Foundation of Cryptography, Lecture 10
Pseudorandom Generator from One-Way Functions

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Section 1

Entropy

Different measures of entropy

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Equality iff X is **uniform**.

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Conditional Entropy

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Example: let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a 2^k regular function. Let X be uniform over $\{0,1\}^n$ and let $Y = f(X)$. Then
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Flattening Shannon entropy

Lemma 1

Let X be a rv over $\mathcal{U}$, let $t \in \mathbb{N}$ and let $\varepsilon > 0$. Then $\exists$ rv Z that is $(\varepsilon + 2^{-t})$-close to X^t, and $\mathrm{H}_\infty(Z) \geq t \cdot \mathrm{H}(X) - O(\sqrt{t \cdot \log(1/\varepsilon)} \cdot \log(|\mathcal{U}| \cdot t)$.

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Proof: ?

Pairwise independent hashing

Definition 2 (pairwise independent function family)

A function family $\mathcal{H} = \{h \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ is pairwise independent, if $\forall$ $x \neq x' \in \{0,1\}^n$ and $y, y' \in \{0,1\}^m$, it holds that $\Pr_{h \leftarrow \mathcal{H}}[h(x) = y \wedge h(x') = y')] = 2^{-2m}$.

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Example $\mathcal{H} = \{(A, b) \in \{0,1\}^{m \times n} \times \{0,1\}^m\}$ with $(A, b)(x) = A \times x + b$.

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Lemma 3 (leftover hash lemma)

Let X be a rv over $\{0,1\}^n$ with $\mathrm{H}_2(X) \geq k$ and let $\mathcal{H} = \{h\colon \{0,1\}^n \mapsto \{0,1\}^m\}$ be pairwise independent, then

$$\mathrm{SD}((H, H(X)), (H, U_m)) \leq 2^{(m-k-2))/2},$$

where H is uniformly distributed over $\mathcal{H}$ and U_m is uniformly distributed over $\{0,1\}^m$.

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- Examples
- Repeated sampling

Section 2

PRG from Regular OWF

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Definition 5

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Claim 6

Let f be a $2^{k=k(n)}$-regular OWF, $\mathcal{H} = \{\mathcal{H}_n\colon \{0,1\}^n \mapsto \{0,1\}^{m(n)=k(n)+\log n}\}$ be efficient family of pairwise independent hash function family, and let $g = g(f,\mathcal{H})$. Then

1. $\mathrm{H}(g(U_n,H_n)) \geq n + \mathrm{H}(H_n) - \frac{1}{n}$, where H_n is uniform over $\mathcal{H}_n$.
2. g is one-way.

g has high entropy

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\begin{aligned}
\mathrm{CP}(g(U_n, H_n)) &:= \Pr_{w,w' \leftarrow \{0,1\}^n \times \mathcal{H}_n}[g(w) = g(w')] \\
&= \Pr_{h,h' \leftarrow \mathcal{H}_n}[h = h'] \cdot \Pr_{x,x' \leftarrow \{0,1\}^n}[f(x) = f(x')] \\
&\cdot \Pr_{h \leftarrow \mathcal{H}_n; x,x' \leftarrow zn}[h(x) = h(x') \mid f(x) = f(x')] \\
&= \mathrm{CP}(\mathrm{H}_n) \cdot \mathrm{CP}(f(U_n)) \cdot (2^{-k} + (1 - 2^{-k}) \cdot 2^{-m}) \\
&\leq \mathrm{CP}(H_n) \cdot \mathrm{CP}(f(U_n)) \cdot (2^{-k} + 2^{-m}) \\
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Hence, $\mathrm{H}_2(g(U_n, H_n)) \geq \mathrm{H}_2(\mathcal{H}_n) + \mathrm{H}_2(U_n) + \log \frac{1}{1+\frac{1}{n}} \geq \mathrm{H}(H_n) + n - \frac{1}{n}$.

