The Discrete Fractional Fourier Transform

Signal Processing Project (Final Report)

Group-33(C_K)

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Overview

This project tries to provide a definition and determine the discrete fractional Fourier transform of a given signal by a method named eigen decomposition.

Goal

1. Consolidate a definition of the discrete fractional Fourier transform (DFrFT) that generalizes the discrete Fourier transform (DFT) just along the lines as continuous fractional Fourier transform generalizes the continuous ordinary Fourier transform.

Introduction

In mathematics, the fractional Fourier transform (FrFT) is a family of linear transformations generalizing the Fourier transform. It can be thought of as the Fourier transform to the n^{th} power, where *n* need not be an integer — thus, it can transform a function to any intermediate domain between time and frequency. Its applications range from filter design and signal analysis to phase retrieval and pattern recognition.

Problem Description

Fractional Fourier transform has attracted a considerable amount of attention, resulting in many applications. There are 2 kinds of fractional Fourier transform, namely continuous and discrete. The continuous FrFT has had a consistent definition along the lines of continuous ordinary Fourier transform. But as time has passed a satisfactory definition of discrete FrFT has been lacking.

Background Theory

The interpretation of the FrFT is a rotation of signals in the time–frequency plane.

Continuous Fractional Fourier Transform

The a-th order continuous FRT is defined as

$$\boxed{\{\mathcal{F}^a f\}(t_a) = \int_{-\infty}^{\infty} K_a(t_a, t) f(t)\, dt}$$

The kernel is $\boxed{K_a(t_a, t) = K_\phi e^{j\pi(t_a^2 \cot\phi - 2t_a t \csc\phi + t^2 \cot\phi)}}$

The above formula of the kernel is said to have the following spectral expansion

$\boxed{K_a(t_a, t) = \sum_{k=0}^{\infty} \psi_k(t_a) e^{-j\frac{\pi}{2}ka} \psi_k(t)}$ which has been defined with respect to the kth Hermite-Gaussian function and the fractional power of the eigenvalue.

A few properties of the FrFT are

1. Unitarity $(\mathcal{F}^a)^{-1} = \mathcal{F}^{-a} = (\mathcal{F}^a)^\dagger$
2. Index additivity $\mathcal{F}^{a_1}\mathcal{F}^{a_2} = \mathcal{F}^{a_2}\mathcal{F}^{a_1} = \mathcal{F}^{a_1+a_2}$
3. Reduction to the ordinary Fourier transform when a=1

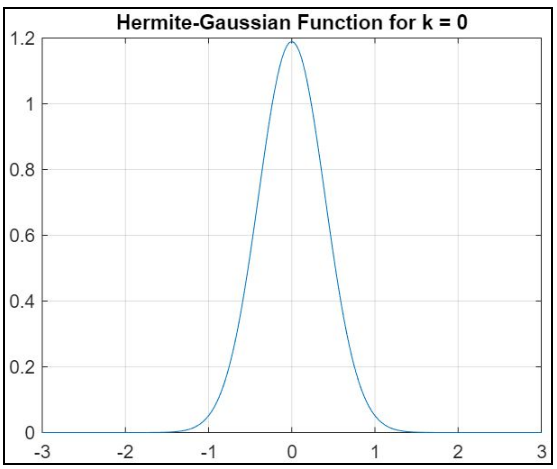
The discrete fractional Fourier transform will also be derived along these three properties.

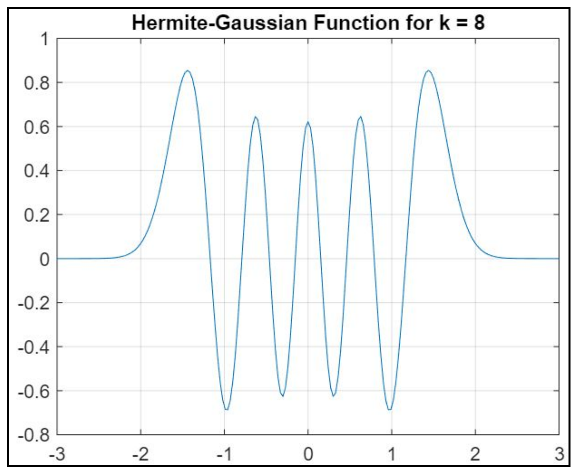
The Hermite-Gaussian Functions

The *k*-th order Hermite-Gaussian is defined as (k = 0, 1, 2, …)

$$\boxed{\psi_k(t) = \frac{2^{1/4}}{\sqrt{2^k k!}} H_k(\sqrt{2\pi}t)e^{-\pi t^2}}$$

where H_k is the *k*-th Hermite polynomial having *k* real zeroes. They form a complete and orthonormal set in L_2 (the set of square integrable L_2-functions is an L^2-space) The Hermite-Gaussian functions are well known to be the eigenfunctions of the Fourier transform operator.





