MATH 201: Coordinate Geometry and Vector Analysis

Chapter: 14.5

"Triple Integrals"

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► **Example 1** Evaluate the triple integral

$$\iiint_G 12xy^2z^3\,dV$$

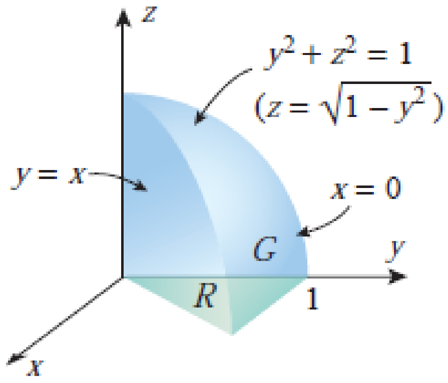
over the rectangular box G defined by the inequalities $-1 \le x \le 2, 0 \le y \le 3, 0 \le z \le 2$.

Solution. Of the six possible iterated integrals we might use, we will choose the one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned}\iiint_G 12xy^2z^3\,dV &= \int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} 12xy^2z^3\,dz\,dy\,dx \\ &= \int_{-1}^{2}\int_{0}^{3} \left[3xy^2z^4\right]_{z=0}^{2}\,dy\,dx = \int_{-1}^{2}\int_{0}^{3} 48xy^2\,dy\,dx \\ &= \int_{-1}^{2} \left[16xy^3\right]_{y=0}^{3}\,dx = \int_{-1}^{2} 432x\,dx \\ &= 216x^2\Big]_{-1}^{2} = 648\end{aligned}$$ ◄

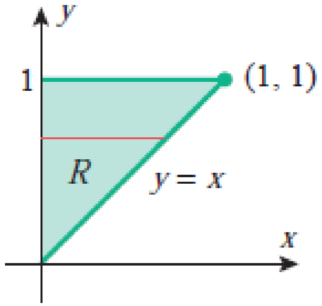
▸ **Example 2** Let G be the wedge in the first octant that is cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\,dV$$



$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA \tag{4}$$

For the double integral over R, the x- and y-integrations can be performed in either order, since R is both a type I and type II region. We will integrate with respect to x first. With this choice, (4) yields



$$\iiint_G z\,dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1 \int_0^y \frac{1}{2}z^2\bigg]_{z=0}^{\sqrt{1-y^2}} dx\,dy$$

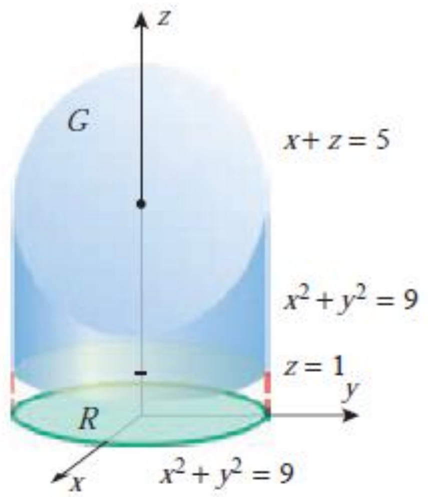
$$= \int_0^1 \int_0^y \frac{1}{2}(1-y^2)\,dx\,dy = \frac{1}{2}\int_0^1 (1-y^2)x\bigg]_{x=0}^{y} dy$$

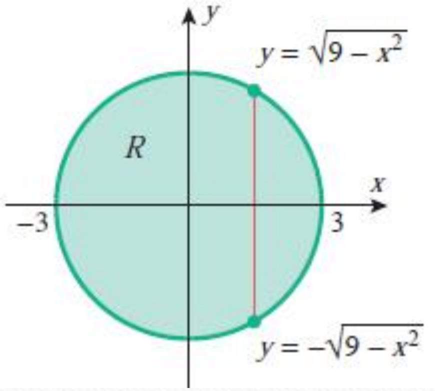
$$= \frac{1}{2}\int_0^1 (y-y^3)\,dy = \frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{8}$$ ◂

▸ **Example 3** Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution. The solid G and its projection R on the xy-plane are shown in Figure 14.5.5. The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$. Thus, from (3) and (5)

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_1^{5-x} dz\right] dA \tag{6}$$





For the double integral over R, we will integrate with respect to y first. Thus, (6) yields

$$\text{volume of } G = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-x} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big]_{z=1}^{5-x} dy\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4 - x)\,dy\,dx = \int_{-3}^{3}(8 - 2x)\sqrt{9 - x^2}\,dx$$

$$= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - x)\, dy\, dx = \int_{-3}^{3} (8 - 2x)\sqrt{9 - x^2}\, dx$$

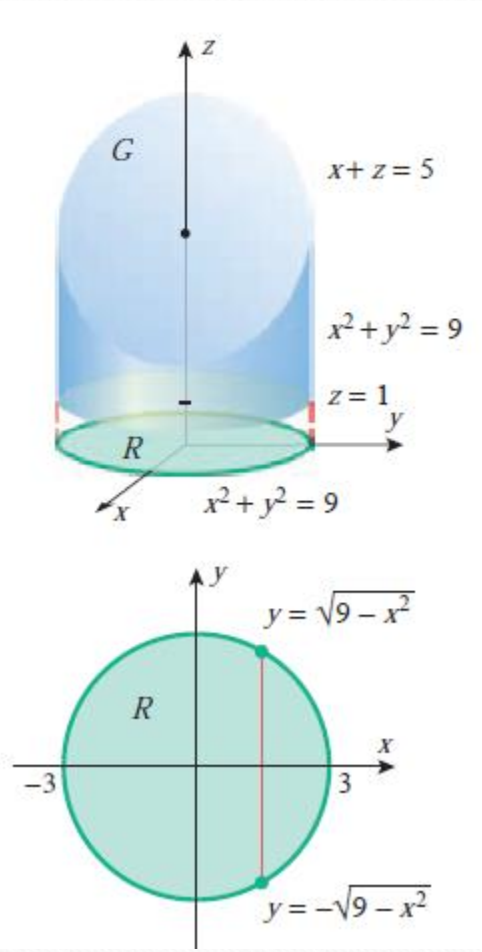
$$= 8\int_{-3}^{3} \sqrt{9 - x^2}\, dx - \int_{-3}^{3} 2x\sqrt{9 - x^2}\, dx$$

