

# Review of the Inverse $\mathbb{Z}$ -Transform in Discrete Option Pricing

Final Year Project Report

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<sup>1</sup>**Disclaimer:** This report is submitted as part requirement for the MEng degree in Mathematical Computation at UCL. It is substantially the result of my own work except where explicitly indicated in the text. The report may be freely copied and distributed provided the source is explicitly acknowledged.

## **Abstract**

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# Chapter 1

## Introduction

### 1.1 Motivation

We attribute the first recorded option contract to that of the Greek philosopher Thales of Miletus (Thompson, 1994). Believing that the olive harvest would be plentiful, Thales secured the right to use the olive presses at a low price. This agreement is considered a call option as it gives the holder the right, but not the obligation, to execute the contract at a predetermined price.

The formal study of pricing options began much later with Bachelier (1900)'s work in modelling stock prices as a Brownian motion and his derivation of an option pricing formula. However, we deem the seminal work of Black and Scholes (1973) in laying out the foundation for modern option pricing theory. The introduction of the Black-Scholes model coincided with the establishment of the Chicago Board Options Exchange (CBOE) in 1973, which standardised option contracts and provided a platform for trading. The rapid growth (Fig 1.1) in the options market sparked a demand and increased interest in option pricing research.

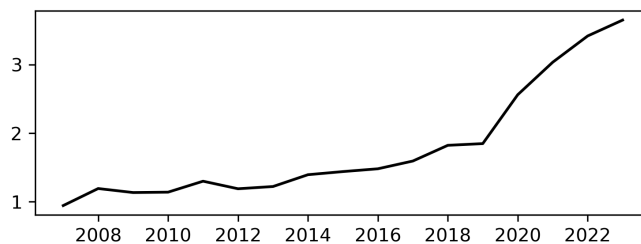


Figure 1.1: Visualisation of the volume of options (in billions) traded over the past 17 years on exchanges CBOE, BATS, C2 and EDGX.

However, the assumptions of the Black-Scholes model does not lend itself to an accurate representation of real-world asset price data. Standard (vanilla) options are the most basic and commonly traded type of options which give the holder the opportunity to buy or sell the asset at a later point in time. Exotic options differ in terms of their structure, payoff and underlying asset to meet specific investor needs or market conditions. Such an example are path-dependent options in which the payoff is determined by the path of the underlying asset price over the life of the option, rather than just the price at expiration. The price of the underlying asset is monitored at discrete points in time. One could say that in practice most, if not all, path-dependent options in markets are discrete path-dependent options (Kou, 2007).

The pricing of discrete path-dependent options has been a longstanding area of research in the financial mathematics literature. The importance and value of these options results in a vast array of research including that around the Black-Scholes model assumption (Lu and Jin, 2017; Guardasoni et al., 2020), general Lévy processes (Fang and Oosterlee, 2009a; Fusai et al., 2016; Phelan, 2018; Chen et al., 2021; Boyarchenko and Levendorskiĭ, 2022), general Markov processes (Cui and Taylor, 2021; Zhang and Li, 2023) and stochastic volatility (Soleymani and Barfeie, 2019; Kirkby and Nguyen, 2020).

A common problem faced by those in the literature is the linear dependency on the monitoring dates of the underlying asset price. Fusai et al. (2016) demonstrated that the iteration on the monitoring dates can be avoided by working in the  $z$ -domain, extending on work by that of Carr and Madan (1999). Applying the Spitzer (1957) identities and solving the resulting equations requires an inverse operation to obtain the option price. Reverting from the  $z$ -domain to the time domain requires a numerical approximation of the inverse  $z$ -transform. Despite its relation to the Fourier transform, the literature lacks in comparison; the performance of existing approximations, in terms of accuracy and efficiency, necessitates a specific setup and choice of parameters.

## 1.2 Aims and Objectives

Given the importance of the numerical inverse  $z$ -transform (NIZT), the focus of this project will be **(i)** studying the effect of the parameters on the varying methods, and **(ii)** testing on a group of well-defined transform pairs. Thus, our aim is to try and discover which methods provide the best fit in terms of efficiency and accuracy and what is the best setup to achieve this. The measurable objectives (O1-O4) associated with this aim are as follows:

- (O1) Analyse the existing NIZT methods in literature
- (O2) Research Boyarchenko and Levendorskiĭ's new NIZT method
- (O3) Find the optimal parameter setup for each method
- (O4) Test the implementations against well-defined transform pairs

## 1.3 Overview

The upcoming chapters of this report aims to provide a detailed description of the different stages taken through this project. In Chapter 2, we provide a self-contained technical background on the key concepts and methods used in this project. Given the breadth of the topic, references are included for those looking to delve deeper into the subject. In Chapter 3, we outline the experiments conducted in Chapter 4 to evaluate the performance of the numerical approximation methods. We detail the implementation of the methods and the experimental setup, including parameter selection and the benchmarking process. We then provide the results on a list of transform pairs and discuss the implications of the findings. Finally, Chapter 5 summarises the steps taken and the findings of this project with suggested work for further continuation.

## Chapter 2

# Background

In Chapter 2, we establish a foundational understanding of the topic in hand. This section is designed to be self-contained, providing essential background for all readers, while references are included for those seeking a deeper exploration.

We conduct an analysis into the mathematical concepts used in option pricing, with a focus on Fourier-based methods. While a large body of theoretical work exists in this area, the practical implementation of these methods is often computationally expensive, especially when evaluating high-dimensional integrals involved in the pricing formulas. We study literature on numerical approximation methods and the application in the pricing of exotic options, specifically lookback and barrier options. Subsequently, we explore the use-case of machine learning techniques to optimise parameters during our experimentation in Chapter 3.

### 2.1 The $\mathcal{Z}$ -Transform

The  $z$ -transform is a transformation of a real or complex time function  $x(n)$ , often used for analysing discrete-time signals and systems. It is a generalisation of the discrete-time Fourier transform (DTFT) that extends the analysis to the complex plane. The bilateral  $Z$ -transform is formally defined as:

$$X(z) = \mathcal{Z}_{n \rightarrow z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (2.1)$$

“By definition, a complex number  $z$  is an ordered pair  $(x, y)$  of real numbers  $x$  and  $y$ , written  $z = (x, y)$ ” (Kreyszig, 2010). In practice, complex numbers are written in the form  $z = x + iy$ , where  $x$  and  $y$  are real numbers and  $i$  is the imaginary unit. We may find it easier to represent complex numbers in their polar form,  $z = re^{i\theta}$ , where  $r$  represents the magnitude of  $z$  and  $\theta$  represents the angle of  $z$  with respect to the positive real axis.