The code 'Hermite_Gaussian.m' gives the output for the above, for different inputs of 'k'.

Theorem 1: If two operators *A* and *B* commute i.e., *AB* = *BA*, there exists a common eigenvector set between *A* and *B*.

Using the above theorem and to tackle our problem we will define an operator

$$(\mathcal{D}^2 + \mathcal{F}\mathcal{D}^2\,\mathcal{F}^{-1})f(t) = \mathcal{S}f(t) = \lambda f(t).$$

where *D* and *F* denote the differentiation and the ordinary Fourier transform operations.

We can prove that the above defined operator *S* is commutable with *F*. Hence, the Hermite-Gaussian functions, which are unique finite energy eigenfunctions of *S*, are also eigenfunctions of *F*.

Solution Approach

The best method to determine the discrete fractional Fourier transform can be by deriving along the similar line as that of continuous fractional Fourier transform.
By using the spectral expansion formula, we can satisfy the three requirements.

$$\boxed{\mathbf{F}^a[m,n] = \sum_{k=0}^{N-1} p_k[m](\lambda_k)^a p_k[n]}$$

Here, $p_k[n]$ is an arbitrary orthonormal eigenvector set and $(\lambda_k)^a$ are the associated eigenvalues.

Computing Approximate Discrete Hermite-Gaussian Functions

This step is an approximation process for determining the matrix. The discrete Gauss-Hermite function defined has to be approximately equal to the continuous Hermite-Gaussian function. The formula

$$\boxed{\hat{H} = \sum_{p=1}^{m} (-1)^{p-1} \frac{[(p-1)!]^2}{(2p)!} (\hat{C}_p + \hat{D}_p)}$$

This formula has to be computed for an approximate value of *m*, which by default we have kept as N/2.

Theorem 2: The matrix H and the DFT matrix F commute.

Using the above theorem, we dig deeper to this final formula above, the discrete equivalent of Hermite-Gaussian is defined as $H = \pi (U^2 + D^2)$, where D is the differentiation operator [$(Df)(x) = (i2\pi)^{-1}\, df(x)/dx$] and U is the shift operator [$(Uf)(x) = xf(x)$, also defined as $U = FDF^{-1}$].

Hence, we also provide an approximation to the defined operators.

$$\boxed{\mathcal{D}^2 \approx \sum_{p=1}^{m} (-1)^{p-1} \frac{[(p-1)!]^2}{(2p)!} (\delta^2)^p} \quad \boxed{\mathcal{U}^2 \approx \sum_{p=1}^{m} (-1)^{p-1} \frac{[(p-1)!]^2}{(2p)!} F(\delta^2)^p F^{-1}}$$

$(\delta^2)^p = d_p(S)$, where d_p is the trigonometric polynomial $d_p(x) = (x - 2 + x^{-1})^p$. The coefficients of d^p can be computed by p successive convolutions of the vector [1 −2 1]. This $d_p(S)$ is represented by a (symmetric) circulant matrix C_p in the sense that the diagonal elements of C_p correspond to the constant term c_0 in the polynomial $d_p(x)$.

$$\boxed{H = \sum_{p=1}^{m} (-1)^{p-1} \frac{[(p-1)!]^2}{(2p)!} (\hat{C}_p + \hat{D}_p + c_0 I_N)}$$

where $\hat{C}_p$ is the circulant matrix C_p whose diagonal is removed (and written separately as $c_0 I_N$) and $\hat{D}_p$ is the diagonal matrix whose elements are given by *Re(FFT(d_p)).* Since we are interested in the eigenvectors of *H*, and the constant diagonal $c_0 I_N$ will influence the eigenvalues, but not the eigenvectors, it can be removed for the computations.

Compute the Transformation Matrix V

Theorem 3: Eigenvectors of the DFT matrix are either even or odd sequences.

