Equations

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1 磁化した降着円盤の軸対称・定常解

1.1 基礎方程式

軸対称・定常磁気流体方程式 (Oda et al. 2009)

$$\frac{\partial}{r\partial r}\left(r\rho v_r\right)+\frac{\partial}{\partial z}=0, \tag{1}$$

$$\rho v_r\frac{\partial v_r}{\partial r}+\rho v_z\frac{\partial v_r}{\partial z}-\frac{\rho v_\varphi^2}{r}=-\rho\frac{\psi}{\partial r}-\frac{\partial p_{\mathrm{tot}}}{\partial r}-\frac{\langle B_\varphi^2\rangle}{4\pi r^2}, \tag{2}$$

$$\rho v_r\frac{\partial v_\varphi}{\partial r}+\rho v_z\frac{\partial v_\varphi}{\partial z}+\frac{\rho v_r v_\varphi}{r}=\frac{\partial}{r^2\partial r}\left(r^2\frac{\langle B_r B_\varphi\rangle}{4\pi}\right)+\frac{\partial}{\partial z}\left(\frac{\langle B_\varphi B_z\rangle}{4\pi}\right), \tag{3}$$

$$0=-\frac{\partial\psi}{\partial z}-\frac{1}{\rho}\frac{\partial p_{\mathrm{tot}}}{\partial z}, \tag{4}$$

$$\frac{\partial}{\partial r}\left[\left(\rho\epsilon+p_{\mathrm{gas}}+p_{\mathrm{rad}}\right)v_r\right]+\frac{v_r}{r}\left(\rho\epsilon+p_{\mathrm{g}}+p_{\mathrm{r}}\right)+\frac{\partial}{\partial z}\left[\left(\rho\epsilon+p_{\mathrm{gas}}+p_{\mathrm{rad}}v_z\right)\right]-v_r\frac{\partial}{\partial r}(p_{\mathrm{gas}}+p_{\mathrm{rad}})-v_z\frac{\partial}{\partial z}(p_{\mathrm{gas}}+p_{\mathrm{rad}})=q^+ \tag{5}$$

$$0=-\frac{\partial}{\partial z}\left[v_z\langle B_z\rangle\right]-\frac{\partial}{\partial r}\left[v_r\langle B_\varphi\rangle\right]+\left[\nabla\times\langle\delta v\times\delta B\rangle\right]_\varphi-\left[\eta\nabla\times\left(\nabla\times\langle B\rangle\right)\right]. \tag{6}$$

ここで、$\psi=-GM_{\mathrm{BH}}/r^3, \rho\epsilon=p_{\mathrm{gas}}/(\gamma-1)+3p_{\mathrm{rad}}, q^+$, and q^- はそれぞれ、重力ポテンシャル、加熱率、冷却率。

また、鉛直方向にはポリトロピック関係 $p_{\mathrm{tot}}=K\rho^{(1+1/N)}$ を仮定。ここで、

$$\rho(r,z)=\rho_0(r)\exp\left(1-\frac{z^2}{H^2}\right)^N, \tag{7}$$

$$p_{\mathrm{tot}}(r,z)=p_{\mathrm{tot},0}(r)\exp\left(1-\frac{z^2}{H^2}\right)^{N+1}, \tag{8}$$

$$T(r,z)=T_0(r)\exp\left(1-\frac{z^2}{H^2}\right). \tag{9}$$

添字の 0 は赤道面での値、また $H^2 = \dfrac{2(N+1)}{\Omega_{\mathrm{K0}}^2}\dfrac{p_{\mathrm{tot},0}}{\rho_0}$ は円盤の半厚み、$\Omega_{\mathrm{K0}} = \sqrt{\dfrac{GM_{\mathrm{BH}}}{r^3}}$ はケプラー角速度。以上を鉛直方向に積分すると面密度、鉛直積分した圧力は

$$\Sigma = \int_{-H}^{+H} \rho dz = 2\rho_0 I_{\mathrm{N}} H, \tag{10}$$

$$W_{\mathrm{tot}} = \int_{-H}^{+H} p_{\mathrm{tot}} dz = 2p_{\mathrm{tot},0} I_{\mathrm{N+1}} H. \tag{11}$$

