Foundation of Cryptography (0368-4162-01), Lecture 1
One-Way Functions
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Section 1

Notation

Notation I

- For $t \in \mathbb{N}$, let $[t] := \{1, \ldots, t\}$.
- Given a string $x \in \{0,1\}^*$ and $0 \leq i < j \leq |x|$, let $x_{i,\ldots,j}$ stands for the substring induced by taking the $i, \ldots, j$ bit of x (i.e., $x[i] \ldots, x[j]$).
- Given a function f defined over a set $\mathcal{U}$, and a set $\mathcal{S} \subseteq \mathcal{U}$, let $f(\mathcal{S}) := \{f(x) \colon x \in \mathcal{S}\}$, and for $y \in f(\mathcal{U})$ let $f^{-1}(y) := \{x \in \mathcal{U} \colon f(x) = y\}$.
- poly stands for the set of all polynomials.
- The worst-case running-time of a *polynomial-time algorithm* on input x, is bounded by $p(|x|)$ for some $p \in$ poly.
- A function is *polynomial-time computable*, if there exists a polynomial-time algorithm to compute it.
- PPT stands for probabilistic polynomial-time algorithms.
- A function $\mu \colon \mathbb{N} \mapsto [0,1]$ is negligible, denoted $\mu(n) = \mathrm{neg}(n)$, if for any $p \in$ poly there exists $n' \in \mathbb{N}$ with $\mu(n) \leq 1/p(n)$ for any $n > n'$.

Distribution and random variables I

- The support of a distribution P over a finite set $\mathcal{U}$, denoted $\text{Supp}(P)$, is defined as $\{u \in \mathcal{U} \colon P(u) > 0\}$.
- Given a distribution P and en event E with $\Pr_P[E] > 0$, we let $(P \mid E)$ denote the conditional distribution P given E (i.e., $(P \mid E)(x) = \frac{D(x) \wedge E}{\Pr_P[E]}$).
- For $t \in \mathbb{N}$, let let U_t denote a random variable uniformly distributed over $\{0,1\}^t$.
- Given a random variable X, we let $x \leftarrow X$ denote that x is distributed according to X (e.g., $\Pr_{x \leftarrow X}[x = 7]$).
- Given a final set $\mathcal{S}$, we let $x \leftarrow \mathcal{S}$ denote that x is uniformly distributed in $\mathcal{S}$.
- We use the convention that when a random variable appears twice in the same expression, it refers to a *single* instance of this random variable. For instance, $\Pr[X = X] = 1$ (regardless of the definition of X).

Distribution and random variables II

- Given distribution P over $\mathcal{U}$ and $t \in \mathbb{N}$, we let P^t over $\mathcal{U}^t$ be defined by $D^t(x_1, \ldots, x_t) = \Pi_{i \in [t]} D(x_i)$.
- Similarly, given a random variable X, we let X^t denote the random variable induced by t independent samples from X.

Section 2

One Way Functions

One-Way Functions

Definition 1 (One-Way Functions (OWFs))

A polynomial-time computable function $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ is one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] = \operatorname{neg}(n)$$

for any PPT A.

polynomial-time computable: there exists a polynomial-time algorithm F, such that $F(x) = f(x)$ for every $x \in \{0,1\}^*$

PPT : probabilistic polynomial-time algorithm

neg: a function $\mu\colon \mathbb{N} \mapsto [0,1]$ is a *negligible* function of n, denoted $\mu(n) = \operatorname{neg}(n)$, if for any $p \in \operatorname{poly}$ there exists $n' \in \mathbb{N}$ such that $g(n) < 1/p(n)$ for all $n > n'$

We typically omit 1^n from the input list of A

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6. Non uniform OWFs

Definition 2 (Non-uniform OWF))

A polynomial-time computable function $f : \{0,1\}^* \mapsto \{0,1\}^*$ is non-uniformly one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[C_n(f(x)) \in f^{-1}(f(x))\right] = \operatorname{neg}(n)$$

for any polynomial-size family of circuits $\{C_n\}_{n\in\mathbb{N}}$.

