Application of Information Theory, Lecture 1
Basic Definitions and Facts

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- Entropy is a function of p (sometimes refers to as $H(p)$).

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   - $h(p) := H(p, 1-p)$ is continuous

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Examples

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   (i.e., for some $x_1 \neq x_2 \neq x_3$, $\mathsf{P}_X(x_1) = \frac{1}{2}, \mathsf{P}_X(x_2) = \frac{1}{4}, \mathsf{P}_X(x_3) = \frac{1}{4}$)
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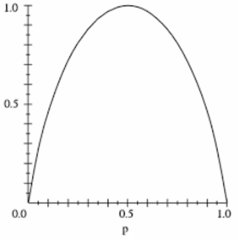

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And all are rather simple to prove

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Shannon function is the only symmetric function (over probability distributions) satisfying the following three axioms:

A1 Continuity: $H(p, 1-p)$ is continuous function of p.

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A3 Grouping axiom:
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We prove (assuming additional axiom) that H^* is the Shannon function H.

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Claim follows by combining the above equations. □

Further generalization of the grouping axiom

Let $1 = k_1 < k_2 < \ldots < k_q < m$ and let $C_t = \sum_{i=k_t}^{k_{t+1}-1} p_i$ (letting $k_{q+1} = m+1$).

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Implication: Let $f(m) := H^*(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m})$

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Proof: Follow by the extended group axiom and the symmetry of H $\square$

Implication: Let $f(m) := H^*(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{m})$

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Further generalization of the grouping axiom

Let $1 = k_1 < k_2 < \ldots < k_q < m$ and let $C_t = \sum_{i=k_t}^{k_{t+1}-1} p_i$ (letting $k_{q+1} = m+1$).

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- Proof extends to any integer (not only 3)

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- By continuity axiom, holds for every p_1, p_2, p_3.

□

Section 1

Basic Properties

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$\implies$ (Jensen inequality): $\mathrm{E}\, f(X) \leq f(\mathrm{E}\, X)$ for any random variable X.

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- Hence, $H(p_1, \ldots, p_m) = \sum_i p_i \log \frac{1}{p_i} \leq \log \sum_i p_i \frac{1}{p_i} = \log m$
- Alternatively, for X over $\{1, \ldots, m\}$,
  $H(X) = \mathrm{E}_X \log \frac{1}{\mathrm{P}_X(X)} \leq \log \mathrm{E}_X \frac{1}{\mathrm{P}_X(X)} = \log m$

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- Tight bounds
  - $H(p_1, \ldots, p_m) = 0$ for $(p_1, \ldots, p_m) = (1, 0, \ldots, 0)$.
  - $H(p_1, \ldots, p_m) = \log m$ for $(p_1, \ldots, p_m) = (\frac{1}{m}, \ldots, \frac{1}{m})$.
- Non negativity is clear.
- A function f is concave ("keura") if $\forall\ t_1, t_2,\ \lambda \in [0, 1] \leq 1$ $\lambda f(t_1) + (1 - \lambda) f(t_2) \leq f(\lambda t_1 + (1 - \lambda) t_2)$

$\implies$ (by induction) $\forall\ t_1, \ldots, t_k$, $\lambda_1, \ldots, \lambda_k \in [0, 1]$ with $\sum_i \lambda_i = 1$ $\sum_i \lambda_i f(t_i) \leq f(\sum_i \lambda_i t_i)$

$\implies$ (Jensen inequality): $\mathrm{E}\, f(X) \leq f(\mathrm{E}\, X)$ for any random variable X.

- $\log(x)$ is (strictly) concave for $x > 0$, since its second derivative $(-\frac{1}{x^2})$ is always negative.
- Hence, $H(p_1, \ldots, p_m) = \sum_i p_i \log \frac{1}{p_i} \leq \log \sum_i p_i \frac{1}{p_i} = \log m$
- Alternatively, for X over $\{1, \ldots, m\}$, $H(X) = \mathrm{E}_X \log \frac{1}{\mathrm{P}_X(X)} \leq \log \mathrm{E}_X \frac{1}{\mathrm{P}_X(X)} = \log m$
- What if $\mathrm{Supp}(X) := \{x \colon \mathrm{P}_X(x) > 0\} \subsetneq [m]$?

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