The *H* matrix was computed in the previous step and by defining the transformation matrix *V*, we will have a result as follows

$$\boxed{V H V^{\mathrm{T}} = \begin{bmatrix} Ev & \\ & Od \end{bmatrix}}$$

By appropriate derivations, we get

$$\boxed{V = \frac{1}{\sqrt{2}} \left[\begin{array}{c|cc} \sqrt{2} & & \\ \hline & I_r & J_r \\ & J_r & -I_r \end{array} \right], \quad r = (N-1)/2 \text{ if } N \text{ is odd}} \quad \boxed{V = \frac{1}{\sqrt{2}} \left[\begin{array}{c|ccc} \sqrt{2} & & & \\ \hline & I_r & & J_r \\ & & 1 & \\ & J_r & & -I_r \end{array} \right], \quad r = (N-2)/2 \text{ if } N \text{ is even.}}$$

I_r is the r×r unit matrix (1's on the main diagonal) and J_r is the r×r antiunit matrix (with 1's on the main antidiagonal).

Compute blocks Ev and Od from Previous Step

$$\boxed{V H V^{\mathrm{T}} = \begin{bmatrix} Ev & \\ & Od \end{bmatrix}}$$

We have to compute the eigenvalue decompositions of the symmetric matrices *Ev* and *Od*.

$$\boxed{Ev = V_e \Lambda_e V_e^{\mathrm{T}} \quad \text{and} \quad Od = V_o \Lambda_o V_o^{\mathrm{T}}}$$

Transforming *Ev* and *Od* with the matrix *V* to get the eigenvectors in *E*

$$\boxed{E H E^{\mathrm{T}} = \left[\begin{array}{c|c} \Lambda_e & \\ \hline & \Lambda_o \end{array} \right], \quad E = V \left[\begin{array}{c|c} V_e & \\ \hline & V_o \end{array} \right]}$$

The columns of *E* are the eigenvectors, which have the desired symmetry properties by construction. The first ones are the evens, the trailing ones are the odds. It remains to interlace them appropriately.

Ordering the Eigenvectors of *E* by Zero Crossing Method

The previous step will return the eigenvalues and eigenvectors in the reverse order. Hence by flipping the matrices will give the required correct order. Then we form an interlacing eigenvector to correspond to the ordering of the eigenvalues $(-i)^n$.

Compute The Fractional Fourier Transform

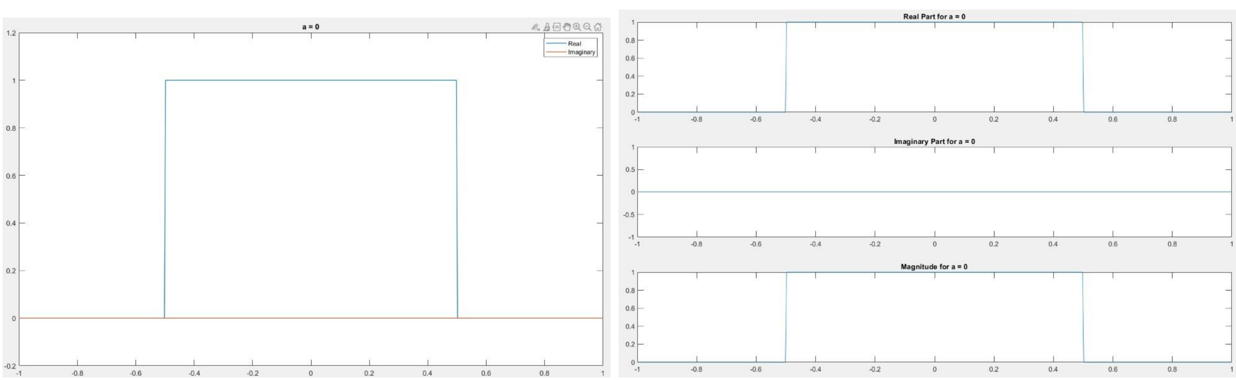
$$\boxed{F^{a} = E\,\Lambda^{a}\,E^{\mathrm{T}}}$$

To compute the DFrFT as $f_a = F^a f = E\,\Lambda_a E^T f$, then the multiplication of the N×N matrix F^a with the vector f requires $O(N^2)$ operations. However, computing $F^a = E\,\Lambda_a E^T$, given E and Λ_a requires $O(N^3)$ operations. Thus, if only one DFrFT has to be computed, it is more efficient to compute $f_a = E\,(\Lambda_a (E^T f))$. Multiplication with E^T requires $O(N^2)$ operations, and multiplication with Λ_a takes another $O(N)$ and, finally, the multiplication with E is again $O(N^2)$, which is cheaper than first evaluating F_a.

