Foundation of Cryptography (0368-4162-01), Lecture 4

Interactive Proofs and Zero Knowledge

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Part I

Interactive Proofs

Interactive Vs. Interactive Proofs

Definition 1 (NP)

$\mathcal{L} \in \mathrm{NP}$ iff $\exists \ell \in$ poly and poly-time algorithm V such that:

- $\forall x \in \mathcal{L} \cap \{0,1\}^n$ there exists $w \in \{0,1\}^{\ell(n)}$ s.t. $\mathsf{V}(x, w) = 1$
- $\mathsf{V}(x, \cdot) = 0$ for every $x \notin \mathcal{L}$

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- Interactive proofs?

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- Interactive algorithm

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Interactive protocols

- Interactive algorithm
- Protocol $\pi = (\mathsf{A}, \mathsf{B})$
- RV describing the parties joint output $\langle \mathsf{A}(i_\mathsf{A}), \mathsf{B}(i_\mathsf{B}))(i)\rangle$
- *m*-round algorithm, *m*-round protocol

Interactive Proofs

Definition 2 (Interactive Proof (IP))

A protocol (P, V) is an interactive proof for $\mathcal{L}$, if V is $\mathtt{PPT}$ and the following hold:

Completeness $\forall x \in \mathcal{L}$, $\Pr[\langle(\mathsf{P}, \mathsf{V})(x)\rangle = \mathtt{Accept}] \geq 2/3$

Soundness $\forall x \notin \mathcal{L}$, and *any* algorithm P^*
$\Pr[\langle(\mathsf{P}^*, \mathsf{V})(x)\rangle = \mathtt{Accept}] \leq 1/3$

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- soundness only against $\texttt{PPT}$: *computationally sound proofs/interactive arguments*.

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- $\mathrm{IP} = \mathrm{PSPACE}$
- We typically consider (and achieve) perfect completeness
- Negligible "soundness error" achieved via repetition.
- soundness only against PPT: *computationally sound proofs/interactive arguments*.
- efficient provers via "auxiliary input"

Section 1

IP for GNI

graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $G_0 = ([m], E_0)$ and $G_1 = ([m], E_1)$ are *isomorphic*, denoted $G_0 \equiv G_1$, if $\exists \pi \in \Pi_m$ such that $(u, v) \in E_0$ iff $(\pi(u), \pi(v)) \in E_1$.
$\mathsf{GI} = \{(G_0, G_1) \colon G_0 \equiv G_1\}$.

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- $\mathsf{GI} \in \mathrm{NP}$

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- $\mathrm{GI} \in \mathrm{NP}$
- Does $\mathrm{GNI} = \{(G_0, G_1) \colon G_0 \not\equiv G_1\} \in \mathrm{NP}$?

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- We will show a simple interactive proof for GNI

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- $\mathsf{GI} \in \mathrm{NP}$
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- We will show a simple interactive proof for GNI Idea: Beer tasting...

IP for GNI

Protocol 4 ((P, V))

Common input $G_0 = ([m], E_0), G_1 = ([m], E_1)$

1. V chooses $b \leftarrow \{0, 1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b) = \{(\pi(u), \pi(v)) \colon (u, v) \in E_b\}$ to P
2. P send b' to V (tries to set $b' = b$)
3. V accepts iff $b' = b$

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Claim 5

The above protocol is IP for GNI, with perfect completeness and soundness error $\frac{1}{2}$.

Proving Claim 5

- Graph isomorphism is an equivalence relation (separates the set of all graph pairs into separate subsets)

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Hence,

$G_0 \equiv G_1$**:** $\Pr[b' = b] \leq \frac{1}{2}$.

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Hence,

$G_0 \equiv G_1$**:** $\Pr[b' = b] \leq \frac{1}{2}$.

$G_0 \not\equiv G_1$**:** $\Pr[b' = b] = 1$ (i.e., i can, possibly inefficiently, extracted from $\pi(E_i)$)

$\square$

Part II

Zero knowledge Proofs

The concept of zero knowledge

- Proving w/o revealing any addition information.

The concept of zero knowledge

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- What does it mean?

The concept of zero knowledge

- Proving w/o revealing any addition information.
- What does it mean?
  Simulation paradigm.

Zero knowledge Proof

Definition 6 (computational ZK)

An interactive proof (P, V) is computational zero-knowledge proof (CZKP) for $\mathcal{L}$, if $\forall$ PPT V^*, $\exists$ PPT S such that $\{\langle(\mathsf{P}, \mathsf{V}^*)(x)\rangle\}_{x\in\mathcal{L}} \approx_c \{\mathsf{S}(x)\}_{x\in\mathcal{L}}$.

