MATH 201: Coordinate Geometry and Vector Analysis

Chapter: 15.1 & 15.2

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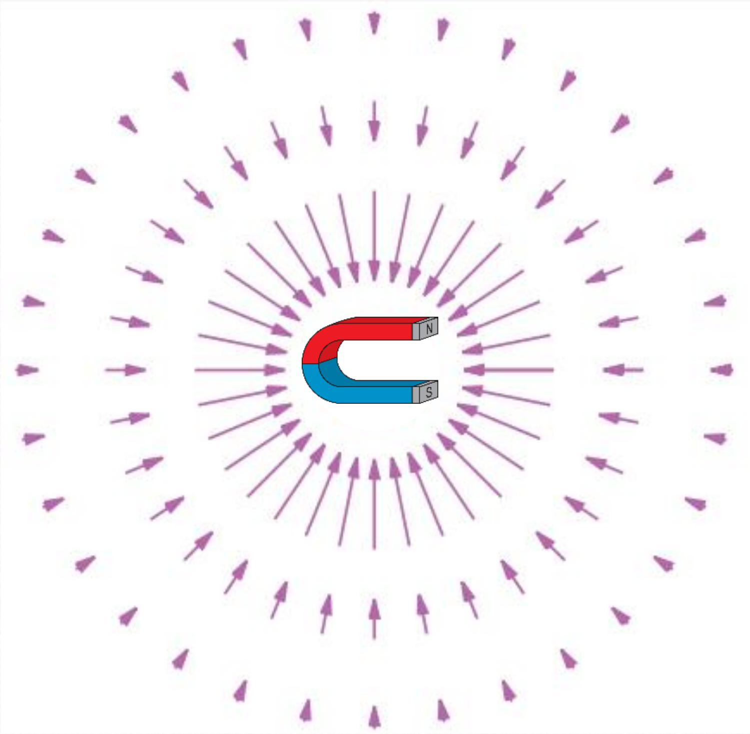
Chapter: 15.1

Topic: Vector Fields

❑ Vector field:

15.1.1 DEFINITION A ***vector field*** in a plane is a function that associates with each point P in the plane a unique vector $\mathbf{F}(P)$ parallel to the plane. Similarly, a vector field in 3-space is a function that associates with each point P in 3-space a unique vector $\mathbf{F}(P)$ in 3-space.



❑ Vector field(continued….):

15.1.1 DEFINITION A ***vector field*** in a plane is a function that associates with each point P in the plane a unique vector $\mathbf{F}(P)$ parallel to the plane. Similarly, a vector field in 3-space is a function that associates with each point P in 3-space a unique vector $\mathbf{F}(P)$ in 3-space.

Expression of vector field in 2-D system:

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

Expression of vector field in 3-D system:

$$\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$

❑ Inverse square field:

According to Newton's Law of Universal Gravitation, particles with masses m and M attract each other with a force $\mathbf{F}$ of magnitude

Magnitude of $\vec{F}$ depends on the inverse of r^2

$$\|\mathbf{F}\| = \frac{GmM}{r^2} \tag{1}$$

where r is the distance between the particles and G is a constant. If we assume that the particle of mass M is located at the origin of an xyz-coordinate system and $\mathbf{r}$ is the radius vector to the particle of mass m, then $r = \|\mathbf{r}\|$, and the force $\mathbf{F}(\mathbf{r})$ exerted by the particle of mass M on the particle of mass m is in the direction of the unit vector $-\mathbf{r}/\|\mathbf{r}\|$. Thus, it follows from (1) that

$$\mathbf{F}(\mathbf{r}) = -\frac{GmM}{\|\mathbf{r}\|^2}\frac{\mathbf{r}}{\|\mathbf{r}\|} = -\frac{GmM}{\|\mathbf{r}\|^3}\mathbf{r} \tag{2}$$

If m and M are constant, and we let $c = -GmM$, then this formula can be expressed as

$$\mathbf{F}(\mathbf{r}) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}$$

Vector $\vec{F}$ depends on the inverse of r^3

Example 3: Consider a vector field $\vec{F}$ and a potential function $\emptyset$ as shown below. Show that $\vec{F}$ is conservative for $\emptyset$.

$$\mathbf{F}(x, y) = \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j})$$

$$\phi(x, y) = -\frac{c}{(x^2 + y^2)^{1/2}}$$

$$\begin{aligned}\nabla\phi(x, y) &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} \\ &= \frac{cx}{(x^2 + y^2)^{3/2}}\mathbf{i} + \frac{cy}{(x^2 + y^2)^{3/2}}\mathbf{j} \\ &= \frac{c}{(x^2 + y^2)^{3/2}}(x\mathbf{i} + y\mathbf{j}) \\ &= \mathbf{F}(x, y)\end{aligned}$$

$$\boldsymbol{F}(x, y) = \nabla\emptyset$$

❑ Divergence and Curl:

The *divergence* relates to the way in which fluid flows toward or away from a point .

