Application of Information Theory, Lecture 9

Parallel Repetition of Interactive Arguments

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Part I

Interactive Proofs and Arguments

$\mathcal{NP}$ as a Non-interactive Proofs

Definition 1 ($\mathcal{NP}$)

$\mathcal{L} \in \mathcal{NP}$ iff $\exists$ and poly-time algorithm V such that:

- $\forall x \in \mathcal{L}$ there exists $w \in \{0,1\}^*$ s.t. $\mathsf{V}(x,w) = 1$
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- Soundness holds unconditionally

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A protocol (P,V) is an interactive proof for $\mathcal{L}$, if V is a PPT and:

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- Games — no-input protocols.

Section 1

Interactive Proof for Graph Non-Isomorphism

Graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $\mathsf{G}_0 = ([m], E_0)$ and $\mathsf{G}_1 = ([m], E_1)$ are isomorphic, denoted $\mathsf{G}_0 \equiv \mathsf{G}_1$, if $\exists \pi \in \Pi_m$ such that
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  Idea: Beer tasting...

Interactive proof for $\mathcal{GNI}$

Protocol 4 ($(\mathsf{P},\mathsf{V})(\mathsf{G}_0 = ([m], E_0), \mathsf{G}_1 = ([m], E_1))$)

1. V chooses $b \leftarrow \{0,1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b)$ to P.[a]
2. P send b' to V (tries to set $b' = b$).
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Claim 5

The above protocol is IP for $\mathcal{GNI}$, with perfect completeness and soundness error $\frac{1}{2}$.

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Hence,

$\mathsf{G}_0 \equiv \mathsf{G}_1$: $\Pr[b' = b] \leq \frac{1}{2}$.

$\mathsf{G}_0 \not\equiv \mathsf{G}_1$: $\Pr[b' = b] = 1$ (i.e., P can, possibly inefficiently, extracted from $\pi(E_i)$)

□

Part II

Hardness Amplification

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- Parallel repetition does achieve optimal amplification rate for interactive proofs and public-coin interactive arguments
- Public-coin interactive proof/argument — in each round the verifier flips coins and sends them to the prover. To compute its output, the verifier applies some (fixed) function to the protocol's transcript.

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- In the following we focus on games (no input protocols)

Section 2

Parallel repetition of public-coin interactive argument

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Theorem 6

Let $\pi = (\mathsf{P}, \mathsf{V})$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \leq \varepsilon$ for any t-time $\widetilde{\mathsf{P}}$, then $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] \leq \varepsilon^{k/4}$ for any $t \cdot \frac{\varepsilon^{k/4}}{mk^3 t_{\mathsf{V}}}$-time $\widetilde{\mathsf{P}^{(k)}}$, where t_{V} is V's running time.

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Proof plan: Let $\widetilde{\mathsf{P}^{(k)}}$ be $t^{(k)}$-time algorithm with $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $t^{(k)} \cdot \frac{mk^3 t_\mathsf{V}}{\varepsilon^{(k)}}$-time $\widetilde{\mathsf{P}}$ with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq (\varepsilon^{(k)})^{4/k}$.

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- Assume for simplicity that $\widetilde{\mathsf{P}^{(k)}}$ is deterministic

Parallel repetition of public-coin interactive argument

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Let $\pi = (\mathsf{P}, \mathsf{V})$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \leq \varepsilon$ for any t-time $\widetilde{\mathsf{P}}$, then $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] \leq \varepsilon^{k/4}$ for any $t \cdot \frac{\varepsilon^{k/4}}{mk^3 t_{\mathsf{V}}}$-time $\widetilde{\mathsf{P}^{(k)}}$, where t_{V} is V's running time.

Proof plan: Let $\widetilde{\mathsf{P}^{(k)}}$ be $t^{(k)}$-time algorithm with $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $t^{(k)} \cdot \frac{mk^3 t_{\mathsf{V}}}{\varepsilon^{(k)}}$-time $\widetilde{\mathsf{P}}$ with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq (\varepsilon^{(k)})^{4/k}$.

- The $k/4$ in the exponent can be pushed to be almost k.
- Assume for simplicity that $\widetilde{\mathsf{P}^{(k)}}$ is deterministic
- Assume wlg. that V sends the first message in π and that in each round it samples and sends ℓ coins.

