LMECA2550-HW2 : Turbojet cycle analysis

Reminders

We will use the nomenclature used in lecture to characterize the propulsion systems. As a reminder, we define the following variables :

- $\pi_d = \frac{p_{t2}}{p_{t0}}$
- $\pi_c = \frac{p_{t3}}{p_{t2}}$
- $\pi_b = \frac{p_{t4}}{p_{t3}}$
- $\pi_t = \frac{p_{t5}}{p_{t4}}$
- $\pi_n = \frac{p_{te}}{p_{t7}}$
- $\pi_{AB} = \frac{p_{t7}}{p_{t5}}$
- $\pi_r = \frac{p_{t0}}{p_0}$

- $\tau_d = \frac{T_{t2}}{T_{t0}}$
- $\tau_c = \frac{T_{t3}}{T_{t2}}$
- $\tau_b = \frac{T_{t4}}{T_{t3}}$
- $\tau_t = \frac{T_{t5}}{T_{t4}}$
- $\tau_n = \frac{T_{te}}{T_{t7}}$
- $\tau_{AB} = \frac{T_{t7}}{T_{t5}}$
- $\tau_r = \frac{T_{t0}}{T_0}$
- $\tau_\lambda = \frac{c_{p,t}T_{t4}}{c_{p,c}T_{t0}}$

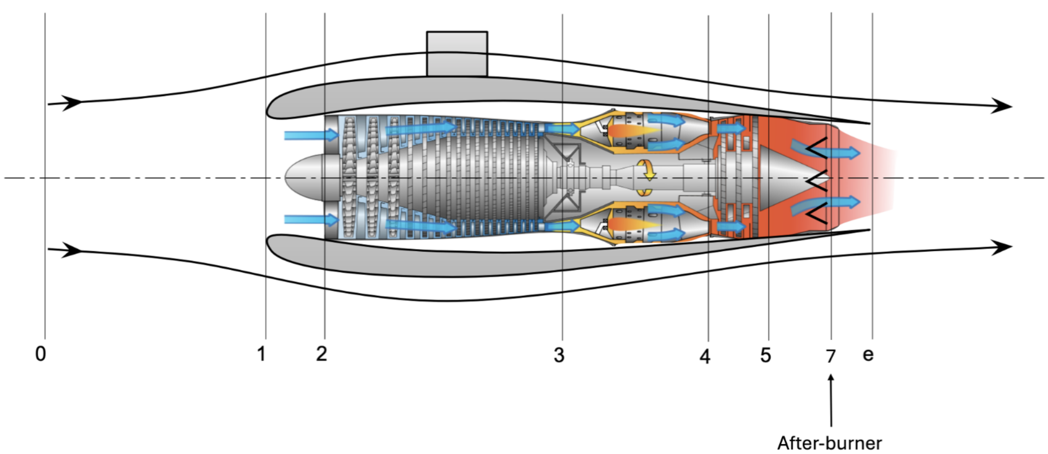

FIGURE 1 – Turbojet

1 Turbojet optimum compressor ratio

We want to derive the optimum pressure and temperature ratio across the compressor of an ideal turbojet without after burner. We consider an ideal turbojet : $\tau_d = 1$, $\pi_d = 1$ $p_{t,e} = p_{t,5}$ $\tau_n = 1$, $\pi_n = 1$, $\pi_b = 1$, $\pi_{AB} = 1$

1.1 Specific thrust

We start from the thrust formula and derive the specific thrust formula from it.

$$\begin{aligned}
F &= \dot{m}(u_e - u_0) \\
\frac{F}{\dot{m}} &= u_e - u_0 && \text{We know that } u = Mc \\
\frac{F}{\dot{m}} &= \left(\frac{c_e}{c_0}M_e - M_0\right)c_0 && c = \sqrt{\gamma RT} \\
\frac{F}{\dot{m}} &= \left(\frac{\sqrt{\gamma RT_e}}{\sqrt{\gamma RT_0}}M_e - M_0\right)\sqrt{\gamma RT_0}
\end{aligned}$$

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{T_e}{T_0}}M_e - M_0\right)c_0 \tag{1}$$

1.1.1 Find M_e

We take the total enthalpy and get M_e from it using the engine parameters.

$$\begin{aligned}
h_{t,e} &= h_e + \frac{u_e^2}{2} \\
T_{te} &= T_e + \frac{u_e^2}{2c_p} \\
\frac{T_{t,e}}{T_e} &= 1 + \frac{\gamma - 1}{2}M_e^2 \\
M_e^2 &= \frac{2}{\gamma - 1}\left(\left(\frac{p_{t,e}}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right) && \text{We that } \frac{p_{t,e}}{p_e} = \pi_r\pi_c\pi_t \\
M_e^2 &= \frac{2}{\gamma - 1}\left((\pi_r\pi_c\pi_t)^{\frac{\gamma-1}{\gamma}} - 1\right) && \pi^{\frac{\gamma-1}{\gamma}} = \tau
\end{aligned}$$

$$M_e^2 = \frac{2}{\gamma - 1}(\tau_r\tau_c\tau_t - 1) \tag{2}$$

1.1.2 Find $\frac{T_e}{T_0}$

We want to describe $\frac{T_e}{T_0}$ as a function of engine parameters.