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Perfect ZK (PZKP)/statistical ZK (SZKP) – the above dist. are identicallly/statistically close, even for *unbounded* V^*.

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5. Extension: auxiliary input

Zero knowledge Proof

Definition 6 (computational $\mathcal{ZK}$)

An interactive proof (P, V) is computational zero-knowledge proof ($\mathcal{CZKP}$) for $\mathcal{L}$, if $\forall$ PPT V^*, $\exists$ PPT S such that $\{\langle(\mathsf{P}, \mathsf{V}^*)(x)\rangle\}_{x\in\mathcal{L}} \approx_c \{\mathsf{S}(x)\}_{x\in\mathcal{L}}$.

Perfect $\mathcal{ZK}$ ($\mathcal{PZKP}$)/statistical $\mathcal{ZK}$ ($\mathcal{SZKP}$) – the above dist. are identicallly/statistically close, even for *unbounded* V^*.

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6. The “standard" NP proof is typically not zero knowledge
7. Next class — ZK for all NP

Section 2

ZK Proof for GI

ZK Proof for Graph Isomorphism

Idea: route finding

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Idea: route finding

Protocol 7 ((P, V))

Common input $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

P's input a permutation π such that $\pi(E_1) = E_0$

1. P chooses $\pi' \leftarrow \Pi_m$ and sends $E = \pi'(E_0)$ to V
2. V sends $b \leftarrow \{0, 1\}$ to P
3. if $b = 0$, P sets $\pi'' = \pi'$, otherwise, it sends $\pi'' = \pi' \circ \pi$ to V
4. V accepts iff $\pi''(E_b) = E$

ZK Proof for Graph Isomorphism

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Claim 8

The above protocol is $\mathcal{SZKP}$ for GI, with perfect completeness and soundness $\frac{1}{2}$.

Proving Claim 8

Completeness Clear

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Completeness Clear

Soundness If exist $j \in \{0,1\}$ for which $\nexists \pi' \in \Pi_m$ with $\pi'(E_j) = E$, then V rejects w.p. at least $\frac{1}{2}$.

Proving Claim 8

Completeness Clear

Soundness If exist $j \in \{0,1\}$ for which $\nexists \pi' \in \Pi_m$ with $\pi'(E_j) = E$, then V rejects w.p. at least $\frac{1}{2}$.
Assuming V rejects w.p. less than $\frac{1}{2}$ and lett π_0 and π_1 be the values guaranteed by the above observation (i.e., mapping E_0 and E_1 to E respectively).

Proving Claim 8

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Then $\pi_0^{-1}(\pi_1(E_1)) = \pi_0$

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Then $\pi_0^{-1}(\pi_1(E_1)) = \pi_0 \implies (G_0, G_1) \in \mathrm{GI}$.

Proving Claim 8

Completeness Clear

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Assuming V rejects w.p. less than $\frac{1}{2}$ and lett π_0 and π_1 be the values guaranteed by the above observation (i.e., mapping E_0 and E_1 to E respectively).
Then $\pi_0^{-1}(\pi_1(E_1)) = \pi_0 \implies (G_0, G_1) \in \mathrm{GI}$.

ZK Idea: for $(G_0, G_1) \in \mathrm{GI}$, it is easy to generate a random transcript for Steps 1-2, and to be able to open it with prob $\frac{1}{2}$.

The simulator

For a start we consider a deterministic cheating verifier V^* that never aborts.

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Algorithm 9 (S)

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$
Do $|x|$ times:

1. Choose $b' \leftarrow \{0,1\}$ and $\pi \leftarrow \Pi_m$, and "send" $\pi(E_{b'})$ to $V^*(x)$.
2. Let b be V^*'s answer. If $b = b'$, send π to V^*, output V^*'s output and halt.
   Otherwise, rewind the simulation to its first step.

Abort

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Abort

Claim 10

$\{\langle(\mathsf{P},\mathsf{V}^*)(x)\rangle\}_{x\in\mathsf{GI}} \approx \{\mathsf{S}(x)\}_{x\in\mathsf{GI}}$

Proving Claim 10

Algorithm 11 (S′)

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$
Do $|x|$ times:

1. Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $V^*(x)$.
2. Let b be V^*'s answer.
   w.p. $\frac{1}{2}$, find π' such that $E = \pi'(E_b)$ and send it to V^*, output V^*'s output and halt.
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Abort

Proving Claim 10

Algorithm 11 (S')

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$
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Abort

Claim 12

$S(x) \equiv S'(x)$ for any $x \in GI$.

