Foundation of Cryptography (0368-4162-01), Lecture 8

Encryption Schemes

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January 3 – 17, 2012

Section 1

Definitions

Correctness

Definition 1 (encryption scheme)

A trippet of PPT's $(G, \mathrm{E}, \mathrm{D})$ such that

1. $G(1^n)$ outputs a key $(e, d) \in \{0,1\}^* \times \{0,1\}^*$
2. $\mathrm{E}(e, m)$ outputs a string in $c \in \{0,1\}^*$
3. $\mathrm{D}(d, c)$ outputs $m \in \{0,1\}^*$

Correctness: $\mathrm{D}(d, \mathrm{E}(e, m)) = m$, for any $(e, d) \in \mathrm{Supp}(G(1^n))$ and $m \in \{0,1\}^*$

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- e – encryption key, d – decryption key
- m – plaintext, $c = \mathrm{E}(e, m)$ – ciphertext
- $E_e(m) \equiv E(e, m)$ and $D_d(c) \equiv D(d, c)$,

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- e – encryption key, d – decryption key
- m – plaintext, $c = \mathsf{E}(e, m)$ – ciphertext
- $E_e(m) \equiv E(e, m)$ and $D_d(c) \equiv D(d, c)$,
- public/private key

Security

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- Other concerns, e.g., multiple encryptions, active adversary

Semantic Security

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2. Formulate via the simulation paradigm
3. Cannot hide the message length

Semantic security – private-key model

Definition 2 (Semantic Security – private-key model)

An encryption scheme $(G, \mathsf{E}, \mathsf{D})$ is semantically secure in the private-key model, if for any PPT A, $\exists$ PPT A$'$ s.t. $\forall$ poly-bounded dist. ensemble $\mathcal{M} = \{\mathcal{M}_n\}_{n\in\mathbb{N}}$ and poly-bounded functions $h, f \colon \{0,1\}^* \mapsto \{0,1\}^*$

$$\Big|\Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathsf{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] - \Pr_{m\leftarrow\mathcal{M}_n}[\mathsf{A}'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)]\Big| = \operatorname{neg}(n)$$

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- poly-bounded?

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- Reflection to ZK
- public-key variant – A gets e

Indistinguishablity of encryptions

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- Less intuitive than semantic security, but easier to work with

Indistinguishablity of encryptions – private-key model

Definition 3 (Indistinguishablity of encryptions – private-key model)

An encryption scheme $(G, \mathrm{E}, \mathrm{D})$ has indistinguishable encryptions in the private-key model, if for any $p, \ell \in \mathrm{poly}$, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n \in \mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$ and poly-time B,

$$\left|\Pr_{e \leftarrow G(1^n)_1}[\mathrm{B}(z_n, E_e(x_n)) = 1] - \Pr_{e \leftarrow G(1^n)_1}[\mathrm{B}(z_n, E_e(y_n)) = 1]\right| = \mathrm{neg}(n)$$

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- Non-uniform definition
- Public-key variant

Equivalence of definitions

Theorem 4

An encryption scheme $(G, \mathsf{E}, \mathsf{D})$ *is semantically secure iff is has indistinguishable encryptions.*

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We prove the private key case

Indistinguishablity $\Longrightarrow$ Semantic Security

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Fix $\mathcal{M}$, A, f and h, be as in Definition 2.

Indistinguishablity $\implies$ Semantic Security

Fix $\mathcal{M}$, A, f and h, be as in Definition 2. We construct A′ as

Algorithm 5 (A′)

Input: 1^n, $1^{|m|}$ and $h(m)$

1. $e \leftarrow G(1^n)_1$
2. $c = E_e(1^{|m|})$
3. Output $A(1^n, 1^{|m|}, h(m), c)$

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Claim 6

A′ is a good simulator for A (according to Definition 2)

Proving Claim 6

Assume exists infinite $\mathcal{I} \subseteq \mathbb{N}$ and $p \in \text{poly}$ s.t. for any $n \in \mathcal{I}$:

$$\left|\Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1}[\mathsf{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \right. \quad (1)$$
$$\left. - \Pr_{m \leftarrow \mathcal{M}_n}[\mathsf{A}'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)] \right| > 1/p(n)$$

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Fix $n \in \mathcal{I}$ and let $x_n \in \mathrm{Supp}(\mathcal{M}_n)$ be a value that maximize Equation (3).

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Assume exits algorithm B that contradicts the indistinguishability of the scheme with respect to
$\{(x_n, y_n = 1^{|x_n|})\}_{n\in\mathbb{N}}$ and
$\{z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n))\}_{n\in\mathbb{N}}$.

