

Foundation of Cryptography
(0368-4162-01), Lecture 6
More on Zero Knowledge

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Part I

Non-Interactive Zero Knowledge

Interaction is crucial for ZK

Claim 1

Assume that $\mathcal{L} \subseteq \{0, 1\}^*$ has a one-message ZK proof (even computational), with standard completeness and soundness,^a then $\mathcal{L} \in \text{BPP}$.

^aThat is, the completeness is $\frac{2}{3}$ and soundness error is $\frac{1}{3}$.

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 - 1 Witness Indistinguishability

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 - 3 Non-interactive “zero knowledge”

Definition

Non-Interactive Zero Knowledge (NIZK)

Definition 2 (NIZK)

The *non interactive* PPT's (P, V) is a NIZK for $\mathcal{L} \in \text{NP}$, if $\exists \ell \in \text{poly}$ s.t.

- **Completeness:**

$\Pr_{c \leftarrow \{0,1\}^{\ell(|x|)}} [V(x, c, P(x, w(x), c)) = 1] \geq 2/3$,
 where $w(x) \in R_{\mathcal{L}}(x)$ for any $x \in \mathcal{L}$ (w is an arbitrary function)

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NIZK in HBM

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Let c^H be the “hidden” CRSS:

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- Verifier only sees the bits in c^H that are indexed by \mathcal{I}
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We give a NIZK for HC - Directed Graph Hamiltonicity, in the HBM, and then transfer it into a NIZK in the standard model.

Implies a (standard model) NIZK for all NP

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Claim 3

Let T be a random $n^3 \times n^3$ Boolean matrix where each entry is 1 w.p n^{-5} . Hence, $\Pr[T \text{ is useful}] \in \Omega(n^{-3/2})$.

Proving Claim 3

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Hence, wp at least $1 - 2 \cdot n^3 \cdot n^{-4} = 1 - O(n^{-1})$, no row or column of T contains more than a single one entry.
- Hence, wp $\theta(1/\sqrt{n})$ the matrix T contains a permutation matrix and all its other entries are zero.
- A random permutation matrix forms a cycle wp $1/n$ (there are $n!$ permutation matrices and $(n-1)!$ of them form a cycle)

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Algorithm 4 (P)

Input: G and a cycle C in G . A CRS $T \in \{0, 1\}_{n^3 \times n^3}$

- 1 If T not useful, set $\mathcal{I} = n^3 \times n^3$ (i.e., reveal all T) and $\phi = \perp$
Otherwise, let H be the (generalized) $n \times n$ sub matrix containing the hamiltonian cycle in T .
- 2 Set $\mathcal{I} = T \setminus H$ (i.e., , reveal the bits of T outside of H)
- 3 Choose $\phi \leftarrow \Pi_n$, s.t. C is mapped to the cycle in H
- 4 Add all the entries in H corresponding to non edges in G (with respect to ϕ) to \mathcal{I}
- 5 Output $\pi = (\mathcal{I}, \phi)$

NIZK for Hamiltonicity in HBM cont.

Algorithm 5 (V)

Input: a graph G , index set $\mathcal{I} \subseteq [n^3] \times [n^3]$, ordered set $\{T_i\}_{i \in \mathcal{I}}$ and a mapping ϕ

- ➊ If all the bits of T are revealed and T is not useful, accept. Otherwise,
- ➋ Verify that $\exists n \times n$ submatrix $H \subseteq T$ with all entries in $T \setminus H$ are zeros.
- ➌ Verify that $\phi \in \Pi_n$, and that all the entries of H not corresponding (according to ϕ) to edges of G are zeros

NIZK for Hamiltonicity in HBM cont.

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Claim 6

The above protocol is a perfect NIZK for HC in the HBM, with perfect completeness and soundness error $1 - \Omega(n^{-3/2})$

NIZK for Hamiltonicity in HBM

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- Zero knowledge?

