Application of Information Theory, Lecture 4

Asymptotic Equipartition Property, Data Compression & Gambling

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Part I

Asymptotic Equipartition Theorem

Entropy as # of bits to describe random variable

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- It takes about $n \cdot h(k/n)$ bits to describe a string of k zeros in $\{0,1\}^n$.

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- $\log \mathbf{p}(x_1, \ldots, x_n) = \log \prod_i p(x_i) = \sum_i \log p(x_i)$

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- $\log \mathbf{p}(x_1, \ldots, x_n) = \log \prod_i p(x_i) = \sum_i \log p(x_i)$
- Hence,
  $\mathrm{E}_{X_1,\ldots,X_n}\left[-\log \mathbf{p}(X_1, \ldots, X_n)\right] = -\sum_i \mathrm{E}\left[\log p(X_i)\right] = H(X_1, \ldots, X_n)$

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- We will show that w.h.p. $-\log \mathbf{p}(X_1, \ldots, X_n)$ is close to its expectation

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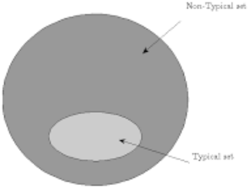

Part II

Data Compression

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- So $H(X_1, \ldots, X_n)$ is approximately the number of bits it takes to describe $X_1, \ldots, X_n$

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- We focus on binary prefix codes ($\Sigma = \{0, 1\}$)

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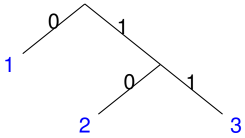

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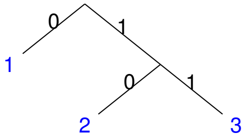

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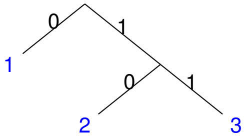

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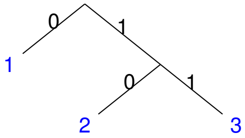

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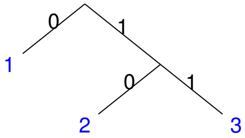

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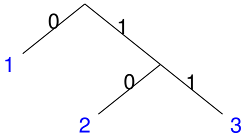

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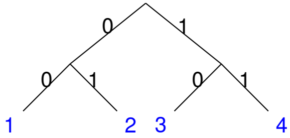

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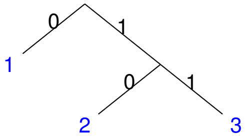

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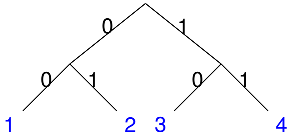

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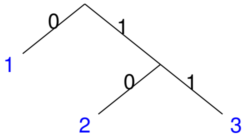

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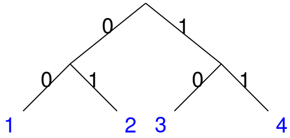

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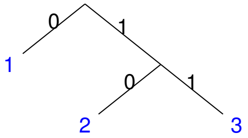

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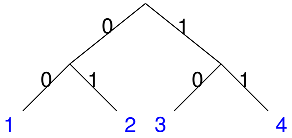

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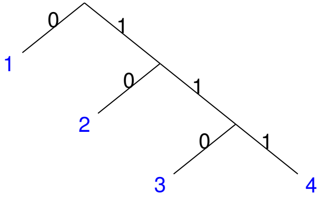

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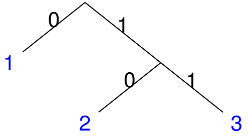

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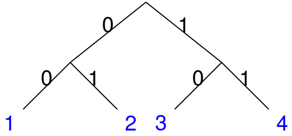

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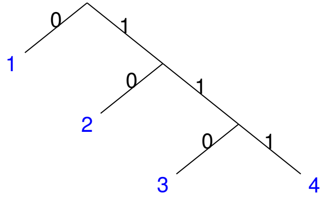

- All are prefix codes: no codeword is a prefix of another codeword

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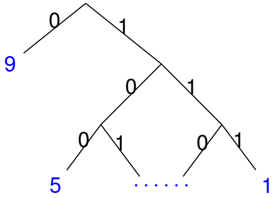

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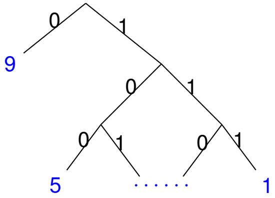

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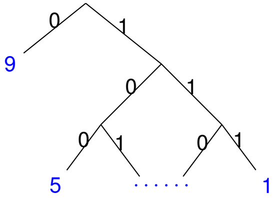

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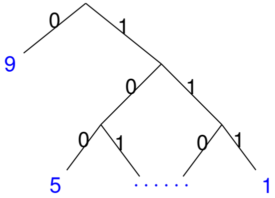

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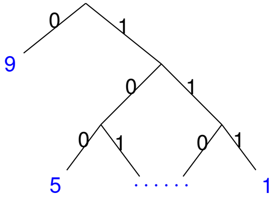

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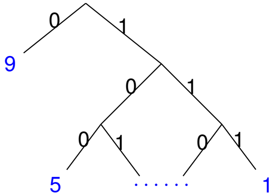

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- Encoding/decoding is clear (and highly efficient)

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- It turns out that $H(X) \leq L(X) \leq H(X) + 1$!