$$\begin{aligned}
\frac{T_{t,e}}{T_0} &= \frac{T_{t,e}}{T_0}\frac{T_{t,2}}{T_{t,0}}\frac{T_{t,3}}{T_{t,2}}\frac{T_{t,4}}{T_{t,3}}\frac{T_{t,5}}{T_{t,4}}\frac{T_{t,e}}{T_{t,5}} \\
\frac{T_{t,e}}{T_0} &= \tau_r\tau_c\tau_b\tau_t
\end{aligned}$$

$$\frac{T_{t,e}}{T_0} = (\frac{p_{t,e}}{p_e})^{\frac{\gamma-1}{\gamma}} = (\pi_r \pi_c \pi_t)^{\frac{\gamma-1}{\gamma}}$$
$$\frac{T_{t,e}}{T_0} = \tau_r \tau_c \tau_t$$

$$\frac{T_e}{T_0} = \frac{T_{t,e}/T_0}{T_{t,e}/T_e}$$
$$\frac{T_e}{T_0} = \tau_b \tag{3}$$

1.1.3 Find τ_t

We use the equilibrium between the compressor and the turbine :

$$\dot{W}_c = \dot{W}_t$$
$$\dot{m} C_p T_{t,4} (1 - \frac{T_{t,5}}{T_{t,4}}) = \dot{m} C_p T_{t,2} (\frac{T_{t,3}}{T_{t,2}} - 1)$$
$$(1 - \tau_t) = \frac{T_{t,2}}{T_{t,4}} (\tau_c - 1)$$

$$\tau_t = 1 - \frac{\tau_r}{\tau_\lambda} (\tau_c - 1) \tag{4}$$

1.1.4 Find τ_b

$$\tau_b = \frac{T_{t,4}}{T_{t,3}} = \frac{T_0}{T_{t,0}} \frac{T_{t,0}}{T_{t,2}} \frac{T_{t,2}}{T_{t,3}} \frac{T_{t,4}}{T_{t,0}} = \frac{1}{\tau_b} \frac{1}{\tau_d} \frac{1}{\tau_c} \tau_\lambda$$

$$\tau_b = \frac{\tau_\lambda}{\tau_r \tau_c} \tag{5}$$

1.1.5 Back to the Specific thrust

We inject the terms 2 3 4 5 computed up above in the specific thrust equation 1 :

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{T_e}{T_0}} M_e - M_0 \right) c_0$$
$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{\tau_\lambda}{\tau_r \tau_c} \frac{2}{\gamma - 1} (\tau_r \tau_c \tau_t - 1)} - M_0 \right) c_0$$

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{\tau_\lambda}{\tau_r \tau_c} \frac{2}{\gamma - 1} ((1 - \frac{\tau_r}{\tau_\lambda} (\tau_c - 1)) \tau_r \tau_c - 1)} - M_0 \right) c_0 \tag{6}$$

1.2 Specific thrust optimization

Now that we retrieve an equation characterizing the specific thrust 6, we can find τ_c and π_c optimizing the specific thrust.
We first need to reshape the function 6 to simplify the calculation :

$$(\frac{u_e}{c_0})^2 = \frac{2}{\gamma - 1}\frac{\tau_\lambda}{\tau_r \tau_c}((1 - \frac{\tau_r}{\tau_\lambda}(\tau_c - 1))\tau_r \tau_c - 1) \tag{7}$$

We can now easily calculate $\frac{\partial}{\partial \tau_c}(\frac{u_e}{c_0})^2 = 0$ that maximizes $\frac{F}{\dot{m}}$

$$\frac{\partial}{\partial \tau_c}(\frac{2}{\gamma - 1}\frac{\tau_\lambda}{\tau_r \tau_c}((1 - \frac{\tau_r}{\tau_\lambda}(\tau_c - 1))\tau_r \tau_c - 1)) = 0$$

$$-\frac{\tau_r^2}{\tau_\lambda} + \frac{1}{\tau_c^2} = 0$$

$$\tau_c = \frac{\pm\sqrt{\tau_\lambda}}{\tau_r} \qquad \text{We reject the negative solution}$$

$$\tau_c = \frac{\sqrt{\tau_\lambda}}{\tau_r} \tag{8}$$

We know that π_c and τ_c are linked by $\pi_c = \tau_c^{\frac{\gamma}{\gamma-1}}$ so we can find π_c

$$\pi_c = (\frac{\sqrt{\tau_\lambda}}{\tau_r})^{\frac{\gamma}{\gamma-1}} \tag{9}$$

1.3 Comparaison with Brayton cycle

For a Brayton cycle, the optimum compressor temperature ratio is given by :

$$\tau_c = \sqrt{\tau_\lambda} \tag{10}$$

We can see that the optimum expression is very close to the one found for the turbojet, the only difference is the presence of the ram effect τ_r in the denominator. This is logical since the Brayton cycle doesn't take into account the ram effect while the turbojet does.

