LMECA2550-HW2 : Turbojet cycle analysis

Reminders

We will use the nomenclature used in lecture to characterize the propulsion systems. As a reminder, we define the following variables :

- $\pi_d = \frac{p_{t2}}{p_{t0}}$
- $\pi_c = \frac{p_{t3}}{p_{t2}}$
- $\pi_b = \frac{p_{t4}}{p_{t3}}$
- $\pi_t = \frac{p_{t5}}{p_{t4}}$
- $\pi_n = \frac{p_{te}}{p_{t7}}$
- $\pi_{AB} = \frac{p_{t7}}{p_{t5}}$
- $\pi_r = \frac{p_{t0}}{p_0}$

- $\tau_d = \frac{T_{t2}}{T_{t0}}$
- $\tau_c = \frac{T_{t3}}{T_{t2}}$
- $\tau_b = \frac{T_{t4}}{T_{t3}}$
- $\tau_t = \frac{T_{t5}}{T_{t4}}$
- $\tau_n = \frac{T_{te}}{T_{t7}}$
- $\tau_{AB} = \frac{T_{t7}}{T_{t5}}$
- $\tau_r = \frac{T_{t0}}{T_0}$
- $\tau_\lambda = \frac{c_{p,t}T_{t4}}{c_{p,c}T_{t0}}$

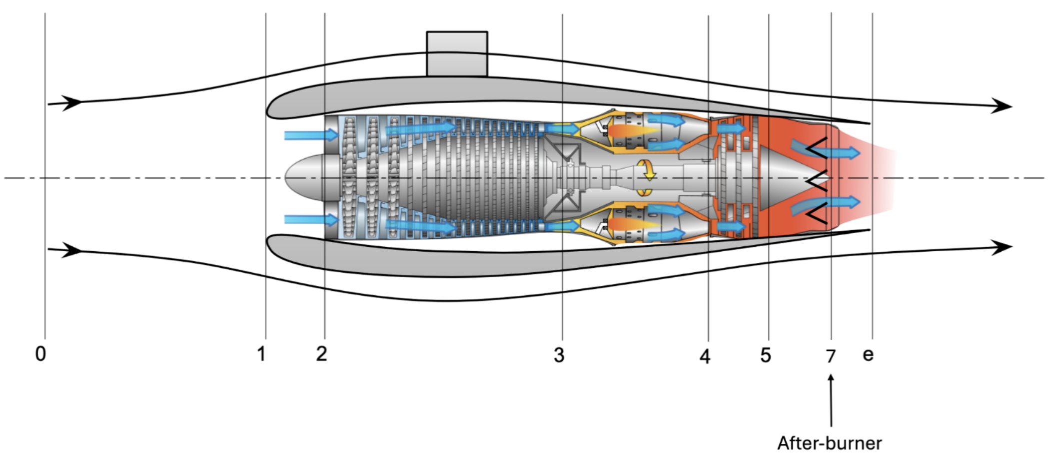

Figure 1 – Turbojet

1 Turbojet optimum compressor ratio

We want to derive the optimum pressure and temperature ratio across the compressor of an ideal turbojet without after burner. We consider an ideal turbojet : $\tau_d = 1$, $\pi_d = 1$ $p_{t,e} = p_{t,5}$ $\tau_n = 1$, $\pi_n = 1$, $\pi_b = 1$, $\pi_{AB} = 1$

1.1 Specific thrust

We start from the thrust formula and derive the specific thrust formula from it.

$$
\begin{aligned}
F &= \dot{m}(u_e - u_0) \\
\frac{F}{\dot{m}} &= u_e - u_0 && \text{We know that } u = Mc \\
\frac{F}{\dot{m}} &= \left(\frac{c_e}{c_0}M_e - M_0\right) c_0 && c = \sqrt{\gamma R T} \\
\frac{F}{\dot{m}} &= \left(\frac{\sqrt{\gamma R T_e}}{\sqrt{\gamma R T_0}}M_e - M_0\right) \sqrt{\gamma R T_0}
\end{aligned}
$$

$$
\frac{F}{\dot{m}} = \left(\sqrt{\frac{T_e}{T_0}}M_e - M_0\right) c_0 \tag{1}
$$

