

Application of Information Theory, Lecture 2

Joint & Conditional Entropy, Mutual Information

Handout Mode

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Part I

Joint and Conditional Entropy

Joint entropy

- Recall that the entropy of rv X , is defined by

$$H(X) = - \sum_{x \in \mathcal{X}} P_X(x) \log P_X(x)$$

- Shorter notation: for $X \sim p$, let $H(X) = - \sum_x p(x) \log p(x)$ (where the summation is over the domain of X).
- The **joint entropy** of (jointly distributed) rvs X and Y with $(X, Y) \sim p$, is

$$H(X, Y) = - \sum_{x,y} p(x, y) \log p(x, y)$$

This is simply the entropy of the rv $Z = (X, Y)$.

- Example:

$X \backslash Y$	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{2}$	0

$$\begin{aligned} H(X, Y) &= -\frac{1}{2} \log \frac{1}{2} - \frac{1}{4} \log \frac{1}{4} - \frac{1}{4} \log \frac{1}{4} \\ &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = 1\frac{1}{2} \end{aligned}$$

Joint entropy, cont.

- ▶ The joint entropy of $(X_1, \dots, X_n) \sim p$, is

$$H(X_1, \dots, X_n) = - \sum_{x_1, \dots, x_n} p(x_1, \dots, x_n) \log p(x_1, \dots, x_n)$$

Conditional entropy

- ▶ Let $(X, Y) \sim p$, let $p_X = \sum_y p(x, y)$, $p_Y = \sum_x p(x, y)$ and $p_{Y|X}(y|x) = \frac{p(x,y)}{p_X(x)}$.
- ▶ For $x \in \text{Supp}(X)$, the random variable $Y|_{X=x}$ is well defined (distributed according to $q(y) = p_{Y|X}(y|x)$).
- ▶ The entropy of Y **conditioned on** X , is defined by

$$H(Y|X) := \mathbb{E}_{x \leftarrow X} H(Y|_{X=x})$$

- ▶ Measures the **uncertainty** in Y given X .

- ▶
$$\begin{aligned} H(Y|X) &= \sum_{x \in \mathcal{X}} p_X(x) \cdot H(Y|_{X=x}) \\ &= - \sum_{x \in \mathcal{X}} p_X(x) \sum_{y \in \mathcal{Y}} p_{Y|X}(y|x) \log p_{Y|X}(y|x) \\ &= - \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} p(x, y) \log p_{Y|X}(y|x) \\ &= - \mathbb{E}_{(X,Y)} \log p_{Y|X}(Y|X) = - \mathbb{E}_{Z=p_{Y|X}(Y|X)} \log Z \end{aligned}$$

Conditional entropy, cont.

► Example

$X \backslash Y$	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{2}$	0

What is $H(Y|X)$ and $H(X|Y)$?

$$\begin{aligned} H(Y|X) &= \mathbb{E}_{x \leftarrow X} H(Y|_{X=x}) \\ &= \frac{1}{2} H(Y|_{X=0}) + \frac{1}{2} H(Y|_{X=1}) \\ &= \frac{1}{2} H\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{2} H(1, 0) = \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} H(X|Y) &= \mathbb{E}_{y \leftarrow Y} H(X|_{Y=y}) \\ &= \frac{3}{4} H(X|_{Y=0}) + \frac{1}{4} H(X|_{Y=1}) \\ &= \frac{3}{4} H\left(\frac{1}{3}, \frac{2}{3}\right) + \frac{1}{4} H(1, 0) = 0.6887 \neq H(Y|X). \end{aligned}$$

Conditional entropy, cont..



$$\begin{aligned} H(X|Y, Z) &= \mathbb{E}_{(y,z) \leftarrow (Y,Z)} H(X|_{Y=y, Z=z}) \\ &= \mathbb{E}_{y \leftarrow Y} \mathbb{E}_{z \leftarrow Z|_{Y=y}} H(X|_{Y=y, Z=z}) \\ &= \mathbb{E}_{y \leftarrow Y} \mathbb{E}_{z \leftarrow Z|_{Y=y}} H((X|_{Y=y})|_{Z=z}) \end{aligned}$$

