Foundation of Cryptography, Lecture 4
Pseudorandom Functions

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Motivation Discussion

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Solution

Function families

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4. We identify function with their description

Random functions

Definition 1 (random functions)

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- The truth table of $\pi \xleftarrow{R} \Pi_n$ is a uniform string of length $2^n \cdot n$
- For integer function m, we will consider the function family $\{\Pi_{n,m(n)}\}$.

Efficient function families

Definition 2 (efficient function family)

An ensemble of function families $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is efficient, if:

Samplable. $\mathcal{F}$ is samplable in polynomial-time: there exists a PPT that given 1^n, outputs (the description of) a uniform element in $\mathcal{F}_n$.

Efficient. There exists a polynomial-time algorithm that given $x \in \{0,1\}^n$ and (a description of) $f \in \mathcal{F}_n$, outputs $f(x)$.

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$$\left|\Pr[\mathsf{D}^{\mathcal{F}_n}(1^n) = 1] - \Pr[\mathsf{D}^{\Pi_{m(n),\ell(n)}}(1^n) = 1\right| = \operatorname{neg}(n),$$

for any oracle-aided PPT D.

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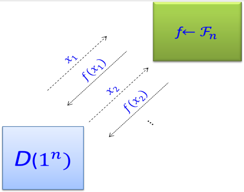

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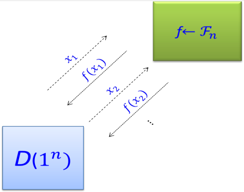

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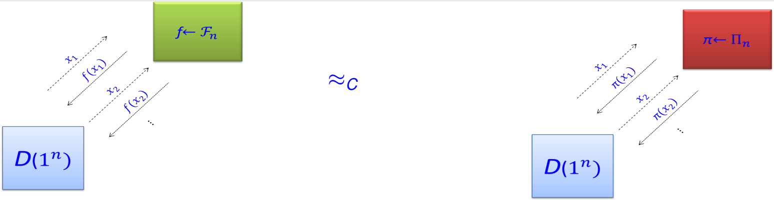

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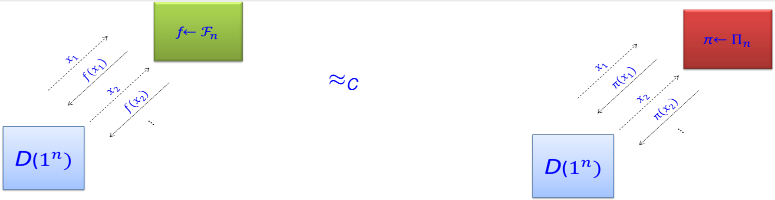

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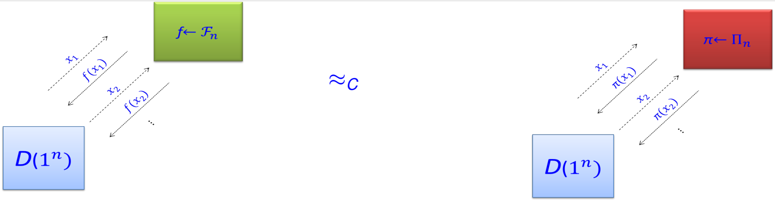

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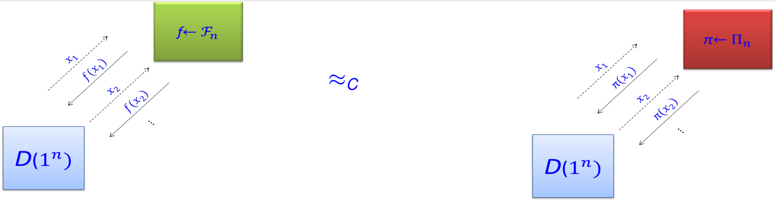

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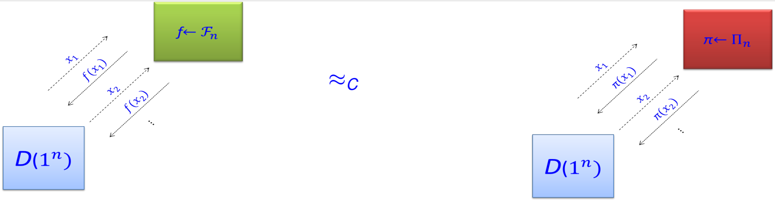

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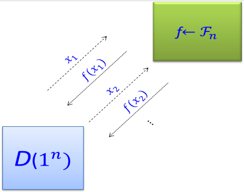

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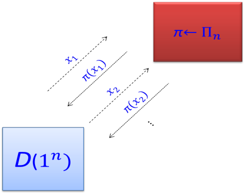

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- We will mainly focus on the case $m(n) = \ell(n) = n$
- Main application: design a scheme assuming that you have random functions, and the realize them using PRFs.