In the analysis of causal systems - systems for which a time origin is defined and is illogical to consider signal values for negative time - the unilateral  $z$ -transform is used. Unlike the bilateral  $z$ -transform in Eq. (2.1), we sum from zero to positive infinity yielding

$$X(z) = \mathcal{Z}_{n \rightarrow z}[x(n)] = \sum_{n=0}^{\infty} x(n)z^{-n} \quad (2.2)$$

The region within the complex  $z$ -plane where the  $z$ -transform converges is known as the Region of Convergence (ROC). The ROC is defined for the set of values of  $z$  for which the  $z$ -transform is

absolutely summable

$$\mathbf{ROC} = \left\{ z : \sum_{n=0}^{\infty} |x(n)z^{-n}| < \infty \right\} \quad (2.3)$$

For causal sequences, the ROC is typically the exterior of the outermost pole in the  $Z$ -plane, denoted as  $|z| > a$ . If we say that  $z_1$  converges, then  $z_1$  exists within the ROC. Thus, all  $z$  such that  $|z| \geq |z_1|$  also converge. This region excludes the poles themselves, as the transform does not converge at those points. For the system to be *stable*, the ROC must include the unit circle,  $|z| = 1$ , implying that all poles must lie within the unit circle (Loveless and Germano, 2021).

**Example 1** Consider the function given by

$$H(z) = \frac{(z-i)(z+i)}{\left(z - \left(-\frac{1}{4} - \frac{1}{2}i\right)\right)\left(z - \left(\frac{1}{2} + \frac{1}{2}i\right)\right)} \quad (2.4)$$

The ROC of  $H(z)$  must exclude the poles at  $z = -\frac{1}{4} - \frac{1}{2}i$  and  $z = \frac{1}{2} + \frac{1}{2}i$ , thus the ROC is  $|z| > \frac{1}{2}$ . The system is stable as the ROC includes the unit circle.

### 2.1.1 Relation to the Fourier Transform

It is useful to note the relationship between the  $z$ -transform and the Fourier transform. Taking the Fourier transform of a sampled function  $x(t)$  results in:

$$\mathcal{F}\left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)\right] = \int_{-\infty}^{\infty} x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) e^{-i\omega t} dt \quad (2.5)$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) \delta(t - n\Delta t) e^{-i\omega t} dt \quad (2.6)$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta t) e^{-i\omega n\Delta t} \quad (2.7)$$

where we make use of the sifting property of the delta function. If we normalise the sampling interval to 1, we get

$$\sum_{n=-\infty}^{\infty} x(n) e^{-in\omega} \quad (2.8)$$

This is the discrete-time Fourier transform (DTFT) of the sequence  $x(n)$ . The sequence  $x(n)$  is sampled at discrete-time intervals  $t_n = n\Delta t$ , where the sampling interval  $\Delta t$  is the time between consecutive samples and the time index  $n$  numbers the samples. The DTFT is a periodic function of  $\omega$  with period  $2\pi$ , and its existence relies on the absolute summability of the sequence  $x(n)$ :

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \quad (2.9)$$

The  $Z$ -transform generalises Eq. (2.8) to the complex plane, not just the unit circle where  $r = 1$  (Schafer and Oppenheim, 1989).

### 2.1.2 Relation to Probability Distribution Functions

Random events and signals refer to situations where the outcome is not deterministic, but can be described by probability. Understanding these concepts involves using Probability Distribution Functions; the Probability Mass Function (PMF) for discrete random variables and the Probability Density Function (PDF) for continuous random variables. Given the nature of this project, we'll be focusing our attention on the PMF.

The PMF is defined for a discrete random variable  $X$  taking on values  $x_i$  with probabilities  $p_i$ , as  $P(X = x_i) = p_i$ . The PMF satisfies the following properties:

$$\sum_{i=0}^n p_i = 1 \quad \text{and} \quad 0 \leq p_i \leq 1 \quad \forall i \quad (2.10)$$

We may find it useful to provide a concise representation of the entire distribution such that we expand upon the PMF,  $p(x)$ , to obtain the Probability Generating Function (PGF),  $G_X(q)$ , defined as

$$G_X(q) = E[q^X] = \sum_{x=0}^{\infty} p(x)q^x, \quad (2.11)$$

where  $E[\cdot]$  denotes the expectation operator, and  $q$  is a complex number. We deliberately use  $q$  to distinguish the PGF from the  $z$ -transform (Eqn. 2.1).

**Example 2** Consider a fair six-sided dice. The PMF for the dice roll is given by

$$p(x) = \begin{cases} \frac{1}{6} & \text{if } x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases} \quad (2.12)$$

where  $p(x)$  is the probability of rolling a number  $x$ . The PGF for the dice roll is then

$$G_X(q) = \sum_{x=0}^{\infty} p(x)q^x = \frac{1}{6} \sum_{x=1}^6 q^x = \frac{q}{6} \cdot \frac{1 - q^6}{1 - q}, \quad (2.13)$$

where we use the formula for the sum of a geometric series. The PGF encapsulates the entire distribution of the dice roll into a single function.

The concept of summarising information is not unique to probability theory. In the analysis of signals, we aim to encapsulate the behaviour of a sequence into a single function. This is akin to the PGF, where the  $z$ -transform is used to analyse discrete-time signals and systems. Drawing on the principles outlined by Ross (2014), we can bridge the gap between probability theory and signal processing, leveraging the  $z$ -transform to analyse the behaviour of signals in the complex plane.