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Input: G

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- For useful T , the location of H is uniform in the real and simulated case.
- ϕ is a random element in Π_n in both cases
- Hence, the simulation is perfect

Section 2

From HBM to Standard NIZK

trapdoor permutations

Definition 8 (trapdoor permutations)

A triplet of PPT's (G, f, Inv) is called (enhanced) family of trapdoor permutation (TDP), if the following holds:

- 1 $G: \{0, 1\}^n \mapsto \{0, 1\}^n$ for every $n \in \mathbb{N}$.
- 2 $f_{pk} = f(pk, \cdot)$ is a permutation over $\{0, 1\}^n$, for every $pk \in \{0, 1\}^n$.
- 3 $\text{Inv}(sk, \cdot) \equiv f_{G(sk)}^{-1}$ for every $sk \in \{0, 1\}^n$
- 4 For any PPT A ,

$$\Pr_{x \leftarrow \{0, 1\}^n, sk \leftarrow \{0, 1\}^n, x = G(sk)} [A(pk, x) = f_{pk}^{-1}(x)] = \text{neg}(n)$$

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- For our purposes, somewhat less restrictive requirements will do

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Definition 9 (RSA)

- $G(p, q)$ sets $pk = (n = pq, e)$ for some $e \in \mathbb{Z}_{\phi(n)}^*$, and $sk = (n, d \equiv e^{-1} \pmod{\phi(n)})$
- $f(pk, x) = x^e \pmod{n}$
- $\text{Inv}(sk, x) = x^d \pmod{n}$

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In particular, $(x^e)^d \equiv x \pmod{n}$, for every $x \in \mathbb{Z}_n^*$, where $d \equiv e^{-1} \pmod{\phi(n)}$

Definition 9 (RSA)

- $G(p, q)$ sets $pk = (n = pq, e)$ for some $e \in \mathbb{Z}_{\phi(n)}^*$, and $sk = (n, d \equiv e^{-1} \pmod{\phi(n)})$
- $f(pk, x) = x^e \pmod{n}$
- $\text{Inv}(sk, x) = x^d \pmod{n}$

Factoring is easy \implies RSA is easy.

example, RSA

In the following $n \in \mathbb{N}$ and all operations are modulo n .

- $\mathbb{Z}_n = [n]$ and $\mathbb{Z}_n^* = \{x \in [n] : \gcd(x, n) = 1\}$
- $\phi(n) = |\mathbb{Z}_n^*|$ (equals $(p-1)(q-1)$ for $n = pq$ with $p, q \in \mathbb{P}$)
- For every $e \in \mathbb{Z}_{\phi(n)}^*$, the function $f(x) \equiv x^e$ is a permutation over \mathbb{Z}_n^* .

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Factoring is easy \implies RSA is easy. Other direction?

The transformation

The transformation

Let (P_H, V_H) be a HBM NIZK for \mathcal{L} , and let $\ell(n)$ be the length of the CRS used for $x \in \{0, 1\}^n$.

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We construct a NIZK (P, V) for \mathcal{L} , with the same completeness and “not too large” soundness error.

The transformation

The protocol

Algorithm 10 (P)

Input: $x \in \mathcal{L}$, $w \in R_{\mathcal{L}}(x)$ and CRS $c = (c_1, \dots, c_{\ell}) \in \{0, 1\}^{n\ell}$, where $n = |x|$ and $\ell = \ell(n)$.

- 1 Choose $sk \leftarrow U_n$, set $pk = G(sk)$ and compute $c^H = (b(z_1 = f_{pk}^{-1}(c_1)), \dots, b(z_{\ell(n)} = f_{pk}^{-1}(c_{\ell})))$
- 2 Let $(\pi_H, \mathcal{I}) \leftarrow P_H(x, w, c^H)$ and output $(\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}})$

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Algorithm 11 (V)

Input: $x \in \mathcal{L}$, CRS $c = (c_1, \dots, c_{\ell}) \in \{0, 1\}^{np}$, and $(\pi_H, \mathcal{I}, pk, \{z_i\}_{i \in \mathcal{I}})$, where $n = |x|$ and $\ell = \ell(n)$.

- 1 Verify that $pk \in \{0, 1\}^n$ and that $f_{pk}(z_i) = c_i$ for every $i \in \mathcal{I}$
- 2 Return $V_H(x, \pi_H, \mathcal{I}, c^H)$, where $c_i^H = b(z_i)$ for every $i \in \mathcal{I}$.

Claim 12

Assuming that (P_H, V_H) is a NIZK for \mathcal{L} in the HBM with soundness error $2^{-n} \cdot \alpha$, then (P, V) is a NIZK for \mathcal{L} with the same completeness, and soundness error α .

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- Zero knowledge:?

The transformation

Proving zero knowledge

Algorithm 13 (S)

Input: $x \in \{0, 1\}^n$ of length n .