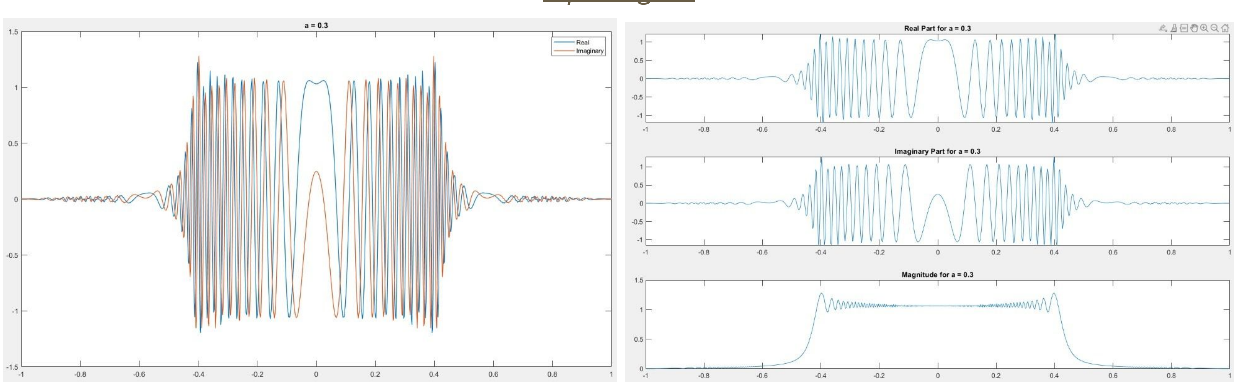
Output

2 video files have been uploaded to show the outputs of the code. Video 'DFrFT' shows the real and imaginary values in the same figure. Video 'DFrFT_Value' shows the real, imaginary and magnitude in different graphs. The input is a rectangular pulse, with samples at 0.002. The videos show the graphs for values of 'a' ranging from 0 to 4, with intervals of 0.1.

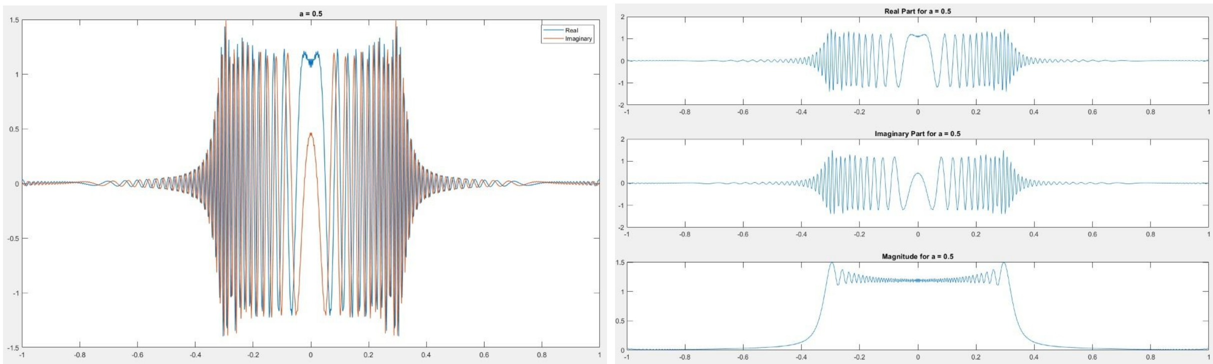
Some instances of the video are shown below.