2 Hybrid cycle after-burning turbojet

2.1 Specific thrust

We carry out the analysis for a turbojet equipped with an after-burner while introducing polytropic efficiencies for the turbine and compressor : η_t, η_c. We need to introfuce a new parameters to unsure the maximum temperature at the after-burner outlet : $\tau_{\lambda,AB} = \frac{T_{t,7}}{T_0}$
We derive the equation in the same fashion as for the case without after-burner. Equation 1, 4 and 5 are not modified.

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{T_e}{T_0}}M_e - M_0\right)c_0$$

$$\tau_t = 1 - \frac{\tau_r}{\tau_\lambda}(\tau_c - 1)$$

$$\tau_b = \frac{\tau_\lambda}{\tau_r \tau_c}$$

2.1.1 Find M_e

We start from find in the first part 2 and introduce the efficiencies : $\pi_c^{\frac{\gamma-1}{\gamma}} = \tau_c^{\eta_c}$ and $\pi_t^{\frac{\gamma-1}{\gamma}} = \tau_t^{1/\eta_t}$.

$$M_e^2 = \frac{2}{\gamma - 1}((\pi_r \pi_c \pi_t)^{\frac{\gamma-1}{\gamma}} - 1)$$

$$M_e^2 = \frac{2}{\gamma - 1}(\tau_r \tau_c^{\eta_c} \tau_t^{1/\eta_t} - 1) \tag{11}$$

2.1.2 Find $\frac{T_e}{T_0}$

We redo all the calculation, taking τ_{AB} into account this time.

$$\frac{T_{t,e}}{T_0} = \frac{T_{t,e}}{T_0} \frac{T_{t,2}}{T_{t,0}} \frac{T_{t,3}}{T_{t,2}} \frac{T_{t,4}}{T_{t,3}} \frac{T_{t,5}}{T_{t,4}} \frac{T_{t,7}}{T_{t,5}} \frac{T_{t,e}}{T_{t,7}}$$
$$\frac{T_{t,e}}{T_0} = \tau_r \tau_c \tau_b \tau_t \tau_{AB}$$

$$\frac{T_{t,e}}{T_0} = (\frac{p_{t,e}}{p_e})^{\frac{\gamma-1}{\gamma}} = (\pi_r \pi_c \pi_t)^{\frac{\gamma-1}{\gamma}}$$
$$\frac{T_{t,e}}{T_0} = \tau_r \tau_c^{\eta_c} \tau_t^{1/\eta_t}$$

$$\frac{T_e}{T_0} = \frac{T_{t,e}/T_0}{T_{t,e}/T_e}$$
$$\frac{T_e}{T_0} = \tau_c^{1-\eta_c} \tau_t^{1-1/\eta_t} \tau_b \tau_{AB} \tag{12}$$

2.1.3 Find τ_{AB}

$$\tau_{AB} = \frac{T_{t,7}}{T_{t,5}} = \frac{T_0}{T_{t,0}} \frac{T_{t,0}}{T_{t,2}} \frac{T_{t,2}}{T_{t,3}} \frac{T_{t,3}}{T_{t,4}} \frac{T_{t,4}}{T_{t,5}} \frac{T_{t,7}}{T_0}$$
$$\tau_{AB} = \frac{1}{\tau_r} \frac{1}{\tau_d} \frac{1}{\tau_c} \frac{1}{\tau_b} \frac{1}{\tau_t} \tau_{\lambda,AB}$$

$$\tau_{AB} = \frac{\tau_{\lambda,AB}}{\tau_\lambda \tau_t} \tag{13}$$

2.1.4 Back to specific thrust

We consider that the fuel mass flow injected in the after-burner is negligible compared to the air mass flow. We inject the terms 11 12 4 5 and 13 computed up above in the specific thrust equation 1 :

$$(\frac{u_e}{c_0})^2 = \frac{T_e}{T_0} M_e^2$$
$$(\frac{u_e}{c_0})^2 = \tau_c^{1-\eta_c} \tau_t^{1-1/\eta_t} \tau_b \tau_{AB} \frac{2}{\gamma - 1}(\tau_r \tau_c^{\eta_c} \tau_t^{1/\eta_t} - 1)$$
$$(\frac{u_e}{c_0})^2 = \frac{2}{\gamma - 1} \tau_{\lambda,AB} (1 - \frac{1}{\tau_r \tau_c{}^{\eta_c} \tau_t{}^{1/\eta_t}})$$