Proving Claim 10

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Do $|x|$ times:

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Abort

Claim 12

$\mathsf{S}(x) \equiv \mathsf{S}'(x)$ for any $x \in \mathrm{GI}$.

Proof: ?

Proving Claim 10 cont.

Algorithm 13 (S'')

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

1. Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $V^*(x)$.
2. Find π' such that $E = \pi'(E_b)$, send it to V^*, output V^*'s output and halt.

Proving Claim 10 cont.

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Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

1. Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $\mathsf{V}^*(x)$.
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Claim 14

$\forall x \in \mathrm{GI}$ it holds that

1. $\langle(\mathsf{P}, \mathsf{V}^*(x))\rangle \equiv \mathsf{S}''(x)$.

Proving Claim 10 cont.

Algorithm 13 (S″)

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

1. Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $\mathsf{V}^*(x)$.
2. Find π' such that $E = \pi'(E_b)$, send it to V^*, output V^*'s output and halt.

Claim 14

$\forall x \in \mathrm{GI}$ it holds that

1. $\langle(\mathsf{P}, \mathsf{V}^*(x))\rangle \equiv \mathsf{S}''(x)$.
2. $\mathrm{SD}(\mathsf{S}''(x), \mathsf{S}'(x)) \leq 2^{-|x|}$.

Proving Claim 10 cont.

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Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

1. Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $\mathsf{V}^*(x)$.
2. Find π' such that $E = \pi'(E_b)$, send it to V^*, output V^*'s output and halt.

Claim 14

$\forall x \in \mathrm{GI}$ it holds that

1. $\langle(\mathsf{P}, \mathsf{V}^*(x))\rangle \equiv \mathsf{S}''(x)$.
2. $\mathrm{SD}(\mathsf{S}''(x), \mathsf{S}'(x)) \leq 2^{-|x|}$.

Proof: ?

Proving Claim 10 cont.

Algorithm 13 (S'')

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

1. Choose $\pi \leftarrow \Pi_m$ and sends $E = \pi(E_0)$ to $V^*(x)$.
2. Find π' such that $E = \pi'(E_b)$, send it to V^*, output V^*'s output and halt.

Claim 14

$\forall x \in GI$ it holds that

1. $\langle (P, V^*(x)) \rangle \equiv S''(x)$.
2. $SD(S''(x), S'(x)) \leq 2^{-|x|}$.

Proof: ? (1) is clear.

Proving Claim 14(2)

Fix (E, π') and let $\alpha = \Pr_{S''}[(E, \pi')]$.

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Hence, $\mathrm{SD}(S''(x), S'(x)) \leq 2^{-|x|}$ □

Remarks

1. Randomized verifiers

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2. Aborting verifiers

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2. Aborting verifiers – Normalize aborting probability

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1. Randomized verifiers
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5. Perfect ZK for “expected time simulators"
6. “Black box" simulation

Section 3

Black-box ZK

Black-box simulators

Definition 15 (Black-box simulator)

(P, V) is CZKP with black-box simulation for $\mathcal{L}$, if $\exists$ oracle-aided PPT S s.t. for every deterministic polynomial-time[a] V*:

$$\{(\mathsf{P}(w_x), \mathsf{V}^*(z))(x)\}_{x \in \mathcal{L}} \approx_c \{\mathsf{S}^{\mathsf{V}^*(x,z_x)}(x)\}_{x \in \mathcal{L}}$$

for any $\{(w_x, z_x) \in R_{\mathcal{L}}(x) \times \{0,1\}^*\}_{x \in \mathcal{L}}$.

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Prefect and statistical variants are defined analogously.

[a] Length of auxiliary input does not count for the running time.

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Prefect and statistical variants are defined analogously.

[a]Length of auxiliary input does not count for the running time.

1. “Most simulators" are black box
2. Strictly weaker then general simulation!

Section 4

Zero Knowledge for all NP

CZKP for 3COL

- Assuming that OWFs exists, we give a CZKP for 3COL .
- We show how to transform it for any $\mathcal{L} \in \mathrm{NP}$ (using that 3COL $\in \mathrm{NPC}$).

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$G = (M, E) \in$ 3COL, if $\exists\, \phi \colon M \mapsto [3]$ s.t. $\phi(u) \neq \phi(v)$ for every $(u, v) \in E$.

