Foundation of Cryptography, Lecture 8
Encryption Schemes

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Section 1

Definitions

Correctness

Definition 1 (encryption scheme)

A trippet of PPTM's $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ such that

1. $\mathsf{G}(1^n)$ outputs $(e, d) \in \{0,1\}^* \times \{0,1\}^*$
2. $\mathsf{E}(e, m)$ outputs $c \in \{0,1\}^*$
3. $\mathsf{D}(d, c)$ outputs $m \in \{0,1\}^*$

Correctness: $\mathsf{D}(d, \mathsf{E}(e, m)) = m$, for any $(e, d) \in \mathsf{Supp}(\mathsf{G}(1^n))$ and $m \in \{0,1\}^*$

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- e – encryption key, d – decryption key
- m – plaintext, $c = \mathsf{E}(e, m)$ – ciphertext
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- public/private key

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  - Other concerns: multiple encryptions, active adversaries, ...

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3. Does not hide the message *length*

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An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ is semantically secure in the private-key model, if $\forall$ PPTM A, $\exists$ PPTM A' s.t. :
$\forall$ poly-length dist. ensemble $\mathcal{M} = \{\mathcal{M}_n\}_{n\in\mathbb{N}}$ and poly-length functions $h, f \colon \{0,1\}^* \mapsto \{0,1\}^*$

$$\Big| \Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathsf{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] - \Pr_{m\leftarrow\mathcal{M}_n}[\mathsf{A}'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)] \Big| = \operatorname{neg}(n)$$

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- Public-key variant — A and A' get e
- Reflection to $\mathcal{ZK}$
- We sometimes omit 1^n and $1^{|m|}$

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An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model, if for any $p, \ell \in \mathrm{poly}$, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and $\{z_n \in \{0,1\}^{p(n)}\}_{n\in\mathbb{N}}$

$$\{(z_n, E_e(x_n))_{e\leftarrow G(1^n)_1}\}_{n\in\mathbb{N}} \approx_c \{(z_n, E_e(y_n))_{e\leftarrow G(1^n)_1}\}_{n\in\mathbb{N}}$$

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Equivalence of Definitions

Theorem 4

An encryption scheme (G, E, D) *is semantically secure* iff *is has indistinguishable encryptions.*

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We prove the private key case

Indistinguishability $\implies$ Semantic Security

Indistinguishability $\Longrightarrow$ Semantic Security

Fix $\mathcal{M}$, A, f and h, as in Definition 2.

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Algorithm 5 (A′)

Input: 1^n, $1^{|m|}$ and $h(m)$

1. $e \leftarrow G(1^n)_1$
2. $c = E_e(1^{|m|})$
3. Output $\mathsf{A}(1^n, 1^{|m|}, h(m), c)$

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Proof: Let

$$\delta(n) := \Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1}[\mathsf{A}(h(m), E_e(m)) = f(m)] - \Pr_{m \leftarrow \mathcal{M}_n}[\mathsf{A}'(h(m)) = f(m)]$$

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We define an algorithm that distinguish two between two ensembles $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$, with advantage $\delta(n)$.

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Hence, the indistinguishability of (G, E, D) yields that $\delta(n) \leq \mathrm{neg}(n)$.

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Claim 7

For every $n \in \mathbb{N}$, exists $x_n \in \mathrm{Supp}(\mathcal{M}_n)$ with
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We define distribution $\mathcal{M}$, functions f, h and algorithm A that has no $\delta(n)/4$ simulator.

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We define distribution $\mathcal{M}$, functions f, h and algorithm A that has no $\delta(n)/4$ simulator. The semantic security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ yields that $\delta(n) \leq \mathsf{neg}(n)$.

Semantic Security $\implies$ Indistinguishability

For PPT B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$, let

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$\Pr_{e \leftarrow G(1^n)_1, t_n \leftarrow \{x_n, y_n\}}[\mathsf{A}(z_n, E_e(t_n)) = f(t_n)] = \frac{1}{2} + \frac{\delta(n)}{4}$

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Semantic Security $\Longrightarrow$ Indistinguishability, cont.

- Let $\mathcal{M}_n$ be x_n w.p. $\frac{1}{2}$, and y_n otherwise.
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By Claim 9:

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Hence, $\delta(n) \leq \text{neg}(n)$.

