George Tso

PHYS 3700 Worksheet 5

1) DTFT of $x[n] = 2\delta[n] - \delta[n-1] + 3\delta[n-2] + \delta[n-4]$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega}$$ $n = 0, 1, 2, 4$ for non-zero values

$$\boxed{= 2 + -e^{-j\Omega} + 3e^{-j2\Omega} + e^{-j4\Omega}}$$

2) Freq. response $= H(\Omega)$

$$y[n] = -0.85y[n-1] + 0.5x[n]$$

Z-trans: $Y(z) = -0.85\,Y(z)z^{-1} + 0.5\,X(z)$

$$Y(z)(1 + 0.85z^{-1}) = 0.5X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.5}{1 + 0.85z^{-1}} \Rightarrow H(\Omega) = \frac{0.5}{1 + 0.85e^{-j\Omega}}$$

3) $$H(z) = \frac{1 - 0.2z^{-2}}{1 + 0.5z^{-1} + 0.4z^{-2}}$$

$$H(\Omega) = \frac{1 - 0.2e^{-j2\Omega}}{1 + 0.5e^{-j\Omega} + 0.4e^{-j2\Omega}}$$

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$$x[n] = 20\cos(1.5n + 12^\circ)$$

$$|X(\Omega)| = 20 \qquad \theta_x(\Omega) = 12^\circ$$

$$|H(\Omega)| = 10^{-\frac{21}{20}} \qquad \theta(\Omega) = 86^\circ$$

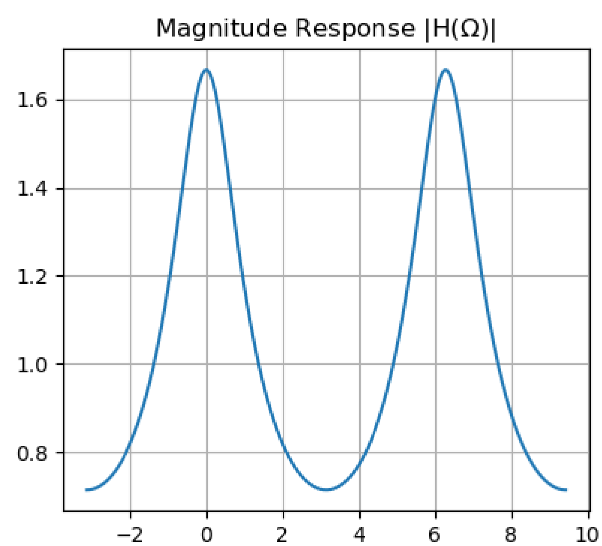
$$\frac{|Y(\Omega)|}{|X(\Omega)|} = |H(\Omega)| \rightarrow |Y(\Omega)| = HX = 20 \cdot 10^{-\frac{21}{20}}$$

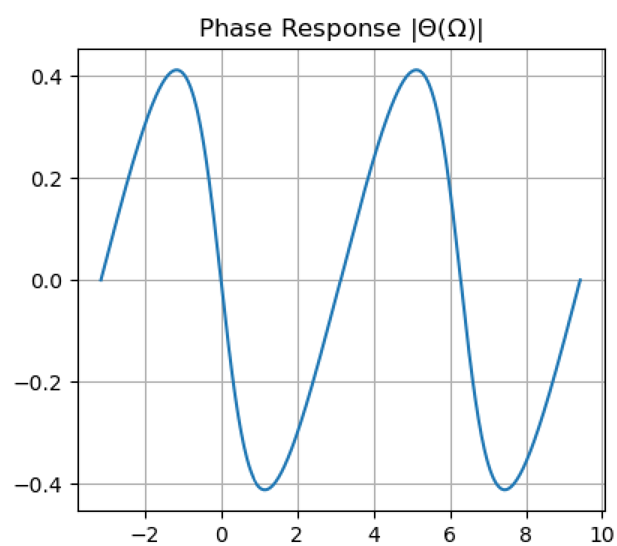
$$\theta = \theta_y - \theta_x \Rightarrow \theta_y = \theta + \theta_x = 98^\circ$$

