Beyond the Circle: Deforming Contours in Inverse Z-Transform

Final Year Project Report
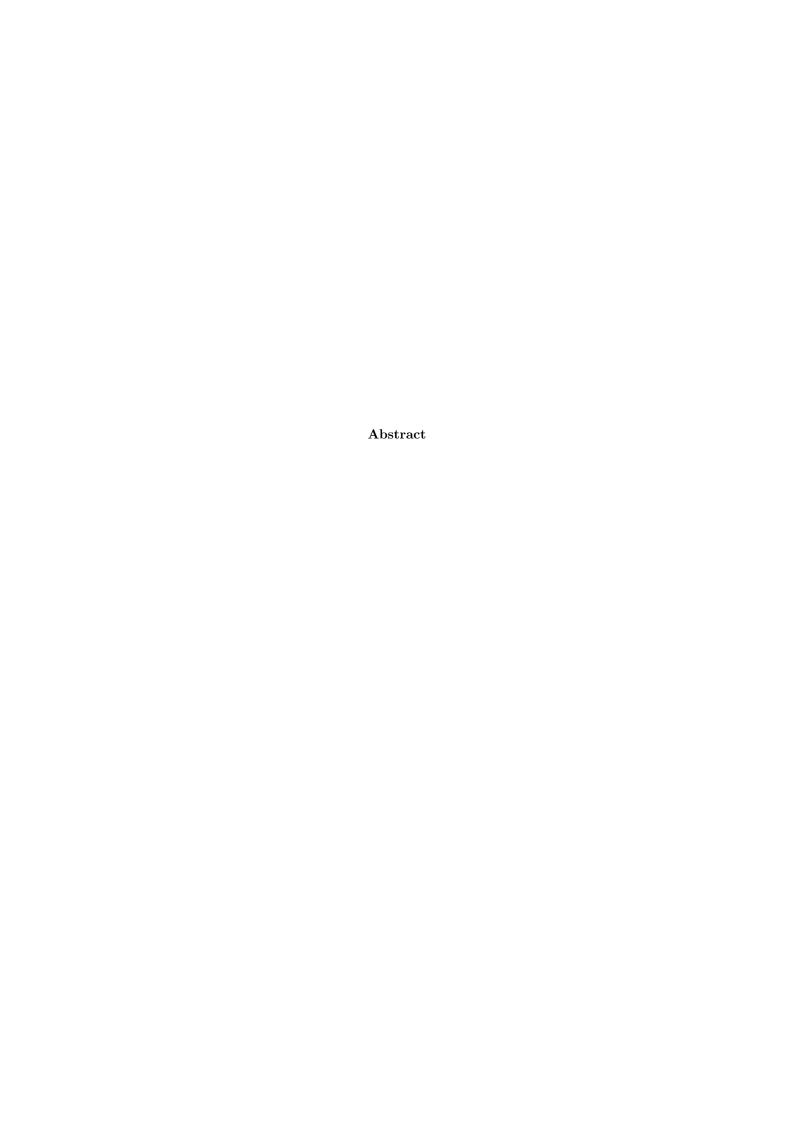
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Abstract

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Introduction

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Background

"By definition, a complex number z is an ordered pair (x, y) of real numbers x and y, written $z = (x, y)$" [Kreyszig, 2010]. In practice, complex numbers are written in the form $z = x + iy$, where x and y are real numbers and i is the imaginary unit. The set of complex numbers is denoted by $\mathbb{C}$.

2.1 The $\mathcal{Z}$-Transform

Come up with a more catchy Z-Transform title

The z-transform is a transformation of a real or complex continuous time function $x(t)$, often used for discrete time signals and is commonly described as the discrete time Fourier transform (DTFT). Taking the Fourier transform of a sampled function results in

$$\mathcal{F}\left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\triangle t)\right] = \int_{-\infty}^{\infty} x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\triangle t)e^{-i\omega t}dt \tag{2.1}$$

We can then make use of the sifting property of the delta function.

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\delta(t - n\triangle t)e^{-i\omega t}dt \tag{2.2}$$

$$= \sum_{n=-\infty}^{\infty} x(n\triangle t)e^{-i\omega nt} \tag{2.3}$$

If we normalise the sampling interval to 1, we get

$$\sum_{n=-\infty}^{\infty} x(n)e^{-in\omega} \tag{2.4}$$

The sequence $x(n)$ is sampled at discrete time intervals $t_n = n\triangle t$, where the sampling interval $\triangle t$ is the time between consecutive samples and the time index n numbers the samples. The DTFT is a periodic function of ω with period 2π. The existence of the DTFT relies on the condition of absolute summability of the sequence $x(n)$; all the terms must converge to a finite value. This requirement is mathematically expressed as

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \tag{2.5}$$

We can extend our analysis to the Z-transform, which generalizes the discrete time Fourier transform to the complex plane, not just the unit circle where $r = 1$. The Z-transform of a sequence $x(n)$ is formally given by

$$X(z) = \mathcal{Z}_{n \to z}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{2.6}$$

For a complete description of z in the complex plane, we tend to its polar form $z = re^{i\theta}$, where r represents the magnitude of z and θ (often written as ω in the context of the unit circle for the DTFT) represents the angle of z with respect to the positive real axis.