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Thus, $\mathrm{H}(g(U_n, H_n)) \geq \mathrm{H}(H_n) + n - \frac{1}{n}$.

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Let $t = t(n) = k(n) - 2\lceil \log(p(n)) \rceil$.

Algorithm 7 (B)

Input: $y \in \{0,1\}^n$.
Sample $h \leftarrow \mathcal{H}_n$ and $z \leftarrow \{0,1\}^t$, and return $\mathsf{D}(y,h,z)$

Algorithm 8 (D)

Input: $y \in \{0,1\}^n$, $h \in \mathcal{H}_n$ and $z_1 \in \{0,1\}^t$.
For all $z_2 \in \{0,1\}^{m-t}$:

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1. Let $(x,h) \leftarrow \mathsf{A}(y,h,z))$.
2. If $f(x) = y$, return x.

$$\Pr_{x \leftarrow \{0,1\}^n; h \leftarrow \mathcal{H}_n}\left[\mathsf{D}(f(x), h, h(x)_{1,\ldots,t}) \in f^{-1}(f(x))\right] \geq \frac{1}{p(n)} \tag{1}$$

g is one-way, cont.

By the leftover hash lemma(?)

$$\mathrm{SD}((f(x), h, h(x)_{1,\ldots,t})_{x \leftarrow \{0,1\}, h \leftarrow \mathcal{H}_n}, (f(x), h, U_t)_{x \leftarrow \{0,1\}, h \leftarrow \mathcal{H}_n}) \leq \frac{1}{2p(n)} \quad (2)$$

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Hence,

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathrm{B}(f(x)) \in f^{-1}(f(x))\right] \geq \frac{1}{p(n)} - \frac{1}{2p(n)} = \frac{1}{2p(n)}.$$

The generator

Claim 9

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^m$ be a OWF with $\mathrm{H}(f(U_n)) \geq n - \frac{1}{2}$, and let b be an hardcore predicate for f. Then $g(x) = f(x) \circ b(x)$ has pseudoentropy $n + \frac{1}{2}$.

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Claim 10

The function $g^{n^2}(x_1, \ldots, x_{n^2}) = g(x_1), \ldots, g(x_{n^2})$ has pseudo min-entropy $n(n+\frac{1}{2}) - O(\sqrt{n \log^2 n} \cdot \log(n^2)) \geq n^2 + n/2 - O(n^{2/3})$.

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Let $f\colon \{0,1\}^n \mapsto \{0,1\}^m$ be a OWF with $\mathsf{H}(f(U_n)) \geq n - \frac{1}{2}$, and let b be an hardcore predicate for f. Then $g(x) = f(x) \circ b(x)$ has pseudoentropy $n + \frac{1}{2}$.

Proof: ?

We call such g a pseudo-entropy generator.

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The function $g^{n^2}(x_1, \ldots, x_{n^2}) = g(x_1), \ldots, g(x_{n^2})$ has pseudo min-entropy $n(n + \frac{1}{2}) - O(\sqrt{n \log^2 n} \cdot \log(n^2)) \geq n^2 + n/2 - O(n^{2/3})$.

Proof: by the flattening lemma, taking $\varepsilon = 2^{-\log^2 n}$ and $t = n$.

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Let $\mathcal{H}\colon \{0,1\}^{n^2+n} \mapsto \{0,1\}^{n^2+n/4}$ be an efficient pairwise hash function, then $G\colon \{0,1\}^{n^2} \times \mathcal{H}_n$ defined by $G(x_1, \ldots, x_{n^2}, h) = (h, h(g^{n^2}(x_1, \ldots, x_{n^2})))$, is a PRG.

