

Application of Information Theory, Lecture 9

Parallel Repetition of Interactive Arguments

Handout Mode

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Part I

Interactive Proofs and Arguments

\mathcal{NP} as a Non-interactive Proofs

Definition 1 (\mathcal{NP})

$\mathcal{L} \in \mathcal{NP}$ iff \exists and poly-time algorithm V such that:

- ▶ $\forall x \in \mathcal{L}$ there exists $w \in \{0, 1\}^*$ s.t. $V(x, w) = 1$
- ▶ $V(x, w) = 0$ for every $x \notin \mathcal{L}$ and $w \in \{0, 1\}^*$

Only $|x|$ counts for the running time of V .

This proof system has

- ▶ Efficient verifier, efficient prover (given the witness)
- ▶ Soundness holds unconditionally

Interactive proofs/arguments

Protocols between **efficient** verifier and **unbounded/efficient** prover.

Definition 2 (Interactive proof)

A protocol (P, V) is an **interactive proof** for \mathcal{L} , if V is a **PPT** and:

Completeness $\forall x \in \mathcal{L}$: $\Pr[(P, V)(x) = 1] \geq 2/3$.

Soundness $\forall x \notin \mathcal{L}$, and **any** algorithm P^* : $\Pr[(P^*, V)(x) = 1] \leq 1/3$.

IP is the class of languages that have interactive proofs.

- ▶ $IP = PSPACE!$
- ▶ The above protocol has **completeness error** $\frac{1}{3}$, and **soundness error** $\frac{1}{3}$
- ▶ We typically consider achieve (directly) perfect completeness.
- ▶ Smaller “soundness error” achieved via repetition.
- ▶ Relaxation: **interactive arguments** [also known as, **Computationally sound proofs**]: soundness only guaranteed against **efficient** (PPT) provers.
- ▶ Games — no-input protocols.

Section 1

Interactive Proof for Graph Non-Isomorphism

Graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $G_0 = ([m], E_0)$ and $G_1 = ([m], E_1)$ are **isomorphic**, denoted $G_0 \equiv G_1$, if $\exists \pi \in \Pi_m$ such that $(u, v) \in E_0$ iff $(\pi(u), \pi(v)) \in E_1$.

- ▶ $\mathcal{GI} = \{(G_0, G_1) : G_0 \equiv G_1\} \in \mathcal{NP}$
- ▶ Does $\mathcal{GNI} = \{(G_0, G_1) : G_0 \not\equiv G_1\} \in \mathcal{NP}$?
- ▶ We will show a simple interactive proof for \mathcal{GNI}
Idea: Beer tasting...

Interactive proof for \mathcal{GNI}

Protocol 4 $((P, V)(G_0 = ([m], E_0), G_1 = ([m], E_1)))$

1. V chooses $b \leftarrow \{0, 1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b)$ to P .^a
2. P send b' to V (tries to set $b' = b$).
3. V accepts iff $b' = b$.

$$^a \pi(E) = \{(\pi(u), \pi(v)) : (u, v) \in E\}.$$

Claim 5

The above protocol is IP for \mathcal{GNI} , with perfect completeness and soundness error $\frac{1}{2}$.

Proving Claim 5

- ▶ Graph isomorphism is an equivalence relation (separates all graph pairs into separate subsets)
- ▶ $([m], \pi(E_i))$ is a random element in $[G_i]$ — the equivalence class of G_i

Hence,

$$G_0 \equiv G_1: \Pr[b' = b] \leq \frac{1}{2}.$$

$$G_0 \not\equiv G_1: \Pr[b' = b] = 1 \text{ (i.e., } P \text{ can, possibly inefficiently, extracted from } \pi(E_i))$$

□

Part II

Hardness Amplification

Hardness amplification

- ▶ In most settings we need **very small** soundness error (i.e., close to 0)
- ▶ Typically done by “amplifying the security” of an interactive proof/argument of **large** soundness error.
- ▶ Two main approaches:
 - ▶ **Sequential** repetition: achieves optimal amplification rate in almost any computation model, but increases the round complexity
 - ▶ **Parallel** repetition: sometimes does not achieve optimal amplification rate and sometimes achieves **nothing**
- ▶ How come parallel repetition might not work? **Example**
- ▶ Parallel repetition **does** achieve optimal amplification rate for interactive proofs and public-coin interactive arguments
- ▶ Public-coin interactive proof/argument — in each round the verifier flips coins and sends them to the prover. To compute its output, the verifier applies some (fixed) function to the protocol’s transcript.

