Application of Information Theory, Lecture 1
Basic Definitions and Facts

Handout Mode

Iftach Haitner

Tel Aviv University.

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The entropy function

X — Discrete random variable (finite number of values) over $\mathcal{X}$ with probability mass $p = p_X$. The entropy of X is defined by:

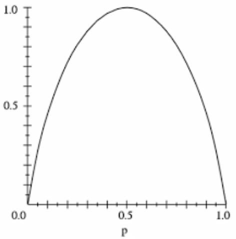
$$H(X) := -\sum_{x\in\mathcal{X}} \Pr[X = x] \cdot \log_2 \Pr[X = x]$$

taking $0 \log 0 = 0$.

- $H(X) = -\sum_x p(x) \log p(x) = \mathrm{E}_X \log \frac{1}{p(X)} = \mathrm{E}_{Y=p(X)} \log \frac{1}{Y}$
- $H(X)$ was introduced by Shannon as a measure for the uncertainty in X — number of bits requited to describe X, information we don't have about X.
- When using the natural logarithm, the quantity is called nats (“natural")
- Entropy is a function of p (sometimes refers to as $H(p)$).

Examples

1. $X \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$:
   (i.e., for some $x_1 \neq x_2 \neq x_3$, $\mathrm{P}_X(x_1) = \frac{1}{2}, \mathrm{P}_X(x_2) = \frac{1}{4}, \mathrm{P}_X(x_3) = \frac{1}{4}$)
   $H(X) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{4}\log\frac{1}{4} = \frac{1}{2} + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 = 1\frac{1}{2}$.
2. $H(X) = H(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.
3. X is uniformly distributed over $\{0,1\}^n$:
   $H(X) = -\sum_{i=1}^{2^n} \frac{1}{2^n}\log\frac{1}{2^n} = -\log\frac{1}{2^n} = n$.
   - n bits are needed to describe X
   - n bits are needed to create X
4. $X = X_1, \ldots, X_n$ where X_i's iid over $\{0,1\}$, with $\mathrm{P}_{X_i}(1) = \frac{1}{3}$. $H(X) = ?$
5. $X \sim (p, q)$, $p + q = 1$
   - $H(X) = H(p,q) = -p\log p - q\log q$
   - $H(1,0) = (0,1) = 0$
   - $H(\frac{1}{2}, \frac{1}{2}) = 1$
   - $h(p) := H(p, 1-p)$ is continuous



Axiomatic derivation of the entropy function

Any other choices for defining entropy?
Shannon function is the only *symmetric* function (over probability distributions) satisfying the following three axioms:

- **A1** Continuity: $H(p, 1-p)$ is continuous function of p.
- **A2** Normalization: $H(\frac{1}{2}, \frac{1}{2}) = 1$
- **A3** Grouping axiom:
  $H(p_1, p_2, \ldots, p_m) = H(p_1 + p_2, p_3, \ldots, p_m) + (p_1 + p_2)H(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2})$

Why $A3$?

Not hard to prove that Shannon's entropy function satisfies above axioms, proving this is the only such function is more challenging.

Let H be a symmetric function that satisfying the above axioms.

We prove (assuming additional axiom) that H is the Shannon function.

Generalization of the grouping axiom

Fix $p = (p_1, \ldots, p_m)$ and let $S_k = \sum_{i=1}^k p_i$.
Grouping axiom: $H(p_1, p_2, \ldots, p_m) = H(S_2, p_3, \ldots, p_m) + S_2 H(\frac{p_1}{S_2}, \frac{p_2}{S_2})$.

