Q3

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September 19, 2020

Question - over all real numbers $\mathbb{R}$ find the minimum value of a positive number y such that

$$y = \sqrt{(x+6)^2+25} + \sqrt{(x-6)^2+121}$$

Both polynomials in the square root have positive values for the unbounded real number range since

$$(x+6)^2 + 25 \geq 0 : -\infty \leq x \leq \infty \; \forall \, x \; \in \; \mathbb{R}$$

$$(x-6)^2 + 121 \geq 0 : -\infty \leq x \leq \infty \; \forall \, x \; \in \; \mathbb{R}$$

Following basic calculus the global minimum will occur at the point $\frac{dy}{dx} = 0$ since the function is smooth, continuous and differentiable at all points

$$\frac{dy}{dx} = 0 = \frac{d}{dx} \cdot \left(\sqrt{(x+6)^2+25} + \sqrt{(x-6)^2+121} \right) \tag{1}$$

$$0 = \frac{d}{dx} \cdot \left(\sqrt{(x+6)^2+25} \right) + \frac{d}{dx} \cdot \left(\sqrt{(x-6)^2+121} \right) \tag{2}$$

$$\frac{d}{dx} \cdot \left(\sqrt{(x+6)^2+25} \right) = \frac{(x+6)}{\sqrt{(x+6)^2+25}} \tag{3}$$

$$\frac{d}{dx} \cdot \left(\sqrt{(x-6)^2+121} \right) = \frac{(x-6)}{\sqrt{(x-6)^2+121}} \tag{4}$$

$$\left(\frac{(x-6)}{\sqrt{(x-6)^2+121}} + \frac{(x+6)}{\sqrt{(x+6)^2+25}} \right) = 0 \tag{5}$$

reducing further

$$\frac{1}{\sqrt{1+\frac{121}{(x-6)^2}}} + \frac{1}{\sqrt{1+\frac{25}{(x+6)^2}}} = \frac{1}{\sqrt{1+\left(\frac{11}{(x-6)}\right)^2}} + \frac{1}{\sqrt{1+\left(\frac{5}{(x+6)}\right)^2}} = 0 \tag{6}$$

$$\frac{1}{\sqrt{1+\frac{121}{(x-6)^2}}}+\frac{1}{\sqrt{1+\frac{25}{(x+6)^2}}}=\frac{1}{\sqrt{1+\left(\frac{11}{(x-6)}\right)^2}}+\frac{1}{\sqrt{1+\left(\frac{5}{(x+6)}\right)^2}}=0 \tag{7}$$

$$\frac{1}{\sqrt{1+\left(\frac{11}{(x-6)}\right)^2}}=\frac{-1}{\sqrt{1+\left(\frac{5}{(x+6)}\right)^2}} \tag{8}$$

$$\sqrt{1+\left(\frac{11}{(x-6)}\right)^2}=-\sqrt{1+\left(\frac{5}{(x+6)}\right)^2} \tag{9}$$

squaring both sides and subtracting 1

$$\left(\frac{11}{(x-6)}\right)^2=\left(\frac{5}{(x+6)}\right)^2 \tag{10}$$

cross multiplying

$$121(x+6)^2-25(x+6)^2=4x^2+73x+144=0 \tag{11}$$

solving the quadratic equation yields the roots $x=-16$ and $x=\frac{-9}{4}$

$$\min y=\min\left(y|_{x=-16}, y|_{x=\frac{-9}{4}}\right)=\min(16\sqrt{3},20)=\underline{\underline{20}} \tag{12}$$