Hardness amplification, cont

- ▶ Give a protocol $\pi = (P, V)$ and $k \in \mathbb{N}$, let $\pi^{(k)} = (P^{(k)}, V^{(k)})$ be the k -fold parallel repetition of π : i.e., k parallel independent copies of π
- ▶ Assume $\Pr[(\tilde{P}, V) = 1] \leq \varepsilon$ for any t -time algorithm \tilde{P} , we would like to prove that $\Pr[(\widetilde{P^{(k)}}^{(k)}, V^{(k)}) = 1^k] \leq f(\varepsilon)$ for any $t^{(k)}$ -time algorithm $\widetilde{P^{(k)}}^{(k)}$.
- ▶ Typically, $t^{(k)} = t \cdot \text{poly}(f(\varepsilon)/k)$
- ▶ If $f(\varepsilon) = \varepsilon^{\Omega(k)}$, the above is an exponential-rate amplification (and hence optimal)
- ▶ If $f(\varepsilon) = \varepsilon^{\delta_1 \cdot k^{\delta_2}}$, the above is a weakly-exponential-rate amplification
- ▶ Why time?
- ▶ Concrete security
- ▶ In the following we focus on games (no input protocols)

Section 2

Parallel repetition of public-coin interactive argument

Parallel repetition of public-coin interactive argument

Theorem 6

Let $\pi = (P, V)$ be m -round, public-coin protocol with $\Pr[(\tilde{P}, V) = 1] \leq \varepsilon$ for any t -time \tilde{P} , then $\Pr[(\widetilde{P^{(k)}}), V^{(k)}] = 1^k] \leq \varepsilon^{k/4}$ for any $t \cdot \frac{\varepsilon^{k/4}}{mk^3 t_V}$ -time $\widetilde{P^{(k)}}$, where t_V is V 's running time.

Proof plan: Let $\widetilde{P^{(k)}}$ be $t^{(k)}$ -time algorithm with $\Pr[(\widetilde{P^{(k)}}), V^{(k)}] = 1^k] = \varepsilon^{(k)}$, we construct $t^{(k)} \cdot \frac{mk^3 t_V}{\varepsilon^{(k)}}$ -time \tilde{P} with $\Pr[(\tilde{P}, V) = 1] \geq (\varepsilon^{(k)})^{4/k}$.

- ▶ The $k/4$ in the exponent can be pushed to be almost k .
- ▶ Assume for simplicity that $\widetilde{P^{(k)}}$ is deterministic
- ▶ Assume wlg. that V sends the first message in π and that in each round it samples and sends ℓ coins.
- ▶ We view the coins of $V^{(k)}$ as a matrix $R \in \{0, 1\}^{m \times (k\ell)}$, letting R_j denote the coins of the j 'th round, and $R_{1, \dots, j}$ the coins of the first j rounds.
- ▶ Let $R \sim \{0, 1\}^{m \times (k\ell)}$

Algorithm \tilde{P}

Let $q = k^2$.

Algorithm 7 (\tilde{P})

1. Let $i^* \leftarrow [k]$.
 2. Upon getting the j 'th message r from V , do:
 - 2.1 Let $R \leftarrow \{0, 1\}^{m \times (k\ell)}$, conditioned that $R_{1,\dots,j-1} = \tilde{R}_{1,\dots,j-1}$ and $R_{j,i^*} = r$.
 - 2.2 If $(\tilde{P}^{(k)}, V^{(k)}(R)) = 1^k$:
 - 2.2.1 Set $\tilde{R}_j = R_j$
 - 2.2.2 Send a_{j,i^*} back to V , for a_j being the j 'th message $\tilde{P}^{(k)}$ send to $V^{(k)}$ in $(\tilde{P}^{(k)}, V^{(k)}(R))$.
 - Else, GOTO Line 2.1
 - 2.3 Abort if the overall number of sampling exceeds $\lceil qm/\varepsilon^{(k)} \rceil$.
- Let \tilde{P}' be the non aborting variant of \tilde{P}' , let \tilde{R} and \tilde{N} be the value of \tilde{R} and $\#$ of samples done in a random execution of $(\tilde{P}', V^{(k)})$.
- $\Pr[(\tilde{P}, V) = 1] \geq \Pr[\text{win}(\tilde{R}, \tilde{N}) := (\tilde{P}^{(k)}, V^{(k)}(\tilde{R})) = 1^k \wedge \tilde{N} \leq qm/\varepsilon^{(k)}]$.

