Application of Information Theory, Lecture 4

Asymptotic Equipartition Property, Data Compression & Gambling

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Part I

Asymptotic Equipartition Theorem

Entropy as # of bits to describe random variable

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- It takes about $n \cdot h(k/n)$ bits to describe a string of k zeros in $\{0,1\}^n$.

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- $\log \mathbf{p}(X_1, \ldots, X_n) = \log \prod_i p(X_i) = \sum_i \log p(X_i)$
- Hence,
  $\mathrm{E}_{X_1,\ldots,X_n}\left[-\log \mathbf{p}(X_1, \ldots, X_n)\right] = -\sum_i \mathrm{E}\left[\log p(X_i)\right] = H(X_1, \ldots, X_n)$

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- We will show that w.h.p. $-\log \mathbf{p}(X_1, \ldots, X_n)$ is close to its expectation

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- The above extends to many variables of different distributions, and not fully independent.

Part II

Data Compression

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- So $H(X_1, \ldots, X_n)$ is approximately the number of bits it takes to describe $X_1, \ldots, X_n$

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- In case $H(X) = nH(X_1)$, then $m \geq n(H(X_1) - \varepsilon) - 1$

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- We focus on binary prefix codes ($\Sigma = \{0, 1\}$)

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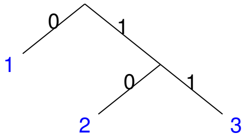

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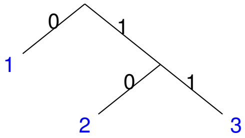

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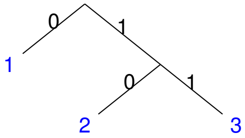

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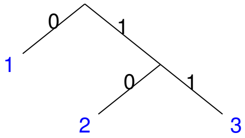

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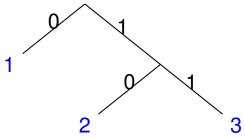

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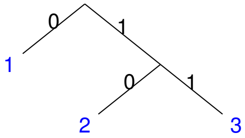

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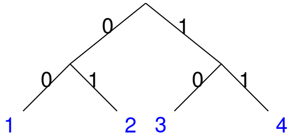

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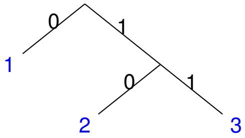

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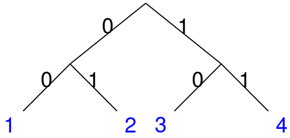

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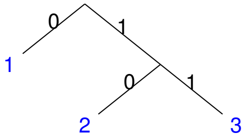

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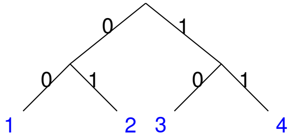

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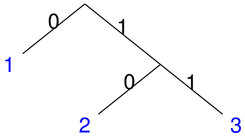

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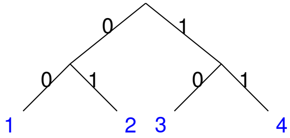

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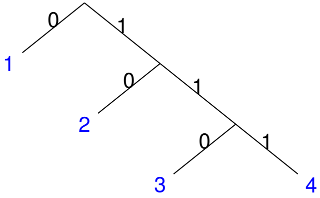

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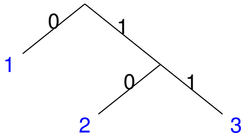

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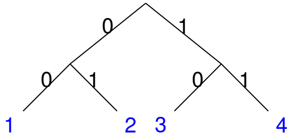

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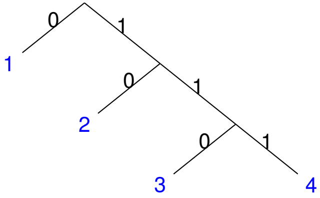

- All are prefix codes: no codeword is a prefix of another codeword

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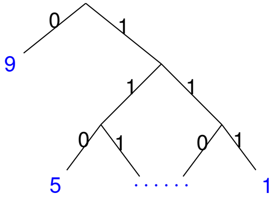

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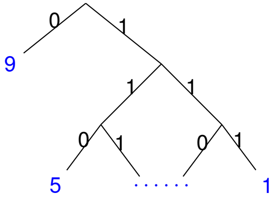

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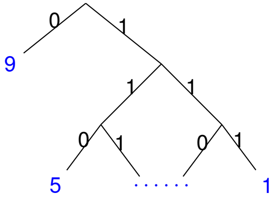

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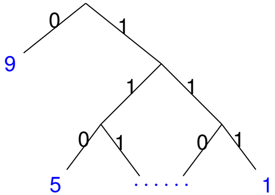

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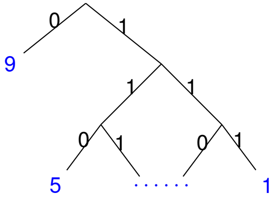

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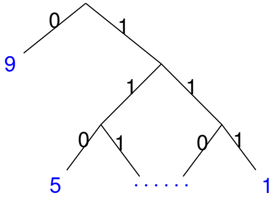

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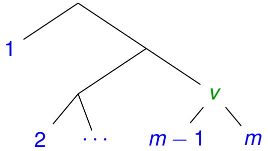

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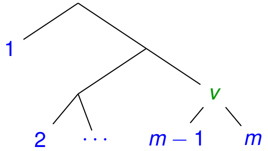