ここで、$I_{\mathrm{N}} = (2^N N!)^2/(2N+1)!$。また、円盤半厚みは

$$H^2 = \frac{2N+3)}{\Omega_{\mathrm{K0}}^2}\frac{W_{\mathrm{tot}}}{\Sigma}. \tag{12}$$

と書き換えられる。

1.2　鉛直積分した基礎方程式

$$\dot{M} = -2\pi r \Sigma v_r = const., \tag{13}$$

$$\dot{M}(l - l_{\mathrm{in}}) = -2\pi r^2 \int_{-H}^{+H} \frac{\langle B_r B_\varphi \rangle}{4\pi} dz, \tag{14}$$

$$\frac{\dot{M}}{2\pi r^2}\frac{W_{\mathrm{rad}} + W_{\mathrm{gas}}}{\Sigma}\xi = Q^+ - Q^-, \tag{15}$$

$$\dot{\Phi} = \int_{-H}^{+H} v_r \langle B_\varphi \rangle dz = \int_r^{r_{\mathrm{out}}} \int_{-H}^{+H} \left[\{\nabla \times \langle \delta v \times \delta B \rangle\}_\varphi - \{\eta \nabla \times (\nabla \times \langle B \rangle)\} \right] dr dz + const. \tag{16}$$

磁場は

$$\Phi = B_\varphi H = \Phi_0 (\Sigma/\Sigma_0)^\zeta. \tag{17}$$

で定義。

1.3　加熱・冷却率

粘性加熱率：

$$Q^+ = \int_{-H}^{+H} \frac{\langle B_r B_\varphi \rangle}{4\pi} r \frac{d\Omega}{dr} dz = -\alpha W_{\mathrm{tot}} r \frac{d\Omega}{dr}. \tag{18}$$

移流冷却率：

$$Q^-_{\mathrm{adv}} = \frac{\dot{M}}{2\pi r^2}\frac{W_{\mathrm{rad}} + W_{\mathrm{gas}}}{\Sigma}\xi \tag{19}$$

光学的に薄い場合の輻射冷却率（制動放射）：

$$Q^-_{\rm thin} = 6.2 \times 10^{20} \frac{I_{2N+1/2}}{2I_N^2} \frac{\Sigma^2}{H} T_0^{1/2}, \tag{20}$$

光学的に厚い場合の輻射冷却率：

$$Q^-_{\rm thick} = \frac{16\sigma I_N T_0^4}{3\tau/2}, \tag{21}$$

ここで、$\tau = \tau_{\rm es} + \tau_{\rm abs}$, $\tau_{\rm es} = 0.5\kappa_{\rm es}\Sigma$。また、吸収に対する光学的厚みは

$$\tau_{\rm abs} = \frac{8I_{\rm N}^2}{3I_{\rm N+4}\tau} \frac{Q^+_{\rm thin}}{Q^-_{\rm thick}}. \tag{22}$$

輻射冷却は、光学的に薄い場合と厚い場合に使える近似解 (e.g., Hubney 1990, Abramowicz et al. 1995)

$$Q^- = \frac{16\sigma I_N T_0^4}{3\tau/2 + \sqrt{3} + \tau_{\rm abs}^{-1}}, \tag{23}$$

を用いる。

コンプトン散乱による冷却率：

$$Q^-_{\rm Comp} = Q^- \kappa_{\rm es} \Sigma \frac{4k_{\rm b}}{m_{\rm e}c^2} \left(\frac{I_{\rm N+1}}{I_{\rm N}} T_{\rm e0} - T_{\rm r} \right). \tag{24}$$

ここで、$T_{\rm e0} = \min(T_0, 10^9\ {\rm K})$ は電子温度、$T_{\rm r} = (Q^-/(a_{\rm r}c))^{1/4}$ は輻射温度。

