Application of Information Theory, Lecture 2
Joint & Conditional Entropy, Mutual Information

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Nov 4, 2014

Part I

Joint and Conditional Entropy

Joint entropy

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Joint entropy, cont.

- The joint entropy of $(X_1, \ldots, X_n) \sim p$, is

$$H(X_1, \ldots, X_n) = -\sum_{x_1, \ldots, x_n} p(x_1, \ldots, x_n) \log p(x_1, \ldots, x_n)$$

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&= -\mathop{\mathrm{E}}_{(X,Y)} \log p_{Y|X}(Y|X) \\
&= -\mathop{\mathrm{E}}_{Z=p_{Y|X}(Y|X)} \log Z
\end{aligned}$$

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$$\begin{aligned} H(X|Y) &= \mathop{\mathrm{E}}_{y \leftarrow Y} H(X|Y=y) \\ &= \frac{3}{4}H(X|Y=0) + \frac{1}{4}H(X|Y=1) \\ &= \frac{3}{4}H(\frac{1}{3},\frac{2}{3}) + \frac{1}{4}H(1,0) = 0.6887 \neq H(Y|X). \end{aligned}$$

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- $H(Y|X) = H(X, Y) - H(X)$ is as an alternative definition for $H(Y|X)$.

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&\quad = H(X) + H(Y|X).
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- Another proof. Let $(X, Y) \sim p$.

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For rvs X, Y, it holds that $H(X,Y) = H(X) + H(Y|X)$.

- Proof immediately follow by the grouping axiom:

$X \backslash Y$			
	$P_{1,1}$	$\dots$	$P_{1,n}$
	$\vdots$	$\vdots$	$\vdots$
	$P_{n,1}$	$\dots$	$P_{n,n}$

Let $q_i = \sum_{j=1}^{n} p_{i,j}$

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Chain rule (for the entropy function), general case

Claim 2

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Applications cont.

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$\Longrightarrow \quad t \geq \log n! = \Theta(n \log n)$

Concavity of entropy function

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Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ be two distributions, and for $\lambda \in [0,1]$ consider the distribution $\tau_\lambda = \lambda p + (1-\lambda)q$.
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We are now certain that we drew the graph of the (two-dimensional) entropy function right...

Part II

Mutual Information

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- $I(X;Y)$ — the "information" that X gives on Y

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- $I(X;Y|Z) = I(Y;X|Z)$ (since $I(X';Y') = I(Y';X')$)

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Chain rule for mutual information

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Claim 4 (Chain rule for mutual information)

For rvs $X_1, \ldots, X_k, Y$, it holds that
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Proof: ? HW

Examples

- Let $X_1,\ldots,X_n$ be iid with $X_i \sim (p, 1-p)$, under the condition that $\bigoplus_i x_i = 0$.
  Compute $I(X_1,\ldots,X_{n-1};X_n)$.

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  By chain rule

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- Let $X_1, \ldots, X_n$ be iid with $X_i \sim (p, 1-p)$, under the condition that $\bigoplus_i x_i = 0$.

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$$\begin{aligned}
I(T; F) &= H(T) - H(T|F) \\
&= \log 6 - \log 4 \\
&= \log 3 - 1.
\end{aligned}$$

Part III

Data processing

Data processing Inequality

Definition 5 (Markov Chain)

Rvs $(X,Y,Z) \sim p$ form a Markov chain, denoted $X \rightarrow Y \rightarrow Z$, if $p(x,y,z) = p_X(x) \cdot p_{Y|X}(y|x) \cdot p_{Z|Y}(z|y)$, for all x, y, z.

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- Since $I(X; Y|Z) \geq 0$, we conclude $I(X; Y) \geq I(X; Z)$.$\square$

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For any rvs X and Y, and any (even random) g, it holds that

$$h(P_e) + P_e \log |\mathcal{X}| \geq H(X|\hat{X}) \geq H(X|Y)$$

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