Section 2

PRF from OWF

Naive Construction

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- Problem, we are constructing the whole truth table, even to compute a single output

The GGM Construction

Construction 5 (GGM)

For $G\colon \{0,1\}^n \mapsto \{0,1\}^{2n}$ and $s \in \{0,1\}^n$,

- $G_0(s) = G(s)_{1,\ldots,n}$
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For $x \in \{0,1\}^k$ let $f_s(x) = G_{x_k}(f_s(x_{1,\ldots,k-1}))$, letting $f_s() = s$.

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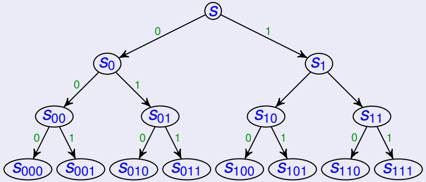

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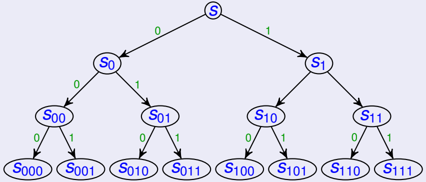

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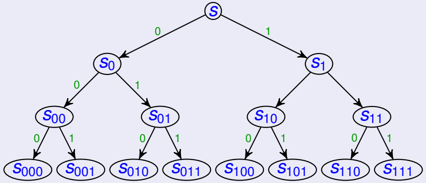

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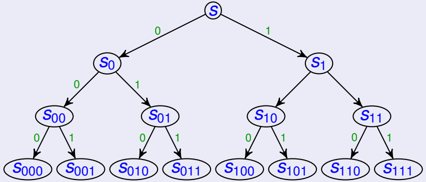

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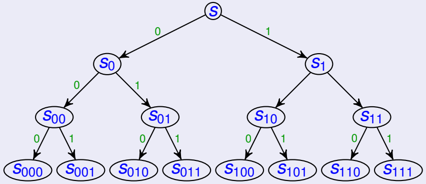

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Theorem 6 (Goldreich-Goldwasser-Micali (GGM))

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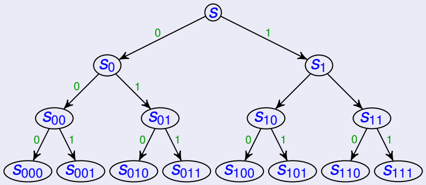
For $x \in \{0,1\}^k$ let $f_s(x) = G_{x_k}(f_s(x_{1,\ldots,k-1}))$, letting $f_s() = s$.



$s_x = f_s(x)$

- Example: $f_s(001) = s_{001} = G_1(s_{00}) = G_1(G_0(s_0)) = G_1(G_0(G_0(s)))$
- G is poly-time $\implies \mathcal{F} := \{\mathcal{F}_n = \{f_s\colon s \in \{0,1\}^n\}\}$ is efficient

Theorem 6 (Goldreich-Goldwasser-Micali (GGM))

If G is a PRG then $\mathcal{F}$ is a PRF.

Corollary 7

OWFs imply PRFs.

Proof Idea

Assume $\exists$ PPT D, $p \in$ poly and infinite set $\mathcal{I} \subseteq \mathbb{N}$ with

$$\left|\Pr[\mathsf{D}^{F_n}(1^n) = 1] - \Pr[\mathsf{D}^{\Pi_n}(1^n) = 1]\right| \geq \frac{1}{p(n)}, \tag{1}$$

for any $n \in \mathcal{I}$.

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$$\left|\Pr[\mathsf{D}'((U_{2n})^t) = 1] - \Pr[\mathsf{D}'(G(U_n))^t) = 1\right| > \frac{1}{np(n)},$$

where $(U_{2n})^t = U_{2n}^{(1)}, \ldots, U_{2n}^{(t)}$ and $G(U_n)^t = G(U_n^{(1)}), \ldots, G(U_n^{(t)})$.