Let $(X_y, Z_y) = (X, Z)|_{Y=y}$. Then

$$\begin{aligned} H(X|Y, Z) &= \mathbb{E}_{y \leftarrow Y} \mathbb{E}_{z \leftarrow Z_y} H(X_y|_{Z=z}) \\ &= \mathbb{E}_{y \leftarrow Y} \mathbb{E}_{z \leftarrow Z_y} H(X_y|_{Z_y=z}) \\ &= \mathbb{E}_{y \leftarrow Y} H(X_y|Z_y) \end{aligned}$$

Relating mutual entropy to conditional entropy

- ▶ What is the relation between $H(X)$, $H(Y)$, $H(X, Y)$ and $H(Y|X)$?
- ▶ Intuitively, $0 \leq H(Y|X) \leq H(Y)$

Non-negativity is immediate. We prove upperbound later.

- ▶ We will also see that $H(Y|X) = H(Y)$ iff X and Y are independent.
- ▶ In our example, $H(Y) = H(\frac{3}{4}, \frac{1}{4}) > \frac{1}{2} = H(Y|X)$
- ▶ Note that $H(Y|X = x)$ might be larger than $H(Y)$ for some $x \in \text{Supp}(X)$.
- ▶ Chain rule (proved next). $H(X, Y) = H(X) + H(Y|X)$
- ▶ Intuitively, uncertainty in (X, Y) is the uncertainty in X plus the uncertainty in Y given X .
- ▶ $H(Y|X) = H(X, Y) - H(X)$ is as an alternative definition for $H(Y|X)$.

Chain rule (for the entropy function)

Claim 1

For rvs X, Y , it holds that $H(X, Y) = H(X) + H(Y|X)$.

- Proof immediately follow by the grouping axiom:

$X \backslash Y$			
	$P_{1,1}$	\dots	$P_{1,n}$
	\vdots	\vdots	\vdots
	$P_{n,1}$	\dots	$P_{n,n}$

Let $q_i = \sum_{j=1}^n p_{i,j}$ ($= \Pr[X = i]$)

$$\begin{aligned} & H(P_{1,1}, \dots, P_{n,n}) \\ &= H(q_1, \dots, q_n) + \sum_i q_i H\left(\frac{P_{i,1}}{q_i}, \dots, \frac{P_{i,n}}{q_i}\right) \\ &= H(X) + H(Y|X). \end{aligned}$$

- Another proof. Let $(X, Y) \sim p$, and recall that $p(x, y) = p_X(x) \cdot p_{Y|X}(y|x)$.

$$\Rightarrow \log p(x, y) = \log p_X(x) + \log p_{Y|X}(y|x)$$

$$\Rightarrow \mathbb{E} \log p(X, Y) = \mathbb{E} \log p_X(X) + \mathbb{E} \log p_{Y|X}(Y|X)$$

$$\Rightarrow H(X, Y) = H(X) + H(Y|X).$$

$$H(Y|X) \leq H(Y)$$

Jensen inequality: for any concave function f , values t_1, \dots, t_k and $\lambda_1, \dots, \lambda_k \in [0, 1]$ with $\sum_i \lambda_i = 1$, it holds that $\sum_i \lambda_i f(t_i) \leq f(\sum_i \lambda_i t_i)$.
Let $(X, Y) \sim p$.

$$\begin{aligned} H(Y|X) &= - \sum_{x,y} p(x, y) \log p_{Y|X}(y|x) \\ &= \sum_{x,y} p(x, y) \log \frac{p_X(x)}{p(x, y)} \\ &= \sum_{x,y} p_Y(y) \cdot \frac{p(x, y)}{p_Y(y)} \log \frac{p_X(x)}{p(x, y)} \\ &= \sum_y p_Y(y) \sum_x \frac{p(x, y)}{p_Y(y)} \log \frac{p_X(x)}{p(x, y)} \\ &\leq \sum_y p_Y(y) \log \sum_x \frac{p(x, y)}{p_Y(y)} \frac{p_X(x)}{p(x, y)} \\ &= \sum_y p_Y(y) \log \frac{1}{p_Y(y)} = H(Y). \end{aligned}$$

$H(Y|X) \leq H(Y)$ cont.

