Foundation of Cryptography, Lecture 8
Encryption Schemes

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April 29, 2014

Section 1

Definitions

Correctness

Definition 1 (encryption scheme)

A trippet of PPTM's $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ such that

1. $\mathsf{G}(1^n)$ outputs $(e, d) \in \{0,1\}^* \times \{0,1\}^*$
2. $\mathsf{E}(e, m)$ outputs $c \in \{0,1\}^*$
3. $\mathsf{D}(d, c)$ outputs $m \in \{0,1\}^*$

Correctness: $\mathsf{D}(d, \mathsf{E}(e, m)) = m$, for any $(e, d) \in \mathsf{Supp}(\mathsf{G}(1^n))$ and $m \in \{0,1\}^*$

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- m – plaintext, $c = \mathsf{E}(e, m)$ – ciphertext
- $E_e(m) \equiv E(e, m)$ and $D_d(c) \equiv D(d, c)$,

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- public/private key

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  - Other concerns: multiple encryptions, active adversaries, ...

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3. Does not hide the message *length*

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Definition 2 (Semantic Security — private-key model)

An encryption scheme $(\mathsf{G},\mathsf{E},\mathsf{D})$ is semantically secure in the private-key model, if $\forall$ PPTM A, $\exists$ PPTM A' s.t. :
$\forall$ poly-length dist. ensemble $\mathcal{M} = \{\mathcal{M}_n\}_{n\in\mathbb{N}}$ and poly-length functions $h, f\colon \{0,1\}^* \mapsto \{0,1\}^*$

$$\Big|\Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathsf{A}(1^n, 1^{|m|}, h(1^n,m), E_e(m)) = f(1^n,m)]$$
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- Public-key variant — A and A' get e
- Reflection to $\mathcal{ZK}$
- We sometimes omit 1^n and $1^{|m|}$

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Definition 3 (Indistinguishablity of encryptions — private-key model)

An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model, if for any $p, \ell \in \text{poly}$, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$

$$\{(z_n, E_e(x_n))_{e \leftarrow G(1^n)_1}\}_{n\in\mathbb{N}} \approx_c \{(z_n, E_e(y_n))_{e \leftarrow G(1^n)_1}\}_{n\in\mathbb{N}}$$

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Equivalence of Definitions

Theorem 4

An encryption scheme (G, E, D) *is semantically secure iff is has indistinguishable encryptions.*

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We prove the private key case

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Fix $\mathcal{M}$, A, f and h, as in Definition 2.

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Algorithm 5 (A′)

Input: 1^n, $1^{|m|}$ and $h(m)$

1. $e \leftarrow G(1^n)_1$
2. $c = E_e(1^{|m|})$
3. Output $\mathsf{A}(1^n, 1^{|m|}, h(m), c)$

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Proof: Let

$$\delta(n) := \Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1}[\mathsf{A}(h(m), E_e(m)) = f(m)] - \Pr_{m \leftarrow \mathcal{M}_n}[\mathsf{A}'(h(m)) = f(m)]$$

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We define an algorithm that distinguish two between two ensembles $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$, with advantage $\delta(n)$.

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Hence, the indistinguishability of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ yields that $\delta(n) \leq \mathsf{neg}(n)$.

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Claim 7

For every $n \in \mathbb{N}$, exists $x_n \in \operatorname{Supp}(\mathcal{M}_n)$ with
$\Pr_{e \leftarrow G(1^n)_1}[\mathsf{A}(h(x_n), E_e(x_n)) = f(x_n)] - \Pr[\mathsf{A}'(h(x_n)) = f(x_n)] \geq \delta(n)$.

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Semantic Security $\implies$ Indistinguishability

For PPT B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$, let

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We define distribution $\mathcal{M}$, functions f, h and algorithm A that has no $\delta(n)/4$ simulator. The semantic security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ yields that $\delta(n) \leq \operatorname{neg}(n)$.

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For PPT B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$, let

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Semantic Security $\implies$ Indistinguishability, cont.

- Let $\mathcal{M}_n$ be x_n w.p. $\frac{1}{2}$, and y_n otherwise.
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Security Under Multiple Encryptions

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Definition 10 (Indistinguishablity for multiple encryptions – private-key model)

An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions for multiple messages in the private-key model, if for any $p, \ell, t \in \text{poly}$, $\{x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$, PPTM B:

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Extensions:

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Extensions:

- Different length messages
- Semantic security version

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Extensions:

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Multiple Encryption in the Public-Key Model

Theorem 11

A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.