Ideal “attacker”

Experiment 8 (\hat{P})

For $j = 1$ to m :

1. Let $R \leftarrow \{0, 1\}^{m \times \ell}$, conditioned that $R_{1,\dots,j-1} = \hat{R}_{1,\dots,j-1}$.
2. If $(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- ▶ Let \hat{R} be the value of \hat{R} in the end of a random execution of \hat{P} .
- ▶ $\hat{R} \sim R|_{(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k}$
- ▶ In particular, $\Pr \left[(\widetilde{P^{(k)}}, V^{(k)}(\hat{R})) = 1^k \right] = 1$
- ▶ Let \hat{N} be # of samples done in \hat{R} .

Lemma 9

$$\Pr \left[\hat{N} \leq qm/\varepsilon^{(k)} \right] \geq 1 - \frac{1}{q}$$

Proving Lemma 9

- ▶ Let $(X_1, \dots, X_m) = \mathbf{R}$ and $(Y_1, \dots, Y_m) = \widehat{\mathbf{R}}$
- ▶ $v(\mathbf{y} = (y_1, \dots, y_j)) := \Pr \left[(\widetilde{\mathbf{P}}^{(k)}, V^{(k)}(X^m)) = 1^k \mid X^j = \mathbf{y} \right]$
(letting $X^j = (X_1, \dots, X_j)$)
- ▶ Conditioned on $Y^j = \mathbf{y} = (y_1, \dots, y_j)$, the expected # of samples done in $(j+1)$ 'th round of $\widehat{\mathbf{P}}$ is $\frac{1}{v(\mathbf{y})}$.
- ▶ We prove Lemma 9 showing that $\mathbb{E} \left[\frac{1}{v(Y^j)} \right] \leq \frac{1}{\varepsilon^{(k)}}$ for every $j \in \{0, \dots, m-1\}$

Claim 10

For $j \in \{0, \dots, m-1\}$ and $\mathbf{y} \in \text{Supp}(Y^j)$ it holds that $\Pr_{Y^j}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

$$\begin{aligned} \text{Hence, } \mathbb{E}_{Y^j} \left[\frac{1}{v(Y^j)} \right] &= \sum_{\mathbf{y} \in \text{Supp}(Y^j)} \Pr[Y^j = \mathbf{y}] \cdot \frac{1}{v(\mathbf{y})} \\ &= \sum_{\mathbf{y}} \Pr[X^j = \mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}} \cdot \frac{1}{v(\mathbf{y})} = \frac{1}{\varepsilon^{(k)}} \cdot \sum_{\mathbf{y} \in \text{Supp}(Y^j)} \Pr[X^j = \mathbf{y}] \leq \frac{1}{\varepsilon^{(k)}}. \quad \square \end{aligned}$$

Proving Claim 10

Note that

$$\begin{aligned}\Pr_{Y_j | Y^{j-1} = \mathbf{y}_{1, \dots, j-1}}[y_j] &= \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{1, \dots, j-1}))^{\ell-1} \cdot \Pr_{X_j | X^{j-1} = \mathbf{y}_{1, \dots, j-1}}[y_j] \cdot v(\mathbf{y}) \quad (1) \\ &= \frac{1}{v(\mathbf{y}_{1, \dots, j-1})} \cdot \Pr_{X_j | X^{j-1} = \mathbf{y}_{1, \dots, j-1}}[y_j] \cdot v(\mathbf{y})\end{aligned}$$

The proof proceeds by induction on j .

