Foundation of Cryptography, Lecture 7&8
Interactive Proofs and Zero Knowledge[1]

Handout Mode

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Part I

Interactive Proofs

$\mathcal{NP}$ as a Non-interactive Proofs

Definition 1 ($\mathcal{NP}$)

$\mathcal{L} \in \mathcal{NP}$ iff $\exists$ and poly-time algorithm V such that:

- $\forall x \in \mathcal{L}$ there exists $w \in \{0,1\}^*$ s.t. $\mathsf{V}(x,w) = 1$
- $\mathsf{V}(x,w) = 0$ for every $x \notin \mathcal{L}$ and $w \in \{0,1\}^*$

Only $|x|$ counts for the running time of V.

A proof system

- Efficient verifier, efficient prover (given the witness)
- Soundness holds unconditionally

Interactive proofs

Protocols between efficient verifier and unbounded provers.

Definition 2 (Interactive proof)

A protocol (P,V) is an interactive proof for $\mathcal{L}$, if V is PPT and:

Completeness $\forall x \in \mathcal{L}$, $\Pr[(\mathsf{P},\mathsf{V})(x) = 1] \geq 2/3$.

Soundness $\forall x \notin \mathcal{L}$, and any algorithm P^*
$\Pr[(\mathsf{P}^*,\mathsf{V})(x) = 1] \leq 1/3$.

IP is the class of languages that have interactive proofs.

- $\mathsf{IP} = \mathrm{PSPACE}$!
- We typically consider (and achieve) perfect completeness.
- Negligible “soundness error" achieved via repetition.
- Sometime we have efficient provers via “auxiliary input".
- Relaxation: *Computationally sound proofs* [also known as, *interactive arguments*]: soundness only guaranteed against efficient (PPT) provers.

Section 1

Interactive Proof for Graph Non-Isomorphism

Graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $G_0 = ([m], E_0)$ and $G_1 = ([m], E_1)$ are isomorphic, denoted $G_0 \equiv G_1$, if $\exists \pi \in \Pi_m$ such that
$(u, v) \in E_0$ iff $(\pi(u), \pi(v)) \in E_1$.

- $\mathcal{GI} = \{(G_0, G_1) \colon G_0 \equiv G_1\} \in \mathcal{NP}$
- Does $\mathcal{GNI} = \{(G_0, G_1) \colon G_0 \not\equiv G_1\} \in \mathcal{NP}$?
- We will show a simple interactive proof for $\mathcal{GNI}$

  Idea: Beer tasting...

Interactive proof for $\mathcal{GNI}$

Protocol 4 ((P, V))

Common input: $G_0 = ([m], E_0), G_1 = ([m], E_1)$.

1. V chooses $b \xleftarrow{R} \{0,1\}$ and $\pi \xleftarrow{R} \Pi_m$, and sends $\pi(E_b)$ to P.[a]
2. P send b' to V (tries to set $b' = b$).
3. V accepts iff $b' = b$.

[a] $\pi(E) = \{(\pi(u), \pi(v) \colon (u,v) \in E\}$.

Claim 5

The above protocol is IP for $\mathcal{GNI}$, with perfect completeness and soundness error $\frac{1}{2}$.

Proving Claim 5

- Graph isomorphism is an equivalence relation (separates the set of all graph pairs into separate subsets)
- $([m], \pi(E_i))$ is a random element in $[\mathrm{G}_i]$ — the equivalence class of G_i

Hence,

$\mathrm{G}_0 \equiv \mathrm{G}_1$: $\Pr[b' = b] \leq \frac{1}{2}$.

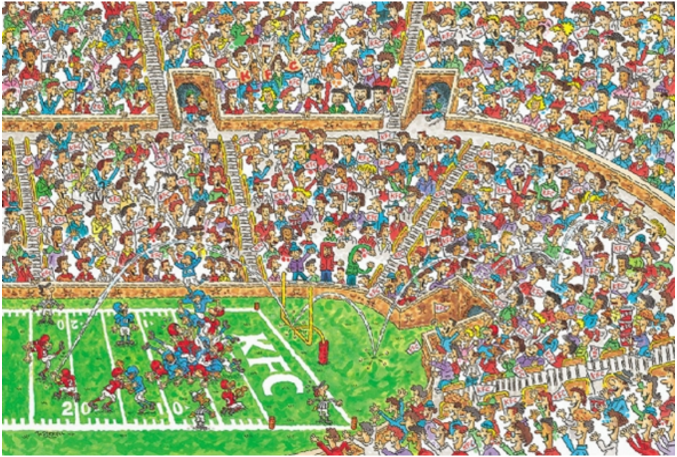
$\mathrm{G}_0 \not\equiv \mathrm{G}_1$: $\Pr[b' = b] = 1$ (i.e., P can, possibly inefficiently, extracted from $\pi(E_i)$)

□

Part II

Zero knowledge Proofs

Where is Waldo?