Security Under Multiple Encryptions

Security Under Multiple Encryptions

Definition 10 (Indistinguishablity for multiple encryptions – private-key model)

An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions for multiple messages in the private-key model, if for any $p, \ell, t \in \text{poly}$, $\{x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n \in \mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$, PPTM B:

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Extensions:

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Extensions:

- Different length messages

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Extensions:

- Different length messages
- Semantic security version

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Extensions:

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- Semantic security version
- Public-key variant

Multiple Encryption in the Public-Key Model

Theorem 11

A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.

Multiple Encryption in the Public-Key Model

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Proof: Let (G, E, D) be a public-key encryption scheme that has no indistinguishable encryptions for multiple messages, with respect to PPT B, $\{x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n \in \mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$.

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Hence, for some function $i(n) \in [t(n)]$:

$$\Big| \Pr_{e \leftarrow G(1^n)_1}\left[B(1^n, e, E_e(x_{n,1}), \ldots, E_e(x_{n,i-1}), E_e(y_{n,i}) \ldots, E_e(y_{n,t(n)})) = 1\right]$$
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Thus, (G, E, D) has no indistinguishable encryptions for single message:

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Hence, for some function $i(n) \in [t(n)]$:

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Thus, $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has no indistinguishable encryptions for single message:

Algorithm 12 (B')

Input: 1^n, $z_n = (i(n), x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)})$, e ,c
Return $\mathsf{B}(c, E_e(x_{n,1}), \ldots, E_e(x_{n,i-1}), c, E_e(y_{n,i+1}) \ldots, E_e(y_{n,t(n)}))$

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Fact 13

Assuming (non uniform) OWFs exists, then $\exists$ encryption scheme that has private-key indistinguishable encryptions for a single messages, but not for multiple messages.

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Proof: Let $g\colon \{0,1\}^n \mapsto \{0,1\}^{n+1}$ be a (non-uniform) PRG, and for $i \in \mathbb{N}$ let g^i be its "iterated extension" to output of length $n+i$ (see Lecture 2).

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Construction 14

- $\mathsf{G}(1^n)$: outputs $e \leftarrow \{0,1\}^n$
- $\mathsf{E}_e(m)$: outputs $g^{|m|}(e) \oplus m$
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Multiple Encryption in the Private-Key Model, cont.

Claim 15

(G, E, D) has private-key indistinguishable encryptions for a single message

Multiple Encryption in the Private-Key Model, cont.

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Proof:

Multiple Encryption in the Private-Key Model, cont.

Claim 15

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Proof: Assume not, and let B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n \in \mathbb{N}}$ and $\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$ be the triplet that realizes it:

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$$\left|\Pr[B(z_n, g^{\ell(n)}(U_n) \oplus x_n) = 1] - \Pr[B(z_n, g^{\ell(n)}(U_n) \oplus y_n) = 1]\right| > \mathrm{neg}(n) \quad (1)$$

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Hence, B yields a (non-uniform) distinguisher for g. (?)

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(G, E, D) does not have a private-key indistinguishable encryptions for multiple messages

Multiple Encryption in the Private-Key Model, cont.

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Proof: Take $x_{n,1} = x_{n,2}$ and $y_{n,1} \neq y_{n,2}$, and let B be the algorithm that on input (c_1, c_2), outputs 1 iff $c_1 = c_2$.□

Section 2

Constructions

Private-Key Indistinguishable Encryptions for Multiple Messages

Suffices to encrypt messages of some fixed length (here the length is n).(?)

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Construction 17

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- $\mathsf{D}_e(r, c)$: output $e(r) \oplus c$

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- $\mathsf{D}_e(r, c)$: output $e(r) \oplus c$

Claim 18

$(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has private-key indistinguishable encryptions for a multiple messages

Private-Key Indistinguishable Encryptions for Multiple Messages

Suffices to encrypt messages of some fixed length (here the length is n).(?)

Let $\mathcal{F}$ be a (non-uniform) length-preserving PRF

Construction 17

- $\mathsf{G}(1^n)$: output $e \leftarrow \mathcal{F}_n$
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Proof: ?