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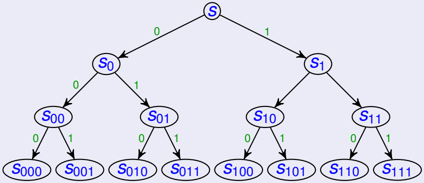
Hence, D′ violates the security of G.(?)

The Hybrid



$s_x = f_s(x)$

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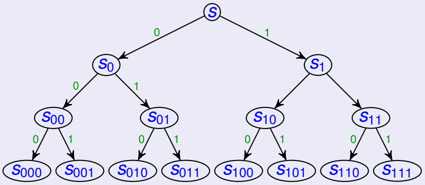

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- Let $\mathcal{T}_i$ be the set of all possible trees, in which the $i+1,\ldots,n$ levels are obtained by "applying GGM" to the i'th level.

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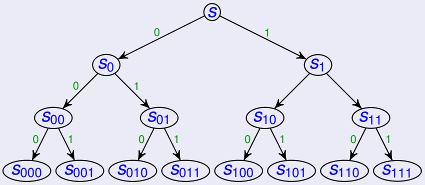

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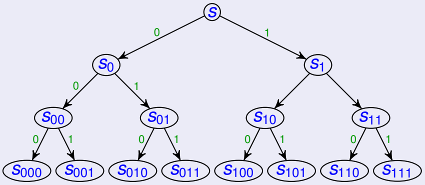

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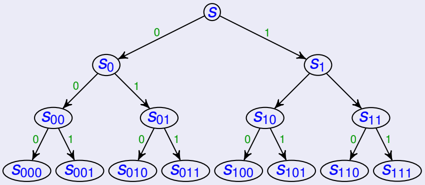

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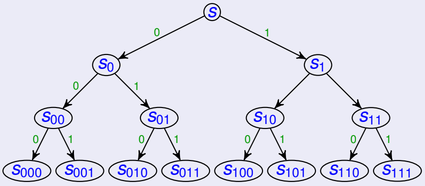

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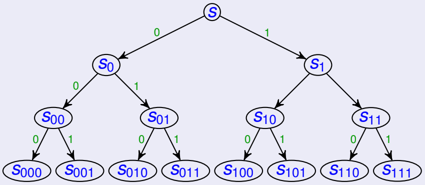

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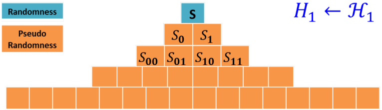

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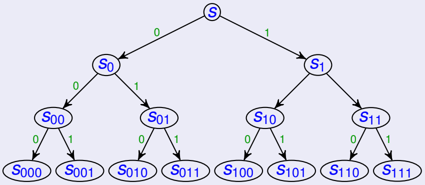

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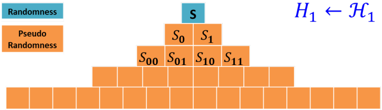

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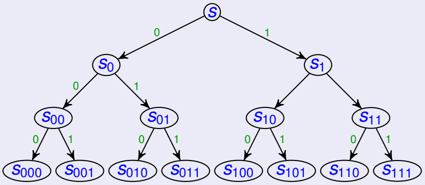

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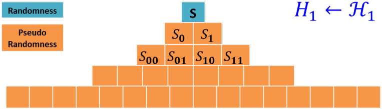

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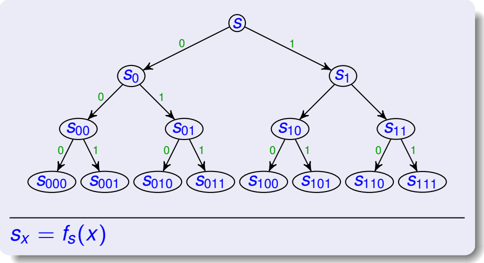
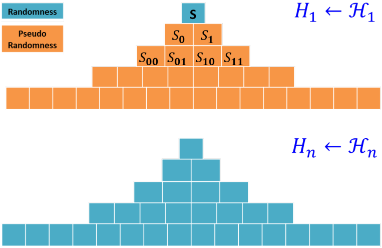
The Hybrid





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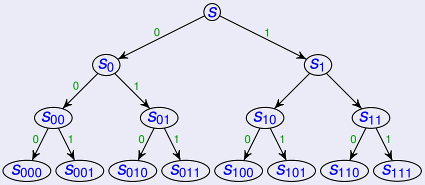
The Hybrid