1.4 解くべき方程式

$f_1 = Q^+ - Q^- - Q^-_{\rm Comp} - Q^-_{\rm adv}$, $f_2 = W_{\rm tot} - W_{\rm gas} - W_{\rm rad} - W_{\rm mag}$ を連立し、ニュートン法で解く。

$$f_1 = \frac{3}{2}\alpha W_{\rm t}\tilde{\Omega} - \frac{16\sigma I_3 T_0^4}{3\tau/2 + \sqrt{3} + \tau_{\rm abs}^{-1}} \frac{r_{\rm s}}{c} - Q^- \kappa_{\rm es}\Sigma \frac{4k_{\rm b}}{m_{\rm e}c^2} \left(\frac{I_{\rm N+1}}{I_{\rm N}} T_0 - T_{\rm r} \right) \frac{r_{\rm s}}{c} - \frac{\dot{m}}{\tilde{r}^2} \frac{W_{\rm tot} - W_{\rm mag}}{\kappa_{\rm es}\Sigma} \xi, \tag{25}$$

$$f_2 = \frac{\dot{m}(\tilde{l} - \tilde{l}_{\rm in})}{\tilde{r}^2 \alpha} \frac{c^2}{\kappa_{\rm es}} - \frac{\Phi_0^2}{8\pi \tilde{H} r_{\rm s}} \left(\frac{\Sigma}{\Sigma_0} \right)^{2\zeta} - \frac{I_4}{I_3} \frac{R}{\mu} \Sigma T_0 - \frac{1}{4c} \frac{16\sigma I_{\rm N} T_0^4}{3\tau/2 + \sqrt{3} + \tau_{\rm abs}^{-1}} \frac{I_4}{I_3} \tilde{H} r_{\rm s} \left(\tau + \frac{2}{\sqrt{3}} \right). \tag{26}$$

ここで、$r = \tilde{r} r_{\rm s}$, $\Omega = \frac{c}{r_{\rm s}} \sqrt{\frac{1}{2\tilde{r}^3}} = \frac{c}{r_{\rm s}} \tilde{\Omega}$, $H = \frac{1}{\Omega} \sqrt{2N+3} (\frac{W_{\rm tot}}{\Sigma})^{1/2} \frac{r_{\rm s}}{c} = \tilde{H} r_{\rm s}$, $\dot{M} = \dot{m} \dot{M}_{\rm Edd}$.

1.5 f_1, f_2 の微分

$\frac{\partial f_1}{\partial \Sigma}$

$$\frac{\partial f_1}{\partial \Sigma} = -\frac{\partial Q^-}{\partial \Sigma}\frac{r_\mathrm{s}}{c} - \frac{\partial Q^-_\mathrm{Comp}}{\partial \Sigma}\frac{r_\mathrm{s}}{c} + \frac{\dot{m}}{\tilde{r}^2}\frac{\xi}{\kappa_\mathrm{es}\Sigma}\frac{\partial W_\mathrm{mag}}{\partial \Sigma} + \xi\frac{\dot{m}}{\tilde{r}^2}\frac{W_\mathrm{tot} - W_\mathrm{mag}}{\kappa_\mathrm{es}\Sigma^2} \tag{27}$$

$$\frac{\partial Q^-_\mathrm{Comp}}{\partial \Sigma} = \frac{4k_\mathrm{b}}{m_\mathrm{e}c^2}\left[(\kappa_\mathrm{es}\Sigma\frac{\partial Q^-}{\partial \Sigma} + \kappa_\mathrm{es}Q^-)(\frac{I_\mathrm{N+1}}{I_\mathrm{N}}T_\mathrm{e0} - T_\mathrm{r}) - \kappa_\mathrm{es}\Sigma Q^-\frac{\partial T_\mathrm{r}}{\partial \Sigma}\right] \tag{28}$$

$$\frac{\partial Q^-}{\partial \Sigma} = -Q^-\frac{\frac{3}{2}\frac{\partial \tau}{\partial \Sigma} - \tau_\mathrm{abs}^{-2}\frac{\tau_\mathrm{abs}}{\partial \Sigma}}{3\tau/2 + \sqrt{3} + \tau_\mathrm{abs}^{-1}} \tag{29}$$

$$\frac{\partial T_\mathrm{r}}{\partial \Sigma} = \frac{T_\mathrm{r}}{4Q^-}\frac{\partial Q^-}{\partial \Sigma} \tag{30}$$

$$\frac{\partial \tau}{\partial \Sigma} = \frac{\partial \tau_\mathrm{abs}}{\partial \Sigma} + 0.5\kappa_\mathrm{es} \tag{31}$$

$$\frac{\partial \tau_\mathrm{abs}}{\partial \Sigma} = \tau_\mathrm{abs}\frac{5/2}{\Sigma} \tag{32}$$

$$\frac{\partial \tilde{H}}{\partial \Sigma} = -\frac{\tilde{H}}{2\Sigma} \tag{33}$$

