\mathsf{P}}$, then $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] \leq \varepsilon^{k/4}$ for any $s \cdot \frac{\varepsilon^{k/4}}{mk^3 s_{\mathsf{V}}}$-size $\widetilde{\mathsf{P}^{(k)}}$, where s_{V} is V's size.

Proof plan: Let $\widetilde{\mathsf{P}^{(k)}}$ be $s^{(k)}$-size algorithm with $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $s^{(k)} \cdot \frac{mk^3 s_{\mathsf{V}}}{\varepsilon^{(k)}}$-size $\widetilde{\mathsf{P}}$ with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq (\varepsilon^{(k)})^{4/k}$.

- The $k/4$ in the exponent can be pushed to be almost k.
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Algorithm 7 ($\widetilde{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. Upon getting the j'th round message r from V, do:

   2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned on $R_{\leq j-1} = \widetilde{R}_{\leq j-1}$ and $R_{j,i^*} = r$.

   2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$:

   2.2.1 Set $\widetilde{R}_j = R_j$

   2.2.2 Send a_{j,i^*} back to V, for a_j being the j'th message $\widetilde{\mathsf{P}^{(k)}}$ send to $\mathsf{V}^{(k)}$ in $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R))$.

   Else, GOTO Line 2.1

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Ideal "attacker"

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For $j = 1$ to m:

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$$\implies \Pr\left[\mathsf{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})\right] = \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widehat{\mathbf{R}})) = 1^k \wedge \widehat{\mathbf{N}} \leq 2qm/\varepsilon^{(k)}\right] \geq 1 - \frac{1}{q}$$

Proving Lemma 9 — $\Pr\left[\widehat{\mathbf{N}} > qm/\varepsilon^{(k)}\right] < \frac{1}{q}$

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For $j \in \{0, \ldots, m-1\}$ and $\mathbf{y} \in \mathrm{Supp}(Y_{\leq j})$, it holds that $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X_{\leq j}}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

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$= \sum_{\mathbf{y}} \Pr[X_{\leq j} = \mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}} \cdot \frac{1}{v(\mathbf{y})} = \frac{1}{\varepsilon^{(k)}} \cdot \sum_{\mathbf{y}\in\mathrm{Supp}(Y_{\leq j})} \Pr[X_{\leq j} = \mathbf{y}] \leq \frac{1}{\varepsilon^{(k)}}$. $\square$

Proving Claim 10 — $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Recall $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) = 1^k \mid X_{\leq j} = \mathbf{y}\right]$.

Proving Claim 10 — $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

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$$\Pr_{Y_j | Y_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{\leq j-1}))^{\ell-1} \cdot \Pr_{X_j | X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y}) \tag{1}$$

$$= \frac{1}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j | X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y})$$

Proving Claim 10 — $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Recall $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) = 1^k \mid X_{\leq j} = \mathbf{y}\right]$.

$$\Pr_{Y_j \mid Y_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{\leq j-1}))^{\ell-1} \cdot \Pr_{X_j \mid X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y}) \tag{1}$$

$$= \frac{1}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j \mid X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

Proving Claim 10 — $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Recall $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) = 1^k \mid X_{\leq j} = \mathbf{y}\right]$.

$$\Pr_{Y_j|Y_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] = \sum_{\ell=1}^{\infty}(1 - v(\mathbf{y}_{\leq j-1}))^{\ell-1} \cdot \Pr_{X_j|X_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y}) \tag{1}$$
$$= \frac{1}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j|X_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{Y_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \Pr_{Y_j|Y_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j]$

Proving Claim 10 — $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Recall $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) = 1^k \mid X_{\leq j} = \mathbf{y}\right]$.

$$\Pr_{Y_j | Y_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{\leq j-1}))^{\ell-1} \cdot \Pr_{X_j | X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y}) \tag{1}$$
$$= \frac{1}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j | X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{Y_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \Pr_{Y_j | Y_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j]$$
$$= \Pr_{X_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \frac{v(\mathbf{y}_{\leq j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j | Y_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \qquad \text{(i.h.)}$$

Proving Claim 10 — $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Recall $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) = 1^k \mid X_{\leq j} = \mathbf{y}\right]$.