Length preserving functions

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A function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is length preserving, if $|f(x)| = |x|$ for every $x \in \{0,1\}^*$

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Theorem 4

Assume that OWFs exit, then there exist length-preserving OWFs

Proof idea: use the assumed OWF to create a length preserving one

Partial domain functions

Definition 5 (Partial domain functions)

For $m, \ell \colon \mathbb{N} \mapsto \mathbb{N}$, let $h \colon \{0,1\}^{m(n)} \mapsto \{0,1\}^{\ell(n)}$ denote a function defined over input lengths in $\{m(n)\}_{n \in \mathbb{N}}$, and maps strings of length $m(n)$ to strings of length $\ell(n)$.

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The definition of one-wayness naturally extends to such functions.

OWFs imply Length Preserving OWFs cont.

Let $f\colon \{0,1\}^* \mapsto \{0,1\}^*$ be a OWF, let $p \in \mathrm{poly}$ be a bound on its computing-time and assume wlg. that p is monotony increasing (can we?).

OWFs imply Length Preserving OWFs cont.

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Construction 6 (the length preserving function)

Define $g\colon \{0,1\}^{p(n)} \mapsto \{0,1\}^{p(n)}$ as

$$g(x) = f(x_{1,\ldots,n}), 0^{p(n)-\left|f(x_{1,\ldots,n})\right|}$$

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Note that g is well defined, length preserving and efficient (why?).

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How can we prove that g is one-way?
Answer: using reduction.

Proving that g is one-way

Proof:
Assume that g is not one-way. Namely, there exists PPT A, $q \in \text{poly}$ and infinite set $\mathcal{I} \subseteq \{p(n) \colon n \in \mathbb{N}\}$, with

$$\Pr_{x \leftarrow \{0,1\}^n} \left[A(y) \in g^{-1}(g(x)) \right] > 1/q(n) \tag{1}$$

for every $n \in \mathcal{I}$.

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We show how to use A for inverting f.

Algorithm 8 (The inverter B)

Input: 1^n and $y \in \{0,1\}^*$

1. Let $x = \mathsf{A}(1^{p(n)}, y, 0^{p(n)-|y|})$
2. Return $x_{1,\ldots,n}$

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Let $\mathcal{I}' := \{n \in \mathbb{N} \colon p(n) \in \mathcal{I}\}$. Then

1. $\mathcal{I}'$ is infinite
2. $\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{B}(1^n, f(x)) \in f^{-1}(f(x))] > 1/q(p(n))$ for every $n \in \mathcal{I}'$

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This contradict the assumed one-wayness of f. $\square$

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Proof: (1) is clear

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&\geq \Pr_{x' \leftarrow \{0,1\}^{p(n)}}[\mathsf{A}(1^{p(n)}, g(x)) \in g^{-1}(g(x))]
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Conclusion

Remark 10

- We directly related the hardness of f to that of g
- The reduction is not "security preserving"

From partial domain functions to all-length functions

Construction 11

Given a function $f\colon \{0,1\}^{\ell(n)} \mapsto \{0,1\}^{\ell(n)}$, define $f_{\mathrm{all}}\colon \{0,1\}^* \mapsto \{0,1\}^*$ as

$$f_{\mathrm{all}}(x) = f(x_{1,\ldots,k}), 0^{n-k}$$

where $n = |x|$ and $k := \max\{\ell(n') \leq n\colon n' \in [n]\}$.

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Clearly, f_{all} is length preserving defined for every input length, and efficient (i.e., poly-time computable) in case f and ℓ are.

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Claim 12

Assume f and ℓ are efficiently computable, f is one-way, and ℓ satisfies $1 \leq \frac{\ell(n+1)}{\ell(n)} \leq p(n)$ for some $p \in \mathrm{poly}$, then f_{all} is one-way function.

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Proof: ?

Weak One Way Functions

Definition 13 (weak one-way functions)

A poly-time computable function $f\colon \{0,1\}^* \mapsto f\colon \{0,1\}^*$ is α-one-way, if

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(1^n, f(x)) \in f^{-1}(f(x))\right] \leq \alpha(n)$$

for any PPT A and large enough $n \in \mathbb{N}$.

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1. (strong) OWF according to Definition 1, are neg(n)-one-way according to the above definition
2. Can we “amplify" weak OWF to strong ones?