Question 6

Can you prove you know where Waldo is without revealing his location?

The concept of zero knowledge

- Proving w/o revealing any addition information.
- What does it mean?

  Simulation paradigm.

Protocols, notation

Let $\Pi = (\mathsf{A}, \mathsf{B})$ be a two-party protocol in which each party has a private input, and the parties has a common input.

- $\langle(\mathsf{A}(a), \mathsf{B}(b))(x)\rangle$: the parties' join view of the in a random execution of Π, in which A has input a, B has input b, and common input x.

  The randomness is over the parties coins.
- For $\mathsf{P} \in \{\mathsf{A}, \mathsf{B}\}$, $\langle(\mathsf{A}(a), \mathsf{B}(b))(x)\rangle_{\mathsf{P}}$ denote P's part of the view in $\langle(\mathsf{A}(a), \mathsf{B}(b))(x)\rangle$.

Distribution ensembles, revisited

We will consider distribution ensembles indexed by arbitrary sets.

Let $\mathcal{L} \subseteq \{0,1\}^*$, and let $P = \{P_x\}_{x\in\mathcal{L}}$ and $Q = \{Q_x\}_{x\in\mathcal{L}}$ be two distribution ensemble.

P is computationally indistinguishable from Q, denoted $P \approx_c Q$, means that

$$\left| \Pr_{y\leftarrow P_x}[\mathsf{D}(x,y)=1] - \Pr_{y\leftarrow Q_x}[\mathsf{D}(x,y)=1] \right| \le \operatorname{neg}(|x|)$$

Zero-knowledge proofs

Definition 7 (zero-knowledge proofs)

An interactive proof (P, V) is computational zero-knowledge ($\mathcal{CZK}$) for $\mathcal{L}$, if $\forall$ PPT V^*, $\exists$ PPT S (i.e., simulator) such that

$$\{\langle(\mathsf{P}, \mathsf{V}^*)(x)\rangle_{\mathsf{V}^*}\}_{x\in\mathcal{L}} \approx_c \{\mathsf{S}(x)\}_{x\in\mathcal{L}} \tag{1}$$

Perfect $\mathcal{ZK}$ ($\mathcal{PZK}$)/statistical $\mathcal{ZK}$ ($\mathcal{SZK}$) — the above distributions are identically/statistically close.

1. $\mathcal{ZK}$ is a property of the prover.
2. $\mathcal{ZK}$ only required to hold wrt. true statements.
3. If P takes input $w \in \mathcal{R}_{\mathcal{L}}(x)$, we consider $\langle(\mathsf{P}(w), \mathsf{V}^*)(x)\rangle_{\mathsf{V}^*}$
4. Trivial to achieve for $\mathcal{L} \in \mathcal{BPP}$.
5. The $\mathcal{NP}$ proof system is typically not zero knowledge.
6. Meaningful also for languages outside $\mathcal{NP}$.
7. Auxiliary input (will give formal def later)

Zero-knowledge proofs, cont.

1. Security parameter
2. ZK for honest verifiers: (1) only holds for $V^* = V$.
3. We sometimes assume for notational convenient, and wlg, that a cheating V^* outputs its view.
4. Statistical ZK proofs are believed to to exists only for a restricted subclass of $\mathcal{NP}$, so to go beyond that we settle for computational ZK (as in this course) or for arguments.
5. Weaker variants: witness hiding and witness indistinguishability

Section 2

Zero-Knowledge Proof for Graph Isomorphism

Zero-knowledge proof for $\mathcal{GI}$

Idea: route finding

Protocol 8 ((P, V))

Common input: $x = (\mathsf{G}_0 = ([m], E_0), \mathsf{G}_1 = ([m], E_1))$

P's input: a permutation π over $[m]$ such that $\pi(E_1) = E_0$.

1. P chooses $\pi' \leftarrow \Pi_m$ and sends $E = \pi'(E_0)$ to V.
2. V sends $b \leftarrow \{0, 1\}$ to P.
3. If $b = 0$, P sets $\pi'' = \pi'$, otherwise, it sends $\pi'' = \pi' \circ \pi$ to V.
4. V accepts iff $\pi''(E_b) = E$.

Claim 9

Protocol 8 is a $\mathcal{SZK}$ for $\mathcal{GI}$, with perfect completeness and soundness $\frac{1}{2}$.

Proving Claim 9

- Completeness: Clear
- Soundness: If exist $j \in \{0,1\}$ for which $\nexists \pi' \in \Pi_m$ with $\pi'(E_j) = E$, then V rejects w.p. at least $\frac{1}{2}$.

  Assuming V rejects w.p. less than $\frac{1}{2}$ and let π_0 and π_1 be the values guaranteed by the above observation (i.e., mapping E_0 and E_1 to E respectively).

