MATH 201: Coordinate Geometry and Vector Analysis

Chapter: 15.4

Green's Theorem

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❑ Green’s theorem:

Green's theorem gives a relationship between the line integral of a two-dimensional vector field over a closed path in the plane and the double integral over the region it encloses.

❑ Green's theorem:

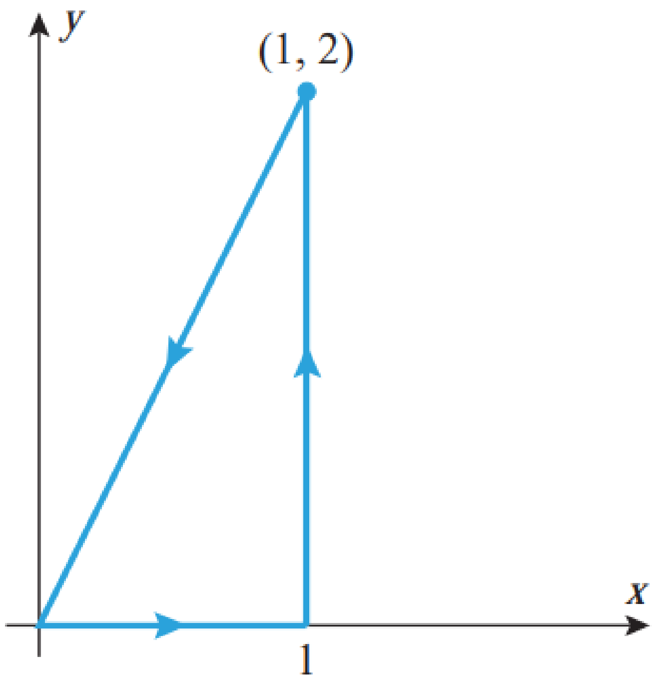
15.4.1 THEOREM (*Green's Theorem*) *Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open set containing R, then*

$$\int_C f(x, y)\, dx + g(x, y)\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA \tag{1}$$

Example 1 Use Green's Theorem to evaluate

$$\int_C x^2 y\,dx + x\,dy$$

along the triangular path shown in Figure .



Solution. Since $f(x, y) = x^2 y$ and $g(x, y) = x$, it follows from (1) that

$$\int_C x^2 y\,dx + x\,dy = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2 y)\right] dA = \int_0^1 \int_0^{2x} (1 - x^2)\,dy\,dx$$

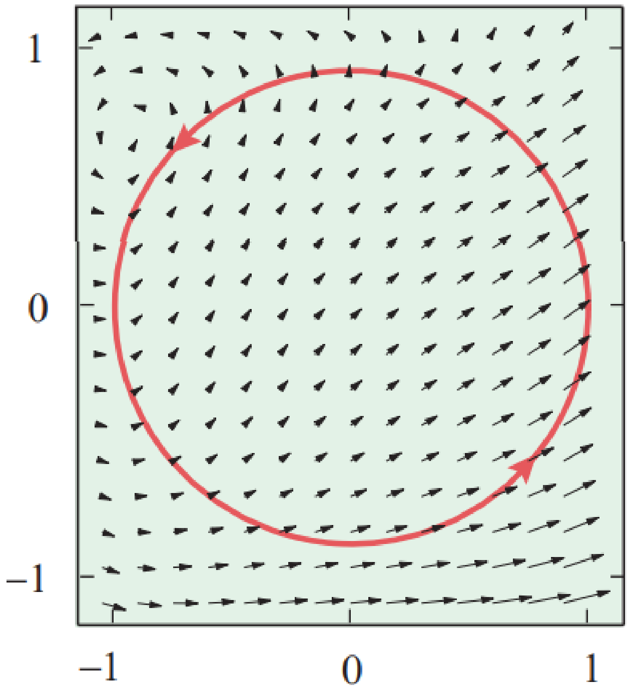
$$= \int_0^1 (2x - 2x^3)\,dx = \left[x^2 - \frac{x^4}{2}\right]_0^1 = \frac{1}{2}$$

$$\boxed{\int_C f(x, y)\,dx + g(x, y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA}$$

▶ **Example 2** Find the work done by the force field

$$\mathbf{F}(x, y) = (e^x - y^3)\mathbf{i} + (\cos y + x^3)\mathbf{j}$$

on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the counterclockwise direction Figure



Solution. The work W performed by the field is

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (e^x - y^3)\, dx + (\cos y + x^3)\, dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3)\right] dA \qquad \boxed{\text{Green's Theorem}}$$

$$= \iint_R (3x^2 + 3y^2)\, dA = 3 \iint_R (x^2 + y^2)\, dA$$

$$= 3 \int_0^{2\pi} \int_0^1 (r^2) r\, dr\, d\theta = \frac{3}{4} \int_0^{2\pi} d\theta = \frac{3\pi}{2} \blacktriangleleft$$