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Proof: by the leftover hash lemma

Section 3

PRG from any OWF

Inefficient construction

Definition 12

Given a function $f\colon \{0,1\}^n \mapsto \{0,1\}^m$ and $x \in \{0,1\}^n$, let
$d_f(x) = \left\lceil \log(\left|f^{-1}(f(x)\right|) + \log n \right\rceil$.
Given $\mathcal{H}\colon \{0,1\}^n \mapsto \{0,1\}^{n+\log n}$, let
$g = g(f,\mathcal{H})\colon \mathcal{H} \times \{0,1\}^n \mapsto \mathcal{H} \times \{0,1\}^n \times \{0,1\}^{n+\log n}$ be defined by
$g(h,x) = f(x), h, h(x)_{1,\ldots,d_f(x)}, 1^{n+\log n - d_f(x)}$.

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Claim 13

Let f be a OWF, $\mathcal{H} = \{\mathcal{H}_n\colon \{0,1\}^n \mapsto \{0,1\}^{n+\log n}\}$ be efficient family of pairwise independent hash function family, and let $g = g(f,\mathcal{H})$. Then

1. $\mathrm{H}(g(U_n,H_n)) \geq n + \mathrm{H}(H_n) - \frac{1}{n}$, where H_n is uniform over $\mathcal{H}_n$.
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Should we expect d_f to be poly-time computable?

Efficient construction, first approach

Definition 14

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ and $\mathcal{H} = \{h\colon \{0,1\}^n \mapsto \{0,1\}^{n+\log n}\}$, let $g = g(f,\mathcal{H})\colon \mathcal{H} \times [n] \times \{0,1\}^n \mapsto \mathcal{H} \times [n] \times \{0,1\}^n \times \{0,1\}^{n+\log n}$ be defined by $g(h,i,x) = f(x), h, i, h(x)_{1,\ldots,i+\log n}, 1^{n+\log n-i}$.

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Claim 15

Assume f is OWF and that $\mathcal{H}$ is the Matrix-based pairwise-independent hash functions. Then the pseudo Shannon-entropy of $g(H_n, I_n, U_n)$, where I_n is uniform over $[n]$, is larger by at least $1/n$ than its (real) Shannon entropy.

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Proof: Define

$$g'(h,i,x) = \begin{cases} f(x), h, i, h(x)_{1,\ldots,i+\log n-1}, U, 1^{n+\log n-i}, & i = d_f(x) \\ g(h,i,x), & \text{otherwise.} \end{cases}$$

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Claim 16

1. $g(H_n, I_n, U_n) \approx_c g'(H_n, I_n, U_n)$
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False-pseudoentropy generator to PRG

1. Using repetition convert the Shannon pseudoentropy of the output of g into min pseudoentropy.

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Very complicated an inefficient construction. Seed length of PRG is $\Theta(n^8)$.

Efficient construction, second approach

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For $f \colon \{0,1\}^n \mapsto \{0,1\}^n$, and the Matrix-based pairwise-independent hash functions $\mathcal{H} = \{h \colon \{0,1\}^n \mapsto \{0,1\}^{n+\log n}\}$, let $g \colon \mathcal{H} \times \{0,1\}^n] \mapsto \mathcal{H} \times \{0,1\}^n \times \{0,1\}^{n+\log n}$ be defined by $g(h,x) = f(x), h, h(x)$.

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But g is invertible and thus its output pseudoentropy is as large as its real entropy.(?)

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Right, but not in the eyes of an online observer.

Next-block pseudoentropy generator

Definition 18 (next-block pseudoentropy)

$X = (X_1, \ldots, X_m)$ has next-block pseudoentropy at least k, $\exists$ rv $Y = (Y_1, \ldots, Y_m)$, (jointly distributed with X), such that:

1. $\forall i$, $(X_1, X_2, \ldots, X_{i-1}, X_i) \approx_c (X_1, X_2, \ldots, X_{i-1}, Y_i)$.
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Quantitative generalization of unpredictability: measures how hard it to predict X_i from $X_1, X_2, \ldots, X_{i-1}$ (for $i \leftarrow [k]$).

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Claim 19

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Continue to Power-point presentation.