Public-key indistinguishable encryptions for multiple messages

Let (G_T, f, Inv) be a (non-uniform) TDP, and let b be hardcore predicate for it.

Public-key indistinguishable encryptions for multiple messages

Let (G_T, f, Inv) be a (non-uniform) TDP, and let b be hardcore predicate for it.

Construction 19 (bit encryption)

- $\mathsf{G}(1^n)$: output $(e, d) \leftarrow \mathsf{G}_T(1^n)$
- $\mathsf{E}_e(m)$: choose $r \leftarrow \{0,1\}^n$ and output $(y = f_e(r), c = b(r) \oplus m)$
- $\mathsf{D}_d(y, c)$: output $b(\mathrm{Inv}_d(y)) \oplus c$

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Claim 20

$(\mathrm{G}, \mathrm{E}, \mathrm{D})$ has public-key indistinguishable encryptions for a multiple messages

Proof:

We believe that public-key encryptions schemes are “more complex" than private-key ones

Section 3

Active Adversaries

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- Chosen plaintext attack (CPA):

  The adversary can ask for encryption and choose the messages to distinguish accordingly

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  The adversary can also ask for decryptions of certain messages

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  The adversary can ask for encryption and choose the messages to distinguish accordingly

- Chosen ciphertext attack (CCA):

  The adversary can also ask for decryptions of certain messages

- In the public-key settings, the adversary is also given the public key
- We focus on indistinguishability, but each of the above definitions has an equivalent semantic security variant.

CPA Security

Let $(\mathsf{G},\mathsf{E},\mathsf{D})$ be an encryption scheme. For a pair of algorithms $\mathsf{A} = (\mathsf{A}_1, \mathsf{A}_2)$, $n \in \mathbb{N}$, $z \in \{0,1\}^*$ and $b \in \{0,1\}$, let:

Experiment 21 ($\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathsf{A}_1^{E_e(\cdot)}(1^n, z)$, where $|m_0| = |m_1|$.
3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

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Definition 22 (private key CPA)

$(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model under CPA attack, if $\forall$ PPT $\mathsf{A}_1, \mathsf{A}_2$, and poly-bounded $\{z_n\}_{n \in \mathbb{N}}$:

$$|\Pr[\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z_n}(0) = 1] - \Pr[\mathsf{Exp}^{\mathsf{CPA}}_{\mathsf{A},n,z_n}(1) = 1]| = \mathrm{neg}(n)$$

CPA Security, cont.

- public-key variant.

CPA Security, cont.

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- The scheme from Construction 17 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)

CPA Security, cont.

- public-key variant.
- The scheme from Construction 17 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)
- The scheme from Construction 19 has indistinguishable encryptions in the public-key model under CPA attack (for short, public-key CPA secure)

CPA Security, cont.

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- The scheme from Construction 17 has indistinguishable encryptions in the private-key model under CPA attack (for short, private-key CPA secure)
- The scheme from Construction 19 has indistinguishable encryptions in the public-key model under CPA attack (for short, public-key CPA secure)
- In both cases, definitions are not equivalent (?)

CCA Security

Experiment 23 ($\mathrm{Exp}^{\mathrm{CCA1}}_{\mathrm{A},n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathrm{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$, where $|m_0| = |m_1|$.
3. $c \leftarrow \mathrm{E}_e(m_b)$
4. Output $\mathrm{A}_2^{E_e(\cdot)}(1^n, s, c)$

CCA Security

Experiment 23 ($\mathrm{Exp}^{\mathrm{CCA1}}_{\mathsf{A},n,z}(b)$)

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3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

Experiment 24 ($\mathrm{Exp}^{\mathrm{CCA2}}_{\mathsf{A},n,z_n}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathsf{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$, where $|m_0| = |m_1|$.
3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot), D_d^{\neg c}(\cdot)}(1^n, s, c)$

CCA Security, cont.

Definition 25 (private key CCA1/CCA2)

$(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model under $x \in \{\mathsf{CCA1}, \mathsf{CCA2}\}$ attack, if $\forall$ PPT $\mathsf{A}_1, \mathsf{A}_2$, and poly-bounded $\{z_n\}_{n\in\mathbb{N}}$:

$$|\Pr[\mathsf{Exp}^{x}_{\mathsf{A},n,z_n}(0) = 1] - \Pr[\mathsf{Exp}^{x}_{\mathsf{A},n,z_n}(1) = 1]| = \mathsf{neg}(n)$$

CCA Security, cont.