- ▶ Assume X and Y are independent (i.e., $p(x, y) = p_X(x) \cdot p_Y(y)$ for any x, y)

$$\Rightarrow p_{Y|X}(y|x) = p_Y(y) \text{ for any } x, y$$

$$\Rightarrow H(Y|X) = H(Y)$$

- ▶ Is the converse also true: $H(Y|X) = H(Y)$ implies X and Y are independent?

Yes, since \log is strictly concave in the range. Equality happens iff all t_i are the same,

- ▶ which happens iff $p(x, y) = p_X(x)p_Y(y)$ for all x, y

Other inequalities

- ▶ $H(X), H(Y) \leq H(X, Y) \leq H(X) + H(Y)$.

Follows from $H(X, Y) = H(X) + H(Y|X)$.

- ▶ Left inequality since $H(Y|X)$ is non negative.
- ▶ Right inequality since $H(Y|X) \leq H(Y)$.
- ▶ $H(X, |Z) = H(X|Z) + H(Y|X, Z)$ (by chain rule)
- ▶ $H(X|Y, Z) \leq H(X|Y)$

Proof:

$$\begin{aligned} H(X|Y, Z) &= \mathbb{E}_{(z,y) \leftarrow (Z,Y)} H(X|_{(Y,Z)=(z,y)}) \\ &= \mathbb{E}_{y \leftarrow Y} \mathbb{E}_{z \leftarrow Z|Y=y} H(X|_{(Y,Z)=(z,y)}) \\ &= \mathbb{E}_{y \leftarrow Y} \mathbb{E}_{z \leftarrow Z|Y=y} H((X|_{Y=y})|_{Z=z}) \\ &\leq \mathbb{E}_{y \leftarrow Y} H(X|_{Y=y}) \\ &= H(X|Y). \end{aligned}$$

Chain rule (for the entropy function), general case

Claim 2

For rvs X_1, \dots, X_k , it holds that

$$H(X_1, \dots, X_k) = H(X_1) + H(X_2|X_1) + \dots + H(X_k|X_1, \dots, X_{k-1}).$$

Proof: ?

- ▶ Extremely useful property!
- ▶ Analogously to the two variables case, it also holds that:
- ▶ $H(X_i) \leq H(X_1, \dots, X_k) \leq \sum_i H(X_i)$
- ▶ $H(X_1, \dots, X_k|Y) \leq \sum_i H(X_i|Y)$

Examples

- ▶ (from last class) Let X_1, \dots, X_n be Boolean iid with $X_i \sim (\frac{1}{3}, \frac{2}{3})$. Compute $H(X_1, \dots, X_n)$
- ▶ As above, but X_n is set to $\bigoplus_{1 \leq i \leq n-1} X_i$?
 - ▶ Via chain rule?
 - ▶ Via mapping?

Applications

- ▶ Let X_1, \dots, X_n be Boolean iids with $X_i \sim (p, 1 - p)$ and let $X = X_1, \dots, X_n$. Let f be such that $\Pr[f(X) = z] = \Pr[f(X) = z']$, for every $k \in \mathbb{N}$ and $z, z' \in \{0, 1\}^k$. Let $K = |f(X)|$.

Prove that $\mathbb{E} K \leq n \cdot h(p)$.

- ▶
$$\begin{aligned}n \cdot h(p) &= H(X_1, \dots, X_n) \\&\geq H(f(X), K) \\&= H(K) + H(f(X) \mid K) \\&= H(K) + \mathbb{E} K \\&\geq \mathbb{E} K\end{aligned}$$

- ▶ Interpretation
- ▶ Upper bounds

Applications cont.