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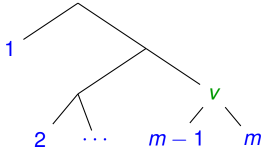

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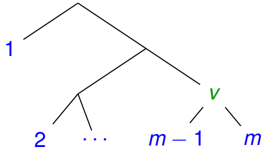

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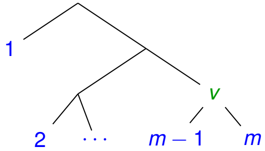

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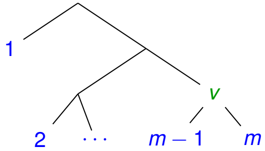

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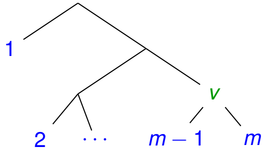

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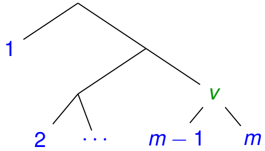

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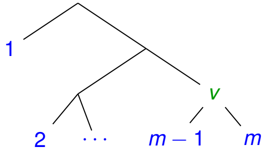

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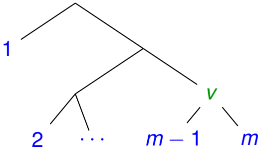

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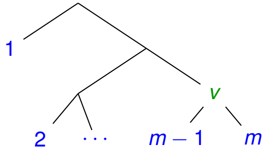

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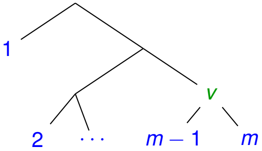

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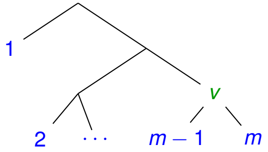

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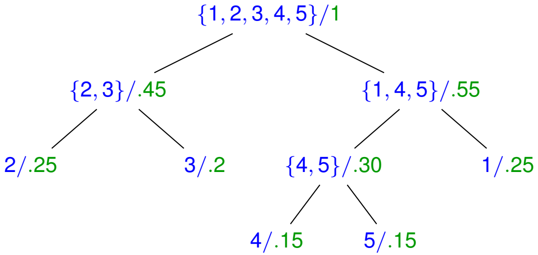

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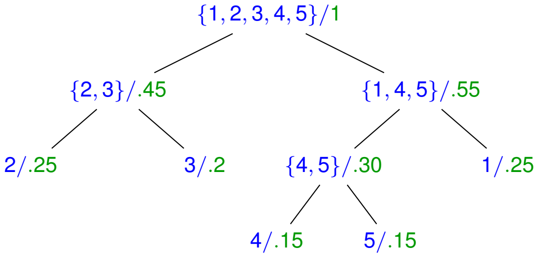

- On board...

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For non-finite codes, proof can be carried using simple induction on code tree depth.

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Discrete distribution generation

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Algorithm G generates the rv $X \sim \{p_1, \ldots, p_m\}$ if the following holds: in each step, G either stops or flips a coin $\sim (q_i, 1 - q_i)$.[a] After it stop, G outputs a value in $\mathbb{N}$. The probability that G outputs i is p_i.

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Let X be rv, and let g be the expected number of coins used by its best generating algorithm. Then $H(X) \leq g(X) \leq H(X) + 1$. If each p_i is a power of 2 (i.e., 2^{-k} for some $k \in \mathbb{Z}$), then $g(X) = H(X)$.

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Proposition 6

Let X be a rv, and let $g_b(X)$ be the expected number of coins used by its best generating algorithm that only flips uniform coins. Then
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- Expected number of coins used is $\sum_i p_i \ell(\mathsf{M}(i)) = -\sum_i p_i \log p_i = H(X)$

Let $X \sim \{p_1, \ldots, p_n\}$

- Let $(p_{i,1}, p_{i,2}, \ldots)$ be the binary representation of p_i and let $p_i^{(j)} = p_{i,j} \cdot 2^{-j}$
- Define K_i over $\mathbb{N}$ by $\Pr[K_i = j] = \frac{p_i^{(j)}}{p_i}$
- Let $Y = (X, K_X)$
- $\Pr[Y = (i,j)] = p_i^{(j)}$. Hence, $\sum_j \Pr[Y = (i,j)] = \Pr[X = i]$.
- $g_b(X) \leq g(Y) = H(Y) = g_b(Y)$
- We conclude the proof showing that $H(Y) \leq H(X) + 2$.

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- Hence, $H(Y) = \sum_i T_i \leq -\sum_i -p_i \log p_i + 2 \sum_i p_i = H(X) + 2$

Part III

Gambling

Horse racing

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- We are interested in $S_n := \prod_{i=1}^{n} S(X_i)$, where X_i's are iid $\sim p$

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- By weak low of large numbers,

$$\frac{1}{n} \log S_n = \frac{1}{n} \sum_i \log(S(X_i)) \xrightarrow{n} \mathrm{E}(\log S(X_1)) = W(\mathbf{b}, \mathbf{p})$$

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where $D(\mathbf{p}||\mathbf{b})$, the relative entropy from $\mathbf{p}$ to $\mathbf{b}$, is known to be non-negative.

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- Hence, $\Delta W = H(X) - H(X|Y) = I(X; Y)$. $\square$