In the analysis of causal systems - systems for which a time origin is defined and is illogical to consider signal values for negative time - the unilateral Z-transform is used. Unlike the bilateral Z-transform in (2.6), summing from zero to positive infinity yields the unilateral Z-transform defined to be

$$X(z) = \mathcal{Z}_{n \to z}[x(n)] = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.7}$$

The region within the complex z-plane where (2.7) converges is known as the Region of Convergence (ROC). If we say that z_1 converges, then z_1 exists within the ROC. Thus, all z such that $|z| > |z_1|$ also converge. Strictly speaking, the ROC is defined for the set of values of z for which the Z-transform is absolutely summable

$$\mathbf{ROC} = \left\{ z : \sum_{n=0}^{\infty} |x(n)z^{-n}| < \infty \right\} \tag{2.8}$$

The ROC for causal sequences is typically the exterior of the outermost pole in the Z-plane, denoted as $|z| > a$. This region excludes the poles themselves, as the transform does not converge at those points. For the system to be *stable*, it's essential that the ROC includes the unit circle, $|z| = 1$. Thus, all poles must lie within the unit circle [Loveless and Germano, 2021].

2.1.1   Probability Generating Functions

May be better to move within Abate and Whitt's method

2.2   The Inverse $\mathcal{Z}$-Transform

The inverse Z-transform aims to find the n-th value of the sequence $x(n)$ given the Z-transform $X(z)$. This is commonly defined as a Cauchy integral around a contour C in the complex plane. The contour C is a counter-clockwise closed path that encloses the region of convergence (ROC). The inverse Z-transform is formally given by

$$x(n) = \mathcal{Z}_{z \to n}^{-1}[X(z)] = \frac{1}{2\pi i} \oint_C X(z)z^{n-1}dz \tag{2.9}$$

Chapter 3

Experiment

Finding different parameters to use for the experiment making use of Machine Learning techniques.

Chapter 4

Results

Chapter 5

Conclusion

5.1 Summary

5.2 Future Work

5.3 Acknowledgements

Bibliography

[Kreyszig, 2010] Kreyszig, E. (2010). *Advanced Engineering Mathematics*. John Wiley & Sons.

[Loveless and Germano, 2021] Loveless, B. and Germano, G. (2021). Review of numerical inversion techniques of the z-transform. Working Paper.

Appendices

Appendix A

Initial Project Plan

Numerical Benchmarking on Inverse Z-Transform and Its Uses in Discrete Pricing Options

Project Plan

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Chapter 1

Aims and Objectives

1.1 Aims

We aim to understand a new efficient method for numerical evaluation of the inverse Z-transform, which states to be faster and more accurate than the standard trapezoid rule. A specific area of applying this method would be to the pricing of discretely monitored exotic options, such as lookback and barrier options, and see how it compares to other methods; Abate and Whitt's approach, C. Cavers' method with Euler, Shanks and epsilon accelerations, etc.

1.2 Objectives

- Understanding Levendorskii's inverse Z-transform and the common numerical evaluation methods

- Implementing the function as a code

- Numerical benchmarking; average error, maximum error and CPU time

- Exploring its uses in discrete pricing options

1.3 Deliverables

- numerical benchmarking results to add to '*Review of numerical inversion techniques of the z-transform*' by Loveless and Germano

- results and implementation in regards to discrete pricing options (*Accurate numerical inverse z-transform and it's use in the Fourier-z pricing of discretely monitored path-dependent options* by Loveless, Phelan and Germano)

Chapter 2

Work Plan

2.1 Project Start $\rightarrow$ 30$^{\text{th}}$ November '23

- background reading on complex numbers & contour integration based methods, fourier transform, z-transform and its inverse, numerical approaches to inverse z-transform and pricing options (barrier and lookback options)

- coding implementation of Levendorskii's inverse z-transform

2.2 1$^{\text{st}}$ December '23 $\rightarrow$ 24$^{\text{th}}$ January '24

- preliminary research on Loveless' and Germano's *'Review of numerical inversion techniques of the z-transform'*

- understanding the other methods; AW, C, CEuler, CShanks and CEpsilon

- going over the different functions; Heaviside Step, Polynomial, Decaying Exp, Sinusodial

- reviewing the code for numerical benchmarking

- implementing it for Levendorskii's method

- begin work on interim report

2.3 24$^{\text{th}}$ January '24 $\rightarrow$ 15$^{\text{th}}$ March '24

- preliminary recap on discrete pricing options (barrier and lookback options) and the need for z-transform

- use-case in discrete pricing options

- start work on project report; however, to be worked on throughout the year/stages

2.4 5$^{\text{th}}$ March '24 $\rightarrow$ 26$^{\text{th}}$ April '24

- extra time to deal with any unexpected problems or delays

- final touches