Application of Information Theory, Lecture 7

Relative Entropy

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Section 1

Definition and Basic Facts

Definition

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$$D(p\|q) = \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i}$$

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- Main interpretation: the information we gained about X, if we originally thought $X \sim q$ and now we learned $X \sim p$

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&= \sum_i p_i \log \frac{p_i}{\frac{1}{m}}
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- Generally, a distribution can change if we condition on event E

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- If p_i is large and q_i is small, then $D(p\|q)$ is large

Additional properties

- $0 \log \frac{0}{0} = 0$, $p \log \frac{p}{0} = \infty$ for $p > 0$
- $\exists i$ s.t. $p_i > 0$ and $q_i = 0$, then $D(p\|q) = \infty$
- If originally $\Pr[X = i] = 0$, then it cannot be more than 0 after we learned something.
- Hence, it make sense to think of it as infinite amount of information learnt
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  (recall that $\Pr[X = i] = 0 \implies \Pr[X = i|E] = 0$, for any event E
- If p_i is large and q_i is small, then $D(p\|q)$ is large
- $D(p\|q) \geq 0$, with equality iff $p = q$ (hw)

Example

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- $p = (p_1, \ldots, p_m)$ — the distribution of q conditioned on the event $i \in [n]$
- $D(p \| q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} = \sum_{i=1}^{n} p_i \log 2^k = \sum_{i=1}^{n} p_i k = k$

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- We gained k bits of information
- Example: $\sum_{i=1}^{n} q_i = \frac{1}{2}$, and we were told that $i \leq n$ or $i > n$, we got one bit of information

Section 2

Axiomatic Derivation

Axiomatic derivation

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Let $\tilde{D}$ is a continuous and symmetric (wrt each distribution) function such that

1. $\tilde{D}(p \| \sim [m]) = \log m - H(p)$
2. $\tilde{D}((p_1, \ldots, p_m) \| (q_1, \ldots, q_m)) = \tilde{D}((p_1, \ldots, p_{m-1}, \alpha p_m, (1-\alpha) p_m) \| (q_1, \ldots, q_{m-1}, \alpha q_m, (1-\alpha) q_m))$, for any $\alpha \in [0, 1]$

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- $\tilde{D}(p\|q) = D((\alpha_{1,1}p_1, \ldots, \alpha_{1,k_1}p_1, \ldots, \alpha_{m,1}p_m, \ldots, \alpha_{m,k_m}p_m)\| (\alpha_{1,1}q_1, \ldots, \alpha_{1,k_1}q_1, \ldots, \alpha_{m,1}q_m, \ldots, \alpha_{m,k_m}q_m))$, for $\sum_j \alpha_{i,j} = 1$ and $\alpha_{i,j\geq 0}$

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- Zeros and non-rational q_i's are dealt by continuity

Section 3

Relation to Mutual Information

Mutual information as expected relative entropy

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- $(X|Y = 1) \sim p_1 = (p_{1,1}, \ldots, p_{1,m})$, $\qquad p_{1,i} = \Pr[X = i | Y = 1]$

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$$\begin{aligned}\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y = 0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\ &+ \Pr[Y = 1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m)\end{aligned}$$

Mutual information as expected relative entropy

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- $(X|Y = 0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X = i|Y = 0]$
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- If we learned $Y = j$, we gained $D(p_j \| q)$

$$\begin{aligned}
\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y = 0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
&\quad + \Pr[Y = 1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m) \\
&= \Pr[Y = 0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y = 1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i}
\end{aligned}$$

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0, 1\}$
- $(X|Y = 0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X = i|Y = 0]$
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- If we learned $Y = j$, we gained $D(p_j \| q)$

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\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y = 0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
&+ \Pr[Y = 1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m) \\
&= \Pr[Y = 0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y = 1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y = 0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y = 1] \cdot \sum p_{1,i} \log p_{1,i} \\
&- \Pr[Y = 0] \cdot \sum p_{0,i} \log q_i - \Pr[Y = 1] \cdot \sum p_{1,i} \log q_i
\end{aligned}$$