$$\begin{aligned}\Pr_{Y^j}[\mathbf{y}] &= \Pr_{Y^{j-1}}[\mathbf{y}_{1, \dots, j-1}] \cdot \Pr_{Y_j | Y^{j-1} = \mathbf{y}_{1, \dots, j-1}}[y_j] \\ &= \Pr_{X^{j-1}}[\mathbf{y}_{1, \dots, j-1}] \cdot \frac{v(\mathbf{y}_{1, \dots, j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j | Y^{j-1} = \mathbf{y}_{1, \dots, j-1}}[y_j] \quad (\text{i.h.}) \\ &= \Pr_{X^{j-1}}[\mathbf{y}_{1, \dots, j-1}] \cdot \frac{v(\mathbf{y}_{1, \dots, j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1, \dots, j-1})} \cdot \Pr_{X_j | X^{j-1} = \mathbf{y}_{1, \dots, j-1}}[y_j] \quad (\text{Eq. (1)}) \\ &= \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}.\end{aligned}$$

Ideal “attacker”, variant

Experiment 11 (\hat{P})

1. Let $i^* \leftarrow [k]$.
 2. For $j = 1$ to m :
 - 2.1 Let $R \leftarrow \{0, 1\}^{m \times \ell}$, conditioned on $R_{1,\dots,j-1} = \hat{R}_{1,\dots,j-1}$.
 - 2.2 If $(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k$, set $\hat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
 - 2.3 Let $R \leftarrow \{0, 1\}^{m \times \ell}$, conditioned on $R_{1,\dots,j-1} = \hat{R}_{1,\dots,j-1}$ and $R_{j,i^*} = \hat{R}_{j,i^*}$.
 - 2.4 If $(\widetilde{P^{(k)}}, V^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 2.3.
- ▶ Let \hat{R} be the final value of \hat{R} in \hat{P} .
 - ▶ $\hat{R} \sim R|_{(\widetilde{P^{(k)}}, V^{(k)}(R))=1^k}$
 - ▶ Let \hat{N} be the # of Step-2.3-samples done in \hat{P} .

Lemma 12

$$\Pr \left[\text{win}(\hat{R}, \hat{N}) \right] \geq 1 - \frac{1}{q}$$

From ideal to real

Let $\tilde{\mathbf{R}}_i = \tilde{\mathbf{R}}|_{i^*=i}$ and $\hat{\mathbf{R}}_i := \hat{\mathbf{R}}|_{i^*=i}$ ($= \hat{\mathbf{R}}$).

Claim 13

$$D(\hat{\mathbf{R}}, \hat{\mathbf{N}} \| \tilde{\mathbf{R}}, \tilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\hat{\mathbf{R}}_i \| \tilde{\mathbf{R}}_i).$$

Claim 14

$$\sum_{i \in [k]} D(\hat{\mathbf{R}}_i \| \tilde{\mathbf{R}}_i) \leq D(\hat{\mathbf{R}} \| \mathbf{R}).$$

- ▶ Thm. 7 in Lecture 7 $\implies D(\hat{\mathbf{R}} \| \mathbf{R}) \leq \log \frac{1}{\Pr[(\tilde{\mathbf{P}}^{(k)}, \mathbf{V}^{(k)}(\mathbf{R}))=1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
- ▶ Hence, $D(\text{win}(\hat{\mathbf{R}}, \hat{\mathbf{N}}) \| \text{win}(\tilde{\mathbf{R}}, \tilde{\mathbf{N}})) \leq D(\hat{\mathbf{R}}, \hat{\mathbf{N}} \| \tilde{\mathbf{R}}, \tilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
- ▶ Claim 12 $\implies \alpha := \Pr[\text{win}(\hat{\mathbf{R}}, \hat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\text{win}(\tilde{\mathbf{R}}, \tilde{\mathbf{N}})]$.
- ▶ Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log(1-\alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
 $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$
- ▶ Recalling $q = k^2$, $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1-\alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$
- ▶ We conclude that $\beta \geq 2^{\frac{4}{k} \log \varepsilon^{(k)}} = \sqrt[k]{\varepsilon^{(k)}}. \square$

Proving Claim 13

HW...

