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Beyond the Circle: Deforming Contours in Inverse Z-Transform

Final Year Project Report

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[1]**Disclaimer:** This report is submitted as part requirement for the MEng degree in Mathematical Computation at UCL. It is substantially the result of my own work except where explicitly indicated in the text. The report may be freely copied and distributed provided the source is explicitly acknowledged.

Abstract

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Background

"By definition, a complex number z is an ordered pair (x, y) of real numbers x and y, written $z = (x, y)$" [Kreyszig, 2010]. In practice, complex numbers are written in the form $z = x + iy$, where x and y are real numbers and i is the imaginary unit. The set of complex numbers is denoted by $\mathbb{C}$.

2.1 The $\mathscr{Z}$-Transform

Come up with a more catchy Z-Transform title

The z-transform is a transformation of a real or complex continuous time function $x(t)$, often used for discrete time signals and is commonly described as the discrete time Fourier transform (DTFT). Taking the Fourier transform of a sampled function results in

$$\mathcal{F}\left[x(t)\sum_{n=-\infty}^{\infty}\delta(t-n\Delta t)\right] = \int_{-\infty}^{\infty} x(t)\sum_{n=-\infty}^{\infty}\delta(t-n\Delta t)e^{-i\omega t}dt \tag{2.1}$$

$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\delta(t-n\Delta t)e^{-i\omega t}dt \tag{2.2}$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta t)e^{-i\omega nt} \tag{2.3}$$

where we make use of the sifting property of the delta function. If we normalise the sampling interval to 1, we get

$$\sum_{n=-\infty}^{\infty} x(n)e^{-in\omega} \tag{2.4}$$

The sequence $x(n)$ is sampled at discrete time intervals $t_n = n\triangle t$, where the sampling interval $\triangle t$ is the time between consecutive samples and the time index n numbers the samples. The DTFT is a periodic function of ω with period 2π and the existence of (2.4) relies on the condition of absolute summability of the sequence $x(n)$; all the terms must converge to a finite value. This requirement is mathematically expressed as

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \tag{2.5}$$

We can extend our analysis to the Z-transform, which generalizes the discrete time Fourier transform to the complex plane, not just the unit circle where $r = 1$. The Z-transform of a sequence $x(n)$ is formally given by

$$X(z) = \mathcal{Z}_{n\to z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{2.6}$$

For a complete description of z in the complex plane, we tend to its polar form $z = re^{i\theta}$, where r represents the magnitude of z and θ (often written as ω in the context of the unit circle for the DTFT) represents the angle of z with respect to the positive real axis.

In the analysis of causal systems - systems for which a time origin is defined and is illogical to consider signal values for negative time - the unilateral Z-transform is used. Unlike the bilateral Z-transform in (2.6), we sum from zero to positive infinity yielding

$$X(z) = \mathcal{Z}_{n\to z}[x(n)] = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.7}$$

The region within the complex z-plane where (2.7) converges is known as the Region of Convergence (ROC). The ROC is defined for the set of values of z for which the Z-transform is absolutely summable

$$\textbf{ROC} = \left\{ z : \sum_{n=0}^{\infty} |x(n)z^{-n}| < \infty \right\} \tag{2.8}$$

The ROC for causal sequences is typically the exterior of the outermost pole in the Z-plane, denoted as $|z| > a$. If we say that z_1 converges, then z_1 exists within the ROC. Thus, all z such that $|z| \geq |z_1|$ also converge. This region excludes the poles themselves, as the transform does not converge at those points. For the system to be *stable*, it's essential that the ROC includes the unit circle, $|z| = 1$. Thus, all poles must lie within the unit circle [Loveless and Germano, 2021].

2.1.1 Probability Generating Functions

May be better to move within Abate and Whitt's method

2.2 The Inverse $\mathcal{Z}$-Transform

The inverse Z-transform aims to find the n-th value of the sequence $x(n)$ given the Z-transform $X(z)$. This is commonly defined as a Cauchy integral around a contour C in the complex plane. The contour C is a counter-clockwise closed path that encloses the region of convergence (ROC). The inverse Z-transform is formally given by

$$x(n) = \mathcal{Z}^{-1}_{z\to n}[X(z)] = \frac{1}{2\pi i} \oint_C X(z)z^{n-1}dz \tag{2.9}$$

In real-world applications, we often require numerical approximation due to computational challenges posed by the Cauchy integral formula. Such approximations enable the effective analysis and processing of complex signals within various technological and financial systems.