## 2.2 The Inverse $\mathcal{Z}$ -Transform

The inverse  $Z$ -transform aims to retrieve the sequence  $x(n)$  from its  $Z$ -transform  $X(z)$ . While this transform is typically defined for finding individual values of  $n$ , it can also be extended to calculate the entire sequence. This is commonly defined by the Cauchy integral around a contour  $C$  in the complex plane. The chosen contour  $C$  is a counter-clockwise closed path that encloses the region of convergence (ROC) of  $X(z)$ . Formally, the inverse  $Z$ -transform is given by



$$x(n) = \mathcal{Z}_{z \rightarrow n}^{-1}[X(z)] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz \quad (2.14)$$

The integral approach enables the extraction of the complete sequence  $x(n)$  for all integer values  $n$ , provided the ROC and the properties of  $X(z)$  allow. In practical settings, numerical approximations are often used due to the computational challenges of evaluating the Cauchy integral. Whilst there are many ways to go about this (Merrikh-Bayat, 2014; Rajković et al., 2004; Horváth et al., 2020), most methods are done on a circular contour. Aligning with the focus of our project, we shift our attention to contour integrals.

### 2.2.1 Abate and Whitt 1992

The numerical approximation formula offered by Abate and Whitt (1992a,b) is based on a Fourier series catering to the inversion of probability generating functions as elucidated in Section 2.1.2. The format is conducive to queuing theory and other probabilistic models where the  $Z$ -transform is defined as  $q = 1/z$ . The authors approximate the inversion using a trapezoidal rule for numerical integration over a complex contour given by

$$x(n) \approx \frac{1}{2nr^n} \left( X(r) + 2 \sum_{k=1}^{n-1} (-1)^k \operatorname{Re} \left( X(re^{\frac{ik\pi}{n}}) \right) + (-1)^n X(-r) \right) \quad (2.15)$$

The parameter  $r$  is used to control the error; setting  $r = 10^{-\lambda/2n}$  yields an accuracy bound of  $10^{-\lambda}$ . The authors leverage the inherent symmetry within the complex plane to enhance computational efficiency by exploiting the complex conjugate symmetry of  $X(z)$ ; each term  $X(re^{\frac{ik\pi}{n}})$  in the upper half has a mirror image in the lower half. The computational load is thus halved by *folding* the problem in this manner.

Given the nature of this project, we may find it easier to use the following definition, where we set the input of  $X(z)$  to that of  $z = 1/q$ , to approximate Eq. (2.14).

$$x(n) \approx \frac{1}{2nr^n} \left( X\left(\frac{1}{r}\right) + 2 \sum_{k=1}^{n-1} (-1)^k \operatorname{Re} \left( X\left(\frac{1}{re^{\frac{ik\pi}{n}}}\right) \right) + (-1)^n X\left(-\frac{1}{r}\right) \right) \quad (2.16)$$

The Nyquist-Shannon sampling theorem states that a signal must be sampled at a rate of at least twice the highest frequency present in the signal to avoid aliasing (Shannon, 1949; Nyquist, 1928). The number of points,  $n$ , used in Eq. (2.16) must be sufficient to capture the information properly. If  $n$  is too small, the approximation may lead to inaccuracies - akin to aliasing in signal processing.

### 2.2.2 Cavers 1978

Extending upon our analysis in Section 2.1.1, Cavers (1978) proposes to sample the  $z$ -transform of a function on a circular contour at equally spaced points and then apply the inverse fast fourier transform (IFFT) to these sampled points to approximate the original time-domain signal. The DTFT suffers from a high computational cost with a time complexity of  $O(N^2)$  given the need to perform  $O(N)$  operations for each of the  $N$  different points. Thus, the author employ a more efficient algorithm by Cooley and Tukey (1965) to what is known as the fast fourier transform (FFT). We can formulate this as:

$$f(n) = r^n \operatorname{IFFT}[f(re^{2\pi ik/N})], \quad (2.17)$$

which can be rewritten into the following:

$$f(n) = \frac{r^n}{N} \text{FFT}[f(re^{2\pi ik/N})], \quad (2.18)$$

where  $r$  denotes the radius of the contour,  $n$  is the time index, and  $N$  is the number of points within the secondary grid. The factor  $r^n$  scales the result appropriately based on the radius of the contour. Loveless and Germano (2021) show that an advantage of being able to increase  $N$  points results in an error tolerance of E-16, *i.e. machine zero*, with double precision; note that this requires some optimisation of the

mention  $r = 10^{**}(\text{gamma} / N)$  and that  $N$  has to be bigger than  $n$

### 2.2.3 Series acceleration techniques

if we have time, otherwise remove this section

## 2.3 Optimisation Techniques

In the context of computational mathematics, optimisations techniques are used to identify the optimal or a sufficiently effective solution to a problem within a given set of constraints. The goal is to minimise or maximise a specific objective function by systematically choosing the values of the variables. The objective function is often referred to as the *cost function* or *loss function* and the variables are referred to as *parameters*. The optimisation problem can be formulated as

$$\text{minimise } f(x) \text{ subject to } x \in \Omega, \quad (2.19)$$

where  $f(x)$  is the objective function and  $\Omega$  is the feasible region defined by the constraints of the problem.

Gradient descent is one of the most popular algorithms for parameter optimisation with success in Deep Learning and Neural Networks employing variants of the algorithm (Lu and Jin, 2017; Zhang, 2019; Zeebaree et al., 2019). The adaptability to diverse problem domains (Tian et al., 2023) parallels our use case, where gradient descent is applied outside traditional deep learning to optimise parameters of a mathematical function (Persson et al., 2022).

### 2.3.1 Gradient Descent

Gradient descent iteratively converges to a local minimum of a function by moving in the direction of the steepest descent, as defined by the negative gradient. This method is expressed mathematically as

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k), \quad (2.20)$$

where  $x_k$  is the parameter vector at iteration  $k$ ,  $\alpha_k$  is the learning rate, and  $\nabla f(k)$  represents the gradient of the function at  $x_k$ . The selection of  $\alpha_k$  determines the size of the step taken towards the minimum; too large can overshoot the minimum, too small can result in a long convergence time. The process repeats until a predetermined termination criterion is met, typically when the change in the value of  $f(k)$  falls below a threshold. This iterative process is showcased in the pseudocode below:

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**Algorithm 1** Gradient Descent

---

```
1: Initialise  $x_0$ , set  $k = 0$ 
2: while termination conditions not met do
3:   Compute gradient  $\nabla f(x_k)$ 
4:   Choose a suitable step size  $\alpha_k$ 
5:   Update  $x_{k+1} = x_k - \alpha_k \nabla f(x_k)$ 
6:    $k = k + 1$ 
7: end while
```

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**Stochastic Gradient Descent**

However, classic Gradient Descent faces limitations, including susceptibility to local minima and potential for overshooting or long convergence times. Stochastic Gradient Descent (SGD) addresses these issues by introducing variability in the optimisation process. It modifies Eq. (2.20) to use a randomly selected subset of data to compute the gradient, to allow for dynamic adjustment of the learning rate and leveraging noise to escape local minima. We define the update rule to

$$x_{k+1} = x_k - \alpha_k \nabla f_{i_k}(x_k) \quad (2.21)$$

where  $\nabla f_{i_k}(x_k)$  is the gradient of the cost function with respect to a random subset  $i_k$ . We thus avoid the pitfalls associated with a static learning rate and promote a quicker convergence time.