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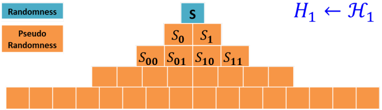

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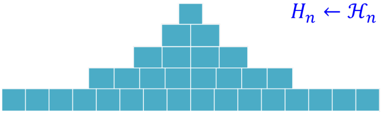

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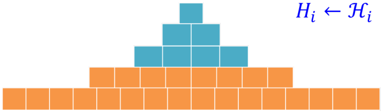
- For some $i \in \{1,\ldots,i-1\}$, algorithm D distinguishes $\mathcal{H}_i$ from $\mathcal{H}_{i+1}$ by $\frac{1}{np(n)}$



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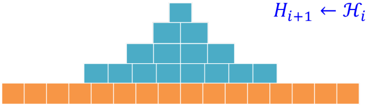
$\not\approx$



$H_{i+1} \leftarrow \mathcal{H}_{i+1}$

The Hybrid cont.

We assume wlg. that D distinguishes between $\mathcal{H}_{n-1}$ and $\mathcal{H}_n$ (?)

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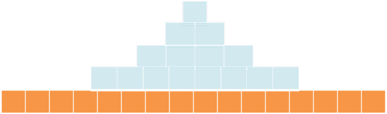

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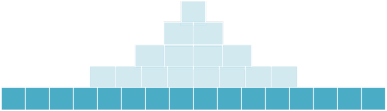

- D distinguishes (via t samples) between
  - R – a uniform string of length $2^n \cdot n$, and
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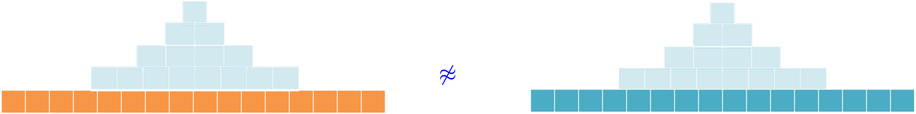

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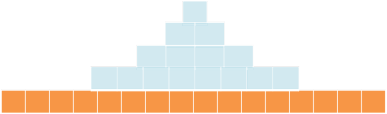

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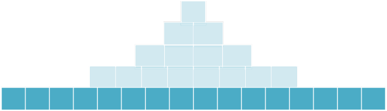

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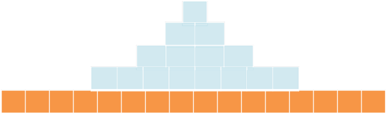

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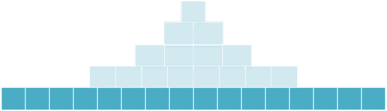

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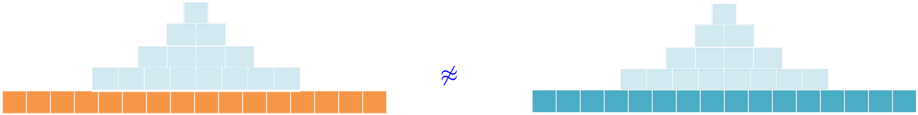

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Algorithm 8 (D′ on $y_1, \ldots, y_t \in (\{0,1\}^{2n})^t$)

Emulate D. On the i'th query q_i made by D:

- If the cell queries by q_i'th is empty, fill it with the next y
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The Hybrid cont.

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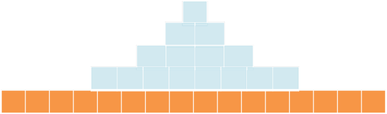

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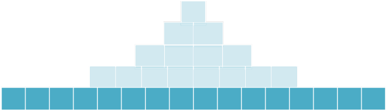

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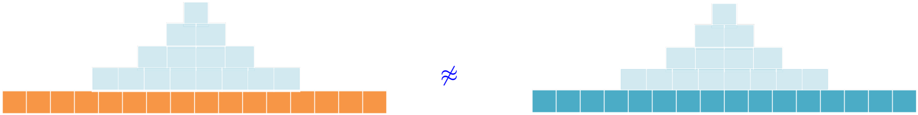

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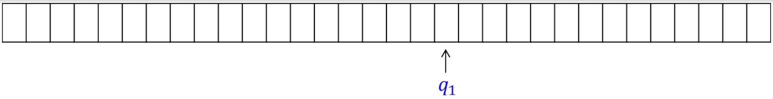

