SIGNAL PROCESSING

LAB PROJECT

DISCRETE FRACTIONAL FOURIER TRANSFORM

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In mathematics, the fractional Fourier transform (FRFT) is a family of linear transformations generalizing the Fourier transform. It can be thought of as the Fourier transform to the n^{th} power, where n need not be an integer — thus, it can transform a function to any intermediate domain between time and frequency. Its applications range from filter design and signal analysis to phase retrieval and pattern recognition.

Fractional Fourier transform has attracted a considerable amount of attention, resulting in many applications. There are 2 kinds of fractional Fourier transform, namely continuous and discrete. The continuous FRFT has had a consistent definition along the lines of continuous Fourier transform. But as time has passed a satisfactory definition of discrete FRFT has been lacking.

The discrete FRFT has been given along theses given lines

- Unitarity
- Index additivity
- Reduction to the DFT when the order is equal to unity
- Approximation of the continuous FRFT

The first two points are essential properties of the continuous Fourier transform, which have to be satisfied exactly by the discrete FRFT. The third is necessary for the discrete fractional Fourier transform to be a consistent generalization of the ordinary discrete Fourier transform. The last point is to get accurate results as that of continuous FRFT.

The first 3 requirements stated above are automatically satisfied when the fractional transform is defined through a spectral expansion analogous to the integral kernel of a continuous Fourier transform.

$$\boxed{\mathbf{F}^a[m,n] = \sum_{k=0}^{N-1} p_k[m](\lambda_k)^a p_k[n]}$$

But there are 2 ambiguities to be resolved in the above equation. Firstly, the eigen structure- the eigenvalues are in general degenerate so the set is not unique. Secondly, it arises in taking the fractional power of the eigenvalues since the fractional power operation is not single valued. Hence, there is a need for a discrete Hermite Gaussian function.

The Discrete Hermite-Gaussians is defined as the solution of a difference equation that is analogous to the defining differential of the continuous Hermite–Gaussian functions. The given equation is the second-order difference equation analogous to the defining differential equation of Hermite–Gaussians. Here, f(n) is the function/signal.

$$\boxed{\begin{aligned} &f[n+1]-2f[n]+f[n-1] \\ &\quad +2\left(\cos\left(\frac{2\pi}{N}n\right)-1\right)f[n]=\lambda f[n] \end{aligned}}$$

The unique orthogonal eigenvector set, which we call u_k, will be taken as the discrete counterpart of the continuous Hermite–Gaussians. The common eigenvector set S and the DFT matrix, which is known to exist since they commute, should also consist of even or odd vectors. Therefore, the problem of finding the common eigenvector set is reduced to finding the eigenvectors of the **Ev** and **Od** matrices, given this tridiagonal matrix

$$\boxed{\mathbf{PSP}^{-1}=\mathbf{PSP}=\begin{bmatrix}\mathbf{Ev} & \mathbf{0} \\ \mathbf{0} & \mathbf{Od}\end{bmatrix}}$$

All eigenvectors have a distinct number of zero crossings and need to establish a convenient method for counting the zero crossings. The exhaustive counting of the zero crossings can be numerically problematic due to the difficulty of determining the sign of a component which is of small magnitude. In conclusion, there is a well-defined procedure for finding and ordering the common eigenvector set of the matrix **S** and the DFT matrix, such that the *n*th member of this eigenvector set has *n* zero crossings and is even or odd depending whether *n* is even or odd.

$$\boxed{\mathbf{e}=\frac{1}{\sqrt{2}}\left[\sqrt{2}\hat{e}_k[0]\hat{e}_k[1]\ldots\hat{e}_k[r]\vdots\hat{e}_k[r]\ldots\hat{e}_k[1]\right]}$$

$$\boxed{\begin{aligned} &[\mathbf{e} \quad e[0]] \\ &\quad=\frac{1}{\sqrt{2}}[\underbrace{\sqrt{2}\hat{e}_k[0]\hat{e}_k(1)\ldots\hat{e}_k[r]}_{k\text{ zero crossings}}\vdots\underbrace{\hat{e}_k[r]\ldots\hat{e}_k[1]\sqrt{2}\hat{e}_k[0]}_{k\text{ zero crossings}}] \end{aligned}}$$

Considering the above conditions of the eigenvector and Hermite-Gaussian functions the definition of discrete FRFT can be given as

$$\boxed{\mathbf{F}^a[m,n]=\sum_{k=0,k\neq(N-1+(N)_2)}^{N} u_k[m]e^{-j\frac{\pi}{2}ka}u_k[n]}$$

0	$f[n]$	$\overset{a}{\longleftrightarrow}$	$f_a[n]$
1	$f[n]+g[n]$	$\overset{a}{\longleftrightarrow}$	$f_a[n]+g_a[n]$
2	$f_a[n]$	$\overset{b}{\longleftrightarrow}$	$f_{a+b}[n]$
3	$f[n]$	$\overset{a=1}{\longleftrightarrow}$	$DFT\{f[n]\}$
4	$f[-n]$	$\overset{a}{\longleftrightarrow}$	$f_a[-n]$
5	$f^*[n]$	$\overset{a}{\longleftrightarrow}$	$f^*_{-a}[n]$
6	$Even\{f[n]\}$	$\overset{a}{\longleftrightarrow}$	$Even\{f_a[n]\}$
7	$Odd\{f[n]\}$	$\overset{a}{\longleftrightarrow}$	$Odd\{f_a[n]\}$
8	$\sum_{n=0}^{N-1}\lvert f[n]\rvert^2$	$=$	$\sum_{n=0}^{N-1}\lvert f_a[n]\rvert^2$

Some properties of discrete FRFT

[REFERENCES]

- ‘Fractional Fourier Transform’ - Wikipedia
- ‘The Discrete Fractional Fourier Transform’ https://ieeexplore.ieee.org/abstract/document/839980
- ‘Discrete Gauss-Hermite Functions and Eigenvectors of the Centered Discrete Fourier Transform’ - https://ieeexplore.ieee.org/document/4217977