## 2.4 Option Pricing

The concept involves determining the value of options, which are financial contracts that give the holder the right, *but not the obligation*, to buy or sell an asset at a set price,  $K$  (strike price), within a specified time-frame,  $T$  (expiration date). The value of an option,  $V$ , is derived from the underlying asset, denoted as  $S$ , which can be a stock, bond, or commodity. This relationship can be mathematically expressed as:

$$V = f(S, K, T, \sigma, r), \quad (2.22)$$

where  $\sigma$  represents the volatility of the underlying asset<sup>1</sup> and  $r$  is the risk-free interest rate<sup>2</sup>.

A pivotal point in option pricing was the introduction of Black and Scholes (1973)'s model in estimating the price of European-style options, which can only be exercised at the expiration date. The Black-Scholes model is based on the assumption that the price of the underlying asset follows a geometric Brownian motion in idealistic conditions. However, many options traded in real markets are American-style, allowing the holder to exercise the option at any time before the expiration date. This complicates the pricing process as it involves solving an *optimum stopping problem*. Merton (1973) extends the Black-Scholes model to American options by expressing the price as the solution to a free boundary problem. While Merton's work provided a theoretical foundation, solving the free boundary problem analytically is challenging. Instead, numerical methods such as binomial trees (Cox et al., 1979) and simulation-based methods (Longstaff and Schwartz, 2001) have been developed to price American options.

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<sup>1</sup>A measure of how much the price of the asset is expected to fluctuate over a certain period of time.

<sup>2</sup>The interest rate at which an invest can lend money with virtually no risk of loss.

### 2.4.1 Path-Dependent Options

We regard American and European options as standard (*vanilla*) options. In contrast, exotic options offer more complex features and conditions that can be tailored to meet specific needs and/or requirements. Such differences are the reason as to why exotic options are traded over-the-counter (OTC)<sup>1</sup>, whereas standard options are typically traded on organised exchanges<sup>2</sup>.

In most cases, the payoff of an option depends on the price of the underlying asset at discrete points in time rather than continuously. This is known as **discrete monitoring**. Two widely traded types of discretely monitored options are lookback and barrier options (Dadachanji, 2015). These options are classified as *path-dependent options* where the payoff is based on the path of the underlying asset price rather than just the price at expiration.

#### Lookback Options

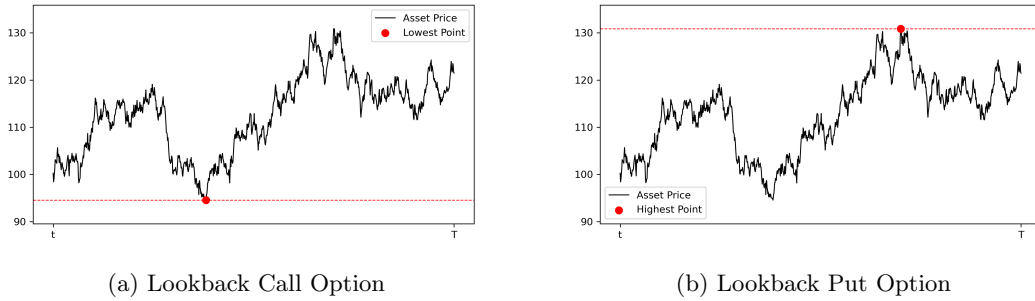


Figure 2.1: Illustration of Lookback Options: Displaying the mechanics of (a) Call Option and (b) Put Option. The red line represents the barrier, while the red dot marks the strike price.

The payoff depends on the maximum and minimum asset price over the life of the option. A *lookback call option* (Figure 2.1a) gives the holder the right to buy the asset at the lowest price during the option period, while a *lookback put option* (Figure 2.1b) allows the holder to sell the asset at the highest price during the option period.

<sup>1</sup>Over-The-Counter refers to the process of trading of financial instruments directly between two parties without the oversight of a formal exchange.

<sup>2</sup>Organised exchanges refer to formal marketplaces that facilitate the buying and selling of financial instruments.

