# MATH 201: Coordinate Geometry and Vector Analysis

## Chapter 13.6: Directional Derivatives and Gradients

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❖ **Directional derivative:** The directional derivative represents the instantaneous rate of change of $\boldsymbol{f(x,y)}$ with respect to distance in the direction of $\boldsymbol{u}$ at the point $(\boldsymbol{x_0}, \boldsymbol{y_0})$.

Parametric equation of a line through $(x_0, y_0)$:

$$x = x_0 + at, y = y_0 + bt; \quad \boldsymbol{u} = (a, b)$$

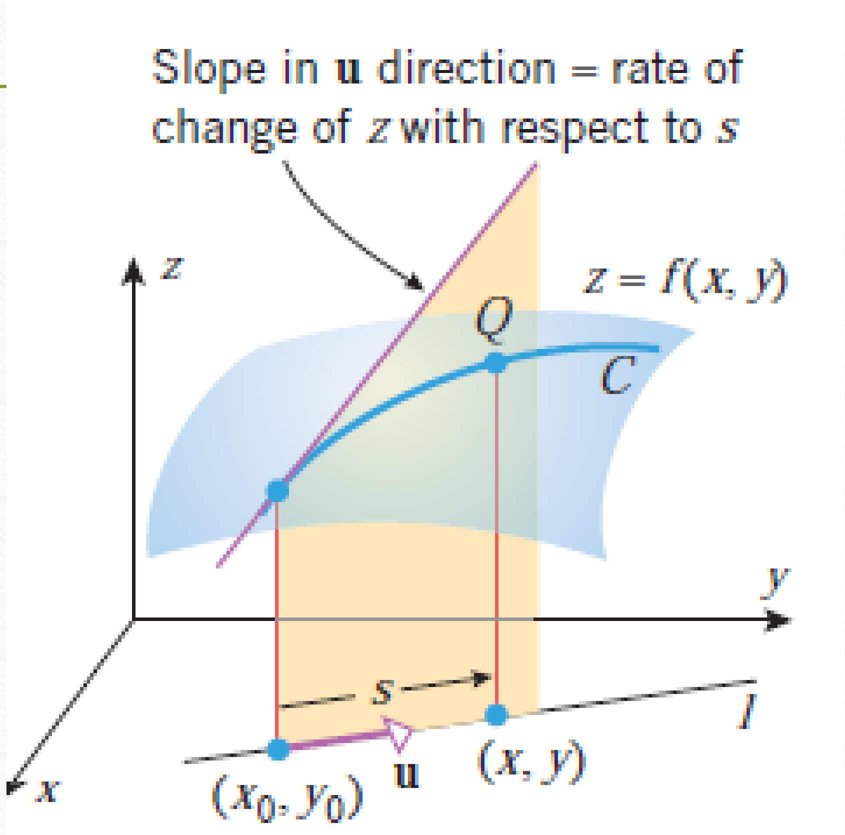
Since $z = f(x, y)$

$$z = f(x_0 + at,\ y_0 + bt)$$

Now,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$$



❖ **Directional derivative:** The directional derivative represents the instantaneous rate of change of $\boldsymbol{f(x,y)}$ with respect to distance in the direction of $\boldsymbol{u}$ at the point $(\boldsymbol{x_0}, \boldsymbol{y_0})$.