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Algorithm 7 (B)

Input: $z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n))$, c
Output 1 iff $\mathsf{A}(1^n, 1^{|x_n|}, h(x+n), c) = f(1^n, x_n)$

Semantic Security $\Longrightarrow$ Indistinguishablity

Assume $\exists$ B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and a $\{z_n\}_{n\in\mathbb{N}}$, such that (wlg.) for infinitely many n's:

$$\Pr_{e\leftarrow G(1^n)_1}[\mathrm{B}(z_n, E_e(x_n)) = 1] - \Pr_{e\leftarrow G(1^n)_1}[\mathrm{B}(z_n, E_e(y_n)) = 1] \geq \frac{1}{p(n)}$$

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- Let $\mathcal{M}_n$ be x_n wp $\frac{1}{2}$ and y_n otherwise.
- Let $f(1^n, x_n) = 1$, $f(1^n, y_n) = 0$ and $h(1^n, \cdot) = z_n$).
- Define A$(1^n, 1^{\ell(n)}, z_n, c)$ to return B(z_n, c).

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$$\Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathrm{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \geq \frac{1}{2} + \frac{1}{p(n)}$$

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For *any* A′

$$\Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathrm{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \leq \frac{1}{2}$$

Security Under Multiple Encryptions

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An encryption scheme $(G, \mathrm{E}, \mathrm{D})$ has indistinguishable encryptions for multiple messages in the private-key model, if for any $p, \ell, t \in \mathrm{poly}$, $\{x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$ and polynomial-time B,

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Extensions:

- Different length messages

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Extensions:

- Different length messages
- Semantic security version

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Extensions:

- Different length messages
- Semantic security version
- Public-key definition

Multiple Encryption in the Public-Key Model

Theorem 9

A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.

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Proof: Assume $(G, \mathrm{E}, \mathrm{D})$ is public-key secure for a single message and not for multiple messages with respect to B, $\{x_{1,t(n)}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$.

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Theorem 9

A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.

Proof: Assume $(G, \mathrm{E}, \mathrm{D})$ is public-key secure for a single message and not for multiple messages with respect to B, $\{x_{1,t(n)}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$.
It follows that for some function $i(n) \in [t(n)]$

$$\begin{aligned}&\Big|\Pr[\mathrm{B}(1^n, e, E_e(x_{n,1}), \ldots, E_e(x_{n,i-1}), E_e(y_{n,i}) \ldots, E_e(y_{n,t(n)})) = 1]\\ &-\Pr[\mathrm{B}(1^n, e, E_e(x_{n,1}), \ldots, E_e(x_{n,i}), E_e(y_{n,i+1}) \ldots, E_e(y_{n,t(n)})) = 1]\Big|\\ &\quad > \mathrm{neg}(n)\end{aligned}$$

where in both cases $e \leftarrow G(1^n)_1$

Algorithm 10 (B′)

Input: 1^n, $z_n = (i(n), x_{1,t(n)}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)}, e\ , c$
Return $\mathrm{B}(c, E_e(x_{n,1}), \ldots, E_e(x_{n,i-1}), c, E_e(y_{n,i+1}) \ldots, E_e(y_{n,t(n)}))$

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B' is critically using the public key

Multiple Encryption in the Private-Key Model

Fact 11

Assuming (non uniform) OWFs exists, there exists an encryption scheme that has private-key indistinguishable encryptions for a single messages, but not *for multiple messages*

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Proof: Let $g\colon \{0,1\}^n \mapsto \{0,1\}^{n+1}$ be a (non-uniform) PRG, and for $i \in \mathbb{N}$ let g^i be its "iterated extension" to output of length i (see Lecture 2, Construction 15).

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Construction 12

- $G(1^n)$ outputs $e \leftarrow \{0,1\}^n$,
- $\mathrm{E}_e(m)$ outputs $g^{|m|}(e) \oplus m$
- $\mathrm{D}_e(c)$ outputs $g^{|c|}(e) \oplus c$

Claim 13

$(G, \mathrm{E}, \mathrm{D})$ has private-key indistinguishable encryptions for a single message

Proof:

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(G, E, D) has private-key indistinguishable encryptions for a single message

Proof: Assume not, and let B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n \in \mathbb{N}}$ and $\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$ be the triplet that realizes it.