1.1.1 Find M_e

We take the total enthalpy and get M_e from it using the engine parameters.

$$
\begin{aligned}
h_{t,e} &= h_e + \frac{u_e^2}{2} \\
T_{te} &= T_e + \frac{u_e^2}{2c_p} \\
\frac{T_{t,e}}{T_e} &= 1 + \frac{\gamma - 1}{2}M_e^2 \\
M_e^2 &= \frac{2}{\gamma - 1}\left(\left(\frac{p_{t,e}}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) && \text{We that } \frac{p_{t,e}}{p_e} = \pi_r \pi_c \pi_t \\
M_e^2 &= \frac{2}{\gamma - 1}\left((\pi_r \pi_c \pi_t)^{\frac{\gamma-1}{\gamma}} - 1\right) && \pi^{\frac{\gamma-1}{\gamma}} = \tau
\end{aligned}
$$

$$
M_e^2 = \frac{2}{\gamma - 1}(\tau_r \tau_c \tau_t - 1) \tag{2}
$$

1.1.2 Find $\frac{T_e}{T_0}$

We want to describe $\frac{T_e}{T_0}$ as a function of engine parameters.

$$
\begin{aligned}
\frac{T_{t,e}}{T_0} &= \frac{T_{t,e}}{T_0}\frac{T_{t,2}}{T_{t,0}}\frac{T_{t,3}}{T_{t,2}}\frac{T_{t,4}}{T_{t,3}}\frac{T_{t,5}}{T_{t,4}}\frac{T_{t,e}}{T_{t,5}} \\
\frac{T_{t,e}}{T_0} &= \tau_r \tau_c \tau_b \tau_t
\end{aligned}
$$

$$\frac{T_{t,e}}{T_0} = (\frac{p_{t,e}}{p_e})^{\frac{\gamma-1}{\gamma}} = (\pi_r \pi_c \pi_t)^{\frac{\gamma-1}{\gamma}}$$
$$\frac{T_{t,e}}{T_0} = \tau_r \tau_c \tau_t$$

$$\frac{T_e}{T_0} = \frac{T_{t,e}/T_0}{T_{t,e}/T_e}$$
$$\frac{T_e}{T_0} = \tau_b \tag{3}$$

1.1.3 Find τ_t

We use the equilibrium between the compressor and the turbine :

$$\dot{W}_c = \dot{W}_t$$
$$\dot{m} C_p T_{t,4} (1 - \frac{T_{t,5}}{T_{t,4}}) = \dot{m} C_p T_{t,2} (\frac{T_{t,3}}{T_{t,2}} - 1)$$
$$(1 - \tau_t) = \frac{T_{t,2}}{T_{t,4}} (\tau_c - 1)$$

$$\tau_t = 1 - \frac{\tau_r}{\tau_\lambda} (\tau_c - 1) \tag{4}$$

1.1.4 Find τ_b

$$\tau_b = \frac{T_{t,4}}{T_{t,3}} = \frac{T_0}{T_{t,0}} \frac{T_{t,0}}{T_{t,2}} \frac{T_{t,2}}{T_{t,3}} \frac{T_{t,4}}{T_{t,0}} = \frac{1}{\tau_b} \frac{1}{\tau_d} \frac{1}{\tau_c} \tau_\lambda$$

$$\tau_b = \frac{\tau_\lambda}{\tau_r \tau_c} \tag{5}$$

1.1.5 Back to the Specific thrust

We inject the terms 2 3 4 5 computed up above in the specific thrust equation 1 :

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{T_e}{T_0}} M_e - M_0 \right) c_0$$
$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{\tau_\lambda}{\tau_r \tau_c} \frac{2}{\gamma - 1} (\tau_r \tau_c \tau_t - 1)} - M_0 \right) c_0$$

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{\tau_\lambda}{\tau_r \tau_c} \frac{2}{\gamma - 1} ((1 - \frac{\tau_r}{\tau_\lambda} (\tau_c - 1)) \tau_r \tau_c - 1)} - M_0 \right) c_0 \tag{6}$$