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&= \Pr[Y=0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y=1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y=0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y=1] \cdot \sum p_{1,i} \log p_{1,i} \\
&- \Pr[Y=0] \cdot \sum p_{0,i} \log q_i - \Pr[Y=1] \cdot \sum p_{1,i} \log q_i \\
&= -H(X|Y) - \sum (\Pr[Y=0] \cdot p_{0,i} + \Pr[Y=1] \cdot p_{1,i}) \log q_i
\end{aligned}
$$

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- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0,1\}$
- $(X|Y=0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\qquad p_{0,i} = \Pr[X=i|Y=0]$
- $(X|Y=1) \sim p_1 = (p_{1,1}, \ldots, p_{1,m})$, $\qquad p_{1,i} = \Pr[X=i|Y=1]$
- If we learned $Y=j$, we gained $D(p_j \| q)$

$$
\begin{aligned}
\mathop{\mathrm{E}}_{Y}[D(p_Y \| q)] &= \Pr[Y=0] \cdot D(p_{0,1}, \ldots, p_{0,m} \| q_1, \ldots, q_m) \\
&\quad + \Pr[Y=1] \cdot D(p_{1,1}, \ldots, p_{1,m} \| q_1, \ldots, q_m) \\
&= \Pr[Y=0] \cdot \sum_i p_{0,i} \log \frac{p_{0,i}}{q_i} + \Pr[Y=1] \cdot \sum_i p_{1,i} \log \frac{p_{1,i}}{q_i} \\
&= \Pr[Y=0] \cdot \sum p_{0,i} \log p_{0,i} + \Pr[Y=1] \cdot \sum p_{1,i} \log p_{1,i} \\
&\quad - \Pr[Y=0] \cdot \sum p_{0,i} \log q_i - \Pr[Y=1] \cdot \sum p_{1,i} \log q_i \\
&= -H(X|Y) - \sum (\Pr[Y=0] \cdot p_{0,i} + \Pr[Y=1] \cdot p_{1,i}) \log q_i \\
&= -H(X|Y) + H(X)
\end{aligned}
$$

Mutual information as expected relative entropy

- Let $X \sim (q_1, \ldots, q_m)$ over $[m]$, and Y be rv over $\{0,1\}$
- $(X|Y=0) \sim p_0 = (p_{0,1}, \ldots, p_{0,m})$, $\quad p_{0,i} = \Pr[X=i|Y=0]$
- $(X|Y=1) \sim p_1 = (p_{1,1}, \ldots, p_{1,m})$, $\quad p_{1,i} = \Pr[X=i|Y=1]$
- If we learned $Y=j$, we gained $D(p_j\|q)$

$$\begin{aligned}
\operatorname*{E}_{Y}[D(p_Y\|q)] &= \Pr[Y=0]\cdot D(p_{0,1},\ldots,p_{0,m}\|q_1,\ldots,q_m)\\
&\quad + \Pr[Y=1]\cdot D(p_{1,1},\ldots,p_{1,m}\|q_1,\ldots,q_m)\\
&= \Pr[Y=0]\cdot\sum_i p_{0,i}\log\frac{p_{0,i}}{q_i} + \Pr[Y=1]\cdot\sum_i p_{1,i}\log\frac{p_{1,i}}{q_i}\\
&= \Pr[Y=0]\cdot\sum p_{0,i}\log p_{0,i} + \Pr[Y=1]\cdot\sum p_{1,i}\log p_{1,i}\\
&\quad - \Pr[Y=0]\cdot\sum p_{0,i}\log q_i - \Pr[Y=1]\cdot\sum p_{1,i}\log q_i\\
&= -H(X|Y) - \sum(\Pr[Y=0]\cdot p_{0,i} + \Pr[Y=1]\cdot p_{1,i})\log q_i\\
&= -H(X|Y) + H(X) = I(X;Y)
\end{aligned}$$

Equivalent definition for mutual information

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- $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$

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$$D(p\|p_X p_Y) = \sum_{x,y} p(x, y) \log \frac{p(x, y)}{p_X(x) p_Y(y)}$$