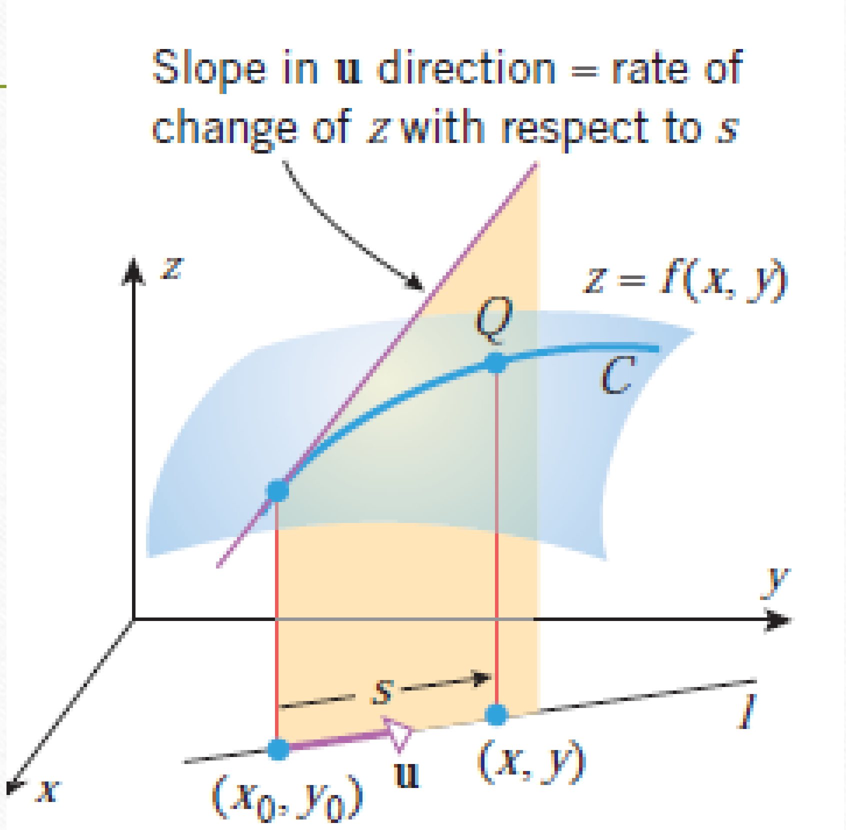
$$\frac{df}{dt} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$$

**Gradient**

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).(a,b)$$

So, we get

$$\boldsymbol{D_u}f = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$$



## ❖ Directional derivative:

**13.6.3 THEOREM**

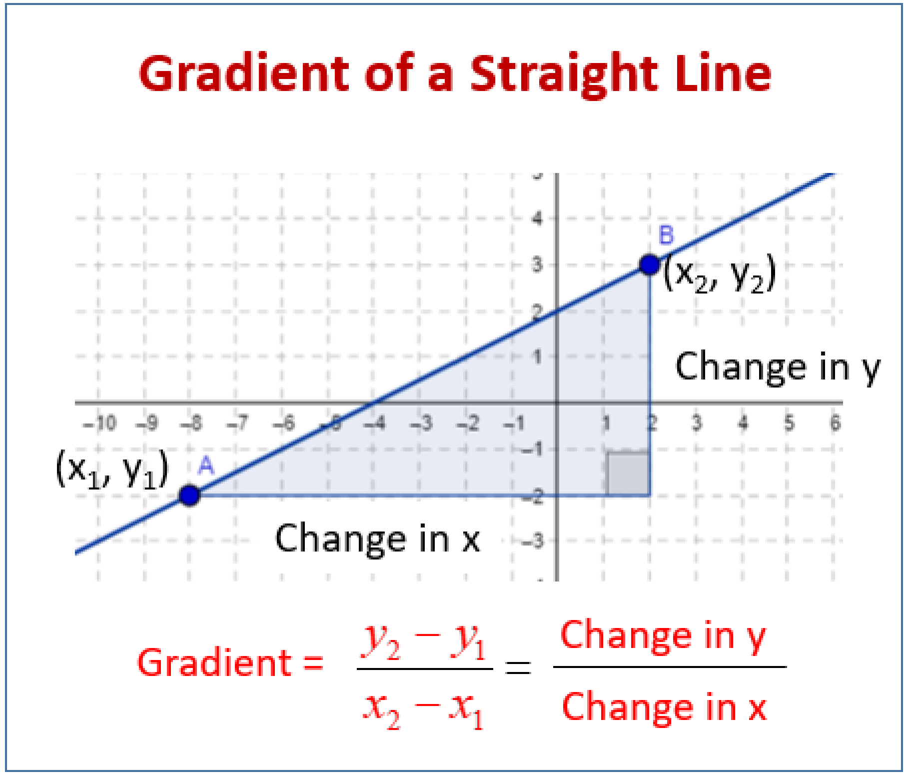
(a) *If* $f(x, y)$ *is differentiable at* $(x_0, y_0)$, *and if* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ *is a unit vector, then the directional derivative* $D_{\mathbf{u}} f(x_0, y_0)$ *exists and is given by*