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$$\left|\Pr[\mathrm{B}(z_n, g^{|x_n|}(U_n) \oplus x_n) = 1] - \Pr[\mathrm{B}(z_n, U_{|x_n|} \oplus x_n) = 1]\right| > \mathrm{neg}(n) \tag{2}$$

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Hence, B implies a (non-uniform) distinguisher for *g*

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$(G, \mathsf{E}, \mathsf{D})$ does not have a private-key indistinguishable encryptions for multiple messages

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Claim 14

$(G, \mathsf{E}, \mathsf{D})$ does not have a private-key indistinguishable encryptions for multiple messages

Proof: Take $x_{n,1} = x_{n,2}$, $y_{n,1} \neq y_{n,2}$ and $\mathsf{D}(c_1, c_2)$ outputs 1 iff $c_1 = c_2$

Section 2

Constructions

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Suffice to encrypt messages of a single length (here the length is n).

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Construction 15

- $G(1^n)$: output $e \leftarrow \mathcal{F}_n$,
- $\mathrm{E}_e(m)$: choose $r \leftarrow \{0,1\}^n$ and output $(r, e(r) \oplus m)$
- $\mathrm{D}_e(r, c)$: output $e(r) \oplus c$

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Proof:

Public key indistinguishable encryptions for multiple messages

Let (G, f, Inv) be a (non-uniform) family of trapdoor permutations (see Lecture 6, Def 8) and let b be an hardcore predicate for f.

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Construction 17 (bit encryption)

- $G(1^n)$: output $(e, d) \leftarrow G(1^n)$
- $\mathrm{E}_e(m)$: choose $r \leftarrow \{0,1\}^n$ and output $(y = f_e(r), c = b(r) \oplus m)$
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- We believe that public-key encryptions are of different complexity than private-key ones

Section 3

Active Adversaries

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- In the public-key settings, the adversary is also given the public key
- We focus on indistinguishability, but each of the above definitions has an equivalent semantic security variant.

CPA Security

Let $(G, \mathsf{E}, \mathsf{D})$ be an encryption scheme. For a pair of algorithms $\mathsf{A} = (\mathsf{A}_1, \mathsf{A}_2)$, $n \in \mathbb{N}$, $z \in \{0,1\}^*$ and $b \in \{0,1\}$, let:

Experiment 19 ($\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathsf{A}_1^{E_e(\cdot)}(1^n, z)$
3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

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Definition 20 (private key CPA)

$(G, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model under CPA attack, if $\forall$ PPT $\mathsf{A}_1, \mathsf{A}_2$, and poly-bounded $\{z_n\}_{n\in\mathbb{N}}$:

$$\left|\Pr[\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z_n}(0) = 1] - \Pr[\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z_n}(1) = 1]\right| = \mathrm{neg}(n)$$

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- public-key variant...
- The scheme from Construction 15 has indistinguishable encryptions in the private-key model under CPA attack(for short, private-key CPA secure)
- The scheme from Construction 17 has indistinguishable encryptions in the public-key model (for short, public-key CPA secure)
- In both cases, definitions are *not* equivalent

CCA Security

Experiment 21 ($\mathrm{Exp}^{\mathrm{CCA1}}_{\mathrm{A},n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathrm{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
3. $c \leftarrow \mathrm{E}_e(m_b)$
4. Output $\mathrm{A}_2^{E_e(\cdot)}(1^n, s, c)$

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4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

Experiment 22 ($\mathrm{Exp}^{\mathrm{CCA2}}_{\mathsf{A},n,z_n}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(x_0, x_1, s) \leftarrow \mathsf{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
3. $c \leftarrow \mathsf{E}_e(x_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot), D_d^{\neg c}(\cdot)}(1^n, s, c)$

Definition 23 (private key CCA1/CCA2)

$(G, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model under $x \in \{\mathsf{CCA1}, \mathsf{CCA2}\}$ attack, if $\forall$ PPT $\mathsf{A}_1, \mathsf{A}_2$, and poly-bounded $\{z_n\}_{n \in \mathbb{N}}$:

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- The public key definition is analogous

Private-key CCA2

- Is the scheme from Construction 15 private-key CCA1 secure?

Private-key CCA2

- Is the scheme from Construction 15 private-key CCA1 secure?
- CCA2 secure?

Let $(G, \mathsf{E}, \mathsf{D})$ be a private key CPA scheme, and let $(\mathsf{Gen}_M, \mathsf{Mac}, \mathsf{Vrfy})$ be an existential unforgeable strong MAC.

Construction 24

- $G'(1^n)$: Output $(e \leftarrow G_E(1^n), k \leftarrow \mathsf{Gen}_M(1^n))$.[a]
- $\mathsf{E}'_{d,k}(m)$: let $c = \mathsf{E}_e(m)$ and output $(c, t = \mathsf{Mac}_k(c))$
- $\mathsf{D}_{e,k}(c, t)$: if $\mathsf{Vrfy}_k(c, t) = 1$, output $\mathsf{D}_e(c)$. Otherwise, output $\perp$

[a] We assume for simplicity that the encryption and decryption keys are the same.