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j) \in [k] \times [m]}$ be iids, let W be an event, and let

$$D_i(z) := \prod_{j=1}^m \Pr[Z_{j,i} = z_{j,i}] \cdot \Pr[Z_{j,-i} = z_{j,-i} | Z_{1,\dots,j-1} = z_{1,\dots,j-1} \wedge Z_{j,i} = z_{j,i} \wedge W].$$

Then $\sum_{i=1}^k D(Z_W || D_i) \leq D(Z_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathbf{P}}^{(k)}, \mathbf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i \in [k]} D(\mathbf{R} | W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R} | W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. \square

Proof: (of Lemma 15) We prove for $m = k = 2$.

- ▶ Let $X = Z_1$ and $Y = Z_2$
- ▶ $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)} [(x_1, x_2, y_1, y_2)]$
- ▶ $C(x_1, x_2, y_1, y_1) := (X|W)(x_1, x_2, y_1, y_1)$
- ▶ $Q(x_1, x_2, y_1, y_1) := \Pr[X_1 = x_1 | W] \cdot \Pr[X_2 = x_2 | W] \cdot \Pr[Y_1 = y_1 | W, X = (x_1, x_2)] \cdot \Pr[Y_2 = y_2 | W, X = (x_1, x_2)]$
- ▶ We write $\frac{C(x_1, x_2, y_1, y_1)}{U(x_1, x_2, y_1, y_1)} = \frac{\Pr[X_1 = x_1 | W] \cdot \Pr[Y_1 = y_1 | W, X = (x_1, x_2)]}{\Pr[X_1 = x_1] \cdot \Pr[Y_1 = y_1]} \cdot \frac{\Pr[X_2 = x_2 | W] \cdot \Pr[Y_2 = y_2 | W, X = (x_1, x_2)]}{\Pr[X_2 = x_2] \cdot \Pr[Y_2 = y_2]} \cdot \frac{C(x_1, x_2, y_1, y_1)}{Q(x_1, x_2, y_1, y_1)}$

Proving Lemma 15, cont.

$$\begin{aligned} D(C||U) = & \mathbb{E}_{(x_1, x_2, y_1, y_2) \leftarrow C} \left[\log \frac{\Pr[X_1 = x_1|W] \cdot \Pr[Y_1 = y_1|W, X = (x_1, x_2)]}{\Pr[X_1 = x_1] \cdot \Pr[Y_1 = y_1]} \right] \\ & + \mathbb{E}_{(x_1, x_2, y_1, y_2) \leftarrow C} \left[\log \frac{\Pr[X_2 = x_2|W] \cdot \Pr[Y_2 = y_2|W, X = (x_1, x_2)]}{\Pr[X_2 = x_2] \cdot \Pr[Y_2 = y_2]} \right] \\ & + \mathbb{E}_{(x_1, x_2, y_1, y_2) \leftarrow C} \left[\log \frac{C(x_1, x_2, y_1, y_2)}{Q(x_1, x_2, y_1, y_2)} \right]. \end{aligned}$$

It follows that

$$\begin{aligned} D(C||U) = & D(X_1|W, X_2|W, X_1, Y_1|W, X, Y_2|W, X, Y_1||X_1, X_2|W, X_1, Y_1, Y_2|W, X, Y_1) \\ & + D(X_2|W, X_1|W, X_2, Y_2|W, X, Y_1|W, X, Y_2||X_2, X_1|W, X_2, Y_2, Y_1|W, X, Y_2) \\ & + D(C||Q), \end{aligned}$$

and the proof follows since $D(\cdot||\cdot) \geq 0$. \square

Parallel repetition of interactive proofs

- ▶ Similar proof to the public-coin proof we gave above.
- ▶ In each round, the attacker \tilde{P} samples **random continuations** of $(\tilde{P}^{(k)}, V^{(k)})$, till he gets an accepting execution.
- ▶ Why fails us to extend this approach for non-public-coin interactive arguments?

Section 3

Parallel amplification for any interactive argument

Parallel amplification theorem for any protocol

- ▶ Can we amplify the security of any interactive argument “in parallel”?
- ▶ Yes we **can**!