$$\boxed{y[n] = 20 \cdot 10^{-\frac{21}{20}} \cos(1.5n + 98^\circ)}$$

5) $$H(\Omega) = \frac{1}{1 - 0.4e^{-j\Omega}}$$

H(3π/4) = 0.743 + 0.164j





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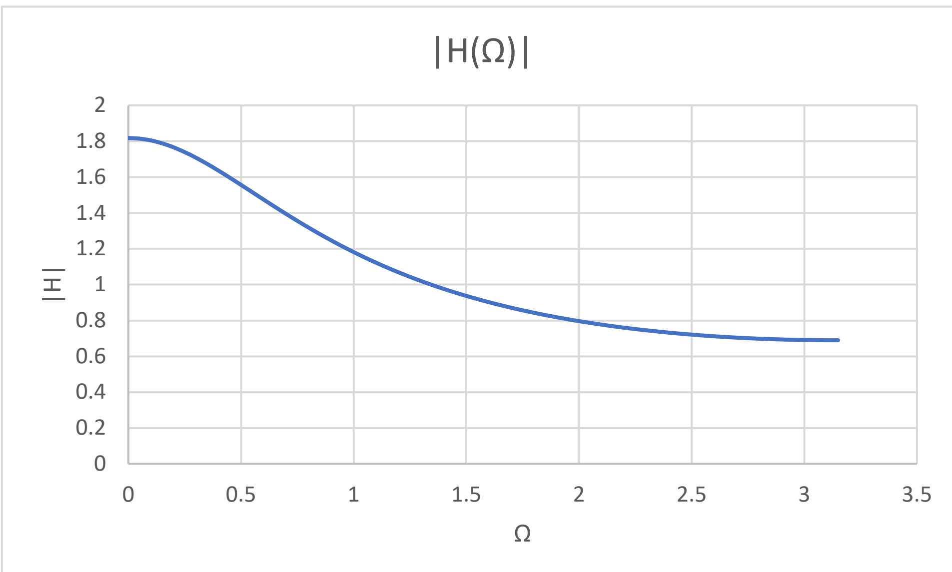
6) $H(z) = \dfrac{1}{z-0.45} \Rightarrow H(\Omega) = \dfrac{1}{e^{j\Omega}-0.45}$

$H\left(\frac{3\pi}{4}\right) \Rightarrow \Omega_{deg} = 135°$

unit circle Cartesian coordinates: $(\cos 135°, \sin 135°)$

$= \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \rightarrow |H(\Omega)| = \dfrac{1}{\text{distance}} = \dfrac{1}{\sqrt{\left(\frac{-1}{\sqrt{2}}-0.45\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}} = 0.7374$



7) $$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$$

$N = 8 \quad k = 1, 5 \quad x[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$

$\cancel{+\delta[n-8]+\delta[n-9]+\delta[n-10]+\delta[n-11]}$

16 17 18 19

$$c_1 = \sum_{n=0}^{7} x[n] e^{-j2\pi \frac{1}{8} n}$$

$$= 1 \cancel{e^{-j2\pi \frac{1}{8} 0}} + e^{-j2\pi \frac{1}{8} 1} + e^{-j2\pi \frac{1}{8} 2} + e^{-j2\pi \frac{1}{8} 3} = 1 - 2.4142j$$

$$c_5 = \sum_{n=0}^{7} x[n] e^{-j2\pi \frac{5}{8} n} = 1 + e^{-j2\pi \frac{5}{8}} + e^{-j2\pi \frac{5}{8} 2} + e^{-j2\pi \frac{5}{8} 3} = 1 + 0.4142j$$

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$c_1 = 1 - 2.4142j \qquad \frac{|c_1|}{8} = 0.3266$

$c_5 = 1 + 0.4142j \qquad \frac{|c_5|}{8} = 0.1353$

8) $x[n] = \sin(n\pi/5) = \sin(2\pi f n)$

$\frac{\pi}{5} \cdot \frac{1}{2\pi} = \frac{1}{10} = \text{freq} \rightarrow$ repeats after 10 n

