

Foundation of Cryptography, Lecture 5

MACs and Signatures¹

Handout Mode

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Part I

Message Authentication Codes (MACs)

Message Authentication Code (MACs)

Definition 1 (MAC)

A triplet of PPT's $(\text{Gen}, \text{Mac}, \text{Vrfy})$ such that:

1. $\text{Gen}(1^n)$ outputs a key $k \in \{0, 1\}^*$
2. $\text{Mac}(k, m)$ outputs a "tag" t
3. $\text{Vrfy}(k, m, t)$ output 1 (YES) or 0 (NO)

Consistency:

$$\forall k \in \text{Supp}(\text{Gen}(1^n)), m \in \{0, 1\}^n \text{ and } t = \text{Mac}_k(m): \quad \text{Vrfy}_k(m, t) = 1$$

Definition 2 (Existential unforgeability)

A MAC $(\text{Gen}, \text{Mac}, \text{Vrfy})$ is **existential unforgeable** (EU), if \forall PPT A :

$$\Pr_{\substack{k \leftarrow \text{Gen}(1^n) \\ (m, t) \leftarrow A^{\text{Mac}_k, \text{Vrfy}_k(1^n)}}} [\text{Vrfy}_k(m, t) = 1 \wedge \text{Mac}_k \text{ was not asked on } m] = \text{neg}(n)$$

Remark: convention

Definition of MAC cont.

- ▶ “Private key” definition
- ▶ Security definition too strong? Any message? Use of Verifier?
- ▶ “Replay attacks”
- ▶ **Strong existential unforgeable MACS** (for short, strong MAC): infeasible to generate **new** valid tag (even for message for which a MAC was asked)

Restricted MACs

Definition 3 (Length-restricted MAC)

Same as in Definition 1, but for $k \in \text{Supp}(G(1^n))$, Mac_k and Vrfy_k only accept messages of length n .

Definition 4 (ℓ -time MAC)

A MAC scheme is **existential unforgeable against ℓ queries** (for short, ℓ -time MAC), if it is existential unforgeable as in Definition 2, but A can only make ℓ queries.

Section 1

Constructions

One-time length-restricted MAC

Construction 5 (One-time MAC)

- ▶ $\text{Gen}(1^n)$: output $k \leftarrow \{0, 1\}^n$.
- ▶ $\text{Mac}_k(m)$: output $h_k(m)$.
- ▶ $\text{Vrfy}_k(m, t)$: output 1 iff $t = h_k(m)$.

Claim 6

The scheme is one-time MAC if $\{h_k\}$ is pairwise-independent.

Subsection 1

Restricted-Length MAC

ℓ -wise independent functions

Definition 7 (ℓ -wise independent)

A function family \mathcal{H} from $\{0, 1\}^n$ to $\{0, 1\}^m$ is ℓ -wise independent, if for every *distinct* $x_1, \dots, x_\ell \in \{0, 1\}^n$ and every $y_1, \dots, y_\ell \in \{0, 1\}^m$, it holds that $\Pr_{h \leftarrow \mathcal{H}} [h(x_1) = y_1 \wedge \dots \wedge h(x_\ell) = y_\ell] = 2^{-\ell m}$.

ℓ -times, restricted-length MAC

Construction 8 (ℓ -time MAC)

Let $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^n \mapsto \{0, 1\}^n\}$ be an efficient $(\ell + 1)$ -wise independent function family.

- ▶ $\text{Gen}(1^n)$: output $h \leftarrow \mathcal{H}_n$.
- ▶ $\text{Mac}(h, m)$: output $h(m)$.
- ▶ $\text{Vrfy}(h, m, t)$: output 1 iff $t = h(m)$.

Claim 9

The above scheme is a length-restricted, ℓ -time MAC

Proof: ?

OWF \implies restricted-length MAC

Construction 10

Same as Construction 8, but uses function $\mathbb{F} = \{\mathbb{F}_n: \{0, 1\}^n \mapsto \{0, 1\}^n\}$ instead of \mathcal{H} .

Claim 11

Assuming that \mathbb{F} is a PRF, then Construction 10 is an existential unforgeable MAC.

Proof: Easy to prove if \mathbb{F} is a family of random functions. Hence, also holds in case \mathbb{F} is a PRF. \square

Subsection 2

Any Length

Collision Resistant Hash Family

Definition 12 (collision resistant hash family (CRH))

A function family $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^* \mapsto \{0, 1\}^n\}$ is **collision resistant**, if

$$\Pr_{h \leftarrow \mathcal{H}_n} [A(1^n, h) = (x, x') \text{ s.t. } x \neq x' \wedge h(x) = h(x')] = \text{neg}(n)$$

for any PPT A .