Definition 25 (private key CCA1/CCA2)

(G, E, D) has indistinguishable encryptions in the private-key model under $x \in \{\text{CCA1}, \text{CCA2}\}$ attack, if $\forall$ PPT A_1, A_2, and poly-bounded $\{z_n\}_{n \in \mathbb{N}}$:

$$|\Pr[\text{Exp}^x_{A,n,z_n}(0) = 1] - \Pr[\text{Exp}^x_{A,n,z_n}(1) = 1]| = \text{neg}(n)$$

- The public key definition is analogous

Private-key CCA2

- Is the scheme from Construction 17 private-key CCA1 secure?

Private-key CCA2

- Is the scheme from Construction 17 private-key CCA1 secure?
- CCA2 secure?

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- Is the scheme from Construction 17 private-key CCA1 secure?
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- Is the scheme from Construction 17 private-key CCA1 secure?
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Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a private-key CPA scheme, and let $(\mathsf{Gen}_M, \mathsf{Mac}, \mathsf{Vrfy})$ be an existential unforgeable strong MAC.

Construction 26

- $\mathsf{G}'(1^n)$: Output $(e \leftarrow \mathsf{G}_E(1^n), k \leftarrow \mathsf{Gen}_M(1^n))$.[a]
- $\mathsf{E}'_{e,k}(m)$: let $c = \mathsf{E}_e(m)$ and output $(c, t = \mathsf{Mac}_k(c))$
- $\mathsf{D}_{e,k}(c, t)$: if $\mathsf{Vrfy}_k(c, t) = 1$, output $\mathsf{D}_e(c)$. Otherwise, output $\perp$

[a] We assume wlg. that the encryption and decryption keys are the same.

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- Is the scheme from Construction 17 private-key CCA1 secure?
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Theorem 27

Construction 26 is a private-key CCA2-secure encryption scheme.

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[a] We assume wlg. that the encryption and decryption keys are the same.

Theorem 27

Construction 26 is a private-key CCA2-secure encryption scheme.

Proof: An attacker on the CCA2-security of (G', E', D') yields an attacker on the CPA security of (G, E, D), or the existential unforgettably of $(\mathrm{Gen}_M, \mathrm{Mac}, \mathrm{Vrfy})$

Public-key CCA1

Public-key CCA1

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a public-key CPA scheme and let (P, V) be a $\mathcal{NIZK}$ for
$\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) \colon \exists (m, z_0, z_1) \quad \text{s.t.} \quad c_0 = \mathsf{E}_{pk_0}(m, z_0) \wedge c_1 = \mathsf{E}_{pk_1}(m, z_1)\}$

Public-key CCA1

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a public-key CPA scheme and let (P, V) be a $\mathcal{NIZK}$ for $\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) \colon \exists (m, z_0, z_1) \quad \text{s.t.} \quad c_0 = \mathsf{E}_{pk_0}(m, z_0) \wedge c_1 = \mathsf{E}_{pk_1}(m, z_1)\}$

Construction 28 (The Naor-Yung Paradigm)

- $\mathsf{G}'(1^n)$:
  1. For $i \in \{0, 1\}$: set $(sk_i, pk_i) \leftarrow \mathsf{G}(1^n)$.
  2. Let $r \leftarrow \{0, 1\}^{\ell(n)}$, and output $pk' = (pk_0, pk_1, r)$ and $sk' = (pk', sk_0, sk_1)$
- $\mathsf{E}'_{pk'}(m)$:
  1. For $i \in \{0, 1\}$: set $c_i = \mathsf{E}_{pk_i}(m, z_i)$, where z_i is a uniformly chosen string of the right length
  2. $\pi \leftarrow \mathsf{P}((c_0, c_1, pk_0, pk_1), (m, z_0, z_1), r)$
  3. Output (c_0, c_1, π).
- $\mathsf{D}'_{sk'}(c_0, c_1, \pi)$: If $\mathsf{V}((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, return $\mathsf{D}_{sk_0}(c_0)$. Otherwise, return $\perp$.