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Proof: Let (G, E, D) be a public-key encryption scheme that has no indistinguishable encryptions for multiple messages, with respect to PPT B, $\{x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$.

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Hence, for some function $i(n) \in [t(n)]$:

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Thus, $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has no indistinguishable encryptions for single message:

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Thus, $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has no indistinguishable encryptions for single message:

Algorithm 12 (B')

Input: 1^n, $z_n = (i(n), x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)})$, e ,c
Return $\mathsf{B}(c, E_e(x_{n,1}), \ldots, E_e(x_{n,i-1}), c, E_e(y_{n,i+1}) \ldots, E_e(y_{n,t(n)}))$

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Assuming (non uniform) OWFs exists, then $\exists$ encryption scheme that has private-key indistinguishable encryptions for a single messages, but not for multiple messages.

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Construction 14

- $\mathsf{G}(1^n)$: outputs $e \leftarrow \{0,1\}^n$
- $\mathsf{E}_e(m)$: outputs $g^{|m|}(e) \oplus m$
- $\mathsf{D}_e(c)$: outputs $g^{|c|}(e) \oplus c$

Multiple Encryption in the Private-Key Model, cont.

Claim 15

(G, E, D) has private-key indistinguishable encryptions for a single message

Multiple Encryption in the Private-Key Model, cont.

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Claim 15

(G, E, D) has private-key indistinguishable encryptions for a single message

Proof: Assume not, and let B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$ be the triplet that realizes it:

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$$\left|\Pr[\mathsf{B}(z_n, g^{\ell(n)}(U_n) \oplus x_n) = 1] - \Pr[\mathsf{B}(z_n, g^{\ell(n)}(U_n) \oplus y_n) = 1]\right| > \operatorname{neg}(n) \quad (1)$$

Multiple Encryption in the Private-Key Model, cont.

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Hence, B yields a (non-uniform) distinguisher for g. (?)

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(G, E, D) does not have a private-key indistinguishable encryptions for multiple messages

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Proof: Take $x_{n,1} = x_{n,2}$ and $y_{n,1} \neq y_{n,2}$, and let B be the algorithm that on input (c_1, c_2), outputs 1 iff $c_1 = c_2$.$\square$

Section 2

Constructions

Private-Key Indistinguishable Encryptions for Multiple Messages

Suffices to encrypt messages of some fixed length (here the length is n).(?)

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- $\mathsf{E}_e(m)$: choose $r \leftarrow \{0,1\}^n$ and output $(r, e(r) \oplus m)$
- $\mathsf{D}_e(r,c)$: output $e(r) \oplus c$

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Construction 17

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- $\mathsf{E}_e(m)$: choose $r \leftarrow \{0,1\}^n$ and output $(r, e(r) \oplus m)$
- $\mathsf{D}_e(r,c)$: output $e(r) \oplus c$

Claim 18

$(\mathsf{G},\mathsf{E},\mathsf{D})$ has private-key indistinguishable encryptions for a multiple messages

Private-Key Indistinguishable Encryptions for Multiple Messages

Suffices to encrypt messages of some fixed length (here the length is n).(?)

Let $\mathcal{F}$ be a (non-uniform) length-preserving PRF

Construction 17

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Proof: ?

Public-key indistinguishable encryptions for multiple messages

Let (G_T, f, Inv) be a (non-uniform) TDP, and let b be hardcore predicate for it.

Public-key indistinguishable encryptions for multiple messages

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Construction 19 (bit encryption)

- $\mathrm{G}(1^n)$: output $(e, d) \leftarrow \mathrm{G}_T(1^n)$
- $\mathrm{E}_e(m)$: choose $r \leftarrow \{0,1\}^n$ and output $(y = f_e(r), c = b(r) \oplus m)$
- $\mathrm{D}_d(y, c)$: output $b(\mathrm{Inv}_d(y)) \oplus c$

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Claim 20

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Claim 20

$(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has public-key indistinguishable encryptions for a multiple messages

Proof:

We believe that public-key encryptions schemes are “more complex" than private-key ones

Section 3

Active Adversaries

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- Chosen plaintext attack (CPA):

  The adversary can ask for encryption and choose the messages to distinguish accordingly

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  The adversary can also ask for decryptions of certain messages

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  The adversary can ask for encryption and choose the messages to distinguish accordingly

- Chosen ciphertext attack (CCA):

  The adversary can also ask for decryptions of certain messages

- In the public-key settings, the adversary is also given the public key
- We focus on indistinguishability, but each of the above definitions has an equivalent semantic security variant.