- **Not** known to implied by OWFs.

Length-restricted MAC \Rightarrow MAC

Construction 13 (Length restricted MAC \Rightarrow MAC)

Let $(\text{Gen}, \text{Mac}, \text{Vrfy})$ be a length-restricted MAC, and let $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^* \mapsto \{0, 1\}^n\}$ be an efficient function family.

- ▶ $\text{Gen}'(1^n)$: Sample $k \leftarrow \text{Gen}(1^n)$ and $h \leftarrow \mathcal{H}_n$. Output $k' = (k, h)$
- ▶ $\text{Mac}'_{k,h}(m) = \text{Mac}_k(h(m))$
- ▶ $\text{Vrfy}'_{k,h}(t, m) = \text{Vrfy}_k(t, h(m))$

Claim 14

Assume \mathcal{H} is an efficient collision-resistant family and $(\text{Gen}, \text{Mac}, \text{Vrfy})$ is existential unforgeable, then $(\text{Gen}', \text{Mac}', \text{Vrfy}')$ is existential unforgeable MAC.

Proof: ?

Part II

Signature Schemes

Signature schemes

Definition 15 (Signature schemes)

A trippet of PPT's $(\text{Gen}, \text{Sign}, \text{Vrfy})$ such that

1. $\text{Gen}(1^n)$: output a pair of keys $(s, v) \in \{0, 1\}^* \times \{0, 1\}^*$
2. $\text{Sign}(s, m)$: output a "signature" $\sigma \in \{0, 1\}^*$
3. $\text{Vrfy}(v, m, \sigma)$: output 1 (YES) or 0 (NO)

Consistency: for any $(s, v) \in \text{Supp}(\text{Gen}(1^n))$, $m \in \{0, 1\}^*$ and $\sigma \in \text{Supp}(\text{Sign}_s(m))$: $\text{Vrfy}_v(m, \sigma) = 1$.

Definition 16 (Existential unforgeability)

A signature scheme is **existential unforgeable** (EU), if \forall PPT A

$$\Pr_{(s,v) \leftarrow \text{Gen}(1^n)} \left[A^{\text{Sign}_s}(1^n, v) = (m, \sigma) \text{ s.t } \text{Vrfy}_v(m, \sigma) = 1 \wedge \text{Sign}_s \text{ didn't query } m \right]$$

is negligible in n .

Signature schemes cont.

- ▶ Signature \Rightarrow MAC
- ▶ “Harder” to construct than MACs: (even restricted forms) require OWF
- ▶ Oracle access to V_{rfy} is not given
- ▶ **Strong existential unforgeable signatures** (for short, strong signatures): infeasible to generate **new** valid signatures (even for message for which a signature was asked)

Theorem 17

OWFs imply strong existential unforgeable signatures.

Section 2

OWFs \Rightarrow Signatures

Subsection 1

One-time signatures

Length-restricted signatures

Definition 18 (length-restricted signatures)

Same as in Definition 15, but for $(s, v) \in \text{Supp}(G(1^n))$, Sign_s and Vrfy_v only accept messages of length n .

Bounded-query signatures

Definition 19 (ℓ -time signatures)

A signature scheme is **existential unforgeable against ℓ -query** (for short, ℓ -time signature), if it is existential unforgeable as in **Definition 16**, but **A** can only ask for ℓ queries.

Claim 20

Assuming CRH exists, then length restricted k -time signatures can be used to construct k -time signatures.

Proof: ?

Proposition 21

Wlg, the signer of a k -time signature scheme, for fixed k , is **deterministic**

Proof: ?

OWF \implies length-restricted one-time signatures

Construction 22 (length-restricted, one-time signature)

Let $f: \{0, 1\}^n \mapsto \{0, 1\}^n$.

1. $\text{Gen}(1^n)$:

1.1 $s_1^0, s_1^1, \dots, s_n^0, s_n^1 \leftarrow \{0, 1\}^n$.

1.2 Secret (signing) key is $s = \{s_i^0, s_i^1\}_{i=1}^n$

1.3 Public (verification) is $v = \{v_i^0, v_i^1\}_{i=1}^n$ for $v_i^b = f(s_i^b)$.

2. $\text{Sign}(s, m)$: $\sigma = (s_1^{m_1}, \dots, s_n^{m_n})$

3. $\text{Vrfy}(v, m, \sigma = (\sigma_1, \dots, \sigma_n))$: check that $f(\sigma_i) = v_i^{m_i}$ for all $i \in [n]$

Lemma 23

If f is a OWF, then *Construction 22* is a length restricted one-time signature scheme.

