Application of Information Theory, Lecture 1
Basic Definitions and Facts

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- Entropy is a function of p (sometimes refers to as $H(p)$).

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   - $h(p) := H(p, 1-p)$ is continuous

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3. X is uniformly distributed over $\{0,1\}^n$:
   $H(X) = -\sum_{i=1}^{2^n} \frac{1}{2^n}\log\frac{1}{2^n} = -\log\frac{1}{2^n} = n$.
   - n bits are needed to describe X
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4. $X = X_1, \ldots, X_n$ where X_i are iid over $\{0,1\}$, with $\mathrm{P}_{X_i}(1) := \Pr[X_i = 1] = \frac{1}{3}$. $H(X) =?$
5. $X \sim (p, q)$, $p + q = 1$
   - $H(X) = H(p, q) = -p\log p - q\log q$
   - $H(1, 0) = (0, 1) = 0$
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Examples

1. $X \sim (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$:
   (i.e., for some $x_1 \neq x_2 \neq x_3$, $\mathrm{P}_X(x_1) = \frac{1}{2}, \mathrm{P}_X(x_2) = \frac{1}{4}, \mathrm{P}_X(x_3) = \frac{1}{4}$)
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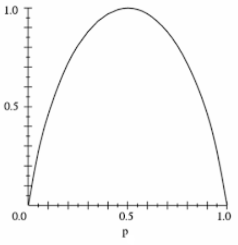

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Can we bound $|Q|$?

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And all are rather simple to prove

Axiomatic derivation of the entropy function

Any other choices for defining entropy?

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Shannon function is the only symmetric function (over probability distributions) satisfying the following three axioms:

A1 Continuity: $H(p, 1-p)$ is continuous function of p.

A2 Normalization: $H(\frac{1}{2}, \frac{1}{2}) = 1$

A3 Grouping axiom:
$H(p_1, p_2, \ldots, p_m) = H(p_1 + p_2, p_3, \ldots, p_m) + (p_1 + p_2)H(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2})$

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Why $A3$?

Not hard to prove that Shannon's entropy function satisfies above axioms, proving this is the only such function is more challenging.

Let H be a function that satisfying the above axioms.

We prove (assuming additional axiom) that H is the Shannon function.

Generalization of the grouping axiom

Fix $p = (p_1, \ldots, p_m)$ and let $S_k = \sum_{i=1}^{k} p_i$.

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Claim follows by combining the above equations. □

Further generalization of the grouping axiom

Let $1 = k_1 < k_2 < \ldots < k_q < m$ and let $C_t = \sum_{i=k_t}^{k_{t+1}-1} p_i$ (letting $k_{q+1} = m + 1$).

Claim 2 (Generalized^{++} grouping axiom)

$H(p_1, p_2, \ldots, p_m) =$
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Implication: Let $f(m) = H(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m})$

- $f(3^2) = 2f(3) = 2H(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

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- $f(mn) = f(m) + f(n)$

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Proof: Follow by the extended group axiom and the symmetry of H $\square$

Implication: Let $f(m) = H(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m})$

- $f(3^2) = 2f(3) = 2H(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

  $\implies f(3^n) = nf(3)$.
- $f(mn) = f(m) + f(n)$

  $\implies f(m^k) = kf(m)$

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- Proof extends to any integer (not only 3)

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- By continuity axiom, holds for every p_1, p_2, p_3.

□

Section 1

Basic Properties

$$0 \leq H(p_1, \ldots, p_m) \leq \log m$$

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$\implies$ (by induction) $\forall\ t_1, \ldots, t_k, \lambda_1, \ldots, \lambda_k \in [0,1]$ with $\sum_i \lambda_i = 1$
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$\Longrightarrow$ (Jensen inequality): $\mathrm{E}\, f(X) \le f(\mathrm{E}\, X)$ for any random variable X.

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- $\log(x)$ is (strictly) concave for $x > 0$, since its second derivative $(-\frac{1}{x^2})$ is always negative.
- Hence, $H(p_1,\ldots,p_m) = \sum_i p_i \log \frac{1}{p_i} \le \log \sum_i p_i \frac{1}{p_i} = \log m$
- Alternatively, for X over $\{1,\ldots,m\}$,
  $H(X) = \mathrm{E}_X \log \frac{1}{\mathrm{P}_X(X)} \le \log \mathrm{E}_X \frac{1}{\mathrm{P}_X(X)} = \log m$

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- $H(X) < H(\cos(X))$, if $0, \pi \in \mathrm{Supp}(X)$.

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