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- We view the coins of $\mathsf{V}^{(k)}$ as a matrix $R \in \{0,1\}^{m \times (k\ell)}$, letting R_j denote the coins of the j'th round, and $R_{1,\ldots,j}$ the coins of the first j rounds.

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- Let $\mathbf{R} \sim \{0,1\}^{m \times (k\ell)}$

Algorithm $\widetilde{\mathsf{P}}$

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Let $q = k^2$.

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Algorithm 7 ($\widetilde{\mathsf{P}}$)

```
1. Let $i^* \leftarrow [k]$.
2. Upon getting the $j$'th message $r$ from $\mathsf{V}$, do:
    2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned that $R_{1,\ldots,j-1} = \widetilde{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = r$.
    2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$:
        2.2.1 Set $\widetilde{R}_j = R_j$
        2.2.2 Send $a_{j,i^*}$ back to $\mathsf{V}$, for $a_j$ being the $j$'th message $\widetilde{\mathsf{P}^{(k)}}$ send to $\mathsf{V}^{(k)}$ in $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R))$.
        Else, GOTO Line 2.1
    2.3 Abort if the overall number of sampling exceeds $\lceil qm/\varepsilon^{(k)} \rceil$.
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- Let $\widetilde{\mathsf{P}}'$ be the non aborting variant of $\widetilde{\mathsf{P}}'$, let $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{N}}$ be the value of $\widetilde{R}$ and $\#$ of samples done in a random execution of $(\widetilde{\mathsf{P}}', \mathsf{V}^{(k)})$.

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- $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq \Pr\left[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) := (\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widetilde{\mathbf{R}})) = 1^k \wedge \widetilde{\mathbf{N}} \leq qm/\varepsilon^{(k)}\right]$.

Ideal “attacker”

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Experiment 8 ($\hat{\mathsf{P}}$)

```
For $j = 1$ to $m$:
    1. Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned that $R_{1,\ldots,j-1} = \hat{R}_{1,\ldots,j-1}$.
    2. If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.
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- In particular, $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\hat{\mathbf{R}}) = 1^k\right] = 1$

Ideal "attacker"

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Ideal "attacker"

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Lemma 9

$\Pr\left[\hat{\boldsymbol{N}} \leq qm/\varepsilon^{(k)}\right] \geq 1 - \frac{1}{q}$

Proving Lemma 9

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- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$

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- We prove Lemma 9 showing that $\mathsf{E}\left[\frac{1}{v(Y^j)}\right] \leq \frac{1}{\varepsilon^{(k)}}$ for every $j \in \{0, \ldots, m-1\}$

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Claim 10

For $j \in \{0, \ldots, m-1\}$ and $\mathbf{y} \in \mathrm{Supp}(Y^j)$ it holds that $\Pr_{Y^j}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

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Hence, $\mathrm{E}_{Y^j}\left[\frac{1}{v(Y^j)}\right] = \sum_{\mathbf{y}\in\mathrm{Supp}(Y^j)} \Pr[Y^j = \mathbf{y}] \cdot \frac{1}{v(\mathbf{y})}$

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$= \sum_{\mathbf{y}} \Pr[X^j = \mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}} \cdot \frac{1}{v(\mathbf{y})} = \frac{1}{\varepsilon^{(k)}} \cdot \sum_{\mathbf{y}\in\operatorname{Supp}(Y^j)} \Pr[X^j = \mathbf{y}] \leq \frac{1}{\varepsilon^{(k)}}$. □

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Note that

$$\begin{aligned}\Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] &= \sum_{\ell=1}^{\infty}(1-v(\mathbf{y}_{1\dots,j-1}))^{\ell-1}\cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j]\cdot v(\mathbf{y}) \qquad (1)\\ &= \frac{1}{v(\mathbf{y}_{1\dots,j-1})}\cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j]\cdot v(\mathbf{y})\end{aligned}$$

Proving Claim 10

Note that

$$\Pr_{Y_j | Y^{j-1} = \mathbf{y}_{1\dots,j-1}}[y_j] = \sum_{\ell=1}^{\infty} (1 - v(\mathbf{y}_{1\dots,j-1}))^{\ell-1} \cdot \Pr_{X_j | X^{j-1} = \mathbf{y}_{1\dots,j-1}}[y_j] \cdot v(\mathbf{y}) \tag{1}$$

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The proof proceeds by induction on j.