Is this a strong signature scheme? With some additional work, it can be turned into a strong one.

Proving Lemma 23

Let a PPT A , $\mathcal{I} \subseteq \mathbb{N}$ and $p \in \text{poly}$ that break the security of Construction 22, we use A to invert f .

Algorithm 24 (Inv)

Input: $y \in \{0, 1\}^n$

1. Choose $(s, v) \leftarrow \text{Gen}(1^n)$ and replace $v_{i^*}^{b^*}$ for a random $i^* \in [n]$ and $b^* \in \{0, 1\}$, with y .
2. Abort, if $A(1^n, v)$ asks to sign message $m \in \{0, 1\}^n$ with $m_{i^*} = b^*$. Otherwise, use s to answer the query.
3. Let (m', σ') be A 's output.
Abort, if σ' is not a valid signature for m' , or $m'_{i^*} \neq b^*$.
Otherwise, return σ_{i^*} .

- ▶ v is distributed as is in the real “signature game”
- ▶ v is independent of i^* and b^* .
- ▶ Therefore Inv inverts f w.p. $\frac{1}{2np(n)}$ for every $n \in \mathcal{I}$.

Subsection 2

Stateful Schemes

Stateful signature schemes²

Definition 25 (Stateful scheme)

Same as in Definition 15, but Sign might keep state which is updated every signature.

- ▶ Make sense in many applications (e.g., smartcards)
- ▶ We'll later use it a building block for building stateless scheme

²Also known as memory-dependant schemes

Stateful schemes — straight-line construction

Let $(\text{Gen}, \text{Sign}, \text{Vrfy})$ be a strong **one-time** signature scheme.

Construction 26 (straight-line construction)

- ▶ $\text{Gen}'(1^n)$: Output $(s', v') = (s_1, v_1) \leftarrow \text{Gen}(1^n)$.
- ▶ $\text{Sign}'_{s_1}(m_i)$, where m_i is i 'th message to sign:
 1. Let $(s_{i+1}, v_{i+1}) \leftarrow \text{Gen}(1^n)$
 2. Let $\sigma_i = \text{Sign}_{s_i}(m_i, v_{i+1})$
 3. Output $\sigma'_i = (\sigma'_{i-1}, m_i, v_{i+1}, \sigma_i)$.^a
- ▶ $\text{Vrfy}'_{v_1}(m, \sigma' = (m_1, v_2, \sigma_1), \dots, (m_i, v_{i+1}, \sigma_i))$:
Check that
 1. $\text{Vrfy}_{v_j}((m_j, v_{j+1}), \sigma_j) = 1$ for every $j \in [i]$
 2. $m_i = m$

^a σ'_0 is the empty string.

Straight-line construction cont.

- ▶ The state of Sign' is used for maintaining the most recent signing key (e.g., s_i), and the last published signature that connects s_i to v_1 .
- ▶ While polynomial time, it is rather inefficient scheme: both running time and signature size are linear in number of published signatures.
- ▶ That $(\text{Gen}, \text{Sign}, \text{Vrfy})$ works for any length (specifically, it is possible to sign message that is longer than the verification key), is critically used.

Lemma 27

$(\text{Gen}', \text{Sign}', \text{Vrfy}')$ is a stateful, strong signature scheme.

Proof: Assume \exists PPT A' , $p \in \text{poly}$ and infinite set $\mathcal{I} \subseteq \mathbb{N}$, such that A' breaks the strong security of $(\text{Gen}', \text{Sign}', \text{Vrfy}')$ with probability $\frac{1}{p(n)}$ for all $n \in \mathcal{I}$. We present PPT A that breaks the security of $(\text{Gen}, \text{Sign}, \text{Vrfy})$.

- ▶ We assume for simplicity that p also bounds the query complexity of A'

Proving Lemma 27 cont.

Let $(m_t, \sigma' = (m_1, v_2, \sigma_1), \dots, (m_t, v_{t+1}, \sigma_t))$ be the pair output by A'

Claim 28

Whenever A' succeeds, $\exists \tilde{i} \in [p]$ such that:

1. Sign' has output $\sigma'_{\tilde{i}-1} = (m_1, v_2, \sigma_1), \dots, (m_{\tilde{i}-1}, v_{\tilde{i}}, \sigma_{\tilde{i}-1})$
2. Sign' has not output $\sigma'_i = (m_1, v_2, \sigma_1), \dots, (m_{\tilde{i}}, v_{\tilde{i}+1}, \sigma_{\tilde{i}})$

Proof: ?

