MATH 201: Coordinate Geometry and Vector Analysis

Chapter: 14.2

"DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS"

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Example 1 Evaluate

$$\text{(a)} \int_0^1 \int_{-x}^{x^2} y^2 x \, dy \, dx \qquad \text{(b)} \int_0^{\pi/3} \int_0^{\cos y} x \sin y \, dx \, dy$$

Solution (a).

$$\int_0^1 \int_{-x}^{x^2} y^2 x \, dy \, dx = \int_0^1 \left[\int_{-x}^{x^2} y^2 x \, dy \right] dx = \int_0^1 \frac{y^3 x}{3} \bigg]_{y=-x}^{x^2} dx$$

$$= \int_0^1 \left[\frac{x^7}{3} + \frac{x^4}{3} \right] dx = \left(\frac{x^8}{24} + \frac{x^5}{15} \right) \bigg]_0^1 = \frac{13}{120}$$

Solution (b).

$$\int_0^{\pi/3} \int_0^{\cos y} x \sin y \, dx \, dy = \int_0^{\pi/3} \left[\int_0^{\cos y} x \sin y \, dx \right] dy = \int_0^{\pi/3} \frac{x^2}{2} \sin y \bigg]_{x=0}^{\cos y} dy$$

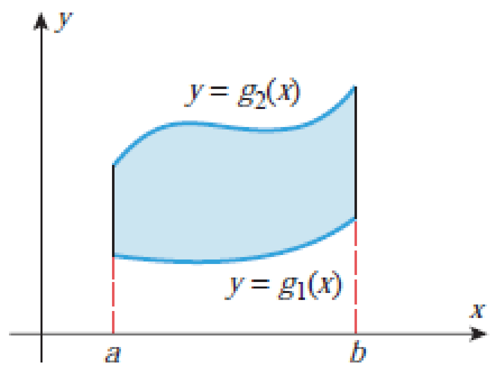
$$= \int_0^{\pi/3} \left[\frac{1}{2} \cos^2 y \sin y \right] dy = -\frac{1}{6} \cos^3 y \bigg]_0^{\pi/3} = \frac{7}{48}$$

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

14.2.1 DEFINITION

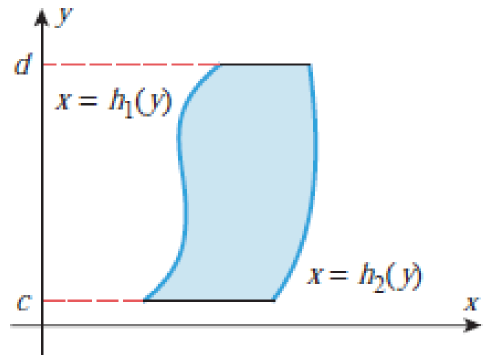
(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$ (Figure 14.2.1*a*).

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$ (Figure 14.2.1*b*).



A type I region

(*a*)



A type II region

14.2.2 THEOREM

(*a*) *If R is a type I region on which $f(x, y)$ is continuous, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{3}$$

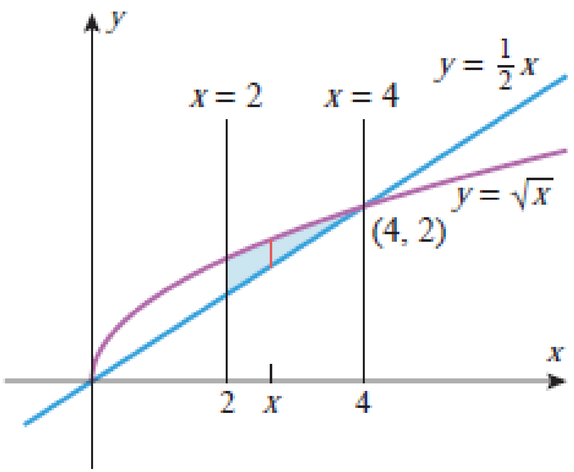
(*b*) *If R is a type II region on which $f(x, y)$ is continuous, then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{4}$$

▸ Example 3 Evaluate

$$\iint_R xy\,dA$$

over the region R enclosed between $y=\frac{1}{2}x$, $y=\sqrt{x}$, $x=2$, and $x=4$.