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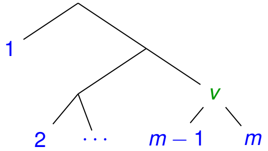

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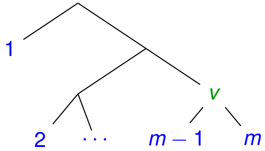

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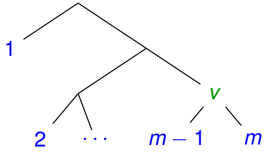

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- $L_X(T) = L_{X'}(T') + (p_{m-1} + p_m) \cdot 1$, for $X' \sim (p_1, \ldots, p_{m-1} + p_m)$

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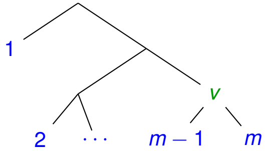

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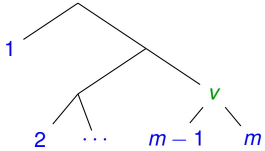

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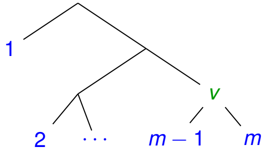

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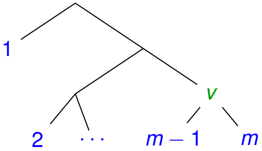

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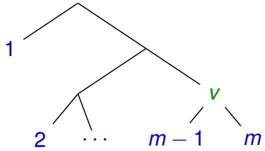

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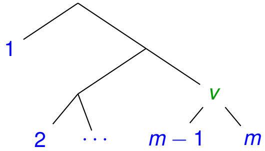

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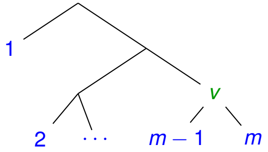

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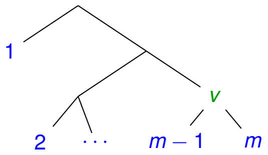

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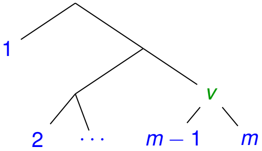

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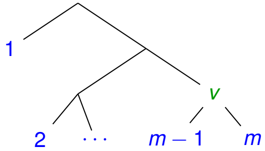

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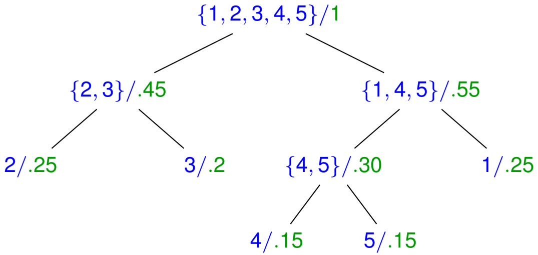

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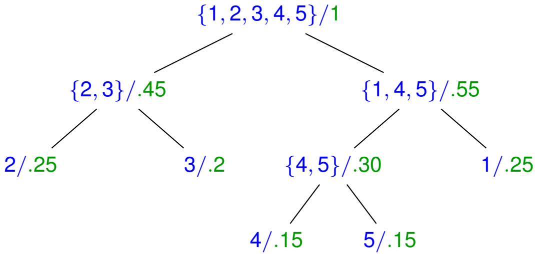

- On board...

Kraft inequality

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$\implies$ At beginning of step i exists an available depth-ℓ_i node.

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Theorem 3

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Algorithm G generates the rv $X \sim \{p_1, \ldots, p_m\}$ if the following holds: in each step, G either stops or flips a coin $\sim (q_i, 1 - q_i)$.[a] After it stop, G outputs a value in $\mathbb{N}$. The probability that G outputs i is p_i.

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Let X be rv, and let $g(X)$ be the expected number of coins used by its best generating algorithm. Then $H(X) \leq g(X) < H(X) + 1$. If each p_i is a power of 2 (i.e., 2^{-k} for some $k \in \mathbb{Z}$), then $g(X) = H(X)$.

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Proposition 6 (proof omitted)

Let X be a rv, and let $g_b(X)$ be the expected number of coins used by its best generating algorithm that only flips uniform coins. Then $H(X) \leq g_b(X) \leq H(X) + 2$.

Part III

Gambling

Horse racing

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For gambling strategy $\mathbf{b} = (b_1, \ldots, b_m)$, and race outcome distribution $\mathbf{p} = (p_1, \ldots, p_m)$, $S_n := \prod_{i=1}^{n} S(X_i) = \prod_{i=1}^{n} \mathbf{b}(X_i)\mathbf{o}(X_i)$, where X_i's are iid $\sim p$

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- $\log S(X_1), \ldots, \log S(X_n)$ are iid
- By weak low of large numbers,

$$\frac{1}{n} \log S_n = \frac{1}{n} \sum_i \log(S(X_i)) \xrightarrow{n} \mathrm{E}(\log S(X_1)) = W(\mathbf{b}, \mathbf{p})$$

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Let $W^*(\mathbf{p}) = \max_{\boldsymbol{b}} W(\boldsymbol{b}, \mathbf{p})$, *then* $W^*(\mathbf{p}) = W(\mathbf{p}, \mathbf{p}) = \sum_i p_i \log o_i - H(\mathbf{p})$

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$$\begin{aligned} W(\mathbf{b}, \mathbf{p}) &= \sum_{i=1^m} p_i \log(b_i o_i) \\ &= \sum_i p_i \log\left(\frac{b_i}{p_i} p_i o_i\right) \end{aligned}$$

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- Hence, $\Delta W = H(X) - H(X|Y) = I(X; Y)$. □