1.2 Specific thrust optimization

Now that we retrieve an equation characterizing the specific thrust 6, we can find τ_c and π_c optimizing the specific thrust.
We first need to reshape the function 6 to simplify the calculation :

$$(\frac{u_e}{c_0})^2 = \frac{2}{\gamma - 1}\frac{\tau_\lambda}{\tau_r\tau_c}((1 - \frac{\tau_r}{\tau_\lambda}(\tau_c - 1))\tau_r\tau_c - 1) \tag{7}$$

We can now easily calculate $\frac{\partial}{\partial\tau_c}(\frac{u_e}{c_0})^2 = 0$ that maximizes $\frac{F}{\dot{m}}$

$$\frac{\partial}{\partial\tau_c}(\frac{2}{\gamma - 1}\frac{\tau_\lambda}{\tau_r\tau_c}((1 - \frac{\tau_r}{\tau_\lambda}(\tau_c - 1))\tau_r\tau_c - 1)) = 0$$

$$-\frac{\tau_r^2}{\tau_\lambda} + \frac{1}{\tau_c^2} = 0$$

$$\tau_c = \frac{\pm\sqrt{\tau_\lambda}}{\tau_r} \qquad \text{We reject the negative solution}$$

$$\tau_c = \frac{\sqrt{\tau_\lambda}}{\tau_r} \tag{8}$$

We know that π_c and τ_c are linked by $\pi_c = \tau_c^{\frac{\gamma}{\gamma-1}}$ so we can find π_c

$$\pi_c = (\frac{\sqrt{\tau_\lambda}}{\tau_r})^{\frac{\gamma}{\gamma-1}} \tag{9}$$

1.3 Comparaison with Brayton cycle

TODO : Add comparaison with Brayton cycle

2 Hybrid cycle after-burning turbojet

2.1 Specific thrust

We carry out the analysis for a turbojet equipped with an after-burner while introducing polytropic efficiencies for the turbine and compressor : η_t, η_c. We need to introfuce a new parameters to unsure the maximum temperature at the after-burner outlet : $\tau_{\lambda,AB} = \frac{T_{t,7}}{T_0}$
We derive the equation in the same fashion as for the case without after-burner. Equation 1, 4 and 5 are not modified.

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{T_e}{T_0}}M_e - M_0\right)c_0$$

$$\tau_t = 1 - \frac{\tau_r}{\tau_\lambda}(\tau_c - 1)$$

$$\tau_b = \frac{\tau_\lambda}{\tau_r\tau_c}$$

2.1.1 Find M_e

We start from find in the first part 2 and introduce the efficiencies : $\pi_c^{\frac{\gamma-1}{\gamma}} = \tau_c^{\eta_c}$ and $\pi_t^{\frac{\gamma-1}{\gamma}} = \tau_t^{1/\eta_t}$.

$$M_e^2 = \frac{2}{\gamma - 1}((\pi_r \pi_c \pi_t)^{\frac{\gamma-1}{\gamma}} - 1)$$

$$M_e^2 = \frac{2}{\gamma - 1}(\tau_r \tau_c^{\eta_c} \tau_t^{1/\eta_t} - 1) \tag{10}$$

2.1.2 Find $\frac{T_e}{T_0}$

We redo all the calculation, taking τ_{AB} into account this time.

$$\frac{T_{t,e}}{T_0} = \frac{T_{t,e}}{T_0}\frac{T_{t,2}}{T_{t,0}}\frac{T_{t,3}}{T_{t,2}}\frac{T_{t,4}}{T_{t,3}}\frac{T_{t,5}}{T_{t,4}}\frac{T_{t,7}}{T_{t,5}}\frac{T_{t,e}}{T_{t,7}}$$
$$\frac{T_{t,e}}{T_0} = \tau_r \tau_c \tau_b \tau_t \tau_{AB}$$

$$\frac{T_{t,e}}{T_0} = (\frac{p_{t,e}}{p_e})^{\frac{\gamma-1}{\gamma}} = (\pi_r \pi_c \pi_t)^{\frac{\gamma-1}{\gamma}}$$
$$\frac{T_{t,e}}{T_0} = \tau_r \tau_c^{\eta_c} \tau_t^{1/\eta_t}$$

$$\frac{T_e}{T_0} = \frac{T_{t,e}/T_0}{T_{t,e}/T_e}$$
$$\frac{T_e}{T_0} = \tau_c^{1-\eta_c} \tau_t^{1-1/\eta_t} \tau_b \tau_{AB} \tag{11}$$