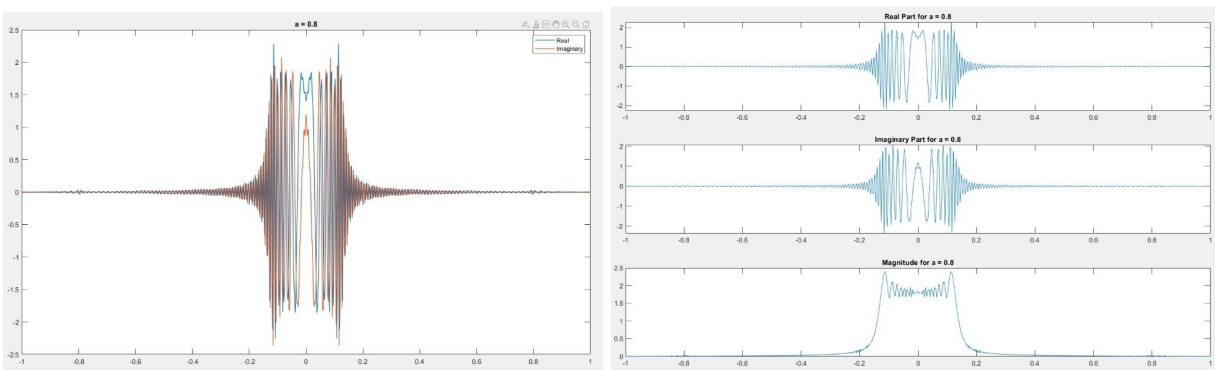
Input signal



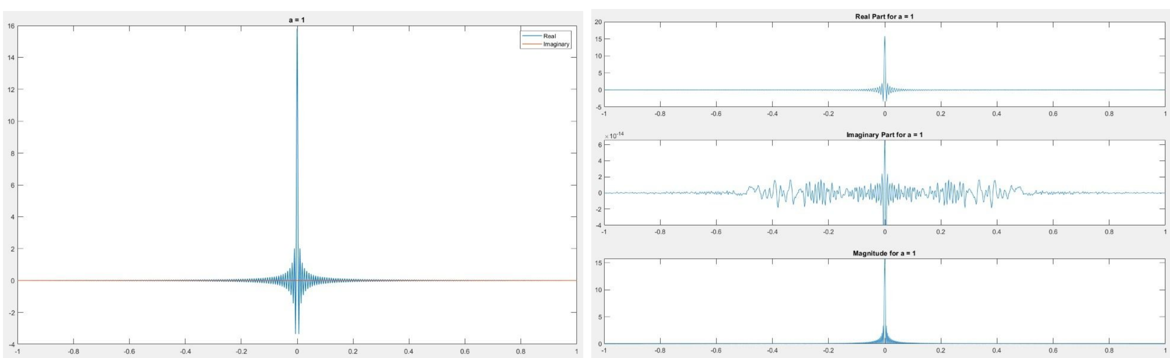
a = 0.3



a = 0.5



a = 0.8



a = 1

The video 'DFrFT_Simple' consists of the same input but defined from -5 to 5 sampled at 0.2. It is just to show the FT of the rectangular pulse is a sinc wave.

The link for the videos:
https://iiitaphyd-my.sharepoint.com/:f:/g/personal/jayant_reddy_students_iiit_ac_in/Ekm8DZGVxWlHs7MaekMyMkAB8x-L-GrJ2biu4aLzZdi8Cg?e=eKcZc9

Conclusions

1. A definition of the discrete FrFT was presented that exactly satisfies the essential operational properties as that of the continuous fractional Fourier transform. This method of determining DFrFT is by eigen decomposition.
2. As a side product, we obtained the discrete counterparts of the Hermite–Gaussian functions during the computation.
3. We notice that periodicity of the outputs with respect to the parameter a. When a=1, the output is the Fourier transform applied on the signal; a=2 results in the inverse signal; a=3 results in the inverse Fourier transform of the signal; and a=4 results back to the input signal.

References

1. 'Fractional Fourier Transform' - Wikipedia
2. 'The Discrete Fractional Fourier Transform' https://ieeexplore.ieee.org/abstract/document/839980
3. 'Computation of the Fractional Fourier Transform' - Adhemar Bultheel and Héctor E. Martínez Sulbaran
4. 'Discrete Gauss-Hermite Functions and Eigenvectors of the Centered Discrete Fourier Transform' - https://ieeexplore.ieee.org/document/4217977
5. MATLAB Documentation - https://in.mathworks.com/help/matlab/
6. 'Signals And Systems' by Alan V. Oppenheim, Alan S. Willsky with S. Hamid
7. 'Digital Signal Processing-Principles, Algorithms and Applications' by John. G. Proakis and Dimitris Manolakis