- Let $(\pi_H, \mathcal{I}, c^H) = S_H(x)$, where S_H is the simulator of (P_H, V_H)
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 - $pk \leftarrow G(U_n)$
 - Each z_i is chosen at random in $\{0, 1\}^n$ such that $b(z_i) = c_i^H$
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- Exists efficient M s.t. $M(S_H(x)) \equiv S(x)$ and $M(P_H(x, w_x)) \approx_c P(x, w_x)$
 - Distinguishing $P(x, w_x)$ from $S(x)$ is hard

Section 3

Adaptive NIZK

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- **Completeness:** $\forall f: \{0, 1\}^{\ell(n)} \mapsto \mathcal{L} \cap \{0, 1\}^n$:
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- **Soundness:** $\forall f: \{0, 1\}^{\ell(n)} \mapsto \{0, 1\}^n$ and P^*
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$$\{(f(c), c, P(f(c), w(f(c))), c \leftarrow \{0, 1\}^{\ell(n)})\}_{n \in \mathbb{N}} \approx_c \{S^f(n)\}_{n \in \mathbb{N}}.$$

where $S^f(n)$ is the output of the following process

- 1 $(c, s) \leftarrow S_1(1^n)$
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Assume TDP exist, then every NP language has an adaptive NIZK with perfect completeness and negligible soundness error.

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In the following, when saying adaptive NIZK, we mean negligible completeness and soundness error.

Section 4

Simulation Sound NIZK

One-time Simulation Sound NIZK

Same as adaptive NIZK, with the following change

- **(Simulation) Soundness:** Let $S = (S_1, S_2)$ is the guaranteed ZK simulator for P and \mathcal{L} . For a pair of PPT's (P_1^*, P_2^*) , let

Experiment 15 (Exp_{V,S,P^*}^n)

- 1 $(c, s) \leftarrow S_1(1^n)$
- 2 $(x, p) \leftarrow P_1^*(1^n, c)$
- 3 $\pi \leftarrow S_2(x, c, s)$
- 4 $(x', \pi') \leftarrow P_2^*(p, \pi)$
- 5 Output (r, x, π, x', π')

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We require $\Pr[(r, x, \pi, x', \pi') \leftarrow \text{Exp}_{V,S,P^*}^n : V(x', \pi', c) = 1 \wedge (x', \pi') \neq (x, \pi)] = \text{neg}(n)$.

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- Does the adaptive NIZK we seen in class have simulation soundness?

Construction

We present a simulation sound NIZK (P, V) for $\mathcal{L} \in \text{NP}$

Ingredients:

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Input: $x \in \mathcal{L}$ and $w \in R_{\mathcal{L}}(x)$, and CRS $r = (r_1, r_2)$

- 1 $(sk, vk) \leftarrow \text{Gen}(1^{|x|})$
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- 4 Output $\pi = (vk, \pi_A, \sigma)$

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Algorithm 17 (V)

Input: $x \in \{0, 1\}^*$, $\pi = (vk, \pi_A, \sigma)$ and a CRS $r = (r_1, r_2)$

Verify that $\text{Vrfy}_{vk}((x, \pi), \sigma) = 1$ and $V_A((x, r_1, vk), r_2, \pi_A) = 1$

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Claim 18

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 - $S_2(x, r, s = (z, sk, vk))$:
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Proof follows by the adaptive WI of (P_A, V_A) and the pseudorandomness of Com

Proving special soundness

Let (P_1^*, P_2^*) be a pair of PPT's attacking (V, S) , and let $r = (r_1, r_2)$, x , π , x' and $\pi' = (vk, \pi_A, \sigma)$ be the values generated by a random execution of $\text{Exp}_{V,S,P^*}^\eta$.

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Assuming that $\text{Vrfy}_{vk}((x', \pi_A), \sigma) = 1$, $x' \notin \mathcal{L}'$ and $(x', \pi') \neq (x, \pi)$, the following holds with save but negligible probability:

- vk is not the signing key of π'

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Assuming that $\text{Vrfy}_{vk}((x', \pi_A), \sigma) = 1$, $x' \notin \mathcal{L}'$ and $(x', \pi') \neq (x, \pi)$, the following holds with save but negligible probability:

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Since r_2 was chosen at random by S_1 , the adaptive soundness of (P_A, V_A) yields that $\Pr[V_A(x_A, r_2, \pi_A) = 1] = \text{neg}(n)$.

Part II

Proof of Knowledge

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