MATH 201: Coordinate Geometry and Vector Analysis

Chapter: 13.7

Tangent Planes and Normal Lines

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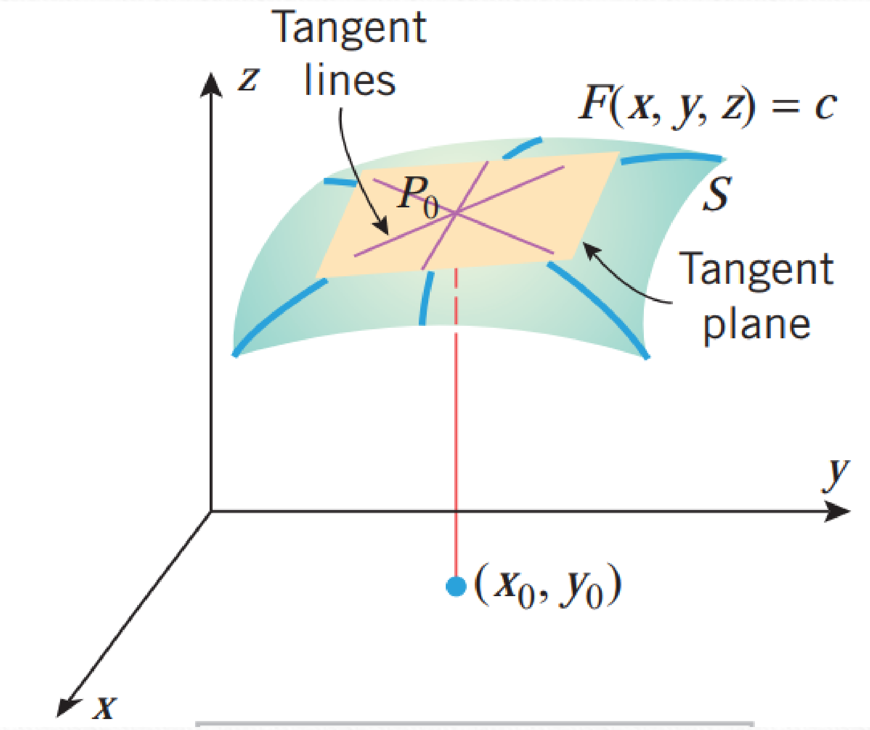
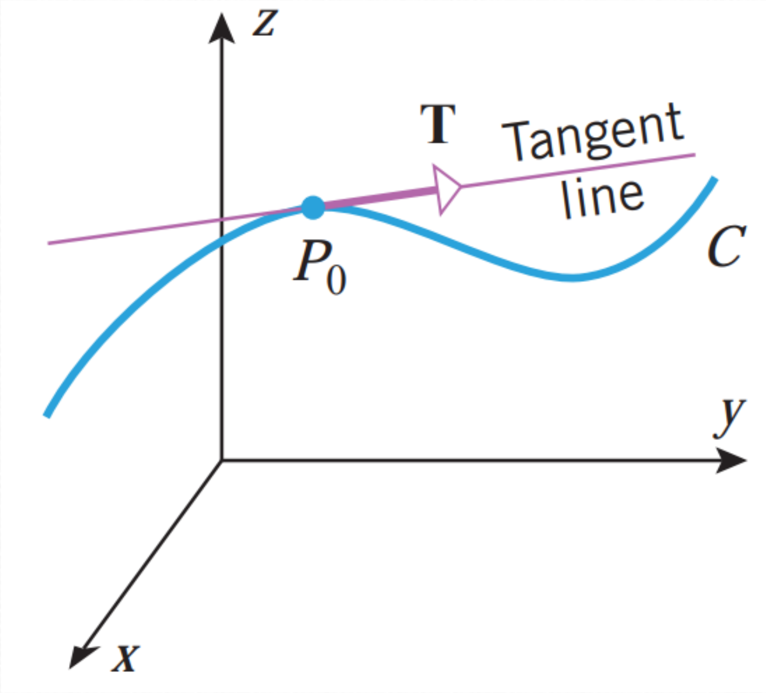
❑ Tangent Planes:

Tangent plane contains all possible tangent lines (*T in figure*) at a particular point (P_0 in figure) of a curve (*C in figure*).





Tangent Planes (continued…..)