Strong to weak OWFs

Claim 14

Assume there exists OWFs, then there exist functions that are $\frac{2}{3}$-one-way, but not (strong) one-way

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Proof: For a OWF f, let

$$g(x) = \begin{cases} (1, f(x)), & x_1 = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Weak to Strong OWFs

Theorem 15

Assume there exists $(1-\alpha)$-weak OWFs with $\alpha(n) > 1/p(n)$ for some $p \in \mathrm{poly}$, then there exists (strong) one-way functions.

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Construction 16 (g – the strong one-way function)

Let $t\colon \mathbb{N} \mapsto \mathbb{N}$ be a poly-time computable function satisfying $t(n) \in \omega(\log n/\alpha(n))$. Define $g\colon (\{0,1\}^n)^{t(n)} \mapsto (\{0,1\}^n)^{t(n)}$ as

$$g(x_1,\ldots,x_t) = f(x_1),\ldots,f(x_t)$$

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Claim 17

g is one-way.

Proving that g is one-way – the naive approach

Let A be a potential inverter for g, and assume that A tries to attacks each of the t outputs of g independently. Then

$$\Pr_{x \leftarrow \{0,1\}^{t(n) \cdot n}}[\mathsf{A}(g(x)) \in g^{-1}(g(x))] \leq (1 - \alpha(n))^{t(n)} \leq e^{-\omega(\log n)} = \mathrm{neg}(n)$$

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Unfortunately, we can assume none of the above.

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Any idea?

Failing Sets

Failing Sets

Definition 18 (failing set)

A function $f\colon \{0,1\}^n \mapsto \{0,1\}^{\ell(n)}$ has a (δ,ε)-failing set for algorithm A, if for large enough n, exists set $\mathcal{S} = \mathcal{S}(n) \subseteq \{0,1\}^{\ell(n)}$ with

1. $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}] \geq \delta(n)$, and
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Definition 18 (failing set)

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1. $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{L}] \geq 1 - \alpha(n)/2$, and
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We'll use A to contradict the hardness of f.

Using A to invert f

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Algorithm 20 (The inverter B)

Input: $y \in \{0,1\}^n$.
Do (with fresh randomness) for $n \cdot p(n)$ times:
If $x = \mathsf{A}(y) \in f^{-1}(y)$, return x

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For every large enough $n \in \mathcal{I}$, it holds that
$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{B}(f(x)) \in f^{-1}(f(x))\right] > 1 - \alpha(n)$

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Hence, f is not $(1-\alpha)$-one-way $\square$

Proof: [of Claim 21]
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&\quad \geq \Pr[\mathsf{B}(y) \in f^{-1}(y) \wedge y \in \mathcal{L}(n)]
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&\quad \geq \quad \Pr[\mathrm{B}(y) \in f^{-1}(y) \wedge y \in \mathcal{L}(n)] \\
&\quad = \quad \Pr[y \in \mathcal{L}(n)] \cdot \Pr[\mathrm{B}(y) \in f^{-1}(y) \mid y \in \mathcal{L}(n)]
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&\quad \geq \ (1 - \alpha(n)/2) \cdot (1 - 2^{-n}) > 1 - \alpha(n),
\end{aligned}
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for large enough n. ♣

Proving that g is one-way

We show that if g is not OWF, then f has no flailing-set of the “right" type.

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$$\Pr_{w \leftarrow \{0,1\}^{t(n) \cdot n}}[\mathsf{A}(g(x)) \in g^{-1}(g(w))] \geq 1/p(n) \tag{2}$$

for every $n \in \mathcal{I}$.

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for every $n \in \mathcal{I}$. Then $\exists$ PPT B and $q \in$ poly s.t.

$$\Pr_{y \leftarrow \mathcal{S}}[\mathsf{B}(y) \in f^{-1}(y)] \geq 1/q(n) \tag{3}$$

for every $n \in \mathcal{I}$ and $\mathcal{S} \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}] \geq \alpha(n)/2$.

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for every $n \in \mathcal{I}$ and $\mathcal{S} \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}] \geq \alpha(n)/2$.

Namely, f does not have a $(\alpha/2, 1/q)$-failing set.

Algorithm B

Algorithm 23 (No failing-set algorithm B)

Input: $y \in \{0,1\}^n$.