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$$\Pr_{Y^j}[\mathbf{y}] = \Pr_{Y^{j-1}}\left[\mathbf{y}_{1\dots,j-1}\right] \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1\dots,j-1}}[y_j]$$

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$$= \Pr_{X^{j-1}}\left[\mathbf{y}_{1\ldots,j-1}\right] \cdot \frac{v(\mathbf{y}_{1\ldots,j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1\ldots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1\ldots,j-1}}[y_j] \qquad \text{(Eq. (1))}$$

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$$\Pr_{Y^j}[\mathbf{y}] = \Pr_{Y^{j-1}}\left[\mathbf{y}_{1,\ldots,j-1}\right]\cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\ldots,j-1}}[y_j]$$
$$= \Pr_{X^{j-1}}\left[\mathbf{y}_{1,\ldots,j-1}\right]\cdot \frac{v(\mathbf{y}_{1,\ldots,j-1})}{\varepsilon^{(k)}}\cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\ldots,j-1}}[y_j] \qquad \text{(i.h.)}$$
$$= \Pr_{X^{j-1}}\left[\mathbf{y}_{1,\ldots,j-1}\right]\cdot \frac{v(\mathbf{y}_{1,\ldots,j-1})}{\varepsilon^{(k)}}\cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1,\ldots,j-1})}\cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\ldots,j-1}}[y_j] \qquad \text{(Eq. (1))}$$
$$= \Pr_{X^j}[\mathbf{y}]\cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}.$$

Ideal “attacker”, variant

Ideal “attacker”, variant

Experiment 11 ($\widehat{\mathsf{P}}$)

```
1. Let $i^* \leftarrow [k]$.
2. For for $j = 1$ to $m$:
    2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.
    2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
    2.3 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and
        $R_{j,i^*} = \widehat{R}_{j,i^*}$.
    2.4 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.
```

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

```
1. Let $i^* \leftarrow [k]$.
2. For for $j = 1$ to $m$:
    2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.
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```

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.

Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

```
1. Let $i^* \leftarrow [k]$.
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    2.4 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.
```

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$

Ideal “attacker”, variant

Experiment 11 ($\widehat{\mathsf{P}}$)

```
1. Let $i^* \leftarrow [k]$.
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    2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.
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        $R_{j,i^*} = \widehat{R}_{j,i^*}$.
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```