13.7.1 DEFINITION Assume that $F(x, y, z)$ has continuous first-order partial derivatives and that $P_0(x_0, y_0, z_0)$ is a point on the level surface $S: F(x, y, z) = c$. If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, then $\mathbf{n} = \nabla F(x_0, y_0, z_0)$ is a ***normal vector*** to S at P_0 and the ***tangent plane*** to S at P_0 is the plane with equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \qquad (3)$$

The line through the point P_0 parallel to the normal vector $\mathbf{n}$ is perpendicular to the tangent plane (3). We will call this the ***normal line***, or sometimes more simply the ***normal*** to the surface $F(x, y, z) = c$ at P_0. It follows that this line can be expressed parametrically as

$$x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t \qquad (4)$$

▶ **Example 1** Consider the ellipsoid $x^2 + 4y^2 + z^2 = 18$.

✓ Find an equation of the tangent plane to the ellipsoid at the point $(1, 2, 1)$.

(b) Find parametric equations of the line that is normal to the ellipsoid at the point $(1, 2, 1)$.

(c) Find the acute angle that the tangent plane at the point $(1, 2, 1)$ makes with the xy-plane.

Solution (a). We apply Definition 13.7.1 with $F(x, y, z) = x^2 + 4y^2 + z^2$ and $(x_0, y_0, z_0) = (1, 2, 1)$. Since

$$\nabla F(x, y, z) = \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)\rangle = \langle 2x, 8y, 2z\rangle$$

we have

$$\mathbf{n} = \nabla F(1, 2, 1) = \langle 2, 16, 2\rangle$$

Hence, from (3) the equation of the tangent plane is

$$2(x-1) + 16(y-2) + 2(z-1) = 0 \quad \text{or} \quad x + 8y + z = 18$$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \qquad (3)$$

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Solution (b). Since $\mathbf{n} = \langle 2, 16, 2 \rangle$ at the point $(1, 2, 1)$, it follows from (4) that parametric equations for the normal line to the ellipsoid at the point $(1, 2, 1)$ are

$$x = 1 + 2t, \quad y = 2 + 16t, \quad z = 1 + 2t$$

$$x = x_0 + F_x(x_0, y_0, z_0)t, \quad y = y_0 + F_y(x_0, y_0, z_0)t, \quad z = z_0 + F_z(x_0, y_0, z_0)t \qquad (4)$$

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Find the acute angle that the tangent plane at the point $(1, 2, 1)$ makes with the xy-plane.

Solution (c). To find the acute angle θ between the tangent plane and the xy-plane, we will apply Formula (9) of Section 11.6 with $\mathbf{n}_1 = \mathbf{n} = \langle 2, 16, 2\rangle$ and $\mathbf{n}_2 = \langle 0, 0, 1\rangle$. This yields

$$\cos\theta = \frac{|\langle 2, 16, 2\rangle \cdot \langle 0, 0, 1\rangle|}{\|\langle 2, 16, 2\rangle\| \, \|\langle 0, 0, 1\rangle\|} = \frac{2}{(2\sqrt{66})(1)} = \frac{1}{\sqrt{66}}$$

Thus,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{66}}\right) \approx 83^\circ$$

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}_2\|}$$

▶ **Example 2** Find an equation for the tangent plane and parametric equations for the normal line to the surface $z = x^2y$ at the point $(2, 1, 4)$.

Same as example 1

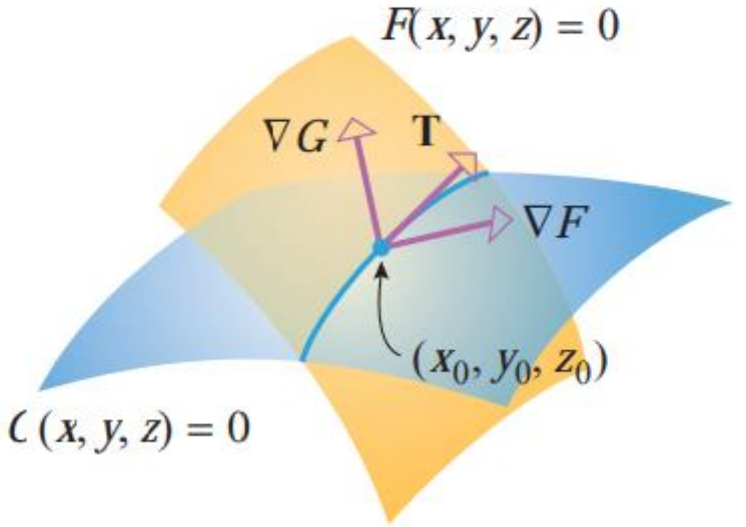
Do it yourself

❑ Using gradients to find tangent lines to intersections of surfaces:

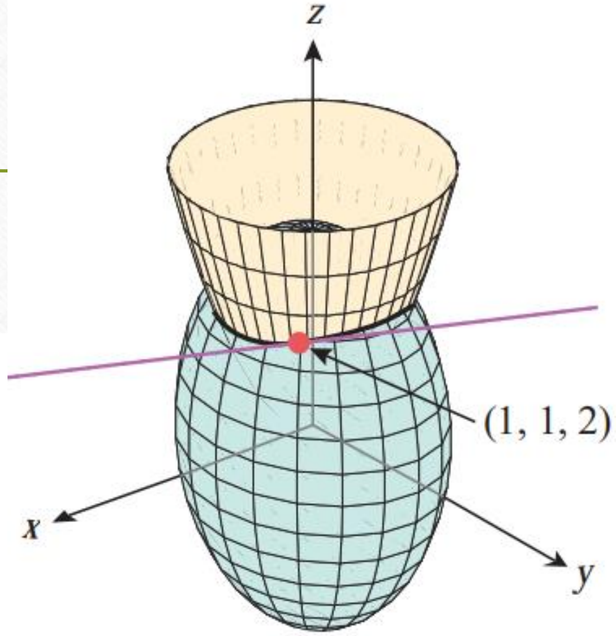
Thus, if the curve of intersection can be smoothly parametrized, then its unit tangent vector **T** at (x_0, y_0, z_0) will be orthogonal to both $\nabla F(x_0, y_0, z_0)$ and $\nabla G(x_0, y_0, z_0)$ (Figure 13.7.6). Consequently, if

$$\nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0) \neq \mathbf{0}$$

then this cross product will be parallel to **T** and hence will be tangent to the curve of intersection. This tangent vector can be used to determine the direction of the tangent line to the curve of intersection at the point (x_0, y_0, z_0).



▶ **Example 3** Find parametric equations of the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $3x^2 + 2y^2 + z^2 = 9$ at the point $(1, 1, 2)$



Solution. We begin by rewriting the equations of the surfaces as

$$x^2 + y^2 - z = 0 \quad \text{and} \quad 3x^2 + 2y^2 + z^2 - 9 = 0$$

and we take

$$F(x, y, z) = x^2 + y^2 - z \quad \text{and} \quad G(x, y, z) = 3x^2 + 2y^2 + z^2 - 9$$

We will need the gradients of these functions at the point $(1, 1, 2)$. The computations are

$$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}, \quad \nabla G(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} + 2z\mathbf{k}$$
$$\nabla F(1, 1, 2) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}, \quad \nabla G(1, 1, 2) = 6\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

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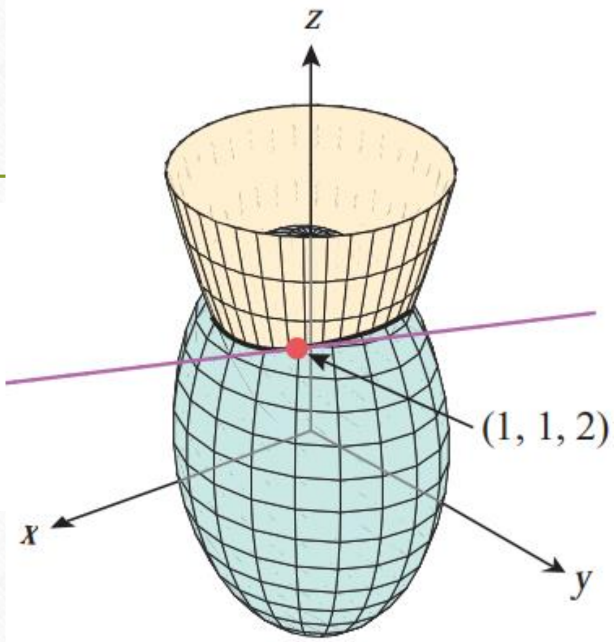

Thus, a tangent vector at $(1, 1, 2)$ to the curve of intersection is

$$\nabla F(1,1,2) \times \nabla G(1,1,2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 6 & 4 & 4 \end{vmatrix} = 12\mathbf{i} - 14\mathbf{j} - 4\mathbf{k}$$

Since any scalar multiple of this vector will do just as well, we can multiply by $\frac{1}{2}$ to reduce the size of the coefficients and use the vector of $6\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$ to determine the direction of the tangent line. This vector and the point $(1, 1, 2)$ yield the parametric equations

$$x = 1 + 6t, \quad y = 1 - 7t, \quad z = 2 - 2t$$ ◀

Exercises: 1, 2, 3-12, 29, 31

THANK YOU