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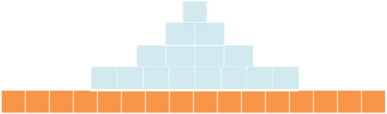

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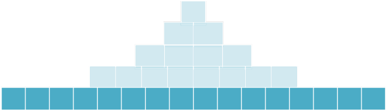

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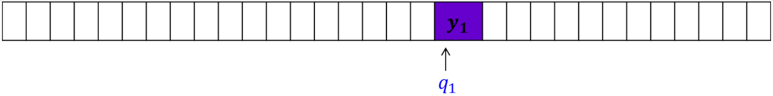

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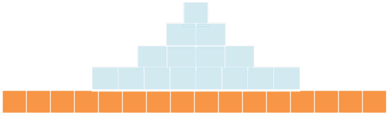

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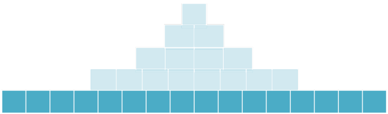

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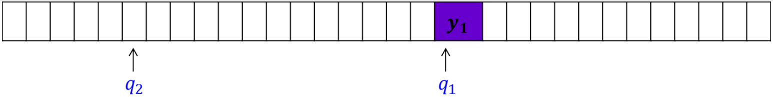

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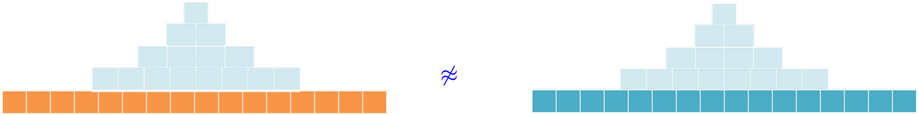

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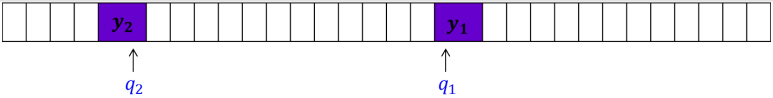

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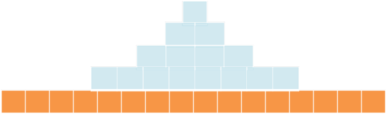

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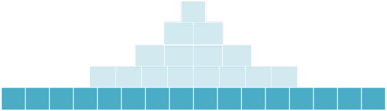

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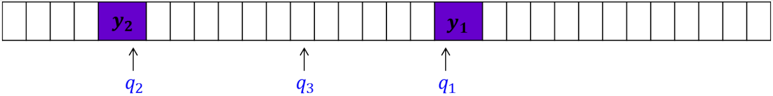

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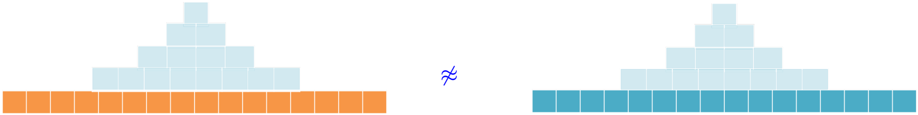

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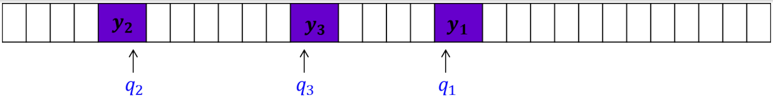

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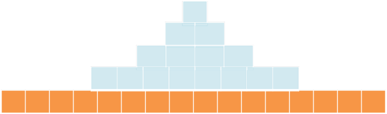

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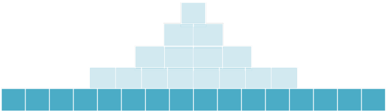

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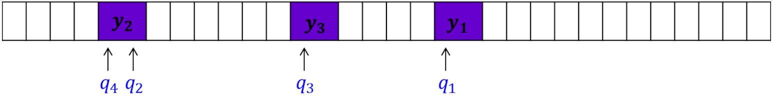

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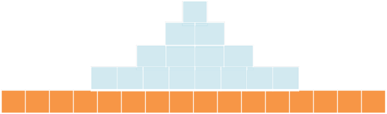