## Barrier Options

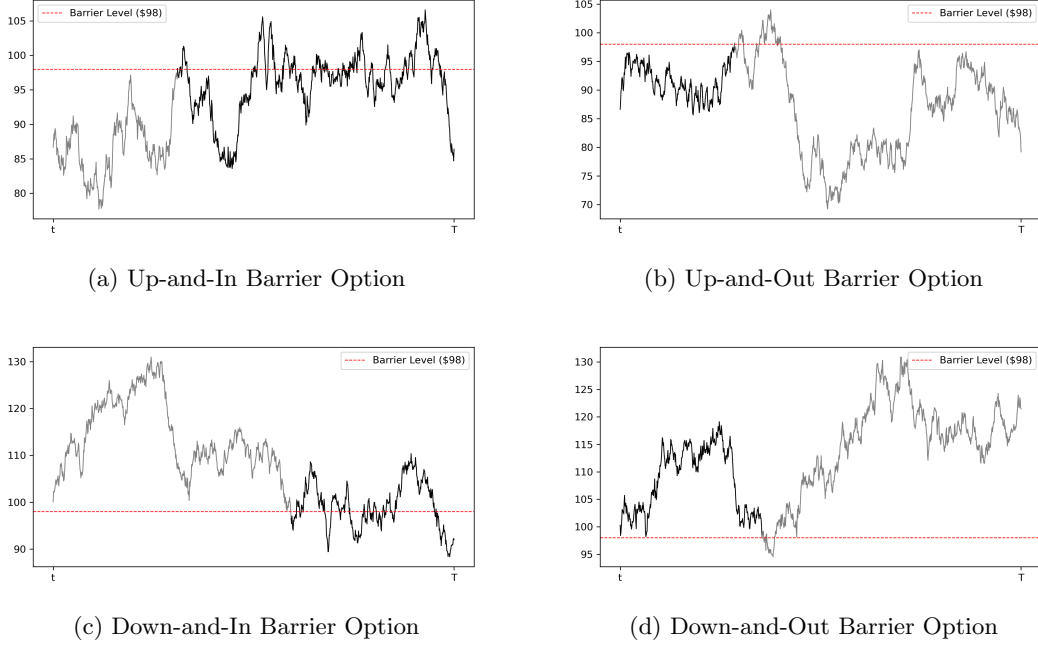


Figure 2.2: Illustration of Barrier Options: Displaying the active (black) and inactive (grey) phases of each option type. (a) Up-and-In Option, (b) Up-and-Out Option (c) Down-and-In Option, and (d) Down-and-Out Option. The red line represents the barrier.

The payoff depends on whether the price of the underlying asset reaches a certain level (the barrier) during the life of the option. A *knock-in* barrier option only come into existence if the barrier has been touched, while a *knock-out* barrier option ceases to exist instead. For example, a *down-and-out* barrier option (Figure 2.2d) is a type of knock-out option that becomes worthless if the price of the underlying asset falls below the barrier level. On the other hand, a *down-and-in* barrier option (Figure 2.2c) is a type of knock-in option that only becomes active if the asset price falls below the barrier level.

### Double Barrier Options

Double barrier option involve two price levels: an upper and lower barrier that define the boundaries for the option's activity. These options can either be *knock-in* or *knock-out*. A double knock-in option becomes active only if the asset price hits either barrier, similar to a combination of Figure 2.2a and Figure 2.2c. Conversely, a double knock-out option ceases to exist if the asset price touches either barrier combining the characteristics of Figure 2.2b and Figure 2.2d.

### 2.4.2 Modelling Asset Prices

Accurate models allow traders, investors and risk managers to assess the fair value of financial products and make informed decisions. The Black-Scholes model assumes that the price of the underlying asset follows a geometric Brownian motion (GBM). Under this assumption, the asset price  $S(t)$  evolves according to the stochastic differential equation:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad (2.23)$$

where  $\mu$  represents the expected return, and  $W(t)$  is a standard Wiener process. The solution to this equation gives the asset price at time  $t$  as:

$$S(t) = S(0)e^{(\mu - 0.5\sigma^2)t + \sigma W(t)} \quad (2.24)$$

The model assumes that the volatility is constant, which is rarely the case as market volatility tends to fluctuate over time due to economics conditions or market sentiment. Additionally, the model doesn't account for jumps or sudden large movements in asset prices, usually seen at times of significant news. These limitations have led to the development of more sophisticated models; given the scope and background of this project, we primarily focus on exponential Lévy processes.

### Exponential Lévy Processes

We extend Eq. (2.23) by incorporating jump processes to account for sudden changes in asset prices. A stochastic process  $X(t)$  is termed a Lévy process if **(i)** the changes in the process are independent of each other, **(ii)** the distribution of the changes in the process depends only on the length of the time interval, not on the starting point, and **(iii)** it starts at zero,  $X(0) = 0$ . The dynamics of a Lévy process can be described by the following stochastic differential equation:

$$dX(t) = \mu dt + \sigma dW(t) + dJ(t), \quad (2.25)$$

where  $J(t)$  represents a jump process (typically a Poisson process) to account for sudden, significant changes. An exponential Lévy process describes the asset price process as:

$$S(t) = S(0)e^{X(t)} \quad (2.26)$$

As the characteristic function of exponential Lévy models are often available in closed form, which facilitates the application of Fourier-based methods, they have been widely used in the pricing of path-dependent options as shown by the works of Fusai et al. (2016); Kwok et al. (2011); Feng and Linetsky (2008); Phelan et al. (2019).

### 2.4.3 Fourier-Based Pricing

The pricing of these options can be computationally expensive due to the high-dimensional integrals involved in the pricing formulas. Whilst Heston (1993) explored the idea of modelling asset prices in the Fourier domain, Carr and Madan (1999) were the first to price European options expressing both the characteristic function and the payoff in the Fourier domain. It involved transforming the pricing problem from the time domain to the frequency domain, where the pricing formula simplifies to a one-dimensional integral. The Fourier-based methods have been widely used for the pricing of options discussed in Section 2.4.1 (Eberlein et al., 2010).

Fang and Oosterlee (2009a) introduced a pricing technique based on the Fourier-cosine expansion approximating the characteristic function of the underlying asset price, which was then expanded to a broader class of exotic options (Fang and Oosterlee, 2009b, 2011) and general Lévy processes (Lord et al., 2008). Whilst the method shows high levels of accuracy with exponential error convergence on the number of terms, this is only the case when the governing probability density function is sufficiently smooth. When modelling the asset price by the variance gamma process (Madan et al., 1998), we achieve only algebraic error convergence due to the discontinuities in the process. Ruijter et al. (2015) showed that this can be remedied using spectral filters

however the issue remains in that the computational time is linearly dependent on the number of monitoring dates for discretely monitored options. This seems to also be the case for Feng and Linetsky (2008)'s employment of the Hilbert transform using backward induction to price barrier options.

In an effort to eliminate the computational inefficiency tied to the number of monitoring dates, Fusai et al. (2006) looked into shifting the pricing problems from the time domain into the  $z$ -domain, where convolution operations become straightforward multiplications. This helped set the stage for Fusai et al. (2016)'s method incorporating the Hilbert and  $z$ -transform to calculate the Wiener-Hopf factors, which decompose the characteristic function of the price process into two parts; positive and negative jumps. The authors demonstrated that the computational cost is independent of the number of monitoring dates, and the error decays exponentially with the number of grid points. Whilst extensions to the method have been made such as those by Phelan (2018)'s new scheme using spectral filters to improve convergence (Phelan et al., 2019), we shift our focus to the use-case of the inverse  $z$ -transform in the pricing of options.