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$$= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)}$$

$$= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y)$$

Equivalent definition for mutual information

- $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$
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- Proof:

$$\begin{aligned} D(p \| p_X p_Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x) p_Y(y)} \\ &= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\ &= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\ &= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \end{aligned}$$

Equivalent definition for mutual information

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D(p \| p_X p_Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x) p_Y(y)} \\
&= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\
&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y)
\end{aligned}$$

Equivalent definition for mutual information

- $(X, Y) \sim p$, then $I(X; Y) = D(p \| p_X p_Y)$
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- Proof:

$$
\begin{aligned}
D(p \| p_X p_Y) &= \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_X(x) p_Y(y)} \\
&= \sum_{x,y} p(x,y) \log \frac{p_{X|Y}(x|y)}{p_X(x)} \\
&= -\sum_{x,y} p(x,y) \log p_X(x) + \sum_{x,y} p(x,y) \log p_{X|Y}(x|y) \\
&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y) = I(X; Y)
\end{aligned}
$$

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&= H(X) + \sum_{y} p_Y(y) \sum_{x} p_{X|Y}(x|y) \log p_{X|Y}(x|y) \\
&= H(X) - H(X|Y) = I(X;Y)
\end{aligned}$$

- We will later see the relation between the above two facts.

Section 4

Relation to Data Compression

Wrong code

Wrong code

Theorem 1

Let p and q be distributions over $[m]$, and let C be code with $\ell(i) = |C(i)| = \left\lceil \log \frac{1}{q_i} \right\rceil$. Then
$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

Wrong code

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$H(p) + D(p\|q) \leq \mathrm{E}_{i \leftarrow p}\left[\ell(i)\right] \leq H(p) + D(p\|q) + 1$

- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}\left[\ell(i)\right] \leq H(q) + 1$.

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- Recall that $H(q) \leq \mathrm{E}_{i \leftarrow q}[\ell(i)] \leq H(q) + 1$.
- Proof of upperbound (upperbound is proved similarly)

$$\mathop{\mathrm{E}}_{i \leftarrow p}[\ell(i)] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil$$

Wrong code

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- Proof of upperbound (upperbound is proved similarly)

$$\mathop{\mathrm{E}}_{i \leftarrow p}[\ell(i)] = \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1)$$

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- Proof of upperbound (upperbound is proved similarly)

$$
\begin{aligned}
\mathop{\mathrm{E}}_{i \leftarrow p}\left[\ell(i)\right] &= \sum_i p_i \left\lceil \log \frac{1}{q_i} \right\rceil < \sum_i p_i (\log \frac{1}{q_i} + 1) \\
&= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i})
\end{aligned}
$$

Wrong code

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$$= 1 + D(p\|q) + H(p)$$

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&= 1 + \sum_i p_i (\log \frac{p_i}{q_i} \frac{1}{p_i}) = 1 + \sum_i p_i (\log \frac{p_i}{q_i}) + \sum_i p_i (\log \frac{1}{p_i}) \\
&= 1 + D(p\|q) + H(p)
\end{aligned}
$$

- Can there be a (close) to optimal code for q that is better for p?

Wrong code

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&= 1 + D(p\|q) + H(p)
\end{aligned}$$

- Can there be a (close) to optimal code for q that is better for p? HW

Section 5

Conditional Relative Entropy

Conditional relative entropy

Conditional relative entropy

Definition 2

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) := \sum_{x \in \mathcal{X}} p_{\mathcal{X}}(x) \cdot \sum_{y \in \mathcal{Y}} p_{\mathcal{Y}|\mathcal{X}}(y|x) \log \frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)}$$

Conditional relative entropy

Definition 2

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

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$$= \mathop{\mathrm{E}}_{(X,Y)\sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

Conditional relative entropy

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$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} [D(X_q | X_p = x \| Y_q | X_q = x)]$

Conditional relative entropy

Definition 2

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

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$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} [D(X_q | X_p = x \| Y_q | X_q = x)]$
- Example: $p =$

