MATH 201: Coordinate Geometry and Vector Analysis

"Lecture 5"

Chapter: 14.6

Triple Integrals in Cylindrical and Spherical Coordinates

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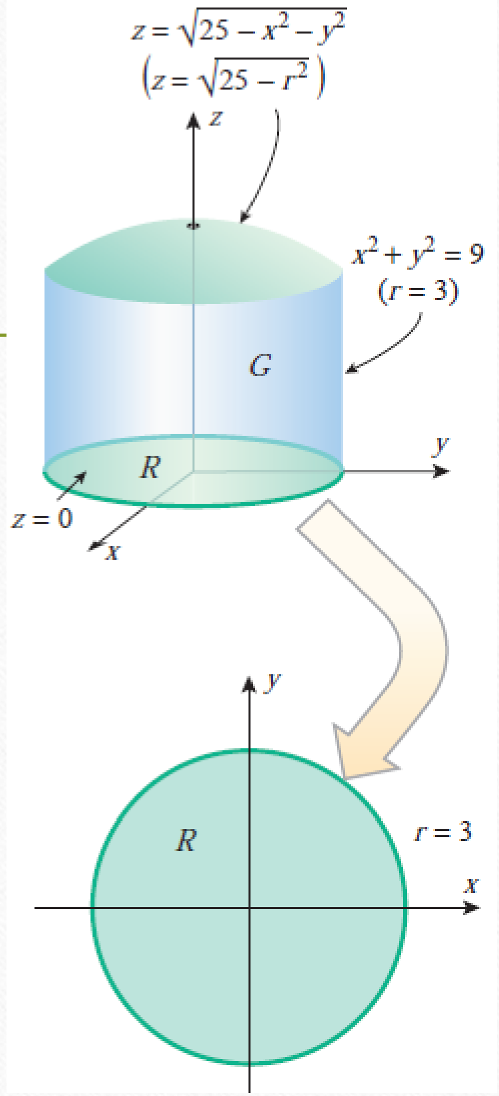
Example 1 Use triple integration in cylindrical coordinates to find the volume of the solid G that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$.



Solution. The solid G and its projection R on the xy-plane are shown in Figure 14.6.5. In cylindrical coordinates, the upper surface of G is the hemisphere $z = \sqrt{25 - r^2}$ and the lower surface is the plane $z = 0$. Thus, from (4), the volume of G is

$$V = \iiint_G dV = \iint_R \left[\int_0^{\sqrt{25-r^2}} dz \right] dA$$

For the double integral over R, we use polar coordinates:

$$V = \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^3 [rz]_{z=0}^{\sqrt{25-r^2}}\, dr\, d\theta$$

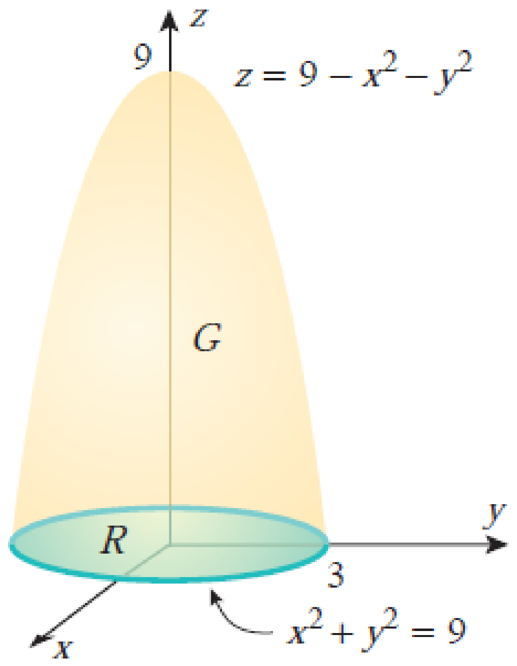
$$= \int_0^{2\pi} \int_0^3 r\sqrt{25 - r^2}\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(25 - r^2)^{3/2} \right]_{r=0}^3 d\theta$$

$$\boxed{\begin{aligned} u &= 25 - r^2 \\ du &= -2r\, dr \end{aligned}}$$

$$= \int_0^{2\pi} \frac{61}{3}\, d\theta = \frac{122}{3}\pi \blacktriangleleft$$

Example 2 Use cylindrical coordinates to evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx$$