1. Choose $w \leftarrow \{0,1\}^{t(n) \cdot n}$, $z = (z_1, \ldots, z_t) = g(w)$ and $i \leftarrow [t]$
2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$
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2. Set $z' = (z_1, \ldots, z_{i-1}, y, z_{i+1}, \ldots, z_t)$
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Fix $n \in \mathcal{I}$ and a set $\mathcal{S} \subseteq \{0,1\}^n$ with $\Pr_{x \leftarrow \{0,1\}^n}[f(x) \in \mathcal{S}] \geq \alpha(n)/2$.
We analyze B's success probability with respect to $\mathcal{S}$, using the following (inefficient) algorithm B^*:

Algorithm B^*

Definition 24 (Bad)

For $z = (z_1, \ldots, z_t) \in Im(g)$ (the image of g), we set $Bad(z) = 1$ iff $\nexists i \in [t]$ with $z_i \in \mathcal{S}$.

Algorithm B^*

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For $z = (z_1, \ldots, z_t) \in Im(g)$ (the image of g), we set $Bad(z) = 1$ iff $\nexists i \in [t]$ with $z_i \in \mathcal{S}$.

B^* differ from B in the way it chooses z': in case $\mathsf{Bad}(z) = 1$, it sets $z' = z$. Otherwise, it sets i to the first index $j \in [t]$ with $z_j \in \mathcal{S}$, and sets z' as B does with respect to this i.

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Claim 25

$\Pr_{y \leftarrow \mathcal{S}}[B^*(y) \in f^{-1}(y)] \geq \frac{1}{p(n)} - \text{neg}(n)$,

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Claim 25

$\Pr_{y \leftarrow \mathcal{S}}[\mathsf{B}^*(y) \in f^{-1}(y)] \geq \frac{1}{p(n)} - \mathrm{neg}(n)$,

Therefore, $\Pr_{y \leftarrow \mathcal{S}}[\mathsf{B}(y) \in f^{-1}(y)] \geq \frac{1}{t(n)p(n)} - \mathrm{neg}(n)$.$\square$

Claim 25 follows from the following two claims,

Claim 26

$\Pr_{w \leftarrow \{0,1\}^{t(n) \cdot n}}[\mathrm{Bad}(g(w))] = \mathrm{neg}(n)$

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Claim 27

- Let $Z = g(W)$ for $W \leftarrow \{0,1\}^{t(n)\cdot n}$
- Let Z' be the value of z' induced by a random execution of $\mathrm{B}^*(f(X)$, for $X \leftarrow \{0,1\}^n \mid f(X) \in \mathcal{S}$.

Then Z and Z' are identically distributed.

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$\Longrightarrow$

$$\Pr_{w \leftarrow \{0,1\}^{t(n)\cdot n}; z=g(w)}\left[\mathsf{A}(z) \in g^{-1}(z)\right] \tag{5}$$

$$\leq \Pr[\mathsf{A}(z) \in g^{-1}(z) \wedge \neg\,\mathsf{Bad}(z)] + \Pr[\mathsf{Bad}(z)]$$

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It follows that

$$\begin{aligned} \Pr_{y \leftarrow \mathcal{S}}[\mathsf{B}^*(y) \in f^{-1}(y)] &\geq \Pr_{w \leftarrow \{0,1\}^{t(n)\cdot n}; z=g(w)}[\mathsf{A}(z) \in g^{-1}(z)] - \mathsf{neg}(n) \\ &\geq \frac{1}{p(n)} - \mathsf{neg}(n). \square \end{aligned}$$

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Algorithm 28 (P)

1. Sample $\ell_1, \ldots, \ell_{t(n)}$, each taking the value 1 with β.
2. Output $z_1, \ldots, z_{t(n)}$, where z_i is sampled according to

$$\begin{cases} f(x) \mid x \leftarrow \{0,1\}^n, f(x) \in \mathcal{S}, & \ell_i = 1; \\ f(x) \mid x \leftarrow \{0,1\}^n, f(x) \notin \mathcal{S}, & \text{otherwise.} \end{cases}$$

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The process for sampling Z' can be described as follows:

1. Choose $\ell_1, \ldots, \ell_{t(n)}$ and $z_1, \ldots, z_{t(n)}$ according to P
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Hence, Z' has the same distribution as of P, and hence as of Z. $\square$

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Remark 29 (hardness amplification via parallel repetition)

- Can we give a more efficient (secure) reduction?

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  What properties of the weak OWF have we used in the proof?