CPA Security

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be an encryption scheme. For a pair of algorithms $\mathsf{A} = (\mathsf{A}_1, \mathsf{A}_2)$, $n \in \mathbb{N}$, $z \in \{0,1\}^*$ and $b \in \{0,1\}$, let:

Experiment 21 ($\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathsf{A}_1^{E_e(\cdot)}(1^n, z)$, where $|m_0| = |m_1|$.
3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

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Definition 22 (private key CPA)

$(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model under CPA attack, if $\forall$ PPT $\mathsf{A}_1, \mathsf{A}_2$, and poly-bounded $\{z_n\}_{n\in\mathbb{N}}$:

$$|\Pr[\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z_n}(0) = 1] - \Pr[\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z_n}(1) = 1]| = \mathsf{neg}(n)$$

CPA Security, cont.

- public-key variant.

CPA Security, cont.

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- The scheme from Construction 17 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)

CPA Security, cont.

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- The scheme from Construction 17 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)
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CPA Security, cont.

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- The scheme from Construction 17 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)
- The scheme from Construction 19 has indistinguishable encryptions in the public-key model under CPA attack (for short, public-key CPA secure)
- In both cases, definitions are not equivalent (?)

CCA Security

Experiment 23 ($\mathrm{Exp}^{\mathrm{CCA1}}_{\mathsf{A},n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathsf{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$, where $|m_0| = |m_1|$.
3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

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3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

Experiment 24 ($\mathrm{Exp}^{\mathrm{CCA2}}_{\mathsf{A},n,z_n}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathsf{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$, where $|m_0| = |m_1|$.
3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot), D_d^{\neg c}(\cdot)}(1^n, s, c)$

CCA Security, cont.

Definition 25 (private key CCA1/CCA2)

$(\mathsf{G},\mathsf{E},\mathsf{D})$ has indistinguishable encryptions in the private-key model under $x \in \{\mathsf{CCA1},\mathsf{CCA2}\}$ attack, if $\forall$ PPT $\mathsf{A}_1,\mathsf{A}_2$, and poly-bounded $\{z_n\}_{n\in\mathbb{N}}$:

$$|\Pr[\mathsf{Exp}^{x}_{\mathsf{A},n,z_n}(0)=1]-\Pr[\mathsf{Exp}^{x}_{\mathsf{A},n,z_n}(1)=1]|=\mathsf{neg}(n)$$

CCA Security, cont.

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- The public key definition is analogous

Private-key CCA2

- Is the scheme from Construction 17 private-key CCA1 secure?

Private-key CCA2

- Is the scheme from Construction 17 private-key CCA1 secure?
- CCA2 secure?

Private-key CCA2

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Let (G, E, D) be a private-key CPA scheme, and let $(\text{Gen}_M, \text{Mac}, \text{Vrfy})$ be an existential unforgeable strong MAC.

Construction 26

- $G'(1^n)$: Output $(e \leftarrow G_E(1^n), k \leftarrow \text{Gen}_M(1^n))$.[a]
- $E'_{e,k}(m)$: let $c = E_e(m)$ and output $(c, t = \text{Mac}_k(c))$
- $D_{e,k}(c,t)$: if $\text{Vrfy}_k(c,t) = 1$, output $D_e(c)$. Otherwise, output $\perp$

[a] We assume wlg. that the encryption and decryption keys are the same.

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- CCA2 secure?

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Theorem 27

Construction 26 is a private-key CCA2-secure encryption scheme.

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Proof:

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- Is the scheme from Construction 17 private-key CCA1 secure?
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Construction 26

- $\mathsf{G}'(1^n)$: Output $(e \leftarrow \mathsf{G}_E(1^n), k \leftarrow \mathsf{Gen}_M(1^n))$.[a]
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- $\mathsf{D}_{e,k}(c,t)$: if $\mathsf{Vrfy}_k(c,t) = 1$, output $\mathsf{D}_e(c)$. Otherwise, output $\perp$

[a] We assume wlg. that the encryption and decryption keys are the same.

Theorem 27

Construction 26 is a private-key CCA2-secure encryption scheme.

Proof: An attacker on the CCA2-security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ yields an attacker on the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$, or the existential unforgettably of $(\mathsf{Gen}_M, \mathsf{Mac}, \mathsf{Vrfy})$.