Claim 1 (Generalized grouping axiom)

$H(p_1, p_2, \ldots, p_m) = H(S_k, p_{k+1}, \ldots, p_m) + S_k \cdot H(\frac{p_1}{S_k}, \ldots, \frac{p_k}{S_k})$

Proof: Let $h(q) = H(q, 1 - q)$.

$$\begin{aligned} H(p_1, p_2, \ldots, p_m) &= H(S_2, p_3, \ldots, p_m) + S_2 h(\frac{p_2}{S_2}) && (1) \\ &= H(S_3, p_4, \ldots, p_m) + S_3 h(\frac{p_3}{S_3}) + S_2 h(\frac{p_2}{S_2}) \\ &\vdots \\ &= H(S_k, p_{k+1}, \ldots, p_m) + \sum_{i=2}^{k} S_i h(\frac{p_i}{S_i}) \end{aligned}$$

Hence,

$$H(\frac{p_1}{S_k}, \ldots, \frac{p_k}{S_k}) = H(\frac{S_{k-1}}{S_k}, \frac{p_k}{S_k}) + \sum_{i=2}^{k-1} \frac{S_i}{S_k} h(\frac{p_i/S_k}{S_i/S_k}) = \frac{1}{S_k} \sum_{i=2}^{k} S_i h(\frac{p_i}{S_i}) \qquad (2)$$

Claim follows by combining the above equations. □

Further generalization of the grouping axiom

Let $1 = k_1 < k_2 < \ldots < k_q < m$ and let $C_t = \sum_{i=k_t}^{k_{t+1}-1} p_i$ (letting $k_{q+1} = m+1$).

Claim 2 (Generalized^{++} grouping axiom)

$H(p_1, p_2, \ldots, p_m) =$
$H(C_1, \ldots, C_q) + C_1 \cdot H(\frac{p_1}{C_1}, \ldots, \frac{p_{k_2-1}}{C_1}) + \ldots + C_q \cdot H(\frac{p_{k_q+1}}{C_q}, \ldots, \frac{p_m}{C_q})$

Proof: Follow by the extended group axiom and the symmetry of H $\square$

Implication: Let $f(m) = H(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m})$

- $f(3^2) = 2f(3) = 2H(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
  $\implies f(3^n) = nf(3)$.
- $f(mn) = f(m) + f(n)$
  $\implies f(m^k) = kf(m)$

$f(m) = \log m$

We give a proof under the additional axiom

A4 $f(m) < f(m+1)$

(you can Google for a proof using only $A1$–$A3$)

- For $n \in \mathbb{N}$ let $k = \lfloor n \log 3 \rfloor$.
- By $A4$, $f(2^k) < f(3^n) < f(2^{k+1})$.
- By grouping axiom, $k < nf(3) < k+1$.

$\implies \quad \frac{\lfloor n \log 3 \rfloor}{n} < f(3) < \frac{\lfloor n \log 3 \rfloor + 1}{n}$ for any $n \in \mathbb{N}$

$\implies \quad f(3) = \log 3$.

- Proof extends to any integer (not only 3)

$H(p,q) = -p\log p - q\log q$

- For rational p,q, let $p = \frac{k}{m}$ and $q = \frac{m-k}{m}$, where m is the smallest common multiplier.
- By grouping axiom, $f(m) = H(p,q) + p \cdot f(k) + q \cdot f(m-k)$.
- Hence,

$$
\begin{aligned}
H(p,q) &= \log m - p\log k - q\log(m-k) \\
&= p(logm - logk) + q(\log m - \log(m-k)) \\
&= -p\log\frac{m}{k} - q\log\frac{m-k}{m} = -p\log p - q\log q
\end{aligned}
$$

- By continuity axiom, holds for every p,q.

$H(p_1, p_2, \ldots, p_m) = -\sum_i^m -p_i \log p_i$

We prove for $m = 3$. Proof for arbitrary m follows the same lines.