It follows that

- ▶ $v_{\tilde{i}}$ was sampled by Sign'

Let $s_{\tilde{i}}$ be the signing key generated by Sign' along with $v_{\tilde{i}}$, and let $\tilde{m} = (m_{\tilde{i}}, v_{\tilde{i}+1})$

- ▶ $\text{Vrfy}_{v_{\tilde{i}}}(\tilde{m}, \sigma_{\tilde{i}}) = 1$
- ▶ $\text{Sign}_{s_{\tilde{i}}}$ was not queried by Sign' on \tilde{m} and output $\sigma_{\tilde{i}}$.
- ▶ $\text{Sign}_{s_{\tilde{i}}}$ was queried at most once by Sign'

Definition of A

Algorithm 29 (A)

Input: $1^n, v$

Oracle: Sign_s

1. Choose $i^* \leftarrow [p = p(n)]$ and $(s', v') \leftarrow \text{Gen}'(1^n)$.
 2. Emulate a random execution of $A'^{\text{Sign}'_{s'}}$ with a single twist:
 - ▶ On the i^* 'th call to $\text{Sign}'_{s'}$, set $v_{j^*} = v$ (rather than choosing it via Gen)
 - ▶ When need to sign using s_{j^*} , use Sign_s .
 3. Let $(m, \sigma = (m_1, v_1, \sigma_1), \dots, (m_q, v_q, \sigma_q)) \leftarrow A'$
 4. Output $((m_{j^*}, v_{j^*}), \sigma_{j^*})$ (abort if $i^* > q$)
-
- ▶ The emulated game $A'^{\text{Sign}'_{s'}}$ has the same distribution as the real game.
 - ▶ Sign_s is called at most once
 - ▶ A breaks $(\text{Gen}, \text{Sign}, \text{Vrfy})$ whenever $i^* = \tilde{i}$.

Subsection 3

Somewhat-Stateful Schemes

A somewhat-stateful scheme

Let $(\text{Gen}, \text{Sign}, \text{Vrfy})$ be a strong one-time signature scheme.

Construction 30 (A somewhat-stateful scheme)

- ▶ $\text{Gen}'(1^n)$: Output $(s', v') = (s_\lambda, v_\lambda) \leftarrow \text{Gen}(1^n)$.
- ▶ $\text{Sign}'_{s_\lambda}(m)$: choose an **unused** $r \in \{0, 1\}^n$
 1. For $i = 0$ to $n - 1$: if $a_{r_{1,\dots,i}}$ **was not** set before:
 - 1.1 For both $j \in \{0, 1\}$, let $(s_{r_{1,\dots,i,j}}, v_{r_{1,\dots,i,j}}) \leftarrow \text{Gen}(1^n)$
 - 1.2 Let $a_{r_{1,\dots,i}} = (v_{r_{1,\dots,i,0}}, v_{r_{1,\dots,i,1}})$.
 - 1.3 Let $\sigma_{r_{1,\dots,i}} = \text{Sign}_{s_{r_{1,\dots,i}}}(a_{r_{1,\dots,i}})$
 2. Output $(r, a_\lambda, \sigma_\lambda, \dots, a_{r_{1,\dots,n-1}}, \sigma_{r_{1,\dots,n-1}}, \sigma_r = \text{Sign}_{s_r}(m))$
- ▶ $\text{Vrfy}'_{v_\lambda}(m, \sigma' = (r, a_\lambda, \sigma_\lambda, \dots, a_{r_{1,\dots,n-1}}, \sigma_{r_{1,\dots,n-1}}, \sigma_r))$

Check that

1. $\text{Vrfy}_{v_{r_{1,\dots,i}}}(a_{r_{1,\dots,i}}, \sigma_{r_{1,\dots,i}}) = 1$ for every $i \in \{0, \dots, n - 1\}$
2. $\text{Vrfy}_{v_r}(m, \sigma_r) = 1$, for $v_r = (a_{r_{1,\dots,n-1}})_{r_n}$

A somewhat-stateful Scheme, cont.

- ▶ Each one-time signature key is used at most once.

Lemma 31

$(\text{Gen}', \text{Sign}', \text{Vrfy}')$ is a stateful strong signature scheme.

Proof: ?