  Then $\pi_0^{-1}(\pi_1(E_1)) = \pi_0 \implies (G_0, G_1) \in \mathcal{GI}$.
- $\mathcal{ZK}$:
  - Honest verifier?
  - Arbitrary verifier? for $(G_0, G_1) \in \mathcal{GI}$, it is easy to generate a random transcript for Steps 1–2, and to be able to open it with prob $\frac{1}{2}$.

The simulator

For a start, consider a deterministic cheating verifier V^* that never aborts.

Algorithm 10 (S)

Input: $x = (G_0 = ([m], E_0), G_1 = ([m], E_1))$

Do $|x|$ times:

1. Choose $b' \leftarrow \{0,1\}$ and $\pi \leftarrow \Pi_m$, and "send" $\pi(E_{b'})$ to $V^*(x)$.
2. Let b be V^*'s answer. If $b = b'$, send π to V^*, output V^*'s view and halt. Otherwise, rewind V^* to its initial step, and go to step 1.

Abort.

Claim 11

$\{\langle(P, V^*)(x)\rangle_{V^*}\}_{x \in \mathcal{L}} \approx \{S(x)\}_{x \in \mathcal{L}}$

Claim 11 implies that Protocol 8 is zero knowledge.

Proving Claim 11

Consider the following inefficient simulator:

Algorithm 12 (S′)

Input: $x = (\mathsf{G}_0 = ([m], E_0), \mathsf{G}_1 = ([m], E_1))$.

Do $|x|$ times:

1. Choose $\pi \leftarrow \Pi_m$ and send $E = \pi(E_0)$ to $\mathsf{V}^*(x)$.
2. Let b be V^*'s answer.

   W.p. $\frac{1}{2}$,

   2.1 Find π' such that $E = \pi'(E_b)$, and send it to V^*.
   2.2 Output V^*'s view and halt.

   Otherwise, rewind V^* to its initial step, and go to step 1.

Abort.

Claim 13

$\mathsf{S}(x) \equiv \mathsf{S}'(x)$ for any $x \in \mathcal{GI}$.

Proof: ?

Proving Claim 11 cont.

Consider a second inefficient simulator:

Algorithm 14 (S'')

Input: $x = (\mathsf{G}_0 = ([m], E_0), \mathsf{G}_1 = ([m], E_1))$

1. Choose $\pi \leftarrow \Pi_m$ and send $E = \pi(E_0)$ to $\mathsf{V}^*(x)$.
2. Find π' such that $E = \pi'(E_b)$ and send it to V^*
3. Output V^*'s view and halt.

Claim 15

$\forall x \in \mathcal{GI}$:

1. $\langle(\mathsf{P}, \mathsf{V}^*)(x)\rangle_{\mathsf{V}^*} \equiv \mathsf{S}''(x)$.
2. $\mathsf{SD}(\mathsf{S}''(x), \mathsf{S}'(x)) \leq 2^{-|x|}$.

Proof: ? (1) is clear.

Proving Claim 15(2)

Fix $t \in \{0,1\}^*$ and let $\alpha = \Pr_{S''(x)}[t]$.
It holds that

$$\Pr_{S'(x)}[t] = \alpha \cdot \sum_{i=1}^{|x|}(1 - \frac{1}{2})^{i-1} \cdot \frac{1}{2}$$
$$= (1 - 2^{-|x|}) \cdot \alpha$$

Hence, $SD(S''(x), S'(x)) \leq 2^{-|x|}\Box$

Remarks

1. Perfect $\mathcal{ZK}$ for “expected polynomial-time" simulators.
2. Aborting verifiers.
3. Randomized verifiers.
   3.1 The simulator first fixes the coins of V^* at random.
   3.2 Same proof goes through.
4. Negligible soundness error?
   4.1 Amplify by repetition
   4.2 But what about the ZK?

“Transcript simulation" might not suffice!

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a public-key encryption scheme and let $\mathcal{L} \in \mathcal{NP}$.

Protocol 16 ((P, V))

Common input: $x \in \{0,1\}^*$

P's input: $w \in \mathcal{R}_{\mathcal{L}}(x)$

1. V samples $(d, e) \stackrel{R}{\leftarrow} \mathsf{G}(1^{|x|})$ and sends e to P
2. P sends $c = \mathsf{E}_e(w)$ to V
3. V accepts iff $\mathsf{D}_d(c) \in \mathcal{R}_{\mathcal{L}}(x)$

- The above protocol has perfect completeness and soundness.
- Is it zero-knowledge?
- It has “transcript simulator" (at least for honest verifiers): $\exists$ PPT S s.t.: $\{\langle(\mathsf{P}(w \in \mathcal{R}_{\mathcal{L}}(x)), \mathsf{V})(x)\rangle_{\mathsf{trans}}\}_{x \in \mathcal{L}} \approx_c \{\mathsf{S}(x)\}_{x \in \mathcal{L}}$,

  where trans stands for the transcript of the protocol (i.e., the messages exchange through the execution).

Section 3

Composition of Zero-Knowledge Proofs

Is zero-knowledge maintained under composition?

