Foundation of Cryptography (0368-4162-01), Lecture 2

Pseudorandom Generators

Iftach Haitner, Tel Aviv University

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Section 1

Distributions and Statistical Distance

Distributions and Statistical Distance

Let P and Q be two distributions over a finite set $\mathcal{U}$. Their *statistical distance* (also known as, variation distance), denoted by $\mathsf{SD}(P,Q)$, is defined as

$$\mathsf{SD}(P,Q) := \frac{1}{2}\sum_{x\in\mathcal{U}} |P(x) - Q(x)| = \max_{\mathcal{S}\subseteq\mathcal{U}} \left(P(\mathcal{S}) - Q(\mathcal{S})\right)$$

We will only consider finite distributions.

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Claim 1

For any pair of (finite) distribution P and Q, it holds that such

$$\mathrm{SD}(P, Q) = \max_{\mathrm{D}} (\Pr_{x \leftarrow P}[\mathrm{D}(x) = 1] - \Pr_{x \leftarrow Q}[\mathrm{D}(x) = 1]),$$

where D is any algorithm.

Some useful facts

Let P, Q, R be finite distributions, then

Triangle inequality:

$$\mathrm{SD}(P, R) \leq \mathrm{SD}(P, Q) + \mathrm{SD}(Q, R)$$

Repeated sampling:

$$\mathrm{SD}((P, P), (Q, Q)) \leq 2 \cdot \mathrm{SD}(P, Q)$$

Random variables

Distribution ensembles and statistical indistinguishability

Definition 2 (distribution ensembles)

$\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$ is a distribution ensemble, if P_n is a (finite) distribution for any $n \in \mathbb{N}$.
$\mathcal{P}$ is efficiently samplable (or just efficient), if $\exists$ PPT *Samp* with $\mathrm{Sam}(1^n) \equiv P_n$.

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Two distribution ensembles $\mathcal{P}$ and $\mathcal{Q}$ are *statistically indistinguishable*, if $\mathrm{SD}(P_n, Q_n) = \mathrm{neg}(n)$.

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Alternatively, if $\left|\Delta^{\mathrm{D}}_{(\mathcal{P},\mathcal{Q})}(n)\right| = \mathrm{neg}(n)$, for *any* algorithm D, where
$\Delta^{\mathrm{D}}_{(\mathcal{P},\mathcal{Q})}(n) := \Pr_{x\leftarrow P_n}[\mathrm{D}(1^n, x) = 1] - \Pr_{x\leftarrow Q_n}[\mathrm{D}(1^n, x) = 1]$.

Section 2

Computational Indistinguishability

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Definition 4 (computational indistinguishability)

Two distribution ensembles $\mathcal{P}$ and $\mathcal{Q}$ are *computationally indistinguishable*, if $\left|\Delta^{\mathsf{D}}_{(\mathcal{P},\mathcal{Q})}(n)\right| = \operatorname{neg}(n)$, for any PPT D.

$$(\Delta^{\mathsf{D}}_{(\mathcal{P},\mathcal{Q})}(n) := \Pr_{x\leftarrow P_n}[\Delta\mathsf{D}(1^n, x) = 1] - \Pr_{x\leftarrow Q_n}[\mathsf{D}(1^n, x) = 1])$$

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- Can it be different from the statistical case?

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Repeated sampling

Question 5

Assume that $\mathcal{P}$ and $\mathcal{Q}$ are computationally indistinguishable, is it always true that $\mathcal{P}^2 = (\mathcal{P}, \mathcal{P})$ and $\mathcal{Q}^2 = (\mathcal{Q}, \mathcal{Q})$ are?

Repeated sampling

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Assume that $\mathcal{P}$ and $\mathcal{Q}$ are computationally indistinguishable, is it always true that $\mathcal{P}^2 = (\mathcal{P}, \mathcal{P})$ and $\mathcal{Q}^2 = (\mathcal{Q}, \mathcal{Q})$ are?

Assume that $\left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2,\mathcal{Q}^2)}(n)\right| = \delta(n)$ for some PPT D, we would like to prove that $\exists$ PPT D′ with $\left|\Delta^{\mathsf{D}}_{(\mathcal{P},\mathcal{Q})}(n)\right| \geq \delta(n)/2$ for every $n \in \mathbb{N}$.