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Ideal “attacker”, variant

Experiment 11 ($\widehat{\mathsf{P}}$)

```
1. Let $i^* \leftarrow [k]$.
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    2.1 Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.
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Ideal "attacker", variant

Experiment 11 ($\widehat{\mathsf{P}}$)

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1. Let $i^* \leftarrow [k]$.
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```

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Lemma 12

$\Pr\left[\mathrm{win}(\widehat{\boldsymbol{R}}, \widehat{\boldsymbol{N}})\right] \geq 1 - \frac{1}{q}$

From ideal to real

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

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Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

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Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

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$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

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$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

- Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

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$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

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- Hence, $D(\mathsf{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathsf{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

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$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

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$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

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- Claim 12 $\implies \alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})] \geq 1 - \frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})]$.

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

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$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

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- Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1 - \alpha) \log(1 - \alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

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- Hence, $D(\text{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \text{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
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- Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1-\alpha) \log(1-\alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

  $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

Claim 13

$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

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- Thus, $\alpha \cdot \log \frac{\alpha}{\beta} + (1 - \alpha) \log(1 - \alpha) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$

  $\implies \beta \geq 2^{\log \alpha + \frac{1-\alpha}{\alpha} \log(1-\alpha) + \frac{1}{\alpha k} \log \varepsilon^{(k)}}$
- Recalling $q = k^2$, $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha} \log(1 - \alpha) \geq -\frac{4 \log k}{k^2} \geq -\frac{1}{k}$

From ideal to real

Let $\widetilde{\mathbf{R}}_i = \widetilde{\mathbf{R}}|_{i^*=i}$ and $\widehat{\mathbf{R}}_i := \widehat{\mathbf{R}}|_{i^*=i} \quad (= \widehat{\mathbf{R}})$.

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$D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq \frac{1}{k} \sum_{i \in [k]} D(\widehat{\mathbf{R}}_i || \widetilde{\mathbf{R}}_i)$.

Claim 14

$\sum_{i \in [k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) \leq D(\widehat{\mathbf{R}} || \mathbf{R})$.

- Thm. 7 in Lecture 7 $\implies$ $D(\widehat{\mathbf{R}} || \mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(\mathbf{R})) = 1^k]} = \log \frac{1}{\varepsilon^{(k)}}$
- Hence, $D(\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}}) || \mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}}, \widehat{\mathbf{N}} || \widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) \leq -\frac{1}{k} \cdot \log \varepsilon^{(k)}$
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- We conclude that $\beta \geq 2^{\frac{4}{k} \log \varepsilon^{(k)}} = \sqrt[k/4]{\varepsilon^{(k)}}$.$\square$

Proving Claim 13

Proving Claim 13

HW...

Proving Claim 14

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids, let* W *be an event, and let*
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\dots,j-1} = z_{1,\dots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.
Then $\sum_{i=1}^{k} D(Z_W || D_i) \leq D(Z_W || Z)$.

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids, let* W *be an event, and let*
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Then $\sum_{i=1}^{k} D(Z_W || D_i) \leq D(Z_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. □

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids, let W be an event, and let
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \land Z_{j,i} = z_{i,j} \land W\right]$*.*
Then $\sum_{i=1}^{k} D(Z_W || D_i) \leq D(Z_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. □

Proof: (of Lemma 15) We prove for $m = k = 2$.

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids, let W be an event, and let
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.
Then $\sum_{i=1}^{k} D(Z_W||D_i) \leq D(Z_W||Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. □

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids, let W be an event, and let