$X \backslash Y$	0	1
0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{4}$	$\frac{1}{2}$

$q =$

$X \backslash Y$	0	1
0	$\frac{1}{8}$	$\frac{1}{4}$
1	$\frac{1}{2}$	$\frac{1}{8}$

Conditional relative entropy

Definition 2

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

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$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

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Conditional relative entropy

Definition 2

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) := \sum_{x \in \mathcal{X}} p_{\mathcal{X}}(x) \cdot \sum_{y \in \mathcal{Y}} p_{\mathcal{Y}|\mathcal{X}}(y|x) \log \frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)}$$

$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} [D(X_q | X_p = x \| Y_q | X_q = x)]$

- Example:

$p =$

X \ Y	0	1
0	$\frac{1}{8}$	$\frac{1}{8}$
1	$\frac{1}{4}$	$\frac{1}{2}$

$q =$

X \ Y	0	1
0	$\frac{1}{8}$	$\frac{1}{4}$
1	$\frac{1}{2}$	$\frac{1}{8}$

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \frac{1}{4} \cdot D((\frac{1}{2}, \frac{1}{2}) \| (\frac{1}{3}, \frac{2}{3})) + \frac{3}{4} \cdot D((\frac{1}{3}, \frac{2}{3}) \| (\frac{4}{5}, \frac{1}{5}))$$

Conditional relative entropy

Definition 2

For two distributions p and q over $\mathcal{X} \times \mathcal{Y}$:

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) := \sum_{x \in \mathcal{X}} p_{\mathcal{X}}(x) \cdot \sum_{y \in \mathcal{Y}} p_{\mathcal{Y}|\mathcal{X}}(y|x) \log \frac{p_{\mathcal{Y}|\mathcal{X}}(y|x)}{q_{\mathcal{Y}|\mathcal{X}}(y|x)}$$
$$= \mathop{\mathrm{E}}_{(X,Y) \sim p(x,y)} \left[\log \frac{p_{\mathcal{Y}|\mathcal{X}}(Y|X)}{q_{\mathcal{Y}|\mathcal{X}}(Y|X)} \right]$$

- Let $(X_p, Y_p) \sim p$ and $(X_q, Y_q) \sim q$, then $D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \mathrm{E}_{x \leftarrow X_p} [D(X_q | X_p = x \| Y_q | X_q = x)]$

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$x \backslash y$	0	1
0	$\frac{1}{8}$	$\frac{1}{8}$
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$x \backslash y$	0	1
0	$\frac{1}{8}$	$\frac{1}{4}$
1	$\frac{1}{2}$	$\frac{1}{8}$

$$D(p_{\mathcal{Y}|\mathcal{X}} \| q_{\mathcal{Y}|\mathcal{X}}) = \frac{1}{4} \cdot D((\frac{1}{2}, \frac{1}{2}) \| (\frac{1}{3}, \frac{2}{3})) + \frac{3}{4} \cdot D((\frac{1}{3}, \frac{2}{3}) \| (\frac{4}{5}, \frac{1}{5}))$$
$$= \ldots$$

Chain rule

Chain rule

Claim 3

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Section 6

Data-processing inequality

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- Hence, $D(f(X)\|f(Y)) \geq D(X\|Y)$.

Section 7

Relation to Statistical Distance

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- Other direction is incorrect: $\mathrm{SD}(p,q)$ might be small but $D(p\|q) = \infty$
- Does $\mathrm{SD}(p, \ [m])$ being small imply $D(p\| \ [m]) = \log m - H(p)$ is small?