#### 2.4.4 NIZT in Option Pricing

We consider the case that of a double barrier option (Section 2.4.1), which are characterised by their dependency on the asset's price staying within a specified range between an upper boundary  $u$  and a lower boundary  $l$ . The probability of the asset's price being at  $x$  at the  $n$ th monitoring without hitting these barriers is represented by  $f(x, n)$ . This forms the basis of our recursive relationship for these options:

$$\int_l^u f(x', n-1)p(x-x', \Delta t) dx', \quad (2.27)$$

where  $p(x-x', \Delta t)$  is the transition density between the states from  $x'$  to  $x$  over a small time increment  $\Delta t$ . The integral expression represents the transition probabilities convolved with the state probabilities from the previous time step, under the condition that the asset remains within the barriers  $u$  and  $l$ . We expand this relationship across all monitoring periods as follows:

$$\sum_{n=1}^{\infty} q^n f(x, n) = \sum_{n=1}^{\infty} q^n \int_l^u f(x', n-1)p(x-x', \Delta t) dx' \quad (2.28)$$

We notice the similarity to that of Eq. (2.2), however we set  $z^{-1} = q$  to remain consistent with that in literature. By shifting the index  $m = n - 1$ , the expression simplifies to

$$= q \int_l^u \sum_{m=0}^{\infty} q^m f(x', m)p(x-x', \Delta t) dx' \quad (2.29)$$

This manipulation restructures the series into a form amenable to the use of transitioning to the  $z$ -domain. We define the  $z$ -transform of  $f(x', n)$  as

$$\tilde{f}(x', q) = \mathcal{Z}_{n \rightarrow q}[f(x', n)] = \sum_{n=0}^{\infty} q^n f(x', n) \quad (2.30)$$

This encapsulation into  $\tilde{f}(x', q)$  allows us to write the relationship as

$$\sum_{n=0}^{\infty} q^n f(x, n) - q^0 f(x, 0) = q \int_l^u \sum_{m=0}^{\infty} q^m f(x', m)p(x-x', \Delta t) dx' \quad (2.31)$$

to which we can see how the  $z$ -transform helps in handling the summation over all  $n$  by turning it into a product of transforms in the  $z$ -domain

$$\tilde{f}(x', q) = q \int_l^u \tilde{f}(x', q) p(x - x', \Delta t) dx' + f(x, 0) \quad (2.32)$$

Such a form enables the use of fourier-based methods; Fusai et al. (2016) demonstrated that the iteration over all monitoring dates can be bypassed using the Spitzer identities, originally detailed by Spitzer (1957) and expanded upon by Kemperman (1963). The identities require the Wiener-Hopf factorisations and decompositions, which were computed with the Plemelj-Sokhotsky relations and the Hilbert transform. Whilst Fusai et al. (2016) provide a comprehensive list of references, we ought to also mention the continuations of Phelan et al. (2018, 2019); Loveless et al. (2023). After such calculations, the inverse  $z$ -transform (Section 2.2) of  $\tilde{f}(x, q)$  is needed to translate the representation back to the time-domain probabilities. Thus, the focus of this work on the inverse  $z$ -transform not only underscores its mathematical importance but also highlights its role in bridging theoretical constructs with real-world financial applications.



## Chapter 3

# Experiment

Having established the theoretical foundation, we now turn our attention to the practical implementation of methods proposed by Abate and Whitt (1992a,b) and Cavers (1978). Whilst the former is based on a circular contour, we also explore the idea laid out by Boyarchenko and Levendorskiĭ (2022) in sampling the  $z$ -transform on a sinh-deformation.

This chapter outlines the experimental setup, implementation and evaluation of the methods. We measure our results in terms of accuracy and computational efficiency against well-known transformation pairs (Table 3.1) in an attempt to provide a thorough comparison of the methods to determine the most effective approach.

Function	$x(t)$	$X(z)$
Heaviside Step	1	$z/(z-1)$
Polynomial	$t$	$z/(z-1)^2$
Decaying Exp	$e^{-at}$	$1/(1 - \frac{\exp(-a\Delta t)}{z})$
Sinusoidal	$\sin(\omega t)$	$(\frac{\sin(w\Delta t)}{z})/(1 - \frac{2\cos(w\Delta t)}{z} + \frac{1}{z^2})$

Table 3.1: List of Transform Pairs

### 3.1 Circular Contour

“The inverse at point  $T$  can be obtained from the contour integral where  $C$  is a counter-clockwise closed path encircling the origin and entirely in the region of convergence” (Horváth et al., 2020). A circular contour is commonly used to solve Eq. (2.14), defined as  $z = re^{i\theta}$ , however the setting of  $r$  is important as shown for Example (1) in Figure (3.1).

We thus formulate an algorithmic approach to the implementations of the NIZT outlined in Section (2.2.1) and Section (2.2.2). For each of the methods, we outline a series of experiments and validation tests to carry out.

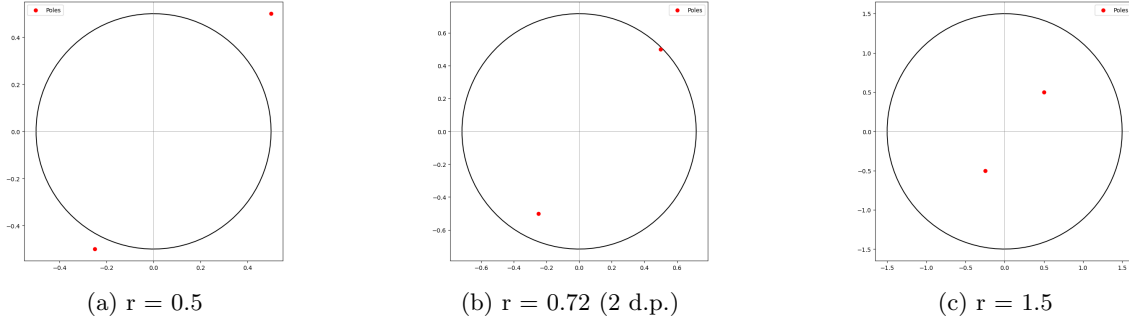


Figure 3.1: Displaying the effect of the radius  $r$  on a circular contour ( $n = 100$ ). The red dots represent the poles of  $H(z)$  in Example 1 with (a) including the poles within the ROC, (b) excluding the poles but inclusive of the unit circle, and (c) excluding the poles but not stable.