The *curl* relates to the rotational properties of the fluid at a point.

❑ Divergence and Curl:

15.1.4 DEFINITION If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***divergence of*** $\mathbf{F}$, written div $\mathbf{F}$, to be the function given by

$$\text{div}\,\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z} \tag{7}$$

15.1.5 DEFINITION If $\mathbf{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then we define the ***curl of*** $\mathbf{F}$, written curl $\mathbf{F}$, to be the vector field given by

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k} \tag{8}$$

▶ **Example 4** Find the divergence and the curl of the vector field

$$\mathbf{F}(x, y, z) = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

Solution.

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(2y^3z) + \frac{\partial}{\partial z}(3z)\\ &= 2xy + 6y^2z + 3\end{aligned}$$

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix}\\ &= \left[\frac{\partial}{\partial y}(3z) - \frac{\partial}{\partial z}(2y^3z)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(3z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(2y^3z) - \frac{\partial}{\partial y}(x^2y)\right]\mathbf{k}\\ &= -2y^3\mathbf{i} - x^2\mathbf{k}\end{aligned}$$ ◀

❑ **Exercise:** For any vector $\mathbf{F} = \mathbf{F}(x, y, z)$ prove that,

37. $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ **38.** $\operatorname{curl}(\nabla\phi) = \mathbf{0}$

Exercise 37: Prove, $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$

Sol^n: We know,

$$\operatorname{curl}\vec{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$$

$$\therefore \operatorname{div}(\operatorname{curl}\vec{F})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$$

g

f

h

$$= \frac{\partial^2 h}{\partial x \partial y} - \frac{\partial^2 g}{\partial x \partial z} + \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 h}{\partial y \partial x} + \frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 f}{\partial z \partial y}$$

$$= 0.$$

[Proved]

Exercise 38: Prove, $\operatorname{curl}(\nabla\phi) = 0$

Sol^n: We know,

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\therefore \operatorname{curl}(\nabla\phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \hat{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right)$$

$$= 0.$$

[Proved].

$$\operatorname{div}\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

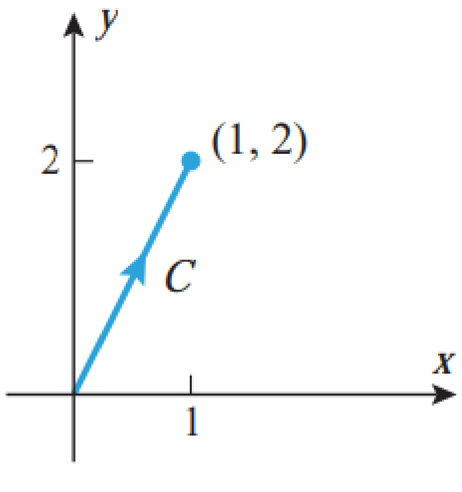
$$\operatorname{curl}\mathbf{F} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}$$

Chapter: 15.2

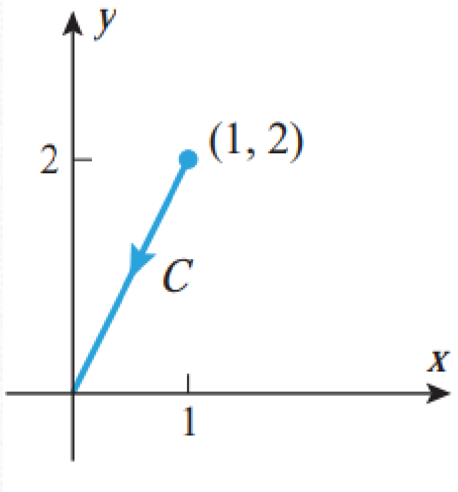
Topic: Line Integrals

Line Integral:

A *line integral* is an integral where the function to be integrated is evaluated along a curve C.