2.1.3 Find τ_{AB}

$$\tau_{AB} = \frac{T_{t,7}}{T_{t,5}} = \frac{T_0}{T_{t,0}}\frac{T_{t,0}}{T_{t,2}}\frac{T_{t,2}}{T_{t,3}}\frac{T_{t,3}}{T_{t,4}}\frac{T_{t,4}}{T_{t,5}}\frac{T_{t,7}}{T_0}$$
$$\tau_{AB} = \frac{1}{\tau_r}\frac{1}{\tau_d}\frac{1}{\tau_c}\frac{1}{\tau_b}\frac{1}{\tau_t}\tau_{\lambda,AB}$$

$$\tau_{AB} = \frac{\tau_{\lambda,AB}}{\tau_\lambda \tau_t} \tag{12}$$

2.1.4 Back to specific thrust

We consider that the fuel mass flow injected in the after-burner is negligible compared to the air mass flow. We inject the terms 10 11 4 5 and 12 computed up above in the specific thrust equation 1 :

$$(\frac{u_e}{c_0})^2 = \frac{T_e}{T_0} M_e^2$$
$$(\frac{u_e}{c_0})^2 = \tau_c^{1-\eta_c} \tau_t^{1-1/\eta_t} \tau_b \tau_{AB} \frac{2}{\gamma - 1}(\tau_r \tau_c^{\eta_c} \tau_t^{1/\eta_t} - 1)$$
$$(\frac{u_e}{c_0})^2 = \frac{2}{\gamma - 1}\tau_{\lambda,AB}(1 - \frac{1}{\tau_r {\tau_c}^{\eta_c} {\tau_t}^{1/\eta_t}})$$

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{2}{\gamma - 1} \tau_{\lambda,AB} (1 - \frac{1}{\tau_r \tau_c{}^{\eta_c} \tau_t{}^{1/\eta_t}})} - M_0 \right) c_0 \tag{13}$$

2.1.5 Specific thrust optimization

We can now easily calculate $\frac{\partial}{\partial \tau_c} (\frac{u_e}{c_0})^2 = 0$ that maximizes $\frac{F}{\dot{m}}$.

$$\frac{\partial}{\partial \tau_c} (\frac{2}{\gamma - 1} \tau_{\lambda,AB} (1 - \frac{1}{\tau_r \tau_c{}^{\eta_c} \tau_t{}^{1/\eta_t}})) = 0$$

$$\tau_c = \frac{\eta_c \eta_t}{1 + \eta_c \eta_t} \left(\frac{\tau_\lambda}{\tau_r} + 1 \right) \tag{14}$$

2.2 Ramjet specific thrust

In the case of a ramjet, there is no compressor and turbine. We have $\tau_c = 1$ and $\tau_t = 1$. We can simplify the specific thrust equation found previously 13 :

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{2}{\gamma - 1} \tau_{\lambda,AB} (1 - \frac{1}{\tau_r})} - M_0 \right) c_0 \tag{15}$$

2.3 Transition Mach number

We want to find the Mach number for which the specific thrust of the turbojet with after-burner equals the specific thrust of the ramjet. We equal equations 13 and 15 :

$$\begin{aligned}
\tau_{\lambda,AB} (1 - \frac{1}{\tau_r \tau_c{}^{\eta_c} \tau_t{}^{1/\eta_t}}) &= \tau_{\lambda,AB} (1 - \frac{1}{\tau_r}) \\
\frac{1}{\tau_r \tau_c{}^{\eta_c} \tau_t{}^{1/\eta_t}} &= \frac{1}{\tau_r} \\
\tau_c{}^{\eta_c} \tau_t{}^{1/\eta_t} &= 1 \\
\tau_c^{\eta_c \eta_t} \tau_t &= 1 \\
\tau_c^{\eta_c \eta_t} (1 - \frac{\tau_r}{\tau_\lambda} (\tau_c - 1)) &= 1 \\
\tau_\lambda \frac{1 - \tau_c^{-\eta_c \eta_t}}{\tau_c - 1} &= \tau_r && \text{We know that } \tau_r = 1 + \frac{\gamma - 1}{2} M^2
\end{aligned}$$

$$M_0 = \sqrt{\frac{2}{\gamma - 1} (\tau_\lambda \frac{1 - \tau_c^{-\eta_c \eta_t}}{\tau_c - 1} - 1)} \tag{16}$$