Public-key CCA1, cont.

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n. (?)
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using ”security parameter" n.

Public-key CCA1, cont.

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Is the scheme CCA1 secure?

Public-key CCA1, cont.

- We assume for simplicity that the encryption key output by $G(1^n)$ is of length at least n. (?)
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Is the scheme CCA1 secure? We need the $\mathcal{NIZK}$ to be adaptive secure.

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Theorem 29

Assuming (P, V) is adaptive secure, then Construction 28 is a public-key CCA1 secure encryption scheme.

Public-key CCA1, cont.

- We assume for simplicity that the encryption key output by $\mathsf{G}(1^n)$ is of length at least n. (?)
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using "security parameter" n.

Is the scheme CCA1 secure? We need the $\mathcal{NIZK}$ to be adaptive secure.

Theorem 29

Assuming (P, V) is adaptive secure, then Construction 28 is a public-key CCA1 secure encryption scheme.

Proof: Given an attacker A' for the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$, we use it to construct an attacker A on the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ or the adaptive security of (P, V).

Proving Thm 29

Let $\mathsf{S} = (\mathsf{S}_1, \mathsf{S}_2)$ be the (adaptive) simulator for $(\mathsf{P}, \mathsf{V}, \mathcal{L})$

Proving Thm 29

Let $S = (S_1, S_2)$ be the (adaptive) simulator for $(P, V, \mathcal{L})$

Algorithm 30 (A)

Input: $(1^n, pk)$

1. Let $j \leftarrow \{0,1\}$, $pk_{1-j} = pk$, $(pk_j, sk_j) \leftarrow G(1^n)$ and $(r, s) \leftarrow S_1(1^n)$
2. Emulate $A'(1^n, pk' = (pk_0, pk_1, r))$:

   On query (c_0, c_1, π) of A' to D':
   If $V((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, answer $D_{sk_j}(c_j)$.
   Otherwise, answer $\perp$.
3. Output the pair (m_0, m_1) that A' outputs
4. On challenge c ($= E_{pk}(m_b)$):
   - Set $c_{1-j} = c$, $c_j = E_{pk_j}(m_a)$ for $a \leftarrow \{0,1\}$, and $\pi \leftarrow S_2((c_0, c_1, pk_0, pk_1), r, s)$
   - Send $c' = (c_0, c_1, \pi)$ to A'
5. Output the value that A' does

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of (G', E', D') w.p. $\delta(n)$, then A breaks the CPA security of (G, E, D) w.p. $(\delta(n) - \mathrm{neg}(n))/2$.

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of (G', E', D') w.p. $\delta(n)$, then A breaks the CPA security of (G, E, D) w.p. $(\delta(n) - \text{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[A' \text{ “makes" } A(1^n) \text{ decrypt an invalid cipher}] = \text{neg}(n) \tag{2}$$

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ w.p. $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ w.p. $(\delta(n) - \operatorname{neg}(n))/2$.

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Assume for simplicity that the above prob is 0.

Proving Thm 29, cont.

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Assume for simplicity that the above prob is 0.
Hence, no information about j has leaked to A through the first stage.

Proving Thm 29, cont.

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Assume A' breaks the CCA1 security of (G', E', D') w.p. $\delta(n)$, then A breaks the CPA security of (G, E, D) w.p. $(\delta(n) - \text{neg}(n))/2$.

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Assume for simplicity that the above prob is 0.
Hence, no information about j has leaked to A through the first stage.

Let $A'(1^n, x, y)$ be A''s output in the emulation induced by $A(1^n)$, conditioned on $a = x$ and $b = y$.

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ w.p. $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ w.p. $(\delta(n) - \operatorname{neg}(n))/2$.

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Assume for simplicity that the above prob is 0.
Hence, no information about j has leaked to A through the first stage.

Let $\mathsf{A}'(1^n, x, y)$ be A''s output in the emulation induced by $\mathsf{A}(1^n)$, conditioned on $a = x$ and $b = y$.

It holds that

1. Since no information about j has leaked, $\mathsf{A}'(1^n, 0, 1) \equiv \mathsf{A}'(1^n, 1, 0)$

Proving Thm 29, cont.