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{2}{\gamma-1}\tau_{\lambda,AB}(1-\frac{1}{\tau_r{\tau_c}^{\eta_c}{\tau_t}^{1/\eta_t}})} - M_0\right) c_0 \tag{14}$$

2.1.5 Specific thrust optimization

We can now easily calculate $\frac{\partial}{\partial \tau_c}(\frac{u_e}{c_0})^2 = 0$ that maximizes $\frac{F}{\dot{m}}$.

$$\frac{\partial}{\partial \tau_c}(\frac{2}{\gamma-1}\tau_{\lambda,AB}(1-\frac{1}{\tau_r{\tau_c}^{\eta_c}{\tau_t}^{1/\eta_t}})) = 0$$

$$\tau_c = \frac{\eta_c\eta_t}{1+\eta_c\eta_t}\left(\frac{\tau_\lambda}{\tau_r}+1\right) \tag{15}$$

2.2 Ramjet specific thrust

In the case of a ramjet, there is no compressor and turbine. We have $\tau_c = 1$ and $\tau_t = 1$. We can simplify the specific thrust equation found previously 14 :

$$\frac{F}{\dot{m}} = \left(\sqrt{\frac{2}{\gamma-1}\tau_{\lambda,AB}(1-\frac{1}{\tau_r})} - M_0\right) c_0 \tag{16}$$

2.3 Transition Mach number

We want to find the Mach number for which the specific thrust of the turbojet with after-burner equals the specific thrust of the ramjet. We equal equations 14 and 16 :

$$\begin{aligned}
\tau_{\lambda,AB}(1-\frac{1}{\tau_r{\tau_c}^{\eta_c}{\tau_t}^{1/\eta_t}}) &= \tau_{\lambda,AB}(1-\frac{1}{\tau_r}) \\
\frac{1}{\tau_r{\tau_c}^{\eta_c}{\tau_t}^{1/\eta_t}} &= \frac{1}{\tau_r} \\
{\tau_c}^{\eta_c}{\tau_t}^{1/\eta_t} &= 1 \\
\tau_c^{\eta_c\eta_t}\tau_t &= 1 \\
\tau_c^{\eta_c\eta_t}(1-\frac{\tau_r}{\tau_\lambda}(\tau_c-1)) &= 1 \\
\tau_\lambda\frac{1-\tau_c^{-\eta_c\eta_t}}{\tau_c-1} &= \tau_r && \text{We know that } \tau_r = 1+\frac{\gamma-1}{2}M^2
\end{aligned}$$

$$M_0 = \sqrt{\frac{2}{\gamma-1}(\tau_\lambda\frac{1-\tau_c^{-\eta_c\eta_t}}{\tau_c-1}-1)} \tag{17}$$

2.3.1 Numerical application

We consider the following parameters :

- $\gamma = 1.4$
- $\pi_c = 20$

- $T_{t,4} = 1400K$
- $T_{t,0} = 216K$
- $\eta_c = \eta_t = 0.9$
- $\tau_\lambda = \frac{T_{t,4}}{T_{t,0}} = \frac{1400}{216} = 6.48$
- $\tau_c = {\pi_c}^{\frac{\gamma-1}{\gamma}\frac{1}{\eta_c}} = 2.588$

We can now compute the transition Mach number using equation 17 :

$$\boxed{M_0 = 2.44}$$

2.4 Graphical representation

We plot the specific thrust of the turbojet with after-burner 14 and the specific thrust of the ramjet 16 as a function of the Mach number to visualize the transition Mach number computed previously. We use the same parameters as in the numerical application part.