$\frac{\partial f_1}{\partial T}$

$$\frac{\partial f_1}{\partial T} = -\frac{\partial Q^-}{\partial T}\frac{r_\mathrm{s}}{c} - \frac{\partial Q^-_\mathrm{Comp}}{\partial T}\frac{r_\mathrm{s}}{c} \tag{34}$$

$$\frac{\partial Q^-_\mathrm{Comp}}{\partial T} = \frac{4k_\mathrm{b}}{m_\mathrm{e}c^2}\kappa_\mathrm{es}\Sigma\left[(\frac{I_\mathrm{N+1}}{I_\mathrm{N}}T_\mathrm{e0} - T_\mathrm{r})\frac{\partial Q^-}{\partial T} + Q^-(\frac{I_\mathrm{N+1}}{I_\mathrm{N}}\frac{\partial T_\mathrm{e0}}{\partial T} - \frac{\partial T_\mathrm{r}}{\partial T})\right] \tag{35}$$

$$\frac{\partial T_\mathrm{r}}{\partial T} = \frac{T_\mathrm{r}}{4Q^-}\frac{\partial Q^-}{\partial T} \tag{36}$$

$$\frac{\partial Q^-}{\partial T} = Q^-(\frac{4}{T_0} - \frac{\frac{3}{2}\frac{\partial \tau}{\partial T} - \tau_\mathrm{abs}^{-2}\frac{\tau_\mathrm{abs}}{\partial T}}{3\tau/2 + \sqrt{3} + \tau_\mathrm{abs}^{-1}}) \tag{37}$$

$$\frac{\partial \tau}{\partial T} = \frac{\partial \tau_\mathrm{abs}}{\partial T} = -\frac{7}{2}\frac{\tau_\mathrm{abs}}{T} \tag{38}$$

$\frac{\partial f_2}{\partial \Sigma}$

$$\frac{\partial f_2}{\partial \Sigma} = -\frac{\partial W_{\rm mag}}{\partial \Sigma} - \frac{I_{\rm N+1}}{4cI_{\rm N}}\tilde{H}r_{\rm s}(\tau+\frac{2}{\sqrt{3}})\frac{\partial Q^-}{\partial \Sigma} - \frac{Q^- I_{\rm N+1}}{4cI_{\rm N}}r_{\rm s}(\tau+\frac{2}{\sqrt{3}})\frac{\partial \tilde{H}}{\partial \Sigma} - \frac{Q^- I_{\rm N+1}}{4cI_{\rm N}}\tilde{H}r_{\rm s}(\tau+\frac{2}{\sqrt{3}})\frac{\partial T_0}{\partial \Sigma} \tag{39}$$

$$\frac{\partial W_{\rm mag}}{\partial \Sigma} = \frac{W_{\rm mag}}{\Sigma}(2\zeta + 0.5) \tag{40}$$

$\frac{\partial f_2}{\partial T}$

$$\frac{\partial f_2}{\partial T} = -\frac{I_{\rm N+1}}{I_{\rm N}}\frac{R}{\mu}\Sigma - \frac{I_{\rm N+1}}{4cI_{\rm N}}\tilde{H}r_{\rm s}(\tau+\frac{2}{\sqrt{3}})\frac{\partial Q^-}{\partial T} - \frac{Q^- I_{\rm N+1}}{4cI_{\rm N}}r_{\rm s}(\tau+\frac{2}{\sqrt{3}})\frac{\partial \tilde{H}}{\partial T} - \frac{Q^- I_{\rm N+1}}{4cI_{\rm N}}\tilde{H}r_{\rm s}(\tau+\frac{2}{\sqrt{3}})\frac{\partial \tau}{\partial T} \tag{41}$$