Solution. In problems of this type, it is helpful to sketch the region of integration G and its projection R on the xy-plane. From the z-limits of integration, the upper surface of G is the paraboloid $z = 9 - x^2 - y^2$ and the lower surface is the xy-plane $z = 0$. From the x- and y-limits of integration, the projection R is the region in the xy-plane enclosed by the circle $x^2 + y^2 = 9$ (Figure 14.6.6). Thus,

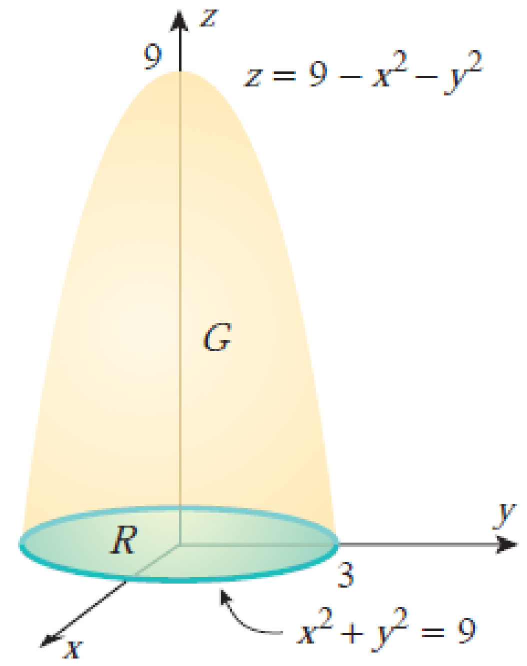
$$\iiint dz\,dy\,dx = \iiint r\,dz\,dr\,d\theta$$

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx = \iiint_G x^2\,dV$$

$$= \iint_R \left[\int_0^{9-r^2} r^2\cos^2\theta\,dz\right] dA = \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} (r^2\cos^2\theta) r\,dz\,dr\,d\theta$$

► Example 2 Use cylindrical coordinates to evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} x^2\,dz\,dy\,dx$$

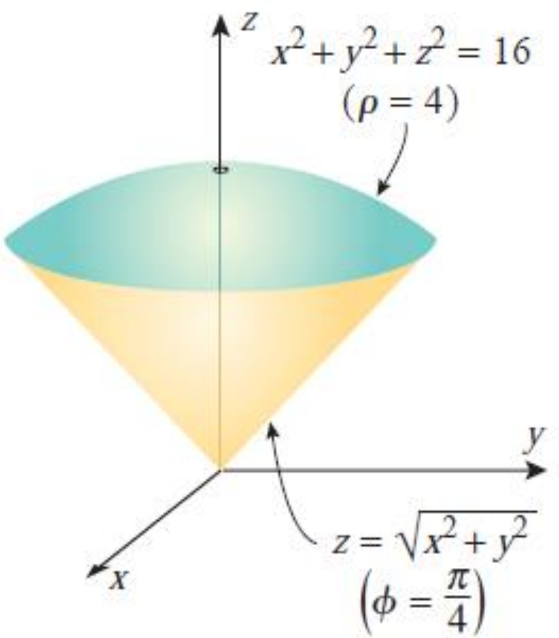


$$= \int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} r^3\cos^2\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{3}\left[zr^3\cos^2\theta\right]_{z=0}^{9-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{3}(9r^3-r^5)\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\left[\left(\frac{9r^4}{4}-\frac{r^6}{6}\right)\cos^2\theta\right]_{r=0}^{3}\,d\theta$$

$$= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{243}{4}\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{243\pi}{4} \blacktriangleleft$$

Example 3 Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.



Solution. The solid G is sketched in Figure 14.6.11. In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone $z = \sqrt{x^2 + y^2}$ is