### 3.1.1 Abate and Whitt 1992

Given the correct setup, we will require, as input, just the function to be inverted,  $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ , and the point  $n \in \mathbb{R}$ . The implementation of such a function follows very easily from Eq. (2.16) as shown in the pseudocode below:

---

#### Algorithm 2 Implementation of Equation 2.16

---

```

1: procedure ABATEWHITT( $\tilde{f}, n$ )
2:    $\lambda \leftarrow x \in \mathbb{R}^+$ 
3:    $\rho \leftarrow 10^{-\lambda/(2 \cdot n)}$ 
4:   summation  $\leftarrow 0$ 
5:   for  $k \leftarrow 1$  to  $n - 1$  do
6:      $z \leftarrow \frac{1}{\rho \cdot \exp(i \cdot k \cdot \pi / n)}$ 
7:     sum  $\leftarrow$  sum  $+$   $(-1)^k \cdot \text{Re}(\tilde{f}(z))$ 
8:   end for
9:   sum  $\leftarrow \tilde{f}(\frac{1}{\rho}) + 2 \cdot$  sum  $+$   $(-1)^n \cdot \tilde{f}(\frac{1}{-\rho})$ 
10:  return  $\frac{\text{sum}}{2 \cdot n \cdot \rho^n}$ 
11: end procedure

```

---

### Computational Accuracy

Abate and Whitt (1992a,b) showed that the absolute error,  $|\epsilon|$ , of their implementation is bounded by

$$|\epsilon| = \frac{1}{r^N - 1} \approx \frac{1}{r^N}, \quad (3.1)$$

where the approximation holds for large  $r^N$ . As discussed in Section (2.2.1), setting  $r = 10^{-\lambda/2n}$  yields an accuracy of  $10^{-\lambda}$ , *i.e.*,  $\lambda$  significant digits. However, we have that if the machine accuracy is  $\epsilon_m = 10^{-\lambda_m}$ , the bound falls apart around  $\lambda > \frac{2}{3}\lambda_m$  (Abate and Whitt, 1992a, Remark 5.8). Loveless et al. (2023) states that “with IEEE 754 double precision representation of floating point numbers,  $\epsilon_m = 2^{-52} \approx 10^{-15.65}$ , and thus it makes little sense to set  $\lambda > 10.5$ , which yields the IZT up to about 10 or 11 significant digits”.

Thus, we investigate the effect of different values of  $\lambda$  on the accuracy of the Abate-Whitt method when applied to the heaviside step and polynomial transform pairs (Table 3.1). We expect to observe an increase in accuracy as  $\lambda$  increases, up to a certain point where the limitations of machine precision and rounding errors become dominant.

## Validity of Parameter Space

### Computational Speed

#### 3.1.2 Cavers 1978

## 3.2 Sinh Deformation

A circular contour integral is commonly used for approximating the inverse  $z$ -transform as we've seen in Section (2.2.1) and (2.2.2). However, this does not mean it may be the best option. Boyarchenko and Levendorskiĭ (2022) propose a new method for a numerical evaluation of Eq. (2.14) by deforming the commonly used circular contour  $\{z = re^{i\theta} | -\pi < \theta < +\pi\}$  through the conformal mapping:

$$\xi(y) = \sigma + ib \sinh(i\theta + y), \quad (3.2)$$

where parameters are chosen such that  $\sigma \in \mathbb{R}$ ,  $b > 0$ , and  $\theta$  is restricted to the interval  $(-\pi/2, \pi/2)$ . The unique selection of parameters allows the contour to be tailored specifically to the characteristics of  $X(z)$ . The authors state that the conformal mapping alleviates typical errors which are discussed in the relevant texts (Boyarchenko and Levendorskiĭ, 2014, 2019; Schmelzer and Trefethen, 2007). Applying a change of variables to Eq. (2.14) yields

$$x(n) = \int_{\mathbb{R}} \frac{b}{2\pi} \xi(y)^{-n-1} \cosh(i\theta + y) X(\xi(y)) dy. \quad (3.3)$$

By denoting the integrand as  $f_n(y)$  and approximating the application of the infinite trapezoidal, we have

$$x(n) \approx 2\epsilon \operatorname{Re} \left( \sum_{j=0}^{M_0} f_n(j\epsilon) (1 - \delta_{j0}/2) \right) \quad (3.4)$$

to look  
at these  
"typical"  
errors

need to look more into it

- mention sufficient conditions under which this works (next paper)
- talk about why he does this

### 3.2.1 Deforming the Contour

In an attempt to recreate the unit circle from the conformal mapping (3.2), we first try to understand the effects of the parameters on the contour. We plot the contour for different values of  $\sigma$ ,  $b$  and  $y$  to observe the deformation.

In relation to the base contour (Figure 3.2a), we observe that increasing  $\sigma$  shifts the contour to the right (Figure 3.2b), while decreasing  $\sigma$  shifts the contour to the left (Figure 3.2c). The parameter  $b$  acts as a scale factor on the sinh deformation as shown in Figure 3.2d and 3.2e. Similarly,  $y$  acts as a scale factor but with a more pronounced vertical effect as seen in Figure 3.2f and 3.2g. Mixing the parameters  $b$  and  $y$  results in a more complex deformation where the counter-opposing effects changes the contour in a non-linear manner (Figure 3.2h and 3.2i).