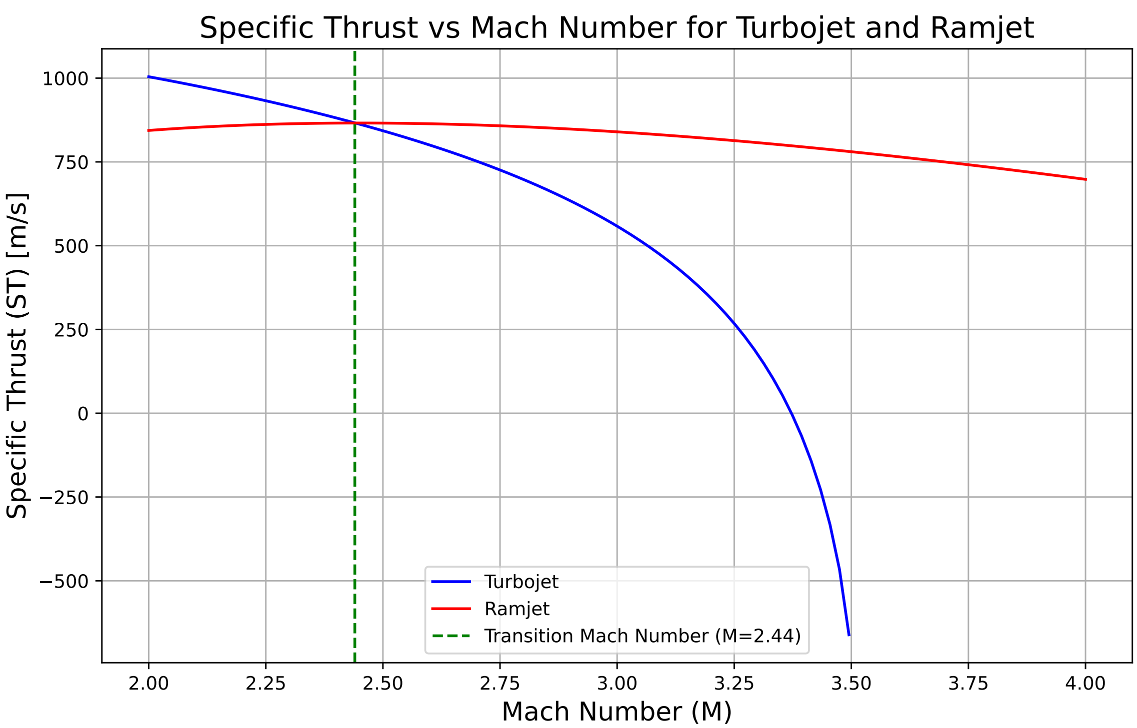

FIGURE 2 – Specific Thrust vs Mach Number for Turbojet and Ramjet

The graph shows that the transition Mach number is around 2.4, which is consistent with the value computed previously. We can also observe that the ramjet specific thrust becomes higher than that of the turbojet equipped with after-burner after this Mach number, confirming the advantage of ramjets at high speeds.
It is interesting to note that past Mach 3.5, the turbojet doesn't produce any more thrust.