think about how to make a smooth transition from IZT to NIZT (and how we only discuss contour integration methods given the nature of this project)

2.2.1 Abate and Whitt 1992

As previously discussed in Section (2.1.1), probability generating functions are an alternative form to the Z-transform by setting $q = 1/z$. This is coherent with the definition for Abate and Whitt's approximate inversion formula based upon a Fourier series approach [Abate and Whitt, 1992a, Abate and Whitt, 1992b]. The authors approximate (2.9) numerically by applying a trapezoidal rule

$$x(n) \approx \frac{1}{2nr^n}\left(X(r) + 2\sum_{k=1}^{n}(-1)^k \mathrm{Re}(X(re^{\frac{ik\pi}{n}})) + (-1)^n X(-r)\right) \tag{2.10}$$

The parameter r is used to control the error; setting $r = 10^{-\lambda/2n}$ yields an accuracy bound of $10^{-\lambda}$.

2.2.2 Cavers 1978

2.2.3 Series acceleration techniques

2.3 Discrete Pricing Options

2.3.1 Lookback and Barrier Options

2.4 Optimization Techniques

2.4.1 Stochastic Gradient Descent

2.4.2 Bayesian Optimization

Chapter 3

Experiment

Finding different parameters to use for the experiment making use of Machine Learning techniques.

Chapter 4

Results

Chapter 5

Conclusion

5.1 Summary

5.2 Future Work

5.3 Acknowledgements

Bibliography

[Abate and Whitt, 1992a] Abate, J. and Whitt, W. (1992a). The fourier-series method for inverting transforms of probability distributions. *Queueing systems*, 10:5–87.

[Abate and Whitt, 1992b] Abate, J. and Whitt, W. (1992b). Numerical inversion of probability generating functions. *Operations Research Letters*, 12(4):245–251.

[Kreyszig, 2010] Kreyszig, E. (2010). *Advanced Engineering Mathematics*. John Wiley & Sons.

[Loveless and Germano, 2021] Loveless, B. and Germano, G. (2021). Review of numerical inversion techniques of the z-transform. Working Paper.

Appendices

Appendix A

Initial Project Plan

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Numerical Benchmarking on Inverse Z-Transform and Its Uses in Discrete Pricing Options

Project Plan

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Chapter 1

Aims and Objectives

1.1 Aims

We aim to understand a new efficient method for numerical evaluation of the inverse Z-transform, which states to be faster and more accurate than the standard trapezoid rule. A specific area of applying this method would be to the pricing of discretely monitored exotic options, such as lookback and barrier options, and see how it compares to other methods; Abate and Whitt's approach, C. Cavers' method with Euler, Shanks and epsilon accelerations, etc.

1.2 Objectives

- Understanding Levendorskii's inverse Z-transform and the common numerical evaluation methods
- Implementing the function as a code
- Numerical benchmarking; average error, maximum error and CPU time
- Exploring its uses in discrete pricing options

1.3 Deliverables

- numerical benchmarking results to add to '*Review of numerical inversion techniques of the z-transform'* by Loveless and Germano
- results and implementation in regards to discrete pricing options (*Accurate numerical inverse z-transform and it's use in the Fourier-z pricing of discretely monitored path-dependent options* by Loveless, Phelan and Germano)

Chapter 2

Work Plan

2.1 Project Start → 30$^{\text{th}}$ November '23

- background reading on complex numbers & contour integration based methods, fourier transform, z-transform and its inverse, numerical approaches to inverse z-transform and pricing options (barrier and lookback options)
- coding implementation of Levendorskii's inverse z-transform

2.2 1$^{\text{st}}$ December '23 → 24$^{\text{th}}$ January '24

- preliminary research on Loveless' and Germano's *'Review of numerical inversion techniques of the z-transform'*
- understanding the other methods; AW, C, CEuler, CShanks and CEpsilon
- going over the different functions; Heaviside Step, Polynomial, Decaying Exp, Sinusodial
- reviewing the code for numerical benchmarking
- implementing it for Levendorskii's method
- begin work on interim report

2.3 24$^{\text{th}}$ January '24 → 15$^{\text{th}}$ March '24

- preliminary recap on discrete pricing options (barrier and lookback options) and the need for z-transform
- use-case in discrete pricing options
- start work on project report; however, to be worked on throughout the year/stages

2.4 5$^{\text{th}}$ March '24 → 26$^{\text{th}}$ April '24

- extra time to deal with any unexpected problems or delays
- final touches