(a)



(b)

▶ **Example 5** Evaluate $\int_C 3xy\,dy$, where C is the line segment joining (0, 0) and (1, 2) with the given orientation.

(a) Oriented from (0, 0) to (1, 2) as in Figure 15.2.6*a*.

(b) Oriented from (1, 2) to (0, 0) as in Figure 15.2.6*b*.

$$(a, b) = (1 - 0, 2 - 0)$$
$$= (1, 2)$$

Solution (a). Using the parametrization

$$x = t, \quad y = 2t \qquad (0 \le t \le 1)$$

$$x = x_0 + at;\ x_0 = 0, a = 1$$
$$y = y_0 + bt;\ y_0 = 0, b = 2$$

we have

$$\int_C 3xy\,dy = \int_0^1 3(t)(2t)(2)\,dt = \int_0^1 12t^2\,dt = 4t^3\Big]_0^1 = 4$$

▸ **Example 5** Evaluate $\int_C 3xy\,dy$, where C is the line segment joining (0, 0) and (1, 2) with the given orientation.

(a) Oriented from (0, 0) to (1, 2) as in Figure 15.2.6*a*.

(b) Oriented from (1, 2) to (0, 0) as in Figure 15.2.6*b*.

$$(a, b) = (0-1, 0-2)$$
$$= (-1, -2)$$

Solution (b). Using the parametrization

$$x = 1 - t, \quad y = 2 - 2t \qquad (0 \le t \le 1)$$

we have

$$x = x_0 + at;\ x_0 = 1, a = -1$$
$$y = y_0 + bt;\ y_0 = 2, b = -2$$

$$\int_C 3xy\,dy = \int_0^1 3(1-t)(2-2t)(-2)\,dt = \int_0^1 -12(1-t)^2\,dt = 4(1-t)^3\Big]_0^1 = -4$$

▸ Example 6 Evaluate

$$\int_C 2xy\,dx + (x^2 + y^2)\,dy$$

along the circular arc C given by $x = \cos t$, $y = \sin t$ $(0 \le t \le \pi/2)$ (Figure 15.2.11).

Do It Yourself

▶ **Example 7** Evaluate

$$\int_C (3x^2 + y^2)\, dx + 2xy\, dy$$

along the circular arc C given by $x = \cos t$, $y = \sin t$ $(0 \leq t \leq \pi/2)$ (Figure 15.2.11).

Solution. From (23) we have

$$\begin{aligned}
\int_C (3x^2 + y^2)\, dx + 2xy\, dy &= \int_0^{\pi/2} [(3\cos^2 t + \sin^2 t)(-\sin t) + 2(\cos t)(\sin t)(\cos t)]\, dt \\
&= \int_0^{\pi/2} (-3\cos^2 t \sin t - \sin^3 t + 2\cos^2 t \sin t)\, dt \\
&= \int_0^{\pi/2} (-\cos^2 t - \sin^2 t)(\sin t)\, dt = \int_0^{\pi/2} -\sin t\, dt \\
&= \cos t\Big]_0^{\pi/2} = -1 \quad ◀
\end{aligned}$$

Some problems from the exercise are solved in the next slides!

13. In each part, evaluate the integral

$$\int_C (3x + 2y)\,dx + (2x - y)\,dy$$

along the stated curve.

(a) The line segment from (0, 0) to (1, 1).
(b) The parabolic arc $y = x^2$ from (0, 0) to (1, 1).
(c) The curve $y = \sin(\pi x/2)$ from (0, 0) to (1, 1).
(d) The curve $x = y^3$ from (0, 0) to (1, 1).

Exercise 13:

(a) C: $x = t,\ y = t,\ 0 \le t \le 1.$

$\therefore \int_0^1 (3t + 2t)\,dt + (2t - t)\,dt$

$= \int_0^1 6t\,dt = 3$ (Ans.)

(b) C: $x = t,\ y = t^2$

$dx = dt,\ dy = 2t\,dt$

$\therefore \int_0^1 (3t + 6t^2 - 2t^3)\,dt = 3$ (Ans.)

(d) C: $x = t^3,\ y = t$

$\therefore dx = 3t^2\,dt,\ dy = dt$

$\therefore \int_0^1 (9t^5 + 6t^3 - t)\,dt = 3.$ (Ans.)