Proving Thm 5, boolean case

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- $g(\alpha,\alpha) = 0$, and hence $g(\alpha,\beta) \geq 0$ for $\beta < \alpha$. $\square$

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- Let $P \sim p$, and let the indicator $\hat{P}$ be 1 iff $P \in \mathcal{S}$.
- Let $Q \sim q$, and let the indicator $\hat{Q}$ be 1 iff $Q \in \mathcal{S}$.
- $\mathrm{SD}(\hat{P}, \hat{Q}) = \Pr[P \in \mathcal{S}] - \Pr[Q \in \mathcal{S}] = \mathrm{SD}(p, q)$

$$
\begin{aligned}
D(p\|q) &\geq D(\hat{P}\|\hat{Q}) && \text{(data-proccessing inequality)}\\
&\geq \frac{2}{\ln 2} \cdot \mathrm{SD}(\hat{P}, \hat{Q})^2 && \text{(the Boolean case)}
\end{aligned}
$$

Proving Thm 5, general case

- Let $\mathcal{U} = \mathrm{Supp}(p) \cup \mathrm{Supp}(q)$
- Let $\mathcal{S} = \{u \in \mathcal{U} \colon p(u) > q(u)\}$
- $\mathrm{SD}(p,q) = \Pr_p[\mathcal{S}] - \Pr_q[\mathcal{S}]$ (by homework)
- Let $P \sim p$, and let the indicator $\hat{P}$ be 1 iff $P \in \mathcal{S}$.
- Let $Q \sim q$, and let the indicator $\hat{Q}$ be 1 iff $Q \in \mathcal{S}$.
- $\mathrm{SD}(\hat{P},\hat{Q}) = \Pr[P \in \mathcal{S}] - \Pr[Q \in \mathcal{S}] = \mathrm{SD}(p,q)$

$$
\begin{aligned}
D(p\|q) &\geq D(\hat{P}\|\hat{Q}) && \text{(data-proccessing inequality)}\\
&\geq \frac{2}{\ln 2}\cdot \mathrm{SD}(\hat{P},\hat{Q})^2 && \text{(the Boolean case)}\\
&= \frac{2}{\ln 2}\cdot \mathrm{SD}(p,q)^2. \ \square
\end{aligned}
$$

Section 8

Conditioned Distributions

Main theorem

Main theorem

Theorem 6

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{U}$*, and let* $Y = (Y_1, \ldots, Y_k)$ *be rv over* $\mathcal{U}^k$*. Then* $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

Main theorem

Theorem 6

Let $X_1, \ldots, X_k$ be iid over $\mathcal{U}$, and let $Y = (Y_1, \ldots, Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

We prove for $k = 2$, general case follows similar lines.

Main theorem

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Let $X_1, \ldots, X_k$ be iid over $\mathcal{U}$, and let $Y = (Y_1, \ldots, Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

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For rv Z, let $Z(z) = \Pr[Z = z]$.

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For rv Z, let $Z(z) = \Pr[Z = z]$. Let $X = (X_1, X_2)$

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We prove for $k = 2$, general case follows similar lines.

For rv Z, let $Z(z) = \Pr[Z = z]$. Let $X = (X_1, X_2)$

$$D(Y \| X) = \sum_{\mathbf{y} \in \mathcal{U}^2} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{X(\mathbf{y})}$$

Main theorem

Theorem 6

Let $X_1,\ldots,X_k$ be iid over $\mathcal{U}$, and let $Y=(Y_1,\ldots,Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^k D(Y_j\|X_j) \leq D(Y\|(X_1,\ldots,X_k))$.

We prove for $k=2$, general case follows similar lines.

For rv Z, let $Z(z)=\Pr[Z=z]$. Let $X=(X_1,X_2)$

$$D(Y\|X)=\sum_{\mathbf{y}\in\mathcal{U}^2} Y(\mathbf{y})\log\frac{Y(\mathbf{y})}{X(\mathbf{y})}=\sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y})\log\frac{Y_1(y_1)}{X_1(y_1)}\frac{Y_2(y_2)}{X_2(y_2)}\frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)}$$

Main theorem

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Let $X_1, \ldots, X_k$ be iid over $\mathcal{U}$, and let $Y = (Y_1, \ldots, Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

We prove for $k = 2$, general case follows similar lines.