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$$= \frac{1}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j | X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{Y_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \Pr_{Y_j | Y_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j]$$

$$= \Pr_{X_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \frac{v(\mathbf{y}_{\leq j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j | Y_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \qquad \text{(i.h.)}$$

$$= \Pr_{X_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \frac{v(\mathbf{y}_{\leq j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j | X_{\leq j-1} = \mathbf{y}_{\leq j-1}}[y_j] \qquad \text{(Eq. (1))}$$

Proving Claim 10 — $\Pr_{Y_{\leq j}}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Recall $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) = 1^k \mid X_{\leq j} = \mathbf{y}\right]$.

$$\Pr_{Y_j|Y_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{\leq j-1}))^{\ell-1} \cdot \Pr_{X_j|X_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y}) \tag{1}$$
$$= \frac{1}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j|X_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\begin{aligned}
\Pr_{Y_{\leq j}}[\mathbf{y}] &= \Pr_{Y_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \Pr_{Y_j|Y_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] \\
&= \Pr_{X_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \frac{v(\mathbf{y}_{\leq j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j|Y_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] && \text{(i.h.)} \\
&= \Pr_{X_{\leq j-1}}\left[\mathbf{y}_{\leq j-1}\right] \cdot \frac{v(\mathbf{y}_{\leq j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{\leq j-1})} \cdot \Pr_{X_j|X_{\leq j-1}=\mathbf{y}_{\leq j-1}}[y_j] && \text{(Eq. (1))} \\
&= \Pr_{X_{\leq j}}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}.
\end{aligned}$$

Ideal "attacker", variant

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:
   - **2.1** Let $R \leftarrow \{0,1\}^{m\times(k\ell)}$, conditioned on $R_{\leq j-1} = \widehat{R}_{\leq j-1}$.
   - **2.2** If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
   - **2.3** Let $R \leftarrow \{0,1\}^{m\times\ell}$, conditioned on $R_{\leq j-1} = \widehat{R}_{\leq j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.
   - **2.4** If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:
   - **2.1** Let $R \leftarrow \{0,1\}^{m\times(k\ell)}$, conditioned on $R_{\le j-1} = \widehat{R}_{\le j-1}$.
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- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

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- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

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- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
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- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
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- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
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- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Lemma 12 (essentially the same proof as of Lemma 9)

$$\Pr\left[\mathrm{win}(\widehat{\boldsymbol{R}}, \widehat{\boldsymbol{N}})\right] = \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widehat{\boldsymbol{R}})) = 1^k \wedge \widehat{\boldsymbol{N}} \le 2qm/\varepsilon^{(k)}\right] \ge 1 - \frac{1}{q}$$

From ideal to real

From ideal to real

Let $\widetilde{\mathsf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

From ideal to real

Let $\widetilde{\mathbf{I}}$ be the value of i^* in $\widetilde{\mathrm{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$.

From ideal to real

Let $\widetilde{\mathsf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

From ideal to real

Let $\widetilde{\mathbf{I}}$ be the value of i^* in $\widetilde{\mathrm{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(\mathbf{R})) = 1^k]} \leq \log \frac{1}{\varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathbf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$.

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1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} \leq \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\mathsf{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathsf{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

From ideal to real

Let $\widetilde{\mathbf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 13

$D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$.

Claim 14

$\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}}||\mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}}||\mathbf{R}) \leq \log \frac{1}{\Pr[\widetilde{(\mathsf{P}^{(k)},\mathsf{V}^{(k)}(\mathbf{R}))}=1^k]} \leq \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\mathrm{win}(\widehat{\mathbf{R}},\widehat{\mathbf{N}})||\,\mathrm{win}(\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq -\frac{1}{k}\cdot\log\varepsilon^{(k)}$
3. Lemma 12 $\implies \alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}},\widehat{\mathbf{N}})] \geq 1-\frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})]$.

From ideal to real

Let $\widetilde{\mathsf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i})$.