- Auxiliary-input zero-knowledge, see next, is maintained under sequential repetition.
- Zero-knowledge might not maintained under parallel repetition (and there seems to be no syntactic way to solve it).

  Examples:

  - Chess game
  - Signature game

Zero-knowledge proof, auxiliary input variant

Definition 17 (zero-knowledge proofs, auxiliary input)

An interactive proof (P, V) is auxiliary-input computational zero-knowledge ($\mathcal{CZK}$) for $\mathcal{L}$, if $\forall$ deterministic poly-time V^*, $\exists$ PPT S s.t.

$$\{\langle(\mathsf{P}, \mathsf{V}^*(z(x))(x)\rangle_{\mathsf{V}^*}\}_{x\in\mathcal{L}} \approx_c \{\mathsf{S}(x, z(x))\}_{x\in\mathcal{L}}.$$

for any poly-output $z \colon \mathcal{L} \mapsto \{0,1\}^*$.

Perfect $\mathcal{ZK}$ ($\mathcal{PZK}$)/statistical auxiliary-input $\mathcal{ZK}$ ($\mathcal{SZK}$) — the above distributions are identically/statistically close.

- Strengthening of the standard definition.
- The protocol for $\mathcal{GI}$ we just saw, is also auxiliary-input $\mathcal{SZK}$
- What about randomized verifiers?
- Necessary for proving that zero-knowledge proof compose sequentially.
- To keep things simple, we will typically prove the non-auxiliary zero-knowledge, but all proofs we present can easily modified to achieve the stronger auxiliary input variant.

Is non-auxiliary-input ZK is auxiliary input ZK?

- Let $\mathcal{L} = \{1^n : n \in \mathbb{N}\}$ and consider the $\mathcal{NP}$-relation for $\mathcal{L}$ defined as $\mathcal{R}_{\mathcal{L}} = \{(1^n, 0), (1^n, 1) : n \in \mathbb{N}\}$.
- Assume exists commitment scheme Com, that is computationally hiding against PPT *uniform* receivers, but not hiding against *non-uniform* PPT receivers.
- The following protocol is $\mathcal{CZK}$, but not auxiliary-input $\mathcal{CZK}$ for $\mathcal{L}$ (it is not even witness hiding).

Protocol 18 ((P, V))

Common input: $x \in \{0,1\}^*$

P's input: $w \in \mathcal{R}_{\mathcal{L}}(x)$

1. P commits to w using $\mathsf{Com}(1^{|x|})$
2. V accepts if $x \in \mathcal{L}$.

Section 4

Black-box Zero Knowledge

Black-box simulators

Definition 19 (Black-box simulator)

(P, V) is $\mathcal{CZK}$ with black-box simulation for $\mathcal{L}$, if $\exists$ oracle-aided PPT S:

$$\{\langle(\mathsf{P}, \mathsf{V}^*(z(x)))(x)\rangle_{\mathsf{V}^*}\}_{x\in\mathcal{L}} \approx_c \{\mathsf{S}^{\mathsf{V}^*(x,z(x))}(x)\}_{x\in\mathcal{L}}$$

for any det. poly-time V^*, and poly-output $z\colon \mathcal{L} \mapsto \{0,1\}^*$.

Prefect and statistical variants are defined analogously.

1. "Most simulators" are black box
2. Strictly weaker then general simulation!

Section 5

Zero-knowledge proofs for all NP

$\mathcal{CZK}$ for 3COL

- Assuming OWFs exists, we give a (black-box) $\mathcal{CZK}$ for 3COL .
- We show how to transform it for any $\mathcal{L} \in \mathcal{NP}$ (using that 3COL $\in \mathcal{NPC}$).

Definition 20 (3COL)

$G = (M, E) \in$ 3COL, if $\exists\ \phi \colon M \mapsto [3]$ s.t. $\phi(u) \neq \phi(v)$ for every $(u, v) \in E$.

We use commitment schemes.

The protocol

Let π_3 be the set of all permutations over $[3]$. We use perfectly binding commitment $\mathsf{Com} = (\mathsf{Snd}, \mathsf{Rcv})$.

Protocol 21 ((P, V))

Common input: graph $\mathsf{G} = (M, E)$.

P's input: a (valid) coloring ϕ of G

1. P chooses $\pi \leftarrow \Pi_3$ and sets $\psi = \pi \circ \phi$
2. $\forall v \in M$: parties interact in $(\mathsf{Snd}(\psi(v)), \mathsf{Rcv})(1^{|G|})$.
   Let c_v and d_v be the resulting commitment and decommitment.
3. V sends $e = (u, v) \leftarrow E$ to P
4. P sends $(d_u, \psi(u)), (d_v, \psi(v))$ to V
5. V verifies that
   5.1 Both decommitments are valid,
   5.2 $\psi(u), \psi(v) \in [3]$, and
   5.3 $\psi(u) \neq \psi(v)$.