For the first integral, see Formula (3) of Section 7.4.

$$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3} 2x\sqrt{9 - x^2}\, dx$$

The second integral is 0 because the integrand is an odd function.

$$= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi$$



▶ **Example 4** Find the volume of the solid enclosed between the paraboloids

$$z = 5x^2 + 5y^2 \quad \text{and} \quad z = 6 - 7x^2 - y^2$$

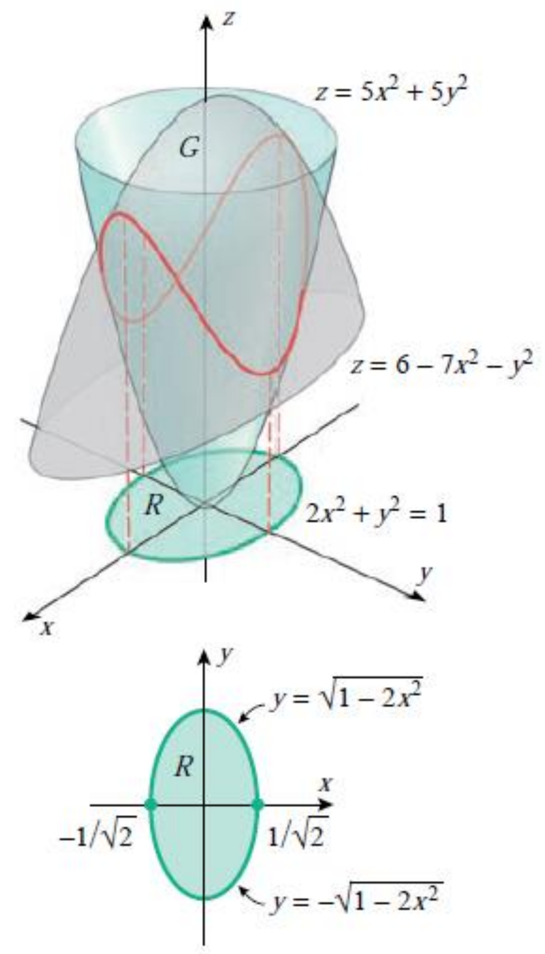
Solution. The solid G and its projection R on the xy-plane are shown in Figure 14.5.6. The projection R is obtained by solving the given equations simultaneously to determine where the paraboloids intersect. We obtain

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

or

$$2x^2 + y^2 = 1 \tag{7}$$

which tells us that the paraboloids intersect in a curve on the elliptic cylinder given by (7).



▲ **Figure 14.5.6**

The projection of this intersection on the xy-plane is an ellipse with this same equation. Therefore,

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\right] dA$$

$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{5x^2+5y^2}^{6-7x^2-y^2} dz\, dy\, dx$$

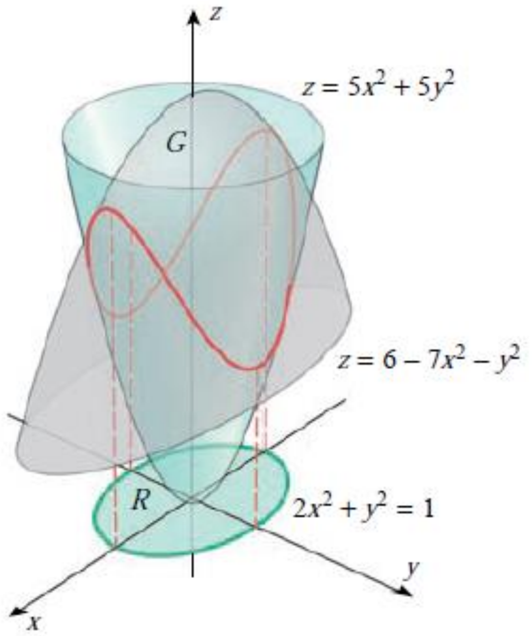
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (6 - 12x^2 - 6y^2)\, dy\, dx$$

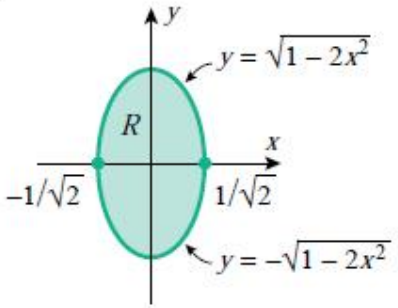
$$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left[6(1-2x^2)y - 2y^3\right]_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= 8\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-2x^2)^{3/2}\, dx = \frac{8}{\sqrt{2}} \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{\sqrt{2}}$$

Let $x = \frac{1}{\sqrt{2}}\sin\theta$.

Use the Wallis cosine formula in Exercise 70 of Section 7.3.





▲ Figure 14.5.6

THANK YOU