- ▶ How many comparisons it takes to sort n elements?

Let S be a sorter for n elements algorithm making t comparisons.

What can we say about t ?

- ▶ Let X be a uniform random permutation of $[n]$ and let Y_1, \dots, Y_t be the answers S gets when sorting X .
- ▶ X is determined by Y_1, \dots, Y_t .

Namely, $X = f(Y_1, \dots, Y_t)$ for some function f .

- ▶ $H(X) = \log n!$

▶

$$\begin{aligned} H(X) &= H(f(Y_1, \dots, Y_t)) \\ &\leq H(Y_1, \dots, Y_t) \\ &\leq \sum_i H(Y_i) \\ &\leq t \end{aligned}$$

$$\Rightarrow t \geq \log n! = \Theta(n \log n)$$

Concavity of entropy function

Let $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$ be two distributions, and for $\lambda \in [0, 1]$ consider the distribution $\tau_\lambda = \lambda p + (1 - \lambda)q$.
(i.e., $\tau_\lambda = (\lambda p_1 + (1 - \lambda)q_1, \dots, \lambda p_n + (1 - \lambda)q_n)$).

Claim 3

$$H(\tau_\lambda) \geq \lambda H(p) + (1 - \lambda)H(q)$$

Proof:

- ▶ Let Y over $\{0, 1\}$ be 1 wp λ
- ▶ Let X be distributed according to p if $Y = 0$ and according to q otherwise.
- ▶ $H(\tau_\lambda) = H(X) \geq H(X | Y) = \lambda H(p) + (1 - \lambda)H(q)$

We are now certain that we drew the graph of the (two-dimensional) entropy function right...

Part II

Mutual Information

Mutual information

- ▶ $I(X; Y)$ — the “information” that X gives on Y

- ▶
$$\begin{aligned} I(X; Y) &:= H(Y) - H(Y|X) \\ &= H(Y) - (H(X, Y) - H(X)) \\ &= H(X) + H(Y) - H(X, Y) \\ &= H(Y) - H(X|Y) \\ &= I(Y; X). \end{aligned}$$

- ▶ The mutual information that X gives about Y equals the mutual information that Y gives about X .
- ▶ $I(X; Y) \geq 0$. When 0?
- ▶ $I(X; X) = H(X)$
- ▶ $I(X; f(X)) = H(f(X))$ (and smaller than $H(X)$ if f is non-injective)
- ▶ $I(X; Y, Z) \geq I(X; Y), I(X; Z)$ (since $H(X | Y, Z) \leq H(X | Y), H(X | Z)$)
- ▶ $I(X; Y|Z) := H(Y|Z) - H(Y|X, Z) \geq 0$
- ▶ $I(X; Y|Z) = I(Y; X|Z)$ (since $I(X'; Y') = I(Y'; X')$)

Numerical example

► Example

$X \backslash Y$	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{2}$	0

$$\begin{aligned} I(X; Y) &= H(X) - H(X|Y) \\ &= 1 - \frac{3}{4} \cdot h\left(\frac{1}{3}\right) \\ &= I(Y; X) \\ &= H(Y) - H(Y|X) \\ &= h\left(\frac{1}{4}\right) - \frac{1}{2}h\left(\frac{1}{2}\right) \end{aligned}$$

Chain rule for mutual information

Claim 4 (Chain rule for mutual information)

For rvs X_1, \dots, X_k, Y , it holds that

$$I(X_1, \dots, X_k; Y) = I(X_1; Y) + I(X_2; Y|X_1) + \dots + I(X_k; Y|X_1, \dots, X_{k-1}).$$

Proof: ? HW

Examples

- ▶ Let X_1, \dots, X_n be iid with $X_i \sim (p, 1 - p)$, under the condition that $\bigoplus_i x_i = 0$.

Compute $I(X_1, \dots, X_{n-1}; X_n)$.