Claim 31

Assume A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ w.p. $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ w.p. $(\delta(n) - \mathsf{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[\mathsf{A}' \text{ “makes" } \mathsf{A}(1^n) \text{ decrypt an invalid cipher}] = \mathsf{neg}(n) \tag{2}$$

Assume for simplicity that the above prob is 0.
Hence, no information about j has leaked to A through the first stage.

Let $\mathsf{A}'(1^n, x, y)$ be A''s output in the emulation induced by $\mathsf{A}(1^n)$, conditioned on $a = x$ and $b = y$.

It holds that

1. Since no information about j has leaked, $\mathsf{A}'(1^n, 0, 1) \equiv \mathsf{A}'(1^n, 1, 0)$
2. The guarantee about A' and the adaptive zero-knowledge of (P, V), yields $|\Pr[\mathsf{A}'(1^n, 1, 1) = 1] - \Pr[\mathsf{A}'(1^n, 0, 0) = 1]| \geq \delta(n) - \mathsf{neg}(n)$

Proving Thm 29, cont..

Let $A(b)$ be A's output on challenge $(1^n, b)$.

Proving Thm 29, cont..

Let $\mathsf{A}(b)$ be A's output on challenge $(1^n, b)$.

$$
\begin{aligned}
&|\Pr[\mathsf{A}(1) = 1] - \Pr[\mathsf{A}(0) = 1]| \\
&= \left| \frac{1}{2}(\Pr[\mathsf{A}'(0,1) = 1] + \Pr[\mathsf{A}'(1,1) = 1]) - \frac{1}{2}(\Pr[\mathsf{A}'(0,0) = 1] + \Pr[\mathsf{A}'(1,0) = 1]) \right|
\end{aligned}
$$

Proving Thm 29, cont..

Let $\mathsf{A}(b)$ be A's output on challenge $(1^n, b)$.

$$
\begin{aligned}
&|\Pr[\mathsf{A}(1) = 1] - \Pr[\mathsf{A}(0) = 1]| \\
&= \left|\frac{1}{2}(\Pr[\mathsf{A}'(0,1) = 1] + \Pr[\mathsf{A}'(1,1) = 1]) - \frac{1}{2}(\Pr[\mathsf{A}'(0,0) = 1] + \Pr[\mathsf{A}'(1,0) = 1])\right| \\
&\geq \frac{1}{2}\left|\Pr[\mathsf{A}'(1,1) = 1] - \Pr[\mathsf{A}'(0,0) = 1]\right| - \frac{1}{2}\left|\Pr[\mathsf{A}'(1,0) = 1] - \Pr[\mathsf{A}'(0,1) = 1]\right|
\end{aligned}
$$

Proving Thm 29, cont..

Let $\mathsf{A}(b)$ be A's output on challenge $(1^n, b)$.

$$
\begin{aligned}
&|\Pr[\mathsf{A}(1) = 1] - \Pr[\mathsf{A}(0) = 1]| \\
&= \left|\frac{1}{2}(\Pr[\mathsf{A}'(0,1) = 1] + \Pr[\mathsf{A}'(1,1) = 1]) - \frac{1}{2}(\Pr[\mathsf{A}'(0,0) = 1] + \Pr[\mathsf{A}'(1,0) = 1])\right| \\
&\geq \frac{1}{2}\left|\Pr[\mathsf{A}'(1,1) = 1] - \Pr[\mathsf{A}'(0,0) = 1]\right| - \frac{1}{2}\left|\Pr[\mathsf{A}'(1,0) = 1] - \Pr[\mathsf{A}'(0,1) = 1]\right| \\
&\geq (\delta(n) - \mathsf{neg}(n))/2 - 0
\end{aligned}
$$

Public-key CCA2

- Is Construction 28 CCA2 secure?

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- **Problem:** Soundness might not hold with respect to the simulated CRS, after seeing a proof for an invalid statement

Public-key CCA2

- Is Construction 28 CCA2 secure?
- **Problem:** Soundness might not hold with respect to the simulated CRS, after seeing a proof for an invalid statement
- **Solution:** use simulation sound $\mathcal{NIZK}$