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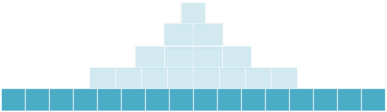

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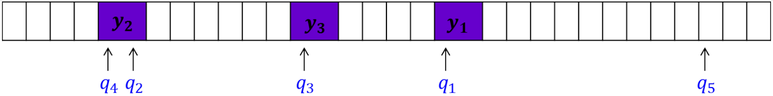

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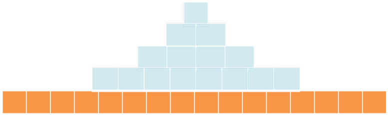

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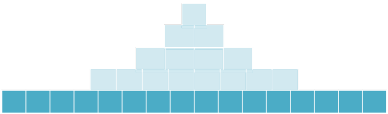

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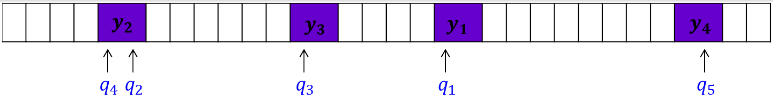

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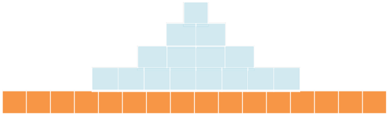

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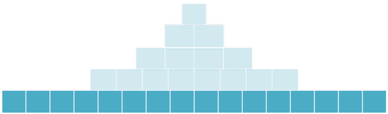

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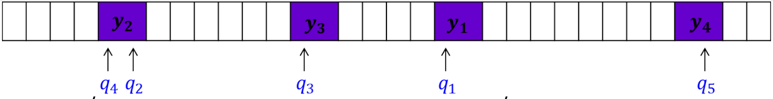

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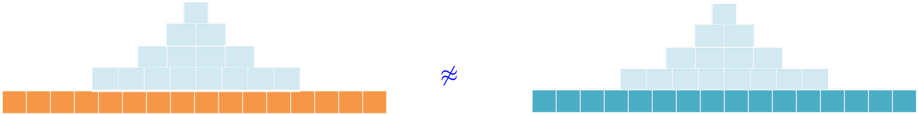

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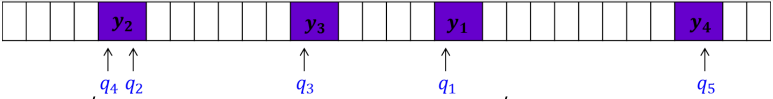

- $D'(U_{2n})^t)$ / $D'(G(U_n))^t)$ emulates D with access to R / P
- Hence, $|\Pr[D'((U_{2n})^t) = 1] - \Pr[D'(G(U_n))^t) = 1| > \frac{1}{np(n)}$

Part I

Pseudorandom Permutations

Formal Definition

Let $\widetilde{\Pi}_n$ be the set of all permutations over $\{0,1\}^n$.

Definition 9 (pseudorandom permutations (PRPs))

A *permutation* ensemble $\mathcal{F} = \{\mathcal{F}_n : \{0,1\}^n \mapsto \{0,1\}^n\}$ is a pseudorandom permutation, if

$$\left| \Pr[\mathsf{D}^{\mathcal{F}_n}(1^n) = 1] - \Pr[\mathsf{D}^{\widetilde{\Pi}_n}(1^n) = 1 \right| = \operatorname{neg}(n), \qquad (2)$$

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  - (partial) Perfect “security"
  - Inversion

Section 3

PRP from PRF

Feistel Permutation

How does one turn a function into a permutation?

Feistel Permutation

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Definition 10 (LR)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, let $\mathsf{LR}_f\colon \{0,1\}^{2n} \mapsto \{0,1\}^{2n}$ be defined by

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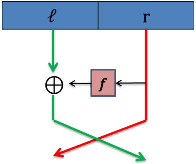

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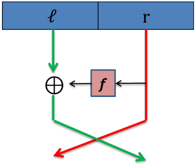

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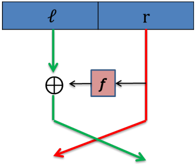

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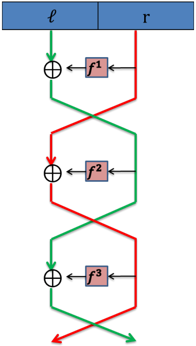

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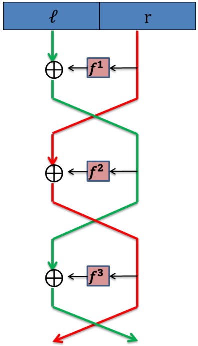

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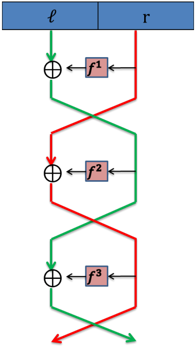

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  (letting $(\ell^0, r^0) = (\ell, r)$)

Luby-Rackoff Thm.