By chain rule

$$\begin{aligned} I(X_1, \dots, X_{n-1}; X_n) \\ &= H(X_1; X_n) + H(X_2; X_n | X_1) + \dots + H(X_{n-1}; X_n | X_1, \dots, X_{n-2}) \\ &= 0 + 0 + \dots + 1 = 1. \end{aligned}$$

- ▶ Let T and F be the top and front side, respectively, of a 6-sided fair dice. Compute $I(T; F)$.

$$\begin{aligned} I(T; F) &= H(T) - H(T|F) \\ &= \log 6 - \log 4 \\ &= \log 3 - 1. \end{aligned}$$

Part III

Data processing

Data processing Inequality

Definition 5 (Markov Chain)

Rvs $(X, Y, Z) \sim p$ form a **Markov chain**, denoted $X \rightarrow Y \rightarrow Z$, if $p(x, y, z) = p_X(x) \cdot p_{Y|X}(y|x) \cdot p_{Z|Y}(z|y)$, for all x, y, z .

Example: random walk on graph.

Claim 6

If $X \rightarrow Y \rightarrow Z$, then $I(X; Y) \geq I(X; Z)$.

► By Chain rule, $I(X; Y, Z) = I(X; Z) + I(X; Y|Z) = I(X; Y) + I(X; Z|Y)$.

► $I(X; Z|Y) = 0$

► $p_{Z|Y=y} \equiv p_{Z|Y=y, X=x}$ for any x, y

►

$$\begin{aligned} I(X; Z|Y) &= H(Z|Y) - H(Z|Y, X) \\ &= \mathbb{E}_{y \leftarrow Y} H(p_{Z|Y=y}) - \mathbb{E}_{(x,y) \leftarrow (Y,X)} H(p_{Z|Y=y, X=x}) \\ &= \mathbb{E}_{y \leftarrow Y} H(p_{Z|Y=y}) - \mathbb{E}_{y \leftarrow Y} H(p_{Z|Y=y}) = 0. \end{aligned}$$

► Since $I(X; Y|Z) \geq 0$, we conclude $I(X; Y) \geq I(X; Z)$. \square

Fano's Inequality

- ▶ How well can we guess X from Y ?
- ▶ Could with **no** error if $H(X|Y) = 0$. What if $H(X|Y)$ is small?

Theorem 7 (Fano's inequality)

For any rvs X and Y , and any (even random) g , it holds that

$$h(P_e) + P_e \log |\mathcal{X}| \geq H(X|\hat{X}) \geq H(X|Y)$$

for $\hat{X} = g(Y)$ and $P_e = \Pr[\hat{X} \neq X]$.

- ▶ Note that $P_e = 0$ implies that $H(X|Y) = 0$
- ▶ The inequality can be weekend to $1 + P_e \log |\mathcal{X}| \geq H(X|Y)$,
- ▶ Alternatively, to $P_e \geq \frac{H(X|Y)-1}{\log |\mathcal{X}|}$
- ▶ Intuition for $\propto \frac{1}{\log |\mathcal{X}|}$
- ▶ We call \hat{X} an **estimator** for X (from Y).

Proving Fano's inequality

Let X and Y be rvs, let $\hat{X} = g(Y)$ and $P_e = \Pr[\hat{X} \neq X]$.

► Let $D = \begin{cases} 1, & \hat{X} \neq X \\ 0, & \hat{X} = X. \end{cases}$

$$\begin{aligned} H(D, X|\hat{X}) &= H(X|\hat{X}) + \underbrace{H(D|X, \hat{X})}_{=0} \\ &= \underbrace{H(D|\hat{X})}_{\leq H(D)=h(P_e)} + \underbrace{H(X|D, \hat{X})}_{\leq P_e \log |\mathcal{X}|(?)} \end{aligned}$$

- It follows that $h(P_e) + P_e \log |\mathcal{X}| \geq H(X|\hat{X})$
- Since $X \rightarrow Y \rightarrow \hat{X}$, it holds that $I(X; Y) \geq I(X; \hat{X})$
 $\implies H(X|\hat{X}) \geq H(X|Y)$