For rv Z, let $Z(z) = \Pr[Z = z]$. Let $X = (X_1, X_2)$

$$D(Y\|X) = \sum_{\mathbf{y} \in \mathcal{U}^2} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{X(\mathbf{y})} = \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(y_1)} \frac{Y_2(y_2)}{X_2(y_2)} \frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)}$$

$$= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(x_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(x_1)}$$

$$+ \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1)Y_2(y_2)}$$

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Let $X_1, \ldots, X_k$ be iid over $\mathcal{U}$, and let $Y = (Y_1, \ldots, Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

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$$\begin{aligned}
D(Y \| X) &= \sum_{\mathbf{y} \in \mathcal{U}^2} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{X(\mathbf{y})} = \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(y_1)} \frac{Y_2(y_2)}{X_2(y_2)} \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(x_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(x_1)} \\
&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= D(Y_1 \| X_1) + D(Y_2 \| X_2) + I(Y_1; Y_2)
\end{aligned}$$

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&= \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_1(y_1)}{X_1(x_1)} + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y_2(y_2)}{X_2(x_1)} \\
&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= D(Y_1 \| X_1) + D(Y_2 \| X_2) + I(Y_1; Y_2) \geq D(Y_1 \| X_1) + D(Y_2 \| X_2)
\end{aligned}
$$

Main theorem

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Let $X_1, \ldots, X_k$ be iid over $\mathcal{U}$, and let $Y = (Y_1, \ldots, Y_k)$ be rv over $\mathcal{U}^k$. Then $\sum_{j=1}^{k} D(Y_j \| X_j) \leq D(Y \| (X_1, \ldots, X_k))$.

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&\quad + \sum_{\mathbf{y}=(y_1,y_2)} Y(\mathbf{y}) \log \frac{Y(\mathbf{y})}{Y_1(y_1) Y_2(y_2)} \\
&= D(Y_1 \| X_1) + D(Y_2 \| X_2) + I(Y_1; Y_2) \geq D(Y_1 \| X_1) + D(Y_2 \| X_2)
\end{aligned}
$$

Conditioning distributions, relative entropy case

Conditioning distributions, relative entropy case

Theorem 7

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event (i.e., Boolean rv). Then $\sum_{j=1}^{k} D((X_j|W)\|X_j) \leq \log \frac{1}{\Pr[W]}$.

Conditioning distributions, relative entropy case

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$$\sum_{j=1}^{k} D((X_j|W)\|X_j) \leq D((X|W)\|X)$$

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$$\sum_{j=1}^{k} D((X_j|W)\|X_j) \leq D((X|W)\|X) \qquad \text{(Thm 6)}$$

$$= \sum_{\mathbf{x} \in \mathcal{X}^k} (X|W)(\mathbf{x}) \log \frac{(X|W)(\mathbf{x})}{X(\mathbf{x})}$$

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$$= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|W)(\mathbf{x}) \log \frac{\Pr\left[W|X=\mathbf{x}]\right)}{\Pr\left[W\right]} \qquad \text{(Bayes)}$$

Conditioning distributions, relative entropy case

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Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event (i.e., Boolean rv). Then $\sum_{j=1}^{k} D((X_j|W)\|X_j) \leq \log \frac{1}{\Pr[W]}$.

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$$= \log \frac{1}{\Pr[W]} + \sum_{\mathbf{x}\in\mathcal{X}^k} (X|W)(\mathbf{x}) \log \Pr[W|X=\mathbf{x}])$$

Conditioning distributions, relative entropy case

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$$= \sum_{\mathbf{x}\in\mathcal{X}^k} (X|W)(\mathbf{x}) \log \frac{\Pr[W|X=\mathbf{x}])}{\Pr[W]} \qquad \text{(Bayes)}$$

$$= \log \frac{1}{\Pr[W]} + \sum_{\mathbf{x}\in\mathcal{X}^k} (X|W)(\mathbf{x}) \log \Pr[W|X=\mathbf{x}])$$

$$\leq \log \frac{1}{\Pr[W]}$$

Conditioning distributions, statistical distance case

Conditioning distributions, statistical distance case

Theorem 8

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{X}$ *and let* W *be an event. Then* $\sum_{j=1}^{k} \mathrm{SD}((X_j|W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Conditioning distributions, statistical distance case

