MATH 201: Coordinate Geometry and Vector Analysis

Chapter: 14.1

"Double Integrals"

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► Example 2 Evaluate

$$\text{(a)}\quad \int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx \qquad \text{(b)}\quad \int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$$

Solution (a).

$$\begin{aligned}
\int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx &= \int_1^3 \left[\int_2^4 (40 - 2xy)\, dy\right] dx \\
&= \int_1^3 (40y - xy^2)\Big]_{y=2}^{4}\, dx \\
&= \int_1^3 [(160 - 16x) - (80 - 4x)]\, dx \\
&= \int_1^3 (80 - 12x)\, dx \\
&= (80x - 6x^2)\Big]_1^3 = 112
\end{aligned}$$

Example 2 Evaluate

$$\text{(a)} \int_1^3 \int_2^4 (40 - 2xy)\, dy\, dx \qquad \text{(b)} \int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy$$

Solution (b).

$$\begin{aligned}
\int_2^4 \int_1^3 (40 - 2xy)\, dx\, dy &= \int_2^4 \left[\int_1^3 (40 - 2xy)\, dx\right] dy \\
&= \int_2^4 (40x - x^2y)\Big]_{x=1}^3 \, dy \\
&= \int_2^4 [(120 - 9y) - (40 - y)]\, dy \\
&= \int_2^4 (80 - 8y)\, dy \\
&= (80y - 4y^2)\Big]_2^4 = 112
\end{aligned}$$

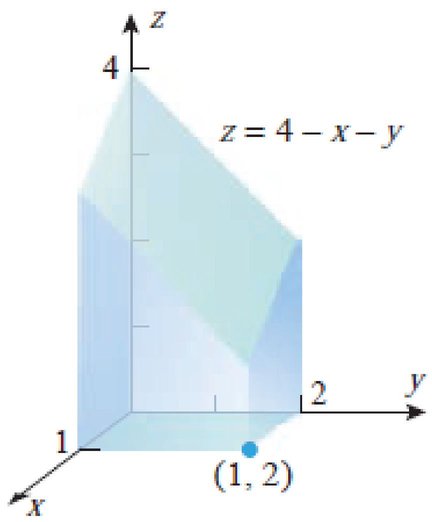
14.1.3 THEOREM (*Fubini's Theorem*) *Let R be the rectangle defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

▶ **Example 3** Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 14.1.6).



Solution. The volume is the double integral of $z = 4 - x - y$ over R. Using Theorem 14.1.3, this can be obtained from either of the iterated integrals

$$\int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \quad \text{or} \quad \int_0^1 \int_0^2 (4 - x - y)\, dy\, dx \tag{8}$$

Using the first of these, we obtain

$$\begin{aligned} V &= \iint_R (4 - x - y)\, dA = \int_0^2 \int_0^1 (4 - x - y)\, dx\, dy \\ &= \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_0^2 \left(\frac{7}{2} - y\right) dy \\ &= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_0^2 = 5 \end{aligned}$$

Example 4: DIY

Evaluate the iterated integrals.

9. $\displaystyle\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$

$\boxed{\text{Exercise : 9}}$

$$\int_0^1 \int_0^1 \frac{x}{(xy+1)^2}\, dy\, dx$$

First we will evaluate

$$\int_0^1 \frac{x}{(xy+1)^2}\, dy$$

Let, $u = xy+1$

$\Rightarrow du = x\,dy$

<u>limit</u>

$y = 0,\ u = 1$

$y = 1,\ u = x+1$

$$= \int_1^{x+1} \frac{du}{u^2}$$

$$= \left[-\frac{1}{u}\right]_1^{x+1}$$

$$= -\left(\frac{1}{x+1} - 1\right)$$

$$= 1 - \frac{1}{x+1}$$

Now, $\displaystyle\int_0^1 \left(1 - \frac{1}{x+1}\right) dx$

$$= \left[x - \ln(x+1)\right]_0^1$$

$$= 1 - \ln(2) \quad \text{(Ans.)}$$

13–16 Evaluate the double integral over the rectangular region R.

14. $$\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\,dA;$$

$R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$

$$\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\,dA$$

$$R = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$

$$\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx$$

Now, $\displaystyle\int_0^1 \frac{y}{\sqrt{x^2+y^2+1}}\,dy$

let, $u = x^2+y^2+1$

$\Rightarrow du = 2y\,dy$

Limits

$y=0,\ u = x^2+1$

$y=1,\ u = x^2+2$

$$= \frac{1}{2}\int_{x^2+1}^{x^2+2} \frac{du}{\sqrt{u}}$$

$$= \frac{1}{2}\left[-\cancel{2}\,u^{1/2}\right]_{x^2+1}^{x^2+2}$$

$$= \left[u^{1/2}\right]_{x^2+1}^{x^2+2}$$

13–16 Evaluate the double integral over the rectangular region R.

14. $\displaystyle\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\, dA;$

$R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$

$$= (x^2+2)^{1/2} - (x^2+1)^{1/2}$$

Now, $\displaystyle\int_0^1 \left(x\sqrt{x^2+2}\, dx\right) - \int_0^1 x\sqrt{x^2+1}\, dx$

$$= \int_2^3 \frac{1}{2}\sqrt{z}\,dz - \int_1^2 \frac{1}{2}\sqrt{v}\,dv$$

$$= \frac{1}{2}\left[\frac{2}{3} z^{3/2}\right]_2^3 - \frac{1}{2}\left[\frac{2}{3} v^{3/2}\right]_1^2$$

$$= \frac{1}{3}\left(3^{3/2} - 2^{3/2}\right) - \frac{1}{3}\left(2^{3/2} - 1\right)$$

$$= \frac{1}{3}\left(\sqrt{27} - \sqrt{8}\right) - \frac{1}{3}\left(\sqrt{8} - 1\right)$$

$$= \frac{1}{3}\left(3\sqrt{3} - 4\sqrt{2} + 1\right)$$ (Ans.)

Let,

$z = x^2 + 2$

$dz = 2x\,dx$

limits

$x = 0,\ z = 2$

$x = 1,\ z = 3$

And

$v = x^2 + 1$

$dv = 2x\,dx$

limits

$x = 0,\ v = 1$

$x = 1,\ v = 2$

THANK YOU