Claim 22

The above protocol is a $\mathcal{CZK}$ for 3COL, with perfect completeness and soundness error $1 - 1/|E|$.

- Completeness: Clear
- Soundness: Let $\{c_v\}_{v \in M}$ be the commitments resulting from an interaction of V with an arbitrary P^*.

  Define $\phi\colon M \mapsto [3]$ as follows:

  $\forall v \in M$: let $\phi(v)$ be the (single) value that it is possible to decommit c_v into (if not in $[3]$, set $\phi(v) = 1$).

  If $G \notin 3COL$, then $\exists (u, v) \in E$ s.t. $\psi(u) = \psi(v)$.

  Hence, V rejects such x w.p. at least $1/|E|$.

Proving $\mathcal{ZK}$

Fix a deterministic, non-aborting V^* that gets no auxiliary input.

Algorithm 23 (S)

Input: $G = (M, E)$.

Do $|G| \cdot |E|$ times:

1. Choose $e' = (u, v) \leftarrow E$.
   - **1.1** Set $\psi(u) \leftarrow [3]$,
   - **1.2** Set $\psi(v) \leftarrow [3] \setminus \{\psi(u)\}$, and
   - **1.3** Set $\psi(w) = 4$ for $w \in M \setminus \{u, v\}$.
2. $\forall v \in M$: commit to $\psi(v)$ to V^* (resulting in c_v and d_v)
3. Let e be the edge sent by V^*.

   If $e = e'$, send $(d_u, \psi(u)), (d_v, \psi(v))$ to V^*, output V^*'s view and halt.

   Otherwise, rewind V^* to its initial step, and go to step 1.

Abort.

Proving $\mathcal{ZK}$ cont.

Algorithm 24 ($\widetilde{\mathsf{S}}$)

Input: $\mathsf{G} = (M, E)$, and a (valid) coloring ϕ of G.

Do for $|G| \cdot |E|$ times:

1. Choose $e' \leftarrow E$.
2. Act like the honest prover does given private input ϕ.
3. Let e be the edge sent by V^*. If $e = e'$
   - 3.1 Send $(\psi(u), d_u), (\psi(v), d_v)$ to V^*,
   - 3.2 Output V^*'s view and halt.

   Otherwise, rewind V^* to its initial step, and go to step 1.

Abort.

Claim 25

$\{\langle(\mathsf{P}(w(x)), \mathsf{V}^*)(x)\rangle_{\mathsf{V}^*}\}_{x\in\mathcal{L}} \approx \{\widetilde{\mathsf{S}}^{\mathsf{V}^*(x)}(x, w(x))\}_{x\in\mathcal{L}}$

Proof: ?

Proving $\mathcal{ZK}$ cont..

Claim 26

$\{S^{V^*(x)}(x)\}_{x\in\mathcal{L}} \approx_c \{\widetilde{S}^{V^*(x)}(x, w(x))\}_{x\in\mathcal{L}}$, for any w with $w(x) \in \mathcal{R}_{\mathcal{L}}(x)$.

Proof: Assume $\exists\ (x, w) \in \mathcal{R}_{\mathcal{L}}$, PPT D, $p \in$ poly and an infinite set $\mathcal{I} \subseteq \mathcal{L}$ s.t.

$$\Pr\left[D(S^{V^*(x)}(x)) = 1\right] - \Pr\left[D(\widetilde{S}^{V^*(x)}(x, w)) = 1\right] \geq \frac{1}{p(|x|)}$$

for all $x \in \mathcal{L}$.

Hence, $\exists$ PPT R^* and $b \in [3]$ such that

$$\Pr\left[\left\langle(\mathsf{Snd}(4), R^*(x, w))\,(1^{|x|})\right\rangle_{R^*} = 1\right] - \Pr\left[\left\langle(\mathsf{Snd}(b), R^*(x, w))\,(1^{|x|})\right\rangle_{R^*} = 1\right] \geq \frac{1}{|x| \cdot p(|x|)}$$

for all $x \in \mathcal{I}$. In contradiction to the (non-uniform) security of Com.

Remarks

- Aborting verifiers
- Auxiliary inputs
- Soundness amplification

Extending to all $\mathcal{NP}$

For $\mathcal{L} \in \mathcal{NP}$, let Map_X and Map_W be two poly-time computable functions s.t.

- $x \in \mathcal{L} \iff \mathsf{Map}_X(x) \in \mathsf{3COL}$
- $w \in \mathcal{R}_{\mathcal{L}}(x) \iff \mathsf{Map}_W(w) \in \mathcal{R}_{\mathsf{3COL}}(\mathsf{Map}_X(x))$.

Let (P, V) be a $\mathcal{CZK}$ for $\mathsf{3COL}$ with black-box simulation.

Protocol 27 ($(\mathsf{P}_{\mathcal{L}}, \mathsf{V}_{\mathcal{L}})$)

Common input: $x \in \mathcal{L}$.

$\mathsf{P}_{\mathcal{L}}$'s input: $w \in \mathcal{R}_{\mathcal{L}}(x)$.

