MATH 201: Coordinate Geometry and Vector Analysis

Chapter: 15.3

Independence of path; conservative vector fields

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❑ Work integral:

We saw in the last section that if $\mathbf{F}$ is a force field in 2-space or 3-space, then the work performed by the field on a particle moving along a parametric curve C from an initial point P to a final point Q is given by the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{or equivalently} \quad \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

Accordingly, we call an integral of this type a ***work integral***. Recall that a work integral can also be expressed in scalar form as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f(x, y)\, dx + g(x, y)\, dy \qquad \boxed{\text{2-space}} \qquad (1)$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz \qquad \boxed{\text{3-space}} \qquad (2)$$

where f, g, and h are the component functions of $\mathbf{F}$.

$$\int \boldsymbol{F}.d\boldsymbol{r} = \int \big(f(x,y), g(x,y)\big).(dx, dy)$$

❑ Fundamental theorem of calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

F **is an anti derivative of** ***f***

Fundamental theorem of line integrals:

15.3.1 THEOREM ***(The Fundamental Theorem of Line Integrals)*** *Suppose that*

$$\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$$

is a conservative vector field in some open region D containing the points (x_0, y_0) and (x_1, y_1) and that $f(x, y)$ and $g(x, y)$ are continuous in this region. If

$$\mathbf{F}(x, y) = \nabla\phi(x, y)$$

and if C is any piecewise smooth parametric curve that starts at (x_0, y_0), ends at (x_1, y_1), and lies in the region D, then

$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{3}$$

or, equivalently,

$$\int_C \nabla\phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0) \tag{4}$$

Independence of Path:

The value of a line integral of a conservative vector field along a piecewise smooth path is independent of the path; that is, the value of the integral depends on the endpoints and not on the actual path C. Accordingly, for line integrals of conservative vector fields,

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0,y_0)}^{(x_1,y_1)} \nabla\phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

▸ Example 2

(a) Confirm that the force field $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ in Example 1 is conservative by showing that $\mathbf{F}(x, y)$ is the gradient of $\phi(x, y) = xy$.

(b) Use the Fundamental Theorem of Line Integrals to evaluate $\int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r}$.

Solution (a).

$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = y\mathbf{i} + x\mathbf{j}$$

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✓ Use the Fundamental Theorem of Line Integrals to evaluate $\int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r}$.

Solution (a).

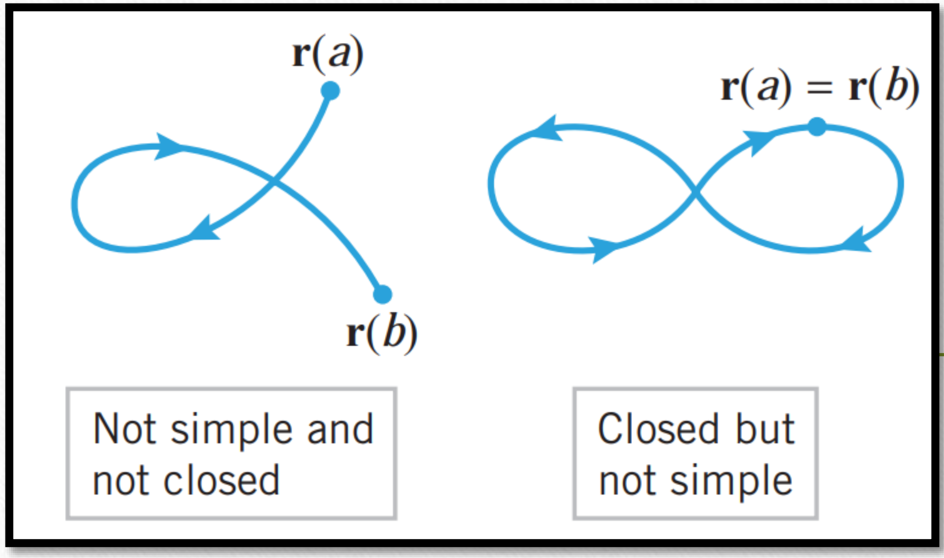
$$\nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = y\mathbf{i} + x\mathbf{j}$$