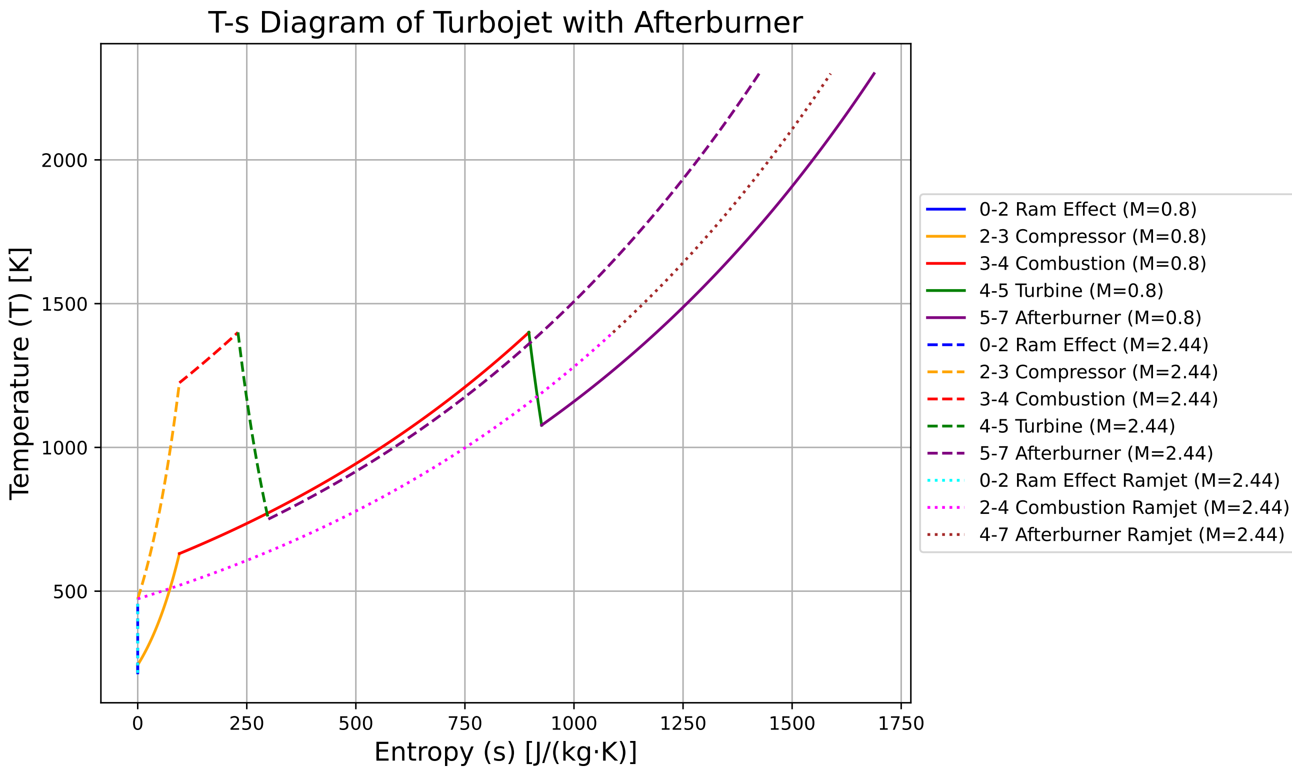

Figure 3 – T-S Diagram of Turbojet and Ramjet at M=0.8 and M=2.44

2.5 T-S Diagram

We plot the T-S diagram of the turbojet with after-burner at a subsonic speed and transition Mach number. We use the same parameters as in the numerical application part.

We can assess the different components of the engine on the T-S diagram. First, we observe that the ram effect (0-2) is more present at M=2.44 than at M=0.8, as expected since the ram effect increases with the Mach number.
We can also observe at that at M=2.44, the compressor generates large amount of heat and need a lot of work from the turbine to compress the air.
At lower Mach number (M=0.8), the temperature increases a lot in the combustion chamber, as well as the entropy while at M=2.44, the temperature and entropy increase are more moderate. This is due to the fact that at higher Mach number, the air entering the combustor is already at a higher temperature due to the ram effect and the compressor.
We can see that the turbine need to produce a huge amount of work at M=2.44 to drive the compressor compared to M=0.8. At the turbine outlet, the air is at lower temperature and much lower entropy at M=2.44 compared with M=0.8
The after-burner increases significantly the temperature and entropy of the air before the nozzle at both Mach numbers, providing additional thrust with huge fuel consumption.
On the other and, the ramjet cycle shows a more straightforward process with a continuous increase in temperature and entropy from inlet to nozzle. We can see the ram effect (0-2) followed by the combustion process (2-5). Pratically all the combustion process is "post combustion" as there is no turbine or compressor but we divide the process in two part is the same fashion as the turbojet.

2.6 Specific fuel thrust consumption

We want to analyse the specific fuel thrust consumption (TSFC) of the turbojet with after-burner. We first need to define the TSFC :

$$TSFC = \frac{\dot{m}_{f,tot}}{F} = \frac{f_{tot}}{F/\dot{m}} \tag{18}$$

We also need to define the fuel-to-air ratio f_{tot} :

2.6.1 Turbojet fuel-to-air ratio

For the burner we have :

$$\begin{aligned}
\dot{m}_0 h_{t,3} + \dot{m}_f LHV &= (\dot{m}_0 + \dot{m}_f) h_{t,4} \\
\dot{m}_0 c_p T_{t,3} + \dot{m}_f LHV &= \dot{m}_0 c_p T_{t,4} \\
f &= \frac{c_p (T_{t,4} - T_{t,3})}{LHV} \\
f &= \frac{c_p T_0 (\tau_\lambda - \tau_r \tau_c)}{LHV}
\end{aligned}$$

We assume that $\dot{m}_0 + \dot{m}_f =\approx \dot{m}_0$

$$f = \frac{\dot{m}_f}{\dot{m}_0}$$

For the after-burner we have :

$$\begin{aligned}
\dot{m}_0 h_{t,5} + \dot{m}_{f,AB} LHV &= (\dot{m}_0 + \dot{m}_{f,AB}) h_{t,7} \\
\dot{m}_0 c_p T_{t,5} + \dot{m}_{f,AB} LHV &= \dot{m}_0 c_p T_{t,7} \\
f_{AB} &= \frac{c_p (T_{t,7} - T_{t,5})}{LHV} \\
f_{AB} &= \frac{c_p T_0 (\tau_{\lambda,AB} - \tau_r \tau_c \tau_b \tau_t)}{LHV}
\end{aligned}$$

We assume that $\dot{m}_0 + \dot{m}_{f,AB} =\approx \dot{m}_0$

$$f_{AB} = \frac{\dot{m}_{f,AB}}{\dot{m}_0}$$

$$\frac{T_{t,5}}{T_0} = \frac{T_{t,0}}{T_0} \frac{T_{t,2}}{T_{t,0}} \frac{T_{t,3}}{T_{t,2}} \frac{T_{t,4}}{T_{t,3}} \frac{T_{t,5}}{T_{t,4}}$$