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$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i}) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

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2. Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
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4. By (2), $\alpha \cdot \log \frac{\alpha}{\beta} + (1 - \alpha) \log \frac{1-\alpha}{1-\beta} \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

From ideal to real

Let $\widetilde{\mathbf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R}))=1^k]} \leq \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\operatorname{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \operatorname{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
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4. By (2), $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log \frac{1-\alpha}{1-\beta} \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

   $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathsf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i})$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i}) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[\widetilde{(\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R}))} = 1^k]} \leq \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\operatorname{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \operatorname{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
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   $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$
5. Since $q = k^2$: $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1 - \alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$

From ideal to real

Let $\widetilde{\mathsf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i})$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i}) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R}))=1^k]} \leq \log \frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\operatorname{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \operatorname{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
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4. By (2), $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log \frac{1-\alpha}{1-\beta} \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

   $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$
5. Since $q = k^2$: $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1-\alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$
6. We conclude that $\beta \geq 2^{\frac{4}{k} \log \varepsilon^{(k)}} = \sqrt[k/4]{\varepsilon^{(k)}}$.$\square$

Proving Claim 14 — $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}}||\mathbf{R})$

Proving Claim 14 — $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}}||\mathbf{R})$

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids and let* W *be an event. For* $z \in \mathrm{Supp}(Z)$, *let*
$\xi_i(z) :=$
$\prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.

Then $\sum_{i=1}^{k} D(Z|_W||\xi_i) \leq D(Z|_W||Z)$.

Proving Claim 14 — $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}}||\mathbf{R})$

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids and let* W *be an event. For* $z \in \operatorname{Supp}(Z)$, *let*
$\xi_i(z) :=$
$\prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.
Then $\sum_{i=1}^{k} D(Z|_W||\xi_i) \leq D(Z|_W||Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_{\widetilde{\mathbf{I}}=i}) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_{\widetilde{\mathbf{I}}=i}) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. $\square$

Proving Lemma 15

We prove for $m = k = 2$.

Proving Lemma 15

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{-i} = x_{-i} \mid X_i = x_i \wedge W] \cdot$
$\Pr[Y_i = y_i] \cdot \Pr[Y_{-i} = Y_{-i} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

Proving Lemma 15

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{-i} = x_{-i} \mid X_i = x_i \wedge W] \cdot$
$\Pr[Y_i = y_i] \cdot \Pr[Y_{-i} = Y_{-i} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

We need to prove that $D(Z|_W || Z) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$.

Proving Lemma 15

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{-i} = x_{-i} \mid X_i = x_i \wedge W] \cdot$
$\Pr[Y_i = y_i] \cdot \Pr[Y_{-i} = Y_{-i} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

We need to prove that $D(Z|_W || Z) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$.

- Let $U = p_Z$ and $C = p_{Z|_W}$.

Proving Lemma 15

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{-i} = x_{-i} \mid X_i = x_i \wedge W] \cdot \Pr[Y_i = y_i] \cdot \Pr[Y_{-i} = Y_{-i} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

We need to prove that $D(Z|_W || Z) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$.

- Let $U = p_Z$ and $C = p_{Z|_W}$.

  (Hence, we need to prove that $D(C||U) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$)

Proving Lemma 15

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{-i} = x_{-i} \mid X_i = x_i \wedge W] \cdot \Pr[Y_i = y_i] \cdot \Pr[Y_{-i} = Y_{-i} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

We need to prove that $D(Z|_W || Z) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$.

- Let $U = p_Z$ and $C = p_{Z|_W}$.

  (Hence, we need to prove that $D(C||U) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$)
- Let $X = (X_0, X_1)$

Proving Lemma 15

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{-i} = x_{-i} \mid X_i = x_i \wedge W] \cdot$
$\Pr[Y_i = y_i] \cdot \Pr[Y_{-i} = Y_{-i} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

We need to prove that $D(Z|_W||Z) \geq \sum_{i=1}^{2} D(Z|_W||\xi_i)$.

- Let $U = p_Z$ and $C = p_{Z|_W}$.

  (Hence, we need to prove that $D(C||U) \geq \sum_{i=1}^{2} D(Z|_W||\xi_i)$)
- Let $X = (X_0, X_1)$
- Let $Q(x_0, x_1, y_0, y_1) := \Pr[X_0 = x_0|W] \cdot \Pr[X_1 = x_1|W] \cdot$
  $\Pr[Y_0 = y_0|W, X = (x_0, x_1)] \cdot \Pr[Y_1 = y_1|W, X = (x_0, x_1)]$

Proving Lemma 15

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{-i} = x_{-i} \mid X_i = x_i \wedge W] \cdot \Pr[Y_i = y_i] \cdot \Pr[Y_{-i} = Y_{-i} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

We need to prove that $D(Z|_W || Z) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$.