1. The two parties interact in $(\mathsf{P}(\mathsf{Map}_W(w)), \mathsf{V})(\mathsf{Map}_X(x))$,

   where $\mathsf{P}_{\mathcal{L}}$ and $\mathsf{V}_{\mathcal{L}}$ taking the role of P and V respectively.
2. $\mathsf{V}_{\mathcal{L}}$ accepts iff V accepts in the above execution.

Claim 28

$(\mathsf{P}_{\mathcal{L}}, \mathsf{V}_{\mathcal{L}})$ is a $\mathcal{CZK}$ for $\mathcal{L}$ with the same completeness and soundness as (P, V) as for $\mathsf{3COL}$.

Completeness and soundness are clear (?)

Proving zero knowledge of $(\mathsf{P}_{\mathcal{L}}, \mathsf{V}_{\mathcal{L}})$

- Let S be a black-box simulator of (P, V).
- The oracle-aided $\mathsf{S}_{\mathcal{L}}$ is defined by $\mathsf{S}_{\mathcal{L}}^{(\cdot)}(x) := \mathsf{S}^{(\cdot)}(\mathsf{Map}_X(x))$.

Claim 29

$\forall$ poly-time $\mathsf{V}_{\mathcal{L}}^*$ and $(x, w) \in \mathcal{R}_{\mathcal{L}}$:
$\{\langle(\mathsf{P}_{\mathcal{L}}(w), \mathsf{V}_{\mathcal{L}}^*)(x)\rangle_{\mathsf{V}^*}\}_{x\in\mathcal{L}} \approx_c \{\mathsf{S}_{\mathcal{L}}^{\mathsf{V}_{\mathcal{L}}^*(x)}(x)\}_{x\in\mathcal{L}}$

Proof:

- Assume for simplicity that Map_X is invective.
- Let $w_{\mathcal{L}}$ be some witness function for $\mathcal{L}$.
- Assume $\{\langle(\mathsf{P}_{\mathcal{L}}(w_{\mathcal{L}}(x)), \mathsf{V}_{\mathcal{L}}^*)(x)\rangle_{\mathsf{V}^*}\}_{x\in\mathcal{L}} \not\approx_c \{\mathsf{S}^{\mathsf{V}_{\mathcal{L}}^*(x)}(x)\}_{x\in\mathcal{L}}$.

$\Longrightarrow$ $\{\langle(\mathsf{P}(\mathsf{Map}_W(w_{\mathcal{L}}(x)), \mathsf{V}^*)(\mathsf{Map}_X(x))\rangle_{\mathsf{V}^*}\}_{x\in\mathcal{L}} \not\approx_c \{\mathsf{S}^{\mathsf{V}^*}(\mathsf{Map}_X(x))\}_{x\in\mathcal{L}}$
for $\mathsf{V}^*(x) := \mathsf{V}_{\mathcal{L}}^*(\mathsf{Map}_X^{-1}(x))$.

$\Longrightarrow$ $\{\langle(\mathsf{P}(w(x)), \mathsf{V}^*)(x)\rangle_{\mathsf{V}^*}\}_{x\in 3\mathrm{COL}} \not\approx_c \{\mathsf{S}^{\mathsf{V}^*(x)}(x)\}_{x\in 3\mathrm{COL}}$
for the appropriate witness function w of $3\mathrm{COL}$.

Part III

Proof of Knowledge

Proof of Knowledge

The protocol (P, V) is a proof of knowledge for $\mathcal{L} \in \mathcal{NP}$, if a P^* convinces V to accept x, then P^* “knows" $w \in \mathcal{R}_{\mathcal{L}}(x)$.

Definition 30 (Knowledge extractor)

Let (P, V) be an interactive proof for $\mathcal{L} \in \mathcal{NP}$. A probabilistic algorithm E is a knowledge extractor for (P, V) and $\mathcal{R}_{\mathcal{L}}$ with error $\eta \colon \mathbb{N} \mapsto \mathbb{R}$, if $\exists t \in \text{poly}$ s.t.

$\forall x \in \mathcal{L}$ and deterministic algorithm P^*, $\mathsf{E}^{\mathsf{P}^*}(x)$ runs in expected time bounded by $\frac{t(|x|)}{\delta(x)-\eta(|x|)}$ and outputs $w \in \mathcal{R}_{\mathcal{L}}(x)$, where $\delta(x) = \Pr[(\mathsf{P}^*, \mathsf{V})(x) = 1]$.

(P, V) is a proof of knowledge for $\mathcal{L}$ with error η.

- A property of V
- Why do we need it? Authentication schemes
- Randomized P^*?

Examples

Claim 31

The $\mathcal{ZK}$ proof we've seen in class for $\mathcal{GI}$, has a knowledge extractor with error $\frac{1}{2}$.

Proof: ?