- For rational p_1, p_2, p_3, let $p_1 = \frac{k_1}{m}$, $q = \frac{k_2}{m}$ and $p_3 = \frac{k_3}{m}$, where $m = k_1 + k_2 + k_3$ is the smallest common multiplier.
- $f(m) = H(p_1, p_2, p_3) + p_1 f(k_1) + p_2 f(k_2) + p_3 f(k_3)$
- Hence,

$$\begin{aligned} H(p_1, p_2, p_3) &= \log m - p_1 \log k_1 - p_2 \log k_2 - p_3 \log k_3 \\ &= -p_1 \log \frac{k_1}{m} - p_2 \log \frac{k_2}{m} - p_3 \frac{k_3}{m} \\ &= -p_1 \log p_1 - p_2 \log p_2 - p_3 \log p_3 \end{aligned}$$

- By continuity axiom, holds for every p_1, p_2, p_3.

$0 \leq H(p_1, \ldots, p_m) \leq \log m$

- Tight bounds
  - $H(p_1, \ldots, p_m) = 0$ for $(p_1, \ldots, p_m) = (1, 0, \ldots, 0)$.
  - $H(p_1, \ldots, p_m) = \log m$ for $(p_1, \ldots, p_m) = (\frac{1}{m}, \ldots, \frac{1}{m})$.
- Non negativity is clear.
- A function f is concave if $\forall\ t_1, t_2, \lambda \in [0,1] \leq 1$
  $\lambda f(t_1) + (1-\lambda) f(t_2) \leq f(\lambda t_1 + (1-\lambda) t_2)$

$\Longrightarrow$ (by induction) $\forall\ t_1, \ldots, t_k, \lambda_1, \ldots, \lambda_k \in [0,1]$ with $\sum_i \lambda_i = 1$
$\sum_i \lambda_i f(\lambda_i t_i) \leq f(\sum_i \lambda_i t_i)$

$\Longrightarrow$ (Jensen inequality): $\mathrm{E}\, f(X) \leq f(\mathrm{E}\, X)$ for any random variable X.

- $\log(x)$ is (strictly) concave for $x > 0$, since its second derivative $(-\frac{1}{x^2})$ is always negative.
- Hence, $H(p_1, \ldots, p_m) = \sum_i p_i \log \frac{1}{p_i} \leq \log \sum_i p_i \frac{1}{p_i} = \log m$
- Alternatively, for X over $\{1, \ldots, m\}$,
  $H(X) = \mathrm{E}_X \log \frac{1}{\mathrm{P}_X(X)} \leq \log \mathrm{E}_X \frac{1}{\mathrm{P}_X(X)} = \log m$

$H(g(X)) \leq H(X)$

Let X be a random variable, and let g be over $\mathrm{Supp}(X) := \{x \colon \mathrm{P}_X(x) > 0\}$.

- $H(Y = g(X)) \leq H(X)$.

  Proof:

$$\begin{aligned} H(X) &= -\sum_x \mathrm{P}_X(x) \log \mathrm{P}_X(x) = -\sum_y \sum_{x \colon g(x)=y} \mathrm{P}_X(x) \log \mathrm{P}_X(x) \\ &\geq -\sum_y \mathrm{P}_Y(y) \max_{x \colon g(x)=y} \log \mathrm{P}_X(x) \\ &\geq -\sum_y \mathrm{P}_Y(y) \log \mathrm{P}_Y(y) = H(Y) \end{aligned}$$

- If g is injective, then $H(Y) = H(X)$.

  Proof: $p_X(X) = P_Y(Y)$.

- If g is non-injective (over $\mathrm{Supp}(X)$), then $H(Y) < H(X)$.

  Proof: ?

- $H(X) = H(2^X)$.
- $H(X) < H(\cos(X))$, if $0, 2\pi \in \mathrm{Supp}(X)$.

Notation

- $[n] = \{1, \ldots, n\}$
- $\mathrm{P}_X(x) = \Pr[X = x]$
- $\mathrm{Supp}(X) := \{x \colon \mathrm{P}_X(x) > 0\}$
- For random variable X over $\mathcal{X}$, let $p(x)$ be its density function: $p(x) = \mathrm{P}_X(x)$.

  In other words, $X \sim p(x)$.
- For random variable Y over $\mathcal{Y}$, let $p(y)$ be its density function: $p(y) = \mathrm{P}_Y(y)$...