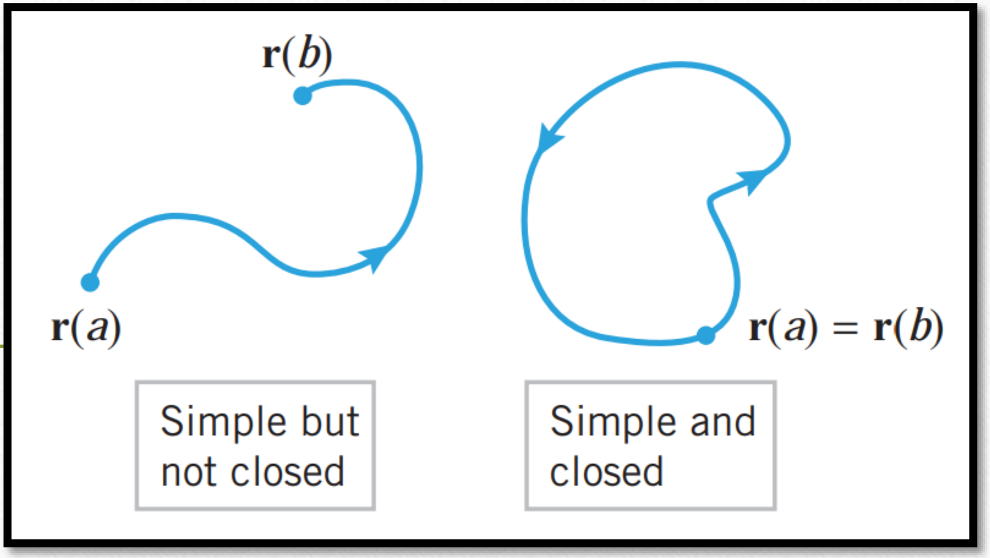
Solution (b). From (5) we obtain

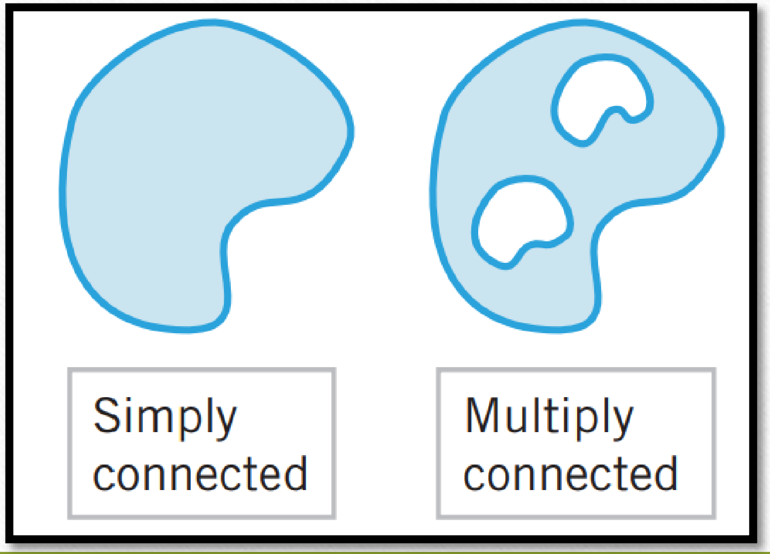
$$\int_{(0,0)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = \phi(1, 1) - \phi(0, 0) = 1 - 0 = 1$$

which agrees with the results obtained in Example 1 by integrating from (0, 0) to (1, 1) along specific paths. ◀

$$\int_{(x_0, y_0)}^{(x_1, y_1)} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0, y_0)}^{(x_1, y_1)} \nabla\phi \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$


r(a)
r(b)
r(a) = r(b)
Not simple and not closed
Closed but not simple


r(b)
r(a)
r(a) = r(b)
Simple but not closed
Simple and closed


Simply connected
Multiply connected

❑ Conservative field test:

15.3.3 THEOREM (*Conservative Field Test*) *If $f(x, y)$ and $g(x, y)$ are continuous and have continuous first partial derivatives on some open region D, and if the vector field $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is conservative on D, then*

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \tag{9}$$

at each point in D. Conversely, if D is simply connected and (9) holds at each point in D, then $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$ is conservative.

Example 3 Use Theorem 15.3.3 to determine whether the vector field

$$\mathbf{F}(x, y) = (y + x)\mathbf{i} + (y - x)\mathbf{j}$$

is conservative on some open set.

Solution. Let $f(x, y) = y + x$ and $g(x, y) = y - x$. Then

$$\frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = -1$$

Thus, there are no points in the xy-plane at which condition (9) holds, and hence $\mathbf{F}$ is not conservative on any open set. ◀

Example 4 Let $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + (1 + 3x^2y^2)\mathbf{j}$.