Recall $\mathsf{LR}_f(\ell, r) = (r, f(r) \oplus \ell)$.

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Luby-Rackoff Thm.

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Assuming that $\mathcal{F}$ is a PRF, then $\mathrm{LR}^3_{\mathcal{F}}$ is a PRP

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Proving Luby-Rackoff

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- To do that, we show both distributions are $O(q^2/2^n)$ close to $\mathit{Distinct} := \left((z_1, \ldots z_q) \xleftarrow{R} (\{0,1\}^{2n})^q \mid \forall i \neq j \colon (z_i)_0 \neq (z_j)_0\right)$.

Reminder: Statistical Distance

Definition 14

The statistical distance between distributions P and Q over $\mathcal{U}$, is defined by

$$\mathsf{SD}(P,Q) = \frac{1}{2} \cdot \sum_{u \in \mathcal{U}} |P(u) - Q(u)| = \max_{\mathcal{S} \subseteq \mathcal{U}} \{\Pr_Q[\mathcal{S}] - \Pr_P[\mathcal{S}]\}$$

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Fact 15

Let $\mathcal{E}$ be an event (i.e., set) and assume $\mathsf{SD}(P|_{\neg\mathcal{E}}, Q) \leq \delta_1$ and $\Pr_P[\mathcal{E}] \leq \delta_2$. Then $\mathsf{SD}(P,Q) \leq \delta_1 + \delta_2$

Proving Fact 15

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For any set $\mathcal{S}$, it holds that

$$\begin{aligned}\Pr_{P}[\mathcal{S}] &= \Pr_{P}[\mathcal{E}] \cdot \Pr_{P|_{\mathcal{E}}}[\mathcal{S}] + \Pr_{P}[\neg\mathcal{E}] \cdot \Pr_{P|_{\neg\mathcal{E}}}[\mathcal{S}] \\ &\geq (1-\delta_2) \cdot \Pr_{P|_{\neg\mathcal{E}}}[\mathcal{S}]\end{aligned} \tag{3}$$

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Hence,

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Thus,

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$(f(x_0),\ldots,f(x_q))_{f \stackrel{R}{\leftarrow} \widetilde{\Pi}}$ is close to *Distinct*

$\left(f(x_0),\ldots,f(x_q)\right)_{f \xleftarrow{\mathrm{R}} \widetilde{\Pi}}$ is close to *Distinct*

Recall *Distinct* $:= \left((z_1,\ldots z_q) \xleftarrow{\mathrm{R}} (\{0,1\}^{2n})^q \mid \forall i \neq j \colon (z_i)_0 \neq (z_j)_0\right)$.

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For $f \in \widetilde{\Pi}$, let $Bad(f) := \exists i \neq j\colon f(x_i)_0 = f(x_j)_0$.

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Claim 16

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Claim 17

$\left((f(x_0),\ldots,f(x_q)); f \xleftarrow{\mathrm{R}} \widetilde{\Pi} \mid \neg\,\mathrm{Bad}(f)\right) \equiv Distinct$

$(f(x_0), \ldots, f(x_q))_{f \xleftarrow{R} \widetilde{\Pi}}$ is close to *Distinct*

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By Fact 15, $(f(x_0),\ldots,f(x_q))_{f \xleftarrow{R} \widetilde{\Pi}}$ is $\frac{q^2}{2^n}$ close to *Distinct*

$(f(x_0), \ldots, f(x_q))_{f \stackrel{R}{\leftarrow} \mathrm{LR}^3(\Pi_n)}$ is close to *Distinct*

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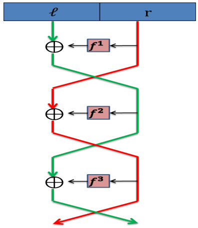
Let $(\ell_1^0, r_1^0),\ldots,(\ell_q^0, r_q^0) = (x_1,\ldots,x_k)$.