Public-key CCA1

Public-key CCA1

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a public-key CPA scheme and let (P, V) be a $\mathcal{NIZK}$ for
$\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) \colon \exists (m, z_0, z_1) \quad \text{s.t.} \quad c_0 = \mathsf{E}_{pk_0}(m, z_0) \land c_1 = \mathsf{E}_{pk_1}(m, z_1)\}$

Public-key CCA1

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a public-key CPA scheme and let (P, V) be a $\mathcal{NIZK}$ for
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Construction 28 (The Naor-Yung Paradigm)

- $\mathsf{G}'(1^n)$:
  1. For $i \in \{0,1\}$: set $(sk_i, pk_i) \leftarrow \mathsf{G}(1^n)$.
  2. Let $r \leftarrow \{0,1\}^{\ell(n)}$, and output $pk' = (pk_0, pk_1, r)$ and $sk' = (pk', sk_0, sk_1)$
- $\mathsf{E}'_{pk'}(m)$:
  1. For $i \in \{0,1\}$: set $c_i = \mathsf{E}_{pk_i}(m, z_i)$, where z_i is a uniformly chosen string of the right length
  2. $\pi \leftarrow \mathsf{P}((c_0, c_1, pk_0, pk_1), (m, z_0, z_1), r)$
  3. Output (c_0, c_1, π).
- $\mathsf{D}'_{sk'}(c_0, c_1, \pi)$: If $\mathsf{V}((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, return $\mathsf{D}_{sk_0}(c_0)$. Otherwise, return $\perp$.

Public-key CCA1, cont.

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n. (?)
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using ”security parameter" n.

Public-key CCA1, cont.

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n. (?)
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using ”security parameter" n.

Is the scheme CCA1 secure?

Public-key CCA1, cont.

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n. (?)
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Is the scheme CCA1 secure?

Theorem 29

Assuming (P, V) is adaptive secure, then Construction 28 is a public-key CCA1 secure encryption scheme.

Public-key CCA1, cont.

- We assume for simplicity that the encryption key output by $\mathsf{G}(1^n)$ is of length at least n. (?)
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using "security parameter" n.

Is the scheme CCA1 secure?

Theorem 29

Assuming (P, V) is adaptive secure, then Construction 28 is a public-key CCA1 secure encryption scheme.

Proof: Given an attacker A' for the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$, we use it to construct an attacker A on the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ or the adaptive security of (P, V).

Proving Thm 29

Let $S = (S_1, S_2)$ be the (adaptive) simulator for $(P, V, \mathcal{L})$

Proving Thm 29

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Algorithm 30 (A)

Input: $(1^n, pk)$

1. Let $j \leftarrow \{0,1\}$, $pk_{1-j} = pk$, $(pk_j, sk_j) \leftarrow G(1^n)$ and $(r, s) \leftarrow S_1(1^n)$
2. Emulate $A'(1^n, pk' = (pk_0, pk_1, r))$:

   On query (c_0, c_1, π) of A' to D':
   If $V((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, answer $D_{sk_j}(c_j)$.
   Otherwise, answer $\perp$.
3. Output the pair (m_0, m_1) that A' outputs
4. On challenge c ($= E_{pk}(m_b)$):
   - Set $c_{1-j} = c$, $c_j = E_{pk_j}(m_a)$ for $a \leftarrow \{0,1\}$, and $\pi \leftarrow S_2((c_0, c_1, pk_0, pk_1), r, s)$
   - Send $c' = (c_0, c_1, \pi)$ to A'
5. Output the value that A' does

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ w.p. $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ w.p. $(\delta(n) - \mathsf{neg}(n))/2$.

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ w.p. $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ w.p. $(\delta(n) - \operatorname{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[\mathsf{A}' \text{ “makes" } \mathsf{A}(1^n) \text{ decrypt an invalid cipher}] = \operatorname{neg}(n) \qquad (2)$$

Proving Thm 29, cont.

Claim 31

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Hence, in the first the emulation of A' is perfect and leaks no information about j.

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Let $A'(1^n, x, y)$ be A''s output in the emulation induced by $A(1^n)$, conditioned on $a = x$ and $b = y$.

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ w.p. $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ w.p. $(\delta(n) - \operatorname{neg}(n))/2$.

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Assume for simplicity that the above prob is 0.
Hence, in the first the emulation of A' is perfect and leaks no information about j.

Let $\mathsf{A}'(1^n, x, y)$ be A''s output in the emulation induced by $\mathsf{A}(1^n)$, conditioned on $a = x$ and $b = y$.