- Let $U = p_Z$ and $C = p_{Z|_W}$.

  (Hence, we need to prove that $D(C||U) \geq \sum_{i=1}^{2} D(Z|_W || \xi_i)$)
- Let $X = (X_0, X_1)$
- Let $Q(x_0, x_1, y_0, y_1) := \Pr[X_0 = x_0 | W] \cdot \Pr[X_1 = x_1 | W] \cdot \Pr[Y_0 = y_0 | W, X = (x_0, x_1)] \cdot \Pr[Y_1 = y_1 | W, X = (x_0, x_1)]$
- We write $\frac{C(x_0,x_1,y_0,y_1)}{U(x_0,x_1,y_0,y_1)} = \frac{\Pr[X_0=x_0|W]\cdot\Pr[Y_0=y_0|W,X=(x_0,x_1)]}{\Pr[X_0=x_0]\cdot\Pr[Y_0=y_0]} \cdot \frac{\Pr[X_1=x_1|W]\cdot\Pr[Y_1=y_1|W,X=(x_0,x_1)]}{\Pr[X_1=x_1]\cdot\Pr[Y_1=y_1]} \cdot \frac{C(x_0,x_1,y_0,y_1)}{Q(x_0,x_1,y_0,y_1)}$

Proving Lemma 15, cont.

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = & \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr\left[X_0 = x_0 | W\right] \cdot \Pr\left[Y_0 = y_0 | W, X = (x_0,x_1)\right]}{\Pr\left[X_0 = x_0\right] \cdot \Pr\left[Y_0 = y_0\right]}\right] \\
& + \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr\left[X_1 = x_1 | W\right] \cdot \Pr\left[Y_1 = y_1 | W, X = (x_0,x_1)\right]}{\Pr\left[X_1 = x_1\right] \cdot \Pr\left[Y_1 = y_1\right]}\right] \\
& + \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{C(x_0,x_1,y_0,y_1)}{Q(x_0,x_1,y_0,y_1)}\right].
\end{aligned}
$$

Proving Lemma 15, cont.

$$D(C||U) = \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_0 = x_0|W]\cdot \Pr[Y_0 = y_0|W, X = (x_0,x_1)]}{\Pr[X_0 = x_0]\cdot \Pr[Y_0 = y_0]}\right]$$
$$+ \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_1 = x_1|W]\cdot \Pr[Y_1 = y_1|W, X = (x_0,x_1)]}{\Pr[X_1 = x_1]\cdot \Pr[Y_1 = y_1]}\right]$$
$$+ \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{C(x_0,x_1,y_0,y_1)}{Q(x_0,x_1,y_0,y_1)}\right].$$

It follows that

$$D(C||U) = D(X_0|_W, X_1|_{W,X_0}, Y_0|_{W,X}, Y_1|_{W,X,Y_0}||X_0, X_1|_{W,X_0}, Y_0, Y_1|_{W,X,Y_0})$$
$$+ D(X_1|_W, X_0|_{W,X_1}, Y_1|_{W,X}, Y_0|_{W,X,Y_1}||X_1, X_0|_{W,X_1}, Y_1, Y_0|_{W,X,Y_1})$$
$$+ D(C||Q)$$

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = & \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_0 = x_0|W]\cdot\Pr[Y_0 = y_0|W, X = (x_0,x_1)]}{\Pr[X_0 = x_0]\cdot\Pr[Y_0 = y_0]}\right] \\
& + \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_1 = x_1|W]\cdot\Pr[Y_1 = y_1|W, X = (x_0,x_1)]}{\Pr[X_1 = x_1]\cdot\Pr[Y_1 = y_1]}\right] \\
& + \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{C(x_0,x_1,y_0,y_1)}{Q(x_0,x_1,y_0,y_1)}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D(C||U) &= D(X_0|_W, X_1|_{W,X_0}, Y_0|_{W,X}, Y_1|_{W,X,Y_0}||X_0, X_1|_{W,X_0}, Y_0, Y_1|_{W,X,Y_0}) \\
&\quad + D(X_1|_W, X_0|_{W,X_1}, Y_1|_{W,X}, Y_0|_{W,X,Y_1}||X_1, X_0|_{W,X_1}, Y_1, Y_0|_{W,X,Y_1}) \\
&\quad + D(C||Q) \\
&= \sum_{i=1}^{2} D(Z|_W||\xi_i) + D(C||Q)
\end{aligned}
$$