Show that $\mathbf{F}$ is a conservative vector field on the entire xy-plane.

(b) Find ϕ by first integrating $\partial\phi/\partial x$.

(c) Find ϕ by first integrating $\partial\phi/\partial y$.

Solution (a). Since $f(x, y) = 2xy^3$ and $g(x, y) = 1 + 3x^2y^2$, we have

$$\frac{\partial f}{\partial y} = 6xy^2 = \frac{\partial g}{\partial x}$$

so (9) holds for all (x, y).

▶ **Example 4** Let $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + (1 + 3x^2y^2)\mathbf{j}$.

(a) Show that $\mathbf{F}$ is a conservative vector field on the entire xy-plane.
(b) Find ϕ by first integrating $\partial\phi/\partial x$.
(c) Find ϕ by first integrating $\partial\phi/\partial y$.

Solution (b). Since the field $\mathbf{F}$ is conservative, there is a potential function ϕ such that

$$\frac{\partial\phi}{\partial x} = 2xy^3 \quad \text{and} \quad \frac{\partial\phi}{\partial y} = 1 + 3x^2y^2 \tag{11}$$

Integrating the first of these equations with respect to x (and treating y as a constant) yields

$$\phi = \int 2xy^3\,dx = x^2y^3 + k(y) \tag{12}$$

where $k(y)$ represents the "constant" of integration. We are justified in treating the constant of integration as a function of y, since y is held constant in the integration process. To find $k(y)$ we differentiate (12) with respect to y and use the second equation in (11) to obtain

$$\frac{\partial\phi}{\partial y} = 3x^2y^2 + k'(y) = 1 + 3x^2y^2$$

$$f(x,y) = \frac{\partial\emptyset}{\partial x}; g(x,y) = \frac{\partial\emptyset}{\partial y}$$

▶ Example 4 Let $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + (1 + 3x^2y^2)\mathbf{j}$.

(a) Show that $\mathbf{F}$ is a conservative vector field on the entire xy-plane.

(b) Find ϕ by first integrating $\partial\phi/\partial x$.

(c) Find ϕ by first integrating $\partial\phi/\partial y$.

$$\frac{\partial \phi}{\partial y} = 3x^2y^2 + k'(y) = 1 + 3x^2y^2$$

from which it follows that $k'(y) = 1$. Thus,

$$k(y) = \int k'(y)\,dy = \int 1\,dy = y + K$$

where K is a (numerical) constant of integration. Substituting in (12) we obtain

$$\phi = x^2y^3 + y + K$$

The appearance of the arbitrary constant K tells us that ϕ is not unique. As a check on the computations, you may want to verify that $\nabla\phi = \mathbf{F}$.

$$\frac{dk}{dy} = 1$$

Or, $\int dk = \int dy$

Or, $k = y + K$

Acti

▶ **Example 4** Let $\mathbf{F}(x, y) = 2xy^3\mathbf{i} + (1 + 3x^2y^2)\mathbf{j}$.

(a) Show that $\mathbf{F}$ is a conservative vector field on the entire xy-plane.

(b) Find ϕ by first integrating $\partial\phi/\partial x$.

✓ Find ϕ by first integrating $\partial\phi/\partial y$.

Solution (c). Integrating the second equation in (11) with respect to y (and treating x as a constant) yields

$$\phi = \int (1 + 3x^2y^2)\,dy = y + x^2y^3 + k(x) \tag{13}$$

where $k(x)$ is the "constant" of integration. Differentiating (13) with respect to x and using the first equation in (11) yields

$$\frac{\partial\phi}{\partial x} = 2xy^3 + k'(x) = 2xy^3$$

from which it follows that $k'(x) = 0$ and consequently that $k(x) = K$, where K is a numerical constant of integration. Substituting this in (13) yields

$$\phi = y + x^2y^3 + K$$

which agrees with the solution in part (b). ◀

Example 5 Use the potential function obtained in Example 4 to evaluate the integral

$$\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^2)\,dy$$

Solution. The integrand can be expressed as $\mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}$ is the vector field in Example 4. Thus, using Formula (3) and the potential function $\phi = y + x^2y^3 + K$ for $\mathbf{F}$, we obtain

$$\begin{aligned}\int_{(1,4)}^{(3,1)} 2xy^3\,dx + (1 + 3x^2y^2)\,dy &= \int_{(1,4)}^{(3,1)} \mathbf{F} \cdot d\mathbf{r} = \phi(3,1) - \phi(1,4)\\ &= (10 + K) - (68 + K) = -58\end{aligned}$$