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$$\begin{aligned}
\delta(n) &= \left|\Pr_{x\leftarrow P_n^2}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow Q_n^2}[\mathsf{D}(x)=1]\right| \\
&\leq \left|\Pr_{x\leftarrow P_n^2}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow (P_n,Q_n)}[\mathsf{D}(x)=1]\right| \\
&\quad + \left|\Pr_{x\leftarrow (P_n,Q_n)}[\mathsf{D}(x)=1] - \Pr_{x\leftarrow Q_n^2}[\mathsf{D}(x)=1]\right|
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&= \left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2,(\mathcal{P},\mathcal{Q}))}(n)\right| + \left|\Delta^{\mathsf{D}}_{((\mathcal{P},\mathcal{Q}),\mathcal{Q}^2)}(n)\right|
\end{aligned}$$

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&= \left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2,(\mathcal{P},\mathcal{Q}))}(n)\right| + \left|\Delta^{\mathsf{D}}_{((\mathcal{P},\mathcal{Q}),\mathcal{Q}^2)}(n)\right|
\end{aligned}
$$

So either $|\Delta^{\mathsf{D}}_{(\mathcal{P}^2,(\mathcal{P},\mathcal{Q}))}(n)| \geq \delta(n)/2$, or $|\Delta^{\mathsf{D}}_{((\mathcal{P},\mathcal{Q}),\mathcal{Q}^2)}(n)| \geq \delta/2$

- Assume that $\left|\Delta^{\mathsf{D}}_{(\mathcal{P}^2,\mathcal{Q}^2)}(n)\right| \geq 1/p(n)$ for some $p \in$ poly and infinitely many n's, and assume wlg. that $\left|\Delta^{\mathsf{D}}_{\mathcal{P}^2,(\mathcal{P},\mathcal{Q})}(n)\right| \geq 1/2p(n)$ for infinitely many n's.

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- Can we use D to contradict the fact that $\mathcal{P}$ and $\mathcal{Q}$ are computationally close?
- Assuming that $\mathcal{P}$ and $\mathcal{Q}$ are efficiently samplable
- Non-uniform settings

Repeated sampling cont.

Given $t = t(n) \in \mathbb{N}$ and a distribution ensemble $\mathcal{P} = \{P_n\}_{n\in\mathbb{N}}$, let $\mathcal{P}^t = \{P_n^{t(n)}\}_{n\in\mathbb{N}}$

Question 6

Let $t = t(n) \leq \text{poly}(n)$ be an eff. computable integer function. Assume that $\mathcal{P}$ and $\mathcal{Q}$ are eff. samplable and computationally indistinguishable , does it mean that $\mathcal{P}^t$ and $\mathcal{Q}^t$ are?

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Proof:

- Induction?

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Proof:

- Induction?
- Hybrid

Hybrid argument

Let D be an algorithm, and for $n \in \mathbb{N}$ let
$\delta(n) = \left|\Delta^{\mathsf{D}}_{(\mathcal{P}^{t(n)},\mathcal{Q}^{t(n)})}(t(n))\right|$.

- For $i \in \{0, \ldots, t = t(n)\}$, let $H^i = (p_1, \ldots, p_i, q_{i+1}, \ldots, q_t)$, where the p's [resp., q's] are uniformly (and independently) chosen from P_n [resp., from Q_n].

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- Since $\delta(n) = \left|\Delta^{\mathsf{D}}_{H^n,H^0}(t)\right| = \left|\sum_{i\in[t]} \Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right|$, there exists $i \in [t]$ with

$$\left|\Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right| \geq \delta(n)/t(n)$$

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$$\left|\Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right| \geq \delta(n)/t(n)$$
- How do we use it?

Using hybrid argument via estimation

Algorithm 7 (D′)

Input: 1^n and $x \in \{0,1\}^*$

1. Find $i \in [t]$ with $\left|\Delta^{\mathsf{D}}_{H^i,H^{i-1}}(t)\right| \geq \delta(n)/2t(n)$
2. Let $(p_1,\ldots,p_i,q_{i+1},\ldots,q_t) \leftarrow H^i$
3. Return $\mathsf{D}(1^t,p_1,\ldots,p_{i-1},x,q_{i+1},\ldots,q_t)$, .

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1. how do we find i?
2. Easy in the non-uniform case

Using Hybrid argument via sampling

Algorithm 8 (D′)

Input: 1^n and $x \in \{0,1\}^*$

1. Sample $i \leftarrow [t = t(n)]$
2. Let $(p_1, \ldots, p_i, q_{i+1}, \ldots, q_t) \leftarrow H^i$
3. Return $D(1^t, p_1, \ldots, p_{i-1}, x, q_{i+1}, \ldots, q_t)$.