Theorem 8

Let $X_1, \ldots, X_k$ *be iid over* $\mathcal{X}$ *and let* W *be an event. Then* $\sum_{j=1}^{k} \mathsf{SD}((X_j|W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Proof: follows by Thm 5, and Thm 6.□

Conditioning distributions, statistical distance case

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Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathrm{SD}((X_j|W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

Proof: follows by Thm 5, and Thm 6.□

Using $(\sum_{j=1}^{k} a_i)^2 \leq k \cdot \sum_{j=1}^{k} a_i^2$, it follows that

Corollary 9

$\sum_{j=1}^{k} \mathrm{SD}((X_j|W), X_j) \leq \sqrt{k \log(\frac{1}{\Pr[W]})}$, *and*

$\mathrm{E}_{j \leftarrow k]} \mathrm{SD}((X_j|W), X_j) \leq \sqrt{\frac{1}{k} \log(\frac{1}{\Pr[W]})}$

Conditioning distributions, statistical distance case

Theorem 8

Let $X_1, \ldots, X_k$ be iid over $\mathcal{X}$ and let W be an event. Then $\sum_{j=1}^{k} \mathrm{SD}((X_j|W), X_j)^2 \leq \log \frac{1}{\Pr[W]}$.

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Extraction

Numerical example

Numerical example

- Let $X = (X_1, \ldots, X_k) \leftarrow \{0,1\}^{40}$ and let $f\colon \{0,1\}^{40} \mapsto 0$ be such that $\Pr[f(X) = 0] = 2^{-10}$.

Numerical example

- Let $X = (X_1, \ldots, X_k) \leftarrow \{0,1\}^{40}$ and let $f\colon \{0,1\}^{40} \mapsto 0$ be such that $\Pr[f(X) = 0] = 2^{-10}$.
- $\mathrm{E}_{j \leftarrow [40]} \mathrm{SD}((X_j | f(X) = 0), \sim \{0,1\}) \leq \sqrt{\frac{1}{40} \cdot 10} = \frac{1}{2}$

Numerical example

- Let $X = (X_1, \ldots, X_k) \leftarrow \{0,1\}^{40}$ and let $f \colon \{0,1\}^{40} \mapsto 0$ be such that $\Pr\left[f(X) = 0\right] = 2^{-10}$.
- $\mathrm{E}_{j \leftarrow [40]} \mathrm{SD}((X_j | f(X) = 0), \sim \{0,1\}) \leq \sqrt{\frac{1}{40} \cdot 10} = \frac{1}{2}$
- Typical bits are not too biassed, even when conditioning on a very unlikely event.

Extension

Extension

Theorem 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively. Let W be an event and assume that the X_i's are iid conditioned on T. Then $\sum_{j=1}^{k} D((TVX_j|W) \| (TV|W)X'_j(T)) \leq \log \frac{1}{\Pr[W]} + \log |\mathrm{Supp}(V|W)|$, where $X'_j(t)$ is distributed according to $X_j|T = t$.

Extension

Theorem 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively. Let W be an event and assume that the X_i's are iid conditioned on T. Then $\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X'_j(T)) \leq \log \frac{1}{\Pr[W]} + \log|\mathrm{Supp}(V|W)|$, where $X'_j(t)$ is distributed according to $X_j|T = t$.

Interpretation.

Proving Thm 10

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|T = t$.

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|T = t$.

$$\sum_{j=1}^{k} D((TVX_j|W) \| (TV|W)X'_j(T))$$
$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t \| (X_j|T = t)\big)\Big]$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|T = t$.

$$\begin{aligned}
&\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X'_j(T)) \\
&= \mathop{\mathsf{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t\|(X_j|T = t)\big)\Big] && \text{(chain rule)}
\end{aligned}$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|T = t$.

$$\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X'_j(T))$$

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$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t\|(X_j|T = t)\big)\Big]$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X_j'(t)$ be distributed according to the distribution of $X_j|T = t$.