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \tag{4}$$

(b) *If* $f(x, y, z)$ *is differentiable at* $(x_0, y_0, z_0)$, *and if* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ *is a unit vector, then the directional derivative* $D_{\mathbf{u}} f(x_0, y_0, z_0)$ *exists and is given by*

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = f_x(x_0, y_0, z_0)u_1 + f_y(x_0, y_0, z_0)u_2 + f_z(x_0, y_0, z_0)u_3 \tag{5}$$

❖ **Gradient:** Gradient represents the slope of a straight line. If $\boldsymbol{y = mx + c}$ is a equation of a straight line, then $\boldsymbol{m}$ represents the gradient (slope).




Gradient of $f = \boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}$

## ❖ Gradient:

**13.6.4 DEFINITION**

(a) If $f$ is a function of $x$ and $y$, then the *gradient of $f$* is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} \tag{8}$$

(b) If $f$ is a function of $x$, $y$, and $z$, then the *gradient of $f$* is defined by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k} \tag{9}$$

**EXERCISE 13:** Find the directional derivative of f at P in the direction of a:

$f(x, y) = \tan^{-1}(y/x)$; $P(-2, 2)$; $\mathbf{a} = -\mathbf{i} - \mathbf{j}$

Exercise 13: Given,

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right), \quad P(-2,2)$$

Gradient,

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

$$= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{1}{x^2}\cdot y\right)\hat{i} + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)\hat{j}$$

$$= -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

Now, $\nabla f(-2,2) = -\frac{2}{8}\hat{i} - \frac{2}{8}\hat{j}$

$$= -\frac{1}{4}\hat{i} - \frac{1}{4}\hat{j}$$

The unit vector,

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{-\boldsymbol{i}-\boldsymbol{j}}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\boldsymbol{i} - \frac{1}{\sqrt{2}}\boldsymbol{j}$$

**EXERCISE 13:** Find the directional derivative of f at P in the direction of a:

$f(x, y) = \tan^{-1}(y/x)$; $P(-2, 2)$; $\mathbf{a} = -\mathbf{i} - \mathbf{j}$

Hence the directional derivative of f at P,

$$D_{\underline{u}} f(-2,2) = f_x(-2,2)\, u_1 + f_y(-2,2)\, u_2$$

$$= \left(-\frac{1}{4}\right)\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{4}\right)\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}} = \frac{1}{2\sqrt{2}} \quad \text{(Ans.)}$$

**EXERCISE 55:** Find a unit vector in the direction in which $f$ increases most rapidly at $P$, and find the rate of change of $f$ at $P$ in that direction:

$$f(x, y) = \sqrt{x^2 + y^2}; \;\; P(4, -3)$$

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$\boxed{55}$ $f(x,y) = \sqrt{x^2+y^2}$ , $P(4,-3)$

Sol$^n$: $\nabla f(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}(2x)\,\hat{i} + \frac{1}{2}(x^2+y^2)^{-1/2}(2y)\,\hat{j}$

$\nabla f(4,-3) = \frac{1}{2}(16+9)^{-1/2}(8)\,\hat{i} + \frac{1}{2}(16+9)^{-1/2}(-6)\,\hat{j}$

$= \frac{8}{2\cdot 5}\,\hat{i} + \frac{-6}{2\cdot 5}\,\hat{j}$

$= \frac{4}{5}\,\hat{i} - \frac{3}{5}\,\hat{j}$

$\|\nabla f(4,-3)\| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2} = 1$

$\therefore$ The unit vector, $\underline{u} = \dfrac{\nabla f(4,-3)}{\|\nabla f(4,-3)\|}$

$= \frac{4}{5}\,\hat{i} - \frac{3}{5}\,\hat{j}$

The rate of change of 'f' at $P(4,-3)$ in the direction of $\underline{u}$ is $\|\nabla f(4,-3)\| = 1$

**EXERCISE 55:** Find a unit vector in the direction in which $f$ ~~increases~~ most rapidly at $P$, and find the rate of change of $f$ at $P$ in that direction:

$f(x, y) = \sqrt{x^2 + y^2}$; $P(4, -3)$

Decreases

**Exercise:** 9, 11, 13, 53, 55, 61, 63

# THANK YOU