Solution. We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure 14.2.6. This line meets the region R at the lower boundary $y=\frac{1}{2}x$ and the upper boundary $y=\sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x=2$ and $x=4$. Thus,

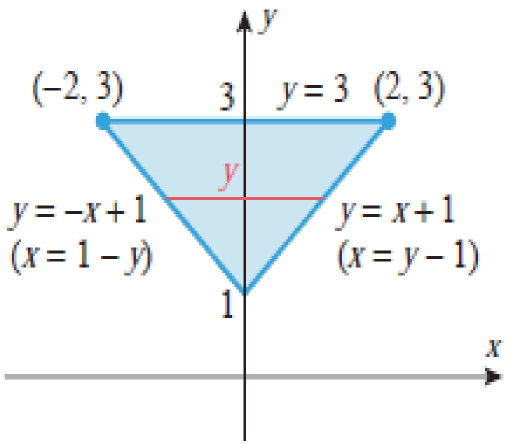
$$\iint_R xy\,dA=\int_2^4\int_{x/2}^{\sqrt{x}} xy\,dy\,dx=\int_2^4\left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}}dx=\int_2^4\left(\frac{x^2}{2}-\frac{x^3}{8}\right)dx$$

$$=\left[\frac{x^3}{6}-\frac{x^4}{32}\right]_2^4=\left(\frac{64}{6}-\frac{256}{32}\right)-\left(\frac{8}{6}-\frac{16}{32}\right)=\frac{11}{6}$$ ◀

▶ **Example 4** Evaluate

$$\iint_R (2x - y^2)\, dA$$

over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.



Solution. We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in Figure 14.2.8. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\iint_R (2x - y^2)\, dA = \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2x\right]_{x=1-y}^{y-1} dy$$

$$= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy$$

$$= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}$$ ◀

Example 5 Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane

$$z = 4 - 4x - 2y \tag{5}$$

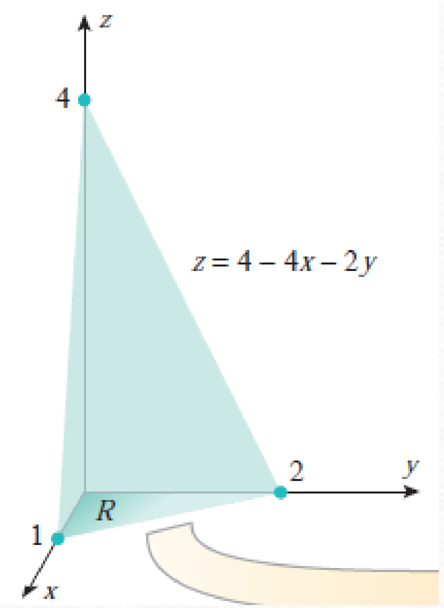
and below by the triangular region R shown in Figure 14.2.10. Thus, the volume is given by

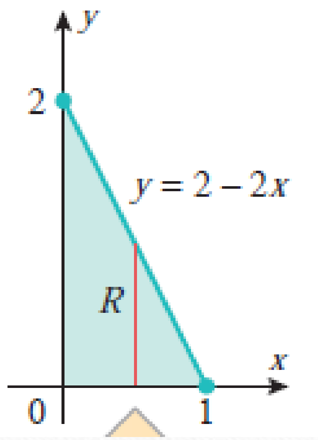
$$V = \iint_R (4 - 4x - 2y)\, dA$$

The region R is bounded by the x-axis, the y-axis, and the line $y = 2 - 2x$ [set $z = 0$ in (5)], so that treating R as a type I region yields

$$V = \iint_R (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_{y=0}^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\, dx = \frac{4}{3}$$





▶ **Example 6** Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution. The solid shown in Figure 14.2.11 is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