The total fuel-to-air ratio is :

$$f_{tot} = f + f_{AB} = \frac{c_p T_0 (\tau_\lambda - \tau_r \tau_c)}{LHV} + \frac{c_p T_0 (\tau_{\lambda,AB} - \tau_r \tau_c \tau_b \tau_t)}{LHV} \tag{19}$$

2.6.2 Ramjet fuel-to-air ratio

For the ramjet we have :

$$\begin{aligned}
\dot{m}_0 h_{t,2} + \dot{m}_f LHV &= (\dot{m}_0 + \dot{m}_f) h_{t,7} \\
\dot{m}_0 c_p T_{t,2} + \dot{m}_f LHV &= \dot{m}_0 c_p T_{t,7} \\
f &= \frac{c_p (T_{t,7} - T_{t,2})}{LHV} \\
f &= \frac{c_p T_0 (\tau_{\lambda,AB} - \tau_r)}{LHV}
\end{aligned}$$

We assume that $\dot{m}_0 + \dot{m}_f =\approx \dot{m}_0$

$$f = \frac{\dot{m}_f}{\dot{m}_0}$$

2.6.3 Numerical application

We have plotted the TSFC of the turbojet with after-burner and the ramjet as a function of the Mach number using the same parameters as in the previous part.

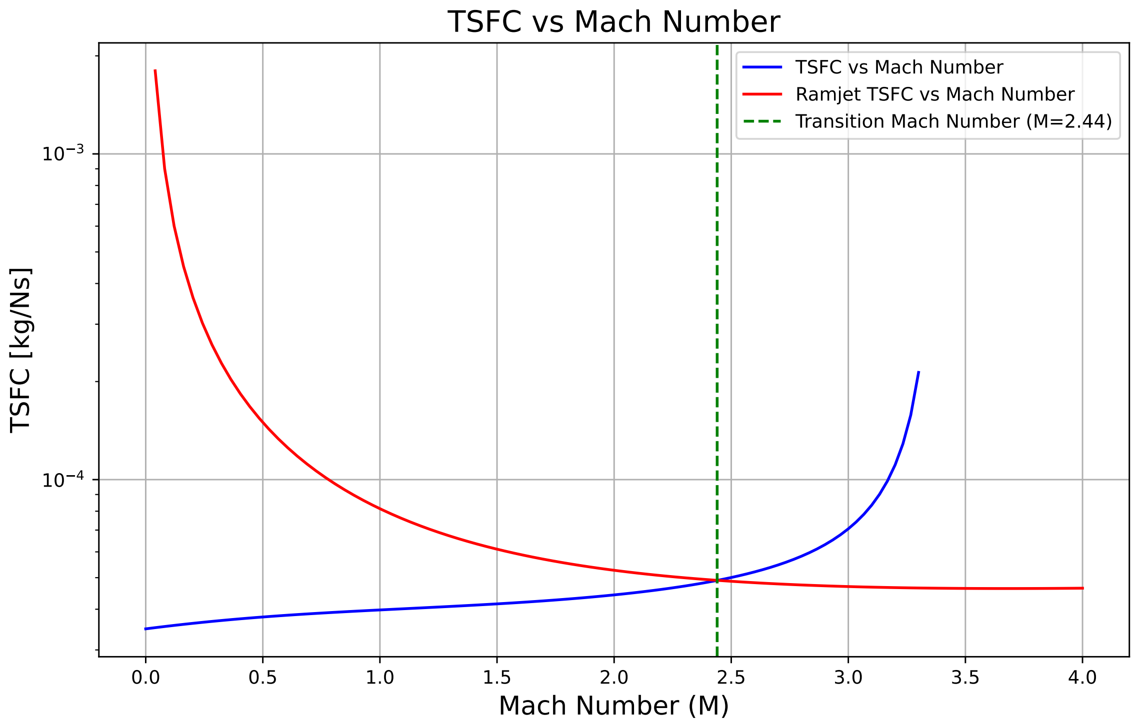

FIGURE 4 – TSFC vs Mach Number for Turbojet and Ramjet

We observe that the TSFC of the ramjet is lower than that of the turbojet with after-burner for Mach numbers higher than the transition Mach number (M=2.44). This confirms the advantage of ramjets at high speeds.
We can also observe that the TFSC of the turbojet with after-burner increases significantly with the Mach number, indicating that the fuel consumption becomes less efficient at higher speeds. On the other hand, the TSFC of the ramjet decreases then remains relatively constant across the high Mach number range.