Private-key CCA2

- Is the scheme from Construction 15 private-key CCA1 secure?
- CCA2 secure?

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Theorem 25

Construction 24 is a private-key CCA2*-secure encryption scheme.*

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Proof: ?

Public-key CCA1

Public-key CCA1

Let $(G, \mathrm{E}, \mathrm{D})$ be a public-key CPA scheme and let (P, V) be a NIZK for $\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) \colon \exists (m, r_0, r_1) \ \ s.t. \ c_0 = \mathrm{E}_{pk_0}(m, r_0) \wedge c_1 = \mathrm{E}_{pk_1}(m, r_1)\}$

Public-key CCA1

Let $(G, \mathsf{E}, \mathsf{D})$ be a public-key CPA scheme and let (P, V) be a NIZK for $\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) \colon \exists (m, r_0, r_1) \ \ s.t. \ c_0 = \mathsf{E}_{pk_0}(m, r_0) \land c_1 = \mathsf{E}_{pk_1}(m, r_1)\}$

Construction 26 (The Naor-Yung Paradigm)

- $G'(1^n)$:
  1. For $i \in \{0,1\}$: set $(sk_i, pk_i) \leftarrow G(1^n)$.
  2. Let $r \leftarrow \{0,1\}^{\ell(n)}$, and output $pk' = (pk_0, pk_1, r)$ and $sk' = (pk', sk_0, sk_1)$
- $\mathsf{E}'_{pk'}(m)$:
  1. For $i \in \{0,1\}$: $c_i = \mathsf{E}_{pk_i}(m, r_i)$, where r_i is a uniformly chosen string of the right length
  2. $\pi \leftarrow \mathsf{P}((c_0, c_1, pk_0, pk_1), (m, r_0, r_1), r)$
  3. Output (c_0, c_1, π).
- $\mathsf{D}'_{sk'}(c_0, c_1, \pi)$: If $\mathsf{V}((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, return $\mathsf{D}_{sk_0}(c_0)$. Otherwise, return $\perp$

Omitted details:

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n.
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using ”security parameter" n.

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Is the scheme CCA1 secure?

Omitted details:

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n.
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Is the scheme CCA1 secure? We need the NIZK to be “adaptive secure".

Theorem 27

Assuming that (P, V) *is adaptive secure, then Construction 26 is a public-key* CCA1 *secure encryption scheme.*

Omitted details:

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n.
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Is the scheme CCA1 secure? We need the NIZK to be "adaptive secure".

Theorem 27

Assuming that (P,V) *is adaptive secure, then Construction 26 is a public-key* CCA1 *secure encryption scheme.*

Proof: Given an attacker A' for the CCA1 security of $(G',\mathsf{E}',\mathsf{D}')$, we use it to construct an attacker A on the CPA security of $(G,\mathsf{E},\mathsf{D})$.
Let $\mathsf{S} = (\mathsf{S}_1,\mathsf{S}_2)$ be the (adaptive) simulator for $(\mathsf{P},\mathsf{V},\mathcal{L})$

Algorithm 28 (A)

1. On $(1^n, pk)$:
   - let $j \leftarrow \{0, 1\}$
   - Let $pk_{1-j} = pk$, $(pk_j, sk_j) \leftarrow G(1^n)$ and $r \leftarrow \mathrm{S}_1(1^n)$
   - Send $pk' = (pk_0, pk_1, r)$ to A′
2. On query (c_0, c_1, π) of A′ to D′: if π is a valid proof for $(c_0, c_1, pk_0, pk_1) \in \mathcal{L}$, return $\mathrm{D}_{sk_j}(c_j)$. Otherwise, return $\perp$.
3. Output the same pair (m_0, m_1) as A′ does
4. On challenge c ($= \mathrm{E}_{sk}(m_b)$):
   - Set $c_{1-j} = c$, $a \leftarrow \{0, 1\}$, $c_j = \mathrm{E}_{pk_j}(m_a)$, and $\pi \leftarrow \mathrm{S}_2((c_0, c_1, pk_0, pk_1), r)$
   - Send $c' = (c_0, c_1, \pi)$ to A′
5. Output the same value that A′ does

Claim 29

Assume that A$'$ breaks the CCA1 security of $(G', \mathrm{E}', \mathrm{D}')$ with probability $\delta(n)$, then A breaks the CPA security of $(G, \mathrm{E}, \mathrm{D})$ with probability $(\delta(n) - \mathrm{neg}(n))/2$.