include difference of  $b$  and  $y$  parameters

### 3.2.2 Parameter Grid Search

- use of grid search to find optimal parameters

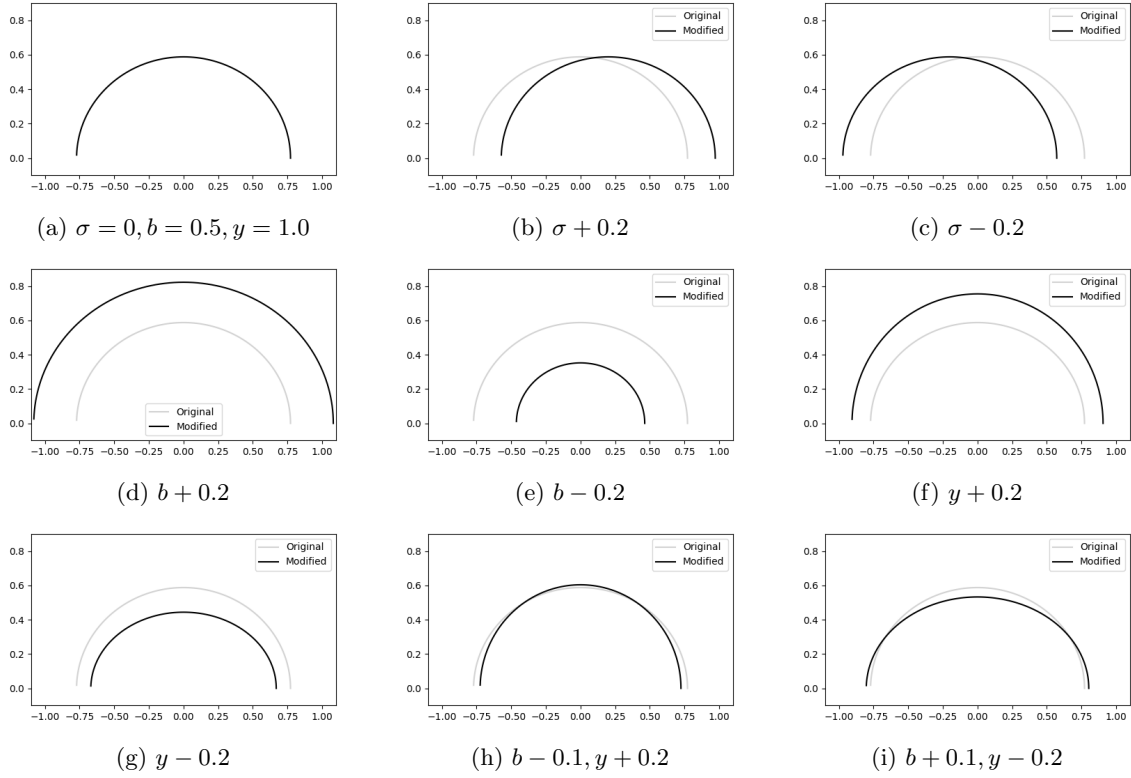
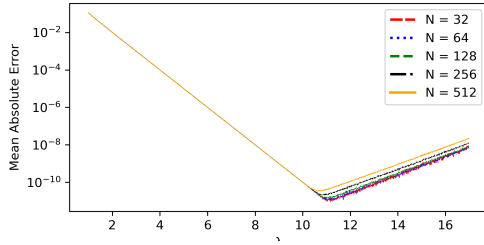


Figure 3.2: Displaying the effect of the parameters on the contour for the conformal mapping (3.2) with  $\{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$  and  $n = 100$  points. Plots (b, c, d, e, f, g, h, i) highlights the changes in parameter the effects on the contour in comparison to the base contour (a). The black line represents the created contour, with the grey contour representing the base contour with parameters  $\sigma = 0, b = 0.5, y = 1.0$ .

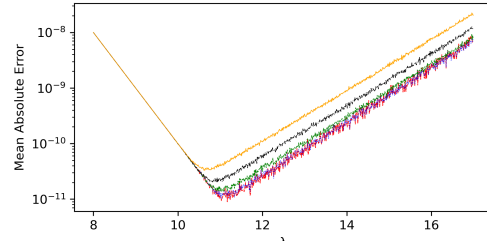
- different search criteria's (*e.g. area, x-axis perimeter, etc*)
- might also be useful to try find parameters that give the results?
- use of SGD and loss function

## Chapter 4

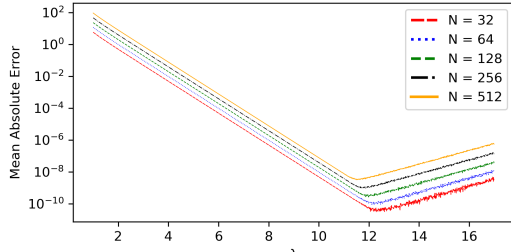
# Results



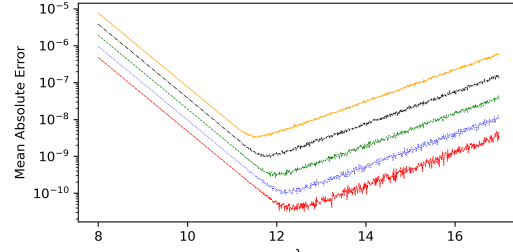
(a) Heaviside Step ( $1 \leq \lambda \leq 17$ )



(b) Heaviside Step ( $8 \leq \lambda \leq 17$ )



(c) Polynomial ( $1 \leq \lambda \leq 17$ )



(d) Polynomial ( $8 \leq \lambda \leq 17$ )

Figure 4.1: Mean absolute error (MAE) of the Abate-Whitt method for different values of  $\lambda$  when applied to the (a, b) Heaviside Step and (c, d) Polynomial transform pairs. The MAE is plotted on a logarithmic scale against  $\lambda$  values, where  $\Delta\lambda = 0.01$ . The different lines represent the MAE for subsets of  $n$  values:  $N = 32, 64, 128, 256, 512$ . **EXPERIMENT ABATE-WHITT COMPUTATIONAL ACCURACY**

## Chapter 5

# Conclusion

### 5.1 Summary

### 5.2 Future Work

#### 5.2.1 DFT vs FFT vs QFT

- Discrete Fourier Transform:  $O(N^2)$
- Fast Fourier Transform:  $O(N \log N)$
- Quantum Fourier Transform:  $O(\log(N)^2)$

#### 5.2.2 Parameter Selection

During the course of the project, there was no set method for selecting the parameters for Equation 3.2, hence, our effort of employing numerous techniques in an attempt to find the optimal parameters. However, as of recently at the time of writing, Boyarchenko and Levendorskiĭ (2024) have released a paper detailing parameter selection for the conformal mapping (3.2).

### 5.3 Acknowledgements

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# Appendices