Claim 32

The $\mathcal{ZK}$ proof we've seen in class for $3\mathrm{COL}$, has a knowledge extractor with error $1 - \frac{1}{|E|}$.

Proof: ?

Part IV

Schnorr Proofs

The settings

- Fix a multiplicative group ensemble $\mathcal{G} = \{\mathcal{G}_n\}_{n\in\mathbb{N}}$, were the each $\mathcal{G}_n$ is a (cyclic) group of prime order p_n.
- wlg. the $\mathcal{G}_n$'s do not intersect. (?)
- $\mathcal{G}$ is efficient: computing the order p_n, the group operations (multiplication and inverse) in $\mathcal{G}_n$, and operation over the filed $\mathbb{F}_{p_n}$ are computable in time $\mathrm{poly}(n)$.
- We will view $\mathcal{G}$ also as a language.
- Let $\mathcal{R}_{\mathcal{G}} := \{(x,(n,G,X))\colon G,X \in \mathcal{G}_n \wedge G^x = X\}$
- For ease of notation, we will mostly focus on a single n, and omit it from the notation
- Scalar operations are carried in $\mathbb{F}_p$.
- Note that $(G^x)^{-1} = G^{-x}$.

Efficient ZK-POK

- Given $X = G^x$, we would like to prove in ZK-POK the knowledge of x.
- Since an $\mathcal{NP}$ statement, we could use generic tools, but would like to have a more efficient proof.

The ZK-POK protocol

Protocol 33 ((P, V))

Common input: $G, X = G^x$.

P's input: x.

1. P: send $A = G^a$ for $a \stackrel{R}{\leftarrow} \mathbb{Z}_p$ to V.
2. V: send $e \stackrel{R}{\leftarrow} \mathbb{Z}_p$ to P.
3. P: send $z \leftarrow ex + a$ to V
4. V: accept if $G^z = X^e \cdot A$.

Claim 34

(P, V) is semi-honest ZK for $\mathcal{G}$, with knowledge extractor for $\mathcal{R}_\mathcal{G}$ of error $1/p$.

Correctness:

$$G^z = G^{ex+a} = G^{ex} \cdot G^a = X^e \cdot A.$$

Semi-honest zero knowledge

Algorithm 35 (S)

Input: $G, X = G^x$.

1. $z, e \stackrel{R}{\leftarrow} \mathbb{Z}_p$
2. $A \leftarrow G^z / X^e$.
3. Output (A, e, z).

1. In the real and emulated executions, (e, z) are identically distributed
2. In both executions, A is the same deterministic function of (X, e, z).
3. Hence, the protocol is perfect semi-honest ZK for $\mathcal{G}$.

Special soundness

1. We will prove special soundness: there exists an efficient algorithm that given two accepting transcripts (A, e_0, z_0) and (A, e_1, z_1) with $e_0 \neq e_1$, outputs x.
2. Implies the claimed POK (hw).
3. Let $e \leftarrow e_1 - e_0$ and $z \leftarrow z_1 - z_0$ and $x \leftarrow z \cdot e^{-1}$.
4. $$\begin{aligned} G^x &= G^{(z_1 - z_0)\cdot e^{-1}} = X^{e_1} A \cdot (X^{e_0} A)^{-1} \cdot G^{e^{-1}} \\ &= X^{e_1 - e_0} \cdot G^{e^{-1}} = G^{x(e_1 - e_0)\cdot e^{-1}} \\ &= X. \end{aligned}$$

ElGammal commitments

Is $X = G^x$ a good commitment scheme?
Let G_n be a fixed generator for $\mathcal{G}_n$ (assume it can be found efficiently (given 1^n).

Definition 36 (ElGamal commitments $(\mathsf{G}, \mathsf{S}, \mathsf{R})$)

- $\mathsf{G}(1^n)$: Output $E \leftarrow G_n^e$ for $e \stackrel{R}{\leftarrow} \mathbb{F}_{p_n}$.
- $\mathsf{S}_E(1^n, m \in \mathbb{F}_{p_n})$: Output $(G_n^r, G_n^m \cdot E^r)$ for $r \stackrel{R}{\leftarrow} \mathbb{F}_{p_n}$.

- Hiding and binding are defined as usual, but with respect to to an honest key generator algorithm.
- Similar to ElGamal encryption but the message is in the exponent.
- What if $m \notin \mathbb{F}_{p_n}$?
- ElGamal is perfectly binding, and hiding under the right hardness assumption (HW)
- Additively homomrphic: $\mathsf{S}_E(m_0; r_0) \cdot \mathsf{S}_E(m_1; r_1) = \mathsf{S}_E(m_0 + m_1; r_0 + m_1)$
  ($\cdot$ stands for point-wise multiplication)

ZK-POK protocol for EG commitments

Protocol 37 ((P, V))

Common input: $G, E, X = \mathsf{S}_E(m; r)$.

P's input: m, r.