Claim 29

Assume that A' breaks the CCA1 security of (G', E', D') with probability $\delta(n)$, then A breaks the CPA security of (G, E, D) with probability $(\delta(n) - \text{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[A' \text{ “makes" } A(1^n) \text{ decrypt an invalid cipher}] = \text{neg}(n) \quad (3)$$

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Ignoring negligible terms, we assume that A always encrypts using sk_0.

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Ignoring negligible terms, we assume that A always encrypts using sk_0.

Let $\mathsf{A}'(1^n, a^*, b^*)$ be the output of $\mathsf{A}'(1^n)$ in the emulation induced by A, where $a = a^*$ and $b = b^*$.

Claim 29

Assume that A′ breaks the CCA1 security of $(G', \mathrm{E}', \mathrm{D}')$ with probability $\delta(n)$, then A breaks the CPA security of $(G, \mathrm{E}, \mathrm{D})$ with probability $(\delta(n) - \mathrm{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[\mathrm{A}' \text{ “makes" } \mathrm{A}(1^n) \text{ decrypt an invalid cipher}] = \mathrm{neg}(n) \quad (3)$$

Ignoring negligible terms, we assume that A always encrypts using sk_0.
Let $\mathrm{A}'(1^n, a^*, b^*)$ be the output of $\mathrm{A}'(1^n)$ in the emulation induced by A, where $a = a^*$ and $b = b^*$. It holds that

1. $\mathrm{A}'(1^n, 0, 1) \equiv \mathrm{A}'(1^n, 1, 0)$

Claim 29

Assume that A' breaks the CCA1 security of $(G', \mathsf{E}', \mathsf{D}')$ with probability $\delta(n)$, then A breaks the CPA security of $(G, \mathsf{E}, \mathsf{D})$ with probability $(\delta(n) - \operatorname{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[\mathsf{A}' \text{ “makes" } \mathsf{A}(1^n) \text{ decrypt an invalid cipher}] = \operatorname{neg}(n) \quad (3)$$

Ignoring negligible terms, we assume that A always encrypts using sk_0.
Let $\mathsf{A}'(1^n, a^*, b^*)$ be the output of $\mathsf{A}'(1^n)$ in the emulation induced by A, where $a = a^*$ and $b = b^*$. It holds that

1. $\mathsf{A}'(1^n, 0, 1) \equiv \mathsf{A}'(1^n, 1, 0)$
2. The adaptive zero-knowledge of (P, V) yields that $|\Pr[\mathsf{A}'(1^n, 1, 1) = 1] - \Pr[\mathsf{A}'(1^n, 0, 0) = 1]| \geq \delta(n) - \operatorname{neg}(n)$

Let A(b) be the outputs of A when the challenge is b.

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$$
\begin{aligned}
&|\Pr[A(1) = 1] - \Pr[A(0) = 1]| \\
&\quad = |\frac{1}{2}(\Pr[A'(0,1) = 1] + \Pr[A'(1,1) = 1]) \\
&\qquad - \frac{1}{2}(\Pr[A'(0,0) = 1] + \Pr[A'(1,0) = 1])\Big|
\end{aligned}
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&\quad \geq \frac{1}{2}|\Pr[A'(1,1) = 1] - \Pr[A'(0,0) = 1]| \\
&\qquad - \frac{1}{2}|\Pr[A'(1,0) = 1] - \Pr[A'(0,1) = 1]|
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$$
\begin{aligned}
&|\Pr[\mathsf{A}(1)=1]-\Pr[\mathsf{A}(0)=1]| \\
&\quad = \Big|\frac{1}{2}(\Pr[\mathsf{A}'(0,1)=1]+\Pr[\mathsf{A}'(1,1)=1]) \\
&\qquad -\frac{1}{2}(\Pr[\mathsf{A}'(0,0)=1]+\Pr[\mathsf{A}'(1,0)=1])\Big| \\
&\quad \geq \frac{1}{2}\big|\Pr[\mathsf{A}'(1,1)=1]-\Pr[\mathsf{A}'(0,0)=1]\big| \\
&\qquad -\frac{1}{2}\big|\Pr[\mathsf{A}'(1,0)=1]-\Pr[\mathsf{A}'(0,1)=1]\big| \\
&\quad \geq (\delta(n)-\mathrm{neg}(n))/2
\end{aligned}
$$

Public-key CCA2

- Is Construction 26 CCA2 secure?

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- **Solution:** use *simulation sound* NIZK