- ▶ Note that Sign' does not keep track of the message history.
- ▶ More efficient scheme — Enough to construct tree of depth $\omega(\log n)$ (i.e., to choose $r \in \{0, 1\}^{\ell \in \omega(\log n)}$)

Subsection 4

Stateless Schemes

Stateless Scheme

Let $\tilde{\Pi}_k$ be the set of all functions from $\{0, 1\}^*$ to $\{0, 1\}^k$, let $q \in \text{poly}$ be “large enough”, and let $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^* \mapsto \{0, 1\}^n\}$ be a CRH.

Construction 32 (Inefficient stateless Scheme)

- ▶ $\text{Gen}'(1^n)$: Sample $(s_\lambda, v_\lambda) \leftarrow \text{Gen}(1^n)$, $\pi \leftarrow \tilde{\Pi}_{q(n)}$ and $h \leftarrow \mathcal{H}_n$.
Output $(s' = (s_\lambda, \pi, h), v' = v_\lambda)$.
 - ▶ $\text{Sign}'_s(m)$: Set $r = \pi(h(m))_{1,\dots,n}$.
 1. For $i = 0$ to $n - 1$:
 - 1.1 For both $j \in \{0, 1\}$, let $(s_{r_{1,\dots,i,j}}, v_{r_{1,\dots,i,j}}) \leftarrow \text{Gen}(1^n; \pi(r_{1,\dots,i,j}))$
 - 1.2 Let $\sigma_{r_{1,\dots,i}} = \text{Sign}_{s_{r_{1,\dots,i}}}(a_{r_{1,\dots,i}} = (v_{r_{1,\dots,i},0}, v_{r_{1,\dots,i},1}))$
 2. Output $(r, a_\lambda, \sigma_\lambda, \dots, a_{r_{1,\dots,n-1}}, \sigma_{r_{1,\dots,n-1}}, \sigma_r = \text{Sign}_{s_r}(m))$
 - ▶ Vrfy' : unchanged
-
- ▶ One one-time signature key might be used **several times**, but always on **the same message**.
 - ▶ Efficient scheme: use PRF (?)

Subsection 5

“CRH free” Schemes

Target collision-resistant functions

Definition 33 (target collision-resistant functions (TCR))

A function family $\mathcal{H} = \{\mathcal{H}_n: \{0, 1\}^* \mapsto \{0, 1\}^n\}$, if

$$\Pr_{(x,a) \leftarrow A_1(1^n); h \leftarrow \mathcal{H}_n; x' \leftarrow A_2(a,h)} [x \neq x' \wedge h(x) = h(x')] = \text{neg}(n)$$

for any pair of PPT's A_1, A_2 .

Theorem 34

OWFs imply efficient compressing TCRs.

Proof: not that trivial...

Target one-time signatures

For simplicity we will focus on non-strong schemes.

Definition 35 (target one-time signatures)

A signature scheme $(\text{Gen}, \text{Sign}, \text{Vrfy})$ is **target one-time existential unforgeable** (for short, target one-time signature), if

$$\Pr_{\substack{m \leftarrow A(1^n) \\ (s, v) \leftarrow \text{Gen}(1^n) \\ (m', \sigma) \leftarrow A(\text{Sign}_s(m))}} [m' \neq m \wedge \text{Vrfy}_v(m', \sigma) = 1] = \text{neg}(n)$$

for any PPT A

Claim 36

OWFs imply target one-time signatures.

Random one-time signatures

Definition 37 (random one-time signatures)

A signature scheme $(\text{Gen}, \text{Sign}, \text{Vrfy})$ is **random one-time existential unforgeable** (for short, random one-time signature), if

$$\Pr_{\substack{m \leftarrow \mathcal{M}_n; (s, v) \leftarrow \text{Gen}(1^n) \\ (m', \sigma) \leftarrow \text{A}(m, \text{Sign}_s(m))}} [m' \neq m \wedge \text{Vrfy}_v(m', \sigma) = 1] = \text{neg}(n)$$

for any PPT A and any efficiently samplable string ensemble $\mathcal{M} = \{\mathcal{M}_n\}_{n \in \mathbb{N}}$.

Claim 38

Assume $(\text{Gen}, \text{Sign}, \text{Vrfy})$ is target one-time signature scheme, then it is random one-time signature scheme.

“CRH free” schemes

Lemma 39

If $(\text{Gen}, \text{Sign}, \text{Vrfy})$ and \mathcal{H} in Construction 32 are target-one-time signature scheme and TCR respectively, then it is a signature scheme.

Proof:

Focus on the target-one-time signatures.

Show that

- ▶ An adversary cannot make the signer use the *same* r for signing two **different** messages.
- ▶ Random-one-time signatures suffice for the **nodes** signatures
- ▶ Target-one-time signatures suffice for the **leaves** signatures