Application of Information Theory, Lecture 2
Joint & Conditional Entropy, Mutual Information

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Nov 4, 2014

Part I

Joint and Conditional Entropy

Joint entropy

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Joint entropy, cont.

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&= -\mathop{\mathrm{E}}_{Z = p_{Y|X}(Y|X)} \log Z
\end{aligned}$$

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What is $H(Y|X)$ and $H(X|Y)$?

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What is $H(Y|X)$ and $H(X|Y)$?

$$\begin{aligned} H(Y|X) &= \mathop{\mathrm{E}}_{x \leftarrow X} H(Y|X=x) \\ &= \frac{1}{2} H(Y|X=0) + \frac{1}{2} H(Y|X=1) \\ &= \frac{1}{2} H(\frac{1}{2}, \frac{1}{2}) + \frac{1}{2} H(1,0) = \frac{1}{2}. \end{aligned}$$

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$$\begin{aligned} H(X|Y) &= \mathop{\mathrm{E}}_{y \leftarrow Y} H(X|Y = y) \\ &= \frac{3}{4} H(X|Y = 0) + \frac{1}{4} H(X|Y = 1) \\ &= \frac{3}{4} H(\frac{1}{3}, \frac{2}{3}) + \frac{1}{4} H(1, 0) = 0.6887 \neq H(Y|X). \end{aligned}$$

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- $$H(X|Y,Z) = \mathop{\mathrm{E}}_{(y,z)\leftarrow(Y,Z)} H(X|Y=y, Z=z)$$

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  for $(X_y,Z_y)=(X,Z)|Y=y$

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- Note that $H(Y|X = x)$ might be larger than $H(Y)$ for some $x \in \mathrm{Supp}(X)$.

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- $H(Y|X) = H(X, Y) - H(X)$ is as an alternative definition for $H(Y|X)$.

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$$\begin{aligned} & H(P_{1,1},\ldots,P_{n,n}) \\ &= H(q_1,\ldots,q_n) + \sum q_i H(\frac{P_{i,1}}{q_i},\ldots,\frac{P_{i,n}}{q_i}) \\ &= H(X) + H(Y|X). \end{aligned}$$

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- Another proof.

Chain rule (for the entropy function)

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Jensen inequality: for any concave function f, values $t_1, \ldots, t_k$ and $\lambda_1, \ldots, \lambda_k \in [0,1]$ with $\sum_i \lambda_i = 1$, it holds that $\sum_i \lambda_i f(t_i) \leq f(\sum_i \lambda_i t_i)$.

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Chain rule (for the entropy function), general case

Claim 2

For rvs $X_1, \ldots, X_k$, it holds that
$H(X_1, \ldots, X_k) = H(X_i) + H(X_2|X_1) + \ldots + H(X_k|X_1, \ldots, X_{k-1})$.

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Examples

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$\implies \quad t \geq \log n! = \Theta(n \log n)$

Concavity of entropy function

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Let $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ be two distributions, and for $\lambda \in [0, 1]$ consider the distribution $\tau_\lambda = \lambda p + (1 - \lambda)q$.
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We are now certain that we drew the graph of the (two-dimensional) entropy function right...

Part II

Mutual Information

Mutual information

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&= h(\frac{1}{4}) - \frac{1}{2} h(\frac{1}{2})
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For rvs $X_1, \ldots, X_k, Y$, it holds that
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Examples

- Let $X_1, \ldots, X_n$ be iid with $X_i \sim (p, 1-p)$, under the condition that $\bigoplus_i x_i = 0$.
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$$\begin{aligned}
I(T; F) &= H(T) - H(T|F) \\
&= \log 6 - \log 4 \\
&= \log 3 - 1.
\end{aligned}$$

Part III

Data processing

Data processing Inequality

Definition 5 (Markov Chain)

Rvs $(X, Y, Z) \sim p$ form a Markov chain, denoted $X \to Y \to Z$, if $p(x, y, z) = p_X(x) \cdot p_{Y|X}(y|x) \cdot p_{Z|Y}(z|y)$, for all x, y, z.

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- Since $I(X; Y|Z) \geq 0$, we conclude $I(X; Y) \geq I(X; Z)$.$\square$

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For any rvs X and Y, and any (even random) g, it holds that

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