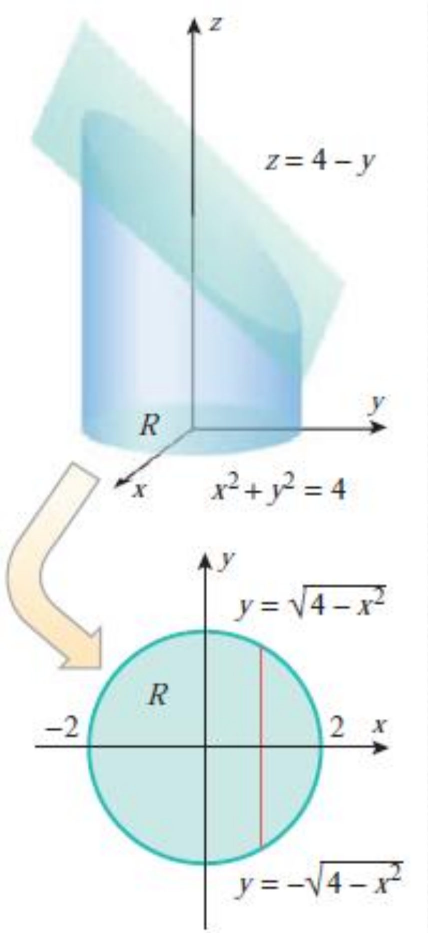
$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi$$

See Formula (3) of Section 7.4. ◀



$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = \frac{1}{2}\pi \times 2^2 = 2\pi$$

Let, $x^2 = u$

$$2x\,dx = du$$

$$\text{Then} \int_0^1 2xe^{x^2}dx = \int_0^1 e^u du$$

▶ **Example 7** Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$$

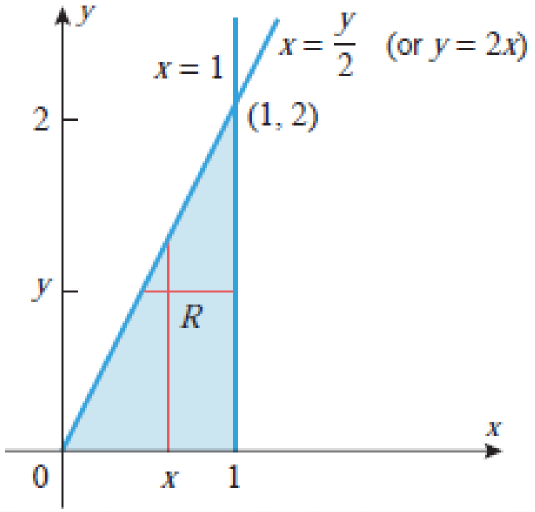
cannot be evaluated by performing the x-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

Solution. For the inside integration, y is fixed and x varies from the line $x = y/2$ to the line $x = 1$ (Figure 14.2.12). For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R in Figure 14.2.12.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \iint_R e^{x^2}\,dA = \int_0^1 \int_0^{2x} e^{x^2}\,dy\,dx = \int_0^1 \left[e^{x^2}y\right]_{y=0}^{2x} dx$$

$$= \int_0^1 2xe^{x^2}\,dx = e^{x^2}\Big]_0^1 = e - 1$$ ◀



► Example 8 Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution. The region R may be treated equally well as type I (Figure 14.2.14*a*) or type II (Figure 14.2.14*b*). Treating R as type I yields

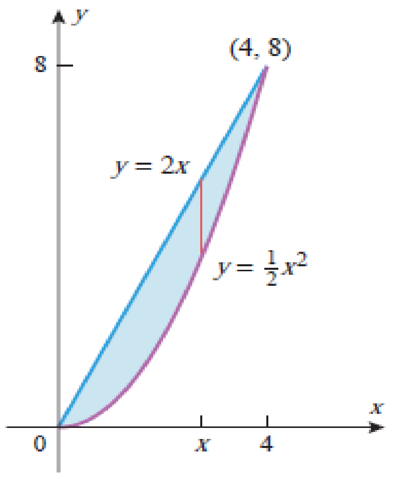
$$\text{area of } R = \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 [y]_{y=x^2/2}^{2x}\,dx$$

$$= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}$$

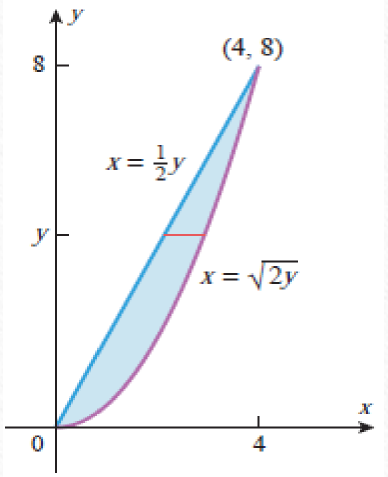
Treating R as type II yields

$$\text{area of } R = \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\,dy$$

$$= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3} \quad ◄$$



(a)



(b)

THANK YOU