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$*.*
Then $\sum_{i=1}^{k} D(Z_W||D_i) \leq D(Z_W||Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. □

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}\left[(x_1, x_2, y_1, y_2)\right]$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids, let* W *be an event, and let*
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.
Then $\sum_{i=1}^{k} D(Z_W || D_i) \leq D(Z_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}(\mathbf{R}))} = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. □

Proof: (of Lemma 15) We prove for $m = k = 2$.

- Let $X = Z_1$ and $Y = Z_2$
- $U(x_1, x_2, y_1, y_2) := \Pr_{(X,Y)}\left[(x_1, x_2, y_1, y_2)\right]$
- $C(x_1, x_2, y_1, y_1) := (X|_W)(x_1, x_2, y_1, y_1)$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ *be iids, let* W *be an event, and let*
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$*.*
Then $\sum_{i=1}^{k} D(Z_W || D_i) \leq D(Z_W || Z)$*.*

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. □

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- $Q(x_1, x_2, y_1, y_1) := \Pr\left[X_1 = x_1 | W\right] \cdot \Pr\left[X_2 = x_2 | W\right] \cdot \Pr\left[Y_1 = y_1 | W, X = (x_1, x_2)\right] \cdot \Pr\left[Y_2 = y_2 | W, X = (x_1, x_2)\right]$

Proving Claim 14

Lemma 15

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids, let W be an event, and let
$D_i(z) := \prod_{j=1}^{m} \Pr\left[Z_{j,i} = z_{i,j}\right] \cdot \Pr\left[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W\right]$.
Then $\sum_{i=1}^{k} D(Z_W || D_i) \leq D(Z_W || Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 15 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}} || \widetilde{\mathbf{R}}_i) = \sum_{i\in[k]} D(\mathbf{R}|_W || \widetilde{\mathbf{R}}_i) \leq D(\mathbf{R}|_W || \mathbf{R}) = D(\widehat{\mathbf{R}} || \mathbf{R})$. □

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- $Q(x_1, x_2, y_1, y_1) := \Pr\left[X_1 = x_1 | W\right] \cdot \Pr\left[X_2 = x_2 | W\right] \cdot \Pr\left[Y_1 = y_1 | W, X = (x_1, x_2)\right] \cdot \Pr\left[Y_2 = y_2 | W, X = (x_1, x_2)\right]$
- We write $\frac{C(x_1,x_2,y_1,y_1)}{U(x_1,x_2,y_1,y_1)} = \frac{\Pr[X_1=x_1|W]\cdot\Pr[Y_1=y_1|W,X=(x_1,x_2)]}{\Pr[X_1=x_1]\cdot\Pr[Y_1=y_1]} \cdot \frac{\Pr[X_2=x_2|W]\cdot\Pr[Y_2=y_2|W,X=(x_1,x_2)]}{\Pr[X_2=x_2]\cdot\Pr[Y_2=y_2]} \cdot \frac{C(x_1,x_2,y_1,y_1)}{Q(x_1,x_2,y_1,y_1)}$

Proving Lemma 15, cont.

Proving Lemma 15, cont.

$$\begin{aligned}
D(C||U) = & \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr\left[X_1 = x_1|W\right]\cdot\Pr\left[Y_1 = y_1|W, X = (x_1, x_2)\right]}{\Pr\left[X_1 = x_1\right]\cdot\Pr\left[Y_1 = y_1\right]}\right] \\
& + \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr\left[X_2 = x_2|W\right]\cdot\Pr\left[Y_2 = y_2|W, X = (x_1, x_2)\right]}{\Pr\left[X_2 = x_2\right]\cdot\Pr\left[Y_2 = y_2\right]}\right] \\
& + \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{C(x_1, x_2, y_1, y_2)}{Q(x_1, x_2, y_1, y_2)}\right].
\end{aligned}$$

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = &\mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr\left[X_1 = x_1|W\right]\cdot\Pr\left[Y_1 = y_1|W, X = (x_1,x_2)\right]}{\Pr\left[X_1 = x_1\right]\cdot\Pr\left[Y_1 = y_1\right]}\right]\\
&+ \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{\Pr\left[X_2 = x_2|W\right]\cdot\Pr\left[Y_2 = y_2|W, X = (x_1,x_2)\right]}{\Pr\left[X_2 = x_2\right]\cdot\Pr\left[Y_2 = y_2\right]}\right]\\
&+ \mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log \frac{C(x_1,x_2,y_1,y_2)}{Q(x_1,x_2,y_1,y_2)}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D(C||U) = &D(X_1|_W, X_2|_{W,X_1}, Y_1|_{W,X}, Y_2|_{W,X,Y_1}||X_1, X_2|_{W,X_1}, Y_1, Y_2|_{W,X,Y_1})\\
&+ D(X_2|_W, X_1|_{W,X_2}, Y_2|_{W,X}, Y_1|_{W,X,Y_2}||X_2, X_1|_{W,X_2}, Y_2, Y_1|_{W,X,Y_2})\\
&+ D(C||Q),
\end{aligned}
$$

Proving Lemma 15, cont.

$$
\begin{aligned}
D(C||U) = &\mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr[X_1 = x_1|W]\cdot\Pr[Y_1 = y_1|W, X = (x_1,x_2)]}{\Pr[X_1 = x_1]\cdot\Pr[Y_1 = y_1]}\right]\\
&+\mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{\Pr[X_2 = x_2|W]\cdot\Pr[Y_2 = y_2|W, X = (x_1,x_2)]}{\Pr[X_2 = x_2]\cdot\Pr[Y_2 = y_2]}\right]\\
&+\mathop{\mathrm{E}}_{(x_1,x_2,y_1,y_2)\leftarrow C}\left[\log\frac{C(x_1,x_2,y_1,y_2)}{Q(x_1,x_2,y_1,y_2)}\right].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D(C||U) = &D(X_1|_W, X_2|_{W,X_1}, Y_1|_{W,X}, Y_2|_{W,X,Y_1}||X_1, X_2|_{W,X_1}, Y_1, Y_2|_{W,X,Y_1})\\
&+D(X_2|_W, X_1|_{W,X_2}, Y_2|_{W,X}, Y_1|_{W,X,Y_2}||X_2, X_1|_{W,X_2}, Y_2, Y_1|_{W,X,Y_2})\\
&+D(C||Q),
\end{aligned}
$$

and the proof follows since $D(\cdot||\cdot) \geq 0$. □

Parallel repetition of interactive proofs

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- Similar proof to the public-coin proof we gave above.

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- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $(\widetilde{\mathsf{P}^{(k)}, \mathsf{V}^{(k)}})$, till he gets an accepting execution.

Parallel repetition of interactive proofs

- Similar proof to the public-coin proof we gave above.
- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $\widetilde{(\mathsf{P}^{(k)}}, \mathsf{V}^{(k)})$, till he gets an accepting execution.
- Why fails us to extend this approach for non-public-coin interactive arguments?

Section 3

Parallel amplification for any interactive argument

Parallel amplification theorem for any protocol

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument “in parallel”?

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument “in parallel”?
- Yes we can!