Application of Information Theory, Lecture 5

Channel Capacity, Isoperimetric inequality

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November 25, 2014

Part I

Channel Capacity

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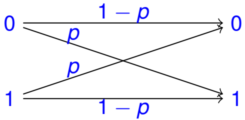

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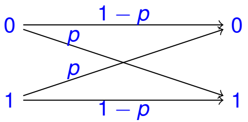

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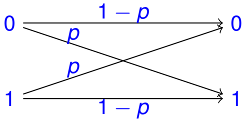

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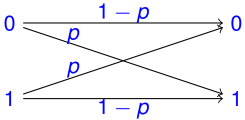

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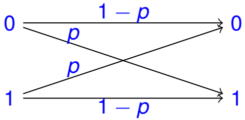

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- Can we send bits with smaller error?

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- Before Shannon it was believed that very small error rate requires very small transamination rate.

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- A revolution in EE and the whole world

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Shannon's theorem

Theorem 1

$\forall p \quad \exists C_p,$ *s.t.* $\forall \varepsilon > 0 \quad \exists m_\varepsilon,$ *s.t.* $\forall m > m_\varepsilon$ *and* $n > m(\frac{1}{C_p} + \varepsilon)$,
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- We prove a weaker variant that holds w.h.p. over $\mathbf{x} \leftarrow \{0,1\}^m$

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- $|\mathbf{y} - \mathbf{y}'| = |\mathbf{y} \oplus \mathbf{y}'|$ — Hamming distance of $\mathbf{y}$ from $\mathbf{y}'$; # of places differ.

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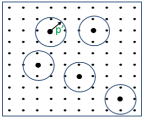

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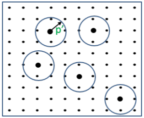

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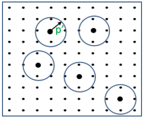

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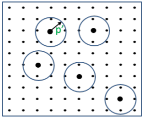

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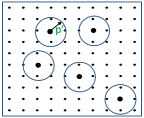

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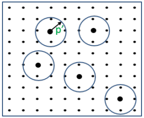

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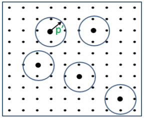

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- Hence, for $m > m_\varepsilon = \max\{m', n'\}$ and $n > m(\frac{1}{C_p} + \varepsilon)$, it holds that $\Pr_{\mathbf{x} \leftarrow \{0,1\}^m}\left[g(f(\mathbf{x}) \oplus Z) \neq \mathbf{x}\right] \le \varepsilon$ $\square$

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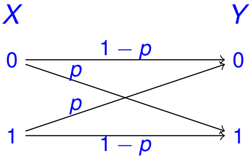

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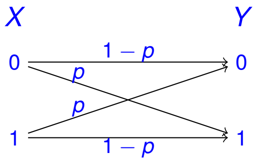

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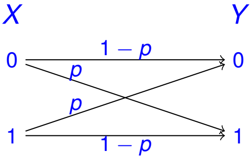

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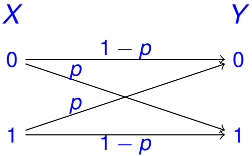
Why $C_p = 1 - h(p)$?

- Let $X \leftarrow \{0,1\}$, $Z \sim (1-p, p)$ and $Y = X \oplus Z$



- $I(X;Y) = H(Y) - H(Y|X) = H(Y) - H(Z) = 1 - h(p) = C_p$
- Received bit “gives" C_p information about transmitted bit
- Hence, to recover m bits, we need to send at least $m \cdot \frac{1}{C_p}$ bits

Size of bounding ball

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Claim 2

For $p \in [0, \frac{1}{2}]$ and $n \in \mathbb{N}$: it holds that $\sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{n \cdot h(p)}$

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For $y \in \{0,1\}^n$ and $p \in [0, \frac{1}{2}]$, let $B_p(y) = \{y \in \{0,1\}^n \colon |y' - y| \leq pn\}$. Then $|B_p(y)| = \sum_{k=0}^{\lfloor pn \rfloor} \binom{n}{k} \leq 2^{n \cdot h(p)}$

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Very useful estimation. Weaker variants follows by AEP or Stirling,

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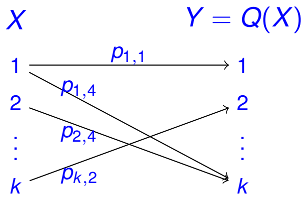
$Q\colon [k] \mapsto [k]$ that channel (a probabilistic function)

$p_{i,j} = \Pr[Q(i) = j]$



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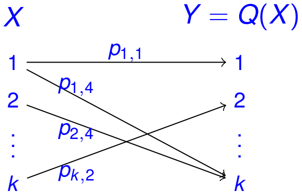

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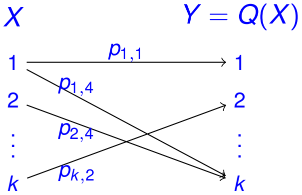

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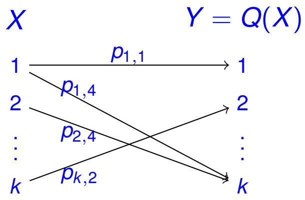

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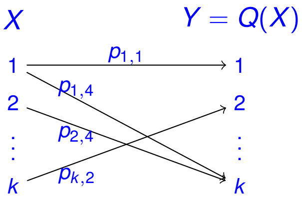

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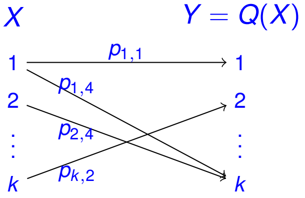

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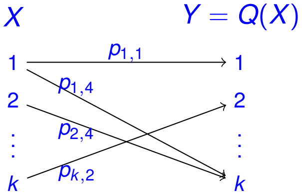

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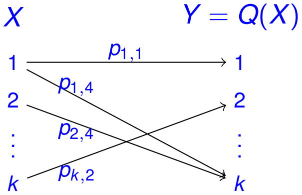

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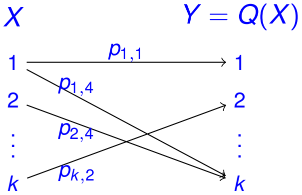

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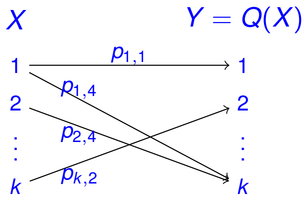

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- Proof: similar lines to the binary case, but more subtle distribution for f

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- Combination of data compression and ECC

Part II

Combinatorial Applications

Movies

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- Every pair of movies was seen by at least 90% of the people
- Claim: there exist two people who saw exactly the same set of movies
- $X \leftarrow [2^n]$ — a random person
- $g_i(x) = \begin{cases} 1, & x \text{ saw movie } i; \\ 0, & \text{otherwise.} \end{cases}$
- $Y_i = g_i(X)$
- $\forall i, j$: $H(Y_i, Y_j) \leq H(.9, \frac{.1}{3}, \frac{.1}{3}, \frac{.1}{3}) \leq \frac{2}{3}$
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- Hence, X is not determined by Y

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- Very useful inequality. No Chernoff, just IT

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Lemma 5

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