The following rv's are defined w.r.t. $(f^1, f^2, f^3) \stackrel{\mathrm{R}}{\leftarrow} \Pi_n^3$.

ℓ_1^0	r_1^0	ℓ_2^0	r_2^0	$\ldots$	ℓ_q^0	r_q^0
ℓ_1^1	r_1^1	ℓ_2^1	r_2^1	$\ldots$	ℓ_q^1	r_q^1
ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
ℓ_1^3	r_1^3	ℓ_2^3	r_2^0	$\ldots$	ℓ_q^3	r_q^3

where $\ell_b^j = r_b^{j-1}$ and $r_b^j = f^j(r_b^{j-1}) \oplus \ell_b^{j-1}$.



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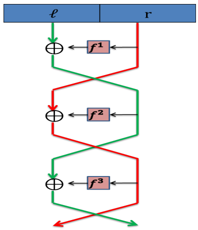

Claim 18

$$\Pr_{f^1 \xleftarrow{R} \Pi_n}\left[\mathrm{Bad}^1 := \exists i \neq j \colon r_i^1 = r_j^1\right] \leq \frac{\binom{q}{2}}{2^n}$$

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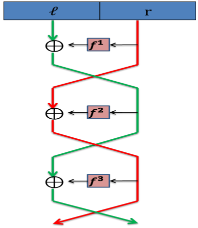

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Proof:

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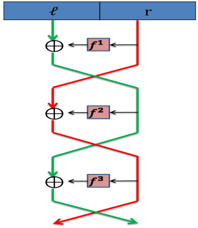

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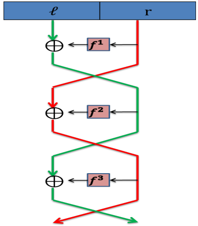

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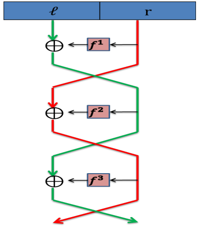

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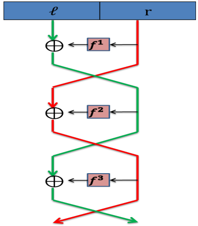

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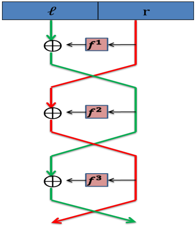

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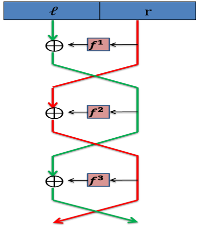

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Proof: similar to the above

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$$\left((\ell_1^3, r_1^3),\ldots,(\ell_q^3, r_q^3) \mid \neg \mathrm{Bad}^2\right) \equiv \mathit{Distinct}$$

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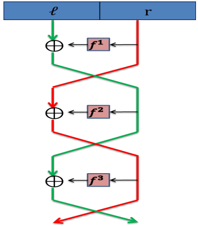

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Proof: ?

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$\left((\ell_1^3, \ldots, \ell_q^3) \mid \neg\,\mathsf{Bad}^2\right)$ is uniform over $\mathcal{S}$.

Proof: For any $\mathbf{z} = (z_1, \ldots, z_q) \in (\{0,1\}^n)^q$ and $\pi \in \Pi_n$:

$$\Pr\left[(\ell_1^3, \ldots, \ell_q^3) = \mathbf{z}\right] = \Pr\left[(\ell_1^3, \ldots, \ell_q^3) = \pi(\mathbf{z}) := (\pi(z_1), \ldots, \pi(z_q))\right] \square$$

Section 4

Applications

General paradigm

Design a scheme assuming that you have random functions, and the realize them using PRFs.

Private-key Encryption

Construction 22 (PRF-based encryption)

Given an (efficient) PRF $\mathcal{F}$, define the encryption scheme $(\mathsf{Gen}, \mathsf{E}, \mathsf{D}))$:

Key generation: $\mathsf{Gen}(1^n)$ returns $k \stackrel{R}{\leftarrow} \mathcal{F}_n$

Encryption: $\mathsf{E}_k(m)$ returns $U_n, k(U_n) \oplus m$

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- Advantages over the PRG based scheme?
- Proof of security?

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- We constructed PRFs and PRPs from length-doubling PRG (and thus from one-way functions)

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  or at least, a less adaptive one