Using Hybrid argument via sampling

Algorithm 8 (D')

Input: 1^n and $x \in \{0,1\}^*$

1. Sample $i \leftarrow [t = t(n)]$
2. Let $(p_1, \ldots, p_i, q_{i+1}, \ldots, q_t) \leftarrow H^i$
3. Return $\mathsf{D}(1^t, p_1, \ldots, p_{i-1}, x, q_{i+1}, \ldots, q_t)$.

$$\left|\Delta^{\mathsf{D}'}_{(\mathcal{P},\mathcal{Q})}(n)\right| = \left|\Pr_{p \leftarrow P_n}[\mathsf{D}'(p) = 1] - \Pr_{q \leftarrow Q_n}[\mathsf{D}'(q) = 1]\right|$$

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&= \left|\frac{1}{t}\left(\Pr_{x \leftarrow (H_t}[\mathrm{D}(x) = 1] - \Pr_{x \leftarrow H_0}[\mathrm{D}(x) = 1]\right)\right| \\
&= \delta(n)/t(n)
\end{aligned}$$

Section 3

Pseudorandom Generators

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An efficiently computable function $g : \{0,1\}^n \mapsto \{0,1\}^{\ell(n)}$ is a pseudorandom generator, if

- g is length extending (i.e., $\ell(n) > n$ for any n)
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- Imply one-way functions (homework)
- Do they have any use?

Section 4

Hardcore Predicates

Hardcore predicates

- Building blocks in constructions of PRGS from OWF

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$$\Pr[P(f(U_n)) = b(U_n)] \leq \frac{1}{2} + \operatorname{neg}(n),$$

for any PPT P.

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- Does the existence of a hardcore predicate for f, implies that f is one way?

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Hardcore predicates

- Building blocks in constructions of PRGS from OWF

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- Fact: any PRG has HCP (homework).
- Fact: any OWF has a hardcore predicate (next class)

Section 5

PRGs from OWPs

OWP to PRG

Claim 12

Let $f : \{0,1\}^n \mapsto \{0,1\}^n$ be a permutation and let $b : \{0,1\}^n \mapsto \{0,1\}$ be a hardcore predicate for f, then $g(x) = (f(x), b(x))$ is a PRG.

OWP to PRG

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Proof: Assume $\exists$ a PPT D, and infinite set $\mathcal{I} \subseteq \mathbb{N}$ and $p \in$ poly with

$$\left|\Delta^{\mathsf{D}}_{g(U_n),U_{n+1}}\right| > \varepsilon(n) = 1/p(n)$$

for any $n \in \mathcal{I}$. We use D for breaking the hardness of b.

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- We assume wlg. that $\Pr[\mathsf{D}(g(U_n)) = 1] - \Pr[\mathsf{D}(U_{n+1}) = 1] \geq \varepsilon(n)$ for any $n \in \mathcal{I}$ (can we do it?), and fix $n \in \mathcal{I}$.

OWP to PRG cont.

- Let $\delta(n) = \Pr[\mathrm{D}(U_{n+1}) = 1]$ (note that $\Pr[\mathrm{D}(g(U_n)) = 1] = \delta + \varepsilon$).

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\delta &= \Pr[\mathrm{D}(f(U_n), U_1) = 1] \\
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Hence,

$$\Pr[\mathsf{D}(f(U_n), \overline{b(U_n)}) = 1] = \delta - \varepsilon \tag{1}$$

OWP to PRG cont.

- $\Pr[D(f(U_n), b(U_n)) = 1] = \delta + \varepsilon$
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OWP to PRG cont.

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- Consider the following algorithm for predicting b:

Algorithm 13 (P)

Input: $y \in \{0,1\}^n$

1. Flip a random coin $c \leftarrow \{0,1\}$.
2. If $\mathrm{D}(y, c) = 1$ output c, otherwise, output $\overline{c}$.

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OWP to PRG cont.

Remark 14

- Prediction to distinguishing (homework)

OWP to PRG cont.

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- Prediction to distinguishing (homework)
- PRG from any OWF: (1) Regular OWFs, first use pairwise hashing to convert into “almost" permutation. (2) Any OWF, harder

Section 6

PRG Length Extension

PRG Length Extension

Construction 15 (iterated function)

Given a length increasing function $g\colon \{0,1\}^n \mapsto \{0,1\}^\ell$ and $i \in \mathbb{N}$, define $g^i\colon \{0,1\}^n \mapsto \{0,1\}^{n+i(\ell-n)}$ as

$$g^i(x) = x^{i-1}_{n+1,\ldots,\left|x^{i-1}\right|}, g(x^{i-1}_{1,\ldots,n}),$$

where $x^{i-1} = g^{i-1}(x)$ and $g^0(x) = x$.

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Let $g\colon \{0,1\}^n \mapsto \{0,1\}^{n+1}$ be a PRG, then $g^t\colon \{0,1\}^n \mapsto \{0,1\}^{n+t(n)}$ is a PRG, for any $t \in$ poly.

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PRG Length Extension cont.

- Fix $n \in \mathbb{N}$, for $i \in \{0, \ldots, t = t(n)\}$, let $H^i = U_{t-i}, g^i(U_n)$ (i.e., the distribution of H^i is $\left(x, g^i(x')\right)_{x \leftarrow \{0,1\}^{t-i}, x' \leftarrow \{0,1\}^n}$)

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Input: 1^n and $y \in \{0,1\}^{n+1}$

1. Sample $i \leftarrow [t]$
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