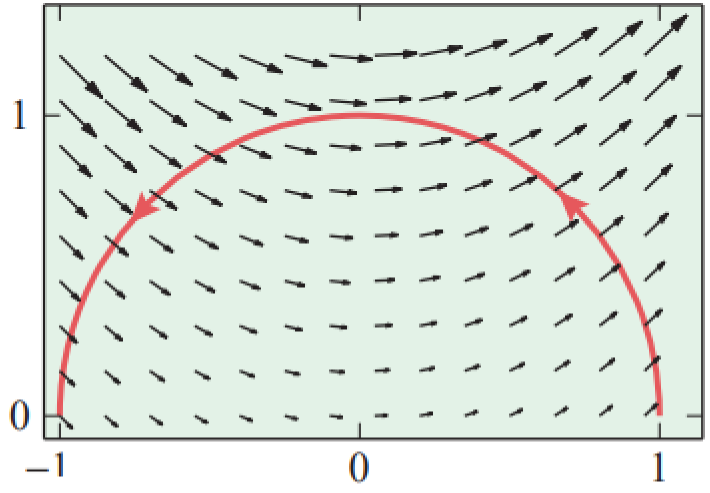
$$\int_C \mathbf{F}(x, y) \cdot d\mathbf{r} = \phi(x_1, y_1) - \phi(x_0, y_0)$$

Example 6 Let $\mathbf{F}(x, y) = e^y\mathbf{i} + xe^y\mathbf{j}$ denote a force field in the xy-plane.

(a) Verify that the force field $\mathbf{F}$ is conservative on the entire xy-plane.

(b) Find the work done by the field on a particle that moves from $(1, 0)$ to $(-1, 0)$ along the semicircular path C shown in Figure 15.3.8.

Vectors not to scale



Solution (a). For the given field we have $f(x, y) = e^y$ and $g(x, y) = xe^y$. Thus,

$$\frac{\partial}{\partial y}(e^y) = e^y = \frac{\partial}{\partial x}(xe^y)$$

so (9) holds for all (x, y) and hence $\mathbf{F}$ is conservative on the entire xy-plane.

▶ **Example 6** Let $\mathbf{F}(x, y) = e^y\mathbf{i} + xe^y\mathbf{j}$ denote a force field in the xy-plane.

(a) Verify that the force field $\mathbf{F}$ is conservative on the entire xy-plane.

✓ Find the work done by the field on a particle that moves from $(1, 0)$ to $(-1, 0)$ along the semicircular path C shown in Figure 15.3.8.

Solution (b). From Formula (34) of Section 15.2, the work done by the field is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C e^y\,dx + xe^y\,dy \tag{14}$$

However, the calculations involved in integrating along C are tedious, so it is preferable to apply Theorem 15.3.1, taking advantage of the fact that the field is conservative and the integral is independent of path. Thus, we write (14) as

$$W = \int_{(1,0)}^{(-1,0)} e^y\,dx + xe^y\,dy = \phi(-1, 0) - \phi(1, 0) \tag{15}$$

As illustrated in Example 4, we can find ϕ by integrating either of the equations

$$\frac{\partial \phi}{\partial x} = e^y \quad \text{and} \quad \frac{\partial \phi}{\partial y} = xe^y \tag{16}$$

$$dr = dx\,\boldsymbol{i} + dy\,\boldsymbol{j}$$

▸ **Example 6** Let $\mathbf{F}(x, y) = e^y\mathbf{i} + xe^y\mathbf{j}$ denote a force field in the xy-plane.

(a) Verify that the force field $\mathbf{F}$ is conservative on the entire xy-plane.

✓ Find the work done by the field on a particle that moves from $(1, 0)$ to $(-1, 0)$ along the semicircular path C shown in Figure 15.3.8.

We will integrate the first. We obtain

$$\phi = \int e^y\,dx = xe^y + k(y) \tag{17}$$

Differentiating this equation with respect to y and using the second equation in (16) yields

$$\frac{\partial \phi}{\partial y} = xe^y + k'(y) = xe^y$$

from which it follows that $k'(y) = 0$ or $k(y) = K$. Thus, from (17)

$$\phi = xe^y + K$$

and hence from (15)

$$W = \phi(-1, 0) - \phi(1, 0) = (-1)e^0 - 1e^0 = -2 \blacktriangleleft$$

Exercise

Do it yourself

THANK YOU