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = & \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_0 = x_0|W]\cdot\Pr[Y_0 = y_0|W, X = (x_0,x_1)]}{\Pr[X_0 = x_0]\cdot\Pr[Y_0 = y_0]}\right] \\
& + \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_1 = x_1|W]\cdot\Pr[Y_1 = y_1|W, X = (x_0,x_1)]}{\Pr[X_1 = x_1]\cdot\Pr[Y_1 = y_1]}\right] \\
& + \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{C(x_0,x_1,y_0,y_1)}{Q(x_0,x_1,y_0,y_1)}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D(C||U) &= D(X_0|_W, X_1|_{W,X_0}, Y_0|_{W,X}, Y_1|_{W,X,Y_0}||X_0, X_1|_{W,X_0}, Y_0, Y_1|_{W,X,Y_0}) \\
&\quad + D(X_1|_W, X_0|_{W,X_1}, Y_1|_{W,X}, Y_0|_{W,X,Y_1}||X_1, X_0|_{W,X_1}, Y_1, Y_0|_{W,X,Y_1}) \\
&\quad + D(C||Q) \\
&= \sum_{i=1}^{2} D(Z|_W||\xi_i) + D(C||Q) \\
&\geq \sum_{i=1}^{2} D(Z|_W||\xi_i). \square
\end{aligned}
$$

Proving Claim 13 — $D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathrm{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathrm{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

Proving Claim 13 — $D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)}, \widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)}, \widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}}, \widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}, \widehat{\mathbf{I}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{I}})$

Proving Claim 13 — $D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)}, \widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)}, \widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}}, \widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}, \widehat{\mathbf{I}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}, \widetilde{\mathbf{I}})$ (data-processing)

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}
$$

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}
\end{aligned}
$$

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}
$$

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{i},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}
$$

For $i \in [k]$, it holds that

$$
D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_{(i)}}\left[D(\widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r}||\widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r})\right]
$$

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathrm{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathrm{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{i},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}
$$

For $i \in [k]$, it holds that

$$
D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_{(i)}}\left[D(\widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r}||\widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r})\right] \quad \text{(chain ru}
$$

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \le \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathrm{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathrm{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\le D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{i},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}$$

For $i \in [k]$, it holds that

$$\begin{aligned}
D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) &= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_{(i)}}\left[D(\widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r}||\widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r})\right] && \text{(chain ru}\\
&= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)})
\end{aligned}$$

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathsf{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathsf{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_i,\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}$$

For $i \in [k]$, it holds that

$$\begin{aligned}
D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) &= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_{(i)}}\left[D(\widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r}||\widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r})\right] && \text{(chain ru}\\
&= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) \quad \text{(since } \widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r} \equiv \widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r}\text{)}
\end{aligned}$$

Proving Claim 13 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathrm{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathrm{P}}$).

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$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{i},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}
$$

For $i \in [k]$, it holds that

$$
\begin{aligned}
D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) &= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_{(i)}}\left[D(\widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r}||\widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r})\right] && \text{(chain ru}\\
&= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) \quad \text{(since } \widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r} \equiv \widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r})
\end{aligned}
$$

Hence, $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)})$ $\square$

Parallel repetition of interactive proofs

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- Similar proof to the public-coin proof we gave above.

Parallel repetition of interactive proofs

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- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}})$, till he gets an accepting execution.

Parallel repetition of interactive proofs

- Similar proof to the public-coin proof we gave above.
- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)})$, till he gets an accepting execution.
- What fails us to extend this approach for non-public-coin interactive arguments?

Section 3

Parallel amplification for any interactive argument

Parallel amplification theorem for any protocol

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument “in parallel”?

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument “in parallel”?
- Yes we can!

Relevant papers

- Kai-Min Chung and Rafael Pass: Tight Parallel Repetition Theorems for Public-Coin Arguments using KL-divergence.

  The proof given in class is in the spirit of this paper.