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We use commitment schemes.

The protocol

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Protocol 17 ((P, V))

Common input: Graph $G = (M, E)$ with $n = |G|$
P's input: a (valid) coloring ϕ of G

1. P chooses $\pi \leftarrow \Pi_3$ and sets $\psi = \pi \circ \phi$
2. $\forall v \in M$: P commits to $\psi(v)$ using $\mathrm{Com}(1^n)$.
   Let c_v and d_v be the resulting commitment and decommitment.
3. V sends $e = (u, v) \leftarrow E$ to P
4. P sends $(d_u, \psi(u)), (d_v, \psi(v))$ to V
5. V verifies that (1) both decommitments are valid, (2) $\psi(u), \psi(v) \in [3]$ and (3) $\psi(u) \neq \psi(v)$.

Claim 18

The above protocol is a CZKP for 3COL, with perfect completeness and soundness $1/|E|$.

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Soundness: Let $\{c_v\}_{v \in M}$ be the commitments resulting from an interaction of V with an arbitrary P*.
Define $\phi\colon M \mapsto [3]$ as follows:
$\forall v \in M$: let $\phi(v)$ be the (single) value that it is possible to decommit c_v into (if not in [3], set $\phi(v) = 1$).

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If $G \notin$ 3COL, then $\exists (u, v) \in E$ s.t. $\psi(u) = \psi(v)$.

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If $G \notin$ 3COL, then $\exists (u, v) \in E$ s.t. $\psi(u) = \psi(v)$.
Hence V rejects such x w.p. a least $1/|E|$

Proving ZK

Fix a deterministic, non-aborting V^* that gets no auxiliary input.

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Fix a deterministic, non-aborting V* that gets no auxiliary input.

Algorithm 19 (S)

Input: A graph $G = (M, E)$ with $n = |G|$
Do $n \cdot |E|$ times:

1. Choose $e' = (u, v) \leftarrow E$. Set $\psi(u) \leftarrow [3]$, $\psi(v) \leftarrow [3] \setminus \{\psi(u)\}$, and $\psi(w) = 1$ for $w \in M \setminus \{u, v\}$
2. $\forall v \in M$: commit to $\psi(v)$ to V* (resulting in c_v and d_v)
3. Let e be the edge sent by V*.
   If $e = e'$, send $(d_u, \psi(u)), (d_v, \psi(v))$ to V*, output V*'s output and halt.
   Otherwise, rewind the simulation to its first step.

Abort

Proving ZK cont.

Claim 20

$\{(\mathsf{P}(w_x), \mathsf{V}^*)(x)\}_{x \in 3\mathrm{COL}} \approx_c \{\mathsf{S}^{\mathsf{V}^*(x)}(x)\}_{x \in 3\mathrm{COL}}$, for any $\{w_x \in R_{3\mathrm{COL}}(x)\}_{x \in 3\mathrm{COL}}$.

Consider the following (inefficient simulator)

Algorithm 21 (S')

Input: $G = (V, E)$ with $n = |G|$
Find (using brute force) a valid coloring ϕ of G
Do $n \cdot |E|$ times

1. Act as the honest prover does given private input ϕ
2. Let e be the edge sent by V^*.
   w.p $1/|E|$, S' sends $(\psi(u), d_u), (\psi(v), d_v)$ to V^*, output V^*'s output and halt.
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Abort

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Claim 22

$$\{\mathsf{S}^{\mathsf{V}^*(x)}(x)\}_{x \in 3\mathrm{COL}} \approx_c \{\mathsf{S}'^{\mathsf{V}^*(x)}(x)\}_{x \in 3\mathrm{COL}}$$

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Abort

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Proof: ?

Proving Claim 22

Assume $\exists$ PPT D, $p \in$ poly and an infinite set $\mathcal{I} \subseteq$ 3COL s.t.

$$\left|\Pr[\mathsf{D}(|x|, \mathsf{S}^{\mathsf{V}^*(x)}(x)) = 1] - \Pr[\mathsf{D}(|x|, \mathsf{S}'^{\mathsf{V}^*(x)}(x)) = 1]\right| \geq 1/p(|x|)$$

for all $x \in \mathcal{I}$.

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for all $x \in \mathcal{I}$.
Hence, $\exists$ PPT R^* and $b \neq b' \in [3]$ such that

$$\{\mathrm{View}_{\mathrm{R}^*}(\mathrm{S}(b), \mathrm{R}^*(x))(1^{|x|})\}_{x \in \mathcal{I}} \not\approx_c \{\mathrm{View}_{\mathrm{R}^*}(\mathrm{S}(b'), \mathrm{R}^*(x))(1^{|x|})\}_{x \in \mathcal{I}}$$

where S is the sender in Com.

