Application of Information Theory, Lecture 6
Counting

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December 2, 2014

Section 1

Graph Homomorphisms

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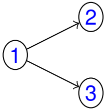

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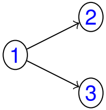

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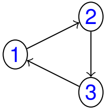

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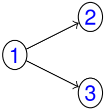

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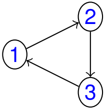

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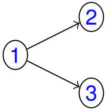

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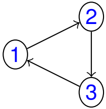

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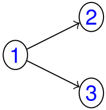

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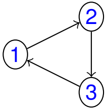

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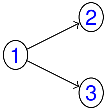

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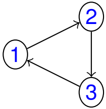

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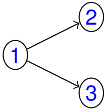

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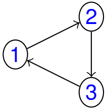

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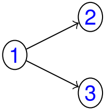

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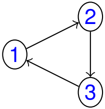

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- Special case of a more general theorem

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Section 2

Perfect Matchings

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Let $G = (A, B, E)$ be bi-partite graph with $|A| = |B|$. Then $P(G)$ — the number of perfect matching in G — is at most $\prod_{v \in A} (d(v)!)^{1/d(v)}$.

- Let $A = B = [n] = \{1, \ldots, n\}$
- It is clear that $P(G) \leq \prod_{i \in [n]} d(i)$:
- Let $\mathcal{M}$ be the perfect matchings in G.
- For $m \in \mathcal{M}$ let $m(i)$ be the node in B matched with i by m.
- Let $M \leftarrow \mathcal{M}$. Hence,

$$\begin{aligned}
\log|\mathcal{M}| = H(M) &= H(M(1)) + H(M(2)|M(1)) + \ldots + H(M(n)|M(1), \ldots, M(n-1)) \\
&\leq H(M(1)) + H(M(2)) + \ldots + H(M(n)) \\
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Bregman's theorem

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Proving Bregman's theorem

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&\leq \prod_{i\in[n]} \log\left((d(i)!)^{1/d(i)}\right). \qquad \square
\end{aligned}$$

Section 3

Shearer's Lemma

$H(X_1, X_2, X_3)$ **Vs.** $H(X_1, X_2) + H(X_2, X_3) + H(X_3, X_1)$

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- If $dk << n$, then a typical X_F is close to the uniform distribution

Section 4

Statistical Distance

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Theorem 4 (Next lecture)

Let X rv over $[m]$. Assume $H(X) \geq \log m - \varepsilon$, then
$\mathrm{SD}(X, \sim [m]) \leq \sqrt{\varepsilon \cdot 2 \cdot \ln 2} = O(\varepsilon)$

Section 5

Gold Coins

of gold coins in a cube

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of gold coins in a cube

- Q — (finite) set of points in $\mathbb{R}^3$
  - Projection of Q on xy — 6
  - Projection of Q on xz — 8
  - Projection of Q on yz — 12
- Can we bound $|Q|$?
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of gold coins, the hyperspace case

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Section 6

Intersecting Graphs

Another corollary of Shearer's lemma

Corollary 5

Let $\mathcal{A}$ and $\mathcal{F}$ be collections of subsets of $[n]$, and for $F \in \mathcal{F}$ let $\mathcal{A}_F$ be the collection $\{A \cap F : A \in \mathcal{A}\}$. Assume that each element of $[n]$ appears in at least m subsets of $\mathcal{F}$, then $|\mathcal{A}|^m \leq \prod_{F \in \mathcal{F}} |\mathcal{A}_F|$.

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- By Shearer's lemma, $H(X) \leq \frac{1}{m} \sum_{F \in \mathcal{F}} H(X_F)$. $\square$

of intersecting graphs

of intersecting graphs

Theorem 6

Let $\mathcal{G}$ be a family of graphs over $[n]$, s.t. $G \cap G'$ contains a triangle for each $G, G' \in \mathcal{G}$. Then $|\mathcal{G}| \leq 2^{\binom{n}{2}-2}$.

of intersecting graphs

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- Hence, $|\mathcal{G}| \leq 2^{m - \frac{m}{m'}} \leq 2^{\binom{n}{2}-2}$

Section 7

Independent Sets

of independent sets in bi-partite graphs

of independent sets in bi-partite graphs

Theorem 7

Let $G = (A, B, E)$ be an n-regular graph with $|A| = |B| = m$. Then the number of independent sets in G is at most $(2^{n+1} - 1)m$.

of independent sets in bi-partite graphs

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Proof: $\mathcal{I}$ — set of independent sets in G.

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- Let $I \leftarrow \mathcal{I}$, let $X_v = 1$ iff $v \in I$, and $X_{\mathcal{S}} = \{X_v \colon v \in \mathcal{S}\}$.

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- Let $I \leftarrow \mathcal{I}$, let $X_v = 1$ iff $v \in I$, and $X_{\mathcal{S}} = \{X_v : v \in \mathcal{S}\}$.
- $$H(I) = H(X_A|X_B) + H(X_B)$$

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&\leq \sum_{v \in A} \left(H(X_v|N(v)) + \frac{1}{n} H(X_{N(v)}) \right)
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of independent sets in bi-partite graphs

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Let $G = (A, B, E)$ be an n-regular graph with $|A| = |B| = m$. Then the number of independent sets in G is at most $(2^{n+1} - 1)m$.

Proof: $\mathcal{I}$ — set of independent sets in G.

- Let $I \leftarrow \mathcal{I}$, let $X_v = 1$ iff $v \in I$, and $X_{\mathcal{S}} = \{X_v \colon v \in \mathcal{S}\}$.
- $$\begin{aligned} H(I) &= H(X_A|X_B) + H(X_B) \\ &\leq \sum_{v \in A} H(X_v|X_B) + \frac{1}{n} \sum_{v \in A} H(X_{N(v)}) \qquad \text{(rhs by Sherer's Lemma)} \\ &\leq \sum_{v \in A} \left(H(X_v|N(v)) + \frac{1}{n} H(X_{N(v)}) \right) \end{aligned}$$
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- Hence $H(I) \leq \sum_{v\in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1 - p(v))\log(2^n - 1)\right)$

of independent sets in bi-partite graphs, cont.

- $\log|\mathcal{I}| = H(I) \leq \sum_{v\in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1-p(v))\log(2^n-1)\right)$

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- $\log |\mathcal{I}| = H(I) \leq \sum_{v \in A} p(v) + \frac{1}{n}\left(h(p(v)) + (1 - p(v)) \log(2^n - 1)\right)$
- Let $f(t) := t + \frac{1}{n}\left(h(t) + (1 - p(t)) \log(2^n - 1)\right)$

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- Hence, $\log|\mathcal{I}| \leq \frac{m}{n}\log(2^{n+1}-1)$. $\square$