2.3.1 Numerical application

We consider the following parameters :

- $\gamma = 1.4$
- $\pi_c = 20$

- $T_{t,4} = 1400K$
- $T_{t,0} = 216K$
- $\eta_c = \eta_t = 0.9$
- $\tau_\lambda = \frac{T_{t,4}}{T_{t,0}} = \frac{1400}{216} = 6.48$
- $\tau_c = \pi_c^{\frac{\gamma-1}{\gamma}\frac{1}{\eta_c}} = 2.588$

We can now compute the transition Mach number using equation 16 :

$$\boxed{M_0 = 2.44}$$

2.4 Graphical representation

We plot the specific thrust of the turbojet with after-burner 13 and the specific thrust of the ramjet 15 as a function of the Mach number to visualize the transition Mach number computed previously. We use the same parameters as in the numerical application part.

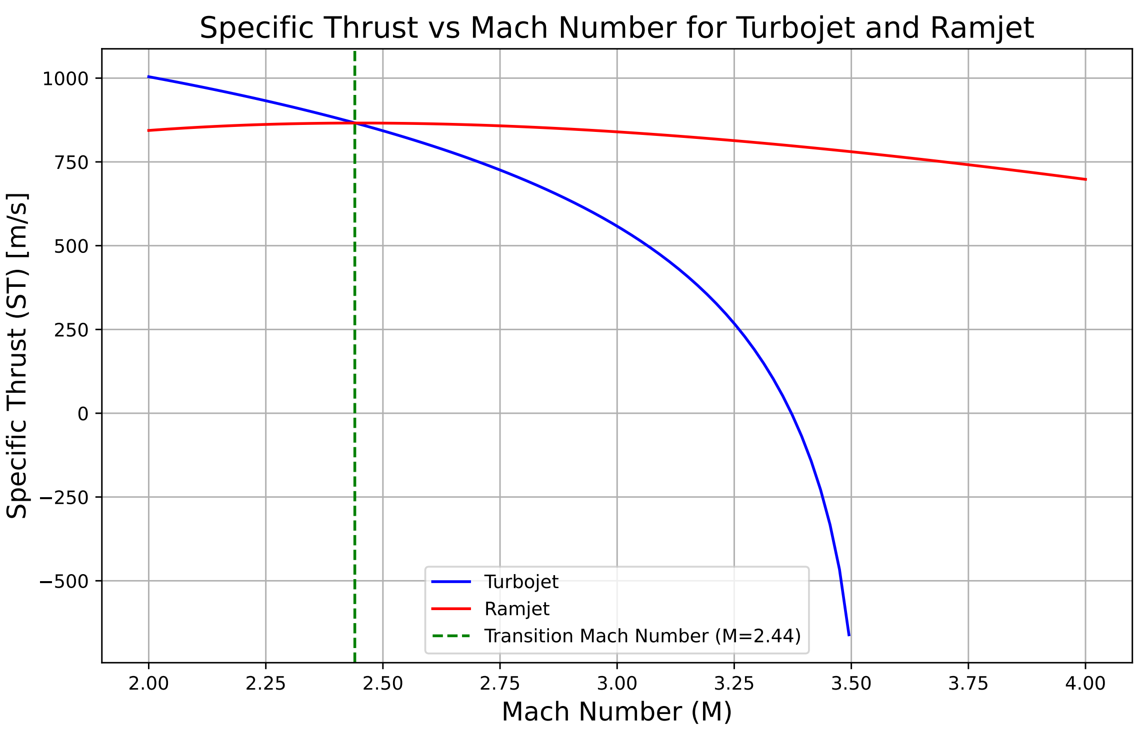

FIGURE 2 – Specific Thrust vs Mach Number for Turbojet and Ramjet

The graph shows that the transition Mach number is around 2.4, which is consistent with the value computed previously. We can also observe that the ramjet specific thrust becomes higher than that of the turbojet equipped with after-burner after this Mach number, confirming the advantage of ramjets at high speeds.
It is interesting to note that past Mach 3.5, the turbojet doesn't produce any more thrust.