3 Off-design analysis of turbojet

We have assumed until now that the engine was operating at design conditions. We want to analyse the performance of the turbojet when operating off-design conditions.
We consider a turbojet with a design inlet total temperature ratio, $\theta_{0,R} = 1.3$.
First we need to define the off-design inlet total temperature ratio :

$$\theta_0 = \frac{T_{t,0}}{T_{SL}} = \frac{T_0}{T_{SL}}\tau_r = \frac{T_0}{T_{SL}}(1 + \frac{\gamma - 1}{2}M_0^2) \tag{20}$$

We have $\theta_{0,R} = \frac{T_{t,0R}}{T_{SL}} = \frac{T_0}{T_{SL}}\tau_{r,R}$

3.1 Effect of off-design on τ_c and τ_λ

First, we recall that at θ_0 we have :

$$\tau_c = 1 + \frac{T_{t,4}}{\theta_0 T_{SL}}(1 - \tau_{t,R}) \tag{21}$$

$$\tau_\lambda = \frac{T_{t,4}}{T_{t,0}} = \frac{T_{t,4}}{\theta_0 T_{SL}} \tag{22}$$

3.1.1 Find $\tau_{t,R}$

We need to find $\tau_{t,R}$ to compute τ_c at off-design conditions. We start from the equilibrium between the compressor and the turbine at design conditions 4 :

$$\tau_{t,R} = 1 - \frac{\tau_{r,R}}{\tau_{\lambda,R}}(\tau_{c,R} - 1) \qquad \text{At design point we have that : } \tau_{r,R} = \theta_{0,R}\frac{T_{SL}}{T_0}$$

$$\tau_{t,R} = 1 - \frac{\theta_{0,R}\frac{T_{SL}}{T_0}}{\tau_{\lambda,R}}(\tau_{c,R} - 1) \qquad \text{At design point : } \tau_{\lambda,R} = \frac{T_{t,4}}{T_{t,0R}} = \frac{T_{t,4}}{\theta_{0,R} T_{SL}}$$

We consider that $\tau_{c,R} = \pi_{c,R}^{\frac{\gamma-1}{\gamma\eta_c}}$ with $\pi_{c,R}$ the compression ratio defined previously.

3.1.2 Numerical application

We have plotted τ_c and τ_λ as a function of the Mach number using the same parameters as in the previous part.

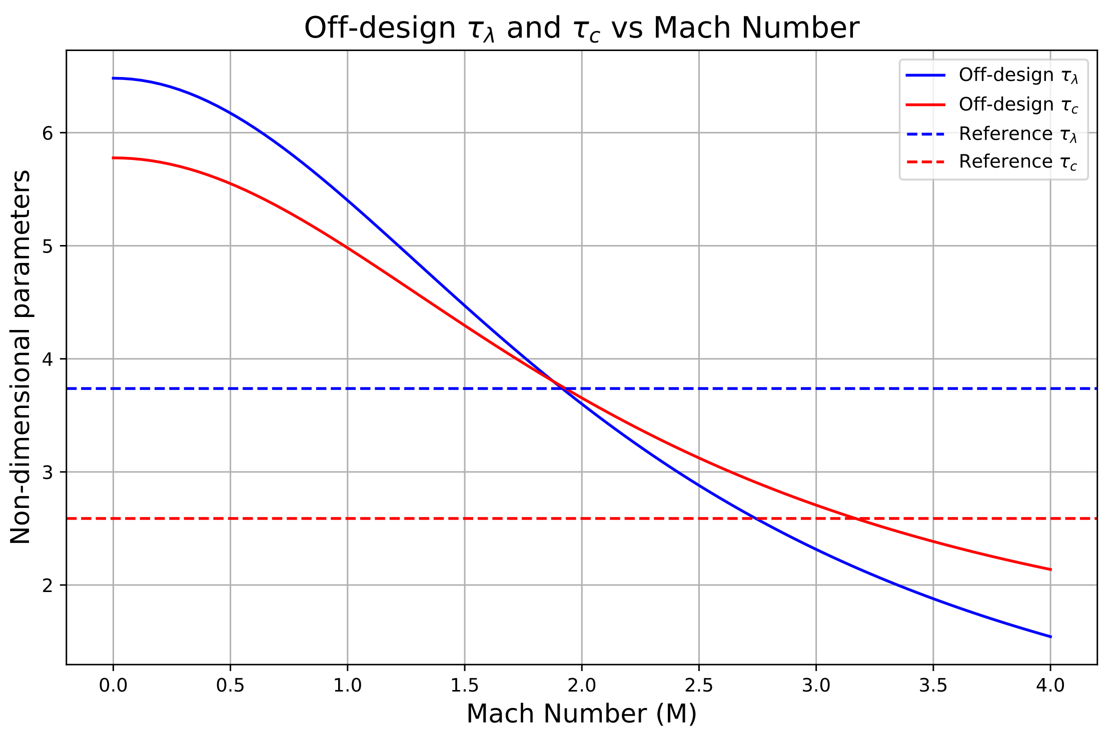

FIGURE 5 – τ_c and τ_λ vs Mach Number for Turbojet