$N = 10$

$$c_k = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n}$$

$k = 1, 5$

$$c_1 = \sin\left(\tfrac{0\pi}{5}\right)e^{-j2\pi\frac{1}{10}0} + \sin\left(\tfrac{1\pi}{5}\right)e^{-j2\pi\frac{1}{10}1} + \sin\left(\tfrac{2\pi}{5}\right)e^{-j2\pi\frac{1}{10}2} +$$

$$\sin\left(\tfrac{3\pi}{5}\right)e^{-j2\pi\frac{1}{10}3} + \sin\left(\tfrac{4\pi}{5}\right)e^{-j2\pi\frac{1}{10}4} + \sin\left(\tfrac{5\pi}{5}\right)e^{-j2\pi\frac{1}{10}5} +$$

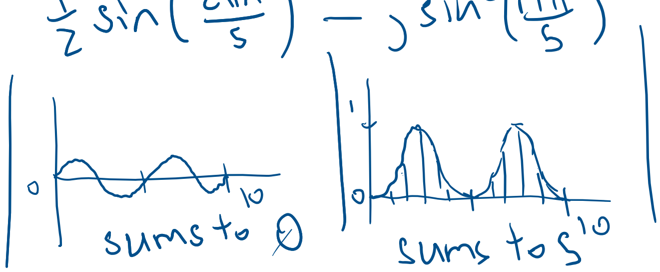
$$\sin\left(\tfrac{6\pi}{5}\right)e^{-j2\pi\frac{1}{10}6} + \sin\left(\tfrac{7\pi}{5}\right)e^{-j2\pi\frac{1}{10}7} + \sin\left(\tfrac{8\pi}{5}\right)e^{-j2\pi\frac{1}{10}8} +$$

$$\sin\left(\tfrac{9\pi}{5}\right)e^{-j2\pi\frac{1}{10}9} = 0 - 5j$$

or: $\sin\left(\frac{n\pi}{5}\right)e^{-j2\pi\frac{1}{10}n} = \sin\left(\frac{n\pi}{5}\right)\left(\cos\left(2\pi\frac{1}{10}n\right) - j\sin\left(2\pi\frac{1}{10}n\right)\right)$

$= \sin\left(\frac{n\pi}{5}\right)\cos\left(\frac{n\pi}{5}\right) - j\sin^2\left(\frac{n\pi}{5}\right)$

$= \frac{1}{2}\sin\left(\frac{2\pi n}{5}\right) - j\sin^2\left(\frac{n\pi}{5}\right)$



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$C_5 =$ same as above, but $k = 5$

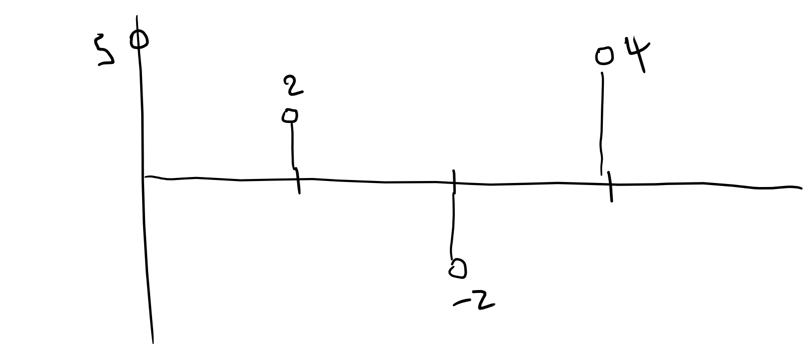
sums to $0 + j0$

Magnitudes: $\frac{|C_1|}{10} = 0.5$ $\qquad \frac{|C_5|}{10} = 0$

Phase: $\tan^{-1} \frac{\mathbb{R} C_1}{\mathbb{I} C_1} = \tan^{-1}\left(\frac{0}{5}\right) = 0$ for C_1