1. P: Send $A \leftarrow \mathsf{S}_E(a; r')$ for $a, r' \overset{R}{\leftarrow} \mathbb{Z}_p$ to V.
2. V: Send $e \overset{R}{\leftarrow} \mathbb{Z}_p$ to P.
3. P: Send $(z \leftarrow ex + a, r'' \leftarrow er + r') \bmod p$ to V
4. V: Accept iff $\mathsf{S}_E(z; r'') = X^e \cdot A$.

Let $\mathcal{R} := \{((m, r), (G, E, \mathsf{S}_{G,E}(m; r)\}$.

Claim 38

(P, V) is semi-honest ZK for $\mathcal{L}(\mathcal{R})$, with knowledge extractor for $\mathcal{R}$ of error $1/p$.

Proof: HW

Part V

Succinct Interactive Arguments

Succinct interactive arguments

Theorem 39

Assume collision-resistant family (CRH) exists, then $\forall \mathcal{L} \in \mathcal{NP}$ *exists* 4*-message, public-coin, interactive argument that on security parameter* 1^κ *and input* $x \in \mathcal{L}$*, the parties communicate* $O(\kappa \cdot \log(|x|))$ *bits.*

- Prover is efficient given the witness.
- Protocol can be made ZK and POK.

PCP theorem

Theorem 40 (PCP theorem, informal)

For every $\mathcal{L} \in \mathcal{NP}$ exists a one-message interactive proof (P, V) with perfect completeness s.t. for any $(x, w) \in \mathcal{R}_{\mathcal{L}}$:

1. P *is efficient (poly-time) given w.*
2. V *reads $O(\log(|x|)$ random locations in the proof.*

Commitment with local decommitment

Definition 41 (Commitment with local decommitment)

An efficient two-stage protocol (S, R):

- **Commit**. S has private input $\sigma \in \{0,1\}^*$ and the common input is 1^κ. The commitment stage results in a joint output c, the commitment, and a private output d to S.
- **Local opening**. $\mathsf{S}(d, i)$ sends (i, σ_i, ℓ) to R, and R either accepts or rejects.

- **Completeness.** R always accepts in honest execution, $\forall i \in [|\sigma|]$.
- **Binding.** With save but $\mathtt{neg}(\kappa)$ probability, a non-uniform PPT sender S^* cannot make the receiver accept two local openings (for same index i).

- Interesting if $\ell \ll |\sigma|$.
- No hiding requirement.

Commitment with local decommitment from CRH

Definition 42 (collision resistant hash family (CRH), non-unifrom variant)

A function family $\mathcal{H} = \{\mathcal{H}_\kappa : \{0,1\}^* \mapsto \{0,1\}^\kappa\}$ is collision resistant, if

$$\Pr_{h \leftarrow \mathcal{H}_\kappa}[\mathsf{A}(1^\kappa, h) = (x, x') \text{ s.t. } x \neq x' \wedge h(x) = h(x')] = \mathsf{neg}(\kappa)$$

for any non-uniform PPT A.

We assume it takes κ bits to describe $h \in \mathcal{H}_\kappa$.

Theorem 43

Assume CRH exist, then exits two-message, public-coin, commitment with local decommitment, with commitment length κ and local opening length $O(\kappa \cdot \log |\sigma|)$.

Proof: Via Merkle tree (board).

Proving Thm 39

Let $\mathcal{L} \in \mathcal{NP}$, $(\mathsf{P}_{\mathsf{PCP}}, \mathsf{V}_{\mathsf{PCP}})$ be a PCP for $\mathcal{L}$ and (S, R) be a commitment with local decommitment.

Protocol 44 ($(\mathsf{P}(w), \mathsf{V})(1^\kappa, x)$)

1. P compute $\pi = \mathsf{P}_{\mathsf{PCP}}(x, w)$.
2. The parties interact in $(\mathsf{S}(\pi), \mathsf{R})(1^\kappa)$.
3. V sends the queries of $\mathsf{V}_{\mathsf{PCP}}(x)$ to P.
4. P locally decommits the required locations in π.
5. V verifies the openings. If all valid, outputs $\mathsf{V}_{\mathsf{PCP}}$'s decision on them.

Soundness proof? Given a cheating prover P^* that makes V accept $x \notin \mathcal{L}$:

1. Extract a proof π^* from P^*.
2. Use π^* to break the soundness of $(\mathsf{P}_{\mathsf{PCP}}, \mathsf{V}_{\mathsf{PCP}})$.

Making the protocol ZK

Protocol 45 ($(\mathsf{P}_{ZK}(w), \mathsf{V}_{ZK})(1^\kappa, x)$)

In each round i, rather than sending the message m_i, the prover

1. Commits to m_i using a perfectly binding commitment.
2. Proves it knows the opening.

After the protocol ends, the prover proves in a ZK proof that the values in the commitments make V accept (on the same challenges).

- Soundness? In each round, extract message from P^*_{ZK} and send it to V.
- Zero knowledge? Commit to garbage, and simulate the zero knowledge proofs.