1. Since no information about j has leaked, $\mathsf{A}'(1^n, 0, 1) \equiv \mathsf{A}'(1^n, 1, 0)$

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ w.p. $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ w.p. $(\delta(n) - \operatorname{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[\mathsf{A}' \text{ “makes" } \mathsf{A}(1^n) \text{ decrypt an invalid cipher}] = \operatorname{neg}(n) \tag{2}$$

Assume for simplicity that the above prob is 0.
Hence, in the first the emulation of A' is perfect and leaks no information about j.

Let $\mathsf{A}'(1^n, x, y)$ be A''s output in the emulation induced by $\mathsf{A}(1^n)$, conditioned on $a = x$ and $b = y$.

1. Since no information about j has leaked, $\mathsf{A}'(1^n, 0, 1) \equiv \mathsf{A}'(1^n, 1, 0)$
2. The adaptive zero-knowledge of (P, V) yields that $|\Pr[\mathsf{A}'(1^n, 1, 1) = 1] - \Pr[\mathsf{A}'(1^n, 0, 0) = 1]| \geq \delta(n) - \operatorname{neg}(n)$

Proving Thm 29, cont..

Let $A(x)$ be A's output on challenge $E_{pk}(m_x)$ (and security parameter 1^n).

Proving Thm 29, cont..

Let $\mathsf{A}(x)$ be A's output on challenge $\mathsf{E}_{pk}(m_x)$ (and security parameter 1^n).

$$
\begin{aligned}
&\left|\Pr[\mathsf{A}(1)=1]-\Pr[\mathsf{A}(0)=1]\right| \\
&=\left|\frac{1}{2}(\Pr[\mathsf{A}'(0,1)=1]+\Pr[\mathsf{A}'(1,1)=1])-\frac{1}{2}(\Pr[\mathsf{A}'(0,0)=1]+\Pr[\mathsf{A}'(1,0)=1])\right|
\end{aligned}
$$

Proving Thm 29, cont..

Let $\mathsf{A}(x)$ be A's output on challenge $\mathsf{E}_{pk}(m_x)$ (and security parameter 1^n).

$$
\begin{aligned}
&|\Pr[\mathsf{A}(1)=1]-\Pr[\mathsf{A}(0)=1]| \\
&= \left|\frac{1}{2}(\Pr[\mathsf{A}'(0,1)=1]+\Pr[\mathsf{A}'(1,1)=1]) - \frac{1}{2}(\Pr[\mathsf{A}'(0,0)=1]+\Pr[\mathsf{A}'(1,0)=1])\right| \\
&\geq \frac{1}{2}\left|\Pr[\mathsf{A}'(1,1)=1]-\Pr[\mathsf{A}'(0,0)=1]\right| - \frac{1}{2}\left|\Pr[\mathsf{A}'(1,0)=1]-\Pr[\mathsf{A}'(0,1)=1]\right|
\end{aligned}
$$

Proving Thm 29, cont..

Let $\mathsf{A}(x)$ be A's output on challenge $\mathsf{E}_{pk}(m_x)$ (and security parameter 1^n).

$$
\begin{aligned}
&|\Pr[\mathsf{A}(1) = 1] - \Pr[\mathsf{A}(0) = 1]| \\
&= \left|\frac{1}{2}(\Pr[\mathsf{A}'(0,1) = 1] + \Pr[\mathsf{A}'(1,1) = 1]) - \frac{1}{2}(\Pr[\mathsf{A}'(0,0) = 1] + \Pr[\mathsf{A}'(1,0) = 1])\right| \\
&\geq \frac{1}{2}\left|\Pr[\mathsf{A}'(1,1) = 1] - \Pr[\mathsf{A}'(0,0) = 1]\right| - \frac{1}{2}\left|\Pr[\mathsf{A}'(1,0) = 1] - \Pr[\mathsf{A}'(0,1) = 1]\right| \\
&\geq (\delta(n) - \mathrm{neg}(n))/2 - 0
\end{aligned}
$$

Public-key CCA2

- Is Construction 28 CCA2 secure?

Public-key CCA2

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- **Problem:** Soundness might not hold with respect to the simulated CRS, after seeing a proof for an invalid statement

Public-key CCA2

- Is Construction 28 CCA2 secure?
- **Problem:** Soundness might not hold with respect to the simulated CRS, after seeing a proof for an invalid statement
- **Solution:** use simulation sound $\mathcal{NIZK}$