$$\rho \cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$$

which simplifies to

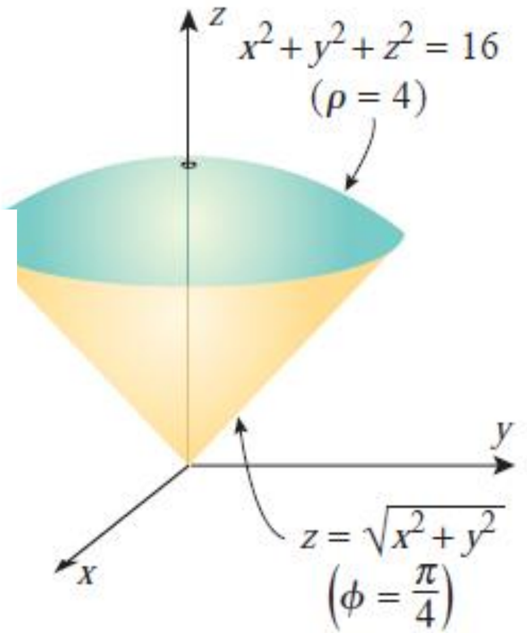
$$\rho \cos\phi = \rho \sin\phi$$

Dividing both sides of this equation by $\rho \cos\phi$ yields $\tan\phi = 1$, from which it follows that

$$\phi = \pi/4$$

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

Example 3 Use spherical coordinates to find the volume of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.



Thus, it follows from the second entry in Table 14.6.1 that the volume of G is

$$V = \iiint_G dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3} \sin\phi \right]_{\rho=0}^{4} d\phi \, d\theta$$

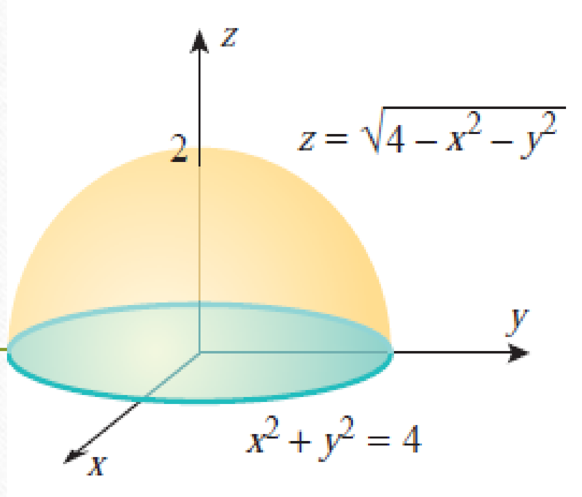
$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{64}{3} \sin\phi \, d\phi \, d\theta$$

$$= \frac{64}{3} \int_0^{2\pi} [-\cos\phi]_{\phi=0}^{\pi/4} d\theta = \frac{64}{3} \int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right) d\theta$$

$$= \frac{64\pi}{3}(2 - \sqrt{2}) \approx 39.26$$

Example 4 Use spherical coordinates to evaluate

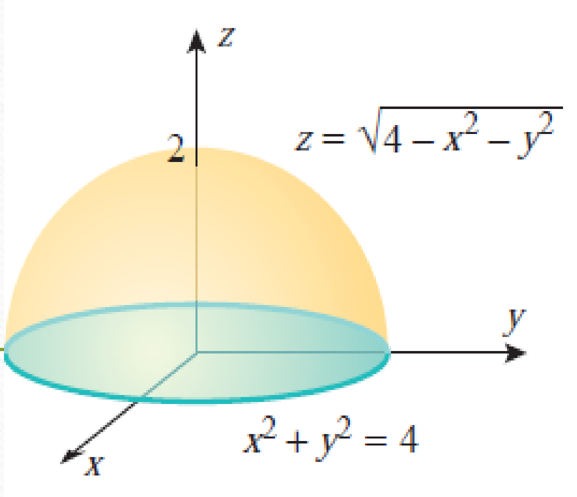
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$



Solution. In problems like this, it is helpful to begin (when possible) with a sketch of the region G of integration. From the z-limits of integration, the upper surface of G is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the lower surface is the xy-plane $z = 0$. From the x- and y-limits of integration, the projection of the solid G on the xy-plane is the region enclosed by the circle $x^2 + y^2 = 4$. From this information we obtain the sketch of G in Figure 14.6.12. Thus,

▶ **Example 4** Use spherical coordinates to evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$$

$$= \iiint_G z^2\sqrt{x^2+y^2+z^2}\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho^5\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{32}{3}\cos^2\phi\sin\phi\,d\phi\,d\theta$$

$$= \frac{32}{3}\int_0^{2\pi}\left[-\frac{1}{3}\cos^3\phi\right]_{\phi=0}^{\pi/2} d\theta = \frac{32}{9}\int_0^{2\pi} d\theta = \frac{64}{9}\pi \blacktriangleleft$$

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

THANK YOU