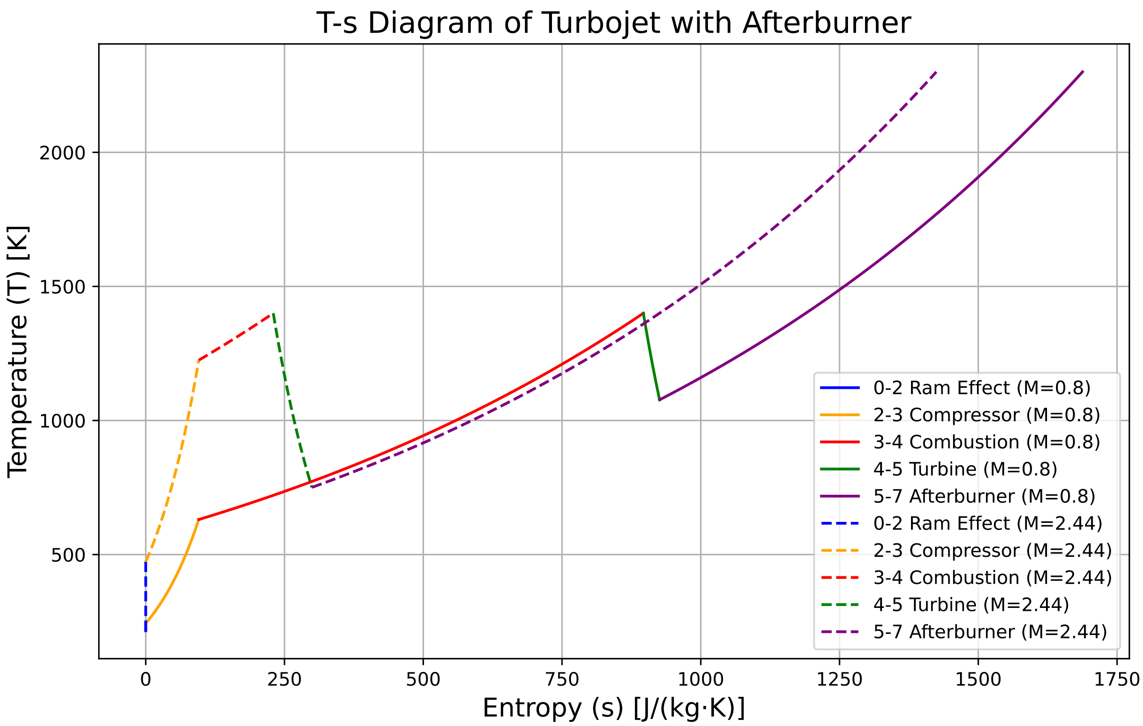

FIGURE 3 – T-S Diagram of Turbojet with After-Burner at M=0.8

2.5 T-S Diagram

We plot the T-S diagram of the turbojet with after-burner at a subsonic speed and transition Mach number. We use the same parameters as in the numerical application part.

We can assess the different components of the engine on the T-S diagram. First, we observe that the ram effect (0-2) is more present at M=2.44 than at M=0.8, as expected since the ram effect increases with the Mach number.
We can also observe at that at M=2.44, the compressor generates large amount of heat and need a lot of work from the turbine to compress the air.
At lower Mach number (M=0.8), the temperature increases a lot in the combustion chamber, as well as the entropy while at M=2.44, the temperature increase is more moderate. This is due to the fact that at higher Mach number, the air entering the combustor is already at a higher temperature due to the ram effect and the compressor, so less fuel is needed to reach the same turbine inlet temperature.
TODO : Add more analysis

2.6 Specific fuel thrust consumption