25. Evaluate the line integral along the curve C.

$$\int_C -y\,dx + x\,dy$$

$C: y^2 = 3x$ from $(3, 3)$ to $(0, 0)$

$C:\quad y = 3-t \Rightarrow dy = -dt$

$\therefore\ x = \dfrac{(3-t)^2}{3} \Rightarrow dx = \frac{1}{3}\cdot 2(3-t)(-dt)$

$\Rightarrow dx = \frac{2}{3}(t-3)dt$

Now, $\displaystyle\int_0^3 -y\,dx + x\,dy$

$= \displaystyle\int_0^3 (t-3)\,\frac{2}{3}(t-3)\,dt + \frac{1}{3}(3-t)^2(-dt)$

$= \displaystyle\int_0^3 \frac{2}{3}(t-3)^2 dt + \frac{1}{3}(t-3)^2(-dt)$

$= \displaystyle\frac{1}{3}\int_0^3 (t-3)^2\,dt$

$= \displaystyle\frac{1}{3}\int_0^3 (t^2 - 6t + 9)\,dt$

$= \displaystyle\frac{1}{3}\left[\frac{t^3}{3} - 3t^2 + 9t\right]_0^3$

$= \frac{1}{3}\left[\frac{1}{3}\cdot 27 - 27 + 27\right]$

$= 3.$ (Ans.)

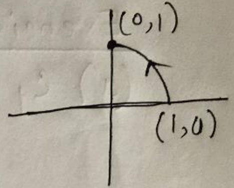
27. Evaluate the line integral along the curve C.

$$\int_C (x^2 + y^2)\,dx - x\,dy$$

$C : x^2 + y^2 = 1$, counterclockwise from $(1, 0)$ to $(0, 1)$

Exercise 27: let, $x = \cos t$

$y = \sin t$



$dx = -\sin t\,dt$

$dy = \cos t\,dt$

When, $x = 1$, $t = 0$

$x = 0$, $t = \frac{\pi}{2}$

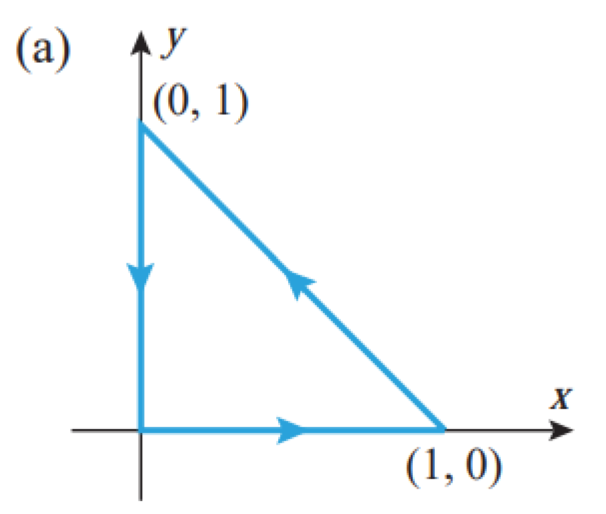
$$\int_C (x^2+y^2)\,dx - x\,dy$$

$$= \int_0^{\pi/2} -\sin t\,dt - \cos t(\cos t)\,dt$$

$$= \int_0^{\pi/2} (-\sin t - \cos^2 t)\,dt$$

$$= -1 - \pi/4 \quad \text{(Ans.)}$$

33– Evaluate $\int_C y\,dx - x\,dy$ along the curve C shown in the figure.



Exercise 33:

C_1: From $(0,0)$ to $(1,0)$

$x = t, \quad y = 0$

$\therefore \int_0^1 0 - x \cdot 0 = \underline{\underline{0}}$

C_2: From $(1,0)$ to $(0,1)$

$x = 1-t, \quad y = t$

$dx = -dt, \quad dy = dt$

$\therefore \int_0^1 -t\,dt - (1-t)\,dt = \int_0^1 (\cancel{-t} - 1 + \cancel{t})\,dt$

$= \cancel{\left[\frac{t^2}{2}\right]} - [t]_0^1 = 0 - 1$

$= \underline{\underline{-1}}$

C_3: From $(0,1)$ to $(0,0)$

$x = 0, \quad y = 1-t$

$dx = 0, \quad dy = -dt$

$\therefore \int_1^0 (1-t) \cdot 0 + 0 \cdot dt = \underline{\underline{0}}$

$\therefore C_1 + C_2 + C_3 = 0 - 1 + 0 = -1$ (Ans.)

THANK YOU