$\tan^{-1} \frac{\mathbb{R} C_5}{\mathbb{I} C_5} =$ undefined for C_5

9) $x[n]$



$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{k}{N} n} \qquad N = 4$$

$$X[0] = 5e^{-j2\pi\frac{0}{4}0} + 2e - 2 + 4 = 9$$

$$X[1] = 5e^{-j2\pi\frac{0}{4}} + 2e^{-j2\pi\frac{1}{4}} - 2e^{-j2\pi\frac{2}{4}} + 4e^{-j2\pi\frac{3}{4}} = 7 + 2j$$

$$X[2] = 5e^{-j2\pi\frac{0}{4}} + 2e^{-j2\pi\frac{2}{4}} - 2e^{-j2\pi\frac{4}{4}} + 4e^{-j2\pi\frac{6}{4}} = -3$$

$$X[3] = 5e^{-j2\pi\frac{0}{4}} + 2e^{-j2\pi\frac{3}{4}} - 2e^{-j2\pi\frac{6}{4}} + 4e^{-j2\pi\frac{9}{4}} = 7 - 2j$$

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Inverse DFT check

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi\frac{k}{N}n}$$

$\frac{1}{4}(9+7-3+7+2j-2j)$

$$x[0] = \frac{1}{4}\left[X[0]\cancel{e^{j2\pi 0}} + X[1]e^{j2\pi 0} + X[2] + X[3]\right] = \frac{20}{4} = 5$$

$$x[2] = \frac{1}{4}\left[9\cancel{e^{j2\pi 0}} + 2e^{j2\pi\frac{2}{4}} - 2e^{j2\pi\frac{4}{4}} + 4e^{j2\pi\frac{6}{4}}\right] = \frac{1}{4}(-8) = -2$$

10) $x[n] = 2\delta[n] - \delta[n-1] + 3\delta[n-2] + 3\delta[n-3]$

DTFT: $X(\Omega) = 2 - e^{-j\Omega} + 3e^{-j2\Omega} + 3e^{-j3\Omega}$

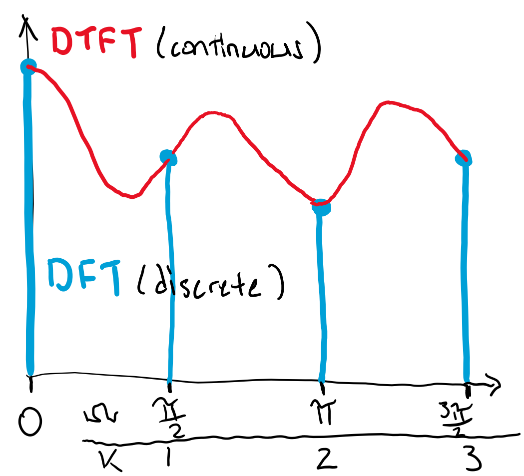
$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi\frac{k}{N}n}, \quad N=4$$

DTF: $X[0] = -1+3+3 = 7$

$X[1] = 2\cancel{e^{-j2\pi\frac{1}{4}0}} - e^{-j2\pi\frac{1}{4}} + 3e^{-j2\pi\frac{2}{4}} + 3e^{-j2\pi\frac{3}{4}} = -1+4j$

$X[2] = 2 - e^{-j2\pi\frac{2}{4}} + 3e^{-j2\pi\frac{4}{4}} + 3e^{-j2\pi\frac{6}{4}} = 3$

$X[3] = 2 - e^{-j2\pi\frac{3}{4}} + 3e^{-j2\pi\frac{6}{4}} + 3e^{-j2\pi\frac{9}{4}} = -1-4j$

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[19]: fig = plt.figure()
      ax = fig.add_axes([0,0,1,1])

      omega = np.linspace(0,3*np.pi/2,150)
      DTFT = [0]*len(omega)
      for i in range(len(omega)):
          DTFT[i] = np.abs(2 - cmath.exp(complex(0,-omega[i])) + 3*cmath.exp(complex(0,-2*omega[i])) + 3*cmath.exp(complex(0,-3*omega[i])))

      ax.plot(omega, DTFT, label = 'Discrete Time Fourier Transform')


      DFT = [7, np.abs(complex(-1,4)), 3, np.abs(complex(-1,-4))]
      k = np.linspace(0,3*np.pi/2,4)

      ax.stem(k, DFT, label = 'Discrete Fourier Transform',
             linefmt = 'C1-',
             markerfmt = 'C1o')

      ax.set_title('Magnitude Spectrum of x[n]')
      ax.set_ylabel('Magnitude')
      ax.set_xlabel('Frequency')
      ax.legend()

[19]: <matplotlib.legend.Legend at 0x19bee373460>
```