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where S is the sender in Com.
We critically used the non-uniform security of Com

S′ is a good simulator

Claim 23

$\{(\mathsf{P}(w_x), \mathsf{V}^*)(x)\}_{x \in 3\mathrm{COL}} \approx_c \{\mathsf{S}'^{\mathsf{V}^*(x)}(x)\}_{x \in 3\mathrm{COL}}$, for any $\{w_x \in R_{\mathrm{GI}}(x)\}_{x \in 3\mathrm{COL}}$.

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Proof: ?

Remarks

- Aborting verifiers
- Auxiliary inputs
- Soundness amplification

Remarks

- Aborting verifiers
- Auxiliary inputs
- Soundness amplification
- Non-uniform hiding guarantee

Extending to all $\mathcal{L} \in \mathrm{NP}$

Let (P, V) be a CZKP for 3COL, and let Map_X and Map_W be two poly-time functions s.t.

- $\forall x \in \{0,1\}^*$: $x \in \mathcal{L} \longleftrightarrow \mathrm{Map}_X(x) \in \mathrm{3COL}$,
- $\forall x \in \mathcal{L}$ and $w \in R_L(x)$: $\mathrm{Map}_W(x, w) \in R_{\mathrm{3COL}}(\mathrm{Map}_X(x))$

Protocol 24 ($(\mathrm{P}_\mathcal{L}, \mathrm{V}_\mathcal{L})$)

Common input: $x \in \{0,1\}^*$
$\mathrm{P}_\mathcal{L}$'s input: $w \in R_\mathcal{L}(x)$

1. The two parties interact in $\langle(\mathrm{P}(\mathrm{Map}_W(x, w)), \mathrm{V})(\mathrm{Map}_X(x))\rangle$, where $\mathrm{P}_\mathcal{L}$ and $\mathrm{V}_\mathcal{L}$ taking the role of P and V respectively.
2. $\mathrm{V}_\mathcal{L}$ accepts iff V accepts in the above execution.

Extending to all $\mathcal{L} \in \mathrm{NP}$ cont.

Claim 25

$(\mathsf{P}_{\mathcal{L}}, \mathsf{V}_{\mathcal{L}})$ is a CZKP for $\mathcal{L}$ with the same completeness and soundness as (P, V) as for $3\mathsf{COL}$.

Extending to all $\mathcal{L} \in \mathrm{NP}$ cont.

Claim 25

$(\mathsf{P}_{\mathcal{L}}, \mathsf{V}_{\mathcal{L}})$ is a CZKP for $\mathcal{L}$ with the same completeness and soundness as (P, V) as for $3\mathrm{COL}$.

- **Completeness and soundness:** Clear.

Extending to all $\mathcal{L} \in \mathrm{NP}$ cont.

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$(\mathsf{P}_{\mathcal{L}}, \mathsf{V}_{\mathcal{L}})$ is a CZKP for $\mathcal{L}$ with the same completeness and soundness as (P, V) as for 3COL.

- **Completeness and soundness:** Clear.
- **Zero knowledge:** Let S (an efficient) ZK simulator for (P, V) (for 3COL).
  Define $\mathsf{S}_{\mathcal{L}}(x)$ to output $\mathsf{S}(\mathsf{Map}_X(x))$, while replacing the string $\mathsf{Map}_X(x)$ in the output of S with x.

Extending to all $\mathcal{L} \in \mathrm{NP}$ cont.

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  $\{(\mathsf{P}(w_x), \mathsf{V}^*)(x)\}_{x \in \mathcal{L}} \not\approx_c \{\mathsf{S}_{\mathcal{L}}^{\mathsf{V}^*(x)}(x)\}_{x \in \mathcal{L}}$ for some $\mathsf{V}^*_{\mathcal{L}}$, implies $\{(\mathsf{P}(\mathsf{Map}_W(x, w_x)), \mathsf{V}^*)(x)\}_{x \in 3\mathrm{COL}} \not\approx_c \{\mathsf{S}^{\mathsf{V}^*(x)}(x)\}_{x \in 3\mathrm{COL}}$,
- $\mathsf{V}^*(x)$: find $x^{-1} = \mathsf{Map}_X^{-1}(x)$ and act like $\mathsf{V}^*_{\mathcal{L}}(x^{-1})$