$$
\begin{aligned}
&\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X_j'(T)) \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t\|(X_j|T = t)\big)\Big] && \text{(chain rule)} \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t\|(X_j|T = t)\big)\Big] \\
&\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\log \frac{1}{\Pr[W \wedge V = v|T = t]}\Big]
\end{aligned}
$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X_j'(t)$ be distributed according to the distribution of $X_j|T = t$.

$$
\begin{aligned}
&\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X_j'(T)) \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t\|(X_j|T = t)\big)\Big] && \text{(chain rule)} \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t\|(X_j|T = t)\big)\Big] \\
&\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\log \frac{1}{\Pr[W \wedge V = v|T = t]}\Big] && \text{(Thm 7)}
\end{aligned}
$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X'_j(t)$ be distributed according to the distribution of $X_j|T = t$.

$$\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X'_j(T))$$

$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t\|(X_j|T = t)\big)\Big] \qquad \text{(chain rule)}$$

$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t\|(X_j|T = t)\big)\Big]$$

$$\le \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\log \frac{1}{\Pr[W \wedge V = v|T = t]}\Big] \qquad \text{(Thm 7)}$$

$$\le \log \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)} \frac{1}{\Pr[W \wedge V = v|T = t]}$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X_j'(t)$ be distributed according to the distribution of $X_j|T = t$.

$$
\begin{aligned}
&\sum_{j=1}^{k} D((TVX_j|W) \| (TV|W)X_j'(T)) \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t \| (X_j|T = t)\big)\Big] && \text{(chain rule)} \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t \| (X_j|T = t)\big)\Big] \\
&\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\log \frac{1}{\Pr[W \wedge V = v|T = t]}\Big] && \text{(Thm 7)} \\
&\leq \log \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)} \frac{1}{\Pr[W \wedge V = v|T = t]} && \text{(Jensen's inequality)}
\end{aligned}
$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X_j'(t)$ be distributed according to the distribution of $X_j | T = t$.

$$
\begin{aligned}
&\sum_{j=1}^{k} D((TVX_j|W) \| (TV|W)X_j'(T)) \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t \| (X_j|T = t)\big)\Big] && \text{(chain rule)} \\
&= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t \| (X_j|T = t)\big)\Big] \\
&\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\Big[\log \frac{1}{\Pr[W \wedge V = v | T = t]}\Big] && \text{(Thm 7)} \\
&\leq \log \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)} \frac{1}{\Pr[W \wedge V = v | T = t]} && \text{(Jensen's inequality)} \\
&= \log \sum_{(t,v)\in \mathrm{Supp}(TV|W)} \frac{\Pr[T = t]}{\Pr[W]}
\end{aligned}
$$

Proving Thm 10

Let $X = (X_1, \ldots, X_k)$, T and V be rv's over $\mathcal{X}^k$, $\mathcal{T}$ and $\mathcal{V}$ respectively, such that X_i's are iid conditioned on T. Let W be an event and let $X_j'(t)$ be distributed according to the distribution of $X_j|T = t$.

$$\sum_{j=1}^{k} D((TVX_j|W)\|(TV|W)X_j'(T))$$

$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\left[\sum_{j=1}^{k} D\big(X_j|W, V = v, T = t\|(X_j|T = t)\big)\right] \qquad \text{(chain rule)}$$

$$= \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\left[\sum D\big((X_j|\overbrace{W, V = v}^{W'})|T = t\|(X_j|T = t)\big)\right]$$

$$\leq \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)}\left[\log \frac{1}{\Pr[W \wedge V = v|T = t]}\right] \qquad \text{(Thm 7)}$$

$$\leq \log \mathop{\mathrm{E}}_{(t,v)\leftarrow(TV|W)} \frac{1}{\Pr[W \wedge V = v|T = t]} \qquad \text{(Jensen's inequality)}$$

$$= \log \sum_{(t,v)\in\mathrm{Supp}(TV|W)} \frac{\Pr[T = t]}{\Pr[W]} \leq \log \frac{||\mathrm{Supp}(V|W)||}{\Pr[W]}. \qquad \square$$