We converted to polar coordinates.

$$\int_C f(x, y)\, dx + g(x, y)\, dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

❑ Finding areas using Green's theorem:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

▸ **Example 3** Use a line integral to find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution. The ellipse, with counterclockwise orientation, can be represented parametrically by

$$x = a\cos t, \quad y = b\sin t \qquad (0 \le t \le 2\pi)$$

If we denote this curve by C, then from the third formula in (6) the area A enclosed by the ellipse is

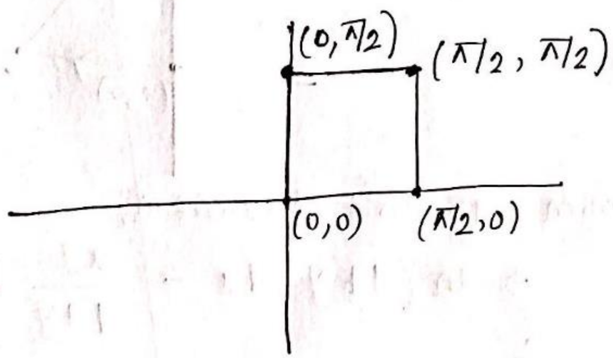
$$\begin{aligned} A &= \frac{1}{2}\oint_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} [(-b\sin t)(-a\sin t) + (a\cos t)(b\cos t)]\,dt \\ &= \frac{1}{2}ab\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = \frac{1}{2}ab\int_0^{2\pi} dt = \pi ab \end{aligned}$$ ◂

$$\boxed{A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C -y\,dx + x\,dy}$$

- **Exercise -5:** Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise.

$\oint_C x\cos y\,dx - y\sin x\,dy$, where C is the square with vertices $(0,0)$, $(\pi/2, 0)$, $(\pi/2, \pi/2)$, and $(0, \pi/2)$.

Exercise: 5



By Green's theorem,

$$\oint_C x\cos y\,dx - y\sin x\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(-y\sin x) - \frac{\partial}{\partial y}(x\cos y)\right] dA$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{\pi/2} (-y\cos x + x\sin y)\,dy\,dx$$

$$= \int_0^{\pi/2} \left(-\frac{y^2}{2}\cos x - x\cos y\right)_0^{\pi/2} dx$$

$$= -\int_0^{\pi/2} \left(\cos x \cdot \frac{\pi^2}{8} - x\right) dx$$

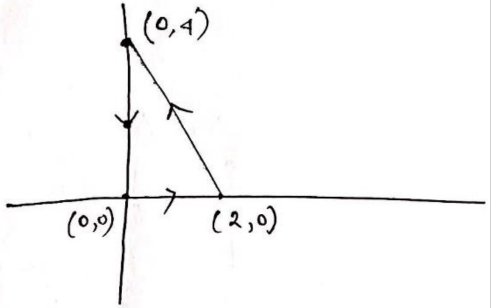
$$= -\left[\frac{\pi^2}{8}\sin x - \frac{x^2}{2}\right]_0^{\pi/2}$$

$$= -\left[\frac{\pi^2}{8} - \frac{1}{2}\cdot\frac{\pi^2}{4}\right] = -\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) = 0 \text{ (Ans.)}$$

Exercise -9: Use Green's Theorem to evaluate the integral. In each exercise, assume that the curve C is oriented counterclockwise.

$$\oint_C \ln(1+y)\,dx - \frac{xy}{1+y}\,dy,$$ where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(0,4)$.

Exercise 9:



From Green's theorem,

$$\oint_C \ln(1+y)\,dx - \frac{xy}{1+y}\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}\left(-\frac{xy}{1+y}\right) - \frac{\partial}{\partial y}\left[\ln(1+y)\right]\right] dA$$

$$= \iint_R \left(-\frac{y}{1+y} - \frac{1}{1+y}\right) dA$$

$$= -\iint_R \left(\frac{1+y}{1+y}\right) dA$$

$$= -\iint_R dA$$

$$= -\int_{x=0}^{2}\int_{y=0}^{4-2x} dy\,dx$$

$$= -\int_0^2 [y]_0^{4-2x}\,dx$$

$$= -\int_0^2 (4-2x)\,dx$$

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

$(x_1,y_1) \to (2,0)$

$(x_2,y_2) \to (0,4)$

$$\boldsymbol{y = 4 - 2x}$$

$$= -\left[4x - x^2\right]_0^2$$

$$= -\left[4\cdot 2 - 4\right]$$

$$= -(8-4)$$

$$= -4 \text{ (Ans.)}$$

THANK YOU